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Minimal triangulations of Kummer varieties

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Abstract

By an (abstract) Kummer variety K^d we mean the d-dimensional torus T^d modulo the involution $x \leftrightarrow -x$. The 2^d elements in T^d of order two are fixed points of the involution and therefore K^d has 2^d isolated singularities (for $d \geqq 3$). Any simplicial decomposition of K^d must have at least as many vertices. In this paper we describe a highly symmetrical simplicial decomposition of K^d with 2^d vertices such that the link of each vertex is a combinatorial real projective space $\mathbb{R}P^{d-1}$ with 2^d-1 vertices. The automorphism group of order $(d+1)!\,2^d$ admits a natural representation in the affine group of dimension d over the field with two elements. A particular case is the classical Kummer surface with 16 nodes ($d=4$). In this case our 16-vertex triangulation has a close relationship with the abstract Kummer configuration 16_6.

1. Introduction and result

Let us recall the notion of a *combinatorial manifold* of dimension d. This is a simplicial complex whose underlying set is a topological manifold M such that the link of each vertex is a combinatorial $(d-1)$-sphere. This may be understood as a recursive definition. For $d \leqq 3$ it is known that any simplicial d-manifold is combinatorial in this sense (see [21]). For $d=5$ a counterexample is due to EDWARDS [8].

A neighborhood of each vertex is a cone over a combinatorial $(d-1)$ -sphere. If the underlying manifold M has isolated singularities we have to modify our definition by saying that a *combinatorial pseudomanifold* of dimension d is a simplicial complex whose underlying set is a topological manifold with isolated singularities such that the link of each vertex is a combinatorial $(d-1)$ -manifold. In this case a neighborhood of each vertex is a cone over a $(d-1)$-manifold, and it is really singular if this manifold is different from the sphere.

These manifolds with cone-like singularities occur in different branches of mathematics and they have been successfully applied to certain problems in the analysis of differential operators (see [4]). In particular algebraic varieties can have such cone-like singularities. A famous example is the complex Kummer surface in $\mathbb{C}P^3$ with the maximal number of 16 isolated singularities (also called *double points* or *nodes*, see [17], [12], [20]). Its real dimension is four, and its topology is known to be that of the 4-dimensional torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$ modulo the involution $\sigma = T^4 \to T^4$, $\sigma(x) = -x$. The 16 singularities are just the 16 elements of order 2 in T^4 regarded as an abelian group. A neighborhood of each singular point is a cone over the real projective 3-space $\mathbb{R}P^3$. Generalized complex Kummer varieties have been considered by W. WIRTINGER [27]. We will include also the case of odd real dimension.

Definition: The d-dimensional torus T^d modulo the involution $\sigma = T^d \to T^d, \sigma(x) = -x$ is called (real) d-dimensional *Kummer variety* K^d. For $d = 2n$ this is the complex n-dimensional Kummer variety. In this case its desingularization is called Kummer manifold (see [24]), for $n = 2$ also $K3$-surface or simply Kummer surface (see [2]).

Remarks: (1) We are going to consider all these spaces only from the topological and combinatorial point of view. For questions concerning complex structures, Kähler metrics, moduli spaces etc. the reader may consult [2], [12].

(2) For odd d no local desingularization is possible because in this case the singularities are cones over an even-dimensional real projective space which does not bound in the tense of bordism. For even d there is the well known *blowing-up* or *dilatation process* (cf. [24]).

Let us recall that a simplicial complex is called *2-neighborly* if each two of its vertices are joined by an edge.

Now we can formulate our main result.

Theorem: *For each $d \geqq 2$ there exists a 2-neighborly combinatorial pseudomanifold with 2^d vertices and $d! \cdot 2^{d-1}$ d-dimensional simplices such that its underlying set is homeomorphic to the Kummer variety K^d. The automorphism group of order $(d+1)! \cdot 2^d$ acts transitively on the vertices and on the d-dimensional simplices. It admits a natural representation in the affine group of dimension d over the field with two elements.*

Remarks: (3) The proof will show that there is a certain triangulation of euclidean d-space such that there is a branched simplicial covering onto our triangulated Kummer variety K^d. The branch locus consists of isolated points. For a remarkable branched simplicial covering of $\mathbb{C}^2 \cong E^4$ onto $\mathbb{C}P^2$ compare [22]. In this case the branch locus is 2-dimensional, and the quotient is the minimal 9-vertex triangulation of $\mathbb{C}P^2$ (cf. [13]).

