TOPOLOGY HW 8

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55.1

Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

Proof. Suppose $r : B^2 \to A$ is a retraction. Then$r|_A$ is the identity map on A. Let $f : A \to A$ be continuous and let $i : A \to B^2$ be the inclusion map. Then, since i, r and f are continuous, so is $F = i \circ f \circ r$. Furthermore, by the Brouwer fixed-point theorem, F has a fixed point x_0. In other words,

$$x_0 = F(x_0) = (i \circ f \circ r)(x_0) = i(f(r(x_0))) \in A \subseteq B^2.$$

Since $x_0 \in A$, $r(x_0) = x_0$ and so

$$x_0 F(x_0) = i(f(r(x_0))) = i(f(x_0)) = f(x_0)$$

since $i(x) = x$ for all $x \in A$. Hence, x_0 is a fixed point of f. $\square$

55.4

Suppose that you are given the fact that for each n, there is no retraction $f : B^{n+1} \to S^n$. Prove the following:

(a) The identity map $i : S^n \to S^n$ is not nulhomotopic.

Proof. Suppose i is nulhomotopic. Let $H : S^n \times I \to S^n$ be a homotopy between i and a constant map. Let $\pi : S^n \times I \to B^{n+1}$ be the map

$$\pi(x, t) = (1 - t)x.$$

π is certainly continuous and closed. To see that it is surjective, let $x \in B^{n+1}$. Then

$$\pi\left(\frac{x}{||x||}, 1 - ||x||\right) = (1 - (1 - ||x||))\frac{x}{||x||} = x.$$

Hence, since it is continuous, closed and surjective, π is a quotient map that collapses $S^n \times 1$ to 0 and is otherwise injective. Since H is constant on $S^n \times 1$, it induces, by the quotient map π, a continuous map $k : B^{n+1} \to S^n$ that is an extension of i. Since it is an extension of i, $k(x) = x$ for all $x \in S^n$, so k is a retraction of B^{n+1} onto S^n. From this contradiction, then, we conclude that i is not nulhomotopic. $\square$

(b) The inclusion map $j : S^n \to R^{n+1} \backslash \{0\}$ is not nulhomotopic.

Proof. Suppose j is nulhomotopic. Let $H : S^n \times I \to \mathbb{R}^{n+1}\backslash\{0\}$ be a homotopy between j and a constant map which maps everything to $x_0 \in \mathbb{R}^{n+1}\backslash\{0\}$. Let $r : \mathbb{R}^{n+1}\backslash\{0\} \to S^n$ be given by

$$x \mapsto \frac{x}{||x||}.$$

Then r is a retraction of $\mathbb{R}^{n+1}\backslash\{0\}$ onto S^n, as we've shown in class. Now, define $H' : S^n \times I \to S^n$ by

$$H'(x,t) = (r \circ H)(x,t).$$

Then, since H is a homotopy between j and a constant map, H' is continuous and, for all $x \in S^n$,

$$H'(x,0) = (r \circ H)(x,0) = r(H(x,0)) = r(j(x))$$

and

$$H'(x,1) = (r \circ H)(x,1) = r(H(x,1)) = r(x_0).$$

Since j is merely the inclusion map, $j(x) \in S^n$, so $r(j(x)) = i(x)$. Hence, H' is a homotopy between the identity map i and the constant map which maps everything to the point $r(x_0) \in S^n$. In other words, i is nulhomotopic. However, as we showed in part (a) above, i is not nulhomotopic. From this contradiction, then, we conclude that j is not nulhomotopic. □

(c) Every nonvanishing vector field on B^{n+1} points directly outward at some point of S^n, and directly inward at some point of S^n.

