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Section 11

16 Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

slu.

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_{12} &\simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \text{ Since gcd}(4,3) = 1 \\ &\simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \text{ By rearrangement} \\ &\simeq \mathbb{Z}_4 \times \mathbb{Z}_6 \qquad \text{Since gcd } (2,3) = 1 \quad \blacklozenge \end{aligned}$$

18 Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic? Why or why not?

slu.

$$\begin{aligned} \mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} &\simeq \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \quad \text{Since gcd}(2,5) = 1 \text{ and gcd}(8,3) = 1 \\ &\simeq \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_{40} \times \mathbb{Z}_3 \text{ Since gcd}(8,5) = 1 \\ &\simeq \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{40} \text{ By rearrangement} \\ &\text{and} \\ \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} &\simeq \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{40} \quad \text{Since gcd}(4,3) = 1 \\ &\text{but} \\ \text{lcm}(8,2) = 8 \text{ and lcm}(4,4) = 4 &\implies |\mathbb{Z}_8 \times \mathbb{Z}_2| = 8 \text{ and } |\mathbb{Z}_4 \times \mathbb{Z}_4| = 4 \\ &\implies \mathbb{Z}_8 \times \mathbb{Z}_2 \not\simeq \mathbb{Z}_4 \times \mathbb{Z}_4 \end{aligned}$$

So, no $\blacklozenge$

24 Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

slu.

$$4 = 2^2, 18 = 2 \cdot 3^2, 15 = 3 \cdot 5 \text{ and } 3 = 3, 36 = 2^2 \cdot 3^2, 10 = 2 \cdot 5$$
$$\implies$$
$$\begin{aligned} \mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} &\simeq \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\ &\simeq \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_2 \times \mathbb{Z}_5 \\ &\simeq \mathbb{Z}_3 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_2 \times \mathbb{Z}_5 \\ &\simeq \mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10} \end{aligned}$$

So, yes $\blacklozenge$

49 Find a counterexample of Exercise 47 with the hypotheses of that G is abelian omitted.

slu.

$$S_3 = \{\iota, (2\ 3), (1\ 2), (1\ 2\ 3), (1\ 3\ 2), (1\ 3)\}$$

If you square each element you get

$$(2\ 3)^2 = \iota, (1\ 2)^2 = \iota, (1\ 2\ 3)^2 = (1\ 3\ 2), (1\ 3\ 2)^2 = (1\ 2\ 3), (1\ 3)^2 = \iota.$$

Then the set of squares and ι is, $S = \{\iota,\ \ (2\ 3),\ \ (1\ 2),\ \ (1\ 3)\}$.

$(2\ 3)(1\ 2) = (1\ 3\ 2) \notin S \implies S$ is not closed $\implies S$ is not a subgroup of S_4 $\quad \triangleright\triangleleft$

I did the previous by brute force with the computer.

However, I found the smallest example up to isomorphism, consider D_3:

D_3	1	A	B	C	D	E
1	1	A	B	C	D	E
A	A	B	1	D	E	C
B	B	1	A	E	C	D
C	C	E	D	1	B	A
D	D	C	E	A	1	B
E	E	D	C	B	A	1

$\implies S = \{1, C, D, E\} \wedge CD = B \notin S \implies S \nleq D_3 \triangleright\triangleleft$

Ok, this is annoying. $D_3 \simeq S_3$... Looking at the table I've realized they are the same thing. I wanted to consider a smaller example so I googled what is the smallest non-abelian group. So, I had already found the smallest example.

Section 13

In Exercises 1 through 15, determine whether the given map ϕ is a homomorphism. [*Hint:* the straightforward way to proceed is to check whether $\phi(ab) = \phi(a)\phi(b)$ for all a and b in the domain of ϕ. However, if we should happen to notice that $\phi^{-1}[\{e'\}]$ is not a subgroup whose left and right cosets coincide, or that ϕ does not satisfy the properties given in Exercises 44 or 45 for finite groups, then we can say at once that ϕ is not a homomorphism.]

2 Let $\phi : \mathbb{R} \to \mathbb{Z}$ under addition be given by $\phi(x) =$ the greatest integer $\leq x$.

slu. No, $\phi(1.5 + 1.5) = \phi(3) = 3 \neq 2 = 1 + 1 = \phi(1.5) + \phi(1.5)$ ♦

12 Let M_n be the additive group of all $n \times n$ matrices with real entries, and let $\mathbb{R}$ be the additive group of real numbers. Let $\phi(A) = \det(A)$

slu. No, $\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1 \neq 0 = 0+0 = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + \det\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ ♦

13 Let M_n and R be as in Exercise 12. Let $\phi(A) = \text{tr}(A)$ for $A \in M_n$, where the **trace** of A is the sum of the elements on the main diagonal of A from the upper-left to the lower-right corner.