(4) The 2^d points of order two in T^d form a subgroup of T^d isomorphic to $(\mathbb{Z}_2)^d$. This can be regarded to be an affine geometry of dimension d over $\mathbb{Z}_2$ (in classical notation: the euclidean geometry $EG(d, 2)$). The corresponding group of affine transformations of this geometry acts on T^d and on K^d as well. The order of this group is 2^d times the order of $GL(d, 2)$, the latter is known to be $(2^d - 1) \cdot (2^d - 2) \cdot (2^d - 4) \ldots (2^d - 2^{d-1})$ (see [7], 7.4).

(5) The number of 2^d vertices is best possible: for $d \geqq 3$ every singularity of K^d must be a vertex in any simplicial decomposition of K^d. These are *absolute vertices* in the sense of [9]. For $d = 2$ four is the minimal number, too.

(6) For $d = 2$ we have the boundary complex of the tetrahedron with 4 vertices. Its automorphism group is the symmetric group S_4 of order $24 = 3! \cdot 2^2$. For $d = 3$ this 8-vertex combinatorial 3-pseudomanifold has been found by A. ALTSHULER who called it $\mathscr{P}$. He also determined the order of the automorphism group to be $192 = 4! \cdot 2^3$ (see [1]). The first interesting case is $d = 4$ which seems to be not yet known in the literature. In the following we are going to present these three cases first, and then we turn to the proof of our theorem in general.

2. The case of dimension 2

In the case $d = 2$ it is known that the quotient of a torus (Riemann surface of genus 1) by the involution $\sigma(x) = -x$ is a sphere (Riemann surface at genus 0) and that the covering is branched at exactly four points with index 2. A polyhedral model is shown in the following figure 1.

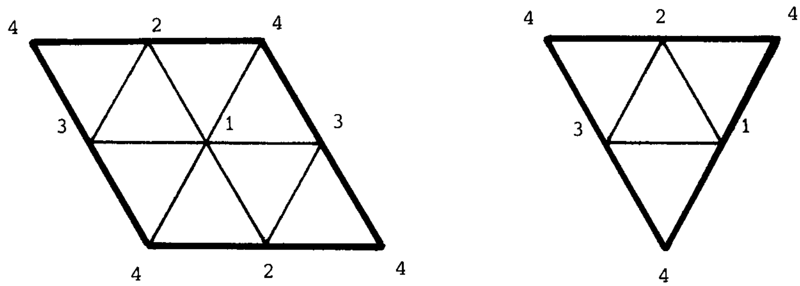

fig. 1

The left hand side of figure 1 shows a decomposition of the torus into 8 triangles, 6 around each of the 4 vertices. The right hand side shows the image of the 2-sheeted covering which is nothing but an ordinary tetrahedron.

From [6] we adapt the modification shown in figure 2. Here the torus decomposes into 4 squares, each subdivided by a diagonal. Pairs of numbers denote vertices in terms of euclidean coordinates.

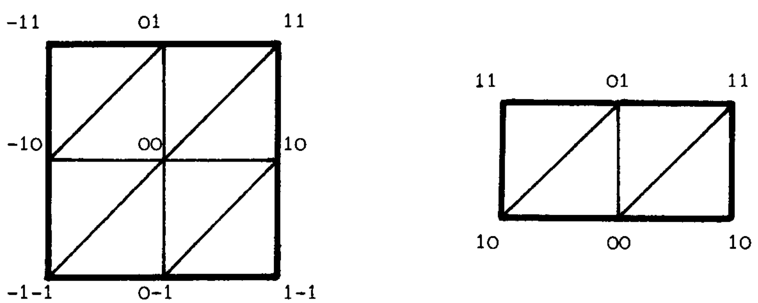

fig. 2

This emphasizes the affine geometric aspect: The four vertices are the four order-2- elements of the torus regarded as an abelian group. These four elements may be considered as an affine plane $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$.