Lemma 0.1. *If $h : S^n \to \mathbb{R}^{n+1}\backslash\{0\}$ extends to a continuous map $k : B^{n+1} \to \mathbb{R}^{n+1}\backslash\{0\}$, then h is nulhomotopic.*

Proof. Let $H : S^n \times I \to \mathbb{R}^{n+1}\backslash\{0\}$ be given by

$$(x,t) \mapsto k((1-t)x).$$

Then H is certainly continuous, since k is. Furthermore, for $x \in S^n$,

$$H(x,0) = k((1-0)x) = k(x) = h(x)$$

and

$$H(x,1) = k((1-1)x) = k(0).$$

Therefore, h is nulhomotopic. □

Now, we are ready to prove the desired result

Proof. Let $(x, v(x))$ be a nonvanishing vector field on B^{n+1}. Since this vector field is non-vanishing, v is a map from B^{n+1} into $\mathbb{R}^{n+1}\backslash\{0\}$. Suppose that $v(x)$ does not point directly inward at any point $x \in S^n$ in order to derive a contradiction. Consider the map $v : B^{n+1} \to \mathbb{R}^{n+1}\backslash\{0\}$ and let w be its restriction to S^n. w extends to a continuous map of B^{n+1} into $\mathbb{R}^{n+1}\backslash\{0\}$ (namely v) so, by the above lemma, w is nulhomotopic.

On the other hand, we want to show that w is homotopic to the inclusion map $j : S^n \to \mathbb{R}^{n+1}\backslash\{0\}$. Let $F : S^n \times I \to \mathbb{R}^{n+1}\backslash\{0\}$ be given by

$$(x, t) \mapsto tx + (1 - t)w(x)$$

for $x \in S^n$. In order to be sure that F is well-defined, we must make sure that $F(x, t) \neq 0$ for all $x \in S^n$, $t \in I$. Since x is always non-zero, $F(x, 0) \neq 0$ and, since the image of w is contained in $\mathbb{R}^{n+1}\backslash\{0\}$, $F(x, 1) \neq 0$. Now, if $F(x, t) = 0$ for some $0 < t < 1$, then

$$tx + (1 - t)w(x) = 0$$

or

$$w(x) = \frac{-t}{1-t}x.$$

However, this means $w(x)$ (and hence $v(x)$) points directly inward at x. From this contradiction, we see that $F(x, t) \neq 0$, so F really maps $S^n \times I \to \mathbb{R}^{n+1}\backslash\{0\}$.

Now, F is certainly continuous and

$$F(x, 0) = x = j(x)$$

and

$$F(x, 1) = w(x),$$

so j is homotopic to w. Since w is nulhomotopic, so is j. This, however, contradicts part (b) above. From this contradiction we conclude that, in fact, v points directly inward at some point.

To show that v points directly outward at some point of S^n, we simply notice that this is equivalent to saying that $-v$ points directly inward at some point. Since $(x, -v(x))$ defines a vector field, we know by the above proof that, in fact, $-v$ does point directly inward at some point, so v points directly outward at that same point. □

(d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

Proof. By contradiction. Suppose $f(x) \neq x$ for all $x \in B^{n+1}$. Then we define $v(x) = f(x) - x$, which gives a non-vanishing vector field $(x, v(x))$ on B^{n+1}. This vector field v cannot point directly outward at any point $x \in S^n$, as that would mean

$$f(x) - x = ax$$

for some positive real a. This in turn would imply that $f(x) = (1 + a)x \notin B^{n+1}$. However, this contradicts the fact proved in part (c) above that, since it is non-vanishing, the vector field v must point directly outward at some point of S^n. From this contradiction, then, we conclude that f has a fixed point. □

(e) Every $n+1$ by $n+1$ matrix A with positive real entries has a positive eigenvalue.

Proof. Let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the linear transformation whose matrix is A. Let $B = S^n \cap \{(x_1, \ldots, x_{n+1}) : x_1, \ldots, x_{n+1} \geq 0\}$. Then this B is surely homeomorphic to B^n, so that the result from part (d) holds for continuous maps from B to itself.

Let $(x_1, \ldots, x_{n+1}) \in B$. Then each component of x is non-negative and at least one is positive. Also, since all entries of A are positive, all the components of the vector $T(x)$ are positive. Hence, the map

$$x \mapsto T(x)/||T(x)||$$

is a continuous map of B into itself, so it has a fixed point x_0. Then

$$T(x_0) = ||T(x_0)||x_0,$$

so T (and, hence, A) has the positive real eigenvalue $||T(x)||$. □

(f) If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. Since h is nulhomotopic, we can, just as we did in part (a), extend h to a continuous map $f : B^{n+1} \to S^n \subset B^{n+1}$. By part (d), then, h has a fixed point x_0. Moreover, $x_0 \in S^n$, since $\text{Im}(f) = S^n$. Therefore, since f and h agree on S^n, $h(x_0) = x_0$.

