MATH 145B—HOMEWORK 3

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NOTE: All functions under discussion are considered continuous, unless that's a property to be proved.

6.3

1 Each of the spaces below is either contractible or homotopy equivalent to S^1 or neither. For each example, determine which alternative holds. You do not need to give detailed proofs but please give a short explanations.
Note: I know I don't have to prove any of this, but I did it already so feel free to skip to the conclusion.

(a) The solid torus $D^2 \times S^1$

Pf.

$D^2 \times S^1 =$
$\{(x, y) \in \mathbb{R}^2 | \quad ||(x, y)|| \leq 1\} \times \{(x, y) \in \mathbb{R}| \quad ||(x, y)|| = 1\} =$
$\{[(x, y), (z, w)] \in \mathbb{R}^2 \times \mathbb{R}^2| \quad ||(x, y)|| \leq 1$ and $||(z, w)|| = 1\}$

$f : D^2 \times S^1 \to S^1; [(x, y), (z, w)] \mapsto (z, w)$ and $g : S^1 \to D^2 \times S^1; (z, w) \mapsto [(0, 0), (z, w)]$

$f \circ g((z, w)) = f(g((z, w))) = f([(0, 0), (z, w)]) = (z, w) \implies f \circ g = 1_{S^1}$

$g \circ f([(x, y), (z, w)]) = g(f([(x, y), (z, w)])) = g((z, w)) = [(0, 0), (z, w)]$

Consider $H : (D^2 \times S^1) \times [0, 1] \to D^2 \times S^1; ([(x, y), (z, w)], t) \mapsto [t(x, y), (z, w)]$

H is continuous since the first coordinate of the output is the multiplication of two continuous functions, and the second coordinate is the same.

$H([(x, y), (z, w)], 1) = [(x, y), (z, w)] \implies H([(x, y), (z, w)], 1) = 1_{D^2 \times S^1}$
$H([(x, y), (z, w)], 0) = [(0, 0), (z, w)] \implies H([(x, y), (z, w)], 0) = g \circ f$

$\implies g \circ f \sim 1_{D^2 \times S^1}$

$D^2 \times S^1 \sim S^1$

■

(c) The cylinder $S^1 \times \mathbb{R}$

Pf.

$S^1 \times \mathbb{R} = \{(x, y) \in \mathbb{R}^2| \quad ||(x, y)|| = 1\} \times \mathbb{R} = \{[(x, y), z] \in \mathbb{R}^2 \times \mathbb{R}|||(x, y)|| = 1\}$

$f : S^1 \times \mathbb{R} \to S^1; [(x, y), z] \mapsto (x, y)$ and $g : S^1 \to S^1 \times \mathbb{R}; (x, y) \mapsto [(x, y), 0]$

$f \circ g((x, y)) = f(g((x, y))) = f([(x, y), 0]) = (x, y) \implies f \circ g = 1_{S^1}$

$g \circ f([(x, y), z]) = g(f([(x, y), z])) = g((x, y)) = [(x, y), 0]$

Consider $H : (S^1 \times \mathbb{R}) \times [0, 1] \to S^1 \times \mathbb{R}; ([(x, y), z], t) \mapsto [(x, y), tz]$

H is continuous since the second coordinate of the output is the multiplication of two continuous functions, and the first coordinate is the same.

$H([(x, y), z], 1) = [(x, y), z] \implies H([(x, y), z], 1) = 1_{S^1 \times \mathbb{R}}$
$H([(x, y), z], 0) = [(x, y), 0] \implies H([(x, y), z], 0) = g \circ f$

$\implies g \circ f \sim 1_{S^1 \times \mathbb{R}} \implies S^1 \times \mathbb{R} \sim S^1$

■

(e) The set of all points $(x, y) \in \mathbb{R}^2$ such that $||(x, y)|| > 1$

Pf .

