MATH 145B—HOMEWORK 1

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»n« := Statement number n

; := Reads 'defined by' if preceded by a function type declaration—i.e. $f : X \to Y; x \mapsto x^2$. reads f from X to Y defined by x maps to x squared
1_X := $\forall$ sets $X : X \neq \emptyset$, $1_X : X \to X; x \mapsto x$ denotes the identity function on X
Crossley := ISBN 978-1-85233-782-7
|_| := The cardinality of _
$\simeq$:= Homeomorphic
$\sim$:= Homotopic, or Homotopy Equivalent depending whether it is between spaces or maps
$||\mathbf{x}||_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$

NOTE: All functions under discussion are considered continuous, unless that's a property to be proved.

Intentionally left blanK

6.2

1 Prove that a discrete space consisting of m points is homotopy equivalent to a discrete space consisting of n points if and only if $m = n$.

Pf.

($\Longleftarrow$) Ass. $|X| = m = n = |Y|$, for some spaces $(X, \mathcal{T}_X) = \mathcal{X}$ and $(Y, \mathcal{T}_Y) = \mathcal{Y}$,
both $\mathcal{T}_X$ and $\mathcal{T}_Y$ are discrete topologies.

Index, both sets with $I = \{1, ..., n\} : X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$.

Consider the maps $f : \mathcal{X} \to \mathcal{Y}; x_i \mapsto y_i, i \in I$ and $g : \mathcal{Y} \to \mathcal{X}; y_i \mapsto x_i, i \in I$

$f \circ g : \mathcal{X} \to \mathcal{X}$ and $f(g(y_i)) = f(x_i) = y_i \implies f \circ g = 1_{\mathcal{Y}}$
$g \circ f : \mathcal{Y} \to \mathcal{Y}$ and $g(f(x_i)) = g(y_i) = x_i \implies g \circ f = 1_{\mathcal{X}}$

$\Rightarrow \mathcal{X} \simeq \mathcal{Y} \Rightarrow \mathcal{X} \sim \mathcal{Y}$

($\Longrightarrow$) Ass. with the notation above, $\mathcal{X} \sim \mathcal{Y}$

Ass. too, $|X| = m \neq n = |Y|$. And, without loss of generality ass., $0 < m < n$.
Note, $\emptyset$ is trivially false, since maps are not defined from and to.

Index with $I : X = \{x_1, ..., x_m\}$, and keep the indexing of Y above. And, let $J := \{m + 1, ..., n\}$.

One can define, surjective maps from $\mathcal{Y}$ to $\mathcal{X}$, but at most injective maps from $\mathcal{X}$ to $\mathcal{Y}$.

Suppose, $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{X}$ give a homotopy equivalence between $\mathcal{X}$ and $\mathcal{Y}$.

$\implies f \circ g \sim 1_{\mathcal{Y}}$

Consider, $f \circ g[\mathcal{Y}] = f[g[\mathcal{Y}]]$, now g is at best surjective, so if g is surjective. Then $f[g[\mathcal{Y}]] = f[\mathcal{X}]$, and if f is injective, then $f \circ g[\mathcal{Y}] \subset \mathcal{Y}$. That's the best case scenario, because if g is not surjective, then $f \circ g[\mathcal{Y}]$ would have a smaller cardinality than if g was surjective. Also, would have a smaller cardinality if f was not injective. So in any case, $f \circ g[\mathcal{Y}] \subset \mathcal{Y}$. Then, $\mathcal{Y} \backslash f \circ g[\mathcal{Y}] \neq \emptyset$.

$\implies f \circ g \neq 1_{\mathcal{Y}}$

But, can we find a homotopy between them? Suppose we can.

$\implies \exists H : \mathcal{Y} \times [0, 1] \to \mathcal{Y}$ and $H(y, 0) = f \circ g(y)$ and $H(y, 1) = 1_{\mathcal{Y}}(y)$

Since, $\mathcal{Y} \backslash f \circ g[\mathcal{Y}] \neq \emptyset$. Choose, $y_0 \in \mathcal{Y} \backslash f \circ g[\mathcal{Y}] \neq \emptyset$.

