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Here we consider properties of $1-x$ viewed as an element of different rings.

(a) Show that $f = 1-x$ is a unit in $\mathbb{Z}[[x]]$.

(b) Show that $f(x) = 1-x$ is not a unit in the ring of continuous functions on $\mathbb{R}$.

(c) Show that $f(x) = 1-x$ is a unit in the ring of continuous functions on $[0, 1/2]$.

slu. of (a)

$$\text{Let } S = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

$$\implies xS = x\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+1}$$

$$\text{Let } n = k+1 \implies xS = \sum_{n=1}^{\infty} x^n$$

$$S = 1 + \sum_{n=1}^{\infty} x^n \implies S = 1 + xS \implies S - xS = 1$$

$$\implies (1-x)\sum_{n=0}^{\infty} x^n = 1 \implies 1-x \text{ is a unit in } R[[x]] \quad \blacklozenge$$

slu. of (b)

f would be a unit if there exists g in $C(\mathbb{R})$ where,

$$(fg)(x) = f(x)g(x) = 1$$

That is if $f(x) = 1-x$, then,

$$(1-x)g(x) = 1 \iff g(x) = \frac{1}{1-x} \ .$$

But, $g(x) \notin C(\mathbb{R})$, since $g(1)$ is an essential discontinuity of g. That is we can't find some other function that is continuous on $\mathbb{R}$, since neither the left hand limit nor the right hand limit of g as $x \to 0$ exist. So, $1-x$ is not a unit in the ring of continuous functions on $\mathbb{R}$ $\quad\lozenge$

slu. of (c)

$1 \notin [0, \frac{1}{2}]$ so $\frac{1}{1-x}$ is continuous on $[0, \frac{1}{2}]$. So by the computation in the solution of (b) we have that $1-x$ is a unit in $C([0, \frac{1}{2}])$ $\quad\lozenge$