21603 Model Theory I

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Contents

Chapter 1

1.1   random info

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1. Set Theory

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1973 book by Chang and Keisler - Model Theory - Highly recommended for elementary model theory.
What is model thoery? Model Theory = logic + universal algebra
1984 - W. Hodges - Shorter Model Theory
model theory = algebraic geometry - field theory
Algebraic structures:

1. groups

2. rings

3. vector spaces

4. fields

5. graphs - (V, E)

6. ordered structures

Around 1870, mathematicians started to layout the foundations for mathematics. One of the ideas was axiomatization. One example was Euclidean axioms for plane geometry.

1.2   Structures and Languages

> **Definition 1.2.1: Language**
>
> L is a language if $L = F \cup R \cup C$ are parameter disjoint.

> **Definition 1.2.2: L-structure**
>
> Let L be a language (similarity type/signature). Then $\mathcal{M}$ is an L-structure provided:
>
> $$\mathcal{M} = (U, \{f^{\mathcal{M}} \mid f \in F\}, \{r^{\mathcal{M}} \mid r \in R\}, \{c^{\mathcal{M}} \mid c \in C\})$$
>
> where U is a nonempty set. U is also called the universe of $\mathcal{M}$.
> For any $f \in F$ there is $U(f)$ natural number such that $f^{\mathcal{M}} : U^{n(F)} \to U$, $R^{\mathcal{M}} \subseteq U^{n(R)}$, $C^{\mathcal{M}} \subseteq U$, $\forall c \in C$.

Notation: $|\mathcal{M}| = U$. The cardinal of $\mathcal{M}$ is $|U|$. $\|\mathcal{M}\|$ denotes the cardinality of $\mathcal{M}$.

Definition 1.2.3: Theory

Let L be a language. A theory T is a set of sentences in L. A sentence is a finite set of symbols from L.

Example 1.2.1 (Sentences)

$L_{\text{gr}} = \{e, \cdot\}$. $e \in C$, $\cdot \in F$. $T_{\text{gr}} = \{\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \forall x (x \cdot e = x, e \cdot x = x), \forall x \exists y (x \cdot y = e, y \cdot x = e)\}$. These are the group axioms (associativity, identity, existence of inverse).

Definition 1.2.4: Term

Let L be a language. A term is:

1. c is a term for any $c \in C$.

2. x when x is a variable.

3. $\tau_1, \ldots, \tau_k$ terms, $f \in F$, $n(f) = k$, then $f(\tau_1, \ldots, \tau_k)$ is a term.

Definition 1.2.5: Term

$\text{Term}(L)$ is a minimal set of finite strings of symbols from $L \cup \{(,)\} \cup X$ that contains $C \cup x$ and closed under the following rule:

$$\tau_1, \ldots, \tau_k \in \text{Term}(L), f\,k-\text{place function symbol, then} f(\tau_1, \ldots, \tau_k) \in \text{Term}(L)$$

Example 1.2.2 (L_r)

$L_r = \{0, 1, +, -\}$. $\text{Term}(L_r) \supseteq \{\sum a_j x_1^{n_j} \mid a_j \in \mathbb{Z}, n_j \in \mathbb{N}\}$.

Example 1.2.3 (L_{gr})

$\text{Term}(L_{\text{gr}}) \supseteq \{x_1 \cdot x_n \cdots x_n \mid x_i \in X, n \in \omega\}$.

Definition 1.2.6: AFml

Let L be a language. The set of atomic fomrulas denotes by $\text{AFml}(L)$ is the smallest set of formulas in L that contains $L \cup \{(,), =\} \cup X$ such that:

1. If $\tau_1, \tau_2 \in \text{Term}(L)$, then $\tau_1 = \tau_2 \in \text{AFml}(L)$.

2. Given $R(x_1, \ldots, x_n)$ relation symbol and $\tau_1, \ldots, \tau_n \in \text{Term}(L)$, then $R(\tau_1, \ldots, \tau_n) \in \text{AFml}(L)$.

Definition 1.2.7: Fml

Fml(L) is the set of (first order) formulas in L. Which is the mininmal set of finite strings of symbols from $L \cup \{(,), =, \neg, \vee, \wedge, \implies, \impliedby, \iff, \forall, \exists\} \cup X$ such that:

1. Fml(L) $\supseteq$ AFml(L).

2. If φ is a formula, then $\neg\varphi$ is a formula.

3. If $x \in \{\wedge, \vee, \implies, \iff\}$ and $\varphi, \psi \in$ Fml(L), then $(\varphi x \psi) \in$ Fml(L).

4. If $\varphi \in$ Fml(L), $Q \in \{\forall, \exists\}$, and $x \in X$, then $Qx\varphi \in$ Fml(L).

5. If $\varphi \in$ Fml(L), FV(φ) is the set of free variables in φ defined by induction on the structure of φ.
   Case 1: $\varphi \in$ AFml(L).

   (a) φ is $\tau_1 = \tau_2$. FV(φ) = FV(τ_1) $\cup$ FV(τ_2).
   (b) φ is $R(\tau_1, \ldots, \tau_n)$. FV($\varphi$) = FV($\tau_1$) $\cup \ldots \cup$ FV(τ_n).

   Case 2:

   (a) if φ is $\neg\psi$, then FV(φ) = FV(ψ).
   (b) if $\varphi = \psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then FV($\varphi$) = FV($\psi_1$) $\cup$ FV(ψ_2).