The list of triangles in the quotient is just

<00 01 11>

<00 10 11>

<00 01 10>

<01 10 11>

Note that the full affine group of this 2-plane over $\mathbb{Z}_2$ is isomorphic to the full symmetry group S_4 of the tetrahedron.

3. The case of dimension 3

In this case the involution σ on the 3-torus has 8 fixed points (just the element of order 2 of this torus) and the quotient modulo σ will have 8 singularities. These are of the type *cone over a real projective plane* $\mathbb{R}P^2$.

A polyhedral model can be constructed as follows. Regard the 3-torus as the cube $[-1,1]^3$ with the obvious identifications along the boundary and decompose it into 8 cubes, one of them being $[0,1]^3$.

Now introduce the diagonal <000 111> of this cube and take the standard triangulation of the cube into 6 tetrahedra (cf. [18], [19]). Then take translated copies of this triangulated cube. This leads to a decomposition of the 3-torus into 48 abstract tetrahedra. After factorizing by σ we get 24 tetrahedra which triangulate K^3. These are shown in figure 3, and the complete list is given in table I (the triples denote elements of $(\mathbb{Z}_2)^3$).

Table I:

000	001	011	111
000	011	010	111
000	010	110	111
000	110	100	111
000	100	101	111
000	101	001	111

010	011	001	101
010	001	000	101
010	000	100	101
010	100	110	101
010	110	111	101
010	111	011	101

100	101	111	011
100	111	110	011
100	110	010	011
100	010	000	011
100	000	001	011
100	001	101	011

001	000	010	110
001	010	011	110
001	011	111	110
001	111	101	110
001	101	100	110
001	100	000	110

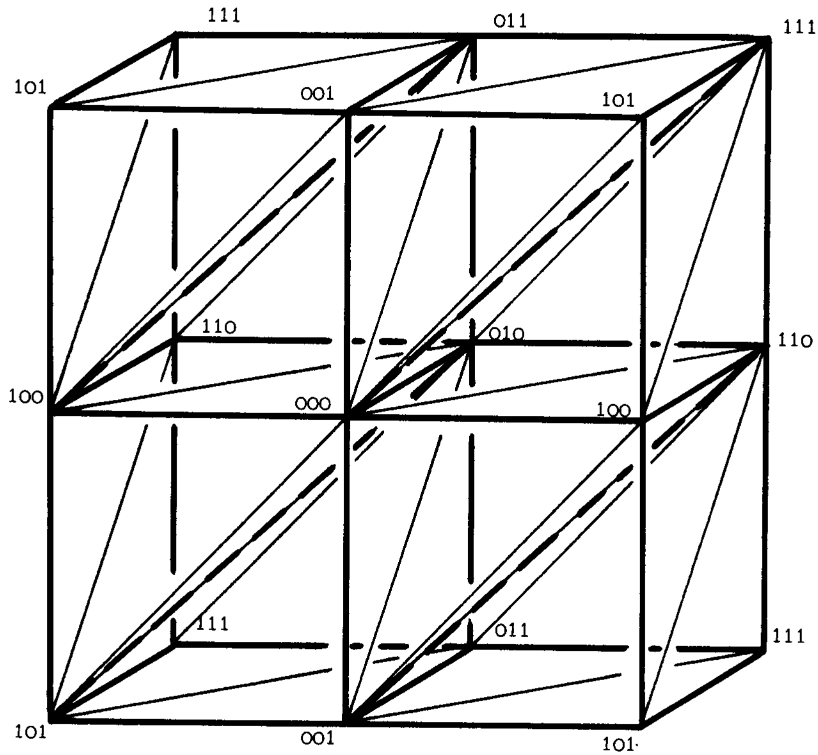

fig. 3

The full automorphism group of this simplicial complex K^3 consists of

- the 8 translations in $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
- the 3! permutations of the 3 coordinates,
- the linear transformation by the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

which is an element of order 4.

This group of order 192 is a subgroup of the affine group in 3 dimensions over $\mathbb{Z}_2$. Its linear part is isomorphic to the symmetric group S_4.

