

# 21-235 Math Studies Analysis I

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# Chapter 1

## 1.1 Ordered Fields (Review)

### Definition 1.1.1: Order

Let  $E$  be a set. An *order* on  $E$  is a relation  $<$  on  $E$  such that for all  $x, y, z \in E$ :

1. (Trichotomy) Exactly one of the following holds:  $x < y$ ,  $x = y$ , or  $x > y$ .
2. (Transitivity) If  $x < y$  and  $y < z$ , then  $x < z$ .

### Example 1.1.1 (Examples of Ordered Sets)

1. This definition develops orders on basic number systems: e.g.  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ .
2. Define  $\lesssim$  on  $\mathbb{Z}$  as follows: We say that  $m \lesssim n$  for  $m, n \in \mathbb{Z}$  if:
  - (a)  $m$  is even and  $n$  is odd
  - (b)  $m, n$  are even and  $m < n$
  - (c)  $m, n$  are odd and  $m < n$ .

Key Concepts:

- upper/lower bounds of sets
- bounded sets
- max/min
- supremum/infimum
- supremum/infimum property: An ordered set  $E$  satisfies such a property if every nonempty set  $A \subseteq E$  that's bounded above/below has a supremum/infimum in  $E$ .
- Fact:  $\sup \text{ prop} \implies \inf \text{ prop}$

### Definition 1.1.2: Ordered Field

Let  $\mathbb{F}$  be a field with order  $<$ . We say that  $\mathbb{F}$  is an *ordered field* provided that:

1. For all  $x, y, z \in \mathbb{F}$ , if  $x < y$ , then  $x + z < y + z$ .
2. For all  $x, y \in \mathbb{F}$ , if  $0 < x$  and  $0 < y$ , then  $0 < x \cdot y$ .

**Example 1.1.2**

$\mathbb{Q}$  is a field.

Facts of any ordered field:

1.  $0 < 1$
2.  $\nexists x \in \mathbb{F}$  such that  $x^2 = -1$ .

**Definition 1.1.3: Ordered Subfield, Homomorphism, Isomorphism**

Let  $\mathbb{F}$  be an ordered field.

1. A set  $\mathbb{K} \subseteq \mathbb{F}$  is called an *ordered subfield* if  $\mathbb{K}$  is an algebraic subfield and  $\mathbb{K}$  is an ordered field equipped with  $<$  from  $\mathbb{F}$ .
2. Let  $\mathbb{G}$  be an ordered field and let  $f : \mathbb{F} \rightarrow \mathbb{G}$ . We say that  $f$  is an *ordered field homomorphism* if it's a field homomorphism and  $f(x) < f(y)$  whenever  $x < y$ .
3.  $f$  is an *ordered field isomorphism* if  $f$  is an ordered field homomorphism and  $f$  is bijective.

**Note:**

1. If  $f : \mathbb{F} \rightarrow \mathbb{G}$  is an ordered field homomorphism,  $f(\mathbb{F})$  is an ordered subfield of  $\mathbb{G}$ .
2. OF property  $\implies f$  is injective.
3.  $\therefore$  every ordered field homomorphism  $f : \mathbb{F} \rightarrow \mathbb{G}$  is such that  $f$  induces a bijection  $f : \mathbb{F} \rightarrow f(\mathbb{F}) \subseteq \mathbb{G}$ .

**Theorem 1.1.1**  $\mathbb{Q}$  is the smallest ordered field. More precisely, if  $\mathbb{F}$  is an ordered field, then there exists a canonical ordered field homomorphism  $f : \mathbb{Q} \rightarrow \mathbb{F}$ .

Upshot/notation abuse: We identify  $f(\mathbb{Q}) = \mathbb{Q}$  to view  $\mathbb{Q} \subseteq \mathbb{F}$ . In turn,  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{F}$ .

## 1.2 Types of Ordered Fields

**Definition 1.2.1: Archimedean, Dedekind complete**

Let  $\mathbb{F}$  be an ordered field.

1. We say that  $\mathbb{F}$  is Archimedean if  $\forall 0 < x \in \mathbb{F}, \exists n \in \mathbb{N}$  such that  $n > x$ .
2. We say that  $\mathbb{F}$  is Dedekind complete if it satisfies the supremum property.

Facts:

1.  $\mathbb{Q}$  is Archimedean.
2. If  $\mathbb{F}$  is Dedekind complete, then  $\forall 0 < x \in \mathbb{F}$  and  $\forall 0 < n \in \mathbb{N}$ ,  $\exists! 0 < y \in \mathbb{F}$  such that  $y^n = x$ .
3.  $\mathbb{Q}$  is not Dedekind complete. ( $\sqrt{2}$  is a counterexample.)

**Theorem 1.2.1**

Suppose  $\mathbb{F}$  is a Dedekind complete ordered field. Then  $\mathbb{F}$  is Archimedean.

*Proof.* If not, then  $\mathbb{N} \subset \mathbb{F}$  is bounded above, and so the supremum property provides  $x \in \mathbb{F}$  such that  $x = \sup \mathbb{N}$ . But then  $x - 1$  is an upper bound for  $\mathbb{N}$ , so there exists  $n \in \mathbb{N}$  such that  $x - 1 < n$ . Hence  $x < n + 1$ , which contradicts the definition of  $x$  as an upper bound. Therefore,  $\mathbb{F}$  is Archimedean.  $\odot$

## 1.3 Dedekind Completion

Throughout this section, let  $\mathbb{F}$  be an Archimedean ordered field.

### Definition 1.3.1: Dedekind cut

We say a set  $C \subseteq \mathbb{F}$  is *Dedekind cut* if:

1.  $C \neq \emptyset$  and  $C \neq \mathbb{F}$ .
2. If  $p \in C$  and  $q \in \mathbb{F}$  such that  $q < p$ , then  $q \in C$ .
3. If  $p \in C$ , then  $\exists r \in C$  such that  $p < r$ .

We will write  $\mathbb{F}^*$  for the set of all Dedekind cuts in  $\mathbb{F}$ . It is called the *Dedekind completion* of  $\mathbb{F}$ .

### Note:

Let  $C \subseteq \mathbb{F}$  be a cut. Then:

1. If  $p \in C$ , then  $q \notin C$ , then  $p < q$ .
2. If  $r \notin C$ , and  $r < s \in \mathbb{F}$ , then  $s \notin C$ .

### Example 1.3.1 (Cut examples)

1. Let  $q \in \mathbb{F}$  and define  $C_q = \{p \in \mathbb{F} \mid p < q\}$ . Then  $C_q$  is a cut.

*Proof.* (a)  $q - 1 < q \implies q - 1 \in C_q$ .  $q \not< q \implies q \notin C_q \implies C_q \neq \mathbb{F}$ .

(b) Let  $p \in C_q$ . Suppose  $s \in \mathbb{F}$  such that  $s < p$ . Then  $s < q \implies s \in C_q$ .

(c) Let  $p \in C_q$ . Then  $p < \frac{p+q}{2} < q \implies \frac{p+q}{2} \in C_q$ . ☺

2. Suppose  $\mathbb{F}$  is such that  $\nexists x \in \mathbb{F}$  such that  $x^2 = 2$ . Let  $C = \{p \in \mathbb{F} \mid p \leq 0 \text{ or } 0 < p^2 < 2\}$ . Then  $C$  is a cut.

*Proof.* (a)  $1 \in C$  and  $1^2 = 1 < 2$ .  $2 \notin C$  and  $2^2 = 4 > 2$ .

(b) Let  $p \in C$  and  $q \in \mathbb{F}$  such that  $q < p$ . If  $q \leq 0$ , then  $q \in C$  trivially. Suppose  $0 < q < p$ . Then  $0 < q^2 < p^2 < 2$ , so  $q \in C$ .

(c) Let  $p \in C$ . If  $p \leq 0$ , then  $1 \in C$  and  $p < 1$ , so we're done. Suppose  $0 < p^2 < 2$ . Note,  $0 < 2 - p^2$ , so  $\frac{2p+1}{2-p^2} > 0$ . Then we can define  $r = 1 + \frac{2p+1}{2-p^2} \geq \max(1, \frac{2p+1}{2-p^2})$ . Then  $(p + 1/r)^2 = p^2 + \frac{2p}{r} + \frac{1}{r^2}$ . We have:

$$\begin{aligned} p^2 + \frac{2p}{r} + \frac{1}{r^2} &< p^2 + \frac{2p}{r} + \frac{1}{r} \\ &= p^2 + \frac{2p+1}{r} \\ &\leq p^2 + 2 - p^2 \\ &= 2. \end{aligned}$$

So,  $p < p + 1/r < 2$  and  $p + 1/r \in C$ . ☺

### 1.3.1 Ordering $\mathbb{F}^*$

#### Lemma 1.3.1

The following hold:

1. If  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ , then exactly one holds:
  - $\mathcal{A} \subset \mathcal{B}$
  - $\mathcal{A} = \mathcal{B}$
  - $\mathcal{B} \subset \mathcal{A}$
2. If  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$  and  $\mathcal{A} \subset \mathcal{B}$  and  $\mathcal{B} \subset \mathcal{C}$ , then  $\mathcal{A} \subset \mathcal{C}$ .

