MATH 145B—**HOMEWORK** 4

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NOTE: All functions under discussion are considered continuous, unless that's a property to be proved.

6.4

3 Let $f : S^1 \to S^1$ be a continuous mapping such that $\deg(f) \neq 1$. Prove that f has a fixed point.

Pf .

Ass. $\deg(f) \neq 1$ $\quad »\triangle«$

Let $S^1 := \{z \in \mathbb{C} | \quad |z| = 1\}$

Suppose f doesn't have a fixed point—i.e. $\forall z \in \mathbb{C} : f(z) \neq z \quad »\Diamond«$

Consider $H : S^1 \times [0, 1] \to S^1; (z, t) \mapsto \frac{(1-t)f(z)-tz}{||(1-t)f(z)-tz||}$

Since H is a rational multi-variable function, H is continuous iff it's denominator is never 0. And, $||x||$ is 0 iff $x = 0$. So we only need to check that case.

$(1 - t)f(z) - tz \stackrel{?}{=} 0$

$\implies (1 - t)f(z) = tz \quad » * «$

$\stackrel{||\cdot||}{\implies} ||(1 - t)f(z)|| = ||tz||$

$\implies |(1 - t)| \cdot ||f(z)|| = |t| \cdot ||z||$

$f(z), z \in S^1 \implies ||f(z)|| = ||z|| = 1$

$\implies |(1 - t)| \cdot 1 = |t| \cdot 1$

$t < 1 \implies 1 - t = |t|$

$\implies 1 - t = t$

$\implies 1 = 2t \implies t = \frac{1}{2}$

$» * « \implies \frac{1}{2}f(z) = \frac{1}{2}z$

$\stackrel{\cdot 2}{\implies} f(z) = z \quad »\blacklozenge«$

$»\Diamond« \implies H$ is continuous since $»\blacklozenge«$ can't happen.

$H(z, 0) = \frac{f(z)}{||f(z)||} = \frac{f(z)}{1} = f(z)$

$H(z, 1) = \frac{z}{||z||} = \frac{z}{1} = z = 1_{s^1}(z)$

$\implies f \sim 1_{S^1}$

$\implies \deg(f) = \deg(1_{S^1}) = 1$

This contradicts $»\triangle«$

$\implies \exists z_0 \in \mathbb{C} : f(z_0) = z_0$

So, f has a fixed point.

$\blacksquare$

6 Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.

Pf .

Let A be a nonsingular 3 by 3 matrix having nonnegative entries.

$\implies$ The entries of A are positive real numbers, because there's no such thing as a complex negative numbers.

Denote the set of all 3 by 3 real valued matrices by $\mathrm{Mat}_{3\times 3}(\mathbb{R})$.

And, the set of all linear endomorphisms of $\mathbb{R}^3$ by $\mathcal{L}(\mathbb{R}^3)$.

$\mathrm{Mat}_{3\times 3}(\mathbb{R}) \simeq \mathcal{L}(\mathbb{R}^3)$ (by Linear Algebra)

$\implies \exists$ ($T : \mathbb{R}^3 \circlearrowleft$) such that A is the matrix representation of T

$S^2_{\geq 0} := (\mathbb{R}^2 \times (\{0\} \cup \mathbb{R}^+)) \cap S^2$

Consider $T^* : \mathbb{R}^3 \to S^2_{\geq 0}; (x, y, z) \mapsto \frac{T((x,y,z))}{||T((x,y,z))||}$

T^* is continuouse since $||T((x, y, z))|| = 0$ iff $T((x, y, z)) = (0, 0, 0)$ since, T is linear $T((x, y, z)) = 0$ iff $(x, y, z) = (0, 0, 0)$. So $||T((x, y, z))|| = 0 \implies (0, 0, 0)/0 = (0, 0, 0)$. So T^* is well defined.