If we regard the triples to be the binary representations of the integers 0, 1, . . ., 7 then the 24 tetrahedra are the following listed in table II:

Table II:

0 1 3 7	2 3 1 5	4 5 7 3	1 0 2 6
0 3 2 7	2 1 0 5	4 7 6 3	1 2 3 6
0 2 6 7	2 0 4 5	4 6 2 3	1 3 7 6
0 6 4 7	2 4 6 5	4 2 0 3	1 7 5 6
0 4 5 7	2 6 7 5	4 0 1 3	1 5 4 6
0 5 1 7	2 7 3 5	4 1 5 3	1 4 0 6

This scheme satisfies an additional *duality condition*: for each tetrahedron, the 4 complementary vertices span a tetrahedron, too. This combinatorial 3-pseudomanifold K^3 and a couple of its geometrical properties has been described by A. ALTSHULER in [1].

To see that it is in fact a pseudomanifold we have to examine the link of each vertex. By symmetry it is sufficient to consider the link of the vertex 000.

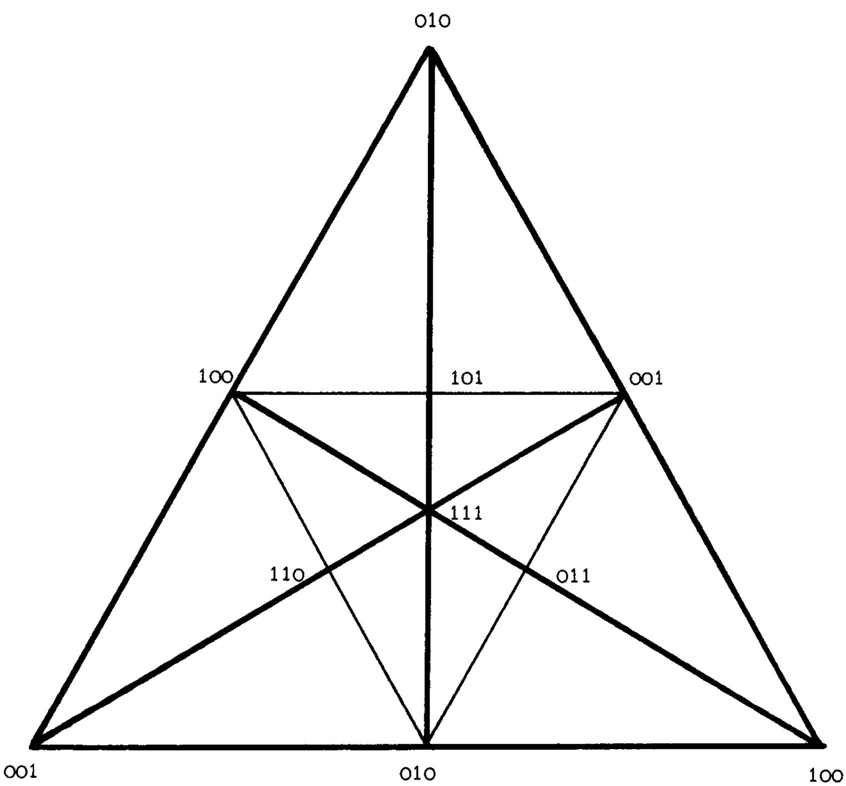

fig. 4a

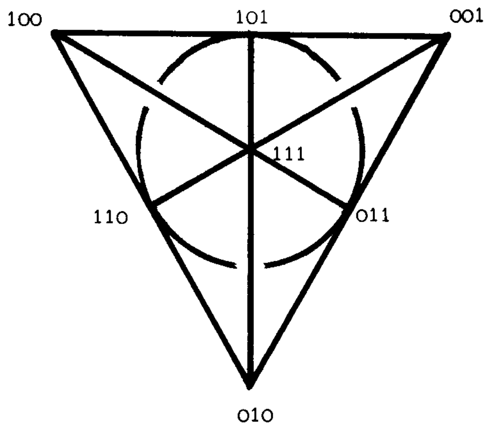

fig. 4b

This is a triangulated real projective plane as shown in figure 4a. Its universal covering is the so-called *tetrakis hexahedron* (i.e. a cube where each square is subdivided into 4 triangles, cf. [11]). It is invariant under the octahedral group of order 24. Observe that this tetrakis hexahedron is combinatorially equivalent to the rhombidodecahedron where each rhombus is subdivided by the short diagonal (cf. [14]). Note also that figure 4a contains 6 of the 7 lines of the projective plane $PG(2,2) = (\mathbb{Z}_2)^3 \setminus \{000\}$. Adding the circle in figure 4b we get the familiar picture of Fano's projective plane.

