21603 Model Theory I

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Chapter 1

1.1 random info

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1. Set Theory
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1973 book by Chang and Keisler - Model Theory - Highly recommended for elementary model theory.
What is model thoery? Model Theory = logic + universal algebra
1984 - W. Hodges - Shorter Model Theory
model theory = algebraic geometry - field theory
Algebraic structures:

1. groups
2. rings
3. vector spaces
4. fields
5. graphs - (V, E)
6. ordered structures

Around 1870, mathematicians started to layout the foundations for mathematics. One of the ideas was axiomatization. One example was Euclidean axioms for plane geometry.

1.2 Structures and Languages

Definition 1.2.1: Language

L is a language if $L = F \cup R \cup C$ are parameter disjoint.

Definition 1.2.2: L-structure

Let L be a language (similarity type/signature). Then $\mathcal{M}$ is an L-structure provided:

$$\mathcal{M} = (U, \{f^{\mathcal{M}} \mid f \in F\}, \{r^{\mathcal{M}} \mid r \in R\}, \{c^{\mathcal{M}} \mid c \in C\})$$

where U is a nonempty set. U is also called the universe of $\mathcal{M}$.
For any $f \in F$ there is $U(f)$ natural number such that $f^{\mathcal{M}} : U^{n(F)} \to U$, $R^{\mathcal{M}} \subseteq U^{n(R)}$, $C^{\mathcal{M}} \subseteq U$, $\forall c \in C$.

Notation: $|\mathcal{M}| = U$. The cardinal of $\mathcal{M}$ is $|U|$. $\|\mathcal{M}\|$ denotes the cardinality of $\mathcal{M}$.

Definition 1.2.3: Theory

Let L be a language. A theory T is a set of sentences in L. A sentence is a finite set of symbols from L.

Example 1.2.1 (Sentences)

$L_{\text{gr}} = \{e, \cdot\}$. $e \in C$, $\cdot \in F$. $T_{\text{gr}} = \{\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \forall x (x \cdot e = x, e \cdot x = x), \forall x \exists y (x \cdot y = e, y \cdot x = e)\}$. These are the group axioms (associativity, identity, existence of inverse).

Definition 1.2.4: Term

Let L be a language. A term is:

1. c is a term for any $c \in C$.
2. x when x is a variable.
3. $\tau_1, \ldots, \tau_k$ terms, $f \in F$, $n(f) = k$, then $f(\tau_1, \ldots, \tau_k)$ is a term.

Definition 1.2.5: Term

Term(L) is a minimal set of finite strings of symbols from $L \cup \{(,)\} \cup X$ that contains $C \cup x$ and closed under the following rule:

$$\tau_1, \ldots, \tau_k \in \text{Term}(L), f\, k - \text{place function symbol, then} f(\tau_1, \ldots, \tau_k) \in \text{Term}(L)$$

Example 1.2.2 (L_r)

$L_r = \{0, 1, +, -\}$. Term$(L_r) \supseteq \{\sum a_j x_1^{n_j} \mid a_j \in \mathbb{Z}, n_j \in \mathbb{N}\}$.

Example 1.2.3 (L_{gr})

Term$(L_{\text{gr}}) \supseteq \{x_1 \cdot x_n \cdots x_n \mid x_i \in X, n \in \omega\}$.

Definition 1.2.6: AFml

Let L be a language. The set of atomic fomrulas denotes by AFml(L) is the smallest set of formulas in L that contains $L \cup \{(,), =\} \cup X$ such that:

1. If $\tau_1, \tau_2 \in \text{Term}(L)$, then $\tau_1 = \tau_2 \in \text{AFml}(L)$.
2. Given $R(x_1, \ldots, x_n)$ relation symbol and $\tau_1, \ldots, \tau_n \in \text{Term}(L)$, then $R(\tau_1, \ldots, \tau_n) \in \text{AFml}(L)$.

Definition 1.2.7: Fml

Fml(L) is the set of (first order) formulas in L. Which is the mininmal set of finite strings of symbols from $L \cup \{(,), =, \neg, \vee, \wedge, \implies, \impliedby, \iff, \forall, \exists\} \cup X$ such that:

1. $\mathrm{Fml}(L) \supseteq \mathrm{AFml}(L)$.
2. If φ is a formula, then $\neg\varphi$ is a formula.
3. If $x \in \{\wedge, \vee, \implies, \iff\}$ and $\varphi, \psi \in \mathrm{Fml}(L)$, then $(\varphi x \psi) \in \mathrm{Fml}(L)$.
4. If $\varphi \in \mathrm{Fml}(L)$, $Q \in \{\forall, \exists\}$, and $x \in X$, then $Qx\varphi \in \mathrm{Fml}(L)$.
5. If $\varphi \in \mathrm{Fml}(L)$, $\mathrm{FV}(\varphi)$ is the set of free variables in φ defined by induction on the structure of φ.
   Case 1: $\varphi \in \mathrm{AFml}(L)$.
   (a) φ is $\tau_1 = \tau_2$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \mathrm{FV}(\tau_2)$.
   (b) φ is $R(\tau_1, \ldots, \tau_n)$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \ldots \cup \mathrm{FV}(\tau_n)$.

