6.1. Homotopy

1. Let $f, g : X \to \mathbb{S}^n$ be continuous functions such that $f(x)$ and $g(x)$ are never antipodal (ie $f(x) \neq -g(x)$ for any $x \in X$). Prove that
$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$
is a homotopy between f and g.

2. Show that the map $f : S^1 \to S^1$ given by $f(x, y) = (-x, -y)$ is homotopic to the identity map.

3. Show that if $f, g : X \to Y$ are homotopic and $h, k : Y \to Z$ are homotopic, then $h \circ f$ and $k \circ g$ are homotopic.

4. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y. Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

5. Let X be a topological space, and let P be a topological space consisting of exactly one point (it has a unique topology). Explain why the set of homotopy classes $[P, X]$ is in $1-1$ correspondence with the set of arc components of X.

6. Let W, X and Y be topological spaces, and let $u \in [W, X]$ and $v \in [X, Y]$ be homotopy classes of continuous mappings. Prove that there is a well-defined homotopy class $v \circ u \in [W, Y]$ with the following property: If f and g are representatives for the equivalence classes u and v, then $g \circ f$ is a representative for $v \circ u$. [Hint: Use Exercise 1 from Munkres.]

7. Let W be a topological space, and let $f : X \to Y$ be continuous.
   - (i) Using Problem 6, show that there is a well defined map of sets $f_* : [W, X] \to [W, Y]$ such that if $v \in [W, X]$ is represented by $g : W \to X$, then $f_*(v)$ is represented by $f \circ g$. Also, explain why f_* is the identity map if $f = id_X$.
   - (ii) Suppose we also have a continuous mapping $h : Y \to Z$. Prove that $(h \circ f)_* = h_* \circ f_*$.
   - (iii) Similarly, show that there is a well defined map of sets $f^* : [Y, W] \to [X, W]$ such that if $v \in [W, Y]$ is represented by $g : W \to Y$, then $f^*(v)$ is represented by $g \circ f$. Also, explain why f^* is the identity map if $f = id_Y$.
   - (iv) Suppose we also have a continuous mapping $h : Z \to X$. Prove that $(f \circ h)^* = h^* \circ f^*$.

8. Let X be any space and $f : X \to S^n$ a continuous map. Using Proposition 6.5, show that if f is not surjective, then f is homotopic to a constant map.

9. Let Y be a nonempty space with the discrete topology (all subsets are open), and let X be a nonempty connected space. Prove that there is a $1-1$ correspondence between $[X, Y]$ and Y.

10. *(Borsuk-Ulam Theorem) Suppose you are given the fact that for each n, no continuous antipode-preserving map $h : S^n \to S^n$ is nullhomotopic. Show that given a continuous map $f : \mathbb{S}^{n+1} \to \mathbb{R}^{n+1}$, there is a point x of $\mathbb{S}^{n+1}$ such that $f(x) = f(-x)$.

    Definition: A map which is homotopic to the constant map is said to be *null-homotopic*.

11. Show that if $g : \mathbb{S}^2 \to \mathbb{S}^2$ is continuous and $g(x) \neq g(-x)$ for all $x \in \mathbb{S}^2$, then g is surjective. (Hint: If $p \in \mathbb{S}^2$, then $\mathbb{S}^2 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^2$. Then use Borsuk-Ulam Theorem)

12. At any given moment in time, there exists a pair of antipodal points on the surface of the earth at which both the temperature and the barometric pressure are equal.