Homework 7 Solutions

Section 14:

2. Find the order of $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)$.

   Solution. In $\mathbb{Z}_4$, $\langle 2\rangle = \{2, 0\}$, so $|\langle 2\rangle| = 2$. In $\mathbb{Z}_{12}$, $\langle 2\rangle = \{2, 4, 6, 8, 10, 0\}$, so $|\langle 2\rangle| = 6$. Hence

$$|(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)| = \frac{|\,\mathbb{Z}_4 \times \mathbb{Z}_{12}\,|}{|\langle 2\rangle \times \langle 2\rangle|} = \frac{|\,\mathbb{Z}_4\,| \cdot |\,\mathbb{Z}_{12}\,|}{|\langle 2\rangle| \cdot |\langle 2\rangle|} = \frac{4 \cdot 12}{2 \cdot 6} = 4.$$

6. Find the order of $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle (4, 3)\rangle$.

   Solution. In $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$, $\langle (4, 3)\rangle = \{(4, 3), (8, 6), (0, 9), (4, 12), (8, 15), (0, 0)\}$, so

$$|(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle (4, 3)\rangle| = \frac{|\,\mathbb{Z}_{12} \times \mathbb{Z}_{18}\,|}{|\langle (4, 3)\rangle|} = \frac{|\,\mathbb{Z}_{12}\,| \cdot |\,\mathbb{Z}_{18}\,|}{6} = \frac{12 \cdot 18}{6} = 36.$$

10. Find the order of $26 + \langle 12\rangle$ in $\mathbb{Z}_{60}\,/\langle 12\rangle$.

    Solution. We must find the smallest integer n such that $n(26 + \langle 12\rangle) = \langle 12\rangle$. $\langle 12\rangle = \{12, 24, 36, 48, 0\}$ in $\mathbb{Z}_{60}$, so $26 + \langle 12\rangle = \{38, 50, 2, 14, 26\} = 2 + \langle 12\rangle$ (this makes computations easier). We now compute:

$$\begin{aligned}
1 \cdot (2 + \langle 12\rangle) &= 2 + \langle 12\rangle \neq \langle 12\rangle \\
2 \cdot (2 + \langle 12\rangle) &= 4 + \langle 12\rangle \neq \langle 12\rangle \\
3 \cdot (2 + \langle 12\rangle) &= 6 + \langle 12\rangle \neq \langle 12\rangle \\
4 \cdot (2 + \langle 12\rangle) &= 8 + \langle 12\rangle \neq \langle 12\rangle \\
5 \cdot (2 + \langle 12\rangle) &= 10 + \langle 12\rangle \neq \langle 12\rangle \\
6 \cdot (2 + \langle 12\rangle) &= 12 + \langle 12\rangle = \langle 12\rangle.
\end{aligned}$$

    Hence $|26 + \langle 12\rangle| = 6$ in $\mathbb{Z}_{60}\,/\langle 12\rangle$.

14. Find the order of $(3, 3) + \langle (1, 2)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1, 2)\rangle$.

    Solution. $\langle (1, 2)\rangle = \{(1, 2), (2, 4), (3, 6), (0, 0)\}$ in $\mathbb{Z}_4 \times \mathbb{Z}_8$.

$$\begin{aligned}
1 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (3, 3) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
2 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (2, 6) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
3 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (1, 1) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
4 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (0, 4) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
5 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (3, 7) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
6 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (2, 2) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
7 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (1, 5) + \langle (1, 2)\rangle \neq \langle (1, 2)\rangle \\
8 \cdot ((3, 3) + \langle (1, 2)\rangle) &= (0, 0) + \langle (1, 2)\rangle = \langle (1, 2)\rangle
\end{aligned}$$

    Hence $|(3, 3) + \langle (1, 2)\rangle| = 8$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1, 2)\rangle$.

