Given an indexed family of sets $\{X_i\}_{i \in I}$, the Cartesian product is defined as,

$$\prod_{i \in I} X_i := \{f : I \to \bigcup_{i \in I} X_i | (\forall i \in I)(f(i) \in X_i)\}$$

.

That is to say that each $i \in I$, has a corresponding value $f(i) \in X_i$. We define the projection maps as an indexed family of functions $\{\pi_j : \prod_{i \in I} X_i \to X_j; f \mapsto f(j)\}_{j \in I}$. For some element $\phi \in \prod_{i \in I} X_i$, $\pi_k(\phi) = \phi(k)$ gives the component of ϕ in the k direction. These family of projection maps is also sometimes called the coordinate functions.

Thinking about the coordinate functions we can give notation for the elements of the Cartesian product. Then $\psi \in \prod_{i \in I} X_i$, can be interpreted as tuple $(x_i)_{i \in I}$, where $x_j := \pi_j(\psi)$ is the value of the j^{th} coordinate of $(x_i)_{i \in I}$. In particular if I is at most countable, then we can write $(x_i)_{i \in I}$ like $(x_1, x_2, x_3, \dots)$. When I is finite with cardinality n, we write $(x_1, x_2, \dots, x_n)$ and say it is an n-tuple.

If $X_i = X_j \forall i \neq j$, then we call the Cartesian product a Cartesian power and we label $X := X_i$ and write $\prod_{i \in I} X_i = X^I$. If I is finite with cardinality n, we write X^n.

An operation on a set S is a map $* : S^n \to S$. The size n of the input vector is the arity of the operation. We give special names to the first few naturals 1, 2, and 3-arity operations are called unary, binary, and ternary operations respectively. For operations with arity n we say they are n-ary.

Now, 0 arity operations are special, they are called nullary operations. First we need to give meaning to S^0. We know the only set with cardinality 0 is the empty set, so $I = \emptyset$. Plugging in we see,

$$X^0 = \{f : \emptyset \to \bigcup_{i \in \emptyset} X_i | (\forall i \in \emptyset)(f(i) \in X_i)\}$$

First let us examine the condition, $(\forall i \in \emptyset)(f(i) \in X_i)$. We can see there is nothing in the empty set, it follows that the statement does not impose any restrictions on f.

Now, $\bigcup_{i \in \emptyset} X_i = \emptyset \implies \forall f \in X^0 : f : \emptyset \to \emptyset$. Since functions are subsets of the Cartesian product of their domain and range. And, $\emptyset^2 = \emptyset \implies \forall f \in X^0 : f = \emptyset \implies X^0 = \{\emptyset\}$. As a consequence, we get that $X^0 = \{()\}$, where $()$ is the empty tuple.

So,$|X^0| = 1$. Therefore, all nullary operations $S^0 \to S$ are constant functions with values in S. So, the set of all nullary operations on S is in natural bijection with S.

We define the unary operation $+1 : \mathbb{N} \to \mathbb{N}; n \mapsto n \cup \{n\}$. As notation we write $n + 1$ for $+1(n)$. The natural numbers $\mathbb{N}$ is the set that satisfies the Peano axioms,

$$(I) 1 := \emptyset \in \mathbb{N}$$
$$(II)(\forall x \in \mathbb{N})(x + 1 \in \mathbb{N})$$
$$(III)(\forall x, y \in \mathbb{N})(x = y \iff x + 1 = y + 1)$$
$$(III)(\forall z \in \mathbb{N})(z + 1 \neq 1)$$

In most books, people when people drop the adjective for the arity of the operation it means that they are speaking of a binary operation. We will adopt said convention.

In Algebra when we talk about an algebraic structure we are talking about a pair $\langle S, \bullet \rangle$, where S is either a set, and $\bullet$ is a n-ary operation.

$\langle \mathbb{N}, u \rangle$, $u : \mathbb{N}^0 \to \mathbb{N}; u() = 1$, is one of the simple structure in the sense that the arity of u is less than 1. In that sense $\mathbb{N}$ is called a pointed set with base point 1.

$\langle \mathbb{N}, +1 \rangle$, is also simple. In that sense $\mathbb{N}$ is called an unary system.

$\langle \mathbb{N}, + \rangle$, $\langle \mathbb{N}, \cdot \rangle$, and $\langle \mathbb{N}, n^m \rangle$ are algebraic structures. Usually, if it is understood from context, we will refer to all of them by their underlying set $\mathbb{N}$. An algebraic structures with respect to an operation is called a magma. If the operation is associative, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, then it is called a semigroup. All of them are semigroups.

In an algebraic structure $\langle S, \bullet \rangle$. If,

$$(\exists e \in S)(\forall y \in S)(e \bullet y = y = y \bullet e)$$

then we say S has an identity element e, with respect to the operation $\bullet$.

Neither $\langle \mathbb{N}, + \rangle$, nor $\langle \mathbb{N}, n^m \rangle$ have identity elements, so they are just semigroups. However, 1 is the multiplicative identity of $\mathbb{N}$. A semigroup with an identity element is called a monoid. So, $\mathbb{N}$ is a monoid with respect to multiplication.