*Proof.* Proof of 2 is trivial, as well as the equality part for 1.

- If  $\mathcal{A} = \mathcal{B}$ , we're done.
- Suppose  $\exists b \in \mathcal{B} \setminus \mathcal{A}$ . If  $a \in \mathcal{A}$ , then  $a < b$ , but  $\mathcal{B}$  is a cut so  $a \in \mathcal{B}$ , so  $\mathcal{A} \subset \mathcal{B}$ .
- Suppose  $\exists a \in \mathcal{A} \setminus \mathcal{B}$ . Then  $a < b$  for all  $b \in \mathcal{B}$ , so  $a \in \mathcal{B}$ , so  $\mathcal{B} \subset \mathcal{A}$ .

⊕

#### Definition 1.3.2: Order on cuts

Given  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ , we say that  $\mathcal{A} < \mathcal{B}$  if  $\mathcal{A} \subset \mathcal{B}$ . The lemma above shows that this is in fact an order.

#### Lemma 1.3.2

Let  $E \subseteq \mathbb{F}^*$  be nonempty and bounded above. Then  $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$  is a cut.

*Proof.* 1. Since  $E \neq \emptyset$ , there exists  $\mathcal{A} \in E$ . So  $\mathcal{A} \neq \emptyset$ , hence  $\mathcal{B} \neq \emptyset$ .

Since  $E$  is bounded above, there exists  $\mathcal{C} \in \mathbb{F}^*$  such that  $\mathcal{A} \subset \mathcal{C}$  for all  $\mathcal{A} \in E$ . Since  $\mathcal{C}$  is a cut, there is  $q \in \mathbb{F}$  such that  $q \notin \mathcal{C}$ . Then  $q \notin \mathcal{A}$  for all  $\mathcal{A} \in E$ , so  $q \notin \mathcal{B}$ .

2. Let  $p \in \mathcal{B}$  and  $q \in \mathbb{F}$  such that  $q < p$ . Since  $\mathcal{B}$  is a union of cuts, it follows that  $p \in \mathcal{A}$  for some  $\mathcal{A} \in E$ . Since  $\mathcal{A}$  is a cut,  $q \in \mathcal{A} \subseteq \mathcal{B}$ .

3. Let  $p \in \mathcal{B}$ . Then  $p \in \mathcal{A}$  for some  $\mathcal{A} \in E$ . Since  $\mathcal{A}$  is a cut, there exists  $r \in \mathcal{A}$  such that  $p < r$ . Since  $\mathcal{A} \subset \mathcal{B}$ , we have  $r \in \mathcal{B}$ .

⊕

#### Theorem 1.3.1

$\mathbb{F}^*$  equipped with the order  $<$  satisfies the supremum property.

*Proof.* Let  $E \subseteq \mathbb{F}$  be a nonempty set that is bounded above. From last time, we know that  $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$  is a cut. We claim that  $\mathcal{B} = \sup E$ .

If  $\mathcal{A} \in E$ , then  $\mathcal{A} \subseteq \mathcal{B}$ . And so  $\mathcal{A} \leq \mathcal{B}$ , so  $\mathcal{B}$  is an upper bound for  $E$ .

Next, suppose that  $\mathcal{C} \in \mathbb{F}^*$  is an upper bound of  $E$ . This means that  $\mathcal{A} \leq \mathcal{C}$  for every  $\mathcal{A} \in E$ , meaning  $\mathcal{A} \subseteq \mathcal{C} \forall \mathcal{A} \in E$ . So  $\mathcal{B} \subseteq \mathcal{C}$ . As such,  $\mathcal{B} \leq \mathcal{C}$ , so  $\mathcal{B} = \sup E$ .

⊕

Remark: In none of the results leading up to this theorem did we use that  $\mathbb{F}$  is anything other than an ordered set. This shows that the cut construction of Dedekind works in general for ordered sets and yields  $\mathbb{F}^*$  that satisfies the supremum property. Also,  $\{C_p \mid p \in \mathbb{F}\} \subseteq \mathbb{F}^*$ .

### 1.3.2 Addition

Idea:  $\mathbb{F} \cong \{C_p \mid p \in \mathbb{F}\}$ .

#### Lemma 1.3.3

Let  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ . Then  $C = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$  is a cut.

*Proof.* Claim:  $\mathcal{A}, \mathcal{B} \neq \emptyset \implies C \neq \emptyset$ .

$\mathcal{A}, \mathcal{B}$  are cuts, so  $\exists M_1, M_2 \in \mathbb{F}$  such that  $a < M_1$  for all  $a \in \mathcal{A}$  and  $b < M_2$  for all  $b \in \mathcal{B}$ . Then  $a + b < M_1 + M_2$  for all  $a \in \mathcal{A}, b \in \mathcal{B}$ , so  $a + b < M_1 + M_2$ , meaning  $M_1 + M_2 \notin C$ .

Also, let  $c = a + b \in C$  for  $a \in \mathcal{A}, b \in \mathcal{B}$ . Let  $q < c \implies q - a < b \implies q - a \in \mathcal{B}$ . Hence,  $q = a + (q - a) \in C$ .

Thirdly, let  $c = a + b \in C$  for  $a \in \mathcal{A}, b \in \mathcal{B}$ . Since  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ ,  $\exists r_a, r_b$  such that  $a < r_a \in \mathcal{A}, b < r_b \in \mathcal{B}$ . Then  $c = a + b < r_a + r_b$ , so  $r_a + r_b \in C$ .

As such,  $C$  is a cut.  $\odot$

Before we define addition, we need to define the negative of a cut.

Heuristic: What we want is that  $-C_1 = C_{-1}$ . The way we do this is by defining  $C_{-p} = \{q \in \mathbb{F} \mid \exists p > q : p \in -C_p^C\}$ . This is the same as  $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C_p\}$ .

Now we study  $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C\}$ .

#### Lemma 1.3.4

Let  $C \in \mathbb{F}^*$ . Then  $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C\}$  is a cut.

#### Definition 1.3.3: Addition

For  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ , we define  $\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$  and  $-\mathcal{A} = \{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{A}\}$ .

#### Theorem 1.3.2

Define  $0 = C_0 \in \mathbb{F}^*$ . The following hold:

1.  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} \in \mathbb{F}^*$ .
2.  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$ .
3.  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^* \implies (\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$ .
4.  $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + 0 = \mathcal{A}$ .
5.  $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + (-\mathcal{A}) = 0$ .

*Proof.* Easy proof, too lazy to write out.  $\odot$

Also:  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$  and  $\mathcal{A} < \mathcal{B} \implies \mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$ .

Important Remark: The Archimedean property is actually needed for the above theorem in order to prove the 5th condition.

### 1.3.3 Multiplication

#### Lemma 1.3.5

Let  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$  such that  $\mathcal{A}, \mathcal{B} > 0$ . Then  $C = \{p \in \mathbb{F} \mid p \leq 0\} \cup \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\}$  is a cut.

#### Lemma 1.3.6

Let  $\mathcal{A} \in \mathbb{F}^*$  be such that  $\mathcal{A} > 0$ . Then  $C = \{p \in \mathbb{F}^* \mid p \leq 0\} \cup \{0 < q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$  is a cut.

#### Definition 1.3.4: Multiplication

Let  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ . We define multiplication as:

1. If  $\mathcal{A}, \mathcal{B} > 0$ , then  $\mathcal{A} \cdot \mathcal{B} = \{ab \mid 0 < a \in \mathcal{A}, 0 < b \in \mathcal{B}\}$ .
2. If  $\mathcal{A} = 0$  or  $\mathcal{B} = 0$ , then  $\mathcal{A} \cdot \mathcal{B} = 0$ .
3. If  $\mathcal{A} > 0$  and  $\mathcal{B} < 0$ , then  $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B}))$ .
4. If  $\mathcal{A} < 0$  and  $\mathcal{B} > 0$ , then  $\mathcal{A} \cdot \mathcal{B} = -((- \mathcal{A}) \cdot \mathcal{B})$ .
5. If  $\mathcal{A}, \mathcal{B} < 0$ , then  $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B})$ .

We define multiplication inversion via:

1. If  $\mathcal{A} > 0$ , then  $\mathcal{A}^{-1} = \{q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$ .
2. If  $\mathcal{A} < 0$ , then  $\mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$ .

