MATH 145B—**HOMEWORK** 6

Ricardo J. Acuña

(862079740)

1. (a) Reading from the picture:

$C_1(S^1) = \{\lambda_0 e_0 +_1 e_1 + \lambda_1 e_2 | \lambda_i \in \mathbb{Z}\}$

$C_0(S^1) = \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_1 v_2 | \lambda_i \in \mathbb{Z}\}$

$C_n(S^1) = 0, \forall n \geq 2$

$\forall n \in \mathbb{N} : n \geq 2 \implies \textbf{Ker } \partial_n = \textbf{Im } \partial_n = 0$

$\implies \forall n \in \mathbb{N} : n \geq 2 : H_n(S^1) = \frac{0}{0} = 0$

$\partial_1(e_0) = v_2 - v_0$

$\partial_1(e_1) = v_1 - v_2$

$\partial_1(e_2) = v_0 - v_1$

$\implies \textbf{Im } \partial_1 =< v_2 - v_0, v_1 - v_2, v_0 - v_1 >$

$v_0 - v_1 = -(v_2 - v_0 + v_1 - v_2) \implies$ one of the generators is redundant

$\implies \textbf{Im } \partial_1 =< v_2 - v_0, v_1 - v_2 >$

Solve $\partial_1(\lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2) = \lambda_0(v_2 - v_0) + \lambda_1(v_1 - v_2) + \lambda_2(v_0 - v_1) = 0$

$\implies \lambda_0(v_2 - v_0) + \lambda_1(v_1 - v_2) + \lambda_2(v_0 - v_1) = (\lambda_2 - \lambda_0)v_0 + (\lambda_1 - \lambda_2)v_1 + (\lambda_0 - \lambda_1)v_2 = 0$

$\implies \lambda_2 - \lambda_0 = 0$ and $\lambda_1 - \lambda_2 = 0$ and $\lambda_0 - \lambda_1 = 0 \implies \lambda_2 = \lambda_0$ and $\lambda_1 = \lambda_2$ and $\lambda_0 = \lambda_1$

$\implies \lambda_0 = \lambda_1 = \lambda_2 := \lambda$

$\implies \partial_1(\lambda(e_0 + e_1 + e_2)) = 0$

$\implies \textbf{Ker } \partial_1 =< e_0 + e_1 + e_2 >$

$\implies H_1(S^1) = \frac{\textbf{Ker } \partial_1}{\textbf{Im } \partial_2} = \frac{< e_0 + e_1 + e_2 >}{0} = \mathbb{Z}$

$\partial_0(v_0) = \partial_0(v_1) = \partial_0(v_2) = 0$

$\implies \textbf{Im } \partial_0 = 0$ and $\textbf{Ker } \partial_0 =< v_0, v_1, v_2 >$

$\implies H_0(S^1) = \frac{\textbf{Ker}\partial_0}{\textbf{Im}\partial_1} = \frac{< v_0, v_1, v_2 >}{< v_2 - v_0, v_1 - v_2 >}$

One can change the basis of $C_0(S^1) = \textbf{Ker}\partial_0$ as such: $< v_0, v_1, v_2 >=< v_0, v_1 - v_2, v_2 - v_0 >$

$\implies H_0(S^1) = \frac{< v_0, v_1 - v_2, v_2 - v_0 >}{< v_2 - v_0, v_1 - v_2 >} =< v_0 >= \mathbb{Z}$

So,

$$H_n(Annulus) = \begin{cases} \mathbb{Z} \, , n = \in \{0, 1\} \\ 0 \, , \text{else} \end{cases}$$

5. (a) $\forall n \in \mathbb{N} : n \geq 3 \implies C_n(S^2) = 0$

$C_2(S^2) = \{\lambda f | \lambda \in \mathbb{Z}\}$

$C_1(S^2) = 0,$

$C_0(S^2) = \{\lambda v | \lambda \in \mathbb{Z}\}$

$\forall n \in \mathbb{N} : n \geq 3 \implies$ **Im** $\partial_n = 0 =$ **Ker** ∂_n. Because they're all 0.

