8. Homotopy Groups

1. Compute the fundamental group of the following spaces.
   (a) The "solid torus", $D^2 \times \mathbb{S}^1$.
   (b) The cylinder $\mathbb{S}^1 \times I$.
   (c) The infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$.
   (d) $\{(x, y) \in \mathbb{R}^2 : \|(x, y)\| < 1\}$
   (e) $\{(x, y) \in \mathbb{R}^2 : \|(x, y)\| > 1\}$
   (f) $\{(x, y) \in \mathbb{R}^2 : \|(x, y)\| \geq 1\}$
   (g) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup (\mathbb{R}_+ \times \{0\})$.
   (h) The subset of $\mathbb{R}^2$ given by $\mathbb{R}^2 - (\mathbb{R}_+ \times \{0\})$.
   (i) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup (\mathbb{R}_+ \times \mathbb{R})$.
   (j) $\mathbb{R}^3$ with the nonnegative x and y axes deleted.

2. For each of the following spaces, show that the fundamental group is isomorphic to the fundamental group of the figure eight.
   (a) The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with a point removed.
   (b) $\mathbb{R}^3$ with the nonnegative x, y, and z axes deleted.
   (c) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup (\mathbb{R} \times \{0\})$.

3. Let $A \subset \mathbb{R}^n$ be a subset, and $h : A \longrightarrow Y$ be a continuous map with $h(a_0) = y_0$. If h extends to a continuous map of $\mathbb{R}^n$, then the induced map $h_* : \pi_1(A, a_0) \longrightarrow \pi_1(Y, y_0)$ is trivial.

4. Prove that there is no continuous surjection from a path connected space to a space which is not path connected.

5. Prove that $\mathbb{S}^n$ is path connected for $n \geq 1$.

6. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a map of degree n. Describe the induced homomorphism
$$f_* : \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^1).$$

7. Calculate the homotopy groups of the complement $\mathbb{R}^2 - \mathbb{S}^1$.

8. * Show that the n-sphere $\mathbb{S}^n$ is simply connected for $n \geq 2$.

9. Let X be the union of two copies of $\mathbb{S}^2$ with a point in common. What is the fundamental group of X?