#### Theorem 1.3.3

Set  $1 = C_1$ . The following hold:

1. If  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ , then  $\mathcal{A} \cdot \mathcal{B} \in \mathbb{F}^*$ .
2. If  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ , then  $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ .
3. If  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ , then  $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$ .
4. If  $\mathcal{A} \in \mathbb{F}^*$ , then  $\mathcal{A} \cdot 1 = \mathcal{A}$ .
5. If  $\mathcal{A} \in \mathbb{F}^*$ , then  $\mathcal{A} \cdot \mathcal{A}^{-1} = 1$ .

Also if  $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$  and  $\mathcal{A}, \mathcal{B} > 0$ , then  $\mathcal{A} \cdot \mathcal{B} > 0$ .

#### Theorem 1.3.4

If  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ , then  $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ .

We now know that  $\mathbb{F}^*$  is an ordered field.

## 1.4 Robert Rec

#### Theorem 1.4.1

$\mathbb{Q}$  is the smallest ordered field.

*Proof.* Let  $\mathbb{F}$  be any ordered field. Let  $1 \in \mathbb{F}$ . Let  $\iota : \mathbb{N} \rightarrow \mathbb{F}$ ,  $n \mapsto 1 + \dots + 1$   $n$  times. Then  $\iota(-n) = -\iota(n)$  for  $n \in \mathbb{N}_0$  and  $-n \in \mathbb{Z}^-$ .



Then we say  $\iota(p/q) = \iota(p)\iota(q)^{-1}$  for  $p/q \in \mathbb{Q}$ . ⊕

**Corollary 1.4.1** Every ordered field is infinite

$\iota[\mathbb{Q}] \subseteq \mathbb{F}$  is infinite.

**Roots**

Let  $\mathbb{F}$  be a Dedekind complete ordered field,  $0 < x \in \mathbb{F}$ ,  $n \in \mathbb{N}$ . Then  $\exists! y \in \mathbb{F}$  such that  $y > 0$  and  $y^n = x$ .

*Proof.*  $n = 1$  is silly. Assume  $n \geq 2$ . Let  $E = \{z \in \mathbb{F} \mid z > 0 \text{ and } z^n < x\}$ . Then  $E$  is nonempty and bounded above by  $x$ . Let  $y = \sup E$ . We claim that  $y^n = x$ .

We want to show that  $y^n \not> x$  and  $y^n \not< x$ .

**Lemma 1.4.1**

In any commutative ring  $R$ ,  $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$ .

And hence for  $0 < a < b$  in  $\mathbb{F}$ , we have  $0 < b^n - a^n = (b - a)nb^{n-1}$ .

Suppose  $y^n < x$ , so  $x - y^n > 0$ . We define  $h = \frac{1}{2} \min\left(1, \frac{x - y^n}{n(y+1)^{n-1}}\right)$ .  $0 < h < 1$ , also  $0 < h < \frac{x - y^n}{n(y+1)^{n-1}}$ .

Then, by the inequality below the lemma, we have

$$\begin{aligned} 0 &< (y + h)^n - y^n \\ &< hn(y + h)^{n-1} \\ &< hn(y + 1)^{n-1} \\ &< x - y^n, \end{aligned}$$

so  $(y + h)^n < x$ , which contradicts the definition of  $y$  as the supremum. ⊕

**Definition 1.4.1: Ring\***

A ring is a field where actually we don't care about inverses anymore.

**Definition 1.4.2: Domain**

$R$  is a domain when  $xy = 0 \implies x = 0 \wedge y = 0$ .

Let  $R$  be a ring. For  $(r, s) \in R \times R \setminus \{0\}$ , we say  $(r, s) \sim (r', s')$  if  $rs' = r's$ .

The field of fractions,  $\text{Frac}(R)$  is the set of equivalence classes of  $R \times R \setminus \{0\}$  under  $\sim$  equipped with the operations  $[(r, s)] + [(r', s')] = [(rs' + r's, ss')]$  and  $[(r, s)] \cdot [(r', s')] = [(rr', ss')]$ .

We check that  $[(r, s)] \cdot [(s, r)] = [(rs, sr)] = [(1, 1)]$ .

Let  $\mathbb{F}$  a field,  $\mathbb{F}^x$  its polynomial ring. Let  $\mathbb{F}(x)$  be the field of fractions of  $\mathbb{F}^x$ . Then  $\mathbb{F}(x) := \text{Frac}(\mathbb{F}^x)$  is the field of rational functions in  $x$  with coefficients in  $\mathbb{F}$ .

Given  $p, q \in \mathbb{F}^x$ , say  $p/q > 0$  if  $p$  and  $q$  have the same sign. Say  $f, g \in \mathbb{F}(x)$ , that  $f > g$  when  $f - g > 0$ .

**Theorem 1.4.2**

$\mathbb{F}(x)$  is never Archimedean.

*Proof.*  $x$  is an upper bound for all  $n \in \mathbb{N}$ . ⊕

**Note:**

If  $\mathbb{F}$  is Archimedean,  $|\mathbb{F}| \leq 2^{\aleph_0}$ .

**Theorem 1.4.3**

Let  $\lambda$  be an infinite cardinal. Then there is an ordered field of cardinality  $\lambda$ .

**Corollary 1.4.2**

The Archimedean property is not a first-order property.

## 1.5 Completeness

**Lemma 1.5.1**

Suppose  $\mathbb{F}$  is an ordered field that is not Dedekind complete. Then  $\exists$  an infinite  $E \subseteq \mathbb{F}$  such that:

1.  $E$  bounded above,  $\emptyset \neq U(E)$  is open,  $\emptyset \neq U(E)^C$  is open.
2.  $a \in U(E)^C, b \in U(E) \implies a < b$ .
3.  $f : \mathbb{F} \rightarrow \mathbb{F}$  with  $f(x) = \begin{cases} 1 & x \in U(E) \\ 0 & x \in U(E)^C \end{cases}$  is differentiable with  $f' = 0$ .

**Theorem 1.5.1 Characteristics of Dedekind Completeness**

Let  $\mathbb{F}$  be an ordered field. The following are equivalent:

1.  $\mathbb{F}$  is Dedekind complete.
2.  $\mathbb{F}$  has the intermediate value property: If  $f : [a, b] \rightarrow \mathbb{F}$  is continuous and  $\min(f(a), f(b)) < c < \max(f(a), f(b))$ , then  $\exists x \in [a, b]$  such that  $f(x) = c$ .
3.  $\mathbb{F}$  satisfies the mean value property: If  $f : [a, b] \rightarrow \mathbb{F}$  is continuous and differentiable on  $(a, b)$ , then  $\exists x \in (a, b)$  such that  $f'(x) = \frac{f(b)-f(a)}{b-a}$ .
4.  $\mathbb{F}$  satisfies Cauchy mean value property: If  $f, g : [a, b] \rightarrow \mathbb{F}$  are both continuous and differentiable on  $(a, b)$ , then  $\exists x \in (a, b)$  such that  $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ .
5.  $\mathbb{F}$  satisfies the extreme value property: If  $f : [a, b] \rightarrow \mathbb{F}$  is continuous, then  $f$  attains a maximum and minimum on  $[a, b]$ .

*Proof.*  $1 \implies 2$ : Let  $f : [a, b] \rightarrow \mathbb{F}$  and continuous. WLOG, assume  $f(a) < c < f(b)$ . Define  $E = \{x \in [a, b] \mid f(x) < c\}$ .  $E$  is nonempty and bounded above by  $b$ . Let  $x = \sup E$ . We claim that  $f(x) = c$ . Since  $f$  is continuous,  $\exists \kappa > 0$  such that  $f(t) < c \forall t \in [a, a + \kappa]$  and  $f(t) > c \forall t \in [b - \kappa, b]$ . So,  $a + \frac{\kappa}{2} < x < b - \frac{\kappa}{2}$ .

Suppose BWOC  $f(x) < c$ . Again by continuity,  $\exists \delta > 0$  such that  $f(t) < c$  for all  $t \in B(x, \delta) \subseteq [a, b]$ . Then  $x + \frac{\delta}{2} \in E$ , contradiction.

Then suppose BWOC  $f(x) > c$ . Again,  $\exists \delta > 0$  such that  $f(t) > c$  for all  $t \in B(x, \delta) \subseteq [a, b]$ . Then  $\exists z \in E$  such that  $x - \frac{\delta}{2} < z \leq x$  and  $f(z) < c$ . But then  $c < f(z) < c$ , contradiction.

So  $f(x) = c$  by trichotomy.