16. Compute $i_{(1,2,3)}[H]$ for the subgroup $H = \{\iota, (2, 3)\}$ of the group S_3.

    Solution. $(1, 2, 3)^{-1} = (1, 3, 2)$, so

$$i_{(1,2,3)}(\iota) = \iota \quad \text{and} \quad i_{(1,2,3,)}((2, 3)) = (1, 2, 3)(2, 3)(1, 3, 2) = (1, 3).$$

    Thus $i_{(1,2,3)}[H] = \{\iota, (1, 3)\}$.

24. Show that A_n is a normal subgroup of S_n and compute S_n/A_n; that is, find a known group to which S_n/A_n is isomorphic.

Proof. Consider the map $\text{sgn} : S_n \to \{-1, 1\}$, which we "showed" in class is a group homomorphism (where $\{-1, 1\}$ is a group under multiplication). Now sgn was defined such that $\text{sgn}(\sigma) = 1$ if and only if σ is even, i.e. $\sigma \in A_n$. Hence $\text{Ker}(\text{sgn}) = A_n$. It follows that A_n is normal in S_n. □

Solution. We consider the case when $n = 1$ separately. $S_1 = A_1 = \{\iota\}$, so S_1/A_1 is the trivial group. For $n \geq 2$, we have previously shown that $(S_n : A_n) = 2$, so $|S_n/A_n| = 2$. There is only one group, up to isomorphism, of order 2, so it must be that $S_n/A_n \simeq \mathbb{Z}_2$.

26. Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G and that G/T is torsion free.
[Note: The **torsion subgroup** of an abelian group is the set of all elements of finite order. A group is **torsion free** if its only element of finite order is the identity element.]

Proof. T was already shown on a previous homework to be a subgroup of G and it is normal due to the fact that G is abelian. We therefore simply show that G/T is torsion free. Suppose $gT \in G/T$ satisfies $(gT)^n = T$ for some $n \in \mathbb{Z}^+$. Then $g^n T = T$, i.e. $g^n \in T$. But then g^n has finite order, so there exists some $m \in \mathbb{Z}^+$ such that $(g^n)^m = e$. Hence $g^{mn} = e$ and so g is of finite order, i.e. $g \in T$. Therefore $gT = T$. We have shown that the identity element of G/T is the only element of finite order in G/T and so G/T is torsion free. □

28. Characterize the normal subgroups of a group G in terms of the cells where they appear in the partition given by the conjugacy relation in the Exercise 27.
[Exercise 27: A subgroup H is **conjugate to a subgroup** K of a group G if there exists an inner automorphism i_g of G such that $i_g[H] = K$. Show that conjugacy is an equivalence relation on the collection of subgroups of G.]

Solution. The normal subgroups are exactly the subgroups in the singleton cells of the partition given by the conjugacy relation.

30. Let H be a normal subgroup of a group G and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

Proof. Note that $|G/H| = (G : H) = m$. We previously showed that any group element raised to the order of the group yields the identity element. Thus $a^m H = (aH)^m = H$ for all $a \in G$ and so $a^m \in H$ for all $a \in G$. □

32. Given any subset S, of a group G, show that it makes sense to speak of the smallest normal subgroup that contains S.
[*Hint*: Use the fact that any intersection of normal subgroups of G is again a normal subgroup of G.]

Proof. Let G be a group and $S \subset G$. Set

$$H := \bigcap_{\substack{S \subseteq K \leq G \\ K \text{ normal}}} K.$$

According to the fact in the hint, H is normal. And since S is contained in every subgroup being intersected, it is also contained in the intersection H itself. Furthermore, if K is any normal subgroup of G containing S, then K is one of the subgroups being intersected and so $H \leq K$. In this sense, H is the smallest normal subgroup of G containing S. □

33. Let G be a group. An element that can be expressed in the form $aba^{-1}b^{-1}$ for some $a, b \in G$ is a **commutator** in G. The preceding exercise shows that there is a smallest normal subgroup C of a group G containing all commutators in G. the subgroup C is the **commutator subgroup** of G. Show that G/C is an abelian group.

