Wilf

1

(a) $a_n = n$

$$a_n = n \implies f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} nx^n$$

Notice $g(x) = \frac{1}{1-x} = \sum x^n$

$$\implies g'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \implies f(x) = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n = xg'(x) = \frac{}{(1-}$$

In [380]:
```
var('x')
f_a(x) = x*(1/(1-x)).derivative(x);f_a
```
Out[380]:
```
x |--> x/(x - 1)^2
```

In [379]:
```
def first_n (f,n):
    return [(f.derivative(x,k)/factorial(k))(0) for k in range(n)]
```

In [381]:
```
first_n (f_a,10)
```
Out[381]:
```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

(b) $a_n = \alpha n + \beta$

$$\sum a_n = \sum(\alpha n + \beta)x^n$$

$$f(x) = \sum a_n = \alpha \sum nx^n + \beta \sum x^n = \alpha\frac{x}{(1-x)^2} + \beta\frac{1}{1-x}$$

$$f(x) = \frac{\alpha x+\beta(1-x)}{(1-x)^2}$$

Put $\alpha = 1, \beta = 11$, then

$$f(x) = \frac{x+11(1-x)}{(1-x)^2}$$

In [204]:
```
[1*n + 11 for n in range(10)]
```
Out[204]:
```
[11, 12, 13, 14, 15, 16, 17, 18, 19, 20]
```

In [382]:
```
f_b(x) = x/(1-x)^2 + 11/(1-x);f_b(0)
```
Out[382]:
```
11
```

In [383]:
```
first_n(f_b,10)
```
Out[383]:
```
[11, 12, 13, 14, 15, 16, 17, 18, 19, 20]
```

(c) $a_n = n^2$

$\sum a_n = \sum n^2 x^n = Li_{-2}(x) = \frac{x(1+x)}{(1-x)^3}$ by wikipedia

In [300]:
```
[n^2 for n in range(10)]
```
Out[300]:
```
[0, 1, 4, 9, 16, 25, 36, 49, 64, 81]
```

In [384]:
```
f_c(x) = (x*(1+x))/((1-1*x)^3)
first_n(f_c,10)
```
Out[384]:
```
[0, 1, 4, 9, 16, 25, 36, 49, 64, 81]
```

(d) $a_n = \alpha n^2 + \beta n + \gamma$

By observation $f(x) = \alpha Li_{-2}(x) + \beta Li_{-1}(x) + \gamma Li_0(x) = \alpha \frac{x(1+x)}{(1-x)^3} + \beta \frac{x}{(1-x)^2} + \gamma \frac{x(1+x)}{1-x}$

Put $\alpha = 3, \beta = 2, \gamma = 5$, then

$f(x) = 3\frac{x(1+x)}{(1-x)^3} + 2\frac{x}{(1-x)^2} + 5\frac{x(1+x)}{1-x}$

In [302]:
```
[3*n^2+2*n +5 for n in range(10)]
```
Out[302]:
```
[5, 10, 21, 38, 61, 90, 125, 166, 213, 266]
```

In [387]:
```
f_d(x) = 3*x*(1+x)/((1-x)^3) + 2*x/((1-x)^2) +5/(1-x)
```

In [388]:
```
first_n(f_d,10)
```
Out[388]:
```
[5, 10, 21, 38, 61, 90, 125, 166, 213, 266]
```

(e) $a_n = P(n)$, where P is a given polynomial, of degre m

If $P(n) = \sum_{k=0}^m b_k n^k$, then $f(x) = \frac{b_0}{1-x} + \sum_{k=1}^m b_k Li_{-k}(x) = b_0 \frac{Li_0(x)}{x} + \sum_{k=1}^m b_k Li_{-k}(x)$

Note the constant term needs to be treated differently.

Since $Li_0(x) = \frac{x}{1-x}$, so $Li_0(0) = 0$, thus the constant term is not taken into account as the first coefficient of the expansion.

Let $P(n) = 7n^4 + 19n^3 + 23$

In [305]:
```
[7*n^8+19*n^3 + n +23 for n in range(10)]
```

Out[305]:
```
[23,
 50,
 1969,
 46466,
 459995,
 2736778,
 11761445,
 40360154,
 117450271,
 301340930]
```

In [344]:
```
def Li_n (n,z):
    if n > 0:
        return polylog(n,z)
    else:
        f = z/(1-z)
        for k in range(-n):
            f = z*f.derivative(z,1)
        return f
```

In [389]:
```
f_e(x) = 7*Li_n(-8,x) +19*Li_n(-3,x)+Li_n(-1,x) +23/(1-x);f_e(0)
```

Out[389]:
```
23
```

In [390]:
```
first_n_app(f_e,10)
```

Out[390]:
```
[23,
 50,
 1969,
 46466,
 459995,
 2736778,
 11761445,
 40360154,
 117450271,
 301340930]
```

(f) $a_n = 3^n$

$\implies f(x) = \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$

In [356]:
```
[3^n  for n in range(10)]
```

Out[356]:
```
[1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683]
```

In [391]:
```
f_f(x) = 1/(1-3*x)
```

In [392]:
```
first_n(f_f,10)
```

Out[392]: [1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683]

(g) $a_n = 5 \cdot 7^n - 3 \cdot 4^n$

$\implies f(x) = \frac{5}{1-7x} - \frac{3}{1-4x}$

In [359]:
```
[5*7^n -3*4^n for n in range(10)]
```

Out[359]: [2, 23, 197, 1523, 11237, 80963, 575957, 4068563, 28627397, 200981603]

In [393]:
```
f_g(x) = 5/(1-7*x) -3/(1-4*x)
```

In [395]:
```
first_n(f_g,10)
```

Out[395]: [2, 23, 197, 1523, 11237, 80963, 575957, 4068563, 28627397, 200981603]