Math 131: Homework 8 Solutions

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November 8, 2015

Problem 1 (Munkres 52.5)

Let $A \subset \mathbb{R}^n$ be a subset, and say that $h : A \to Y$ is a continuous map with $h(a_0) = y_0$. If h extends to a continuous map of $\mathbb{R}^n$, then the induced map $h_* : \pi_1(A, a_0) \to \pi_1(Y, y_))$ is trivial.

Proof. If h has such such an extension then it is equal to the composition

$$A \xrightarrow{\quad i \quad} \mathbb{R}^n \xrightarrow{\quad \tilde{h} \quad} Y$$

So the induced map is equal to the composition

$$\begin{array}{ccccc} \pi_1(A, a_0) & \xrightarrow{\quad i_* \quad} & \pi_1(\mathbb{R}^n, a_0) & \xrightarrow{\quad \tilde{h}_* \quad} & \pi_1(Y, y_0) \\ & & \parallel & & \\ & & 0 & & \end{array}$$

and so the image is 0, as it must be the image of the middle map, which is a map out of the trivial group. □

Problem 2

Let $\pi : P \to B$ be the data of a covering space of a connected space B. If $\pi^{-1}(b)$ has k elements for some k, then it does for every $b \in B$.

Proof. We consider the following useful lemma:

Lemma 1. *A locally constant map is constant on connected components.*

Let X be a space and suppose $f : X \to \mathbb{R}$ is a map that is locally constant. That is, for each $x \in X$ there is a neighborhood U of x so that $f(u) = f(x)$ for all $u \in x$. From this, we see $f^{-1}(c)$ is open for every c, as for x so that $f(x) = c$ then there is a neighborhood U of x so that $f(u) = c$ as well, so there is an open set containing x contained in $f^{-1}(c)$ and it is open. Yet we can write

$$X = \bigsqcup_{c \in \mathbb{R}} f^{-1}(c)$$

where the union is obviously disjoint, as we cannot have some x mapping to two values. Yet by the definition of connected component, each connected component must be contained in one of these sets. In particular, if X is connected, then f is constant. The same argument applies if we allow f to take the value ∞. □

Now define a map $N : B \to \mathbb{R} \cup \{\infty\}$ given by $N(b) = \#$ of pre-images of b under π. Yet this is obviously locally constant as for some $b \in B$ we have an evenly covered neighborhood U of b, so that $\pi^{-1}(U) = \bigsqcup U_i$ and each $U_i \simeq U$, and each $b' \in U$ must have exactly the number of preimages as there are U_i's, so the map is locally constant. The value may be infinite. □

Problem 3

Let $\pi : E \to B$ be the data of a covering map. If B is Hausdorff (resp. regular, locally compact Hausdorff, completely regular, compact and finitely covered) then so is E.

Proof. We consider each case separately.

i) Suppose B is Hausdorff. Let $x, y \in E$ be distinct points. We find neighborhoods separating them. If $\pi(x) = \pi(y)$, then there is U contained $p(x) = p(y)$ so that $\pi^{-1}(U) = \bigsqcup U_i$. But x, y must be in different U_i, so two of the disjoint U_i's will suffice. If $p(x) \neq p(y)$. Then by Hausdorffness of B, there are neighborhoods U_x, U_y, which we can reduce to be evenly covered by intersecting them with evenly covered neighborhoods of $p(x), p(y)$. Then the components of the slices pre-images of U_x, U_y that contain x, y will be open and disjoint, as if they shared a point, so would U_x, U_y.

ii) Suppose B is regular and one point sets are closed .

Note that B being regular means it is Hausdorff so by the above E is Hausdorff. In particular, one point sets are closed in E and for a local homeomorphism ϕ we have $\phi(\overline{C}) = \overline{\phi(C)}$. By Lemma 31.1 it suffices to show for any $x \in E$ there is a neighborhood V so that $\overline{V} \subset U$.

Hence let $x \in E$ be a point. And U be a neighborhood of E. Consider $\pi^{-1}(W)$ for W an openly covered neighborhood of $\pi(x)$. There is a $W_i \simeq W$ that contains x. Reduce W_i to $W_i \cap U$ (without relabelling). We still have $W_i \simeq \pi(W_i)$. But by regularity of $\pi(W_i)$ there is a neighborhood V of $\pi(x)$ so that $\overline{V} \subset \pi(W_i)$. Therefore $\pi^{-1}(\overline{V}) = \overline{\pi^{-1}(V)} \subset W_i \subset E$ and we have found the desired neighborhood.

iii) Suppose B is Hausdorff and locally compact.