Consider $T^*|_{S^2_{\geq 0}} : S^2_{\geq 0} \circlearrowleft$, the restriction of T^* to $S^2_{\geq 0}$

Since the image of the stereographic projection of $S^2_{\geq 0}$ is D^2, and the stereographic projection admits a continuous inverse. It follows that $S^2_{\geq 0} \simeq D^2$.

$\implies \exists (f : D^2 \to S^2_{\geq 0})$ and $(g : S^2_{\geq 0} \to D^2) : f \circ g = 1_{D^2}$ and $g \circ f = 1_{S^3_{\geq 0}}$

$\implies g \circ T^*|_{S^2_{\geq 0}} \circ f : D^2 \to D^2$

$f, g, T^*|_{S^2_{\geq 0}}$ are continuous $\implies g \circ T^*|_{S^2_{\geq 0}} \circ f$ is continuous

$\implies g \circ T^*|_{S^2_{\geq 0}} \circ f$ has a fixed point (by Brawer's)

$\implies \exists (x, y) \in D^2 : g \circ T^*|_{S^2_{\geq 0}} \circ f((x, y)) = (x, y)$

$\stackrel{f}{\implies} f \circ g \circ T^*|_{S^2_{\geq 0}} \circ f((x, y)) = f((x, y)) \in S^2_{\geq 0}$

$\implies 1_{S^2_{\geq 0}} \circ T^*|_{S^2_{\geq 0}} \circ f((x, y)) = f((x, y))$

$\implies T^*|_{S^2_{\geq 0}} \circ f((x, y)) = f((x, y))$

$\implies T^*|_{S^2_{\geq 0}}(f((x, y))) = \frac{T(f(x,y))}{||T(f(x,y))||} = f((x, y))$

$\implies \frac{T(f(x,y))}{||T(f(x,y))||} = f((x, y))$

$\implies T(f(x, y)) = ||T(f(x, y))||f((x, y))$

Let $\lambda := ||T(f(x, y))|| \in \mathbb{R}$ and $(a, b, c) := f(x, y) \in S^2_{\geq 0} \implies (a, b, c) \in R^3$

$\implies A(a, b, c)^T = \lambda(a, b, c)^T$ (by the one-to-one correspondence between A and T)

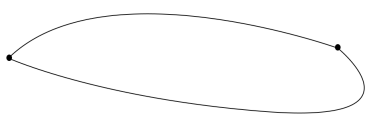
$\implies A$ has a positive real eigenvalue

■

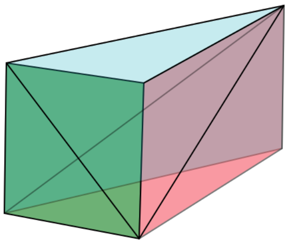
7.2

1 Give an example of space build of two vertices and two edges which is homeomorphic to the simplicial circle but not a simplicial complex.

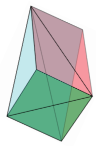
If you normalize the edges you get a circle. However, it is not a simplicial complex, because the intersection of the top and bottom lines is two disconnected points. Which isn't a simplex. So, it isn't a simplicial complex.



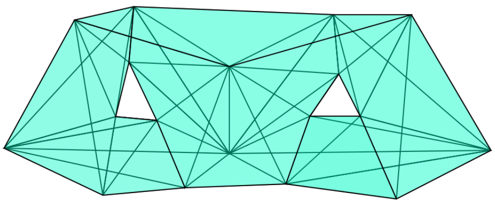
3 (a) Draw a triangulation of the cylinder.



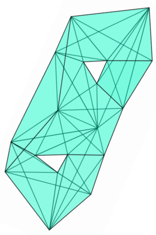
(b) Calculate the Euler number of the cylinder.

$\chi\Bigg($  $\Bigg) = 6 - 12 + 6 = 0$

7 a) Draw a triangulation of the surface of genus two, Σ_2.



(b) Calculate the Euler number of the surface of genus two, Σ_2.

$\chi\Bigg($  $\Bigg) = 18 - 64 + 44 = -2$