$2 \implies 1$ : We'll show  $\neg 1 \implies \neg 2$ . Suppose  $\mathbb{F}$  is not Dedekind complete. Then we can let  $f : \mathbb{F} \rightarrow \mathbb{F}$  be the strange function from the lemma, and we can pick  $a < b$  with  $a \in U(E)^C$  and  $b \in U(E)$ . Then  $f$  is continuous on  $[a, b]$ ,  $f(a) = 0 < 1 = f(b)$ , but there is not  $x \in [a, b]$  with  $f(x) = \frac{1}{2}$ , by construction.

$1 \implies 5$ : First we claim that if  $\mathbb{F}$  is Dedekind and  $f : [a, b] \rightarrow \mathbb{F}$  is continuous, then  $f([a, b]) \subseteq \mathbb{F}$  is a bounded set. We prove the claim.

Consider  $E = \{x \in [a, b] \mid f([a, x]) \text{ is bounded}\}$ .  $a \in E$  and  $E$  is bounded, so we can let  $s = \sup E$ . Next note that by continuity, if  $[c, d] \subseteq [a, b]$  such that  $f([c, d])$  is bounded, then  $\exists \delta > 0$  such that  $f([a, b] \cap [c - \delta, d + \delta])$  is bounded. Using this, deduce in turn that  $a < s$ ,  $s = \max E$ , and  $s = b$ .

So now suppose  $\mathbb{F}$  is Dedekind complete and let  $f : [a, b] \rightarrow \mathbb{F}$  be continuous. The claim establishes that  $f([a, b]) \subseteq \mathbb{F}$  is a bounded set, so we can let  $\begin{cases} \mu = \inf f([a, b]) \\ \lambda = \sup f([a, b]) \end{cases}$ . Suppose BWOC that  $f(x) < \lambda$  for all  $x \in [a, b]$ . Then the function  $g : [a, b] \rightarrow \mathbb{F}$  defined by  $g(x) = \frac{1}{\lambda - f(x)}$  is continuous and positive. So by the claim, there is  $k > 0$  such that  $g(x) \leq k$  for all  $x \in [a, b]$ . But then

$$\frac{1}{\lambda - f(x)} \leq k \implies \frac{1}{k} \leq \lambda - f(x) \implies f(x) \leq \lambda - \frac{1}{k},$$

for all  $x \in [a, b]$ . But this contradicts the definition of  $\lambda$ , as we just found a better upper bound.

Therefore, there does exist  $M \in [a, b]$  such that  $f(M) = \lambda$ , which is  $\max f([a, b])$ .

The min follows from a similar argument.

5  $\implies$  4: Let  $f, g : [a, b] \rightarrow \mathbb{F}$  be continuous and differentiable on  $(a, b)$ . Let  $h : [a, b] \rightarrow \mathbb{F}$  via  $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$ . It suffices to show  $\exists x \in (a, b)$  such that  $h'(x) = 0$ .

By construction,  $h(a) = h(b)$ . If  $h(x) = h(a)$  for all  $x \in [a, b]$ , then  $h' = 0$  and we're done. Suppose then that  $h$  is not constant. Then EVT shows that  $f$  attains its maximal/minimum values, and at least one must occur at the point  $x \in (a, b)$ , therefore  $h'(x) = 0$ .

4  $\implies$  3: Let  $g(x) = x$ . Done.

3  $\implies$  1. We'll show  $\neg 1 \implies \neg 3$ . Suppose  $\mathbb{F}$  is not Dedekind complete. Then we can let  $f : \mathbb{F} \rightarrow \mathbb{F}$  be the function from the lemma, and we can pick  $a < b$  with  $a \in U(E)^C$  and  $b \in U(E)$ . Then consider the restriction  $f : [a, b] \rightarrow \mathbb{F}$ . Then  $1 = 1 - 0 = f(b) - f(a)$ . Then,  $f'(x)(b - a) = 0 \cdot (b - a) = 0$  for all  $x \in \mathbb{F}$ .  $0 \neq 1$  so  $\neg 3$  as desired.  $\odot$

# Chapter 2

## $\mathbb{R}, \mathbb{C}, \bar{\mathbb{R}}$

### Theorem 2.0.1

$\mathbb{R}$  is uncountable.

*Proof.*  $\mathbb{Q} \subseteq \mathbb{R}$ , so  $\mathbb{R}$  is definitely infinite. Suppose BWOC that there was a bijection  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Set  $I_0 = [f(0) + 1, f(0) + 2]$  and not that  $f(0) \notin I_0$ . Suppose we are given closed, nested, non-singleton intervals  $I_n \subseteq I_{n-1} \subseteq \dots \subseteq I_0$  such that  $f(k) \notin I_k$  for  $0 \leq k \leq n$ . If  $f(n+1) \notin I_n$ , then set  $I_{n+1} = I_n$ . Otherwise, set  $I_{n+1}$  to some non-singleton closed interval contained in  $I_n$  such that  $f(n+1) \notin I_{n+1}$ .

Since  $\mathbb{R}$  is Dedekind complete, we have that  $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$ . So, there is an  $x$  such that  $x \in I_n$  for all  $n \in \mathbb{N}$ . But then  $x \neq f(n)$  for all  $n \in \mathbb{N}$ , contradiction since  $f$  is a bijection.  $\odot$

### Note:

Upshot: Most of  $\mathbb{R}$  is transcendental over  $\mathbb{Q}$ .

## 2.1 Extended Reals: $\bar{\mathbb{R}}$

### Definition 2.1.1: Extended Reals

$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ . We endow  $\bar{\mathbb{R}}$  with the following order: We write  $x < y$  for  $x, y \in \bar{\mathbb{R}}$  if:

1.  $x, y \in \mathbb{R}$  and  $x < y$ .
2.  $x = -\infty$  and  $y \in \bar{\mathbb{R}} \setminus \{-\infty\}$ .
3.  $x \in \bar{\mathbb{R}} \setminus \{\infty\}$  and  $y = \infty$ .

Facts:

- $(\bar{\mathbb{R}}, <)$  is an ordered set that satisfies the supremum property.
- All sets in  $\bar{\mathbb{R}}$  are bounded above.
- All sets in  $\bar{\mathbb{R}}$  admit a sup/inf, i.e.
  - $\sup : \mathcal{P}(\bar{\mathbb{R}}) \rightarrow \bar{\mathbb{R}}$ .
  - $\inf : \mathcal{P}(\bar{\mathbb{R}}) \rightarrow \bar{\mathbb{R}}$ .

Note:  $\sup \emptyset = -\infty$  and  $\inf \emptyset = \infty$ . Also,  $A \subseteq B \subseteq \bar{\mathbb{R}}$  implies  $\sup A \leq \sup B$  and  $\inf A \geq \inf B$ . And if  $E \neq \emptyset$ , then  $\inf E \leq \sup E$ .

**Note:**

$\bar{\mathbb{R}}$  isn't an OF because if it were, then it would be Dedekind complete and then there would exist an ordered field isomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x) = \infty$  for some  $x \in \mathbb{R}$ . but then  $f(x+1) = f(x) + f(1) = \infty + 1 = \infty$ , which is not a true statement.

**Definition 2.1.2**

We endow  $\bar{\mathbb{R}}$  with the following “algebra.”

1. If  $x \in \mathbb{R}$ , we set  $x + \infty = \infty + x = \infty$ .
2. If  $x \in \mathbb{R}$ , we set  $x + (-\infty) = (-\infty) + x = -\infty$ .
3.  $\infty + \infty = \infty$ .
4.  $-\infty + (-\infty) = -\infty$ .
5. If  $0 < x \in \bar{\mathbb{R}}$ , we set  $x \cdot \infty = \infty \cdot x = \infty$ .
6. If  $0 < x \in \bar{\mathbb{R}}$ , we set  $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$ .
7. If  $0 > x \in \bar{\mathbb{R}}$ , we set  $x \cdot \infty = \infty \cdot x = -\infty$ .
8. If  $0 > x \in \bar{\mathbb{R}}$ , we set  $x \cdot (-\infty) = (-\infty) \cdot x = \infty$ .
9. If  $x \in \mathbb{R}$ , we set  $\frac{x}{\infty} = \frac{x}{-\infty} = 0$ .
10.  $\infty^{-1} = 0 = (-\infty)^{-1}$ .
11. If  $0 < x \in \bar{\mathbb{R}}$ , we set  $\frac{x}{0} = \infty$ .
12. If  $0 > x \in \bar{\mathbb{R}}$ , we set  $\frac{x}{0} = -\infty$ .

Forbidden/undefined:  $\infty + (-\infty)$ ,  $\infty \cdot 0$ ,  $\frac{0}{0}$ ,  $\frac{\pm\infty}{\pm\infty}$ ,  $\frac{\pm\infty}{\mp\infty}$ .

**2.1.1 Sequences in  $\bar{\mathbb{R}}$** **Definition 2.1.3: Sequence**

A sequence in  $\bar{\mathbb{R}}$  is  $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$  for  $\ell \in \mathbb{Z}$ .

