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1. (a)   Reading from the picture from Homework 6:

$C_1(S^1) = \{\lambda_0 e_0 + \lambda_1 e_1 + \lambda_1 e_2 | \lambda_i \in \mathbb{Z}\}$

$C_0(S^1) = \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_1 v_2 | \lambda_i \in \mathbb{Z}\}$

$C_n(S^1) = 0, \forall n \geq 2$

$\forall n \in \mathbb{N} : n \geq 2 \implies \mathbf{Ker} \; \partial_n = \mathbf{Im} \; \partial_n = 0$

$\implies \forall n \in \mathbb{N} : n \geq 2 : H_n(S^1) = \frac{0}{0} = 0$

$\partial_1(e_0) = v_2 - v_0$

$\partial_1(e_1) = v_1 - v_2$

$\partial_1(e_2) = v_0 - v_1$

$\implies \mathbf{Im} \; \partial_1 = < v_2 - v_0, v_1 - v_2, v_0 - v_1 >$

$v_0 - v_1 = -(v_2 - v_0 + v_1 - v_2) \implies$ one of the generators is redundant

$\implies \mathbf{Im} \; \partial_1 = < v_2 - v_0, v_1 - v_2 >$

Solve $\partial_1(\lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2) = \lambda_0(v_2 - v_0) + \lambda_1(v_1 - v_2) + \lambda_2(v_0 - v_1) = 0$

$\implies \lambda_0(v_2 - v_0) + \lambda_1(v_1 - v_2) + \lambda_2(v_0 - v_1) = (\lambda_2 - \lambda_0)v_0 + (\lambda_1 - \lambda_2)v_1 + (\lambda_0 - \lambda_1)v_2 = 0$

$\implies \lambda_2 - \lambda_0 = 0$ and $\lambda_1 - \lambda_2 = 0$ and $\lambda_0 - \lambda_1 = 0 \implies \lambda_2 = \lambda_0$ and $\lambda_1 = \lambda_2$ and $\lambda_0 = \lambda_1$

$\implies \lambda_0 = \lambda_1 = \lambda_2 := \lambda$

$\implies \partial_1(\lambda(e_0 + e_1 + e_2)) = 0$

$\implies \mathbf{Ker} \; \partial_1 = < e_0 + e_1 + e_2 >$

$\implies H_1(S^1; \mathbb{Z}_2) = \frac{\mathbf{Ker} \; \partial_1}{\mathbf{Im} \; \partial_2} = \frac{<e_0 + e_1 + e_2>}{0} = \mathbb{Z}_2$

$\partial_0(v_0) = \partial_0(v_1) = \partial_0(v_2) = 0$

$\implies \mathbf{Im} \; \partial_0 = 0$ and $\mathbf{Ker} \; \partial_0 = < v_0, v_1, v_2 >$

$\implies H_0(S^1; \mathbb{Z}_2) = \frac{\mathbf{Ker}\partial_0}{\mathbf{Im}\partial_1} = \frac{<v_0, v_1, v_2>}{<v_2 - v_0, v_1 - v_2>}$

One can change the basis of $C_0(S^1) = \mathbf{Ker}\partial_0$ as such: $< v_0, v_1, v_2 > = < v_0, v_1 - v_2, v_2 - v_0 >$

$\implies H_0(S^1; \mathbb{Z}_2) = \frac{<v_0, v_1 - v_2, v_2 - v_0>}{<v_2 - v_0, v_1 - v_2>} = < v_0 > = \mathbb{Z}_2$

So,

$$H_n(Annulus; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \,, n = \in \{0, 1\} \\ 0 \,, \text{else} \end{cases}$$

(c) By The Universal Coefficient Theorem for $\mathbb{Z}_2$:

$\forall n \in \mathbb{N} : n \geq 3 :$

$H_n(Annulus; \mathbb{Z}_2) = (H_n(Annulus) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{n-1}(Annulus), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_2(Annulus; \mathbb{Z}_2) = (H_2(Annulus) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_1(Annulus), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_1(Annulus; \mathbb{Z}_2) = (H_1(Annulus) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_0(Annulus), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

$H_0(Annulus; \mathbb{Z}_2) = (H_0(Annulus) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{-1}(Annulus), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

5. (a) $\forall n \in \mathbb{N} : n \geq 3 \implies C_n(S^2) = 0$

$C_2(S^2) = \{\lambda f | \lambda \in \mathbb{Z}\}$

$C_1(S^2) = 0,$

$C_0(S^2) = \{\lambda v | \lambda \in \mathbb{Z}\}$

$\forall n \in \mathbb{N} : n \geq 3 \implies \mathbf{Im}\, \partial_n = 0 = \mathbf{Ker}\, \partial_n$. Because they're all 0.

f is the only element of $C_2(S^2) \implies \partial_2(f) = 0 \implies \mathbf{Im}\, \partial_2 = 0$ and $\mathbf{Ker}\, \partial_2 = <f>$

$\implies H_2(S^2; \mathbb{Z}_2) = \frac{\mathbf{Ker}\partial_2}{\mathbf{Im}\partial_3} = \frac{<f>}{0} = <f> = \mathbb{Z}_2$