4. The case of dimension 4: the Kummer surface and the Kummer configuration

We construct the 16-vertex combinatorial pseudomanifold K^4 following the pattern of the 3-dimensional case. Let us start with the 4-dimensional cube $[0,1]^4$ and take the well known standard triangulation into 24 simplices by introducing the main diagonal $<0000\ \ 1111>$ (cf. [19]). This is invariant under the group S_4 acting transitively on these 24 simplices by permuting the 4 coordinates. A starting simplex is $<0000\ \ 0001\ \ 0011\ \ 0111\ \ 1111>$ or $<0\ \ 1\ \ 3\ \ 7\ \ 15>$ if we interpret the 4-tuples as binary representations of the integers $0, 1, \ldots, 15$.
Table III shows these 24 simplices.

Table III:

0000	0001	0011	0111	1111
0000	0001	0111	0101	1111
0000	0001	0101	1101	1111
0000	0001	1101	1001	1111
0000	0001	1001	1011	1111
0000	0001	1011	0011	1111

0000	0010	0011	0111	1111
0000	0010	0111	0110	1111
0000	0010	0110	1110	1111
0000	0010	1110	1010	1111
0000	0010	1010	1011	1111
0000	0010	1011	0011	1111

0000	0100	1100	1110	1111
0000	0100	1110	0110	1111
0000	0100	0110	0111	1111
0000	0100	0111	0101	1111
0000	0100	0101	1101	1111
0000	0100	1101	1100	1111

0000	1000	1100	1110	1111
0000	1000	1110	1010	1111
0000	1000	1010	1011	1111
0000	1000	1011	1001	1111
0000	1000	1001	1101	1111
0000	1000	1101	1100	1111

Then we regard the coordinates as integers modulo 2 and take all the 8 translates of this 4-dimensional-triangulated cube (note that it is already invariant under the translation by 1111). This leads to $8 \cdot 24 = 192$ 4-dimensional simplices. By construction this simplicial complex is invariant under

- the 16 translation of $(\mathbb{Z}_2)^4$ (This subgroup consisting of pure translations is called *group of* 16 *operations* by HUDSON [12] Ch. I § 4),
- the 4! permutations of the 4 coordinates
- the linear transformation by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which is an element of order 5.

By construction this group is a subgroup of the affine group in 4 dimensions over $\mathbb{Z}_2$. It acts transitively on the 4-dimensional simplices. Each vertex link is in fact a combinatorial real projective 3-space $\mathbb{R}P^3$ which can be seen by direct calculation. The proof of out theorem in the next section will prove this in general. Therefore we omit the details here.

The Kummer configuration

Let us study how this simplicial complex relates with the classical combinatorial properties of the Kummer surface (see [12], [20]).

First of all the 16 vertices correspond to the 16 singular points of the Kummer surface (cf. [17]). They are also called *nodes* (see [12]).

The set of all $\binom{16}{2} = 120$ pairs of vertices are called *Kummer lines* (see [26]).

These are exactly the 120 lines in the affine geometry of dimension 4 over the field with 2 elements.

According to [26] the set of $\binom{16}{3} = 560$ triples of vertices splits into two different classes. The 320 triples of the first kind determine a singular tangent plane, also called a *trope*, the other triples do not. See below our combinatorial analogues of the 16 tropes. Note that there are exactly 140 2-planes in the affine geometry $(\mathbb{Z}_2)^4$; each containing 4 points and consequently 4 triples. Our triangulation of K^4 contains 400 triangles, namely 160 triples of the first kind and the 240 of the second kind.

According to [26] the set of $\binom{16}{4} = 1820$ 4-tuples (or *tetrads*) splits into four classes:

1. The 80 tetrads of the first kind, also called *Rosenhain tetrads*. These are characterized by the fact that each of its triples (but not the quadruple) is contained in a trope.
2. The 60 tetrads of the second kind, also called *Göpel tetrads*. These have the property that no triple is contained in a trope.