In turn, we define new sequences  $\{a_N\}_{N=\ell}^{\infty}, \{b_N\}_{N=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$ :

- $a_N = \inf\{x_n \mid n \geq N\}$ .
- $b_N = \sup\{x_n \mid n \geq N\}$ .

We then set  $\liminf_{n \rightarrow \infty} x_n = \sup_{N \geq \ell} \inf_{n \geq N} x_n = \sup_{N \geq \ell} a_N$  and  $\limsup_{n \rightarrow \infty} x_n = \inf_{N \geq \ell} \sup_{n \geq N} x_n = \inf_{N \geq \ell} b_N$ .

**Example 2.1.1**

Let  $x_n = \begin{cases} (-1)^n & n \equiv 0 \pmod{2} \\ n & n \equiv 1 \pmod{2} \end{cases}$ . Then,  $\limsup_{n \rightarrow \infty} x_n = \infty$  and  $\liminf_{n \rightarrow \infty} x_n = 1$ .

**Proposition 2.1.1**

Let  $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$ . Then  $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$ .

*Proof.* Let  $M, N \geq \ell$  and  $K = \max(M, N)$ . Then,  $\inf_{n > N} x_n \leq \inf_{n > K} x_n \leq \sup_{n \geq K} x_n \leq \sup_{n \geq M} x_n$ .

Thus,  $\liminf_{n \rightarrow \infty} x_n = \sup_{N \geq \ell} \inf_{n \geq N} x_n \leq \sup_{n \geq M} x_n$  for all  $M \geq \ell$ . So,  $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$ .  $\ominus$

### Proposition 2.1.2

Let  $a_n, b_n \in \bar{\mathbb{R}}$  and suppose  $\exists K \geq \ell$  such that  $a_n \leq b_n$  for all  $n \geq K$ . Then,  $\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n$  and  $\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n$ .

*Proof.* We can claim that if  $k \geq K$ , then

$$\begin{aligned} \inf\{a_n \mid n \geq k\} &\leq \inf\{b_n \mid n \geq k\} \\ \sup\{b_n \mid n \geq k\} &\leq \sup\{a_n \mid n \geq k\}. \end{aligned}$$

Indeed, if  $\exists k \geq K$  such that  $\inf\{a_n \mid n \geq k\} > \inf\{b_n \mid n \geq k\}$ , then  $\exists m \geq k$  such that  $b_m < \inf\{a_n \mid n \geq k\} \leq a_m \leq b_m$ , contradiction. Ditto for sup.

Now define for  $N \geq \ell$ ,  $C_N = \inf_{n \geq N} a_n$ ,  $D_N = \inf_{n \geq N} b_n$ ,  $E_N = \sup_{n \geq N} a_n$ , and  $F_N = \sup_{n \geq N} b_n$ .

The above claims show that  $N \geq K$  then  $C_N \leq D_N$  and  $E_N \leq F_N$ . Then we iterate to learn:

$$\begin{aligned} \liminf_{n \rightarrow \infty} a_n &= \sup_{N \geq \ell} C_N \leq \sup_{N \geq \ell} D_N = \liminf_{n \rightarrow \infty} b_n \\ \limsup_{n \rightarrow \infty} a_n &= \inf_{N \geq \ell} E_N \leq \inf_{N \geq \ell} F_N = \limsup_{n \rightarrow \infty} b_n. \end{aligned}$$

$\ominus$

### Theorem 2.1.1

Suppose  $a_n, b_n \in \bar{\mathbb{R}}$ . The following hold:

1. If  $\limsup_{n \rightarrow \infty} a_n < x \in \bar{\mathbb{R}}$ , then  $\exists N \geq \ell$  such that  $a_n < x$  for all  $n \geq N$ .
2. If  $\liminf_{n \rightarrow \infty} a_n > x \in \bar{\mathbb{R}}$ , then  $\exists N \geq \ell$  such that  $a_n > x$  for all  $n \geq N$ .
3.  $\liminf_{n \rightarrow \infty} a_n = -\limsup_{n \rightarrow \infty} -a_n$ .
4.  $\limsup_{n \rightarrow \infty} a_n = -\liminf_{n \rightarrow \infty} -a_n$ .
5.  $\limsup_{n \rightarrow \infty} a_n + b_n \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$ , provided that all arithmetic operations are well-defined.
6.  $\liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \leq \liminf_{n \rightarrow \infty} a_n + b_n$ , provided that all arithmetic operations are well-defined.

*Proof.* 1. Suppose  $\limsup_{n \rightarrow \infty} a_n = \inf_{N \geq \ell} \sup_{n \geq N} a_n < x$ . This implies that  $\exists N \geq \ell$  such that  $\sup_{n \geq N} a_n < x$ , meaning  $a_n < x$  for all  $n \geq N$ .

2. Similar as above.

3. For any  $\emptyset \neq X \subseteq \mathbb{F}$ , we have that  $-\sup(-X) = \inf X$  and  $-\inf(-X) = \sup X$ . So the result follows.

4. Same as above.

5. We break into cases:

- (a)  $\limsup a_n = \infty$  or  $\limsup b_n = \infty$ . Then  $\limsup a_n + b_n = \infty \geq \limsup a_n + \limsup b_n$ .
- (b) Suppose either  $\limsup a_n = -\infty$  or  $\limsup b_n = -\infty$ . WLOG consider the first option. Since  $\limsup b_n < \infty$ , then there exists  $N_1 \geq \ell$  and  $K \in \mathbb{R}$  such that  $b_n < K$  for  $n \geq N_1$ . Now let  $m \in \mathbb{N}$  and note that  $-\infty < -m - K$ . We can use the first result of the theorem to pick  $N_2 \geq \ell$  such that  $n \geq N_2 \implies a_n < -m - K$ . Then, if  $n \geq \max(N_1, N_2)$ , we have  $a_n + b_n < -m$ , so  $\limsup a_n + b_n = -\infty \leq \limsup a_n + \limsup b_n$ .

- (c)  $\limsup a_n, \limsup b_n \in \mathbb{R}$ . Let  $\epsilon > 0$ , then  $\exists N_1, N_2 \geq \ell$  such that  $n \geq N_1 \implies a_n < \limsup a_n + \frac{\epsilon}{2}$  and  $n \geq N_2 \implies b_n < \limsup b_n + \frac{\epsilon}{2}$ . Then,  $n \geq \max(N_1, N_2) \implies a_n + b_n < \limsup a_n + \limsup b_n + \epsilon$ , so  $\limsup a_n + b_n \leq \limsup a_n + \limsup b_n + \epsilon$  for all  $\epsilon$ .

6. Same as above.

⊕

### Lemma 2.1.1

Let  $x_n \subseteq \mathbb{R}$ . The following are equivalent for  $x \in \mathbb{R}$ :

1.  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .
2.  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$ .

*Proof.* Let  $\epsilon > 0$ . Then  $\exists N \geq \ell$  such that  $n \geq N \implies x - \epsilon < x_n < x + \epsilon$ . Thus,  $x - \epsilon \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq x + \epsilon$  for all  $\epsilon > 0$ . This implies that  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$ .

Now let  $\epsilon > 0$ . Then by the previous theorem, there exists  $N_1, N_2 \geq \ell$  such that  $\begin{cases} x - \epsilon < x_n & n \geq N_1 \\ x_n < x + \epsilon & n \geq N_2 \end{cases}$ .

Thus,  $n \geq \max(N_1, N_2) \implies x - \epsilon < x_n < x + \epsilon$ , so  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

⊕

### Definition 2.1.4

Let  $x_n \in \bar{\mathbb{R}}$  and  $x \in \bar{\mathbb{R}}$ . We say that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  if  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = x$ .

Remarks:

1. The lemma shows this extends the notion of convergence in  $\mathbb{R}$ .
2. Limits are unique, when they exist.

### Example 2.1.2

1.  $\lim_{n \rightarrow \infty} n = \infty$  ( $n \rightarrow \infty$  as  $n \rightarrow \infty$ ).
2. Version of squeeze lemma
3. TFAE:
  - $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ .
  - $\liminf_{n \rightarrow \infty} x_n = \infty$ .
  - $\forall M \in \mathbb{N}$ , there exists  $N \geq \ell$  such that  $n \geq N \implies M \leq x_n$ .

## Chapter 3

# Metric Spaces

### Definition 3.0.1: Metric

Let  $X$  be a nonempty set. A metric on  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  such that:

1.  $d(x, y) \geq 0$  for all  $x, y \in X$ , and  $d(x, y) = 0 \iff x = y$ .
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
3.  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

### Definition 3.0.2

A metric space is  $(X, d)$  for  $X \neq \emptyset$  and  $d$  a metric on  $X$ .