To prove that h maps some point to its antipode, first we prove the following parallel of part (d) above:

Lemma 0.2. *If $f : B^{n+1} \to B^{n+1}$ is continuous, then there exists a point $x \in B^{n+1}$ such that $f(x) = -x$.*

Proof. By contradiction. Suppose $f(x) \neq -x$ for all $x \in B^{n+1}$. Then $v(x) := f(x) + x$ gives a non-vanishing vector field $(x, v(x))$ on B^{n+1}. But the vector field v cannot point directly inward at any point $x \in S^n$, for that would mean

$$-ax = v(x) = f(x) + x$$

for some positive real a. If this were the case, $f(x) = -(1 + a)x$ would lie outside the unit ball.

However, we proved in part (c) above that every nonvanishing vector field on B^{n+1} points directly inward at some point of S^n. From this contradiction, then, we conclude that there exists a point $x \in B^{n+1}$ such that $f(x) = -x$. □

Now, to finish off the proof, we simply note that by this lemma, our extension f must have some point $x_1 \in B^{n+1}$ at which $f(x_1) = -x_1$. Again, since $\text{Im}(f) = S^n$, $x_1 \in S^n$ and therefore, since f and h coincide on S^n, $h(x_1) = -x_1$. □

57.2

Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective.

Proof. Suppose not. Then there exists some point $p \in S^2$ such that $g(x) \neq p$ for all $x \in S^2$. Then we can restrict the range of g to yield the continuous map $g' : S^2 \to S^2 \backslash \{p\}$. Now, if h denotes stereographic projection, then $h : S^2 \backslash \{p\} \to \mathbb{R}^2$ is a homeomorphism and so

$$f = h \circ g' : S^2 \to \mathbb{R}^2$$

is continous. By the Borsuk-Ulam Theorem for S^2, there exists $x \in S^2$ such that $f(x) = f(-x)$. Hence,

$$g(x) = g'(x) = (h^{-1} \circ f)(x) = (h^{-1} \circ f)(-x) = g'(-x) = g(-x).$$

This contradiction implies that, in fact, g is surjective. $\square$

57.4

Suppose you are given the fact that for each n, no continuous antipode-preserving map $h : S^n \to S^n$ is nulhomotopic. Prove the following:

(a) There is no retraction $r : B^{n+1} \to S^n$.

Proof. Suppose there exists a retraction $r : B^{n+1} \to S^n$. Let v be the restriction of r to S^n. Then $v(x) = x$ for all $x \in S^n$, so

$$v(-x) = -x = -v(x)$$

so v is an antipode-preserving map. Now, v can be extended to a continuous map on B^{n+1} (namely r), so, by Lemma 0.1 above, v is nulhomotopic. From this contradiction, we conclude that there is no retraction $r : B^{n+1} \to S^n$. $\square$

(b) There is no continuous antipode-preserving map $g : S^{n+1} \to S^n$.

Proof. Suppose $g : S^{n+1} \to S^n$ is continuous and antipode preserving. Take S^n to be the "equator" of S^{n+1}. Then the restriction of g to S^n is a continuous and antipode-preserving h of S^1 to itself. By our hypothesis, h is not nulhomotopic.

Now, the upper hemisphere E of S^{n+1} is homeomorphic to B^{n+1}. To see this, let $p : E \to B^{n+1}$ be given by

$$p(x) = p(x_1, x_2, \ldots, x_{n+2}) = (x_1, x_2, \ldots, x_{n+1})$$

Then p is certainly continuous and bijective. Furthermore, for any $x \in B^{n+1}$

$$p^{-1}(x) = p^{-1}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1}, 1 - ||x||)$$

is also continuous. Hence, p is a homeomorphism.

Now, g is a continuous extension of h to E and E is homeomorphic to B^{n+1}, so, by Lemma 0.1, h is nulhomotopic. From this contradiction, then, we conclude that there is no continuous antipode-preserving map from S^{n+1} to S^n. $\square$

(c) (Borsuk-Ulam Theorem) Given a continuous map $f : S^{n+1} \to \mathbb{R}^{n+1}$, there is a point x of S^{n+1} such that $f(x) = f(-x)$.

