MATH 145B—HOMEWORK 1

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»n« := Statement number n

; := Reads 'defined by' if preceded by a function type declaration—i.e. $f : X \to Y; x \mapsto x^2$. reads f from X to Y defined by x maps to x squared

$1_X := \forall$ sets $X : X \neq \emptyset$, $1_X : X \to X; x \mapsto x$ denotes the identity function on X

$(\_)^T$:= The transpose of $(\_)$

$[\_]_{m\times n} := m \times n$ decorates the object and denotes the dimension of the object. Used for clarity.

Crossley := ISBN 978-1-85233-782-7

$|\_|$:= The cardinality of _

$\equiv$:= Constantly equals

6.1

2 Show that the map $f : S^1 \to S^1$ given by $f(x, y) = (-x, -y)$ is homotopic to the identity map.

Pf. Want to show $f : S^1 \to S^1; (x, y) \mapsto (-x, -y)$ is homotopic to 1_{S^1}

$$H : S^1 \times [0, 1] \to S^1; ((x, y), t) \mapsto (\begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} (x, y)^T)^T$$

$$H((x, y), 0) = (\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} (x, y)^T)^T = ([I]_{2\times 2}(x, y)^T)^T = (x, y)$$
$$H((x, y), 1) = (\begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} (x, y)^T)^T = (-[I]_{2\times 2}(x, y)^T)^T = (-x, -y)$$

This completes half of the proof, now let's consider the continuity of H:

$$(\begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}_{2\times 2} [(x, y)^T]_{2\times 1})^T =$$

$$\begin{pmatrix} \cos(\pi t)x - \sin(\pi t)y \\ \sin(\pi t)x + \cos(\pi t)y \end{pmatrix}^T =$$

$(\cos(\pi t)x - \sin(\pi t)y, \sin(\pi t)x + \cos(\pi t)y)$

So H is continuous since both, $\pi_x \circ H$ and $\pi_y \circ H$ are sums and products of continuous functions. This completes $\frac{3}{4}$ of the proof.

Now, it just a matter of checking whether $H[S^1 \times [0, 1]] \subset S^1$

$$(\cos(\pi t)x - \sin(\pi t)y)^2 + (\sin(\pi t)x + \cos(\pi t)y)^2 =$$
$$\cos^2(\pi t)x^2 - 2\cos(\pi t)\sin(\pi t)xy + \sin^2(\pi t)y^2 + \sin^2(\pi t)x^2 + 2\cos(\pi t)\sin(\pi t)xy + \cos^2(\pi t)y^2 =$$
$$\cos^2(\pi t)x^2 - \cancel{2\cos(\pi t)\sin(\pi t)xy} + \sin^2(\pi t)y^2 + \sin^2(\pi t)x^2 + \cancel{2\cos(\pi t)\sin(\pi t)xy} + \cos^2(\pi t)y^2 =$$
$$\cos^2(\pi t)x^2 + \sin^2(\pi t)x^2 + \sin^2(\pi t)y^2 + \cos^2(\pi t)y^2 =$$
$$(\cos^2(\pi t) + \sin^2(\pi t))x^2 + (\sin^2(\pi t) + \cos^2(\pi t))y^2 =$$
$$1x^2 + 1y^2 = x^2 + y^2 = 1, \text{ Since } (x, y) \in S^1$$

So, $\forall (x, y) \in S^1 : \forall t \in [0, 1] : H((x, y), t) \in S^1$

■

3 Show that if $f, g : X \to Y$ are homotopic and $h, k : Y \to Z$ are homotopic, then $h \circ f$ and $k \circ g$ are homotopic.

Pf.

$f \sim g \implies \exists(\text{continuous } F : X \times [0,1] \to Y) : F(x,0) = f(x) \text{ and } F(x,1) = g(x)$

$h \sim k \implies k \sim h \implies \exists(\text{continuous } K : Y \times [0,1] \to Z) : K(x,0) = k(x) \text{ and } K(x,1) = h(x)$

Consider $H : X \times [0,1] \to Z; (x,t) \mapsto K(F(x,t),t)$
$\implies H(x,0) = K(F(x,0),0) = K(f(x),0) = h(f(x)) = h \circ f(x)$
and
$\implies H(x,1) = K(F(x,1),1) = K(g(x),1) = k(g(x)) = k \circ g(x)$

Since, both, F and K are continuous, H is continuous as H is the composition of two continuous functions.

So, $h \circ f \sim k \circ g$

■

4 Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y. Let $I = [0,1]$. Show that for any X, the set $[X, I]$ has a single element.

Pf.

