Question: 1

For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

a. $G = S_4$ and $H = A_4$

b. $G = A_5$ and $H = \{(1), (1\,2\,3), (1\,3\,2)\}$

c. $G = S_4$ and $H = D_4$

d. $G = Q_8$ and $H = \{1, -1, i, -i\}$

e. $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution:

a. $|G/H| = 4/12$ which is not an integer. Therefore, H can't be normal in G.

b. Counterexample: $(1\ 5)(2\ 3)(1\ 2\ 3) = (1\ 3\ 5) \neq (1\ 2\ 3)(1\ 5)(2\ 3) = (1\ 5\ 2)$.

c. $|G/H| = 4/8$ which is not an integer. Therefore, H can't be normal in G.

d. Since $|G|/|H| = 2$, we know that $|G : H| = 2$. So if we take any $g \in G$, then if $g \in H$, we have that $gH = H = Hg$. If $g \notin H$, then since there are two left cosets of H in G and g isn't in H, the two cosets should be H and gH. Left cosets are disjoint, so we can determine that $gH = G - H$. Right cosets are also disjoint, though, so $Hg = G - H = gH$, so $gH = Hg$ for all $g \in G$, so H is normal in G because its index is 2.

Since we know there is only one subgroup of order 2, we know that this is isomorphic to $\mathbb{Z}_2$, which has the following Cayley table.

+	0	1
0	0	1
1	1	0

e. H is normal in G because $G = \mathbb{Z}$ is abelian and all subgroups of abelian groups are normal.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Question: 4

Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is, in the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$.

a. Show that U is a subgroup of T.

b. Prove that U is abelian.

c. Prove that U is normal in T.

d. Show that T/U is abelian.

e. Is T normal in $GL_2(\mathbb{R})$?

Solution:

a. To prove that U is a subgroup of T, we have to prove the following criteria: associativity, identity, inverse, and closure.

   (a) Associativity: Let $a, b, c \in \mathbb{R}$. Then $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. Then $A(BC) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = (AB)C$. Therefore, U is associative.

   (b) Identity: Consider $u \in U$ where $x = 0$. Then $u = I_2$ which is the identity for all 2×2 matrices.

   (c) Inverse: The inverse of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. This can be checked by multiplying the two matrices together and getting I_2.

   (d) Closure: Let $u, v \in U$. Then $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Then $uv = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \in U$. Therefore, U is closed.

   Therefore, U is a subgroup of T.

b. U is abelian because if we have $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, then $uv = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix} = vu$. 😄

c. U is normal in T because T is abelian and all subgroups of abelian groups are normal.

d. Consider the following:
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}.$$
From this, we know that every coset in T/U has a representative diagonal matrix. We know that diagonal matrices commute, so we know that T/U is abelian.

e. Counterexample:
$$\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}\begin{pmatrix} n & n \\ 0 & n \end{pmatrix}\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} n^3 & 0 \\ n^3 & n^3 \end{pmatrix}.$$

Question: 5

Show that the intersection of two normal subgroups is a normal subgroup.

Solution: Let H and K be two normal subgroups of G. Then, for $h \in H$ and $k \in K$ and $g \in G$, $ghg^{-1} \in H$ and $gkg^{-1} \in K$. Now, let $T = H \cap K$.

$$\begin{aligned} \text{Let } t \in T \Rightarrow\ & t \in H \text{ and } t \in K \\ \Rightarrow\ & gtg^{-1} \in H \text{ and } gtg^{-1} \in K && \Rightarrow gtg^{-1} \in H \cap K \\ \Rightarrow\ & gtg^{-1} \in T \end{aligned}$$

$\therefore$ for all $g \in G$, $t \in T$, $gtg^{-1} \in T$. Therefore, T is a normal subgroup of G.

Question: 11

If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Solution: For $g \in G$, consider the conjugate subgroup $gHg^{-1} \leqslant G$. We also know that the order of $gHg{-}1$ is the same as the order of H, which we called k. But, since H is the only subgroup of order k, any subgroup that has order k must be H. Therefore, $gHg^{-1} = H$. Therefore, H is normal in G. 😄

Question: 13

Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

a. Calculate the center of S_3.

b. Calculate the center of $GL_2(\mathbb{R})$.

c. Show that the center of any group G is a normal subgroup of G.

d. If $G/Z(G)$ is cyclic, show that G is abelian.

Solution:

a. $Z(S_3) = \{(e)\}$

b. If we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, we can multiply out AB and BA to see the following equality:

$$ae + bg = ae + cf \implies bg = cf$$

However, this equality needs to hold true for all choices of g, f because our B was arbitrary and not related to A. This means that $b = c = 0$. This means that the equation

$$af + bh = be + df$$

reduces to $af = df$ or $a = d$. This means that we can say

$$Z(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \backslash \{0\} \right\}.$$

c. By definition, for any $z \in Z(G)$, the following equation will hold true, $zG = Gz$. By the definition of a normal subgroup, $Z(G)$ is normal in G.

d. By definition $G/Z(G) = \langle xZ(G) \rangle$ for some $xZ(G) \in G/Z(G)$ where x is the representative for the coset $xZ(G)$.

If we let $a \in G$, then we know that $aZ(G) = (xZ(G))^m$ for some m. We can also rewrite $(xZ(G))^m = x^m Z(G)$.

If we take another $b \in G$, then we know that $bZ(G) = (xZ(G))^n$ for some n. We can then rewrite $(xZ(G))^n = x^n Z(G)$.

From these two equations, we gets that $ax^{-m}, bx^{-n} \in Z(G)$. For shorthand purposes, we can say $p = ax^{-m}$ and $q = bx^{-n}$. Then, we get that $a = px^m$ and $b = qx^n$. Multiplying both gives us $ab = (px^m)(qx^n) = pqx^{m+n}$. The last step was done beacuse we know that $Z(G)$ is abelian.

If we multiply the other way, we know see that $ba = (qx^n)(px^m) = pqx^{m+n}$. This means that $ab = ba$. Therefore, $G/Z(G)$ is abelian. ☺

Question: 14

Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$; that is, G' is the subgroup of all finite products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the **commutator subgroup** of G.

a. Show that G' is a normal subgroup of G.

b. Let N be a normal subgroup of G. Prove that G/N is abelian iff N contains the commutator subgroup of G.

Solution:

a. Let $s = aba^{-1}b^{-1}$ be the generator of G'. We can say that for any $g \in G$, $gsg^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$. By this structure, we can see that $gsg^{-1} \in G'$. Effectively, conjugating by g is a homomoprhism G' is normal in G.

b. If $a, b \in G$ and we assume G/N is abelian, then we have $(aN)(bN) = (bN)(aN) \Leftrightarrow Nab = Nba \Leftrightarrow Naba^{-1}b^{-1} = N \Leftrightarrow aba^{-1}b^{-1} \in N$.

   Now, if we assume that $aba^{-1}b^{-1} \in N$, this is the same as $ab(ba)^{-1} \in N$. This means that $Nab = Nba$, or as we showed before, $(aN)(bN) = (bN)(aN)$. Therefore, G/N is abelian. 😄.