In our interpretation these 80 + 60 = 140 4-tuples correspond exactly to the 140 2-planes in the affine geometry $(\mathbb{Z}_2)^4$. A typical 4-tuple of the first kind is $<0000\ 1000\ 0100\ 1100>$, a 4-tuple of the second kind is $<0000\ 1000\ 0110\ 1110>$. These 4-tuples are easily seen to be coplanar.

3. The 1440 tetrads of the third kind containing two triples of a trope and two triples not contained in a trope. The 480 tetrahedra of our triangulation are of this third kind.
4. Finally there are the 240 4-tuples which are completely contained in a 6-tuple spanning a trope.

It seems that no characteristic 5-tuples have been observed in the classical literature [12], [20]. Possibly it has been overlooked that our 192 5-tuples generated by $<0000\ 0001\ 0011\ 0111\ 1111>$ yield a triangulation of the Kummer surface.

Now let us turn to the classical configuration given by the 16 6-tuples of nodes lying in the 16 tropes [12]. Any two such 6-tuples intersect in a pair of nodes, any pair of nodes is contained in two tropes, each node is incident with 6 tropes. This is the famous *Kummer configuration* 16_6 (cf. [25]). In our notation the 6-tuples are generated by

$$<0000\ 0001\ 0010\ 0100\ 1000\ 1111>$$

under the action of all translations. This leads to the 16 6-tuples is invariant under the group of order $16 \cdot 5!$ introduced above, and in addition it is invariant under an element of order 6 lying in the affine group in 4 dimensions over $\mathbb{Z}_2$. This is the linear transformation induced by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The full group consists of the 16 translations permuting the 16 6-tuples and the symmetric group S_6 permuting the 6 points of the 6-tuples. Compare W. Burau [3] for two related configurations with somewhat smaller automorphism group.

A real Kummer variety

Finally let us mention a polyhedral analogue of the real part of a Kummer variety where all the 16 nodes are real. This is a 2-dimensional surface with 16 singular points. A neighborhood of each singularity is a cone over two disjoint closed curves. There is a famous plaster model of this real Kummer variety (see [12], for the history of this model compare [10]).

A polyhedral analogue (see figure 5) is given by the boundary triangles of a suitable choice of 8 tetrads of the first kind (Rosenhain tetrads). There are many such choices, and this polyhedral real Kummer variety has less symmetry than K^4. Figure 5 shows the choice of the following 8 tetrads:

$<0000\ 1100\ 0011\ 1111>$ $<0000\ 1000\ 0001\ 1001>$

$<1000\ 0100\ 1011\ 0111>$ $<1100\ 0100\ 1101\ 0101>$

$<0010\ 1110\ 0001\ 1101>$ $<0011\ 1011\ 0010\ 1010>$

$<1010\ 0110\ 1001\ 0101>$ $<1110\ 0110\ 1111\ 0111>$

5. Proof of the theorem

We regard the d-torus to be $[-1, +1]^d$ with the usual identification defined by the group $(2\mathbb{Z})^d$. This can be decomposed in a natural way into 2^d d-cubes. One such cube is $[0,1]^d$ and the group $(\mathbb{Z}_2)^d$ acts transitively on these cubes. Now let us introduce the standard triangulation of this cube $[0,1]^d$ by $d!$ simplices, all containing the main diagonal $(0,\ldots,0)\ (1,\ldots,1)$. According to [19] these simplices will be of the form

$\{(x_1,\ldots,x_d) | x_{\pi(1)} \leqq \ldots \leqq x_{\pi(d)}\}$ where π runs over all permutations in the full symmetric group S_d. The vertices of the simplex corresponding to $\pi =$ identity are

$$(0,\ldots,0),\ (0,\ldots,0,1),\ (0,\ldots,0,1,1),\ldots (0,1,1,\ldots,1),\ (1,\ldots,1)$$

Regarding the d-tuples of numbers 0,1 as binary representations of integers, then this starting simplex appears as

$$<0\ \ 1\ \ 3\ \ 7\ \ 15\ldots 2^d-1>.$$

Now subdivide all 2^d cubes by translated copies of this cube $[0,1]^d$. The result will be a collection of $2^d \cdot d!$ simplices tessellating the d-torus. (However, this is not a