### Example 3.0.1

1.  $\mathbb{R}$  with  $d(x, y) = |x - y|$ .
2.  $\mathbb{C}$  with  $d(x, y) = |x - y|$ .
3. (Discrete Metric) Let  $X \neq \emptyset$  be any set. Then  $d : X \times X \rightarrow \{0, 1\}$  defined by  $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$  is a metric on  $X$ .
4. Let  $V$  be a normed metric space with norm  $\|\cdot\|$ . Then  $d(x, y) = \|x - y\|$  is a metric on  $V$ .
5. Suppose  $(Y, d)$  is a metric space and suppose  $f : X \rightarrow Y$  is an injection where  $X \neq \emptyset$  is a set. Then  $\sigma : X \times X \rightarrow \mathbb{R}$  defined by  $\sigma(x, y) = d(f(x), f(y))$  is a metric on  $X$ .

*Proof.* We need to show that  $\sigma$  satisfies the three properties of a metric.

- (a)  $\sigma(x, y) \geq 0$  because  $d \geq 0$  and  $\sigma(x, y) = 0 \iff d(f(x), f(y)) = 0 \iff f(x) = f(y) \iff x = y$ .
- (b) The other two are very trivial.

☺

6. Let  $Y$  be a metric space and  $\emptyset \neq X \subseteq Y$ . Then  $d : X \times X \rightarrow \mathbb{R}$  defined by  $d(x, y) = d_Y(x, y)$  is a metric on  $X$ .



7. Consider  $f : (0, \infty) \rightarrow \mathbb{R}$  and  $g : (0, \infty) \rightarrow \mathbb{R}$  with  $f(x) = \log x$  and  $g(x) = \frac{1}{x}$ . Then  $d_f(x, y) = \left| \log \frac{x}{y} \right|$  and  $d_g(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x-y|}{|x||y|}$  are metrics on  $(0, \infty)$ .
8. Let  $V, W$  be finite dimensional vector spaces over  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ . Let  $L(V, W) = \{T : V \rightarrow W : T \text{ linear}\}$ . Then define  $\text{rk}(T) = \dim \text{ran } T$  for  $T \in L(V, W)$ . Note that  $\text{ran}(T+S) = \{Tx+Sx \mid x \in \mathbb{F}\} \subseteq \{Tx+Sy \mid x, y \in \mathbb{F}\} = \text{ran } T + \text{ran } S$ . Then,  $\text{rk}(T+S) \leq \text{rk}(T) + \text{rk}(S)$ . Define  $d(T, S) = \text{rk}(T-S) \in \mathbb{N} \subseteq [0, \infty]$ .
- $d(T, S) = 0 \iff \text{rk}(T-S) = 0 \iff T-S = 0$ .
  - Has symmetry.
  - Triangle inequality:  $d(T-S) = \text{rk}(T-R+R-S) \leq \text{rk}(T-R) + \text{rk}(R-S) = d(T, R) + d(R, S)$ .
9. Let  $f : \bar{\mathbb{R}} \rightarrow [-1, 1]$  via  $f(x) = \begin{cases} 1 & x = \infty \\ -1 & x = -\infty \\ \frac{x}{\sqrt{1+x^2}} & x \in \mathbb{R} \end{cases}$ . Then  $d(x, y) = |f(x) - f(y)|$  is a metric on  $\bar{\mathbb{R}}$ .

### Definition 3.0.3

Let  $X$  be a metric space.

1. For  $x \in X$  and  $r \geq 0$ , we define  $B(x, r) = \{y \in X \mid d(x, y) < r\}$ . And  $B[x, r] = \{y \in X \mid d(x, y) \leq r\}$ .
2. A set  $E \subseteq X$  is bounded if  $\exists(R \geq 0)$  such that  $E \subseteq B(x, R)$  for some  $x \in X$ .
3. Let  $Y$  be any set and  $f : Y \rightarrow X$ . We say  $f$  is a bounded function if  $f(Y) \subseteq X$  is bounded. We write  $\mathcal{B}(Y; X) = \{g : Y \rightarrow X \mid g \text{ is bounded}\}$ .

### Example 3.0.2

1.  $f : \mathbb{R} \rightarrow \mathbb{C}$  via  $f(t) = e^{it} \implies f(t) = 1 \implies f(\mathbb{R}) \subseteq B[0, 1]$  is bounded. So,  $f \in \mathcal{B}(\mathbb{R}; \mathbb{C})$ .
2.  $f : (0, \infty) \rightarrow \mathbb{R}$  via  $f(t) = \frac{\log t}{\sqrt{1+(\log t)^2}}$ . So,  $f \in \mathcal{B}((0, \infty); \mathbb{R})$ .
3. Let  $X$  be a metric space and  $Y$  a nonempty set. Consider  $\mathcal{B}(X; Y)$ . If  $f \in \mathcal{B}(X; Y)$ , then  $\exists y \in Y$  and  $R \geq 0$  such that  $d(f(x), y) \leq R$  for all  $x$ . Thus,  $\sup_{x \in X} d(f(x), y) := \sup\{d(f(x), y) \mid x \in X\} \in [0, R]$ . Similarly, if  $f, g \in \mathcal{B}(X; Y)$ , then exists  $R \geq 0$  and  $y_1, y_2 \in Y$  such that  $d(f(x), y_1) \leq R$  and  $d(g(x), y_2) \leq R$  for all  $x \in X$ . Then,  $d(f(x), g(x)) \leq d(f(x), y_1) + d(y_1, y_2) + d(y_2, g(x)) \leq 2R + d(y_1, y_2) < \infty$  for all  $x \in X$ . So,  $\sup_{x \in X} d(f(x), g(x)) < \infty$ . We now define

$$d : \mathcal{B}(X; Y) \times \mathcal{B}(X; Y) \rightarrow [0, \infty)$$

$$(f, g) \mapsto \sup_{x \in X} d(f(x), g(x)).$$

*Proof.* Consider the properties of a metric:

- $d(f, g) = 0 \iff \sup_{x \in X} d(f(x), g(x)) = 0 \iff d(f(x), g(x)) = 0 \iff f(x) = g(x)$  for all  $x \in X \iff f = g$ .
- Symmetry is trivial.
- Let  $f, g, h \in \mathcal{B}(X; Y)$ . Then,  $d(f, h) = \sup_{x \in X} d(f(x), h(x)) \leq \sup_{x \in X} d(f(x), g(x)) + \sup_{x \in X} d(g(x), h(x)) \leq d(f, g) + d(g, h)$ .

☺

**Definition 3.0.4**

Let  $X$  and  $Y$  be metric spaces:

1. A map  $f : X \rightarrow Y$  is an isometric embedding if  $d_Y(f(x), f(y)) = d_X(x, y)$  for all  $x, y \in X$ . Note, such an  $f$  is injective.
2.  $f$  is an isometry if it's an isometric embedding and surjective.
3.  $X$  and  $Y$  are isometric if there exists an isometry  $f : X \rightarrow Y$ .

**Example 3.0.3**

1. Consider  $\mathbb{R}^n$  with  $|\cdot| = \|\cdot\|_2$ , that is, 2-norm.
2. Recall  $O(n) = \{M \in \mathbb{R}^{n \times n} \mid M^T M = I\}$  and  $R \in O(n) \implies |Rx| = |x|$ .  
Let  $a \in \mathbb{R}^n$ ,  $R \in O(n)$ , and set  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  via  $f(x) = a + Rx$ . Then,

$$|f(x) - f(y)| = |a + Rx - (a + Ry)| = |Rx - Ry| = |R(x - y)|.$$

Also,  $y = f(x) = a + Rx \iff y - a = Rx$ . So,  $f$  is an isometry.

3. Consider  $x \mapsto ix \in \mathbb{C}$  for  $x \in \mathbb{R}$ . This is an isometric embedding but obviously not an isometry for it is not surjective.

The next example is so important that we call it a theorem. Recall  $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{R})$  for  $X \neq \emptyset$  is a set. Note that if  $V$  is a normed vector space, then  $\mathcal{B}(X; V)$  is too:  $\|f\|_{\mathcal{B}} = \sup_{x \in X} \|f(x)\|_V$  is a norm (exercise) and  $d_{\mathcal{B}}(f, g) = \|f - g\|_{\mathcal{B}}$ .

**Theorem 3.0.1**

Let  $X$  be a metric space and fix an arbitrary element  $a \in X$ . For  $x \in X$ , we'll define  $\varphi_x : X \rightarrow \mathbb{R}$  via  $\varphi_x(y) = d(x, y) - d(y, a)$ . The following hold:

1.  $\varphi_x \in \mathcal{B}(X)$  for all  $x \in X$ .
2. Define  $\Phi : X \rightarrow \mathcal{B}(X)$  via  $\Phi(x) = \varphi_x$ . Then,  $\Phi$  is an isometric embedding.