f is the only element of $C_2(S^2) \implies \partial_2(f) = 0 \implies$ **Im** $\partial_2 = 0$ and **Ker** $\partial_2 =< f >$

$\implies H_2(S^2) = \frac{\textbf{Ker}\partial_2}{\textbf{Im}\partial_3} = \frac{<f>}{0} =< f >= \mathbb{Z}$

0 is the only element of $C_1(S^2) \implies \partial_1(0) = 0 \implies$ **Im** $\partial_1 = 0$ and **Ker** $\partial_1 = 0$

v is the only element of $C_0(S^2) \implies \partial_0(v) = 0 \implies$ **Im** $\partial_0 = 0$ and **Ker** $\partial_0 =< v >$

$\implies H_0(S^2) = \frac{\textbf{Ker}\partial_0}{\textbf{Im}\partial_2} = \frac{<v>}{0} =< v >= \mathbb{Z}$

So,

$$H_n(S^2) = \begin{cases} \mathbb{Z} \text{ , } n = \in \{0, 2\} \\ 0 \text{ , else} \end{cases}$$

7. (a) $C_2(S^2) = \{\lambda f | \lambda \in \mathbb{Z}\}$

$C_1(\mathbb{R}P^2) = \{\lambda_0 a + \lambda_1 b | \lambda_i \in \mathbb{Z}\}$

$\downarrow$

$C_0(\mathbb{R}P^2) = \{\lambda_0 v_0 + \lambda_1 v_1 | \lambda_i \in \mathbb{Z}\}$

$\forall n \in \mathbb{N} : n \geq 3 \implies C_n(\mathbb{R}P^2) = 0 \implies$ **Im** $\partial_n = 0 =$ **Ker** ∂_n

f is the only element of $C_2(\mathbb{R}P^2) \implies \partial_2(f) = 2b - 2a$ since you twist each edge once.

$\implies$ **Im** $\partial_2 =< 2(b-a) >$

$x(b-a) = xb - xa = 0 \implies x = 0 \implies$ **Ker** $\partial_2 = 0$

$\implies H_n(\mathbb{R}P^2) = \frac{\textbf{Ker}\partial_n}{\textbf{Im}\partial_{n+1}} = \frac{0}{0} = 0$, for $n \geq 2$

$\partial_1(a) = v - w$ and $\partial_1(b) = v - w$

$\implies$ **Im** $\partial_1 =< v - w >$

$y_1(v-w) + y_2(v-w) = (y_1 + y_2)v - (y_1 + y_2)w = 0 \implies y_1 + y_2 = 0 \implies -y_1 = y_2$

$\implies$ **Ker** $\partial_1 =< b - a >$

$\implies H_1(\mathbb{R}P^2) = \frac{\textbf{Ker}\partial_1}{\textbf{Im}\partial_2} = \frac{<b-a>}{<2(b-a)>} = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2$

$\partial_0(v_0) = \partial_0(v_1) = 0 \implies$ **Im** $\partial_0 = 0$ and **Ker** $\partial_0 =< v_0, v_1 >$

One can change the generators $< v_0, v_1 >$ to $< v_1 - v_0, v_0 >$.

$\implies H_0(\mathbb{R}P^2) = \frac{\textbf{Ker}\partial_0}{\textbf{Im}\partial_1} = \frac{<v_0,v_1>}{<v_1-v_0>} = \frac{<v_1-v_0,v_0>}{<v_1-v_0>} =< v_0 >= \mathbb{Z}$

So,

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, n = 0 \\ \mathbb{Z}_2, n = 1 \\ 0, \text{else} \end{cases}$$

10 Call the picture in the image K

pf.

$C_2(K) = \{\lambda_0 f_0 + \lambda_1 f_1 | \ \lambda_i \in \mathbb{Z}\}$

$C_1(K) = \{\lambda_0 b + \lambda_1 a | \ \lambda_i \in \mathbb{Z}\}$

$C_0(K) = \{\lambda_0 v | \ \lambda_i \in \mathbb{Z}\}$

$\partial_2(f_0) = a - b$

$\partial_2(f_1) = b$

$\implies$ **Im** $\partial_2 =< a - b, b >=< a, b >$

$x_1(a-b) + x_2 b = 0 \implies x_1 a + (x_2 - x_1)b = 0 \implies x_1 = 0, x_2 - x_1 = 0 \implies x_1 = x_2 = 0$

$\implies$ **Ker** $\partial_2 = 0$

Im $\partial_3 = 0$, since $C_3(K) = 0$

$\implies H_2(K) = \frac{0}{0} = 0$

Ker $\partial_n = 0$ and **Im** $\partial_n = 0$,for all $n \geq 3$ since $C_n(K) = 0$ then.

$\implies H_n(K) = \frac{0}{0} = 0$,for all $n \geq 2$

$\partial_1(b) = 0$

$\partial_1(a) = 0$

$\implies$ **Im** $\partial_1 = 0$

$\implies$ **Ker** $\partial_1 =< a, b >$

$\implies H_1(K) = \frac{<a,b>}{\textbf{Im} \ \partial_2} = \frac{<a,b>}{<a,b>} = 0$

$\partial_0(v) = 0$

$\implies$ **Ker** $\partial_0 =< v >$

$H_0(K) = \frac{<v>}{\textbf{Im} \ \partial_1} = \frac{<v>}{0} =< v >= \mathbb{Z}$

So,

$$H_n(K) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & \text{else} \end{cases}$$

For Torus $\chi(T^2) = \sum_{n=0}^{\infty}(-1)^n \textbf{dim} \ C_n(K) = 1 - 2 + 1 + 0 + \cdots = 0$

But, $\chi(K) = \sum_{n=0}^{\infty}(-1)^n \textbf{dim} \ C_n(K) = 1 + 0 + 0 + \cdots = 1$

We didn't need to do any of this... You can get it from Euler's formula $\chi = E - V + F$. $\chi_T = 0$, K has 1 more face so $F' = F + 1 \implies \chi_K = 0 + 1 = 1$.

The homology gets smaller, but the Euler characteristic gets bigger.

■

12. (a) See drawing of X.

$C_2(X) = \{\lambda f |\ \lambda \in \mathbb{Z}\}$

$C_1(X) = \{\lambda_0 a + \lambda_1 b + \lambda_2 c + \lambda_3 d |\ \lambda_i \in \mathbb{Z}\}$

$C_0(X) = \{\lambda v |\ \lambda \in \mathbb{Z}\}$

$\partial_2(\lambda f) = 2\lambda(a + b + c)$

$\implies$ **Im** $\partial_2 = 2 < a + b + c >$

$2\lambda(a + b + c) = 0 \implies \lambda = 0$

$\implies$ **Ker** $\partial_2 = 0$

$C_n(K) = 0$, for $n \geq 3 \implies$ **Im** $\partial_n = 0$

$\implies H_n(X) = \frac{0}{0}$, for $n \geq 2$

$\partial_1(a) = \partial_1(b) = \partial_1(c) = \partial_1(d) = 0$

$\implies$ **Im** $\partial_1 = 0$

$\implies$ **Ker** $\partial_1 =< a, b, c, d >$

$\implies H_1(X) = \frac{<a,b,c,d>}{2<a+b+c>} = \frac{<a+b+c,b,c,d>}{2<a+b+c>} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\partial_0(v) = 0$

$\implies$ **Ker** $\partial_0 =< v >$

$\implies H_0(X) = \frac{<v>}{0} = \mathbb{Z}$

So

$$H_n(X) = \begin{cases} \mathbb{Z} & , n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & , n = 1 \\ 0 & , \text{else} \end{cases}$$