Homework 3 Solutions

Section 4:

40. Let $\langle G, \cdot \rangle$ be a group. Consider the binary operation $*$ on the set G defined by $a * b = b \cdot a$ for $a, b \in G$. Show that $\langle G, * \rangle$ is a group and that $\langle G, * \rangle$ is isomorphic to $\langle G, \cdot \rangle$.

    Proof. First, note that G is closed under $*$ since it is closed under $\cdot$: If $g, h \in G$, then $g * h = h \cdot g \in G$.

    To see that $*$ is associative on G, let $a, b, c \in G$. Then, since $\cdot$ is associative on G, we have
    $$(a * b) * c = c \cdot (a * b) = c \cdot (b \cdot a) = (c \cdot b) \cdot a = (b * c) \cdot a = a * (b * c).$$
    Therefore, $*$ is associative on G.

    Let e be the identity element for $\cdot$ on G. We claim that it is also an identity element for $*$ on G. Indeed, let $g \in G$. Then, since $e \cdot g = g = g \cdot e$, we have
    $$e * g = g \cdot e = g = e \cdot g = g * e.$$
    Therefore, e is an identity element for $*$ on G.

    To see that all elements of G have inverses with respect to $*$, let $g \in G$ and denote by g' the inverse of g with respect to $\cdot$. Then, since $g \cdot g' = e = g' \cdot g$, we have
    $$g * g' = g' \cdot g = e = g \cdot g' = g' * g.$$
    Therefore, g' is also an inverse for g with respect to $*$. Hence all elements of G have inverses with respect to $*$. We conclude that $\langle G, * \rangle$ is a group.

    Define a map $\phi : G \to G$ via $\phi(g) = g'$ for $g \in G$. We showed on the last assignment, that $(g')' = g$, so
    $$(\phi \circ \phi)(g) = \phi(\phi(g)) = \phi(g') = (g')' = g.$$
    Hence ϕ is its own inverse function and it is therefore bijective.

    We claim that it is also the case that $(g \cdot h)' = h' \cdot g'$ for $g, h \in G$. Indeed, we have
    $$(h' \cdot g') \cdot (g \cdot h) = h' \cdot (g' \cdot (g \cdot h)) = h' \cdot ((g' \cdot g) \cdot h) = h' \cdot (e \cdot h) = h' \cdot h = e$$
    and similarly, $(g \cdot h) * (h' \cdot g') = e$. By the uniqueness of inverses, $(g \cdot h)' = h' \cdot g'$, as desired. As a result, we have
    $$\phi(g \cdot h) = (g \cdot h)' = h' \cdot g' = \phi(g) * \phi(h).$$
    Therefore $\langle G, \cdot \rangle$ and $\langle G, * \rangle$ are isomorphic groups. $\square$

Section 5:

8. Determine whether the set of $n \times n$ (real) matrices with determinant 2 is a subgroup of $GL_n(\mathbb{R})$.

    Solution. The set is not a subgroup. One reason (among several) is that the $n \times n$ identity matrix has determinant 1 – not 2 – and thus is not in the set.

13. Determine whether the set of $n \times n$ (real) matrices A such that $A^T A = I_n$ is a subgroup of $GL_n(\mathbb{R})$. [These matrices are called **orthogonal**.]

    Solution. Yes, the set is a subgroup.

    Proof. Let $H = \{A \in GL_n(\mathbb{R}) \mid A^T A = I_n\}$.

    Given $A, B \in H$, we have $A^T A = I_n$ and $B^T B = I_n$, so
    $$(AB)^T (AB) = B^T A^T A B = B^T I_n B = B^T B = I_n.$$
    Since the product of invertible matrices is invertible, this shows that $AB \in H$ and so H is closed under matrix multiplication.

Next, note that $(I_n)^T = I_n$, so $(I_n)^T I_n = (I_n)^2 = I_n$, showing that $I_n \in H$.

