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Section 10

2 Find all the cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

slu.

First notice that since $2\mathbb{Z}$ is abelian, the left cosets of $4\mathbb{Z}$ are equal to the right cosets of $4\mathbb{Z}$.

$0, 2 \in 2\mathbb{Z} \implies 4\mathbb{Z} + 0 = \{\dots, -4, 0, 4, \dots\} \wedge 4\mathbb{Z} + 2 = \{\dots, -6, -2, 2, 6, \dots\}$

$4\mathbb{Z} + 0 \cup 4\mathbb{Z} + 2 = 2\mathbb{Z}$

So, all the cosets of $4\mathbb{Z} \leq 2\mathbb{Z}$, are $4\mathbb{Z} + 0$ and $4\mathbb{Z} + 2$ ♦

4 Find all the cosets of the subgroup $\langle 4 \rangle$ of $\mathbb{Z}_{12}$.

slu.

$\langle 4 \rangle = \{0, 4, 8\}$ and $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

$\mathbb{Z}_{12}$ is abelian, so left cosets are right cosets.

$\langle 4 \rangle + 0 = \{0, 4, 8\}$

$\langle 4 \rangle + 1 = \{1, 5, 9\}$

$\langle 4 \rangle + 2 = \{2, 6, 10\}$

$\langle 4 \rangle + 3 = \{3, 7, 11\}$,

$\mathbb{Z}_{12} = \bigcup_{i=0}^{3} \langle 4 \rangle + i$

So, all the cosets of $\langle 4 \rangle \leq \mathbb{Z}_{12}$ are $\langle 4 \rangle + 0, \langle 4 \rangle + 1, \langle 4 \rangle + 2, \langle 4 \rangle + 3$ ♦

6 Find all the left cosets of the subgroup $\{\rho_0, \mu_2\}$ of the group D_4 given by table 8.12.

slu.

Since ρ_0 is the identity element of D_4, $\forall x \in D_4 x\rho_0 = x$.

Since multiplication is given by a table, multiplying on the left to μ_2 is given by the column labeled μ_2.

So, we can just read the left cosets from the table by taking the ρ_0, and μ_0 columns. Now, we can see that $2 \cdot 4 = 8$, so the first 4 rows exhaust D_4. Also, considering the other 4 rows reverses the order, but they're the same 4 cosets as the previous 4 rows.

Therefore, $\{\rho_0, \mu_2\}, \{\rho_1, \delta_2\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_1\}$ are all the left cosets of $\{\rho_0, \mu_2\} \leq D_4$ ♦

12 Find the index of $\langle 3 \rangle$ in the group $\mathbb{Z}_2 4$

slu. $\langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\} \implies |\langle 3 \rangle| = 8 \wedge |\mathbb{Z}_{24}| = 24 \implies (\mathbb{Z}_{24} : \langle 3 \rangle) = \frac{24}{8} = 3$ ♦

16 Let $\mu = (1,2,4,5)(3,6)$ in S_6. Find the index of $\langle \mu \rangle$ in S_6.

slu.

μ is a product of disjoint cycles.

The square of a transposition is the identity.

Let $\tau = (3,6)$ .So $\tau^2 = \iota$.

Write $\sigma = (1,2,4,5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 1 & 6 \end{pmatrix}$.

$$\begin{bmatrix} \iota & 1 & 2 & 3 & 4 & 5 & 6 \\ \sigma & 2 & 4 & 3 & 5 & 1 & 6 \\ \sigma^2 & 4 & 5 & 3 & 1 & 2 & 6 \\ \sigma^3 & 5 & 1 & 3 & 2 & 4 & 6 \\ \sigma^4 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Then $\sigma^2 = (1,4)(5,2)$, $\sigma^3 = (1,5,4,2)$, and $\sigma^3 = \iota$.