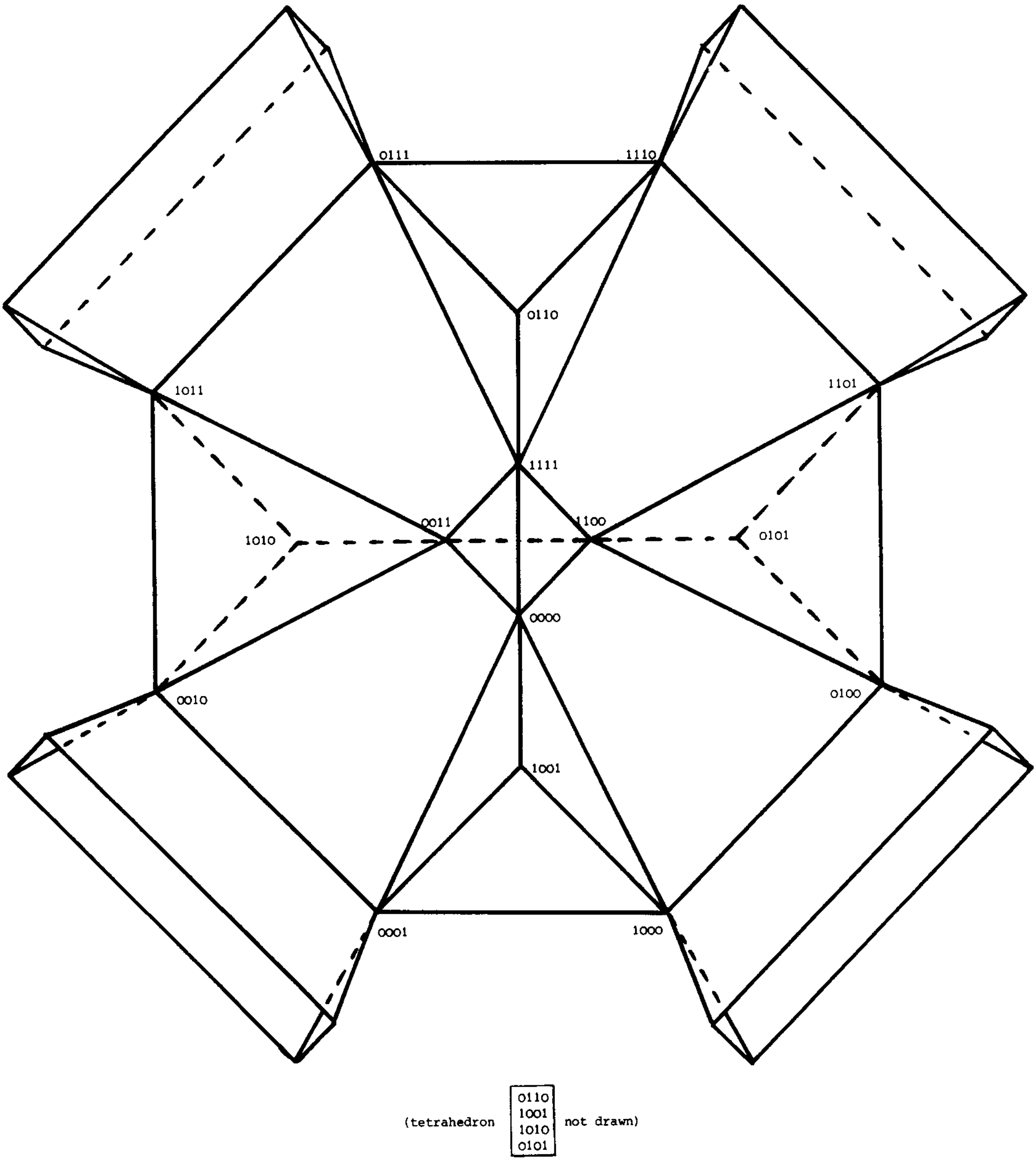
0111
1110
0110
1011
1101
1111
0011
1100
1010
0101
0000
0010
0100
1001
0001
1000
(tetrahedron
0110
1001
1010
0101
not drawn)

fig. 5

combinatorial manifold because of the occurence of double edges). Note that it is invariant under the reflection at the origin $(0,\ldots,0)$. This enables us to form the quotient of this tessellated $[-1,+1]^d$ modulo this reflection leading to a simplicial decomposition of K^d. This will have the 16 vertices corresponding to $\{0,1\}^d$ and $2^{d-1}.\ d!$ d-dimensional simplices.