Proof. Suppose $f(x) \neq f(-x)$ for all $x \in S^{n+1}$. Then the map

$$g(x) = \frac{[f(x) - f(-x)]}{||f(x) - f(-x)||}$$

is a continuous map $g : S^{n+1} \to S^n$ such that $g(-x) = -g(x)$ for all $x \in S^{n+1}$. In other words, g is a continuous antipode-preserving map, contradicting our result in part (b). Hence, there is a point $x \in S^{n+1}$ such that $f(x) = f(-x)$. □

(d) If $A_1, \ldots, A_{n+1}$ are bounded polygonal domains in $\mathbb{R}^{n+1}$, there exists an n-plane in $\mathbb{R}^{n+1}$ that bisects each of them.

Proof. Take $n+1$ bounded polygonal regions $A_1, \ldots A_{n+1}$ in the hyperplane $\mathbb{R}^{n+1} \times 1$ in $\mathbb{R}^{n+2}$ and we show that there is an n-dimensional hyperplane L in this plane that bisects each of them.

Given $u \in S^{n+1}$, let us consider the hyperplane P in $\mathbb{R}^{n+2}$ passing through the origin having u as its unit normal vector. This hyperplane divides $\mathbb{R}^{n+1}$ into two half-spaces; let $f_i(u)$ equal the area of that portion of A_i that lies in the half-space defined by P containing u.

If u is the unit vector defined by $(0, \ldots, 0, 1)$, then $f_i(u) = \text{area } A_i$; and if u is the negative of this vector, then $f_i(u) = 0$. Otherwise, the hyperplane P intersects the hyperplane $\mathbb{R}^2 \times 1$ in a n-dimensional hyperplane L that splits $\mathbb{R}^{n+1} \times 1$ into two half-planes, and $f_i(u)$ is the area of that part of A_i that lies on one side of this hyperplane.

Replacing u by $-u$ gives us the same hyperplane P, but the other half-space, so that $f_i(-u)$ is the area of that part of A_i that lies on the other side of P from u. It follows that

$$f_i(u) + f_i(-u) = \text{area } A_i.$$

Now, consider the map $F : S^{n+1} \to \mathbb{R}^{n+1}$ given by $F(u) = (f_1(u), \ldots, f_{n+1}(u))$. Part (c) gives us the point u of S^{n+1} for which $F(u) = F(-u)$. Then $f_i(u) = f_i(-u)$ for $i = 1, \ldots, n+1$, so that $f_i(u) = \frac{1}{2}\text{area } A_i$, as desired. □

A

Definition 0.1. Let G be a topological group and X a topological space, then an **action** of G on X is a continuous map $G \times X \to X$ with the image of (g, x) denoted by $g(x)$, such that:

a) $e(x) = x$ for any $x \in X$.

b) $g_1(g_2(x)) = (g_1 g_2)(x)$ for any $g_1, g_2 \in G$ and $x \in X$.

Given $x \in X$ the **orbit** of x under the action of G is the set $Gx = \{g(x) : g \in G\}$, and the **isotropy group** of x is the subgroup of G given by $G_x = \{g \in G : g(x) = x\}$. We say that the action is **transitive** if $K = \cap_{x \in X} G_x = \{e\}$; that is, if $g(x) = x \forall x \in X$ if and only if $g = e$. The action is said to be **almost effective** if $K = \cap_{x \in X} G_x$ is finite for all $x \in X$.

The action is said to be **free**, respectively **almost free**, if G_x is trivial, respectively finite, for all $x \in X$.

Prove the following theorem.

Theorem 0.3. *Let G be a compact topological group acting on a Hausdorff space X. Fix $x \in X$ and let G_x be the isotropy group of x and let G/G_x have the quotient topology. Show that the map $\phi : G/G_x \to G_x$ given by $\phi(\overline{g}) = g(x)$ is well-defined and is a homeomorphism, where $\overline{g} \in G/G_x$.*

Proof. Fix x. Let $q : G \to Gx \subseteq X$ be the restriction of the action to Gx. Then q is certainly a surjective continuous map. Also, if $C \subseteq G$ is closed, then, since G is compact, C is compact, meaning, since q is continuous, that $q(C)$ is compact and thus, since Gx is Hausdorff, closed. In other words, q is a closed map and hence a quotient map.

