Question: 2

Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

a. $\phi : \mathbb{R}^* \longrightarrow GL_2(\mathbb{R})$ defined by
$$\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

b. $\phi : \mathbb{R} \longrightarrow GL_2(\mathbb{R})$ defined by
$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

c. $\phi : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by
$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d.$$

d. $\phi : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^*$ defined by
$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$

e. $\phi : \mathbb{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by
$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b.$$

Solution:

a. $\phi(a)\phi(b) = \phi(ab)$. $Ker(\phi) = \phi^{-1}(I_2) = 1$

b. $\phi(a)\phi(b) = \phi(a+b)$. $Ker(\phi) = \phi^{-1}(I_2) = 0$

c. Trace is known to not be a group homomorphism.

d. $\phi(a)\phi(b) = \phi(ab)$. $Ker(\phi) = SL_2(\mathbb{R})$.

e. $\phi(a) + \phi(b) = \phi(a+b)$. $Ker(\phi) = \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R}\right\}$.

Question: 9

If $\phi : G \longrightarrow H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is also abelian.

Solution: If $x, y \in \phi(G)$, then there exist $g, h \in G$ with $x = \phi(g)$ and $y = \phi(h)$. So, $xy = \phi(g)\phi(h) = \phi(gh) = \phi(hg) = \phi(h)\phi(g) = yx$, so $\phi(G)$ is abelian. 😊

Question: 11

Show that a homomorphism defined on a cyclic group is completely determined by its action on the generator of the group.

Solution: Let $G = \langle g \rangle$ for some $g \in G$, a cyclic group. Now, let ϕ be a homomorphism on G. $\phi(\langle g \rangle)$ is a cyclic group and we need to show that $\phi(\langle g \rangle) = \langle \phi(g) \rangle$. If $|G| = n$, then we have the following:

$$\begin{aligned}
\phi(\langle g \rangle) &= \phi\{e, g, \ldots, g^{n-1}\} \\
&= \{\phi(e), \phi(g), \ldots, \phi(g^{n-1})\} \\
&= \{\phi(e), \phi(g), \ldots, \phi(g)^{n-1}\} \\
&= \langle \phi(g) \rangle
\end{aligned}$$

Therfore, our claim is true.

Question: 14

Let G be a finite group and N a normal subgroup of G. If H is a subgroup of G/N, prove that $\phi^{-1}(H)$ is a subgroup of G of order $|H| \cdot |N|$, where $\phi : G \longrightarrow G/N$ is the canonical homomorphism.

Solution: Let $a, b \in \phi^{-1}(H)$, then $\phi(a) = aN$ is an element of H and $a^{-1}N$ is also an element of H. Then we also have that $\phi(a)\phi(b) = aNbN = abN = \phi(ab)$ since N is normal. Hence, $\phi(ab)$ is an element of $H \Longrightarrow ab \in \phi^{-1}(H)$, so $\phi^{-1}(H)$ is a subgroup.

To determine the order of this group, we analyze the cosets. For each $h \in H$, we can have hN. Since there are $|N|$ elements in each coset, $|H|$ cosets, and cosets are disjoint, we have that $|H| \cdot |N|$ elements exist in $\phi^{-1}(H)$. 😄

Question: P1

Let T be the circle group as defined in the text: the set of all complex numbers of modulus one, with operation being multiplication. Let $\psi : T \longrightarrow T$ be the map defined by $\psi(z) = z^2$.

a. Prove that ψ is a homomorphism of T.

b. Determine the kernel K of ψ.

c. Show that $T/K \cong T$.

Solution:

a. This is a homomorphism because $\psi(xy) = \psi(x)\psi(y)$, a commonly known fact about the magnitudes of complex numbers.

b. The kernel is the set of all complex numbers z such that $z^2 = 1$. This is the set of all complex numbers z such that $z = \pm 1$.

c. Given any $z_1 = e^{i\theta}$, there exists a $z_2 = e^{i\theta/2}$ such that $\psi(z_2) = z_1$. Therefore, ψ is a surjective homomorphism and the claim is trivial by the First Isomorphism Theorem.

Question: P2

Determine which, if either, of the diagrams below are correct commutative diagrams. Explain. The map $\imath : G \to G$ represents the identity map where $\imath(x) = x$ and π_1 and π_2 are projection maps.

$$\begin{tikzcd}
& G \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G \arrow[dr, "g \to (g,e)"'] & & H \arrow[dl, "h \to (e,h)"] \\
& G \times H &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[dr, "g \to (g,e)"] \arrow[dd, "\iota"'] & \\
& G \times H \arrow[dl, "\pi_1"] \\
G &
\end{tikzcd}$$

Solution: A diagram is commutative if all paths lead to the same place. So, for the first diagram, if we start at (g, h), we see that the first path will yield us (g, e) while the second path yields (e, h). Therefore, this diagram is not commutative.

For the second diagram, we see that the path from $G \to G$ for any $g \in G$ will yield g. Additionally, the path from $G \to G \times H \to G$ for any $g \in G$ will also yield g. Therefore, this diagram is commutative.