

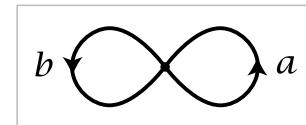
20. Suppose $f_t : X \rightarrow X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \rightarrow X$.]

1.2 Van Kampen's Theorem

The van Kampen theorem gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known. By systematic use of this theorem one can compute the fundamental groups of a very large number of spaces. We shall see for example that for every group G there is a space X_G whose fundamental group is isomorphic to G .

To give some idea of how one might hope to compute fundamental groups by decomposing spaces into simpler pieces, let us look at an example. Consider the space X formed by two circles A and B intersecting in a single point, which we choose as the basepoint x_0 . By our preceding calculations we know that $\pi_1(A)$ is infinite cyclic, generated by a loop a that goes once around A .

Similarly, $\pi_1(B)$ is a copy of \mathbb{Z} generated by a loop b going once around B . Each product of powers of a and b then gives an element of $\pi_1(X)$. For example, the product $a^5b^2a^{-3}ba^2$ is the loop that goes five times around A , then twice around B , then three times around A in the opposite direction, then once around B , then twice around A . The set of all words like this consisting of powers of a alternating with powers of b forms a group usually denoted $\mathbb{Z} * \mathbb{Z}$. Multiplication in this group is defined just as one would expect, for example $(b^4a^5b^2a^{-3})(a^4b^{-1}ab^3) = b^4a^5b^2ab^{-1}ab^3$. The identity element is the empty word, and inverses are what they have to be, for example $(ab^2a^{-3}b^{-4})^{-1} = b^4a^3b^{-2}a^{-1}$. It would be very nice if such words in a and b corresponded exactly to elements of $\pi_1(X)$, so that $\pi_1(X)$ was isomorphic to the group $\mathbb{Z} * \mathbb{Z}$. The van Kampen theorem will imply that this is indeed the case.



Similarly, if X is the union of three circles touching at a single point, the van Kampen theorem will imply that $\pi_1(X)$ is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, the group consisting of words in powers of three letters a , b , c . The generalization to a union of any number of circles touching at one point will also follow.

The group $\mathbb{Z} * \mathbb{Z}$ is an example of a general construction called the *free product* of groups. The statement of van Kampen's theorem will be in terms of free products, so before stating the theorem we will make an algebraic digression to describe the construction of free products in some detail.

Free Products of Groups

Suppose one is given a collection of groups G_α and one wishes to construct a single group containing all these groups as subgroups. One way to do this would be to take the product group $\prod_\alpha G_\alpha$, whose elements can be regarded as the functions $\alpha \mapsto g_\alpha \in G_\alpha$. Or one could restrict to functions taking on nonidentity values at most finitely often, forming the direct sum $\bigoplus_\alpha G_\alpha$. Both these constructions produce groups containing all the G_α 's as subgroups, but with the property that elements of different subgroups G_α commute with each other. In the realm of nonabelian groups this commutativity is unnatural, and so one would like a 'nonabelian' version of $\prod_\alpha G_\alpha$ or $\bigoplus_\alpha G_\alpha$. Since the sum $\bigoplus_\alpha G_\alpha$ is smaller and presumably simpler than $\prod_\alpha G_\alpha$, it should be easier to construct a nonabelian version of $\bigoplus_\alpha G_\alpha$, and this is what the free product $*_\alpha G_\alpha$ achieves.

Here is the precise definition. As a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i} and is not the identity element of G_{α_i} , and adjacent letters g_i and g_{i+1} belong to different groups G_α , that is, $\alpha_i \neq \alpha_{i+1}$. Words satisfying these conditions are called *reduced*, the idea being that unreduced words can always be simplified to reduced words by writing adjacent letters that lie in the same G_{α_i} as a single letter and by canceling trivial letters. The empty word is allowed, and will be the identity element of $*_\alpha G_\alpha$. The group operation in $*_\alpha G_\alpha$ is juxtaposition, $(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$. This product may not be reduced, however: If g_m and h_1 belong to the same G_α , they should be combined into a single letter $(g_m h_1)$ according to the multiplication in G_α , and if this new letter $g_m h_1$ happens to be the identity of G_α , it should be canceled from the product. This may allow g_{m-1} and h_2 to be combined, and possibly canceled too. Repetition of this process eventually produces a reduced word. For example, in the product $(g_1 \cdots g_m)(g_m^{-1} \cdots g_1^{-1})$ everything cancels and we get the identity element of $*_\alpha G_\alpha$, the empty word.

