6.4. Brouwer's Fixed Point Thm.

* Any cont. map from $[0,1] \to [0,1]$ must have a fixed point.

Q: How about maps $D^2 \longrightarrow D^2$

$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ closed disk

36 Thm.: (Brouwer's Fixed-Point Thm.)

Let $f: D^2 \longrightarrow D^2$ continuous, where

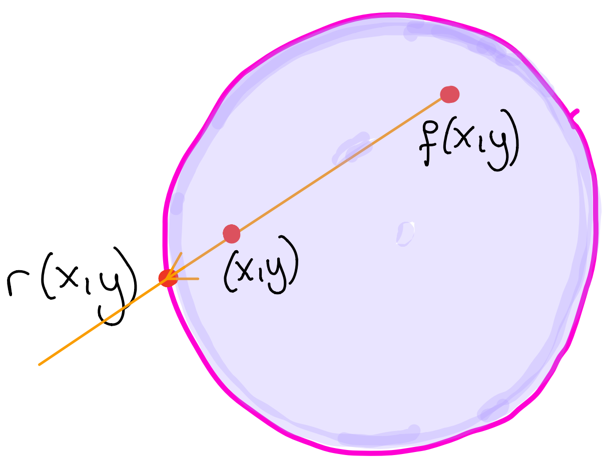
$\Longrightarrow$ f has a fixed point.

i.e. $\exists (x,y) \in D^2$ with $f(x,y) = (x,y)$

Proof: Suppose $f: D^2 \longrightarrow D^2$ doesn't have any fixed point.

i.e. $f(x,y) \neq (x,y) \quad \forall (x,y) \in D^2$.

Define $r: D^2 \longrightarrow S^1$ by



$r(x,y) \in S^1$: the point on S^1 where the line passing through (x,y) and $f(x,y)$ intersect $\partial D^2 = S^1$.

- r is continuous:

Since small perturbations of x produce small perturbations of $f(x,y)$

Hence, also small perturbations of the ray through these two points.

Namely, if (x',y') is sufficiently close to (x,y), then $f(x',y')$ will be close to $f(x,y)$ since f is continuous.

Hence $r(x',y')$ is close to $r(x,y)$.

- $r(x,y) = (x,y)$ if $(x,y) \in S^1$

Now, define $F: S^1 \times I \longrightarrow S^1$

by $F((x,y),t) := r(tx, ty)$

F is continuous.

$$F((x,y),0) = r(0,0) = c_{S^1} \quad \forall (x,y) \in S^1$$

constant.

$$c: S^1 \longrightarrow S^1 \quad c(x,y) = r(0,0) \quad \forall (x,y) \in S^1$$

$$F((x,y),1) = r(x,y) \qquad \forall (x,y) \in S^1$$

$$= (x,y) = id_{S^1}(x,y)$$

$$\Rightarrow \quad id_{S^1} \sim c_{S^1}$$

$$\Rightarrow \quad \underbrace{deg(id_{S^1})}_{\substack{\shortparallel \\ 1}} = \underbrace{deg(c_{S^1})}_{\substack{\shortparallel \\ 0}} \qquad \longrightarrow\longleftarrow$$

$\square$