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Let R be an integral domain, and let $S = R[[x]]$, which is the ring of formal power series in a variable x. (So an arbitrary element of S is $f(x) = \sum_{i=0}^{\infty} a_i x^i$ where $a_i \in R$. In this ring we don't care about convergence issues, so we don't think of $f(x)$ as functions because we can't necessarily "plug in values of x." But we can still add and multiply formal power series, regardless of whether they converge, so S is still a ring.) Show that S is an integral domain.

pf.

Let $f, g \in S : f = \left(\sum_{i=0}^{\infty} a_i x^i\right)$, and $g = \left(\sum_{i=0}^{\infty} b_i x^i\right)$ then,

$$fg := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \ .$$

R is an integral domain. So, $1 \in R$.

$1 \neq 0$ in $R[[x]]$, since $1 \neq 0$ in R.

Let $f = 1$, then $a_0 = 1$, and $a_i = 0 \quad \forall i > 0$. So,

$$\begin{aligned}
1g &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(a_0 b_n + \sum_{k=1}^{n} a_k b_{n-k}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(1 b_n + \sum_{k=1}^{n} 0 b_{n-k}\right) x^n \\
&= \sum_{n=0}^{\infty} b_n x^n \\
&= g \ .
\end{aligned}$$

Let $g = 1$, then $b_0 = 1$, and $b_i = 0 \quad \forall i > 0$. So,

$$\begin{aligned}
f1 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \\
f1 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} a_k b_{n-k} + a_n b_0\right) x^n \\
f1 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} a_k 0 + a_n 1\right) x^n \\
f1 &= \sum_{n=0}^{\infty} a_n x^n \\
&= f \ .
\end{aligned}$$

So $\forall f \in R[[x]] : 1f = 1 = f1$, that is 1 is the multiplicative identity of $R[[x]]$.

$$fg = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

Let $l = n - k \implies k = n - l$.

k starts at 0 and ends at $n \implies l$ starts at n and ends at 0 .

$$\begin{aligned}
\text{So,} fg &= \sum_{n=0}^{\infty} \left(\sum_{l=n}^{0} a_{n-l} b_l \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} a_{n-l} b_l \right) x^n \text{ by commutativity of addition in } R \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} b_l a_{n-l} \right) x^n \text{ by commutativity of multiplication in } R \\
&= gf \qquad\qquad \text{by the definition of multiplication in } R[[x]] \text{ .}
\end{aligned}$$

So, $\forall f, g \in R[[x]], fg = gf$. That is multiplication is commutative in $R[[x]]$.

If $f \neq 0$ and $h = \sum_{j=0}^{\infty} c_j x^j \neq 0$, then there is a first non-zero coefficient of a_i of f, and c_j of h.

$$fh = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k c_{n-k} \right) x^n$$

Consider the coefficient of x^n, when $n = i + j$.

$$\begin{aligned}
n = i + j &\implies \text{The coefficient of } x^n \text{ is } \sum_{k=0}^{n} a_k c_{i+j-k} \\
k < i &\implies a_k = 0 \\
&\implies \sum_{k=0}^{i+j} a_k c_{i+j-k} = \sum_{k=i}^{i+j} a_k c_{i+j-k} \\
k > i \implies i + j - k < j &\implies c_{i+j-k} = 0 \\
&\implies \sum_{k=0}^{i+j} a_k c_{i+j-k} = a_i cj \neq 0 \\
&\implies \text{ The coefficient of } x^{i+j} \text{ is not } 0 \text{ .}
\end{aligned}$$

So, $f \neq 0$ and $h \neq 0 \implies fh \neq 0$. Which is the contrapositive statement to,

$$fh = 0 \implies f = 0 \text{ or } h = 0.$$

So, if R is an integral domain, then $R[[x]]$ is an integral domain. ■