*Proof.* First note,  $|\varphi_x(y)| = |d(x, y) - d(y, a)| \leq d(x, a)$  by the triangle inequality. So,  $\|\varphi_x\|_{\mathcal{B}} = \sup_{y \in X} |\varphi_x(y)| \leq d(x, a) < \infty$ . This shows the first result.

Next, fix  $x, z \in X$  and consider  $\varphi_x(y) - \varphi_z(y) = d(x, y) - d(y, a) - d(z, y) + d(y, a)$ . So,

$$|\varphi_x(y) - \varphi_z(y)| = |d(x, y) - d(y, z)| \leq d(x, z).$$

Thus,  $d_{\mathcal{B}}(\varphi_x, \varphi_y) = \|\varphi_x - \varphi_y\|_{\mathcal{B}} = \sup_{y \in X} |\varphi_x(y) - \varphi_y(y)| \leq d(x, z)$ .

On the other hand,  $|\varphi_x(z) - \varphi_z(z)| = |d(x, z) - \overset{0}{d(z, z)}| = d(x, z)$ . So,  $d_{\mathcal{B}}(\varphi_x, \varphi_z) = d(x, z)$ . ⊙

## Chapter 4

# Basic Metric Space Topology

FILL IN LATER

### Proposition 4.0.1

Let  $Y_1, \dots, Y_n$  be metric spaces and consider  $Y = \prod_{i=1}^n Y_i$ , endowed with a  $p$ -metric from Homework 3. That is,

$$d_p(x, y) = \begin{cases} \left( \sum_{i=1}^n d_{Y_i}^p(x_i, y_i) \right)^{1/p} & 1 \leq p < \infty \\ \max_{1 \leq i \leq n} d_{Y_i}^p(x_i, y_i) & p = \infty \end{cases}.$$

Suppose  $\{y_k\}_{k=\ell}^\infty \subseteq Y$  is given by  $y_k = (y_{k,1}, \dots, y_{k,n})$ . The following hold:

1. Let  $y = (y_1, \dots, y_n) \in Y$ . Then  $y_k \rightarrow y$  in  $Y$  as  $k \rightarrow \infty \iff y_{k,i} \rightarrow y_i$  in  $Y_i$  as  $k \rightarrow \infty$  for all  $1 \leq i \leq n$ .
2.  $\{y_k\}_{k=\ell}^\infty$  is Cauchy in  $Y$  if and only if  $\{y_{k,i}\}_{k=\ell}^\infty$  is Cauchy in  $Y_i$  for all  $1 \leq i \leq n$ .

*Proof.* We'll only prove 1. as 2. is very similar. Suppose  $y_k \rightarrow y$  as  $k \rightarrow \infty$ . Note that for  $1 \leq i \leq n$ ,  $d_i(y_{k,i}, y_i) \leq d_Y(y_k, y)$ . Thus, for  $\epsilon > 0$ , we pick  $K \geq \ell$  such that if  $k \geq K$ , then  $d_Y(y_k, y) \leq \epsilon$ . But then  $k \geq K \implies d_i(y_{k,i}, y_i) \leq d_Y(y_k, y) \leq \epsilon$  for all  $1 \leq i \leq n$ , meaning  $y_{k,i} \rightarrow y_i$  as  $k \rightarrow \infty$  for  $1 \leq i \leq n$ .

Now suppose  $y_{k,i} \rightarrow y_i$  as  $k \rightarrow \infty$  for all  $1 \leq i \leq n$ . Let  $\epsilon > 0$  and pick  $K_i \geq \ell$  such that  $k \geq K_i \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$ . Let  $K = \max K_i \geq \ell$ , and note  $k \geq K \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$  for all  $1 \leq i \leq n$ . This means

$$\begin{cases} \left( \sum_{i=1}^n d_i^p(y_{k,i}, y_i) \right)^{1/p} \leq \left( \sum_{i=1}^n \frac{\epsilon^p}{n} \right)^{1/p} = \epsilon & 1 \leq p < \infty \\ \max_i d_i(y_{k,i}, y_i) < \epsilon & p = \infty \end{cases}$$

So,  $y_k \rightarrow y$  as  $k \rightarrow \infty$ . ☺

### Definition 4.0.1

Let  $X \neq \emptyset$  be a set and  $d_1, d_2$  be metrics on  $X$ . We say  $d_1$  and  $d_2$  are equivalent if  $\exists c_1, c_2 > 0$  such that  $c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$  for all  $x, y \in X$ .

The point is that equivalent metrics give the same notions of convergence, Cauchyness, and boundedness.

**Example 4.0.1** (Equivalent Norms)

1. All norms on  $\mathbb{F}^n$  are equivalent.
2. From recitation,  $\|\cdot\|_p$  are all equivalent on  $\mathbb{F}^n$  for  $1 \leq p \leq \infty$ .
3. Let  $Y_1, \dots, Y_n$  be metric spaces and form  $Y = \prod_{i=1}^n Y_i$ . Then

$$d_p(x, y) = \|(d_1(x, y), \dots, d_n(x, y))\|_p \asymp \|(d_1(x, y), \dots, d_n(x, y))\|_q = d_q(x, y)$$

Therefore,  $d_p \asymp d_q$  in  $Y$ .

Note: This does not mean all metrics on  $Y$  are equivalent.

**Example 4.0.2**

Let  $V_1, \dots, V_n, W$  be normed vector spaces over  $\mathbb{F}$ . We define  $\mathcal{L}(V_1, \dots, V_n; W)$  is the set of  $\{T \in L(V_1, \dots, V_n; W) \mid \|T\|_{\mathcal{L}} < \infty\}$  where  $\|T\|_{\mathcal{L}} := \sup\{\|T(v_1, \dots, v_n)\|_W \mid v_i \in V_i : \|v_i\|_{V_i} < 1\} \in [0, \infty]$ . Facts:

1. This is indeed a norm.
2.  $T \in \mathcal{L} \iff \|T(v_1, \dots, v_n)\|_W \leq c \prod_{i=1}^n \|v_i\|_{V_i}$  for all  $v_i \in V_i$  for some  $0 \leq c < \infty$ .  $c = \|T\|_{\mathcal{L}}$  is the best constant.

**Theorem 4.0.1** Algebra of Sequences

Let  $V_1, \dots, V_n, W$  be normed vector spaces over a common field  $\mathbb{F}$ . The following hold:

1. Let  $\{v_{k,i}\}_{k=\ell}^{\infty} \subseteq V_i$  for  $1 \leq i \leq n$  be such that  $v_{k,i} \rightarrow v_i$  in  $V_i$  as  $k \rightarrow \infty$ . Let  $\{T_k\}_{k=\ell}^{\infty} \subseteq \mathcal{L}(V_1, \dots, V_n; W)$  be such that  $T_k \rightarrow T$  as  $k \rightarrow \infty$ . Then  $T_k(v_{k,1}, \dots, v_{k,n}) \rightarrow T(v_1, \dots, v_n)$  in  $W$  as  $k \rightarrow \infty$ .
2. If  $\{u_k\}, \{v_k\} \subseteq V_1$  are such that  $u_k \rightarrow u, v_k \rightarrow v$  then  $u_k + v_k \rightarrow u + v$  as  $k \rightarrow \infty$ .

*Proof.* We'll only do 1 because 2 is easy. We start with  $n = 2$  for simplicity. Suppose  $\{x_k\} \subseteq V_1, \{y_k\} \subseteq V_2$  such that  $x_k \rightarrow x$  and  $y_k \rightarrow y$  as  $k \rightarrow \infty$ . Then let  $\sup_{k \geq \ell} \max\{\|x_k\|_{V_1}, \|y_k\|_{V_2}, \|T_k\|_{\mathcal{L}}\} = M < \infty$ . Then,

$$\begin{aligned} T_k(x_k, y_k) - T(x, y) &= T_k(x_k, y_k - y) + T_k(x_k, y) - T(x, y) \\ &= T_k(x_k, y_k - y) + T_k(x_k - x, y) + T_k(x, y) - T(x, y). \end{aligned}$$

This shows that

$$\begin{aligned} \|T_k(x_k, y_k) - T(x, y)\|_W &\leq \|T_k\|_{\mathcal{L}} \|x_k\|_{V_1} \|y - y_k\|_{V_2} + \|T_k\|_{\mathcal{L}} \|x - x_k\|_{V_1} \|y_k\|_{V_2} + \|T - T_k\|_{\mathcal{L}} \|x_k\|_{V_1} \|y_k\|_{V_2} \\ &\leq M^2 \|y - y_k\|_{V_2} + M^2 \|x - x_k\|_{V_1} + M^2 \|T - T_k\|_{\mathcal{L}} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ . ☺

**Definition 4.0.2**

1. We say a metric space  $X$  is complete if every Cauchy sequence in  $X$  is convergent in  $X$ .
2. We say a normed vector space is Banach if it's complete.
3. We say an inner product space is a Hilbert space if it's Banach.