Proof. Let $gC, hC \in G/C$. Then $hgh^{-1}g^{-1} \in C$ and so $hgh^{-1}g^{-1}C = C$. Therefore

$$(gC)(hC) = C(gC)(hC) = (hgh^{-1}g^{-1}C)(gC)(hC) = (hgh^{-1}g^{-1}gh)C = (hg)C = (hC)(gC).$$

Hence G/C is abelian. $\square$

34. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

Proof. Let G be a group of finite order and suppose $H \leq G$ is the only subgroup of G with order $|H|$. Given $g \in G$, $i_g[H]$ is a subgroup of G and since i_g is a bijection, $i_g[H]$ has the same order as H. But since H was assumed to be the only subgroup of G with order $|H|$, this implies that $gHg^{-1} = i_g[H] = H$. Since g was arbitrary, this shows that $gHg^{-1} = H$ for all $g \in G$ and so H is a normal subgroup of G. $\square$

<u>Section 15</u>:

4. Classify $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$ according to the fundamental theorem of finitely generated abelian groups.

Solution. $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle \simeq \mathbb{Z}_8$

Proof. $\langle(1,2)\rangle = \{(1,2),(2,4),(3,6),(0,0)\}$ in $\mathbb{Z}_4 \times \mathbb{Z}_8$, so

$$|(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle| = \frac{|\,\mathbb{Z}_4 \times \mathbb{Z}_8\,|}{|\langle(1,2)\rangle|} = \frac{|\,\mathbb{Z}_4\,| \cdot |\,\mathbb{Z}_8\,|}{4} = \frac{4 \cdot 8}{4} = 8.$$

Since $(0,1) + \langle(1,2)\rangle$ must be added to itself 8 times to obtain $\langle(1,2)\rangle$ and $|(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle| = 8$, it must be that $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$ is cyclic of order 8, i.e. it is isomorphic to $\mathbb{Z}_8$. $\square$

8. Classify $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle$ according to the fundamental theorem of finitely generated abelian groups.

Solution. $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle \simeq \mathbb{Z} \times \mathbb{Z}$

Proof. We define a map $\phi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $\phi(\ell, m, n) = (\ell - m, m - n)$. ϕ is easily shown to be a homomorphism and $\phi(1,0,0) = (1,0)$ and $\phi(0,0,-1) = (0,1)$, so ϕ is surjective.

Now $\phi(1,1,1) = (0,0)$, so $\langle(1,1,1)\rangle \subseteq \text{Ker}(\phi)$. If $(\ell, m, n) \in \text{Ker}(\phi)$, then $\ell - m = 0$ and $m - n = 0$, i.e. $\ell = m = n$. Hence $(\ell, m, n) = \ell \cdot (1,1,1) \in \langle(1,1,1)\rangle$. We have show that $\text{Ker}(\phi) \subseteq \langle(1,1,1)\rangle$ and conclude that $\text{Ker}(\phi) = \langle(1,1,1)\rangle$.

Therefore, by the first isomorphism theorem, $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(1,1,1)\rangle \simeq \mathbb{Z} \times \mathbb{Z}$. $\square$

10. Classify $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8)/\langle(0,4,0)\rangle$ according to the fundamental theorem of finitely generated abelian groups.

Solution. $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8)/\langle(0,4,0)\rangle \simeq \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_8$

Proof. The map $\phi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8 \to \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_8$ defined by $\phi(\ell, m, n) \mapsto (\ell, m \cdot 1, n)$ is readily shown to be a surjective homomorphism with kernel $\langle(0,4,0)\rangle$. By the first isomorphism theorem, $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_8)/\langle(0,4,0)\rangle \simeq \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_8$. $\square$

35. Let $\phi : G \to G'$ be a group homomorphism, and let N be a normal subgroup of G. Show that $\phi[N]$ is a normal subgroup of $\phi[G]$.

    Proof. Given $x \in \phi[N]$ and $y \in \phi[G]$, there are some $n \in N$ and $g \in G$ such that $x = \phi(n)$ and $y = \phi(g)$. Then

    $$yxy^{-1} = \phi(g)\phi(n)\phi(g)^{-1} = \phi(g)\phi(n)\phi(g^{-1}) = \phi(gng^{-1}).$$

    Since N is normal in G, $gng^{-1} \in N$ and so $yxy^{-1} = \phi(gng^{-1}) \in \phi[N]$. It follows that $\phi[N]$ is normal in $\phi[G]$. □

36. Let $\phi : G \to G'$ be a group homomorphism and let N' be a normal subgroup of G'. Show that $\phi^{-1}[N']$ is a normal subgroup of G.

