Homework 4 Solutions

Section 8:

12. Find the orbit of 1 under the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}.$$

Solution.

$$\begin{aligned} \tau(1) &= 2 \\ \tau(2) &= 4 \\ \tau(4) &= 3 \\ \tau(3) &= 1 \end{aligned}$$

The orbit of 1 under τ is $\{1, 2, 3, 4\}$.

42. Let A be a set and B a subset of A. Determine whether $\{\sigma \in S_n \mid \sigma[B] \subseteq B\}$ is sure to be a subgroup of S_A under the induced operation, where $\sigma[B] = \{\sigma(x) \mid x \in B\}$.

Solution. No, the set needn't be a subgroup under the induced operation. Consider the counterexample with $A = \mathbb{R}$ and $B = \mathbb{Z}$. $\sigma(x) = 2x$ defines a permutation $\sigma \in S_{\mathbb{R}}$, with inverse $\sigma^{-1}(x) = \frac{x}{2}$. $\sigma[\mathbb{Z}] \subseteq \mathbb{Z}$, so σ is in the given set, but clearly σ^{-1} is not, since (for instance) $1 \in \mathbb{Z}$, but $\sigma^{-1}(1) = \frac{1}{2} \notin \mathbb{Z}$.

46. Show that S_n is nonabelian for $n \geq 3$.

Proof. Given $n \geq 3$, consider the permutations $\sigma, \tau \in S_n$ given by

$$\sigma(k) = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{if } k = 1 \\ k & \text{else} \end{cases} \quad \& \quad \tau(k) = \begin{cases} 2 & \text{if } k = 3 \\ 3 & \text{if } k = 2 \\ k & \text{else} \end{cases}$$

for $k \in \{1, 2, \ldots, n\}$. (Note that σ and τ are clearly permutations, since $\sigma^2 = \tau^2 = \iota$. In fact, these are the cycles $(1, 2)$ and $(2, 3)$ of Section 9.) Then

$$(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(1) = 2 \neq 3 = \tau(2) = \tau(\sigma(1)) = (\tau\sigma)(1).$$

Hence $\sigma\tau \neq \tau\sigma$ and therefore S_n is nonabelian. □

Section 9:

9. Compute $(1, 2)(4, 7, 8)(2, 1)(7, 2, 8, 1, 5)$.

Solution. $(1, 2)(4, 7, 8)(2, 1)(7, 2, 8, 1, 5) = (1, 5, 8)(2, 4, 7)$

12. Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$ as a product of disjoint cycles and then as a product of transpositions.

Solution.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix} = (1, 3, 4, 7, 8, 6, 5, 2) = (1, 2)(1, 5)(1, 6)(1, 8)(1, 7)(1, 4)(1, 3)$$

29. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

Proof. Let H be a subgroup of S_n for some $n \geq 2$. If all elements of H are even, then we are done, so suppose not all elements of H are even. Then there exists some odd $\sigma \in H$. Consider $\lambda_\sigma : H \to H$, $\tau \mapsto \sigma\tau$, which we showed in class is a permutation of H.

Let A be the set of even permutations in H and B the set of odd permutations in H. Given $\tau \in A$, τ is even and σ is odd, so $\sigma\tau$ is odd. Hence $\lambda_\sigma(A) \subseteq B$. Similarly, $\lambda_\sigma(B) \subseteq A$. Since λ_σ is injective, this implies $|A| \leq |B|$ and $|B| \leq |A|$, i.e. $|A| = |B|$.

Since $A \cup B = H$ and $A \cap B = \varnothing$, we have $|H| = |A| + |B| = 2|A|$ and so $|A| = \frac{|H|}{2}$. Therefore half the permutations in H are even. $\square$

33. Consider S_n for a fixed $n \geq 2$ and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.

Proof. Note that, since σ is odd, so is σ^{-1} (as discussed in class). Let $\tau \in S_n$ be an odd permutation. Then, since both σ^{-1} and τ are odd, it must be that $\sigma^{-1}\tau$ is even, i.e. $\sigma^{-1}\tau \in A_n$. Then $\tau = \sigma(\sigma^{-1}\tau)$ and so τ is a product of σ and a permutation in A_n. $\square$

34. Show that $(1, 2, \ldots, m)^2$ is a cycle if m is odd.

Proof. Suppose m is odd. Then $m = 2k + 1$ for some $k \in \mathbb{Z}^+$ and we may explicitly compute:

$$\begin{aligned}(1, 2, \ldots, m)^2 &= (1, 2, \ldots, 2k + 1)^2 \\ &= (1, 3, 5, \ldots, 2k + 1, 2, 4, 6, \ldots, 2k).\end{aligned}$$

Therefore $(1, 2, \ldots, m)^2$ is a cycle if m is odd. $\square$

35. Following the line of thought opened by Exercise 34, complete the following with a condition involving n and r so that the resulting statement is a theorem:

If σ is a cycle of length n, then σ^r is also a cycle if and only if...