$X := \{(x, y) \in \mathbb{R}^2 | \; ||(x, y)|| > 1\}$

$f : X \to S^1; (x, y) \mapsto \frac{(x,y)}{||(x,y)||}$ and $g : S^1 \to X; (x, y) \mapsto 2(x, y)$

$f \circ g((x, y)) = f(g((x, y))) = f(2(x, y)) = \frac{2(x,y)}{||2(x,y)||} = \frac{2(x,y)}{2||(x,y)||} = \frac{(x,y)}{||(x,y)||}$

Since, $(x, y) \in s^1$, $||(x, y)|| = 1$. So, $f \circ g((x, y)) = (x, y)$

$\implies f \circ g = 1_{s^1}$

$g \circ f((x, y)) = g(f((x, y))) = g(\frac{(x,y)}{||(x,y)||}) = 2\frac{(x,y)}{||(x,y)||}$

Consider $H : X \times [0, 1] \to X; [(x, y), t] \mapsto \left(\frac{t||(x,y)||+2(1-t)}{||(x,y)||}\right)(x, y)$

H is continuous since it's a rational function of three variables times a vector, and the denominator is never 0 as $(x, y) \in X$.

$H((x, y), 0) = \left(\frac{0||(x,y)||+2(1-0)}{||(x,y)||}\right)(x, y) = 2\frac{(x,y)}{||(x,y)||} \implies H((x, y), 0) = g \circ f$

$H((x, y), 1) = \left(\frac{1||(x,y)||+2(1-1)}{||(x,y)||}\right)(x, y) = \frac{||(x,y)||}{||(x,y)||}(x, y) = (x, y) \implies H((x, y), 1) = 1_X$

$\implies g \circ f \sim 1_x$

$\implies X \sim S^1$

■

(g) The subset of $\mathbb{R}^2$ given by $S^1 \cup (\mathbb{R}^+ \times \{0\})$.

Pf .

$f : S^1 \cup (\mathbb{R}^+ \times \{0\}) \to S^1$ defined by

$$f((x,y)) = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (1,0), \text{ if } (x,y) \in (\mathbb{R}^+ \times \{0\}) \end{cases}$$

By gluing lemma, f is continuous.
Since, $(1,0)$ is both in S^1 and $(\mathbb{R}^+ \times \{0\})$.
And, $f((1,0)) = 1_{S^1}((1,0)) = (1,0)$.
And, f is the constant map $(1,0)$ on $(\mathbb{R}^+ \times \{0\}) \backslash \{(1,0)\}$.
And, f is the identity map on $S^1 \backslash \{(1,0)\}$.

and $g : S^1 \hookrightarrow S^1 \cup (\mathbb{R}^+ \times \{0\}); (x,y) \mapsto (x,y)$

$$f \circ g((x,y)) = f(g((x,y))) = f((x,y)) = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (1,0), \text{ if } (x,y) \in (\mathbb{R}^+ \times \{0\}) \end{cases}$$

Since the domain of g is S^1 $f \circ g((x,y)) = (x,y) \implies f \circ g = 1_{s^1}$

$$g \circ f((x,y)) = g(f(x,y))$$
$$= g\left(\begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (1,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}\right)$$
$$= \begin{cases} g((x,y)), \text{ if } (x,y) \in S^1 \\ g((1,0)), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}$$
$$= \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (1,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}$$

Consider $H : (S^1 \cup (\mathbb{R}^+ \times \{0\})) \times [0,1] \to S^1 \cup (\mathbb{R}^+ \times \{0\})$ defined by

$$H((x,y),t) = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ t(1,0) + (1-t)(x,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}$$

$$H((x,y),0) = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ 0(1,0) + (1-0)(x,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases} = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (x,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}$$

$$\implies H((x,y),0) = 1_{S^1 \cup (\mathbb{R}^+ \times \{0\})}$$

$$H((x,y),1) = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ 1(1,0) + (1-1)(x,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases} = \begin{cases} (x,y), \text{ if } (x,y) \in S^1 \\ (1,0), \text{ if } (x,y) \in \mathbb{R}^+ \times \{0\} \end{cases}$$

$$\implies H((x,y),1) = f \circ g$$

H is continuous when restricted to $(\mathbb{R}^+ \times \{0\}) \times [0,1]$ since the restricted map is a straight line homotopy. H is also continuous when restricted to $S^1 \times [0,1]$ since it's the identity homotopy. Since $((1,0),t)$ is both on $(\mathbb{R}^+ \times \{0\}) \times [0,1]$ and $S^1 \times [0,1]$, we only need to show both cases agree for all t.