It is still invariant under the 16 translations $x \to x + x_o, x_o \in (\mathbb{Z}_2)^d$ and under the action of the full symmetric group S_d permuting the d coordinates. In addition it will be invariant under an element of order $d+1$ which permutes the vertices

$$(1,0,\ldots,0),\ (0,1,\ldots,0),\ldots,(0,\ldots,0,1),\ (1,\ldots,1)$$

cyclically. This element is the linear transformation of $(\mathbb{Z}_2)^d$ defined by the matrix

$$B = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 1 \\ 0 & 1 & \ldots & 0 & 1 \\ & & & & \vdots \\ 0 & \ldots & 1 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 1 \end{pmatrix}$$

It maps the starting simplex $<0\ \ 1\ \ 3\ \ 7\ldots 2^d-1>$ to $<0\ 2^d-1\ \ 2^d-2\ \ 2^d-4\ldots 2^{d-1}>$.

The order of this group is $(d+1)\cdot d!\cdot 2^d$. By construction it is a subgroup of the group of affine transformations of $(\mathbb{Z}_2)^d$ and it contains all the translations. The linear part is isomorphic to the symmetric group S_{d+1}.

It remains to show that this simplicial complex is really a combinatorial pseudo-manifold, i. e. we have to show that every vertex link is a combinatorial triangulation of the real projective space $\mathbb{R}P^{d-1}$. We will do that by showing that a certain double covering is a combinatorial $(d-1)$-sphere.

Following Coxeter the expanded simplex $e\alpha_d$ is the polytope spanned by the $d(d+1)$ points in euclidean d-space whose coordinates are permutations of $(1,-1,0,\ldots,0)$. Its dual is a polytope with $2^{d+1}-2$ many vertices (cf. [6]). This is not simplicial: its faces are parallelepipeds. However, there is a simplicial decomposition of its boundary complex by the standard triangulation of each parallelepided. This has altogether $(d+1)!$ simplices and it is invariant under the full symmetric group S_{d+1} (acting transitively on the $(d-1)$-simplices) and under the reflection at the origin. In the previous paper [15] it has been shown that one starting simplex may be chosen to be $<1\ \ 3\ \ 7\ \ 15\ldots.\ 2^d-1>$ with a suitable labeling of the vertices.

This implies that after factorizing by the reflection at the origin we get exactly the right starting simplex $<1\ \ 3\ \ 7\ldots 2^d-1>$ in the link of the vertex 0. From [15] it follows that the two different group actions before and after dividing out fit together, and clearly the reflection is a central element. This implies that the link of 0 in K^d is a two-fold quotient of a combinatorial $(d-1)$-sphere where antipodal points are identified. No antipodal points in this $(d-1)$-sphere are joined by edges, and no vertex is joined simultaneously with an antipodal pair. Therefore the quotient triangulation will be a combinatorial manifold whose topological type clearly is that of $\mathbb{R}P^{d-1}$.

6. A quite symmetrical combinatorial real projective space

Proposition: *For any d there is a combinatorial* $\mathbb{R}P^{d-1}$ *with* $2^{d+1}-1$ *vertices and* $\frac{1}{2}(d+1)!$ $(d-1)$-*dimensional simplices which is invariant under the simplex transitive action of the full symmetric group* S_{d+1}.

The proof follows from the last part of the proof of the theorem: Take the link of the vertex 0 in the combinatorial Kummer variety K^d. It is constructed from a triangulation of the dual of the expanded simplex $e\alpha_d$. Note that *half of* $e\alpha_d$ has already been studied by Coxeter [7] who denoted it by $e\alpha_d/2$. Therefore our combinatorial $\mathbb{R}P^{d-1}$ is a triangulated $e\alpha_d/2$.

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