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RJ Acuna

(862079740)

Section 14

In Exercises 1 through 8, find the order of the given factor group.

2 $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)$

slu.

$2 \cdot 2 = 4 \equiv 0$ (mod 4) $\implies$ $|2| = 2$ in $\mathbb{Z}_4$

$2 \cdot 6 = 12 \equiv 0$ (mod 12) $\implies$ $|2| = 6$ in $\mathbb{Z}_4$

$\implies |\langle 2\rangle \times \langle 2\rangle| = 2 \cdot 6 = 12 \quad \Diamond$

$|\langle 2\rangle \times \langle 2\rangle|12 \wedge |\mathbb{Z}_4 \times \mathbb{Z}_{12}| = 4 \cdot 12 \implies |(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)| = 4$

6 $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$

$4 \cdot 3 = 12 \equiv 0$ (mod 12) $\implies$ $|4| = 3$ in $\mathbb{Z}_{12}$

$3 \cdot 6 = 18 \equiv 0$ (mod 12) $\implies$ $|3| = 6$ in $\mathbb{Z}_{18}$

lcm$(3,6) = 6 \implies |\langle(4,3)\rangle| = 6$

$|\langle(4,3)\rangle| = 6 \wedge |\mathbb{Z}_{12} \times \mathbb{Z}_{18}| = 12 \cdot 18 = 2 \cdot 6 \cdot 18 \implies |(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle| = 2 \cdot 18 = 36 \quad \Diamond$

In Exercises 9 through 15, give the order of the element in the factor group.

10 $26 + \langle 12\rangle$ in $\mathbb{Z}_{60}/\langle 12\rangle$

slu.

$26 = 12 \cdot 2 + 2 \implies 26 + \langle 12\rangle = 2 + \langle 12\rangle$

$2 \cdot 6 = 12 \implies (2 + \langle 12\rangle)^6 = 2 \cdot 6 + \langle 12\rangle = 12 + \langle 12\rangle = \langle 12\rangle \implies |26 + \langle 12\rangle| = 6$ in $\mathbb{Z}_{60}/\langle 12\rangle \quad \Diamond$

14 $(3,3) + \langle(1,1)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\langle(1,1)\rangle$

slu.

$(3,3) = 3(1,1) \implies (3,3) \in \langle(1,1)\rangle \implies (3,3) + \langle(1,1)\rangle = \langle(1,1)\rangle$

$\implies |(3,3) + \langle(1,1)\rangle| = 1$ in $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\langle(1,1)\rangle \quad \Diamond$

16 Compute $i_{\rho_1}[H]$ for the subgroup $H = \{\rho_0, \mu_1\}$ of the group S_3 of Example 8.7.

slu.

$\rho_1\rho_0\rho_1^{-1} = \rho_1\rho_1^{-1} = \rho_0$

and

$\rho_1\mu_1\rho_1^{-1} = \rho_1\mu_1\rho_2 = \mu_3\rho_2 = \mu_2$

so, $i_{\rho_1}[H] = \{\rho_0, \mu_2\} \quad \Diamond$

24 Show A_n is a normal subgroup of S_n and compute S_n/A_n; that is find a known group to which S_n/A_n is isomorphic.

pf. $|S_n| = n! \wedge |A_n| = \frac{n!}{2} \implies |S_n/A_n| = \frac{n!}{\frac{n!}{2}} = 2 \implies S_n/A_n \simeq \mathbb{Z}_2$ ■

26 Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion free.

pf.

Note that $T \trianglelefteq G \iff \forall g \in G, t \in T, gtg^{-1} \in T$.

G is abelian, and $T \leq G$. Let $g \in G$,and $t \in T$, then $gtg^{-1} = gg^{-1}t = et = t \in T \implies T \trianglelefteq G$

Suppose for the sake of contradiction $\exists g \in G, g \notin T, n \in \mathbb{Z} : g \neq e \wedge (gT)^n = T$.

$(gT)^n = g^nT = T \implies g^n = e \implies |g| = n$

Which is a contradiction, because T contains all of the elements of G that have finite order.

So, the only element of G/T of finite order is the coset T.

Therefore, G/T is torsion-free ■

30 Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

pf. $H \trianglelefteq G \wedge m = (G : H) = |G/H| \in \mathbb{Z} \implies \forall aH \in G/H, \quad (aH)^m = a^mH = H \implies a^m \in H$ ■

34 Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

pf.

WTS $|G| = m \in \mathbb{Z} \wedge (\exists! H \leq G \wedge n \in \mathbb{Z}) : |H| = n \implies H \trianglelefteq G$.

WTS $\forall g \in G, \forall h \in H, \ ghg^{-1} \in H \iff H \trianglelefteq G$

Let $g \in G$, and $h \in H$, then $(ghg^{-1})^n = ghg^{-1}ghg^{-1} \cdots ghg^{-1} = ghehehe \cdots hehg^{-1} = gh^ng^{-1}$.

$|H| = n \implies h^n = e \implies (ghg^{-1})^n = e \implies ghg^{-1} \in H$ since $\exists! H \leq G : |H| = n$ ■

Section 15

In Exercises 1 through 12, classify the given group according to the fundamental theorem of finitely generated abelian groups.

