Derived Series and Descending/Ascending Central Series

Def: Given a group G & $a, b \in G$, the commutator of a and b is $[a, b] = aba^{-1}b^{-1}$. The subgroup $[G, G] \leq G$ generated by all commutators is called the commutator subgroup of G (the book denotes $[G, G]$ instead by C, but the former is a useful notation for us).

Ex: Consider S_3. We have

$$[(1,3,2),(1,2)] = (1,3,2)(1,2)(1,3,2)^{-1}(1,2)^{-1} = (1,3,2)(1,2)(1,2,3)(1,2) = (1,2,3)$$

so $A_3 = \langle(1,2,3)\rangle \leq [S_3, S_3]$. Note that $S_3/A_3 \simeq \mathbb{Z}_2$, which is abelian, so we must have $[S_3, S_3] \leq A_3$ (by the theorem proved in class and in the book). Thus $[S_3, S_3] = A_3$.

Ex: Since A_3 is abelian, we have

$$[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} = \tau\sigma\sigma^{-1}\tau^{-1} = \iota$$

for all $\sigma, \tau \in A_3$. Therefore, the smallest subgroup of A_3 containing all commutators is $\{\iota\}$. Thus $[A_3, A_3] = \{\iota\}$.

The preceding example illustrates the following proposition.

Prop: If G is an abelian group with identity element e, then $[G, G] = \{e\}$.

So we see that there is a natural sequence of subgroups

$$S_3 \geq [S_3, S_3] = A_3 \geq [A_3, A_3] = \{\iota\}.$$

Def: Let G be a group. We set $G^{(0)} := G$ and inductively define $G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$ for $n \geq 1$, yielding a "series" of subgroups:

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots,$$

where each subgroup is normal in its predecessor (but not necessarily in G). This is called the derived series of G. If $G^{(n)} = \{e\}$ for some $n \geq 0$, then G is said to be solvable.

Ex: S_3 is solvable since $(S_3)^{(2)} = \{\iota\}$.

Def: Given a group G and $H \leq G$, we define $[H, G]$ to be the subgroup of G generated by all commutators of the form $[h, g]$ with $h \in H$ and $g \in G$.

Def: Let G be a group. We set $G_1 := G$ and inductively define $G_n := [G_{n-1}, G]$ for $n > 1$, yielding a "series" of subgroups:

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots,$$

where each subgroup is normal in its predecessor (but not necessarily in G). [Note: can you prove this?] This is called the descending central series of G (or lower central series). If $G_n = \{e\}$ for some $n \geq 1$, then G is said to be nilpotent.

Ex: Consider S_3. Based on our previous computations, $(S_3)_2 = [S_3, S_3] = A_3$. When we did this computation, we noted that $[(1,3,2),(1,2)] = (1,2,3)$. But since $(1,3,2) \in A_3$ and $(1,2) \in S_3$, this shows that $(1,2,3) \in [A_3, S_3]$. Since $[A_3, S_3] \leq A_3$ and $(1,2,3)$ generates A_3, this shows that $(S_3)_3 = [A_3, S_3] = A_3$. Similarly, $(S_3)_n = A_3$ for all $n \geq 3$. So the descending central series of S_3 is

$$S_3 \geq A_3 \geq A_3 \geq A_3 \geq \cdots.$$

It follows that S_3 is not nilpotent.

Def: Let G be a group. We set $Z_0(G) := \{e\}$ (where e is the identity element of G) and inductively define

$$\begin{aligned} Z_n(G) &:= \{g \in G \mid [g,h] \in Z_{n-1}(G)\ \forall h \in G\} \\ &= \pi_{n-1}^{-1}\left[Z(G/Z_{n-1}(G))\right] \end{aligned}$$

(where $\pi_{n-1} : G \to G/Z_{n-1}(G)$ is the natural group homomorphism) for $n > 1$, yielding a "series" of subgroups:

$$\{e\} = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$$

where each subgroup is normal in the group which follows it (but not necessarily in G). [Note: can you prove this?] This is called the ascending central series of G (or upper central series).

It's worth noting here that if G is a group, then $Z_1(G) = Z(G)$, the center of G. It follows that if $Z(G) = \{e\}$, then $Z_n(G) = \{e\}$ for all $n \geq 0$.

Ex: Since the center of S_3 is $Z(S_3) = \{\iota\}$, the ascending central series of S_3 is

$$\{\iota\} \leq \{\iota\} \leq \{\iota\} \leq \cdots .$$