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1. (Book 50.17) Let E be a finite extension of F. Show that if E is a splitting field, then it is a splitting field of a single polynomial $f(x) \in F[x]$.

   pf.

   If the field F is finite of characteristic p a prime number, then every finite extension of F is generated by the polynomial $x^{p^n} - x$ for some n.

   Otherwise, for F is infinite we have.

   Let $F(\alpha, \beta)$, be a finite extension of F.

   Let m_α, and m_β be the minimal polynomials for α and β.

   Let $\alpha_1, \dots, \alpha_n$, be the n distinct roots of m_α.

   Let $\beta_1, \dots, \beta_n$, be the m distinct roots of m_β.

   Since, F is infinite, let $\gamma = \alpha + \kappa\beta : \gamma \neq \alpha_i + \kappa\beta_j$ for $1 \leq i \leq n$, and $1 \leq j \leq m$.

   $h(x) := m_\alpha(\gamma - \kappa x)$.

   $h(\beta) = m_\alpha(\gamma - \kappa\beta) =_\alpha (\alpha) = 0 \implies (x - \beta) | h$.

   So h splits in $F(\gamma)$, and $\beta \in F(\gamma)$. But $\gamma - \kappa\beta = \alpha$, so $\alpha \in F(\gamma)$. So, $F(\gamma) = F(\alpha, \beta)$.

   Then an induction establishes the result ■

2. (Book 50.20) Show that the automorphism group $Aut(\mathbb{Q}(2^{(1/3)})/\mathbb{Q})$ is the trivial group. (Remember that $Aut(E/F)$ is the group of automorphisms of $\varphi : E \to E$ such that $\varphi(a) = a$ for all $a \in F$.)

   slu.

   $x = 2^{1/3} \implies x^3 = 2 \implies x^3 - 2 = 0$.

   $x^3 - 2$ is thus the minimal irreducible polynomial for $2^{1/3}$, by Einsenstein's criterion with the prime being 2.

   $x^3 - 2 = 0 \implies x^3 = 2\exp(i(2n\pi)) \implies x = 2^{1/3}\exp(i\left(\frac{2n\pi}{3}\right)) \quad (n = 0, 12)$.

   So, $x = 2^{1/3}$, or $x = 2^{1/3}\exp(i\left(\frac{2\pi}{3}\right))$, or $x = 2^{1/3}\exp(i\left(\frac{4\pi}{3}\right))$.

   $\mathbb{Q}(2^{1/3})$, only contains one of the roots of $x^3 - 2$, so since permutations in $Aut(\mathbb{Q}(2^{(1/3)}/\mathbb{Q})$ fix $\mathbb{Q}$, any such permutation must map $2^{1/3} \mapsto 2^{1/3}$. Thus, $Aut(\mathbb{Q}(2^{(1/3)}/\mathbb{Q})$, is trivial ♢

3. (Book 51.3) Find an $\alpha \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) =: E$ so that $E = \mathbb{Q}(\alpha)$. Compute explicitly how to write $\sqrt{2}$ and $\sqrt{3}$ in terms of α.

   slu.

$$\begin{aligned}
\alpha = \sqrt{2} + \sqrt{3} &\iff \alpha - \sqrt{2} = \sqrt{3} \\
&\iff \alpha^2 - 2\sqrt{2}\alpha + 2 = 3 \\
&\iff \alpha^2 - 1 = 2\sqrt{2}\alpha \\
&\iff \alpha^4 - 2\alpha^2 + 1 = 8\alpha^2 \\
&\iff \alpha^4 - 10\alpha^2 + 1 = 0
\end{aligned}$$

   By the quadratic formula,

$$\alpha^2 = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6} \implies \alpha = \pm\sqrt{5 \pm 2\sqrt{6}} \notin \mathbb{Q}$$

   So, $x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$. Furthermore, $\mathbb{Q}(\alpha)$ has basis $\{1, \alpha, \alpha^2, \alpha^3\}$.

   $\alpha^2 = 5 + 2\sqrt{6}$, $\alpha^3 = 11\sqrt{2} + 9\sqrt{3}$.

   So $\sqrt{2} = \frac{9\alpha - \alpha^3}{9-11}$, and $\sqrt{3} = \frac{11\alpha - \alpha^3}{11-9}$. So $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ ♢

4. (Book 51.17) Let K be a finite normal extension of F and let $G = Aut(K/F)$. Define the norm and trace of an element $\alpha \in K$ as follows:

$$N(\alpha) := \prod_{\sigma \in G} \sigma(\alpha), \qquad Tr(\alpha) := \sum_{\sigma \in G} \sigma(\alpha)$$

Show that $N(\alpha)$ and $Tr(\alpha)$ are elements of F. (Hint: what is $\sigma(N(\alpha))$ for $\sigma \in G$? Then, look at part 2 of Theorem 53.6 in the special case $E = F$.)

pf.

Let $\tau \in G, \alpha \in K$. Then, since τ is a ring homomorphism,

$$\tau(N(\alpha)) = N(\alpha), \text{ and } \tau(Tr(\alpha)) = Tr(\alpha).$$

So, since τ fixes $N(\alpha)$, and $Tr(\alpha)$,it follows that they must be in F $\quad\Diamond$

5. (Book 51.18 (a,b,e,f)) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Using the definition in the previous problem, compute the norm and trace of $\alpha = \sqrt{2}$ and $\beta = \sqrt{2} + \sqrt{3}$.

slu.

By the definitions in page 450 $Aut(K/\mathbb{Q}) = \{\iota, \sigma_1, \sigma_2, \sigma_3\}$, with the following assignments leaving other elements fixed,

$$\begin{aligned}
&\sigma_1; \sqrt{2} \mapsto -\sqrt{2}, \sqrt{6} \mapsto -\sqrt{6}, \\
&\sigma_2; \sqrt{3} \mapsto -\sqrt{3}, \sqrt{6} \mapsto -\sqrt{6}, \\
&\sigma_3; \sqrt{2} \mapsto -\sqrt{2}, \sqrt{3} \mapsto -\sqrt{3}.
\end{aligned}$$

So,

$$N(\alpha) = \iota(\sqrt{2})\sigma_1(\sqrt{2})\sigma_2(\sqrt{2})\sigma_3(\sqrt{2})) = (\sqrt{2})(-\sqrt{2})(\sqrt{2})(-\sqrt{2}) = 4, \text{ and } Tr(\alpha) = 0.$$

So,

$$\begin{aligned}
N(\beta) &= \iota(\sqrt{2}+\sqrt{3})\sigma_1(\sqrt{2}+\sqrt{3})\sigma_2(\sqrt{2}+\sqrt{3})\sigma_3(\sqrt{2}+\sqrt{3})) \\
&= (\sqrt{2}+\sqrt{3})(-\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})(-\sqrt{2}-\sqrt{3}) \\
&= -(5+2\sqrt{6})(-\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3}) \\
&= -(5+2\sqrt{6})(-(2-\sqrt{6})+(\sqrt{6}-3)) \\
&= -(5+2\sqrt{6})(-5+2\sqrt{6}) \\
&= -(-25+24) = 1, \text{ and } Tr(\beta) = 0.
\end{aligned}$$