Solution. ... $\gcd(n, r) = 1$.

Additional Exercise:

1. Let f be a group isomorphism of S_3 with itself. Prove that there exists some $\sigma \in S_3$ such that $f(\tau) = \sigma\tau\sigma^{-1}$ for all $\tau \in S_3$ (that is, σ does not depend on τ).

   Proof. Set $\mathrm{Aut}(S_3) = \{\phi : S_3 \to S_3 \mid \phi \text{ is a group isomorphism}\}$ and, given $\sigma \in S_3$, define $c_\sigma : S_3 \to S_3$ by $c_\sigma(\tau) = \sigma\tau\sigma^{-1}$. We will show that $\mathrm{Aut}(S_3) = \{c_\sigma \mid \sigma \in S_3\}$ and therefore $f = c_\sigma$ for some $\sigma \in S_3$.

   First, we note that c_σ is indeed a group isomorphism for each $\sigma \in S_3$. It is easy to check that $c_\sigma \circ c_{\sigma^{-1}} = \mathrm{id}_{S_3} = c_{\sigma^{-1}} \circ c_\sigma$ for each $\sigma \in S_3$. Hence each c_σ is bijective. Also, for $\sigma, \tau, \rho \in S_3$, we have

   $$c_\sigma(\tau\rho) = \sigma\tau\rho\sigma^{-1} = \sigma\tau\sigma^{-1}\sigma\rho\sigma^{-1} = c_\sigma(\tau)c_\sigma(\rho),$$

   showing that c_σ is a group isomorphism for each $\sigma \in S_3$, and hence $\{c_\sigma \mid \sigma \in S_3\} \subseteq \mathrm{Aut}(S_3)$.

   Furthermore, if $\sigma, \tau, \rho \in S_3$ and $c_\sigma(\tau) = c_\rho(\tau)$, then $\sigma\tau\sigma^{-1} = \rho\tau\rho^{-1}$. Multiplying on the left by σ^{-1} and on the right by ρ, we have $\tau\sigma^{-1}\rho = \sigma^{-1}\rho\tau$. But since τ was arbitrary, this implies that $\sigma^{-1}\rho$ commutes with every element of S_3. Exercise 47 from Section 8 stated that ι was the only element of S_3 for which this is true, so it must be that $\sigma^{-1}\rho = \iota$, i.e. $\rho = \sigma$. Hence the c_σ are all distinct, i.e. $|\{c_\sigma \mid \sigma \in S_3\}| = |S_3| = 6$.

   Now, if $\tau \in S_3$ is a transposition and $\phi : S_3 \to S_3$ is an isomorphism, then $\phi(\tau)^2 = \phi(\tau^2) = \phi(\iota) = \iota$. Since $\phi(\iota) = \iota$ and ϕ is injective, $\phi(\tau) \neq \iota$. It is easily checked that the transpositions are the only non-identity elements of S_3 which square to ι. Therefore $\phi(\tau)$ must be a transposition and so ϕ permutes the transpositions in S_3. Now, we have

   $$\begin{aligned} \phi(\iota) &= \iota \\ \phi((1,2,3)) &= \phi((1,3)(1,2)) = \phi((1,3))\phi((1,2)) \\ \phi((1,3,2)) &= \phi((1,2)(1,3)) = \phi((1,2))\phi((1,3)), \end{aligned}$$

   so ϕ is completely determined by where it sends the transpositions. Together, these observations yield an *injective* map $\mathrm{Aut}(S_3) \hookrightarrow S_{\{\text{transpositions in } S_3\}}$. Since there are 6 permutations of the 3-element set $\{\text{transpositions in } S_3\}$, we have

   $$|\mathrm{Aut}(S_3)| \leq |S_{\{\text{transpositions in } S_3\}}| = 6.$$

   But now we have shown that

   $$6 = |\{c_\sigma \mid \sigma \in S_3\}| \leq |\mathrm{Aut}(S_3)| \leq 6.$$

   Therefore $\{c_\sigma \mid \sigma \in S_3\} = \mathrm{Aut}(S_3)$, as desired. It follows that $f \in \{c_\sigma \mid \sigma \in S_3\}$ and so there exists some $\sigma \in S_3$ such that $f(\tau) = c_\sigma(\tau) = \sigma\tau\sigma^{-1}$ for all $\tau \in S_3$. $\square$

   [Note: This statement can be restated more simply by saying, "All automorphisms of S_3 are inner." As you may guess, automorphisms of S_3 are simply isomorphisms from S_3 to itself. *Inner* automorphisms are those obtained by *conjugation*, that is, our c_σ's.

   This simpler statement generalizes to "For $n \in \mathbb{Z}^+ \setminus \{6\}$, all automorphisms of S_n are inner." However, the proof does not so easily generalize. Can you see what is different in the general case?]