slu. Yes,

$$\begin{aligned}
\text{tr}(A+B) &= \text{tr}((A_{ij}) + (B_{ij})) \\
&= \text{tr}((A_{ij} + B_{ij})) \\
&= \sum_{i=1}^{n} A_{ii} + B_{ii} \\
&= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} \\
&= \text{tr}((A_{ij})) + \text{tr}((B_{ij})) \\
&= \text{tr}(A) + \text{tr}(B) \quad \blacklozenge
\end{aligned}$$

14 Let $GL(n, \mathbb{R})$ be the multiplicative group of invertible $n \times n$ matrices, and let $\mathbb{R}$ be the additive group of real numbers. Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}$ be given by $\phi(A) = \text{tr}(A)$, where $\text{tr}(A)$ is defined as in Exercise 13.

slu. No,

$$\text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \text{tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0 \neq 2 = 1 + 1 = \text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + \text{tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \quad \blacklozenge$$

In Exercises 16 through 24, compute the indicated quantities for the given homomorphism ϕ. (See Exercise 46.)

21 $\text{Ker}(\phi)$ and $\phi(14)$ for $\phi : \mathbb{Z}_{24} \to S_8$ where $\phi(1) = (2, 5)(1, 4, 6, 7)$

slu.

$\phi(2) = [(1\ 4\ 6\ 7)(2\ 5)]^2 = (1\ 6)(4\ 7), \phi(3) = [(1\ 4\ 6\ 7)(2\ 5)]^3 = (1\ 7\ 6\ 4)(2\ 5), \phi(4) = [(1\ 4\ 6\ 7)(2\ 5)]^4 = \iota.$

We can see from the computation that, multiples of 4 give us the identity. So, $\text{Ker}(\phi) = \{0, 4, 8, 12, 16, 20\}$

$$\begin{aligned}
\phi(14) &= [(1\ 4\ 6\ 7)(2\ 5)]^{14} = [(1\ 4\ 6\ 7)(2\ 5)]^{12+2} \\
&= ([(1\ 4\ 6\ 7)(2\ 5)]^4)^3[(1\ 4\ 6\ 7)(2\ 5)]^2 = \iota^3[(1\ 4\ 6\ 7)(2\ 5)]^2 = (1\ 6)(4\ 7) \quad \blacklozenge
\end{aligned}$$

In Exercises 33 through 43, give an example of a nontrivial homomorphism ϕ for the given groups, if an example exists. If no such homomorphism exists explain why that is so. You may use Excercises 44 and 45.

41 $\phi : D_4 \to S_3$

slu.

From problem 49 we can see $S_3 \simeq D_3$

$$
\begin{aligned}
D_n &= \langle r, s | r^n = s^2 = e, srs = r^{-1} \rangle \\
&\implies \\
D_4 &= \langle r, s | r^4 = s^2 = e, srs = r^{-1} \rangle \\
&\text{and} \\
D_3 &= \langle a, b | a^3 = b^2 = e', bab = a^{-1} \rangle
\end{aligned}
$$

So, let's check,

$$\nu(r) = a \implies \nu(e) = \nu(r^4) = \nu(r)^4 = \nu(r)^3\nu(r) = a^3\phi(r) = e'^3 a = a \neq e'$$

. So, $\nu(r) \neq a$. Check,

$$\nu(r) = b \implies \nu(e) = \nu(r^4) = \nu(r)^4 = b^4 = (b^2)^2 = e'^2 = e'$$

. So, $\nu(s)b\nu(s) = \nu(s)\nu(r)\nu(s) = \nu(srs) = \nu(r^{-1}) = \nu(r)^{-1} = b^{-1} = b$. So, $\nu(s) = e'$.

Since, all the elements of D_4 are, $e, r, r^2, r^3, s, sr, sr^2, sr^3$. Then define ψ by,

$\psi(e) = e'$,

$\psi(r) = b$,

$\psi(r^2) = b^2 = e'$,

$\psi(r^3) = be' = b$,

$\psi(s) = e'$,

$\psi(sr) = e'b = b$,

$\psi(sr^2) = e'b^2 = e'$,

$\psi(sr^3) = e'b = b$,

Now construct an isomorphism μ from D_3 to S_3 like so,

$\mu(e) = \iota$,

$\mu(a) = (123)$,

$\mu(a^2) = (132)$

$\mu(b) = (23)$

$\mu(ba) = (12)$

$\mu(ba^2) = (13)$.

μ is a bijection from D_3 to S_3. It satisfies the homomorphism property by construction, because I just looked at the tables of D_3 and S_3 to write it.