   Case 2:
   (a) if φ is $\neg\psi$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$.
   (b) if $\varphi = \psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi_1) \cup \mathrm{FV}(\psi_2)$.

   Case 3: φ is $Qx\psi$, $Q \in \{\forall, \exists\}$. Then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \setminus \{x\}$.
6. Sent(L) are the sentences in L. $\mathrm{Sent}(L) = \{\varphi \in \mathrm{Fml}(L) \mid \mathrm{FV}(\varphi) = \emptyset\}$.

Example 1.2.4

If $L_f = \{+, \cdot, 0, 1\}$, then $T_f = \{$

- $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$,
- $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$,
- $\forall x \forall y (x + y = y + x)$,
- $\forall x \forall y (x \cdot y = y \cdot x)$,
- $\forall x (x \cdot 1 = x, 1 \cdot x = x)$,
- $\forall x (x + 0 = x, 0 + x = x)$,
- $\forall x \exists y (x \cdot y = 1, y \cdot x = 1)$,
- $\forall x \exists y (x + y = 0, y + x = 0)$,
- $\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

$\}$.

Definition 1.2.8: L-theory

T is an L-theory if $T \subseteq \mathrm{Sent}(L)$.

The example above is “field theory”.

Definition 1.2.9

Let M be an L-structure. $\tau(\bar{x})$ is a term, $\bar{a} \in |M|^{\ell(n)}$. T
Case 1: $\tau(\bar{x}) = c$ for some constant symbol. Then $\tau^M(\bar{a}) = c^M$.
Case 2: $\tau(\bar{x}) = x_i$. Then $\tau^M(\bar{a}) = a_i$.
Case 3: $\tau(\bar{x}) = f(\tau_1, \ldots, \tau_k)$. Then $\tau^M(\bar{a}) = f^M(\tau_1^M(\bar{a}), \ldots, \tau_k^M(\bar{a}))$.

Definition 1.2.10: $\models$

Let L be a language, $\varphi \in \mathrm{Fml}(L)$, M and L-structure, $n = \ell(\bar{x})$, $\bar{a} \in |M|^n$. Define $M \models \varphi(\bar{a})$ at $\bar{a}$ by induction on the structure of φ:

- If φ is atomic,
  - when $\varphi(x)$ is $\tau_1 = \tau_2$, then $M \models \varphi(\bar{a})$ iff $\tau_1(\bar{a}) = \tau_2(\bar{a})$.
  - when $\varphi(x)$ is $R(\tau_1, \ldots, \tau_k)$, then $M \models \varphi(\bar{a})$ iff $(\tau_1(\bar{a}), \ldots, \tau_k(\bar{a})) \in R^M$.
- If φ is not atomic, then:
  - if φ is $\neg\psi$, then $M \models \varphi(\bar{a})$ iff $M \models \psi(\bar{a})$ is false.
  - if φ is $\psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $M \models \varphi(\bar{a})$ iff $M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a})$.
  - if φ is $\exists y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff there is $b \in |M|$ such that $M \models \psi(b, \bar{a})$.
  - if φ is $\forall y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff for all $b \in |M|$, $M \models \psi(b, \bar{a})$.

Definition 1.2.11

Let M be an L-structure and T an L-theory. $M \models T$ iff for every $\varphi \in T$, $M \models \varphi$. We say T "satisfies" M.

Example 1.2.5 (Models)
$M \models T_f \iff (|M|, +^M, \cdot^M, 0^M, 1^M)$ is a field.

Definition 1.2.12: Mod

$\mathrm{Mod}(T) = \{M\ L\text{-structure} \mid M \models T\}$.

Example 1.2.6
$\mathrm{Mod}(T_f)$ is the class of all fields and $\mathrm{Mod}(T_{\mathrm{gr}})$ is the class of all groups.

Definition 1.2.13: Structure Isomorphism

Let M, N both be L-structures. f is an isomorphism from M onto N if $f : |M| \to |N|$ is a bijection such that:

- $f(c^M) = c^N$ for all $c \in C$.
- $G(x_1, \ldots, x_k)$ function symbol. $a_1, \ldots, a_k \in |M|$, then $f(G^M(a_1, \ldots, a_k)) = G^N(f(a_1), \ldots, f(a_k))$.
- $R(x_1, \ldots, x_k)$ predicate symbol. $a_1, \ldots, a_k \in |M|$, then $(a_1, \ldots, a_k) \in R^M$ iff $(f(a_1), \ldots, f(a_k)) \in R^N$.

We write $f : M \cong N$. Also $M \cong N \iff \exists f : M \cong N$.

Definition 1.2.14

Let $\lambda \geqslant \aleph_0$, T an L-theory. T is λ-categorical if for all $M, N \models T$ of cardinality λ, $M \cong N$.

Theorem 1.2.1 Los Conjecture (1954)

Let L be a language, T a first order L-theory, in an at most countable language. If $\exists \lambda > \aleph_0$ such that T is λ-categorical, then for all $\mu > \aleph_0$, T is μ-categorical.

Somewhere around 1961-1965, Morley proved this conjecture.