## 6.2. Homotopy Equivalence

1. Prove that a discrete space consisting of $m$ points is homotopy equivalent to a discrete space consisting of $n$ points if and only if $m = n$.
2. Write down a homotopy equivalence between $(0, 1)$ and $[0, 1]$.
3. Show that a space $X$ is contractible iff every map $f : X \to Y$, for arbitrary $Y$, is nullhomotopic. Similarly, show $X$ is contractible iff every map $f : Y \to X$ is nullhomotopic.
4. List all homotopy classes of maps $(0, 1) \to (0, 1)$.
5. A space $X$ is said to be contractible if the identity map $i_X : X \to X$ is nulhomotopic.
   (a) Show that $I$ and $\mathbb{R}$ are contractible.
   (b) Show that a contractible space is path-connected.
   (c) Show that if Y is contractible, then for any $X$, the set $[X, Y]$ has a single element.
   (d) Show that if $X$ is contractible and $Y$ is path connected, then $[X, Y]$ has a single element.
6. Show that composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Conclude that homotopy equivalence is an equivalence relation.
7. If $X$ and $Y$ are topological spaces and $f, g : X \to Y$ are homotopic homeomorphisms, prove that their inverses $f^{-1}$ and $g^{-1}$ are also homotopic.
8. Suppose that $X$ and $Y$ are nonempty spaces such that $X \times Y$ is contractible. Prove that both $X$ and $Y$ are contractible.
9. Suppose that we are given continuous mappings $f, g : X \to \mathbb{S}^n$ such that $f(x) \neq -g(x)$ for all $x$. Prove that f is homotopic to $g$.
10. Suppose that $0 < a \leq 1$ and consider the off center circle in $\mathbb{C} \setminus \{0\}$ defined by $\phi_a(z) = z + 1$. Prove that if $a < 1$ then $\phi_a$ is homotopic to $\phi_1$ in $\mathbb{C} \setminus \{0\}$.
11. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be homotopy equivalences of topological spaces. Prove that the product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also a homotopy equivalence.