Suppose, $[X, I] = \{f, g\} \Rightarrow f \nsim g$ »0«

$f \in [X, I] \implies \exists$ continuous $F : X \times I \to I$ where $(x,0) \mapsto f(x)$ and $(x,1) \mapsto 1$

$g \in [X, I] \implies \exists$ continuous $G : X \times I \to I$ where$(x,0) \mapsto g(x)$ and $(x,1) \mapsto 1$

Consider, $H : X \times I \to I; H(x,t) = \begin{cases} F(x,2t), t \leq 1/2 \\ G(x,2-2t), t \geq 1/2 \end{cases}$

$\Rightarrow H(x,0) = F(x,0) = f(x)$
$\Rightarrow H(x,1) = G(x,2-2) = G(x,0) = f(x)$

Since, H equals F and G exclusively $\forall t \in [0, \frac{1}{2})$ and $\forall t \in (\frac{1}{2}, 1]$ respectively.
H is automatically continuous on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$

The gluing lemma tells us it is enough to check that H is continuous at $\frac{1}{2}$ is well defined—i.e check

$$F(x, 2\frac{1}{2}) \stackrel{?}{=} G(x, 2 - 2\frac{1}{2})$$

Indeed it is, since $F(x, 2\frac{1}{2}) = F(x,1) = 1 = G(x,1) = G(x, 2-1) = G(x, 2 - 2\frac{1}{2})$

$\Rightarrow H$ is a homotopy from f to $g \Rightarrow f \sim g$ which contradicts »0«

So, $[X, I]$ has one element.

■

11 Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x \in S^2$, then g is surjective. (Hint: If $p \in S^2$, then $S^2 \backslash \{p\}$ is homeomorphic to $\mathbb{R}^2$. Then use Borsuk-Ulam Theorem)

Pf . WTS $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x \in S^2 \implies g$ is surjective.

Suppose $\exists p \in S^2 : \forall x \in S^2 : g(x) \neq p$ »0«

$S^2 \backslash \{p\}$ is homeomorphic to $\mathbb{R}^2$ (by Hint) $\Rightarrow$
$\exists f : S^2 \backslash \{p\} \to \mathbb{R}^2$ and $\exists f^{-1} : \mathbb{R}^2 \to S^2 \backslash \{p\}$:
$f \circ f^{-1} = 1_{\mathbb{R}^2}$ and $f^{-1} \circ f = 1_{S^2 \backslash \{p\}}$ »1«

Induce g into $S^2 \backslash \{p\}$ with $g' : S^2 \to S^2 \backslash \{p\}; x \mapsto g(x)$

Consider $f \circ g' : S^2 \to \mathbb{R}^2$. $f \circ g'$ is continuous since it is a composition of continuous functions.
$\exists x \in S^2 : f \circ g'(x) = f \circ g'(-x)$ (By Borsuk-Ulam)
$\Rightarrow f(g'(x)) = f(g'(-x))$ (by definition of $\circ$)
$\Rightarrow f(g(x)) = f(g(-x))$ (by definition of g')
$\Rightarrow f^{-1}(f(g(x))) = f^{-1}(f(g(-x)))$ (by applying f^{-1})
$\Rightarrow f^{-1} \circ f(g(x)) = f^{-1} \circ f(g(-x))$ (by definition of $\circ$)
$\Rightarrow 1_{S^2 \backslash \{p\}}(g(x)) = 1_{S^2 \backslash \{p\}}(g(-x))$ (by »1«)
$\Rightarrow g(x) = g(-x)$

This contradicts the property that $\forall x \in S^2 : g(x) \neq g(-x)$ so »0« is false.

$\Rightarrow \exists p \in S^2 : \forall x \in S^2 : g(x) = p$
$\Rightarrow g[S^2] = S^2$
$\Rightarrow g$ is surjective

■

6.2

2 Write down a homotopy equivalence between $(0,1)$ and $[0,1]$.

Pf .

$f:(0,1)\to[0,1]; x\mapsto 1/2$
$g:[0,1]\to(0,1); y\mapsto 1/2$

Since, $[0,1]\overset{g}{\to}(0,1)\overset{f}{\to}[0,1]$ and $f\circ g(y)=f(g(y))=f(1/2)=1/2$
All, continuous functions, on $[0,1]$ are homotopic to constant maps.
In particular $1_{[0,1]}\sim 1/2=f\circ g$

Since, $(0,1)\overset{f}{\to}[0,1]\overset{g}{\to}(0,1)$ and $g\circ f(x)=g(f(x))=g(1/2)=1/2$
Consider, $H:(0,1)\times[0,1]\to(0,1); (x,t)\mapsto t/2+(1-t)x$
H is continuous since it is a polynomial.
$H(x,0)=x\Rightarrow$ at $0, H=1_{(0,1)}$
And since $H(x,1)=1/2\Rightarrow 1_{(0,1)}\sim 1/2=g\circ f$
So $(0,1)\simeq[0,1]$

■

3 Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

The answer is in two parts 0 and 1

Pf . 0

WTS X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic.

