Question: 34

Find all the subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Use this information to show that $\mathbb{Z}_3 \times \mathbb{Z}_3 \ncong \mathbb{Z}_9$.

Solution: The subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are:

- $\mathbb{Z}_3 \times \mathbb{Z}_3$
- $\{(0,0)\}$
- $\{(0,0),(1,0),(2,0)\}$
- $\{(0,0),(0,1),(0,2)\}$
- $\{(0,0),(1,1),(2,2)\}$
- $\{(0,0),(1,2),(2,1)\}$

  The subgroups of $\mathbb{Z}_9$ are:

- $\mathbb{Z}_9$
- $\{0\}$
- $\{0,3,6\}$

Since these groups have different sets of subgroups, they are not isomorphic. In other words, $\mathbb{Z}_3 \times \mathbb{Z}_3 \ncong \mathbb{Z}_9$.

Question: 35

Find all the subgroups of the symmetry group of an equilateral triangle

Solution: There are 4 things we have to check for: closed under the operation, identity, inverse, and associativity:

Closed: If we have $a + b\sqrt{2} \in G$ and $c + d\sqrt{2} \in G$, then their product is $(ac + 2bd) + (ad + bc)\sqrt{2}$. We know that this is in G because if $a, b, c, d \in \mathbb{Q}$, then we know that $(ac+2bd), (ad+bc) \in \mathbb{Q}$ too because of closure of rationals. Therefore G is closed under the operation of multiplication.

Identity The identity for G is 1. We know that 1 is in G because $0, 1 \in \mathbb{Q}$ and if $a, b \in \mathbb{Q}$, then $a + b\sqrt{2}$ is in G.

Question: 41

Prove that

$$G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \wedge a, b \neq 0\}$$

is a subgroup of $\mathbb{R}^*$ under the group operation of multiplication.

Solution:

Question: 45

Prove that the intersection of two subgroups of a group G is also a subgroup of G.

Solution:

Question: 46

Prove or disprove: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

Solution:

Question: 47

Prove or disprove: If H and K are subgroups of a group G, then $HK = \{hk : h \in H \wedge k \in K\}$ is a subgroup of G. What is G is abelian?

Solution:

Question: 48

Let G be a group and $g \in G$. Show that

$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$$

is a subgroup of G. This subgroup is called the **center** of G.

Solution:

Question: 53

Let H be a subgroup of G and

$$C(H) = \{g \in G : gh = hg \text{ for all } h \in H\}.$$

Prove that $C(H)$ is a subgroup of G. This group is called the **centralizer** of H in G.

Solution:

Question: 54

Let H be a subgroup of G. If $g \in G$, show that $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is also a subgroup of G.

Solution: