8.5.: The Van Kampen Thm.

$$X = U \cup V$$

$x_0 \in U, V \underset{\text{open}}{\subset} X$ path-connected

$U \cap V$ path-connected

$\Rightarrow \varphi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$

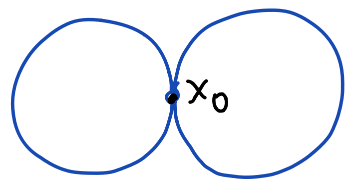
is surjective and

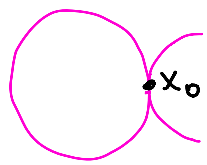
$$N = \mathrm{Ker}(\varphi) = \langle (i_1)_*(\omega)\,(i_2)_*(\omega)^{-1} \rangle$$

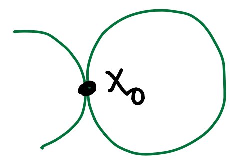
So, $\boxed{\pi_1(X) \approx \dfrac{\pi_1(U) * \pi_1(V)}{N}}$

$$
\begin{tikzcd}
 & U \arrow[dr, hook, "j_1"] & \\
U \cap V \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & U \cap V = X \\
 & V \arrow[ur, hook, "j_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "(j_1)_*"] & \\
\pi_1(U \cap V) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "(j_2)_*"'] &
\end{tikzcd}
$$

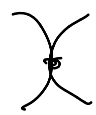
Ex: $X =$ 

Solution: U 



V

U and V are path-connected.

$U \cap V =$  $\sim \bullet$

$$\begin{tikzcd} & U \arrow[dr, "j_1"] & \\ U \cap V \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & U \cup V \\ & V \arrow[ur, "j_2"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "(j_1)_*"] & \\ \pi_1(U \cap V) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] & & \pi_1(U \cup V) \\ & \pi_1(V) \arrow[ur, "(j_2)_*"'] & \end{tikzcd}$$

ω

$$(j_1)_* \circ (i_1)_* = (j_2)_* \circ (i_2)_*$$

$$N = \left\langle (i_1)_*(w)\,(i_2)_*(w)^{-1} \right\rangle = 0$$

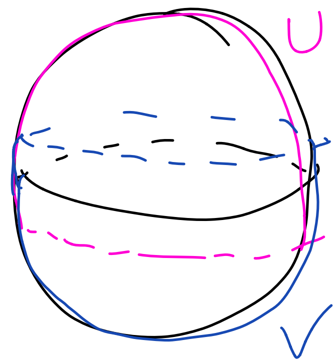
since $w \in U \cap V \sim \{pt\}$.

$$U \sim S^1 \qquad V \sim S^1$$

$$\therefore \quad \pi_1(X) = \frac{\pi_1(S^1) * \pi_1(S^1)}{0} = \mathbb{Z} * \mathbb{Z}$$

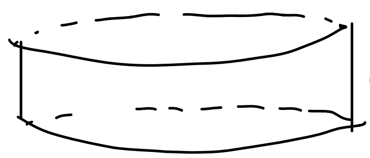
Ex. 25: $\pi_1(S^n) = 0 \quad \forall n \geq 2.$

S^2



path-conn.

$U \cap V =$



$\sim S^1$

path-conn.

$$
\begin{tikzcd}
 & \overset{\sim D^2}{U} \arrow[dr, "j_1"] & \\
S^1 \sim U \cap V \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & U \cup V = S^2 \\
 & \underset{\sim D^2}{V} \arrow[ur, "j_2"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & \\
 & \pi_1(U) \arrow[dr, "(j_1)_*"] & \\
\pi_1(U\cap V) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] \arrow[d, equal] & & \pi_1(S^2) \\
\mathbb{Z} & \pi_1(V) \arrow[ur, "(j_2)_*"'] \arrow[d, equal] & \\
 & 0 &
\end{tikzcd}$$

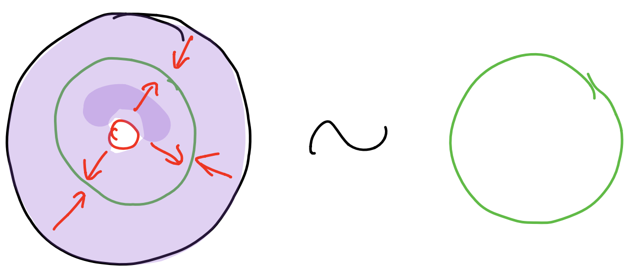
$$\Rightarrow N = \langle (i_1)_*(\omega)(i_2)_*(\omega)^{-1} \rangle = 0$$

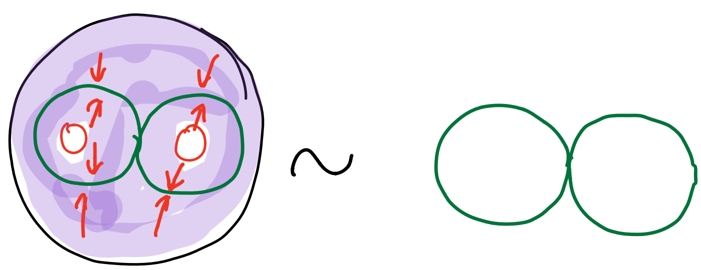
$$\pi_1(S^2) \simeq \frac{\pi_1(U) * \pi_1(V)}{N}$$

$$\simeq \frac{0 * 0}{0} = 0$$

Ex. 26: $\pi_1(D^2) = 0$

Let $U = D^2 - \{(-1,0)\} \sim S^1$
$V = D^2 - \{(1,0)\} \sim S^1$



$U \cap V =$ 

$$\begin{tikzcd}
 & \mathbb{Z} = \langle a \rangle \arrow[d, phantom, "\shortparallel"] & \\
 & \pi_1(U) \arrow[dr, "(j_1)_*"] & \\
\pi_1(U \cap V) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] & & \pi_1(D^2) \\
 & \pi_1(V) \arrow[ur, "(j_2)_*"'] & \\
 & \mathbb{Z} = \langle b \rangle \arrow[u, phantom, "\shortparallel"] &
\end{tikzcd}$$

$\pi_1(U \cap V) = \mathbb{Z} * \mathbb{Z}$

$\langle a, b \rangle$

$$N = \langle (i_1)_*(\omega)(i_2)_*(\omega)^{-1} \rangle = \mathbb{Z} * \mathbb{Z}$$

$$\Rightarrow \pi_1(D^2) = \frac{\pi_1(U) * \pi_1(V)}{N}$$

$$= \frac{\langle a \rangle * \langle b \rangle}{\langle a, b \rangle} = 0$$

Although homotopy groups are hard to calculate, they are very usefull since they contain so much topological information.

27 Thm.: "Whitehead Theorem"

X, Y: connected simplicial complexes

$f: X \to Y$ a map s.t.

$f_*: \pi_i(X) \to \pi_i(Y)$ isomorphism $\forall i$.

$\Rightarrow f$ is an homotopy equivalence between X and Y.

i.e. $\exists g: Y \to X$ s.t. $f \circ g \sim id_Y$
continuous $\qquad g \circ f \sim id_X$

Remark: This doesn't say that

$\pi_n(X) \approx \pi_n(Y) \Rightarrow X \sim Y$