6.3. The Circle

1. Each of the spaces below is either contractible or homotopy equivalent to $\mathbb{S}^1$ or neither. For each example, determine which alternative holds. You do not need to give detailed proofs but please give a short explations.

   (a) The solid torus $D^2 \times \mathbb{S}^1$

   (b) The cylinder $\mathbb{S}^1 \times [0, 1]$

   (c) The cylinder $\mathbb{S}^1 \times \mathbb{R}$

   (d) The set of all points $(x, y) \in \mathbb{R}^2$ such that $\|(x, y)\| \geq 1$

   (e) The set of all points $(x, y) \in \mathbb{R}^2$ such that $\|(x, y)\| > 1$

   (f) The set of all points $(x, y) \in \mathbb{R}^2$ such that $\|(x, y)\| < 1$

   (g) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup (\mathbb{R}^+ \times \{0\})$.

   (h) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup ([0, +\infty) \times \mathbb{R})$.

   (i) The subset of $\mathbb{R}^2$ given by $\mathbb{S}^1 \bigcup (\mathbb{R}^+ \times \mathbb{R})$.

   (j) The torus $\mathbb{S}^1 \times \mathbb{S}^1$

   (k) The sphere $\mathbb{S}^2$

2. (a) Let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous mappings and let's take the complex multiplication operation on $\mathbb{S}^1 \subset \mathbb{C}$. Define $h(z)$ to be the product $h(z) = f(z) \cdot g(z)$. Show that $\deg(h)$ is equal to $\deg(f) + \deg(g)$.

   (b) If $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ are homotopic continuous mappings, then $\deg(f) = \deg(g)$.

3. If $f, g : S^1 \to S^1$ are two continuous maps, express $\deg(f \circ g)$ in terms of $\deg(f)$ and $\deg(g)$. Use this to show that $f \circ g$ is homotopic to $g \circ f$.

4. Find the mistake in the following argument which implies to show that the mappings $f(z) = z$ and $g(z) = z^2$ from $\mathbb{S}^1$ to itself are homotopic: Let $H(z, t) = z^{t+1}$. Then $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$.