6.4. Brauwer's Fixed-Point Theorem

1. A space X is said to have the **Fixed Point Property** if for each continuous mapping $f : X \to X$ there is some $p \in X$ such that $f(p) = p$. By the Brouwer fixed Point Theorem and its consequences, a space X has the Fixed Point Property if X is homeomorphic to D^2 (and more generally for all D^n, but this is not proved in the course. In contrast, if $X = \mathbb{S}^n$ then the antipodal map $T(x) = -x$ has no fixed points, so $\mathbb{S}^n$ does not have the Fixed Point Property.

   (a) Prove that if X has the Fixed Point Property, then X is connected.

   (b) Prove that if X does not have the Fixed Point Property and Y is an arbitrary space, then $X \times Y$ also does not have the Fixed Point Property.

2. Suppose that X and Y are nonempty topological spaces such that $X \times Y$ has the Fixed Point Property. Prove that X and Y have the fixed point property.

3. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous mapping such that $deg(f) \neq 1$. Prove that f has a fixed point.

4. Show that if $h : \mathbb{S}^1 \to \mathbb{S}^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$

5. Let A be a 3 by 3 matrix of positive real numbers. Show that A has a positive real eigenvalue (characteristic value).

6. Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.