So,$\mu = \sigma\tau$ given that disjoint cycles commute, we can write $\mu^n = \sigma^n \tau^n$ for $n \in \mathbb{Z}$. Then,

$$\mu^2 = \sigma^2\tau^2 = \sigma^2\iota = \sigma^2, \mu^3 = \sigma^3\tau^3 = \iota^2\tau = \tau, \mu^4 = \sigma^4\tau^4 = \iota\sigma\iota^2 = \sigma, \mu^5 = \sigma^5\tau^5 = \iota\sigma^2\iota\tau, \mu^6 = \sigma^6\tau^6 = \iota.$$

So, $|\langle \mu \rangle| = 6$, and we have that $|S_6| = 6!$, so $(S_6 : \langle \mu \rangle) = \frac{6!}{6} = 5!$

28 Let H be a subgroup of G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

pf.

$g^{-1}hg \in H \forall g \in G$ and $\forall h \in H$.

$g \in G \wedge g^{-1}hg \in H \implies gg^{-1}hg = ehg = hg \in gH$, but notice $hg \in Hg \implies gH \subset Hg$.

$g^{-1} \in G \implies (g^{-1})^{-1}hg^{-1} = ghg^{-1} \in H \; \forall h \in H$.

$g \in G \wedge ghg^{-1} \in H \implies ghg^{-1}g = ghe = gh \in Hg \implies Hg \subset gH$.

$gH \subset Hg \wedge Hg \subset gH \implies Hg = gH \quad \blacksquare$

34 Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.

pf.

Since, the only divisors of pq are $1, p$, and q. The Lagrange's theorem gives us that, the order of any subgroup of G, must divide the order of G. Therfore, if H is a subgroup of G, it must have order $1, p$ or q. Any, subgroup of order 1 is the trivial subgroup. And any subgroup of prime order is cyclic, so the conclusion follows even if G has no proper subgroups $\blacksquare$

39 Show that if H is a subgroup of index 2 in a finite group G, then every left coset of H is also a right coset of H.

pf.

Since left cosets partition G into sets of equal cardinality, and $(G : H) = 2$, so there are only two distinct left cosets of H.

Right cosets also partition G into sets of equal cardinality, and $(G : H) = 2$ for right cosets too, so there are only two distinct right cosets of H.

Let e be the identity of G, then $eH = H = He$, by the properties of the identity. So, H is both a left and a right coset of H. Let aH,and bH, be the other distinct left and right cosets of H.

Since G is partitioned by the left cosets $G = H \cup aH$, and $H \cap aH = \emptyset$. Similarly, $G = H \cup Hb$, and $H \cap Hb = \emptyset$. Therefore, $aH = G = Hb$. So every left coset of H is also a right coset of H ■

40 Show that if a group G with identity e has finite order n, then $a^n = e$ for all $a \in G$.

pf.

If G is of finite order n, then the order of an element of G, divides n.

Let $|\langle a \rangle| = m \implies n = mk$ some $k \in \mathbb{Z}$. So, $a^n = (a^m)^k = e^k = e$ ■

Section 11

In Excercises 3 through 7, find the order of the given element of the direct product.

5 $(8, 10)$ in $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$ slu.

$$8 = 2^3 \wedge 12 = 2^2 \cdot 3 \implies \text{gcd}(8, 12) = 4 \wedge \frac{12}{4} = 3 \implies |8| = 3 \text{ in } \mathbb{Z}12.$$

$$10 = 2 \cdot 5 \wedge 18 = 2 \cdot 3^2 \implies \text{gcd}(10, 18) = 2 \wedge \frac{18}{2} = 9 \implies |10| = 9 \text{ in } \mathbb{Z}18.$$

$$9 = 3^2 \implies \text{lcm}(3, 9) = 9 \implies |(8, 10)| = 9 \text{ in } \mathbb{Z}_{12} \times \mathbb{Z}_{18} \quad \blacklozenge$$

8 What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$? of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$?

slu.

$|\mathbb{Z}_6 \times \mathbb{Z}_8| = |\mathbb{Z}_6||\mathbb{Z}_8| = 6 \cdot 8 = 48$. The divisors of 48 are, $1, 2, 3, 4, 6, 8, 12, 16, 24$, and 48.