$$H((1,0),t) = \begin{cases} (1,0), \text{ if } (1,0) \in S^1 \\ t(1,0) + (1-t)(1,0), \text{ if } (1,0) \in \mathbb{R}^+ \times \{0\} \end{cases}$$
$$= \begin{cases} (1,0), \text{ if } (1,0) \in S^1 \\ t(1,0) + 1(1,0) - t(1,0), \text{ if } (1,0) \in \mathbb{R}^+ \times \{0\} \end{cases}$$
$$= \begin{cases} (1,0), \text{ if } (1,0) \in S^1 \\ (t-t)(1,0) + (1,0), \text{ if } (1,0) \in \mathbb{R}^+ \times \{0\} \end{cases} = \begin{cases} (1,0), \text{ if } (1,0) \in S^1 \\ (1,0), \text{ if } (1,0) \in \mathbb{R}^+ \times \{0\} \end{cases} = (1,0)$$

$$\implies g \circ f \sim 1_{S^1 \cup (\mathbb{R}^+ \times \{0\})} \implies S^1 \cup (\mathbb{R}^+ \times \{0\}) \sim S^1$$

■

(i) The subset of $\mathbb{R}^2$ given by $S^1 \cup (\mathbb{R}^+ \times \mathbb{R})$

Pf .

Note, $(0,0) \notin s^1 \cup (\mathbb{R}^+ \times \mathbb{R})$
(This observation is implicit because, the question (h) considered $s^1 \cup ([0,+\infty) \times \mathbb{R})$)

$f : s^1 \cup (\mathbb{R}^+ \times \mathbb{R}) \to S^1; \frac{(x,y)}{||(x,y)||}$ and $g : S^1 \hookrightarrow S^1 \cup (\mathbb{R}^+ \times \mathbb{R}); (x,y) \mapsto (x,y)$

$f \circ g((x,y)) = f(g((x,y))) = f((x,y)) = \frac{(x,y)}{||(x,y)||}$

Since the domain of g is S^1 $(x,y) \in S^1 \implies ||(x,y)|| = 1$

$\implies f \circ g((x,y)) = (x,y) \implies f \circ g = 1_{S^1}$

$g \circ f((x,y)) = g(f(x,y)) = g(\frac{(x,y)}{||(x,y)||}) = \frac{(x,y)}{||(x,y)||}$

Consider $H : s^1 \cup (\mathbb{R}^+ \times \mathbb{R}) \times [0,1] \to s^1 \cup (\mathbb{R}^+ \times \mathbb{R}); [(x,y),t] \mapsto \left(\frac{t||(x,y)||+(1-t)}{||(x,y)||}\right)(x,y)$

H is continuous since it's a rational function of three variables times a vector, and the denominator is never 0 as $(x,y) \in s^1 \cup (\mathbb{R}^+ \times \mathbb{R})$.

$H((x,y),0) = \left(\frac{0||(x,y)||+(1-0)}{||(x,y)||}\right)(x,y) = \frac{(x,y)}{||(x,y)||} \implies H((x,y),0) = g \circ f$

$H((x,y),1) = \left(\frac{1||(x,y)||+(1-1)}{||(x,y)||}\right)(x,y) = \frac{||(x,y)||}{||(x,y)||}(x,y) = (x,y) \implies H((x,y),1) = 1_{s^1 \cup (\mathbb{R}^+ \times \mathbb{R})}$

$\implies g \circ f \sim 1_{s^1 \cup (\mathbb{R}^+ \times \mathbb{R})}$

$\implies s^1 \cup (\mathbb{R}^+ \times \mathbb{R}) \sim S^1$

■

The, proof of (g) is ok, but it's somewhat involved as I assumed $(0,0)$ was in $S^1 \cup (\mathbb{R}^+ \times \{0\})$. I'm convinced now it is not the case, and in fact one can replace $\{0\}$ for $\mathbb{R}$ in the previous proof and it proves (g) much easier.