0 is the only element of $C_1(S^2) \implies \partial_1(0) = 0 \implies \mathbf{Im}\, \partial_1 = 0$ and $\mathbf{Ker}\, \partial_1 = 0$

v is the only element of $C_0(S^2) \implies \partial_0(v) = 0 \implies \mathbf{Im}\, \partial_0 = 0$ and $\mathbf{Ker}\, \partial_0 = <v>$

$\implies H_0(S^2; \mathbb{Z}_2) = \frac{\mathbf{Ker}\partial_0}{\mathbf{Im}\partial_2} = \frac{<v>}{0} = <v> = \mathbb{Z}_2$

So,

$$H_n(S^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \,, n = \in \{0, 2\} \\ 0 \,, \text{else} \end{cases}$$

(c) By The Universal Coefficient Theorem for $\mathbb{Z}_2$:

$\forall n \in \mathbb{N} : n \geq 4 :$

$H_n(S^2; \mathbb{Z}_2) = (H_n(S^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{n-1}(S^2), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_3(S^2; \mathbb{Z}_2) = (H_3(S^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_2(S^2), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_2(S^2; \mathbb{Z}_2) = (H_2(S^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_1(S^2), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

$H_1(S^2; \mathbb{Z}_2) = (H_1(S^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_0(S^2), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_0(S^2; \mathbb{Z}_2) = (H_0(S^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{-1}(S^2), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

7. (a) $C_2(S^2) = \{\lambda f | \lambda \in \mathbb{Z}\}$

$C_1(\mathbb{R}P^2) = \{\lambda_0 a + \lambda_1 b | \lambda_i \in \mathbb{Z}\}$

$C_0(\mathbb{R}P^2) = \{\lambda_0 v_0 + \lambda_1 v_1 | \lambda_i \in \mathbb{Z}\}$

$\forall n \in \mathbb{N} : n \geq 3 \implies C_n(\mathbb{R}P^2) = 0 \implies \mathbf{Im}\, \partial_n = 0 = \mathbf{Ker}\, \partial_n$

$\implies H_n(\mathbb{R}P^2) = \frac{\mathbf{Ker}\partial_n}{\mathbf{Im}\partial_{n+1}} = \frac{0}{0} = 0$, for $n \geq 3$

f is the only element of $C_2(\mathbb{R}P^2) \implies \partial_2(f) = 2(b - a) \equiv 0 \pmod 2$

$\implies \mathbf{Im}\, \partial_2 = 0$

$\implies \mathbf{Ker}\, \partial_2 = <f>$

$\implies H_2(\mathbb{R}P^2) = \frac{\mathbf{Ker}\partial_2}{\mathbf{Im}\partial_3} = \frac{<f>}{0} = \mathbb{Z}_2$

$\partial_1(a) = v - w$ and $\partial_1(b) = v - w$

$\implies \mathbf{Im}\, \partial_1 = <v - w>$

$y_1(v - w) + y_2(v - w) = (y_1 + y_2)v - (y_1 + y_2)w = 0 \implies y_1 + y_2 = 0 \implies -y_1 = y_2$

$\implies \mathbf{Ker}\, \partial_1 = <b - a>$

$\implies H_1(\mathbb{R}P^2) = \frac{\mathbf{Ker}\partial_1}{\mathbf{Im}\partial_2} = \frac{<b-a>}{0} = \frac{\mathbb{Z}_2}{0} = \mathbb{Z}_2$

$\partial_0(v_0) = \partial_0(v_1) = 0 \implies \mathbf{Im}\, \partial_0 = 0$ and $\mathbf{Ker}\, \partial_0 = <v_0, v_1>$

One can change the generators $<v_0, v_1>$ to $<v_1 - v_0, v_0>$.

$\implies H_0(\mathbb{R}P^2) = \frac{\mathbf{Ker}\partial_0}{\mathbf{Im}\partial_1} = \frac{<v_0,v_1>}{<v_1-v_0>} = \frac{<v_1-v_0,v_0>}{<v_1-v_0>} = <v_0> = \mathbb{Z}_2$

So,

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}_2, n \in \{0, 1, 2\} \\ 0, \text{ else} \end{cases}$$

(c) By The Universal Coefficient Theorem for $\mathbb{Z}_2$:

$\forall n \in \mathbb{N} : n \geq 3 :$

$H_n(\mathbb{R}P^2; \mathbb{Z}_2) = (H_n(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{n-1}(\mathbb{R}P^2), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_2(\mathbb{R}P^2; \mathbb{Z}_2) = (H_2(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_1(\mathbb{R}P^2), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2$

$H_1(\mathbb{R}P^2; \mathbb{Z}_2) = (H_1(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_0(\mathbb{R}P^2), \mathbb{Z}_2) = (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

$H_0(\mathbb{R}P^2; \mathbb{Z}_2) = (H_0(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{-1}(\mathbb{R}P^2), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$

11 See new drawing.

$C_3(T^3) = \{\lambda U \mid \lambda \in \mathbb{Z}\}$

$C_2(T^3) = \{\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 \mid \lambda_i \in \mathbb{Z}\}$

