Homework 1 Solutions

Section 2:

$*$	a	b	c	d	e
a	a	b	c	b	d
b	b	c	a	e	c
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

1. Compute $b * d$, $c * c$, and $[(a * c) * e] * a$ using the above table.

   Solution. $b * d = e$, $c * c = b$, and $[(a * c) * e] * a = [c * e] * a = a * a = a$.

2. Compute $(a * b) * c$ and $a * (b * c)$ using the above table. Can you say on the basis of this computation whether $*$ is associative?

   Solution. $(a * b) * c = b * c = a$ and $a * (b * c) = a * a = a$. We cannot tell based on this computation whether $*$ is associative. We would need to check the rest of the triple products.

3. Compute $(b * d) * c$ and $b * (d * c)$ using the above table. Can you say on the basis of this computation whether $*$ is associative?

   Solution. $(b * d) * c = e * c = a$ and $b * (d * c) = b * b = c$. Since they are not equal, $*$ is not associative.

4. Is $*$ as defined in the above table commutative? Why?

   Solution. $*$ is not commutative, since $b * e = c \neq b = e * b$.

7. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = a - b$. Determine whether $*$ is commutative and whether it is associative.

   Solution. $*$ is not commutative, since (for instance) $1 * 0 = 1 - 0 = 1 \neq -1 = 0 - 1 = 0 * 1$. $*$ is not associative either, since (for instance)
   $(1 * 1) * 1 = (1 - 1) * 1 = 0 * 1 = -1 \neq 1 = 1 * 0 = 1 * (1 - 1) = 1 * (1 * 1)$

8. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = ab + 1$. Determine whether $*$ is commutative and whether it is associative.

   Solution. $*$ is commutative since, for $a, b \in \mathbb{Q}$, $a * b = ab + 1 = ba + 1 = b * a$. $*$ is not associative since (for instance)
   $(0*0)*1 = (0 \cdot 0+1)*1 = 1*1 = 1 \cdot 1+1 = 2 \neq 1 = 0 \cdot 1+1 = 0*1 = 0*(0 \cdot 1+1) = 0*(0*1)$.

9. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = \dfrac{ab}{2}$. Determine whether $*$ is commutative and whether it is associative.

   Solution. $*$ is commutative since, for $a, b \in \mathbb{Q}$, $a * b = \dfrac{ab}{2} = \dfrac{ba}{2} = b * a$. $*$ is also associative since, for $a, b, c \in \mathbb{Q}$, $(a * b) * c = \dfrac{ab}{2} * c = \dfrac{\frac{ab}{2} \cdot c}{2} = \dfrac{abc}{4} = \dfrac{a \cdot \frac{bc}{2}}{2} = a * \dfrac{bc}{2} = a * (b * c)$.

10. Let $*$ be defined on $\mathbb{Z}^+$ by letting $a * b = 2^{ab}$. Determine whether $*$ is commutative and whether it is associative.

    Solution. $*$ is commutative since, for $a, b \in \mathbb{Z}^+$, $a * b = 2^{ab} = 2^{ba} = b * a$. $*$ is not associative since (for example)
    $(2 * 2) * 1 = 2^{2 \cdot 2} * 1 = 16 * 1 = 2^{16 \cdot 1} = 2^{16} \neq 2^8 = 2^{2 \cdot 4} = 2 * 4 = 2 * 2^{2 \cdot 1} = 2 * (2 * 1)$.

11. Let $*$ be defined on $\mathbb{Z}^+$ by letting $a * b = a^b$. Determine whether $*$ is commutative and whether it is associative.
Solution. $*$ is not commutative since (for instance) $1 * 2 = 1^2 = 1 \neq 2 = 2^1 = 2 * 1$. $*$ is not associative since (for example)
$(2 * 1) * 2 = 2^1 * 2 = 2 * 2 = 2^2 = 4 \neq 2 = 2^1 = 2 * 1 = 2 * 1^2 = 2 * (1 * 2)$.

27. Either prove or give a counterexample to the statement: "Every binary operation on a set consisting of a single element is both commutative and associative."
Claim: The statement is true.

Proof. Let S be a set consisting of one element and $*$ a binary operation on S. First, some observations. Given $x \in S$, we have $S = \{x\}$. By assumption, $x * x \in S$, so it must be that $x * x = x$.

To show commutativity, let $a, b \in S$. Then $a = b = x$ and so

$$a * b = x * x = x = x * x = b * a.$$

Hence $*$ is commutative.

To show associativity, let $a, b, c \in S$. Then $a = b = c = x$ and so

$$(a * b) * c = (x * x) * x = x * x = x = x * x = x * (x * x) = a * (b * c).$$

Hence $*$ is associative. □

[Note: This is not the "slickest" proof, but is instead a naive rigorous proof. There are other, more succinct ways of proving the statement.]

36. Suppose that $*$ is an associative binary operation on a set S. Let

$$H = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}.$$

Show that H is closed under $*$.

Proof. Let $a, b \in H$. Then, by definition, we have $a, b \in S$ and, for all $x \in S$, $a * x = x * a$ and $b * x = x * b$. Using these and the fact that $*$ is associative, we then obtain the following for all $x \in S$:

$$\begin{aligned}
(a * b) * x &= a * (b * x) \\
&= a * (x * b) \\
&= (a * x) * b \\
&= (x * a) * b \\
&= x * (a * b).
\end{aligned}$$

Since $a * b \in S$ and we have shown that $(a * b) * x = x * (a * b)$ for all $x \in S$, it follows that that $a * b \in H$. Hence H is closed under $*$. □

37. Suppose that $*$ is an associative and commutative binary operation on a set S. Show that $H = \{a \in S \mid a * a = a\}$ is closed under $*$.

Proof. Let $a, b \in H$. Then $a * a = a$ and $b * b = b$. Hence,

$$\begin{aligned}
(a * b) * (a * b) &= a * (b * (a * b)) && \text{(associativity)} \\
&= a * ((b * a) * b) && \text{(associativity)} \\
&= a * ((a * b) * b) && \text{(commutativity)} \\
&= a * (a * (b * b)) && \text{(associativity)} \\
&= (a * a) * (b * b) && \text{(associativity)} \\
&= a * b.
\end{aligned}$$

Therefore $a * b \in H$ and so H is closed under $*$. □