Verifying directly that this multiplication is associative would be rather tedious, but there is an indirect approach that avoids most of the work. Let W be the set of reduced words $g_1 \cdots g_m$ as above, including the empty word. To each $g \in G_\alpha$ we associate the function $L_g : W \rightarrow W$ given by multiplication on the left, $L_g(g_1 \cdots g_m) = gg_1 \cdots g_m$ where we combine g with g_1 if $g_1 \in G_\alpha$ to make $gg_1 \cdots g_m$ a reduced word. A key property of the association $g \mapsto L_g$ is the formula $L_{gg'} = L_g L_{g'}$ for $g, g' \in G_\alpha$, that is, $g(g'(g_1 \cdots g_m)) = (gg')(g_1 \cdots g_m)$. This special case of associativity follows rather trivially from associativity in G_α . The formula $L_{gg'} = L_g L_{g'}$ implies that L_g is invertible with inverse $L_{g^{-1}}$. Therefore the association $g \mapsto L_g$ defines a homomorphism from G_α to the group $P(W)$ of all permutations of W . More generally, we can define $L : W \rightarrow P(W)$ by $L(g_1 \cdots g_m) = L_{g_1} \cdots L_{g_m}$ for each reduced word $g_1 \cdots g_m$. This function L is injective since the permutation $L(g_1 \cdots g_m)$ sends the empty word to $g_1 \cdots g_m$. The product operation in W corresponds under L to

composition in $P(W)$, because of the relation $L_{gg'} = L_g L_{g'}$. Since composition in $P(W)$ is associative, we conclude that the product in W is associative.

In particular, we have the free product $\mathbb{Z} * \mathbb{Z}$ as described earlier. This is an example of a *free group*, the free product of any number of copies of \mathbb{Z} , finite or infinite. The elements of a free group are uniquely representable as reduced words in powers of generators for the various copies of \mathbb{Z} , with one generator for each \mathbb{Z} , just as in the case of $\mathbb{Z} * \mathbb{Z}$. These generators are called a *basis* for the free group, and the number of basis elements is the *rank* of the free group. The abelianization of a free group is a free abelian group with basis the same set of generators, so since the rank of a free abelian group is well-defined, independent of the choice of basis, the same is true for the rank of a free group.

An interesting example of a free product that is not a free group is $\mathbb{Z}_2 * \mathbb{Z}_2$. This is like $\mathbb{Z} * \mathbb{Z}$ but simpler since $a^2 = e = b^2$, so powers of a and b are not needed, and $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of just the alternating words in a and b : $a, b, ab, ba, aba, bab, abab, baba, ababa, \dots$, together with the empty word. The structure of $\mathbb{Z}_2 * \mathbb{Z}_2$ can be elucidated by looking at the homomorphism $\varphi: \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ associating to each word its length mod 2. Obviously φ is surjective, and its kernel consists of the words of even length. These form an infinite cyclic subgroup generated by ab since $ba = (ab)^{-1}$ in $\mathbb{Z}_2 * \mathbb{Z}_2$. In fact, $\mathbb{Z}_2 * \mathbb{Z}_2$ is the semi-direct product of the subgroups \mathbb{Z} and \mathbb{Z}_2 generated by ab and a , with the conjugation relation $a(ab)a^{-1} = (ab)^{-1}$. This group is sometimes called the infinite dihedral group.

For a general free product $*_\alpha G_\alpha$, each group G_α is naturally identified with a subgroup of $*_\alpha G_\alpha$, the subgroup consisting of the empty word and the nonidentity one-letter words $g \in G_\alpha$. From this viewpoint the empty word is the common identity element of all the subgroups G_α , which are otherwise disjoint. A consequence of associativity is that any product $g_1 \cdots g_m$ of elements g_i in the groups G_α has a unique reduced form, the element of $*_\alpha G_\alpha$ obtained by performing the multiplications in any order. Any sequence of reduction operations on an unreduced product $g_1 \cdots g_m$, combining adjacent letters g_i and g_{i+1} that lie in the same G_α or canceling a g_i that is the identity, can be viewed as a way of inserting parentheses into $g_1 \cdots g_m$ and performing the resulting sequence of multiplications. Thus associativity implies that any two sequences of reduction operations performed on the same unreduced word always yield the same reduced word.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha: G_\alpha \rightarrow H$ extends uniquely to a homomorphism $\varphi: *_\alpha G_\alpha \rightarrow H$. Namely, the value of φ on a word $g_1 \cdots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$, and using this formula to define φ gives a well-defined homomorphism since the process of reducing an unreduced product in $*_\alpha G_\alpha$ does not affect its image under φ . For example, for a free product $G * H$ the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \rightarrow G \times H$.