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(a) Show that every element of $M_{n\times n}(\mathbb{Q})$ is either a unit or a zero divisor.

(b) Find an element of $M_{2\times 2}(\mathbb{Z})$ that is not a unit or a zero divisor.

pf. of (a)

Let $A \in M_{n\times n}(\mathbb{Q})$

$$A \text{ is invertible } \iff \text{nullspace}(A) = \{\vec{0}\}$$

$$\text{Suppose that nullspace}(A) \neq \{\vec{0}\}$$

$$\implies \exists \vec{x} \in \mathbb{Q}^n : A\vec{x} = \vec{0} \text{ where } \vec{x} \neq \vec{0}\ .$$

Then, form the square$n \times n$ matrix,

$$B = [\vec{x}|\vec{0}|\cdots|\vec{0}]$$

with $\vec{x}$ in the first column and $\vec{0}$ in the remaining $n-1$ columns, for $n \geq 1$.

Then, $AB = 0_{2\times 2}$, so if det$(A) = 0$, A is a zero divisor.

Suppose A is a zero divisor. Then $\exists C = [\vec{c_1}|\ldots|\vec{c_n}] \in M_{n\times n}(\mathbb{Q}) : C \neq 0_{2\times 2}$ and $AC = 0_{2\times 2}$.

$$\implies C = [\vec{c_1}|\vec{0}|\cdots|\vec{0}] + [\vec{0}|\vec{c_2}|\cdots|\vec{0}] + \cdots + [\vec{0}|\vec{0}|\cdots|\vec{c_n}]$$

$$\implies AC = A([\vec{c_1}|\vec{0}|\cdots|\vec{0}]+[\vec{0}|\vec{c_2}|\cdots|\vec{0}]+\ldots\cdots+[\vec{0}|\vec{0}|\cdots|\vec{c_n}]) = A[\vec{c_1}|\vec{0}|\cdots|\vec{0}]+A[\vec{0}|\vec{c_2}|\cdots|\vec{0}]+\cdots+A[\vec{0}|\vec{0}|\cdots|\vec{c_n}] = 0_{n\times n}$$

$$\implies \exists i : \vec{c_i} \neq \vec{0} \text{ and } A\vec{c_i} = 0$$

$$\implies \text{det}(A) = 0$$

If det$(A) \neq 0$, then A is invertible, i.e: $\exists B \in M_{n\times n}(\mathbb{Q}) : AB = BA = I_{n\times n}$. So, A is a unit. Therefore, every element of $M_{n\times n}(\mathbb{Q})$ is either a zero divisor or a unit $\blacksquare$

slu. of (b)

By above if a matrix is a zero divisor it's determinant has to be zero. Therefore, we need a matrix with non-zero determinant that's not a unit.

Consider $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, it has a unique inverse $\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in the ring of matrices with rational entries. However, $\frac{1}{2} \notin \mathbb{Z}$, so one of its entries is not an integer. Therefore, C is not a unit in $M_{2\times 2}(\mathbb{Z})$ $\Diamond$