($\Leftarrow$) Ass. every map $f : X \to Y$, for arbitrary Y, is nullhomotopic.

$\Rightarrow \forall(f_j : X \to Y) : \exists y_i \in Y : \exists(c_i : X \to Y; x \mapsto y_i) : f_j \sim c_i$
$\Rightarrow [X, Y] = \{c_i : X \to Y\}$
$\Rightarrow [X, Y]$ has at most $|Y|$ elements

Since Y is arbitrary, we can choose $Y = \{0\}$.
$\Rightarrow [X, \{0\}]$ has 1 element.
$\Rightarrow [X, \{0\}] = \{c : X \to \{0\}; x \mapsto 0\}$.

Consider $g : \{0\} \to X; 0 \mapsto x_0$
$c \circ g(0) = c(g(0)) = c(x_0) = 0 \Rightarrow c \circ g = 1_{\{0\}}$
$g \circ c(x) = g(c(x)) = g(0) = x_0 \Rightarrow g \circ c \equiv x_0$

Now, since Y is arbitrary, we can choose again $Y = X$.
$\Rightarrow [X, X] = \{c_i : X \to X; x \mapsto x_i | x_i \in X\}$.

So, for some $x_0 \in X$, $g \circ c : X \to X \equiv x_0 \equiv c_0 \sim 1_X$

$\Rightarrow X \simeq \{0\} \Rightarrow X$ is contractible

($\Rightarrow$) Ass. X is contractible

$\Rightarrow X \simeq \{0\}$
Let, Y be an arbitrary topological space.
Then, by Lemma 6.10 in Crossley $[X, Y] = [\{0\}, Y]$

Since,$\{0\}$ has one element, $\forall(g : \{0\} \to Y), g$ has to be a constant map.

$[\{0\}, Y] = \{c_i : \{0\} \to Y; 0 \mapsto y_i | y_i \in Y\}$ and $[X, Y] = [\{0\}, Y]$
$\Rightarrow$
$[X, Y] = \{k_i : X \to Y; x \mapsto y_i | y_i \in Y\}$
$\Rightarrow$
$\forall(f : X \to Y) : \exists y_i \in Y : (k_i : X \to Y; x \mapsto y_i) : f \sim k_i$
So, all maps f from X to Y are nullhomotopic

So, X is contractible iff every map $f : X \to Y$, for arbitrary Y , is nullhomotopic.

■

Pf . 1

Also WTS X is contractible iff every map $f : Y \to X$ is nullhomotopic.

$(\Leftarrow)$ Ass. every map $f : Y \to X$ is nullhomotopic.

$\Rightarrow \forall(f_j : Y \to X) : \exists x_i \in X : \exists(m_i : Y \to X; y \mapsto x_i) : f_j \sim m_i$
$\Rightarrow [Y, X]$ has at most $|X|$ elements

Since, Y is arbitrary, we can choose $Y = \{x_0\}$: $x_0 \in X$.
Now, since $\{x_0\}$ has one element $\forall(g_i : \{x_0\} \to X)$, g_i is constant.
So, $[\{x_0\}, X] = \{g_i : \{x_0\} \to X; x_0 \mapsto x_i | x_i \in X\}$

We can choose again, $Y = X$: $[X, X] = \{m_i : X \to X; x \mapsto x_i\}$

$\Rightarrow \exists x_0 \in X$: $m_0 \sim 1_X$

Consider, $l : X \to \{x_0\}; x \mapsto x_0$
$l \circ g_0(x_0) = l(g_0(x_0)) = l(x_0) = x_0 \Rightarrow l \circ g_0 = 1_{\{x_0\}}$

$g_0 \circ l(x) = g_0(l(x)) = g_0(x_0) = x_0 \Rightarrow g_0 \circ l = m_0 \sim 1_X$

So, $X \simeq \{x_0\} \Rightarrow X$ is contractible.

$(\Rightarrow)$ Ass. X is contractible.

$\Rightarrow X \simeq \{x_0\}$
Let, Y be an arbitrary topological space.
Then, by Lemma 6.10 in Crossley $[Y, X] = [Y, \{x_0\}]$

Since, $\{x_0\}$ has one element, $[Y, \{x_0\}] = \{c : Y \to \{x_0\}; y \mapsto x_0\}$ also has one element c.
And, since $[Y, X] = [Y, \{x_0\}]$ by the lemma.

$\forall(f : X \to Y) : f \sim c$

So, all functions from X to Y are nullhomotopic.

So, X is contractible iff every map $f : Y \to X$ is nullhomotopic.

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