Now, put $\phi = \psi \circ \mu$ which gives,

$\phi(r) = \phi(r^3) = \phi(sr) = \phi(sr^3) = (23)$, $\phi(e) = \phi(r^2) = \phi(s) = \phi(sr^2) = e'$

Which is nontrivial ⧫

47 Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is a prime must either be the trivial homomorphism or a one-to-one map.

pf.

$|G|$ is prime $\implies$ G has no proper nontrivial subgroups.

Ker$(\phi) \leq G \implies$ Ker$(\phi) = \{e\} \vee$ Ker$(\phi) = G$

Ker$(\phi) = \{e\} \implies \phi$ is injective.

Ker$(\phi) = G \implies \phi$ is trivial ■

49 Show that if G, G', and G'' are groups and if $\phi : G \to G'$ and $\gamma : G' \to G''$ are homomorphisms, then the composite map $\gamma\phi : G \to G''$ is a homomorphism.

pf.

WTS $\forall x, y \in G : \gamma\phi(xy) = \gamma\phi(x)\gamma\phi(y)$

Let $x, y \in G$ be arbitrary.

Compute,

$$\begin{aligned} \gamma\phi(xy) &= \gamma(\phi(xy)) \\ &= \gamma(\phi(x)\phi(y)) \\ &= \gamma(\phi(x))\gamma(\phi(y)) \\ &= \gamma\phi(x)\gamma\phi(y) \end{aligned}$$

Additional Exercises

1 Let $\phi : G \to G'$ be a homomorphism of groups and suppose $g \in G$ is an element of finite order.

(a) Prove that $|\phi(g)|$ divides $|g|$.

pf.

Let $|g| = n$, $g^n = e \implies \phi(g)^n = \phi(g^n) = g(e) = e'$.

Now,$\phi(g)^n = e'$ doesn't mean $|\phi(g)| = n$.

But, it does mean that $\exists k, m \in \mathbb{Z}^+ : n = mk \wedge g^m = e'$, where m is the order of $\phi(g)$.

So, the order of $\phi(g)$ divides the order of g ■.

(b) Prove that $|\phi(g)| = |g|$ if ϕ is injective.

pf.

$\phi(g)^n = \phi(g^n) = \phi(e) = e'$.

Since, ϕ is injective no power smaller than n maps g^n to e' under ϕ.

So, the smallest power you need to raise $\phi(g)$ to get e' is n.

So, ϕ is a bijection. So $|\phi(g)| = |g|$ ■

2** Let G and G' be groups. Prove that a homomorphism $\phi : G \to G'$ is an isomorphism if and only if there exists some homomorphisms $\psi : G' \to G$ such that $\psi \circ \phi = \text{id}_G$ and $\phi \circ \psi = \text{id}_{G'}$.

pf.

$(\implies)$ Ass. $\phi : \langle G, \bullet \rangle \to \langle G', * \rangle$ is an isomorphism

Then ϕ is bijective, and

$$\forall x, y \in G : \phi(x \bullet y) = \phi(x) * \phi(y)$$

ϕ is bijective, so it has an inverse mapping ϕ^{-1}.

That is $\phi \circ \phi^{-1} = 1_G$ and $\phi^{-1} \circ \phi = 1'_G$

So, for all $x, y \in G$ we can write $x = \phi^{-1}(\phi(x))$ and $y = \phi^{-1}(\phi(y))$. Let $\phi(x) = x'$ and $\phi(y) = y'$.

$$\begin{aligned}
\phi^{-1}(\phi(x \bullet y)) &= \phi^{-1}(\phi(x) * \phi(y)) \\
x \bullet y &= \phi^{-1}(\phi(x) * \phi(y)) \\
\phi^{-1}(\phi(x)) \bullet \phi^{-1}(\phi(y)) &= \phi^{-1}(\phi(x) * \phi(y)) \\
\phi^{-1}(x') \bullet \phi^{-1}(y') &= \phi^{-1}(x' * y')
\end{aligned}$$

So, ϕ^{-1} satisfies the homomorphism property, thus is a homomorphism as in the theorem.

($\Longleftarrow$) Ass. $\exists$ homomorphisms $\phi, \psi : \phi \circ \psi = 1_{G'}$ and $\psi \circ \phi = 1_G$

$\psi \circ \phi = 1_G \implies \psi = \phi^{-1} \implies \phi$ is bijective.

ϕ is a homomorphism, so ϕ is an isomorphism.

Thus, proven. This holds for binary algebraic structures, so it holds for groups.

■

3 Show that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$ for all $n \in \mathbb{Z}^+$.

pf.

The determinant det:$GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a homomorphism.

$\det^{-1}[\{1\}] = SL_n(\mathbb{R})$ by definition of $SL_n(\mathbb{R})$. So, Ker(det)=$SL_n(\mathbb{R})$.

By 13.20 $SL_n(\mathbb{R})$ is normal for each $n \in \mathbb{Z}^+$ ■