Q2: Let $A \in M_{2\times 2}(\mathbb{C})$ and define a ring homomorphism $\phi : \mathbb{C}[x] \to M_{2\times 2}(\mathbb{C})$ by

$$\phi(f) := f(A)$$

Find a quadratic polynomial $f(x)$ such that $\phi(f) = 0$ (where 0 means the matrix with all 0 entries). (You can assume the entries of A are $a, b, c, d \in \mathbb{C}$, and you can write your answer in terms of these entries if you want to.)

pf.

The characteristic polynomial of A works.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{n\times n}(\mathbb{C})$.

Then $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc)I$.

$$\begin{aligned}
p(A) &= A^2 - (a+d)A + (ad-bc)I \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} + \begin{pmatrix} -a^2-ad & -ba-bd \\ -ca-cd & -da-d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ba-bd \\ ca+dc-ca-cd & cb+d^2-da-d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} bc-ad & 0 \\ 0 & cb-da \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

That is $\phi(p) = 0$