Finally, to see that H contains the inverses (in $GL_n(\mathbb{R})$) of each of its elements, let $A \in H$. Then we have $A^T A = I_n$. Multiplying on the left by $(A^T)^{-1} = (A^{-1})^T$ and on the right by A^{-1}, we obtain $(A^{-1})^T A^{-1} = I_n$, showing that $A^{-1} \in H$.

As shown in class, these together imply that H is a subgroup of $GL_n(\mathbb{R})$. □

22. Describe all elements in the cyclic subgroup of $GL_2(\mathbb{R})$ generated by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Solution. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^2 = I_2$, so the elements of the cyclic subgroup of $GL_2(\mathbb{R})$ generated by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ are simply I_2 and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

23. Describe all elements in the cyclic subgroup of $GL_2(\mathbb{R})$ generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all $n \in \mathbb{Z}$, so these are the elements of the cyclic subgroup generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

35. Find the order of the cyclic subgroup of $GL_4(\mathbb{R})$ generated by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 = I_4$$

The cyclic subgroup of $GL_4(\mathbb{R})$ generated by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ has order 3.

45. Show that a nonempty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

Proof. Let H be a nonempty subset of a group G and let e be the identity element of G.
$(\Rightarrow)$ Suppose H is a subgroup of G. Then, for $a, b \in H$, we must have $b^{-1} \in H$ and H must be closed under the operation of G, so $ab^{-1} \in H$.
$(\Leftarrow)$ Suppose $ab^{-1} \in H$ for all $a, b \in H$.

Since H is nonempty, there is some element $a \in H$. Then $e = aa^{-1} \in H$, so H contains the identity element of G.

Further, since $e \in H$, then for any element $a \in H$, we have $a^{-1} = ea^{-1} \in H$. So H contains the inverses of each of its elements.

Finally, given $a, b \in H$, we know that $b^{-1} \in H$, so $ab = a(b^{-1})^{-1} \in H$. Hence H is closed under the operation of G.

As shown in class, these together imply that H is a subgroup of G. $\square$

54. For sets H and K, we define the **intersection** $H \cap K$ by

$$H \cap K = \{x \mid x \in H \text{ and } x \in K\}.$$

Show that if $H \leq G$ and $K \leq G$, then $H \cap K \leq G$.

Proof. Suppose $H \leq G$ and $K \leq G$ for groups H, K, and G. Then, in particular, H, K, and $H \cap K$ are nonempty (since each contains the identity element of G).

Let $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. By Exercise 45 (above), since H and K are subgroups of G, this implies $ab^{-1} \in H$ and $ab^{-1} \in K$. But then $ab^{-1} \in H \cap K$, so by the same exercise, since $H \cap K$ is nonempty, we have $H \cap K \leq G$. $\square$

57. Show that a group with no proper nontrivial subgroups is cyclic.

Proof. Let G be a group with no proper nontrivial subgroups and let e be its identity element.
Case 1: If $G = \{e\}$, then it is cyclic with generator e.
Case 2: If $G \neq \{e\}$, then G contains at least one element other than e. Let $a \in G$ be such an element. Then $\langle a \rangle$ is a nontrivial subgroup of G (since it contains an element other than e). But G has no proper nontrivial subgroups, so it must be that $\langle a \rangle = G$, i.e. G is cyclic with generator a. $\square$

Section 8:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

2. Compute $\tau^2\sigma$ (where τ & σ are defined above).

Solution.

$$\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$$

$$\tau^2\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix}$$

5. Compute $\sigma^{-1}\tau\sigma$ (where τ & σ are defined above).

Solution.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix}$$

$$\sigma^{-1}\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

$$\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$$

6. Compute $|\langle \sigma \rangle|$ (where σ is defined above).

Solution.

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix}$$

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$$
$$\sigma^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix}$$
$$\sigma^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

So $|\langle \sigma \rangle| = 6$.

7. Compute $|\langle \tau^2 \rangle|$ (where τ is defined above).

   Solution.

$$\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$$
$$(\tau^2)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

So $|\langle \tau^2 \rangle| = 2$.