2 (a) Let $f, g : S^1 \to S^1$ be continuous mappings and let's take the complex multiplication operation on $S^1 \subset \mathbb{C}$. Define $h(z)$ to be the product $h(z) = f(z) \cdot g(z)$. Show that deg(h) is equal to deg(f)+ deg(g)

Pf .

$f : S^1 \to S^1 \implies \exists n \in \mathbb{Z} : f \sim z^n$

$\implies \exists (F : S^1 \times [0,1] \to S^1) : F(z,1) = f(z)$ and $F(s,0) = z^n$

$g : S^1 \to S^1 \implies \exists m \in \mathbb{Z} : g \sim z^m$

$\implies \exists (G : S^1 \times [0,1] \to S^1) : G(z,1) = g(z)$ and $G(z,0) = z^{m}{''}$

"$\cdot$" denotes the complex multiplication operation on $\mathbb{C}$

$h : S^1 \to S^1; z \mapsto f(z) \cdot g(z)$

Consider $H : S^1 \times [0,1] \to S^1$ defined by

$H(z,t) = F(z,t) \cdot G(z,t)$

H is continuous since it's the complex multiplication of two continuous functions.

$H(z,0) = F(z,0) \cdot G(z,0) = z^n \cdot z^m$

and

$H(z,0) = F(z,1) \cdot G(z,1) = f(z) \cdot g(z) = h(z)$

$\implies h \sim z^n \cdot z^m = z^{n+m}$

$\implies$ deg$(h) =$ deg$(z^{n+m}) = n + m =$ deg(f)+ deg(g)

■

(b) If $f, g : S^1 \to S^1$ are homotopic continuous mappings, then deg$(f) =$ deg(g)

Pf .

$f, g : S^1 \to S^1$ are homotopic continuous mappings

$\implies \exists n \in \mathbb{Z} : f \sim z^n$ and $\exists m \in \mathbb{Z} : g \sim z^m$

$\implies$ deg$(f) = n$ and deg$(g) = m$

$f \sim g$ and $g \sim z^m \implies f \sim z^m$

$\implies$ deg$(f) = m$

$\implies$ deg$(f) =$ deg(g)

■

3 If $f, g : S^1 \to S^1$ are two continuous maps, express $\deg(f \circ g)$ in terms of $\deg(f)$ and $\deg(g)$. Use this to show that $f \circ g$ is homotopic to $g \circ f$.

Pf .

$f : S^1 \to S^1 \implies \exists n \in \mathbb{Z} : f \sim z^n$

$\implies \exists (F : S^1 \times [0,1] \to S^1) : F(z,1) = f(z)$ and $F(s,0) = z^n$

$g : S^1 \to S^1 \implies \exists m \in \mathbb{Z} : g \sim z^m$

$\implies \exists (G : S^1 \times [0,1] \to S^1) : G(z,1) = g(z)$ and $G(z,0) = z^m$

Consider $H : S^1 \times [0,1] \to S^1$ defined by

$H(z,t) = F(G(z,t),t)$

H is continuous since it's the composition of two continuous functions.

$H(z,1) = F(G(z,1),1) = F(g(z),1) = f(g(z)) = f \circ g(z)$

and

$H(z,0) = F(G(z,0),0) = F(z^m,1) = (z^m)^n = z^{mn}$

$\implies f \circ g \sim z^{nm}$

$\implies$ deg$(f \circ g) =$ deg$(z^{nm}) = nm =$ deg(f) deg(g)

By the symmetry, one can evaluate the previous sentence for $f = g$ and $g = f$, so it becomes:

deg$(g \circ f) =$ deg$(z^{mn}) = mn =$ deg(g) deg$(f) =$ deg(f) deg(g)

$\implies$ deg$(f \circ g) =$ deg$(g \circ f)$

$\implies f \circ g \sim g \circ f$ (by 35 Corollary in Circle Lecture Notes)

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