Now, take $H^{-1}[\{(y_0, t)\}], t \in [0, 1]$. Consider, $t = 0$, then $H^{-1}[\{(y_0, 0)\}] = \{H(y_0, 0)\} = f \circ g[\{y_0\}] = \emptyset$, since $f \circ g$ is not defined at y_0. But, at $t = 1$, $H^{-1}[\{(y_0, 1)\}] = \{H(y_0, 1)\} = \{1_{\mathcal{Y}}(y_0)\} = \{y_0\}$. Now, the topology on $\mathcal{Y} \times [0, 1]$, has to be the product topology. So, the continuity of H, depends on $[0, 1]$. $\mathcal{Y} \times [0, 1]$ is totally disconnected, since $\mathcal{Y}$ is totally disconnected. However, $\{y_0\}$ is a connected component of $\mathcal{Y}$. So, $\{y_0\} \times [0, 1]$ is connected. If H where continuous, then it would be defined, $\forall (y_0, t) \in \{y_0\} \times [0, 1]$, it is not the case. So, H can't be continuous, so it can't be a homotopy.

$\implies f \circ g \nsim 1_{\mathcal{Y}} \implies \mathcal{X} \nsim \mathcal{Y}$ whenever $m \neq n$

If, $m = n$, then there is no problem as the argument ($\Longleftarrow$) is logically reversible.

So, $\mathcal{X} \sim \mathcal{Y} \Leftrightarrow m = n$

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3 Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic. Note: The answer is in two parts 0 and 1

Pf . 0

WTS X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic.

$(\Leftarrow)$ Ass. every map $f : X \to Y$, for arbitrary Y, is nullhomotopic.

$\Rightarrow \forall(f_j : X \to Y) : \exists y_i \in Y : \exists(c_i : X \to Y; x \mapsto y_i) : f_j \sim c_i$
$\Rightarrow [X, Y] = \{c_i : X \to Y\}$
$\Rightarrow [X, Y]$ has at most $|Y|$ elements

Since Y is arbitrary, we can choose $Y = \{0\}$.
$\Rightarrow [X, \{0\}]$ has 1 element.
$\Rightarrow [X, \{0\}] = \{c : X \to \{0\}; x \mapsto 0\}$.

Consider $g : \{0\} \to X; 0 \mapsto x_0$
$c \circ g(0) = c(g(0)) = c(x_0) = 0 \Rightarrow c \circ g = 1_{\{0\}}$
$g \circ c(x) = g(c(x)) = g(0) = x_0 \Rightarrow g \circ c \equiv x_0$

Now, since Y is arbitrary, we can choose again $Y = X$.
$\Rightarrow [X, X] = \{c_i : X \to X; x \mapsto x_i | x_i \in X\}$.

So, for some $x_0 \in X$, $g \circ c : X \to X \equiv x_0 \equiv c_0 \sim 1_X$

$\Rightarrow X \sim \{0\} \Rightarrow X$ is contractible

$(\Rightarrow)$ Ass. X is contractible

$\Rightarrow X \sim \{0\}$
Let, Y be an arbitrary topological space.
Then, by Lemma 6.10 in Crossley $[X, Y] = [\{0\}, Y]$

Since,$\{0\}$ has one element, $\forall(g : \{0\} \to Y)$, g has to be a constant map.

$[\{0\}, Y] = \{c_i : \{0\} \to Y; 0 \mapsto y_i | y_i \in Y\}$ and $[X, Y] = [\{0\}, Y]$
$\Rightarrow$
$[X, Y] = \{k_i : X \to Y; x \mapsto y_i | y_i \in Y\}$
$\Rightarrow$
$\forall(f : X \to Y) : \exists y_i \in Y : (k_i : X \to Y; x \mapsto y_i) : f \sim k_i$
So, all maps f from X to Y are nullhomotopic

So, X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic.

■

Pf . 1

Also WTS X is contractible iff every map $f: Y \to X$ is nullhomotopic.

$(\Leftarrow)$ Ass. every map $f: Y \to X$ is nullhomotopic.

$\Rightarrow \forall(f_j : Y \to X) : \exists x_i \in X : \exists(m_i : Y \to X; y \mapsto x_i) : f_j \sim m_i$
$\Rightarrow [Y, X]$ has at most $|X|$ elements

Since, Y is arbitrary, we can choose $Y = \{x_0\}$: $x_0 \in X$.
Now, since $\{x_0\}$ has one element $\forall(g_i : \{x_0\} \to X)$, g_i is constant.
So, $[\{x_0\}, X] = \{g_i : \{x_0\} \to X; x_0 \mapsto x_i | x_i \in X\}$

We can choose again, $Y = X$: $[X, X] = \{m_i : X \to X; x \mapsto x_i\}$

$\Rightarrow \exists x_0 \in X$: $m_0 \sim 1_X$

Consider, $l : X \to \{x_0\}; x \mapsto x_0$
$l \circ g_0(x_0) = l(g_0(x_0)) = l(x_0) = x_0 \Rightarrow l \circ g_0 = 1_{\{x_0\}}$

$g_0 \circ l(x) = g_0(l(x)) = g_0(x_0) = x_0 \Rightarrow g_0 \circ l = m_0 \sim 1_X$

So, $X \sim \{x_0\} \Rightarrow X$ is contractible.