By Theorem 29.2 locally compact is equivalent to saying that for each x and neighborhood U of x, there is a smaller precompact neighborhood V. (neighborhood with compact closure) contained in U. The proof is now identical to that for ii), except we note that $\overline{V}$ must now be compact.

iv) Suppose b is completely regular.

Recall a space is completely regular if a point x and closed set C not containing it can be separated by a continuous function taking value 1 at x and 0 on C. Note that completely regular easily implies regular by considering preimages of say $(3/4, 1]$ and $[0, 1/4)$. Completely regular is also Hausdorff by definition so one-point sets are closed.

Suppose B is completely regular, thus by the above E is regular. Let x, C be a point and closed set in C. And let U, U' be disjoint open sets containing them (by regularity). Consider V a neighborhood of $\pi(x)$ evenly covered by $\bigsqcup V)i$ and reduce U to $U \cap V_i$ for the V_i containing x. Now by regularity (and Lemma 31.1). There is a $W \subset \overline{W} \subset U$. And so Consider $\pi(W) \subset V$ which is open and constains $\pi(x)$. Then let f be map that is 1 at x and 0 on the closured $B - \pi(W)$. Now define

$$F(e) = \begin{cases} f \circ p & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

This map is continuous as the two maps agree at the value 0 on the complement of W. And so this is continuous by the pasting Lemma. Clearly $C \cap W = \emptyset$ by our choice of U originally, and so this takes value 0 on C.

v) Suppose B is compact and finitely covered. Then E is also compact.

First note that B has a finite cover V_i by evenly covered neighborhoods. Each V_i is covered by finitely many V_{ij} for j ranging over the number of covering copies of each V_i.

Now let U_α be an open cover of E, and consider $U_\alpha \cap V_{ij} \subset V_{ij}$. Then since U_α covers, the $\bigcup_\alpha U_\alpha \cap V_{ij} = V_{ij}$. But π is a local homeomorphism on each V_{ij} so $\bigcup_{\alpha,i,j} \pi(U_\alpha \cap V_{ij})$ is a cover of B. Hence there are finitely many α that cover by compactness of B, without reducing the already finite indices i, j. Then for these α, $\bigcup U_\alpha$ is a finite subcover. As if $x \in E$ then $\pi(x) \in \pi(U_\alpha \cap V_{ij})$ for some α, so $x \in U_\alpha \cap V_{ij} \subset U_\alpha$ for that α.

$\square$

pg. 347, Problem 3. Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be liftings such that $\widetilde{\alpha}(1) = \widetilde{\beta}(0)$. Show that $\widetilde{\alpha} * \widetilde{\beta}$ is a lifting of $\alpha * \beta$.

By the pasting lemma, both $\alpha * \beta$ and $\widetilde{\alpha} * \widetilde{\beta}$ produce continuous paths. Then it suffices to show $p(\widetilde{\alpha} * \widetilde{\beta}) = \alpha * \beta$. We can pick a representation for $\alpha * \beta$, say going

$$\alpha * \beta = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and similarly for $\widetilde{\alpha}$ and $\widetilde{\beta}$. Then by applying the hypothesis, we get immediately that $p(\widetilde{\alpha} * \widetilde{\beta}) = \alpha * \beta$ as desired.

pg. 366, Problem 2. For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of these holds.

(a) $B^2 \times S^1$
(b) The torus T with a point removed
(c) $S^1 \times I$
(d) $S^1 \times \mathbb{R}$
(e) $\mathbb{R}^3$ with the nonnegative x, y, z axes deleted
(f) $\{x \mid |x| > 1\} \subset \mathbb{R}^2$
(g) $\{x \mid |x| \geq 1\} \subset \mathbb{R}^2$
(h) $\{x \mid |x| < 1\} \subset \mathbb{R}^2$
(i) $S^1 \cup (\mathbb{R}_+ \times 0) \subset \mathbb{R}^2$
(j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R}) \subset \mathbb{R}^2$
(k) $S^1 \cup (\mathbb{R} \times 0) \subset \mathbb{R}^2$
(l) $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$