**Example 4.0.3**

1.  $(\mathbb{R}, |\cdot|)$  is complete.
2.  $X = \prod_{i=1}^n X_i$  with  $p$ -metric is complete if and only if each  $X_i$  is complete. In particular,  $(\mathbb{R}^n, \|\cdot\|)$  is complete.
3.  $\mathbb{F}^n$  is complete with any more.
4.  $\mathbb{R} \setminus \{0\}$  is not complete with  $|\cdot|$  as the metric.
5.  $\mathbb{Q}^n$  with  $|\cdot|$  is not complete.

**Example 4.0.4**

1.  $V$  is a finite dimensional normed vector spaces.  $\varphi : \mathbb{F}^n \rightarrow V$  isomorphism. Then  $\mathbb{F}^n \ni x \mapsto \|\varphi(x)\|_V \in [0, \infty)$  defines a norm on  $\mathbb{F}^n$ , which we call  $\|x\|$ . Then  $(\mathbb{F}^n, \|\cdot\|)$  is isometric to  $(V, \|\cdot\|_V)$ , and hence  $V$  is complete.
2. Let  $\emptyset \neq X$  be a set endowed with the discrete metric. Suppose  $\{x_n\}_{n=\ell}^\infty \subseteq X$  is Cauchy and pick  $N \geq \ell$  such that  $n, m \geq N \implies d(x_n, x_m) < 1$ . Then  $x_n = x_m = x_N$ . So  $x_n \rightarrow x_N$  as  $n \rightarrow \infty$ . Therefore  $X$  is complete.

Note that  $Y = \prod Y_i$  is complete iff each individual  $Y_i$  is complete.

**Theorem 4.0.2**

Let  $V_1, \dots, V_k, W$  be normed vector spaces over  $\mathbb{F}$ . If  $W$  is Banach, then so is  $\mathcal{L}(V_1, \dots, V_k)$ .

*Proof.* Suppose  $\{T_n\}_{n=\ell}^\infty \subseteq \mathcal{L}(V_1, \dots, V_k; W)$  is Cauchy. For fixed  $v_1, \dots, v_k \in \prod_{i=1}^k V_i$ , we bound

$$\|T_n(v_1, \dots, v_k) - T_m(v_1, \dots, v_k)\|_W \leq \|T_n - T_m\|_{\mathcal{L}} \prod_{i=1}^k \|v_i\|_{V_i}.$$

Therefore,  $\{T_n(v_1, \dots, v_k)\}_{n=\ell}^\infty \subseteq W$  is Cauchy and hence convergent. We may thus define  $T : V_1 \times \dots \times V_k \rightarrow W$  via  $T(v_1, \dots, v_k) = \lim_{n \rightarrow \infty} T_n(v_1, \dots, v_k)$ .

1.  $T \in \mathcal{L}(V_1, \dots, V_k; W)$ :

$$T(\alpha x + \beta y, v_2, \dots, v_k) = \alpha T_n(x, v_2, \dots, v_k) + \beta T_n(y, v_2, \dots, v_k)$$

As  $n \rightarrow \infty$ , we get:

$$T(\alpha x + \beta y, v_2, \dots, v_k) = \alpha T(x, v_2, \dots, v_k) + \beta T(y, v_2, \dots, v_k).$$

Repeat in other slots if  $k \geq 2$ . As such, it is multilinear.

2.  $T \in \mathcal{L}(V_1, \dots, V_k; W)$ : Fix  $v_i \in V_i$  with  $\|v_i\|_{V_i} \leq 1$ . Then

$$\begin{aligned} \|T(v_1, \dots, v_k)\|_W &= \lim_{n \rightarrow \infty} \|T_n(v_1, \dots, v_k)\|_W \\ &\leq \left( \limsup_{n \rightarrow \infty} \|T_n\|_{\mathcal{L}} \right) \prod_{i=1}^k \|v_i\|_{V_i} \leq \limsup_{n \rightarrow \infty} \|T_n\|_{\mathcal{L}} < \infty. \end{aligned}$$

3.  $T_n \rightarrow T$  in  $\mathcal{L}$  as  $n \rightarrow \infty$ : Let  $\epsilon > 0$  and pick  $N \geq \ell$  such that  $n, m \geq N \implies \|T_n - T_m\|_{\mathcal{L}} < \frac{\epsilon}{2}$ . Then let  $v_i \in V_i$  with  $\|v_i\|_{V_i} \leq 1$ . Then,

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W = \lim_{m \rightarrow \infty} \|T_m(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leq \lim_{m \rightarrow \infty} \|T_m - T_n\|_{\mathcal{L}} < \frac{\epsilon}{2}.$$

But this implies

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leq \frac{\epsilon}{2}.$$

By taking the supremum, we get that  $\|T - T_n\|_{\mathcal{L}} \leq \frac{\epsilon}{2} < \epsilon$ .

☺

#### Corollary 4.0.1

$V^* = \mathcal{L}(V; \mathbb{F})$  is always Banach.

#### Definition 4.0.3

Let  $X$  be a metric space,  $E \subseteq X$ .

1.  $x \in E$  is an interior point if  $\exists \epsilon > 0$  such that  $B(x, \epsilon) \subseteq E$ .  $E^\circ = \{x \in E \mid x \text{ is an interior point}\}$ .  $E$  is open iff  $E = E^\circ$ .  $E$  is closed iff  $E^c$  is open.
2.  $x \in X$  is a boundary point of  $E$  if  $\forall \epsilon > 0$ ,  $B(x, \epsilon) \cap E \neq \emptyset$  and  $B(x, \epsilon) \cap E^c \neq \emptyset$ . We write  $\partial E = \{x \in X \mid x \text{ is a boundary point of } E\}$ .  $\bar{E} = E^\circ \cup \partial E$ .
3. We say  $x \in X$  is a limit point (accumulation point) of  $E$  if  $\forall \epsilon > 0$   $(B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$ . We write  $E' = \{x \in X \mid x \text{ is a limit point of } E\}$ . If  $x \in E \setminus E'$ , then  $x$  is an isolated point.

#### Example 4.0.5

Let  $(X, \text{disc})$  be given. Claim: all subsets of  $X$  are both open and closed.

*Proof.*  $B(x, 1) = \{x\} \implies E \subseteq X$  can be written as

$$E = \cup_{x \in E} B(x, 1),$$

which is open. Therefore  $E = (E^c)^c$  is also closed.

☺

Any metric space in which all sets are open and closed is called a discrete space.

#### Theorem 4.0.3

Let  $X$  be a metric space and  $C \subseteq X$ . The following are equivalent:

1.  $C$  is closed.
2.  $C$  is sequentially closed; If  $\{x_n\}_{n=\ell}^\infty \subseteq C$  is such that  $x_n \rightarrow x$  in  $X$  as  $n \rightarrow \infty$ , then  $x \in C$ .

*Proof.* 1  $\rightarrow$  2. Let  $\{x_n\} \subseteq C$  be such that  $x_n \rightarrow x \in X$ . Suppose BWOC that  $x \in C^c$ , which is open. Then  $\exists N \geq \ell$  such that  $n \geq N \implies x_n \in C^c \cup C$ , which is a contradiction.

2  $\rightarrow$  1. BWOC, suppose that  $C$  is not closed, which means  $C^c$  is not open. Then  $\exists x \in C^c$  such that we can pick  $\{x_n\}_{n=0}^\infty \subseteq C$  such that  $x_n \in B(x, 2^{-n}) \cap C$ . This means that  $\{x_n\}_{n=0}^\infty \subseteq C$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . But  $x \notin C$ , so we have a contradiction. ☺

### Corollary 4.0.2

Let  $X$  be a complete metric space, and  $\emptyset \neq C \subseteq X$ . Then  $C$  is closed in  $X$  iff  $C$  is a complete metric space with the metric from  $X$ .

*Proof.*  $\implies$  : Let  $\{x_n\}_{n=\ell}^\infty \subseteq C$  be Cauchy. Then  $x_n \rightarrow x \in X$  as  $n \rightarrow \infty$  because  $X$  is complete. By since  $C$  is closed,  $x \in C$ .

$\impliedby$  : Let  $\{x_n\} \subseteq C$  be such that  $x_n \rightarrow x$  in  $X$  as  $n \rightarrow \infty$ . Then  $\{x_n\}$  is cauchy in  $C$ , meaning it's convergent in  $C$ , so  $x \in C$ , so  $C$  is sequentially closed.  $\odot$