# 8.4. Path Connectivity and $\pi_0$:

$$\pi_n(X) = [I^n : X] \qquad n > 0$$

Q: How about $n = 0$?

$\pi_0(X) = [I^0 : X]$ doesn't generally have group structure.

However, it contains some usefull information.

If we have a pointed map $f : I^0 \longrightarrow X$

then $f$ is determined by $f(-1)$,

since $f(1)$ must be the base pt. of $X$.

Two maps $f, g : I^0 \longrightarrow X$ are pointed

homotopic if there is a path in $X$

from $f(-1)$ to $g(-1)$.

**Defn.:** A space $X$ is **path-connected** if for given any two points $x_0, x_1 \in X$, there is a continuous map $p:[0,1] \longrightarrow X$ s.t. $p(0)=x_0$ and $p(1)=x_1$

**Remark:** If a space is not path-connected then we can form an equivalence relation on the points in $X$, where $x \sim y$ if there is a path from $x$ to $y$.
The set of equivalence classes is exactly the set of homotopy classes $[I^0; X]$ which is called **0-th homotopy group** even though it is not a group.

20 Prop.:

X path-connected $\Rightarrow$ X connected

Proof: Suppose X is disconnected.

$\Rightarrow \quad X = U \cup V \qquad U \cap V = \emptyset$

$\emptyset \neq U, V \underset{\text{open}}{\subset} X$

Let $x \in U$ and $y \in V$.

Since X is path-connected,

$\exists\, p : [0,1] \longrightarrow X$ s.t. $p(0) = x$, $p(1) = y$.

* Since U and V are open,

$p^{-1}(U), p^{-1}(V) \subset [0,1]$ open subsets

* $p^{-1}(U), p^{-1}(V) \neq \emptyset$

$0 \in p^{-1}(U)$ since $x \in U$ and $p(0) = x$

$1 \in p^{-1}(V)$ since $y \in V$ and $p^{-1}(1) = y$

* $p^{-1}(U) \cup p^{-1}(V) = [0,1]$

$U \cup V = X \Rightarrow p^{-1}(U \cap V) = p^{-1}(X) \Rightarrow p^{-1}(U) \cap p^{-1}(V) = [0,1]$

* $p^{-1}(U) \cap p^{-1}(V) = \emptyset$

$U \cap V = \emptyset \Rightarrow p^{-1}(U \cap V) = \emptyset \Rightarrow p^{-1}(U) \cap p^{-1}(V) = \emptyset$

Thus, $[0,1]$ is disconnected.

$\rightarrow\leftarrow$

However, $[0,1]$ is connected.

$\therefore$ $X$ is connected. $\square$

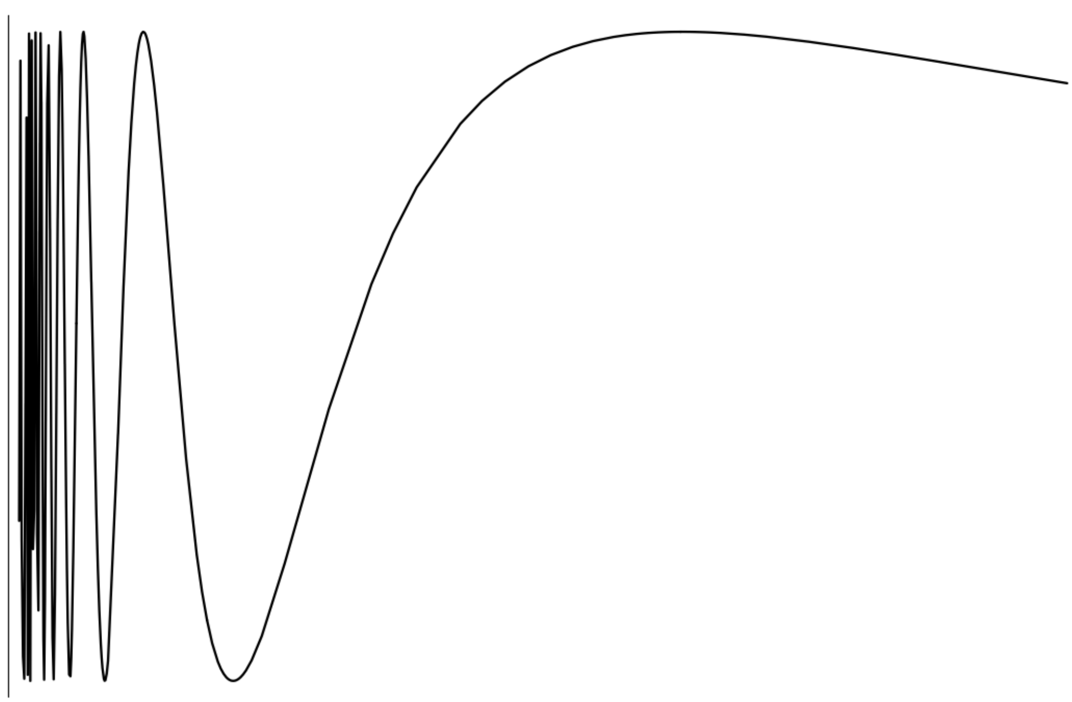
connected $\not\Rightarrow$ path-connected

Example 8.21 "topologist's sine curve"

Let $X \subset \mathbf{R}^2$ be the set

$$X = \{(x, y) : x = 0, -1 \leq y \leq 1\} \cup \{(x, y) : 0 < x \leq 1 \text{ and } y = \sin(1/x)\}.$$

connected but not path-connected.



**Defn.:** $X$ is called simply-connected if it is path-connected and

$$\pi_1(X, x_0) = 0$$

**Ex.:** $\mathbb{R}^n$, $D^2$, $S^n$ $(n \geq 2)$

are simply connected.

## Remarks:

(1) The image of a path connected space under a continuous map is path-connected.

(2) A space can be split into path components since it is based on an equivalence relation.

(3) A space is not generally the disjoint union of its path components.

Ex. 8.21, $\mathbb{Q}$ $\qquad \mathbb{Q} \neq \bigcup_{p \in \mathbb{Q}} \{p\}$

(4) $I^n$ is path-connected.

$\alpha : I^n \to \mathbb{R}$ continuous. $\Rightarrow \alpha(I^n)$ path-conn.

So, it has limited use in studying path non-path connected spaces.

22 **Prop.:** $X$ : any pointed topological space

$$\Longrightarrow \pi_n(X) = \pi_n(X_0) \quad (n>0)$$

where $X_0$ is the path connected component of $X$ which contains the base point.

23 **Corollary:** $\pi_0(\mathbb{Q}) = \mathbb{Q}$

$$\pi_i(\mathbb{Q}) = 0 \quad \forall i>0$$

similarly, $\pi_0(\mathbb{Z}) = \mathbb{Z}$

$$\pi_i(\mathbb{Z}) = 0 \quad \forall i>0$$

$\mathbb{Z} \underset{\text{isom.}}{\approx} \mathbb{Q}$ but $\pi_0(\mathbb{Z}) \not\approx \pi_0(\mathbb{Q})$

So, $\pi_0$ is unable to distinguish isomorphic topological spaces.