4. $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$

$1 \cdot 4 = 4 \equiv 0$ (mod 4) $\implies |1| = 4$ in $\mathbb{Z}_4$

and $2 \cdot 4 = 8 \equiv 0$ (mod 8) $\implies |2| = 4$ in $\mathbb{Z}_8$

lcm$(4,4) = 4 \implies |\langle(1,2)\rangle| = 4$ in $\mathbb{Z}_4 \times \mathbb{Z}_8$

$|\langle(1,2)\rangle| = 4$ and $|\mathbb{Z}_4 \times \mathbb{Z}_8| = 4 \cdot 8 \implies |(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle| = 8 = 2^3$

So, there are three possibilities $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$\langle(1,2)\rangle = \{(1,2),(2,4),(3,6),(0,0)\}$

$4(1,0) = (0,0) \in \langle(1,2)\rangle \implies |(1,0) + \langle(1,2)\rangle| = 4$

$8(0,1) = (0,0) \in \langle(1,2)\rangle \implies |(0,1) + \langle(1,2)\rangle| = 8$

Since $\mathbb{Z}_8$ is the only choice that has an element of order 8, it follows that,$(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle \simeq \mathbb{Z}_8$ ♦

8 $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle$

slu.

$|(1,1,1)| = \aleph_0$ so we can't do simple counting arguments.

However, even though we don't have the notion of dimension. We can see that $\langle(1,1,1)\rangle$ is the one dimensional span of the vector $(1,1,1)$ in the $\mathbb{Z}^3$ vector space. Therefore, a reasonable assumption is that $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle \simeq \mathbb{Z}^2$.

$(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) = \langle(1,0,0),(1,1,0),(1,1,1)\rangle$

$$\begin{aligned}\forall w = (x,y,z) \in \mathbb{Z}^3 : (x,y,z) &= (x-z, y-z, 0) + z(1,1,1) \\ &= (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1) \\ &= (x-y+y-z+z, y-z+z, z) = w\end{aligned}$$

$(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle = \langle(1,0,0),(1,1,0),(1,1,1)\rangle/\langle(1,1,1)\rangle = \langle(1,0,0),(1,1,0)\rangle \simeq \mathbb{Z}^2 \quad \blacksquare$

10 $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8)/\langle(0,4,0)\rangle$

slu.

$\langle(0,4,0)\rangle \simeq 4\mathbb{Z} \implies (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8)/\langle(0,4,0)\rangle \simeq \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_8 \quad \blacksquare$

35 Let $\phi : G \to G'$ be a group homomorphism, and let N be a normal subgroup of G. Show that $\phi[N]$ is normal subgroup of $\phi[G]$.

pf.

$\forall g \in G, n \in \mathbb{N} : gng^{-1} \in N \implies \phi(gng^{-1}) = \phi(g)\phi(n)\phi(g^{-1}) = \phi(g)\phi(n)\phi(g)^{-1} \in \phi[N]$.

But, $\forall g \in G, \phi(g) \in \phi[G] \wedge \forall n \in N, \phi(n) \in N \implies \phi[N] \trianglelefteq \phi[G] \quad \blacksquare$

Additional Exercises

1 You showed on the last homework assignment that, for any $n \in \mathbb{Z}^+, SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$. To what group that we've seen in this class is $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ isomorphic? Prove your claim. [Note: Your answer shouldn't depend on n.]

pf.

$SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R}) \implies \forall G \in GL_n(\mathbb{R}), S \in SL_n(\mathbb{R}), GSG^{-1} \in GL_n(\mathbb{R})/SL_n(\mathbb{R})$.

det : $GL_n(\mathbb{R}) \twoheadrightarrow \mathbb{R}^*$ is a surjective homomorphism.

$\det(GSG^{-1}) = \det(G)\det(S)\det(G^{-1}) = \det(G)\det(G)^{-1}\det(S) = \det(S) = 1$.

$\det(GS) = \det(G)\det(S) = \det(G) = r \in \mathbb{R}^*$.

So, each coset of $GL_n(\mathbb{R})/SL_n(\mathbb{R})$, corresponds to the equivalence class of one $r \in \mathbb{R}^*$.

$\implies GL_n(\mathbb{R})/SL_n(\mathbb{R}) \simeq \mathbb{R}^* \quad \blacksquare$

2 Let G be a simple group and suppose $\phi : G \to G'$ is a homomorphism. Prove that ϕ is either the trivial homomorphism or a one-to-one map.

pf.

Ker$(\phi) \trianglelefteq G$ and G is a simple group.

Note that Ker(ϕ) is always normal. Since G is simple the only normal subgroups of G are $\{e\}$ and G.

If Ker$\phi = G$ then $\phi^{-1}[\{e'\}] = G$, so ϕ is the trivial homomorphism $G \to G'$.

If Ker$\phi = \{e\}$ then $\phi^{-1}[\{e'\}] = \{e\}$, so ϕ is injective $\quad \blacksquare$

3 Let G be a group and N a normal subgroup of G. Provide a bijection between the set of subgroups of G/N and the set of subgroups of G that contain N (i.e. all $H \leq G$ such that $N \subseteq H$). Prove that it is indeed a bijection.

pf.

By hint, consider $\pi : N \subseteq H_\alpha \leq G = S \to \cup_{\alpha \in \mathrm{A}} H_\alpha / N; H \mapsto H/N$

$\pi[S] = \cup_{\alpha \in \mathrm{A}} H_\alpha / N$, since for each H_β/N, there is a H_β that gets mapped to H_β/N.

$\{e\} = N/N$ is the identity element of $\cup H_\alpha/N$.

$\mathrm{Ker}(\pi) = \pi^{-1}[\{e\}] = \{H \in S | \pi(H) = \{e\}\}$

Let $H \in S$, then $\pi(H) = H/N = \{e\} \iff H = N$

$\implies \mathrm{Ker}(\pi) = N$

So, π is a bijection $\blacksquare$