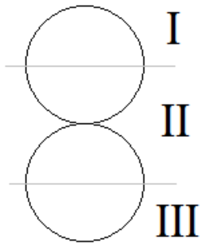
We will write 0 to indicate the trivial group, $\mathbb{Z}$ for the infinite cyclic group, and F_2 for the free group on two generators, which is the fundamental group if the figure eight. Then we get

(a) $\mathbb{Z}$
(b) F_2
(c) $\mathbb{Z}$
(d) $\mathbb{Z}$
(e) F_2
(f) $\mathbb{Z}$
(g) $\mathbb{Z}$
(h) 0
(i) $\mathbb{Z}$
(j) $\mathbb{Z}$
(k) F_2
(l) 0

1 Page 366 Problem 4

Let X be the figure eight and let Y be the theta space. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other.

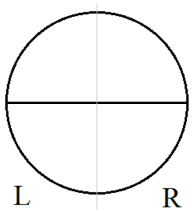
(Solution by Patrick Komiske)
We take the figure eight to be two circles of unit radius with centers on the y axis which touch at the origin and the theta space to be the unit circle unioned with $[-1,1] \times 0$. We will construct f as follows: consider the three regions shown in the figure below.



We will make f translate region I down by 1, map region II onto the x axis, and translate region III up by 1. That is,

$$f(x,y) = \begin{cases} (x, y-1), & y > 1 \\ (x,0), & |y| \le 1 \\ (x, y+1), & y < -1 \end{cases}$$

For mapping the theta space to the figure eight, we have the following picture



and we use the function

$$g(x,y) = \begin{cases} (2\sqrt{|y|(1-|y|)}, 2y), & x > 0 \\ (-2\sqrt{|y|(1-|y|)}, 2y), & x \le 0 \end{cases}$$

which shrinks the x axis to the origin and maps the top and bottom semicircles to circles of unit radius with their centers on the y axis.
Since the maps simply squish the theta space and the figure eight into each other, we can see that $f \circ g$ and $g \circ f$ are homotopic to the identity map. Thus f and g are homotopy inverse to each other.

2

(Solution by Tara Aida)

2.1

Suppose there existed a retraction $r : B^{n+1} \to S^n$. Then, we would have that $r \circ i = id_{S^n}$ where i is the inclusion map from $S^n \to B^{n+1}$. We know that a map from S^n to itself can be extended to a map on B^{n+1} iff it is nullhomotopic. Since r is an extension of $r \circ i$ to B^{n+1}, we conclude that $r \circ i = id_{S^n}$ is nullhomotopic and thus has degree 0. However, we know that id_{S^n} has degree 1, a contradiction.

2.2

Suppose h has no fixed point. Then, for all $u \in S^n$, $h(u) \neq u$. Then, we may construct the homotopy map:

$$H(u,t) = \frac{(1-t)h(u) - tu}{|(1-t)h(u) - tu|}$$

which shows that h is homotopic to the antipodal map. Since the antipodal map is really just the comoposition of $n+1$ reflection maps, the antipodal map has degree $(-1)^{n+1}$. Thus, we have shown the contrapositive of the statement desired.

2.3

Suppose that for all $u \in S^n$, $h(u) \neq -u$. Then, we may construct the homotopy map:

$$H(u,t) = \frac{(1-t)h(u) + tu}{|(1-t)h(u) + tu|}$$

which shows that h is homotopic to the identity map which has degree 1. Thus, we have shown the contrapositive of the statement desired.

2.4

Define the map $q(u) = v(u)/|v(u)|$, which we can do since v is nonvanishing. Since $v(u)$ is a tangent vector field, we may further conclude that $q(u) \neq \pm u$ for any $u \in S^n$ (if this was true, $v(u)$ could not be tangent to u, it would be parallel or anti-parallel). By part (b) we can conclude that q is homotopic to the antipodal map and by part (c) we can conclude that it is homotopic to the identity map–thus, the identity map and antipodal map are homotopic. This implies that $(-1)^{n+1} = 1$ which implies that n is odd.

3

Let X be the union of two copies of S^2 with a point in common. What is the fundamental group of X?

If U and V are the two spheres, since their intersection (a point) is path connected, then $i_U^*(\pi_1(U)) \bigcup i_V^*(\pi_1(V))$ is a generating set for $\pi_1 X$, by Theorem 59.1. But $U = V = S^2$ is simply connected, hence $\pi_1 X$ is trivial. $\square$