8.2 Induced Homomorphism

Defn.: Given a pointed continuous map $f: X \to Y$ there is an induced function (homomorphism)

$$f_*: \pi_n(X) \to \pi_n(Y) \text{ defined by}$$

$$f_*([\alpha]) = [f \circ \alpha]$$

for any pointed map $\alpha: I^n \to X$

Q: Is it well defined.

YES. since if $\alpha \sim \beta \Rightarrow f \circ \alpha \sim f \circ \beta$.

Theorem 8.12

Given a pointed continuous map $f : X \to Y$, the induced function $f_* : \pi_n(X) \to \pi_n(Y)$ is a group homomorphism, with the following properties:

1. If $g : Y \to Z$ is another pointed map, then $(g \circ f)_* = g_* \circ f_*$.
2. If $i : X \to X$ is the identity map, then i_* is the identity homomorphism $\pi_n(X) \to \pi_n(X)$ for each n.
3. If $h : X \to Y$ is (pointed) homotopic to f, then $h_* = f_*$.
4. If $c : X \to Y$ takes every point of X to the base point of Y, then $c_* = 0$, the zero homomorphism.

Proof: First show that f_* is a homomorphism.

Let $\alpha_1, \alpha_2 : I^n \longrightarrow X$ be two pointed maps.

$$f_*\left([\alpha_1] * [\alpha_2]\right) = f_*\left([\alpha_1 * \alpha_2]\right) = \left[f \circ (\alpha_1 * \alpha_2)\right]$$

$$= \left[(f \circ \alpha_1) * (f \circ \alpha_2)\right] = [f \circ \alpha_1] * [f \circ \alpha_2]$$

Think of $\alpha_1, \alpha_2 : (I^n, \partial I^n) \longrightarrow X$

$$\alpha_1 * \alpha_2 (s_1, \dots, s_n) = \begin{cases} \alpha_1(s_1, s_2, \dots, 2s_n) & \text{if } s_n \leq 1/2 \\ \alpha_2(s_1, s_2, \dots, 2s_n - 1) & \text{if } s_n \geq 1/2 \end{cases}$$

$$f(\alpha_1 * \alpha_2)(s_1, \dots, s_n) = \begin{cases} f \circ \alpha_1(s_1, s_2, \dots, 2s_n)) & \text{if } s_n \le 1/2 \\ f \circ \alpha_2(s_1, s_2, \dots, 2s_n - 1) & \text{if } s_n \geq 1/2 \end{cases}$$

$$= (f \circ \alpha_1) * (f \circ \alpha_2)\ (s_1, \dots, s_n)$$

1. $(g \circ f)_* \overset{?}{=} g_* \circ f_*$

Proof: $(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)]$

$$= g_*([f \circ \alpha]) = g_*(f_*(\alpha)) = (g_* \circ f_*)(\alpha)$$

2. $i : X \to X \Rightarrow (i)_* : \pi_1(X) \to \pi_1(X)$

$$i_* = id_{\pi_1(X)}$$

Proof:

$$(i)_*([\alpha]) = [i \circ \alpha] = [\alpha].$$

③ $f \sim h \Rightarrow f_* = h_*$

Proof: $f, h : X \to Y$

Let $\alpha : I^n \to X$ be a pointed map.

$$f_*([\alpha]) = [f \circ \alpha] = [h \circ \alpha] = h_*([\alpha])$$

$$f \sim h \Rightarrow f \circ \alpha \sim h \circ \alpha \Rightarrow [f \circ \alpha] = [h \circ \alpha]$$

④ $c : X \to Y \Rightarrow c_*$ is the zero hom.

$c(x) = y_0 \quad \forall x \in X$

Proof: Want to show that

$$c_* : \pi_1(X) \to \pi_1(Y)$$

$$c_*([\alpha]) = id_{\pi_1(Y)} \quad \forall [\alpha] \in \pi_1(X)$$

where $\alpha : I \to X$ is a pointed map.

$$c_*([\alpha]) = [c \circ \alpha] = [c] = id_{\pi_1(Y)}$$

$$c \circ \alpha(s) = c(s) \quad \forall s \in I$$

$\square$

13 **Prop.:** There is NO continuous map $f: D^2 \to S^1$ s.t. $f(x,y) = (x,y)$ $\forall (x,y) \in S^1$!

Proof: Suppose $\exists f: D^2 \to S^1$ s.t.

$$f(x,y) = (x,y) \quad \forall (x,y) \in S^1$$

Now, consider the inclusion map

$$i: S^1 \hookrightarrow D^2$$

onto the boundary $\partial D^2 = S^1$.

This gives rise to a sequence

$$S^1 \xhookrightarrow{i} D^2 \xrightarrow{f} S^1$$

So, $\pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2) \xrightarrow{f_*} \pi_1(S^1)$

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{f_*} \mathbb{Z}$$

Since we know that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 0$

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{i_*} & 0 & \xrightarrow{f_*} & \mathbb{Z} \\ n & \longmapsto & 0 & \longmapsto & f_*(0) \end{array}$$

On the other hand, $f \circ i = id_{S^1}$

since $f(x,y) = (x,y) \quad \forall (x,y) \in S^1$

so, $(f \circ i)_* = (id_{S^1})_* = id_{\pi_1(S^1)}$

$$\Rightarrow f_* \circ i_* = id_{\pi_1(S^1)} = \mathbb{Z} \quad \rightarrow\leftarrow$$

$$(f_* \circ i_*)(n) = f_*(0) \qquad \forall n \in \mathbb{Z}.$$

which is not identity.

$\square$

(14) Prop.: $f: S^1 \to S^1$

$f \circ f = c_{x_0} \Rightarrow f_*: \pi_1(S^1) \to \pi_1(S^1)$

is the zero homomorphism.

Proof: $f: S^1 \longrightarrow S^1$ s.t.

$f \circ f = c$ constant map

$(f \circ f)_* = f_* \circ f_* = c_* = 0$ zero homomorphism

$\pi_1(S^1) = \mathbb{Z}$

The only group homomorphism $h: \mathbb{Z} \to \mathbb{Z}$

s.t. $h \circ h = 0$ is the zero homomorphism.

Since otherwise,

$h(n) = m \neq 0$ for some $n \in \mathbb{Z}$.

$\Rightarrow h \circ h(n) = h(\underset{\neq 0}{m}) \neq 0 \quad \rightarrow\leftarrow$

$\therefore \; f_* = 0$ $\square$

(15) **Prop.:** X, Y: topological spaces

$$X \underset{\substack{\text{pointed}\\ \text{homotopy}\\ \text{equivalent}}}{\sim} Y \Rightarrow \pi_n(X) \approx \pi_n(Y) \quad \forall n$$

Proof: If $X \sim Y \Rightarrow \exists f: X \to Y$
$\exists g: Y \to X$

s.t. $f \circ g \sim id_Y$ and $g \circ f \sim id_X$

$\Rightarrow (f \circ g)_* = id_{\pi_1(Y)}$ and $(g \circ f)_* = id_{\pi_1(X)}$

$\Rightarrow f_* \circ g_* = id_{\pi_1(Y)}$ and $g_* \circ f_* = id_{\pi_1(X)}$

$\Rightarrow$ Both f_* and g_* are invertible.

$\Rightarrow$ They are isomorphisms.

Recall: isomorphism = bijective homomorphism

$\square$

16 Ex.: $\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$

$\pi_i(\mathbb{R}^2 - \{0\}) = 0 \quad \forall i \neq 1$

Since $\mathbb{R}^2 - \{0\} \sim S^1$.

$\square$