My construction of the natural numbers follows Halmos. However, I claim $0 \notin \mathbb{N}$, minor modifications to the theory are required and are sketched as follows. I leave the details to the interested reader. I worked them out but my computer died so I lost the .tex files, now they only exist in .pdf form in an email I sent to one of my friends.

Put, $\mathbb{N} := \{1, 2, 3, 4, 5, \dots\}$ the natural numbers. Define, $|\emptyset| := 0$. To count is to give an isomorphism between a set and a standard subset of the natural numbers. An isomorphism is a one to one and onto map that preserves the algebraic structure of the domain and the range. At the level of sets there is none, however we still like to talk about isomorphism to keep the language consistent.

Standard subsets of $\mathbb{N}$ are sequences containing 1, increasing by 1 and ending in some element of $\mathbb{N}$. We call sets that cannot be put into one to one correspondence with a standard subset infinite. If we can put a set into one to one correspondence with $\mathbb{N}$ we say it is countable. The concept of cardinality is defined this way for finite and infinite sets. The cardinality of $\mathbb{N}$, is called $|\mathbb{N}| := \aleph_0$ aleph null, or aleph naught. If a set is neither finite, nor countable, we say it is uncountable and it has higher cardinality.

We define an order on $\mathbb{N}$, by the natural chain of standard sets,

$$\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \cdots \subset \mathbb{N}$$

That is $\{1\} \subset \{1, 2\}$, so $|\{1\}| < |\{1, 2\}|$, and so on. In a weaker sense $\leq$ is defined by not requiring the left hand side to be a proper subset.

The Cartesian product has two interpretations,

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} | x_i \in X_i\} = \{f : I \to \bigcup_{i \in I} X_i | (\forall i \in I)(f(i) \in X_i)\}$$

First, it is the set of all tuples, where each tuple has size $|I|$, where x_j is the j^{th} coordinate of $(x_i)_{i \in I}$.

Second, it is also the set of all maps f such that $i \mapsto f(i)$. From the second interpretation we can recover the concept of coordinates by defining the projection maps,

$$\pi_j : \prod_{i \in I} X_i \to X_j; f \mapsto f(j)$$

Then, $\pi_k(f) = f(k)$ is the component of f in the k direction.

When $X_i = X$ for all $i \in I$, and $I \subset \mathbb{N}$, with $n = |I|$, we call this the Cartesian power, and we write X^n.

An operation on a set S is a map $* : S^n \to S$. The size n of the input vector is the arity of the operation. We give special names to the first few naturals 1, 2, and 3-arity operations are called unary, binary, and ternary operations respectively. For operations with arity n we say they are n-ary.

Now, 0 arity operations are special, they are called nullary operations. First we need to give meaning to S^0. To accomplish this we set $I = \emptyset$. We define,

$$X^0 := \prod_{i \in \emptyset} X = \{(x_i)_{i \in \emptyset} | x_i \in X\} = \{f : \emptyset \to \bigcup_{i \in \emptyset} X_i | (\forall i \in \emptyset)(f(i) \in X_i)\}$$

In the first interpretation since there is nothing in the empty set, it follows $X^0 = \{()\}$, where $()$ has no coordinates, it is called the empty tuple.

In the second interpretation, $\forall i \in \emptyset$ is unbound as it determines no values i. Furthermore, $\bigcup_{i \in \emptyset} X_i = \emptyset$ for the same reason. So, $X^0 = \{f : \emptyset \to \emptyset\}$.

Recall from Naïve Set Theory (Halmos), that a function is a subset of the Cartesian product of its domain and range. $\emptyset \times \emptyset = \emptyset \implies X^0 = \{\emptyset\}$.

In both cases $|X^0| = 1$. So, any nullary operation $S^0 \to S$ is just an element of S. The set of all nullary operations on S, is in one to one correspondence with the elements of S.

If $|I| \geq \aleph_0$, then we need to accept the Axiom of Choice in order to ensure that the Cartesian product is not empty.