University of California, Riverside
Math 136: Theory of numbers

June 8, 2019
Spring Quarter

Sample final: short answer

(a) (*6 points*) Use the law of quadratic reciprocity to determine whether or not 12 is a square modulo 31.

Write $12 = \underbrace{m}_{\text{Square-free}} \cdot n^2$ $\quad m = 3$, $n = 2$

$\boxed{\text{No!}}$

$$\Rightarrow \left(\frac{12}{31}\right) = \left(\frac{mn^2}{31}\right) = \left(\frac{m}{31}\right) = \left(\frac{3}{31}\right) = -1$$

By Law of Quad. rec., $\left(\frac{3}{31}\right)\left(\frac{31}{3}\right) = (-1)^{\boxed{\left(\frac{3-1}{2}\right)\left(\frac{31-1}{2}\right)}}$ odd

$$\Rightarrow \left(\frac{3}{31}\right) = -\underbrace{\left(\frac{31}{3}\right) = -\left(\frac{1}{3}\right)}_{\text{Reduce 31 mod 3}} = -1$$

(b) (*6 points*) Use induction to show that $11^n \equiv 1 + 10n \pmod{100}$ for all integers $n \geq 1$.

BASE STEP: $n = 1 \Rightarrow 11 = 1 + 10$

$\Rightarrow 11 \equiv 1 + 10 \pmod{100}$

IND. STEP: ASSUME TRUE FOR n

ie, $11^n \equiv 1 + 10n \pmod{100}$

To SHOW TRUE FOR $n+1$, compute

$$11^{n+1} = 11^n \cdot 11 \equiv (1 + 10n)(1 + 10) \pmod{100}$$

$$\equiv 1 + 10(n+1) + \underbrace{10^2 \cdot n}_{=100n} \pmod{100}$$

$$\equiv 1 + 10(n+1) \pmod{100}$$

(c) (*6 points*) Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a multiplicative function, p is a prime and m is a positive integer for which

$$f(p^m) = \begin{cases} m & \text{if } m \text{ is odd} \\ -m & \text{if } m \text{ is even} \end{cases}$$

Compute $F(56)$, where $F(n) = \sum_{d|n} f(d)$.

$56 = 2^3 \cdot 7$ $\qquad$ $F(n)$ is multiplicative, since f is.

$$F(56) = F(2^3) \cdot F(7)$$

$$= \left(f(1) + f(2) + f(2^2) + f(2^3)\right)\left(f(1) + f(7)\right)$$

$$= (0 - 1 + 2 - 3)(0 - 1) = \boxed{3}$$

(d) (*6 points*) Show that if $n > 1$, then $12^n - 1$ is not prime.

$$12^n - 1 = \underbrace{(12 - 1)}_{= 11}(12^{n-1} + 12^{n-2} + \cdots + 1) \quad \text{for all } n > 1$$

So, $11 \mid 12^n - 1 \Rightarrow 12^n - 1$ is not prime