By 11.16, $\mathbb{Z}_6 \times \mathbb{Z}_8$ is abelian and finite so it has subgroups of orders $1, 2, 3, 4, 6, 8, 12, 16, 24, 48$.

By 11.17, we have the criterion that a group of order m is cyclic if m is square free.

$48 = 2^4 \cdot 3, 24 = 2^3 \cdot 3, 16 = 2^4, 12 = 2^2 \cdot 3, 8 = 2^3, 6 = 2 \cdot 3$.

$1 < 2 < 3 < 4 < 6 \implies 6$ is the largest order of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$.

$|\mathbb{Z}_{12} \times \mathbb{Z}_{15}| = |\mathbb{Z}_{12}||\mathbb{Z}_{15}| = 12 \cdot 15 = 180$. The divisors of 180 are, $2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 90$, and 180. Which are the orders of the subgroups of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

$4 = 2^2$ divides $180, 90, 36, 20, 12$ and 4. So, now consider just $2, 3, 5, 6, 9, 10, 15, 18, 30, 45$.

$9 = 3^2$ divides $45, 18$, and 9. So, now consider just $2, 3, 5, 6, 10, 15, 30$.

$30 = 3 \cdot 2 \cdot 5$, so it's square free and is larger than everything else. So, 30 is the largest order of all the cyclic subgroups of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ ♦

10 Find all the proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

slu.

$$\langle(1,0,0)\rangle, \langle(0,1,0)\rangle, \langle(0,0,1)\rangle, \langle(1,1,0)\rangle, \langle(0,1,1)\rangle, \langle(1,0,1)\rangle \quad \blacklozenge$$

11 Find all the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4. slu. $|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$, $4|8$, so there are subgroups of order 4.

gcd$(0,2) = 2$, and $\frac{2}{2} = 1 \implies |0| = 1$ in $\mathbb{Z}_2$.

gcd$(1,2) = 1$, and $\frac{2}{1} = 2 \implies |1| = 2$ in $\mathbb{Z}_2$.

gcd$(0,4) = 4$, and $\frac{4}{4} = 1 \implies |0| = 1$ in $\mathbb{Z}_4$.

gcd$(1,4) = 1$, and $\frac{4}{1} = 4 \implies |1| = 4$ in $\mathbb{Z}_4$.

gcd$(2,4) = 2$, and $\frac{4}{2} = 2 \implies |2| = 2$ in $\mathbb{Z}_4$.

gcd$(3,4) = 1$, and $\frac{4}{1} = 4 \implies |3| = 4$ in $\mathbb{Z}_4$.

lcm$(1,1) = 1$, lcm$(1,2) = 2$,lcm$(1,4) = 4$. So, $\langle(0,1)\rangle$, and $\langle(0,3)\rangle$ are subgroups of order 4.

lcm$(2,1) = 2$, lcm$(2,2) = 2$,lcm$(2,4) = 4$. So, $\langle(1,1)\rangle$, and $\langle(1,3)\rangle$ are subgroups of order 4. This exhausts the list $\blacklozenge$

12 Find all the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ that are isomorphic to the Klein 4-group

slu.

The Klein 4—group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. So, clearly $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$ is one of them.

By the previous computations. $|0| = 1$, and $|1| = 2$ in $\mathbb{Z}_2$. And, $|0| = 1, |1| = 4, |2| = 2, |3| = 4$ in $\mathbb{Z}_4$. So the orders are 1, or 2 for $\mathbb{Z}_2$, and $1, 2$,or 4 for $\mathbb{Z}_4$

The only element of order 2 in $\mathbb{Z}_4$ is 2, so the other two are,

$$\mathbb{Z}_2 \times \{0\} \times \langle 2\rangle \text{ and } \{0\} \times \mathbb{Z}_2 \times \langle 2\rangle \quad \blacklozenge$$

Additional Excercises

Given a group G and some subset $A \subseteq G$, recall that we defined

$$\langle A\rangle := \bigcap_{A\subseteq H\leq G} H$$

In words $\langle A\rangle$ is the intersection of all subgroups of G that contain A.