$(\Rightarrow)$ Ass. X is contractible.

$\Rightarrow X \sim \{x_0\}$
Let, Y be an arbitrary topological space.
Then, by Lemma 6.10 in Crossley $[Y, X] = [Y, \{x_0\}]$

Since, $\{x_0\}$ has one element, $[Y, \{x_0\}] = \{c : Y \to \{x_0\}; y \mapsto x_0\}$ also has one element c.
And, since $[Y, X] = [Y, \{x_0\}]$ by the lemma.

$\forall(f : X \to Y) : f \sim c$

So, all functions from X to Y are nullhomotopic.

So, X is contractible iff every map $f: Y \to X$ is nullhomotopic.

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7 If X and Y are topological spaces and $f, g : X \to Y$ are homotopic homeomorphisms, prove that their inverses f^{-1} and g^{-1} are also homotopic.

Pf .

Ass. X and Y are topological spaces and $f, g : X \to Y$ are homotopic homeomorphisms.

$\implies f \circ f^{-1} = 1_Y$ and $f^{-1} \circ f = 1_X$ and $g \circ g^{-1} = 1_Y$ and $g^{-1} \circ g = 1_X$ and $f \sim y$

$f^{-1} = f^{-1} \circ 1_Y = f^{-1} \circ (g \circ g^{-1}) = (f^{-1} \circ g) \circ g^{-1} \sim (f^{-1} \circ f) \circ g^{-1} = 1_X \circ g^{-1} = g^{-1}$

$\implies f^{-1} \sim g^{-1}$

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9 Suppose that we are given continuous mappings $f, g : X \to S^n$such that $f(x) \neq -g(x)$ for all x. Prove that f is homotopic to g.

Pf .

Ass. $f, g : X \to S^n$ such that $f(x) \neq -g(x)$ for all x.

Consider $H : X \times [0, 1] \to S^n; (x, t) \mapsto \frac{(1-t)f(x)+tg(x)}{||(1-t)f(x)+tg(x)||_2}$

$H(x, 0) = \frac{(1-0)f(x)+0g(x)}{||(1-0)f(x)+0g(x)||_2} = \frac{f(x)}{||f(x)||_2}$

Since $f(x) \in S^n \implies ||f(x)||_2 = 1$. So, $H(x, 0) = f(x)$.

Similarly, $H(x, 1) = \frac{(1-1)f(x)+1g(x)}{||(1-1)f(x)+1g(x)||_2} = \frac{g(x)}{||g(x)||_2} = g(x)$.

H is a rational function of multi-variable polynomials, so it is continuous whenever the denominator is not 0. So, solve the following equation to determine the continuity of H.

$||(1-t)f(x) + tg(x)||_2 = 0$ and Since, the euclidean norm is 0 $\Leftrightarrow$ the argument to the norm is 0.

$\implies (1-t)f(x) + tg(x) = 0 \implies (1-t)f(x) = -tg(x)$

Take the euclidean norm of both sides, $||(1-t)f(x)||_2 = ||-tg(x)||_2 \implies ||(1-t)||_2||f(x)||_2 = ||-t||_2||g(x)||_2$

Since both $f(x)$ and $g(x)$ are on the n-sphere, their euclidean norm is 1, so $||(1-t)||_2 = ||-t||_2$.

t is positive and less than 1, so $1 - t = t$.

$\implies 1 = 2t \implies t = \frac{1}{2}$ — i.e that's the only point in $[0, 1]$ where the denomiator could be 0.

Now, reconsider $(1-t)f(x) + tg(x) = 0$ with $t = \frac{1}{2}$

$\implies (1 - \frac{1}{2})f(x) + \frac{1}{2}g(x) = 0 \implies \frac{1}{2}f(x) + \frac{1}{2}g(x) = 0 \implies \frac{1}{2}(f(x) + g(x)) = 0 \implies f(x) + g(x) = 0$
$\implies f(x) = -g(x)$

But, $\forall x \in X : f(x) \neq -g(x) \implies ||(1-t)f(x) + tg(x)||_2 \neq 0$

$\implies H$ is continuous.

$\implies f \sim g$

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