    Proof. Given $n \in \phi^{-1}[N']$ and $g \in G$, we have $\phi(n) \in N'$ and $\phi(gng^{-1}) = \phi(g)\phi(n)\phi(g)^{-1}$. Since N' is normal in G' and $\phi(n) \in N'$, it follows that $\phi(gng^{-1}) = \phi(g)\phi(n)\phi(g)^{-1} \in N'$. Thus $gng^{-1} \in \phi^{-1}[N']$. It follows that $\phi^{-1}[N']$ is normal in G. □

Additional Exercises:

1. You showed on the last homework assignment that, for any $n \in \mathbb{Z}^+$, $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$. To what group that we've seen in this class is $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ isomorphic? Prove your claim. [Note: Your answer shouldn't depend on n.]

    Solution. $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \simeq \mathbb{R}^*$ (under multiplication).

    Proof. We've seen that $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a homomorphism with kernel $SL_n(\mathbb{R})$. Since det is obviously surjective, the first isomorphism theorem implies that $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \simeq \mathbb{R}^*$. □

2. Let G be a simple group and suppose $\phi : G \to G'$ is a homomorphism. Prove that ϕ is either the trivial homomorphism or a one-to-one map.

    Proof. First, note that $\text{Ker}(\phi)$ is a normal subgroup of G. Since G is assumed to be simple, it must be that $\text{Ker}(\phi) = \{e\}$ or $\text{Ker}(\phi) = G$. Hence either ϕ is one-to-one or the trivial homomorphism, respectively. □

3. Let G be a group and N a normal subgroup of G. Provide a bijection between the set of subgroups of G/N and the set of subgroups of G that contain N (i.e. all $H \leq G$ such that $N \subseteq H$). Prove that it is indeed a bijection.

    Solution. Define

    $$\phi : \{H \leq G \mid N \subseteq H\} \to \{X \leq G/N\}$$

    by $\phi(H) = \pi[H]$, where $\pi : G \to G/N$ is the natural homomorphism defined by $\pi(g) = gN$. (Pay careful attention to the distinction between parentheses and brackets!)

    Proof. First note that ϕ is indeed well-defined, since the image of a subgroup under a homomorphism was shown to be a subgroup of the codomain.

    To see that ϕ is injective, suppose $\phi(H) = \phi(H')$ for some $H \leq G$ and $H' \leq G$ such that $N \subseteq H$ and $N \subseteq H'$. Then $\pi[H] = \pi[H']$ and so, given $h \in H$, there is some $h' \in H'$ such that $hN = h'N$. But then $(h(h')^{-1})N = (hN)(h'N)^{-1} = N$, so $h(h')^{-1} \in N \subseteq H'$. But then, since $h', h(h')^{-1} \in H'$, we have $h = \left(h(h')^{-1}\right) h' \in H'$. It follows that $H \subseteq H'$. The argument is symmetric in H and H', so we also have $H' \subseteq H$ and therefore $H = H'$. Hence ϕ is injective.

    To see that ϕ is surjective, let $X \leq G/N$. We claim that $\pi^{-1}[X]$ is a subgroup of G containing N. First, note that $N \in X$ since N is the identity element of G/N and X is a subgroup of G/N. Second, note that $\pi(n) = nN = N \in X$ for all $n \in N$. Therefore $N \subseteq \pi^{-1}[X]$. Finally, since the preimage of a subgroup under a homomorphism is a subgroup of the domain, $\pi^{-1}[X]$ is a subgroup of G (again, containing N). It remains to observe that $\pi[\pi^{-1}[X]] = X$ (since π is surjective), so $\phi(\pi^{-1}[X]) = X$ and we have shown that ϕ is surjective.

We conclude that ϕ is a well-defined bijection. $\square$