1** We claimed in class that $\langle A\rangle$ is "the smallest subgroup of G containing A" in the following sense:

$A \subseteq \langle A\rangle \leq G$ and if $A \subseteq K \leq G$, then $\langle A\rangle \leq K$. Prove this claim.

pf.

$\langle A\rangle$ is a group, so it is closed under the group operation. $\langle A\rangle \leq G \implies e \in \langle A\rangle$. $K \leq G \implies e \in K$. So, the identity element of $\langle A\rangle$ is the identity element of K. Now, $\langle A\rangle$ is a group so it has inverses. So, $\langle A\rangle \leq K \quad \blacklozenge$

2 Verify that $\langle(1,2),(2,3)\rangle = S_3$ by expressing every non-identity element of S_3 as a product whose factors are each either $(1,2)$ or $(2,3)$.

pf.

$$\begin{aligned}
S_3 &= \left\{\iota, \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\right\}\\
&= \{\iota, (1,2), (2,3), (1,3), (1,2,3), (3,1,2)\}
\end{aligned}$$

Let $\tau_1 = (1,2)$, and $\tau_2 = (2,3)$.

$$\begin{bmatrix}
\iota & 1 & 2 & 3\\
\tau_1 & 2 & 1 & 3\\
\tau_2 & 1 & 3 & 2\\
\tau_2\tau_1 & 3 & 1 & 2\\
\tau_1\tau_2 & 2 & 3 & 1\\
\tau_2\tau_1\tau_2 & 3 & 2 & 1
\end{bmatrix}$$

$\tau_2\tau_1 = (1,3,2)$,$\tau_1\tau_2 = (1,2,3)$, and $\tau_2\tau_1\tau_2 = (1,3)$ ♦

3 Consider $H = \langle(2,4),(1,2,3,4)\rangle \leq S_4$.

(a) Find 7 distinct non-identity elements of H and express them as products of disjoint cycles.

slu.

Let, $\tau = (2,4) = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 3 & 2\end{pmatrix}, \mu = (1,2,3,4) = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}$. then

$$\begin{bmatrix}
\iota & 1 & 2 & 3 & 4\\
\mu & 2 & 3 & 4 & 1\\
\mu^2 & 3 & 4 & 1 & 2\\
\mu^3 & 4 & 1 & 2 & 3\\
\iota = \mu^4 & 1 & 2 & 3 & 4
\end{bmatrix}$$

$$\begin{bmatrix}
\iota & 1 & 2 & 3 & 4\\
\tau & 1 & 4 & 3 & 2\\
\mu & 2 & 3 & 4 & 1\\
\tau\mu & 4 & 3 & 2 & 1\\
\mu^2 & 3 & 4 & 1 & 2\\
\tau\mu^2 & 3 & 2 & 1 & 4\\
\mu^3 & 4 & 1 & 2 & 3\\
\tau\mu^3 & 2 & 1 & 4 & 3
\end{bmatrix}$$

Since $\tau^2 = \iota$, we only count τ once.

Now, μ, μ^2, and μ^3 give 3 more. And, $\tau\mu, \tau\mu^2$, and $\tau\mu^3$ give the other 3 needed for 6.

So, we only need to express them as disjoint cycles,

$$(2,4), (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1,4)(2,3), (1,3), (1,2)(3,4)$$

(b)* Prove that H has exactly 8 elements. To what group that we've previously seen is H isomorphic?

pf.

Since, $\langle \mu \rangle \leq H$ and $|\langle \mu \rangle| = 4 \implies |H| = 4k$, for some $k \in \mathbb{N}$.

Similarly, $|\langle \tau \rangle| = 2 \implies |H| = 2m$ some $m \in \mathbb{N}$.

