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The <u>center</u> of a ring R is

$$Z(R) := \{x \in R \mid xa = ax \quad \forall a \in R\}$$

1. Show $Z(R)$ is a subring of R.

   pf.

   Let $x, y \in Z(R), a \in R$. Then,

$$a0 = 0 = 0a$$

$$\begin{aligned} a, x, y \in R &\implies a(x-y) = ax - ay \\ y \in Z(R) &\implies a(x-y) = ax - ya \\ x \in Z(R) &\implies a(x-y) = xa - ya \\ a, x, y \in R &\implies xa - ya = (x-y)a \\ &\implies a(x-y) = (x-y)a \end{aligned}$$

$$\begin{aligned} y \in R &\implies xy = yx \\ \text{Multiplication by } a &\implies axy = ayx \\ y \in Z(R) &\implies axy = yax \\ x \in Z(R) &\implies axy = yxa \end{aligned}$$

   So, $x, y \in \mathbb{Z}(R) \implies 0, x-y$, and $xy \in \mathbb{Z}(R)$.

   Thus, by the conclusion of exercise 48 in p. 176 $Z(R)$ is a sub-ring of R ■

2. Show that the center of $M_{2\times 2}(\mathbb{R})$ is spanned (as a vector space) by the identity matrix.

   pf.

   Suppose, $\exists X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in Z\left(M_{2\times 2}(\mathbb{R})\right) : \forall r \in \mathbb{R} \quad X \neq rI$. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\begin{aligned} XA = AX \implies \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} x_1 & 0 \\ x_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ &\implies x_2 = x_3 = 0 \qquad \implies X = \begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} XB = BX \implies \begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_4 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix} \\ &\implies x_1 = x_4 \\ &\implies X = x_1 I \end{aligned}$$

   $X \neq rI \rightarrow\leftarrow X = x_1 I$. So by contradiction $X = rI$.

   Therefore, $Z(M_{2\times 2}(\mathbb{R})) = \text{Span}\,\{I\}$ ■

3. Let $R = \mathbb{R}S_n$ be the group algebra of the symmetric group, and define the element

$$x = \sum_{1 \leq i \neq j \leq n} (i\,j)$$

(For example, if $n = 3$, then $x = (1\,2) + (1\,3) + (2\,3)$.) Show that for any transposition $(k\,\ell) \in S_n$ we have $(k\,\ell)x(k\,\ell) = x$.
(Hint: you can use the fact that if $\sigma \in S_n$ and $(a_1 \cdots a_k)$ is a cycle in S_n, then $\sigma(a_1\,a_2 \cdots a_k)\sigma^{-1} = (\sigma(a_1) \cdots \sigma(a_k))$.)
Note: this actually shows $x \in Z(S_n)$, because any element of S_n is a product of transpositions.
pf.

$$(k\,l)x(k\,l) = \sum_{1 \leq i \neq j \leq n} (k\,l)(i\,j)(k\,l) \quad \text{by left and right distribution}$$

Now either, (I) $i \neq k$ and $j \neq k$, (II) $i = k$ and $j \neq l$, (III)$i \neq k$ and $j = l$, or (IV) $i = k$ and $j = l$.
(I): $(k\,l)(i\,j)(k\,l) = (k\,l)(k\,l)(i\,j) = (i\,j)$
(II): $(k\,l)(k\,j)(k\,l) = (k\,l)(l\,j) = (k\,j)$
(III): $(k\,l)(i\,l)(k\,l) = (k\,l)(k\,i) = (i\,l)$
(I): $(k\,l)(k\,l)(k\,l) = (k\,l)(k\,l)(k\,l) = (k\,l)$
Thus,

$$(k\,l)x(k\,l) = x \quad \blacksquare$$

4. Use Fermat's theorem to compute the remainder of 37^{49} when it is divided by 7.

slu. $37^{49} = 37^{48+1} = 37^{48}37 = 37^{6 \cdot 8}37 = (37^6)^8 37 \equiv_7 1^8 37 \equiv_7 37 \equiv_7 5 \cdot 7 + 2 \equiv_7 2$ So, 2 ♦

5. Let $\phi_a : R[x] \to R$ be the evaluation homomorphism, defined by $\phi_a(f) = f(a)$.
Let $f(x) = (x - 2)(x + 3)$.

(a) If $R = \mathbb{Z}$, find all $a \in R$ with $\phi_a(f) = 0$.
slu. $\mathbb{Z}$ is an integral domain, thus $a = 2$, or $a = -3$ $\Diamond$

(b) If $R = \mathbb{Z}/8\mathbb{Z}$, find all $a \in R$ with $\phi_a(f) = 0$.
slu.

$$\begin{aligned}
f(1) &= (1-2)(1+3) \equiv_8 4 \\
f(2) &= (2-2)(2+3) \equiv_8 0 \\
f(3) &= (3-2)(3+3) \equiv_8 6 \\
f(4) &= (4-2)(4+3) \equiv_8 6 \\
f(5) &= (5-2)(5+3) \equiv_8 0 \\
f(6) &= (6-2)(6+3) \equiv_8 4 \\
f(7) &= (7-2)(7+3) \equiv_8 2
\end{aligned}$$

So, $a = 2$ or $a = 5$ $\Diamond$