$C_1(T^3) = \{\lambda_0 a + \lambda_1 b + \lambda_2 c \mid \lambda_i \in \mathbb{Z}\}$

$C_0(T^3) = \{\lambda v \mid \lambda \in \mathbb{Z}\}$

$\forall n \in \mathbb{N} : n \geq 4 \implies C_n(T^3) = 0 \implies \mathbf{Im}\,\partial_n = 0 = \mathbf{Ker}\,\partial_n$

$\implies H_n(T^3) = \frac{\mathbf{Ker}\partial_n}{\mathbf{Im}\partial_{n+1}} = \frac{0}{0} = 0$, for $n \geq 4$

$\partial(U) = 0$

$\implies \mathbf{Im}\,\partial_3 = 0 \implies \mathbf{Ker}\,\partial_3 = <U>$

$\implies H_3(T^3) = \frac{\mathbf{Ker}\ \partial_3}{\mathbf{Im}\ \partial_4} = \frac{<U>}{0} = <U> = \mathbb{Z}$

$\partial_2(f_1) = a - b - a + b = 0$

$\partial_2(f_2) = a - c - a + c = 0$

$\partial_2(f_3) = c - b - c + b = 0$

$\implies \mathbf{Ker}\,\partial_2 = <f_1, f_2, f_3>$

$\implies \mathbf{Im}\,\partial_2 = 0$

$\implies H_2(T^3) = \frac{\mathbf{Ker}\ \partial_2}{\mathbf{Im}\ \partial_3} = \frac{<f_1, f_2, f_3>}{0} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$\partial_1(a) = \partial_1(b) = \partial_1(a) = 0$

$\implies \mathbf{Ker}\,\partial_1 = <a, b, c>$

$\implies \mathbf{Im}\,\partial_1 = 0$

$\implies H_1(T^3) = \frac{\mathbf{Ker}\ \partial_1}{\mathbf{Im}\ \partial_2} = \frac{<a,b,c>}{0} == \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$\partial_0(v) = 0$

$\implies \mathbf{Ker}\,\partial_0 = <v$

$\implies \mathbf{Im}\,\partial_0 = 0$

$\implies H_0(T^3) = \frac{\mathbf{Ker}\ \partial_0}{\mathbf{Im}\ \partial_1} = \frac{<v>}{0} = <v> = \mathbb{Z}$

So,

$$H_n(T^3) = \begin{cases} \mathbb{Z}, n \in \{0, 3\} \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, n \in \{1, 2\} \\ 0, \text{else} \end{cases}$$

12. (a)   See drawing of X.

$C_2(X) = \{\lambda f \mid \lambda \in \mathbb{Z}\}$

$C_1(X) = \{\lambda_0 a + \lambda_1 b + \lambda_2 c + \lambda_3 d \mid \lambda_i \in \mathbb{Z}\}$

$C_0(X) = \{\lambda v \mid \lambda \in \mathbb{Z}\}$

$\partial_2(\lambda f) = 2\lambda(a + b + c) \equiv 0 \pmod 2$

$\implies$ **Im** $\partial_2 = 0$

$\implies$ **Ker** $\partial_2 = < f >$

$C_n(K) = 0$, for $n \geq 3 \implies$ **Im** $\partial_n = 0$

$\implies H_n(X) = \frac{0}{0}$, for $n \geq 3$

$\implies H_2(X) = \frac{<f>}{0} = \mathbb{Z}_2$

$\partial_1(a) = \partial_1(b) = \partial_1(c) = \partial_1(d) = 0$

$\implies$ **Im** $\partial_1 = 0$

$\implies$ **Ker** $\partial_1 = < a, b, c, d >$

$\implies H_1(X) = \frac{<a,b,c,d>}{0} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\partial_0(v) = 0$

$\implies$ **Ker** $\partial_0 = < v >$

$\implies H_0(X) = \frac{<v>}{0} = \mathbb{Z}_2$

So

$$H_n(X) = \begin{cases} \mathbb{Z}_2 & , n \in \{0, 2\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & , n = 1 \\ 0 & , \text{else} \end{cases}$$

(c) By The Universal Coefficient Theorem for $\mathbb{Z}_2$:

$\forall n \in \mathbb{N} : n \geq 3 :$

$H_n(X; \mathbb{Z}_2) = (H_n(X) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{n-1}(X), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = 0 \oplus 0 = 0$

$H_2(X; \mathbb{Z}_2) = (H_2(X) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_1(X), \mathbb{Z}_2) = (0 \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2) = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2$

$H_1(\mathbb{R}P^2; \mathbb{Z}_2) = (H_1(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_0(\mathbb{R}P^2), \mathbb{Z}_2) = ([\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2] \otimes \mathbb{Z}_2) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_2)$

$= \mathbb{Z} \otimes \mathbb{Z}_2 \otimes \mathbb{Z} \otimes \mathbb{Z}_2 \oplus \mathbb{Z} \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \otimes \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$H_0(X; \mathbb{Z}_2) = (H_0(X) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_{-1}(X), \mathbb{Z}_2) = (\mathbb{Z} \otimes \mathbb{Z}_2) \oplus \text{Tor}(0, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2$