Since, $\langle \tau \rangle \cap \langle \mu \rangle = \{0\}$, and since the left cosets of $\langle \mu \rangle$ partition H, and the right cosets of $\langle \tau \rangle$ partition H. Given that all the elements of H are products of μ,and τ, multiplying $\langle \mu \rangle$ on the left by some product $\sigma_l \mu^n$, will absorb the μ^n part and it will be like multiplying by some σ_l, likewise multiplying $\langle \tau \rangle$ on the right by some $\tau^n \sigma_r$ will yield a multiplication by σ_r on the right. The previous computation shows that the collection of elements of the left cosets of $\langle \mu \rangle$ are the same as the collection of elements of the right cosets of $\langle \tau \rangle$. It follows, that $m = 4$, and $k = 2$. It is not too hard to see that $\mathbb{Z}_2 \times \mathbb{Z}_4 \simeq H$ ■

4** Fix a finite set A and let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a partition of A.

Set $H_{\mathcal{P}} := \{\sigma \in S_A | \sigma[P_i] \subset P_i \text{ for } i = 1, 2, \dots, k\}$.

(a) Prove that $H_{\mathcal{P}}$ is a subgroup of S_A and

$$H_{\mathcal{P}} = \prod_{i=1}^{k} S_{P_i}.$$

pf.

$\iota[P_i] = P_i \implies \iota \in H_{\mathcal{P}}$.

If $\sigma[P_i] \subset P_i$, then $\sigma^{-1}\sigma[P_i] = P_i$ for some i.

If $\sigma[P_i] \subset P_i$ and $\tau[P_i] \subset P_i$, then $\sigma\tau[P_i] \subset P_i$ for some i.

So, $H_{\mathcal{P}} \leq S_A$.

Let $\phi : S_A \to \prod_{i=1}^{k} P_i; \sigma \mapsto (\sigma|_{P_1}, \sigma|_{P_2}, \dots, \sigma|_{P_k})$

Then for each $\sigma|_{P_i}$, define $\sigma_i' : P_i \hookrightarrow S_A; \sigma_i'(x) = \begin{cases} \sigma|_{P_i}(x), x \in P_i \\ x \qquad\quad, x \notin P_i \end{cases}$

Then $\phi^{-1} : \prod_{i=1}^{k} P_i \to S_A; (\sigma|_{P_1}, \sigma|_{P_2}, \dots, \sigma|_{P_k}) \mapsto \sigma_1' \circ \sigma_2' \circ \dots \circ \sigma_k' = \sigma$, which works because $\mathcal{P}$ is a partition.

Now $\phi(\sigma\tau) = (\sigma\tau|_{P_1} \sigma\tau|_{P_2}, \dots, \sigma\tau|_{P_k})$

$= (\sigma|_{P_1}\tau|_{P_1}, \sigma|_{P_2}\tau|_{P_2}, \dots, \sigma|_{P_k}\tau|_{P_k}) = (\sigma|_{P_1}, \sigma|_{P_2}, \dots, \sigma|_{P_k})(\tau|_{P_1}, \tau|_{P_2}, \dots, \tau|_{P_k}) = \phi(\sigma)\phi(\tau)$

So, $H_{\mathcal{P}} = \prod_{i=1}^{k} P_i$ ■

(b) Let $\sigma \in S_A$ and let $\mathcal{O}$ be the set of orbits of σ. Prove that $\sigma \in H_{\mathcal{O}}$.

pf. The orbits of σ are disjoint. $X \in \mathcal{O} \implies \sigma[X] = X \implies \sigma \in H_{\mathcal{O}}$.

(c) Let $\sigma \in S_A$ and prove that $|\sigma|$ is the least common multiple of the sizes of the orbits of σ.

pf. $\mathcal{O}$ partitions S_A. So, $S_A = \prod_{X \in \mathcal{O}} X$, so $|\sigma| = \text{lcm}(|X|)_{X \in \mathcal{O}}$, meaning take the least common multiple of all the orbits $X \in \mathcal{O}$. This works because, S_A is equal to the direct product, and the order of an element in the direct product is equal to the lcm of the orders of the components ■