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1. Let R be a commutative ring with identity element $1_R \in R$, and suppose that R has characteristic $p \in \mathbb{Z}$. (I.e. $pa = 0$ for all $a \in R$.) Show that the map $\phi : R \to R$ defined by $\phi(a) = a^p$ is a ring homomorphism.

   pf.

   Let $a, b \in R$

$$\begin{aligned}\phi(a+b) &= (a+b)^p \\ &= \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k} \\ &= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} a^k b^{p-k} \\ &= \frac{p!}{0!(p-0)!} a^0 b^{p-0} + \sum_{k=1}^{p-1} \frac{(p-1)!}{k!(p-k)!} paa^{k-1} b^{p-k} + \frac{p!}{p!(p-p)!} a^p b^{p-p} \\ &= b^p + \sum_{k=1}^{p-1} \frac{(p-1)!}{k!(p-k)!} 0a^{k-1} b^{p-k} + a^p \text{ by } pa = 0 \\ &= b^p + a^p \\ &= a^p + b^p \\ &= \phi(a) + \phi(b) \\ \phi(ab) &= (ab)^p \\ &= abab \cdots abab \quad \text{p times} \\ &= a^p b^p \quad \text{by commutativity of multiplication} \\ &= \phi(a)\phi(b)\end{aligned}$$

   So, ϕ is a ring homomorphism ■

2. Let R be a commutative ring and let $a \in R$. Show that $I_a := \{x \in R \mid ax = 0\}$ is an ideal of R.

   pf.

   Let $y \in R$, and $x \in I_a$.

$$yxa = yax = y0 = 0 = 0y = axy \quad \text{by commutativity of } R$$

   Shows, that any $x \in I_a$ and y is any element of R, then both $xy, yx \in I_a$ so I_a is an ideal ■

3. Let R be a commutative ring and $I \subset R$ an ideal. Show that the following set is also an ideal of R:

$$\sqrt{I} := \{a \in R \mid a^k \in I \text{ for some } k\}$$

   pf.

   Let $a \in \sqrt{I}$, and $b \in R$.

   $a \in \sqrt{I} \implies \exists k \in \mathbb{Z} : a^k \in I$, then $(ab)^k = a^k b^k = (ba)^k$ by the commutativity of R.

   Since $b^k \in \mathbb{R}$, then $a^k b^k \in I$, so $(ab)^k$ and $(ba)^k$ are in I.

   Therefore ab and ba are in $\sqrt{I}$ and $\sqrt{I}$ is an ideal ■

4. Find $\sqrt{I}$ for the following two ideals:

(a) $I = 500\mathbb{Z} \subset \mathbb{Z}$

slu.

$\forall m \in \mathbb{Z} : \exists n, k \in \mathbb{Z} : m^k = 500n$

$\implies m = (500n)^{1/k} = (5 \cdot 10^2 n)^{1/k} = (5^3 \cdot 2^2 n)^{1/k}$

$k = 0$ doesn't work because $1 \neq 500$.

$k = 1 \implies n = 1 \implies m = 500$.

$k = 2$, and $\implies m = (5^3 \cdot 2^2 n)^{1/2}$

We want to find n such that the equation is solvable over the integers.

The smallest such solutions would be desirable.

$n = 5 \implies m = (5^4 \cdot 2^2)^{1/2} = 50$

$n = 5 \cdot 2^2 \implies m = (5^4 \cdot 2^4)^{1/2} = 100$

So, 50 and 100 are in $\sqrt{I}$.

Now for, $k > 2$ we can see the following, $k = 3$, and $n = 2 \implies m = (5^3 \cdot 2^3)^{1/3} = 10$

$k = 4$, and $n = 5 \cdot 2^2 \implies m = (5^4 \cdot 2^4)^{1/4} = 10$.

$\forall l \in \mathbb{Z} : l \geq 3$ if $k = l$, we can put $n = 5^{l-3} \cdot 2^{l-2}$, when we have $m = (5^3 \cdot 2^2 \cdot 5^{l-3} \cdot 2^{l-2})^{1/k} = 10$.

So m is $500, 100, 50$, or 10. An note 10 divides all the others.

Thus, $(10l)^k = 10^k l^k = 500nl^k$ for some k. Therefore $10\mathbb{Z} \subset \sqrt{500\mathbb{Z}}$.

Suppose for a contradiction $\exists m \in \sqrt{500\mathbb{Z}} : \forall l \in \mathbb{Z} : m \neq 10l$

Since $m \in \sqrt{500\mathbb{Z}}$ we can see that there is a $k \in \mathbb{Z}$ such that,

$m^k = 500n = 10 \cdot 50n \implies 10|m^k \implies 10|m \implies \exists l \in \mathbb{Z} : m = 10l \rightarrow\leftarrow m \neq 10l$

Therefore $\sqrt{500\mathbb{Z}} \subset 10\mathbb{Z}$. And by the previous containment we can see that

$$10\mathbb{Z} = \sqrt{500\mathbb{Z}} \quad \blacksquare$$

(b) $I = \langle x^3 \rangle \subset \mathbb{Q}[x]$.

Remember $\langle f(x) \rangle$ means "the ideal generated by $f(x)$," so

$$\langle f(x) \rangle = \{ f(x)g(x) \mid g(x) \in \mathbb{Q}[x] \}$$

slu.

$\langle x \rangle \subset \sqrt{I}$ since given $x^n \in \langle x \rangle$, $(x^n)^3 = x^{3n} = x^3 x^{3n-3} \in I$, $\forall n \in \mathbb{Z}_{\geq 0}$.

Suppose for the sake of contradiction that $\exists f(x) \in \sqrt{I} : \forall g(x) \in \mathbb{Q}[x] : f(x) \neq xg(x)$

$f(x) \in \sqrt{I} \implies \exists k \in \mathbb{Z} : f(x)^k = h(x) \in I \implies f(x)^k = x^3 d(x) = xx^2 d(x) \implies x|f(x)^k \implies x|f(x)$

Therefore, we arrive at a contradiction. That is if $f(x) \in \sqrt{I}$, then $f(x) \in \langle x \rangle$.

Thus $\sqrt{I} \subset \langle x \rangle$,

$$\sqrt{I} = \langle x \rangle \quad \blacksquare$$