Now, if $z \in Gx$, then $z = g_0(x)$ for some $g_0 \in G$. Now,

$$q^{-1}(z) = q^{-1}(g_0x) = \{g_0a : a \in G_x\} = g_0G_x$$

since, if $a \in G_x$, $q(g_0a) = (g_0a)x = g_0(ax) = g_0(x)$. Therefore,

$$G/G_x = \{q^{-1}(z) : z \in Gx\}.$$

Since we have given G/G_x the quotient topology, Corollary 22.3 tells us that q induces a bijective continuous map $f : G/G_x \to Gx$. Furthermore, f is a homeomorphism since q is a quotient map.

Thus, it only remains to show that, in fact, $f = \phi$. However, this is clear, since, if $gG_x \in G/G_x$,

$$f(gG_x) = q(g) = g(x) = \phi(gG_x).$$

Therefore, ϕ is well-defined and a homeomorphism. □

B

For all $n \in \mathbb{N}$, let $SO(n) = \{A \in GL_n(\mathbb{R}) : AA^t = I, det(A) = 1\}$. Show that S^n is homeomorphic to $SO(n+1)/SO(n)$. Note you can imbed $SO(n)$ inside $SO(n+1)$ as follows:

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Proof. Note that if $A \in SO(n+1)$, then, for any $x \in S^n$,

$$||Ax||^2 = \langle Ax, Ax \rangle = \langle x, x \rangle = ||x||^2,$$

so $Ax \in S^n$. Also, if I_{n+1} is the identity matrix, then $I_{n+1} \in SO(n+1)$ and $I_{n+1}x = x$. Finally, if $A, B \in SO(n+1)$, then

$$(AB)x = A(Bx),$$

so $SO(n+1)$ acts on S^n.

Now, let $a = (1, 0, \ldots, 0) \in S^n$. Then, by part A above,

$$SO(n+1)/SO(n+1)_a \approx (SO(n+1))a, \tag{1}$$

where $SO(n+1)_a$ is the isotropy subgroup of a in $SO(n+1)$ and $(SO(n+1))a$ is the orbit of a in S^n under the action of $SO(n+1)$.

To complete the proof, then, we need to show the following:

(i) $(SO(n+1))a = S^n$.

(ii) $SO(n+1)_a \simeq SO(n)$.

To show (i), let $x \in S^n$. Then there exists a matrix $B \in SO(n+1)$ with x as its first column because x is normal and there exists an orthonormal basis for $\mathbb{R}^{n+1}$ containing x (we just let the other columns of B be the other vectors in this basis). Furthermore,

$$Ba = x,$$

so $x \in (SO(n+1))a$. Since our choice of x was arbitrary, we see that $S^n \subseteq (SO(n+1))a$. We've already shown that $(SO(n+1))a \subseteq S^n$, so we conclude that $(SO(n+1))a = S^n$.

To show (ii), we first identify $SO(n)$ with a subgroup of $SO(n+1)$ *via* the embedding $\phi : SO(n) \to SO(n+1)$ given by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

It is immediately clear that this is an injection and, if $A, B \in SO(n)$,

$$\phi(A+B) = \begin{pmatrix} 1 & 0 \\ 0 & A+B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = \phi(A) + \phi(B).$$

Hence, $SO(n) \simeq \phi(SO(n)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in SO(n) \right\}$.

Now, it just remains to demonstrate that $\phi(SO(n)) = SO(n+1)_a$. Certainly, if $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \phi(SO(n))$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} a = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

so $\phi(SO(n)) \subseteq SO(n+1)_a$. Furthermore, if $B \in SO(n+1)_a$, then it must be the case that $b_{11} = 1$, which means $b_{1i} = 0$ and $b_{j1} = 0$ for all $i, j = 2, \ldots, n+1$. In other words, $B \in \phi(SO(n))$, so $SO(n+1)_a \subseteq \phi(SO(n))$. Therefore, $SO(n+1)_a \simeq SO(n)$.

Applying results (i) and (ii) to equation (1) then gives us the desired homeomorphism,

$$SO(n+1)/SO(n) \approx S^n.$$

□

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