   Case 3: φ is $Qx\psi$, $Q \in \{\forall, \exists\}$. Then FV($\varphi$) = FV($\psi$) $\setminus \{x\}$.

6. Sent(L) are the sentences in L. Sent(L) = $\{\varphi \in$ Fml(L) $\mid$ FV(φ) = $\emptyset\}$.

Example 1.2.4

If $L_f = \{+, \cdot, 0, 1\}$, then $T_f = \{$

- $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$,

- $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$,

- $\forall x \forall y (x + y = y + x)$,

- $\forall x \forall y (x \cdot y = y \cdot x)$,

- $\forall x (x \cdot 1 = x, 1 \cdot x = x)$,

- $\forall x (x + 0 = x, 0 + x = x)$,

- $\forall x \exists y (x \cdot y = 1, y \cdot x = 1)$,

- $\forall x \exists y (x + y = 0, y + x = 0)$,

- $\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

$\}$.

Definition 1.2.8: L-theory

T is an L-theory if $T \subseteq$ Sent(L).

The example above is "field theory".

Definition 1.2.9

Let M be an L-structure. $\tau(\bar{x})$ is a term, $\bar{a} \in |M|^{\ell(n)}$. T
Case 1: $\tau(\bar{x}) = c$ for some constant symbol. Then $\tau^M(\bar{a}) = c^M$.
Case 2: $\tau(\bar{x}) = x_i$. Then $\tau^M(\bar{a}) = a_i$.
Case 3: $\tau(\bar{x}) = f(\tau_1, \ldots, \tau_k)$. Then $\tau^M(\bar{a}) = f^M(\tau_1^M(\bar{a}), \ldots, \tau_k^M(\bar{a}))$.

Definition 1.2.10: $\models$

Let L be a language, $\varphi \in \text{Fml}(L)$, M and L-structure, $n = \ell(\bar{x})$, $\bar{a} \in |M|^n$. Define $M \models \varphi(\bar{a})$ at $\bar{a}$ by induction on the structure of φ:

- If φ is atomic,

    - when $\varphi(x)$ is $\tau_1 = \tau_2$, then $M \models \varphi(\bar{a})$ iff $\tau_1(\bar{a}) = \tau_2(\bar{a})$.
    - when $\varphi(x)$ is $R(\tau_1, \ldots, \tau_k)$, then $M \models \varphi(\bar{a})$ iff $(\tau_1(\bar{a}), \ldots, \tau_k(\bar{a})) \in R^M$.

- If φ is not atomic, then:

    - if φ is $\neg\psi$, then $M \models \varphi(\bar{a})$ iff $M \models \psi(\bar{a})$ is false.
    - if φ is $\psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $M \models \varphi(\bar{a})$ iff $M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a})$.
    - if φ is $\exists y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff there is $b \in |M|$ such that $M \models \psi(b, \bar{a})$.
    - if φ is $\forall y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff for all $b \in |M|$, $M \models \psi(b, \bar{a})$.

Definition 1.2.11

Let M be an L-structure and T an L-theory. $M \models T$ iff for every $\varphi \in T$, $M \models \varphi$. We say T "satisfies" M.

Example 1.2.5 (Models)
$M \models T_f \iff (|M|, +^M, \cdot^M, 0^M, 1^M)$ is a field.

Definition 1.2.12: Mod

$\text{Mod}(T) = \{M \text{ L-structure} \mid M \models T\}$.

Example 1.2.6
$\text{Mod}(T_f)$ is the class of all fields and $\text{Mod}(T_{\text{gr}})$ is the class of all groups.

Definition 1.2.13: Structure Isomorphism

Let M, N both be L-structures. f is an isomorphism from M onto N if $f : |M| \to |N|$ is a bijection such that:

- $f(c^M) = c^N$ for all $c \in C$.

- $G(x_1, \ldots, x_k)$ function symbol. $a_1, \ldots, a_k \in |M|$, then $f(G^M(a_1, \ldots, a_k)) = G^N(f(a_1), \ldots, f(a_k))$.

- $R(x_1, \ldots, x_k)$ predicate symbol. $a_1, \ldots, a_k \in |M|$, then $(a_1, \ldots, a_k) \in R^M$ iff $(f(a_1), \ldots, f(a_k)) \in R^N$.

We write $f : M \cong N$. Also $M \cong N \iff \exists f : M \cong N$.

> **Definition 1.2.14**
>
> Let $\lambda \geqslant \aleph_0$, T an L-theory. T is λ-categorical if for all $M, N \models T$ of cardinality λ, $M \cong N$.

> **Theorem 1.2.1** Los Conjecture (1954)
>
> Let L be a language, T a first order L-theory, in an at most countable language. If $\exists \lambda > \aleph_0$ such that T is λ-categorical, then for all $\mu > \aleph_0$, T is μ-categorical.

Somewhere around 1961-1965, Morley proved this conjecture.

Chapter 2

Basic Concepts

> **Lenma 2.0.1**
>
> 1. $M \cong N \implies N \cong M$.
>
> 2. $M \cong M$, $f = \mathrm{id}_{|M|}$.
>
> 3. Let M_1, M_2, M_3 be all L-structures. Then $f_1 : M_1 \cong M_2$ and $f_2 : M_2 \cong M_3 \implies f_2 \circ f_1 : M_1 \cong M_3$.
>
> In other words, $\cong$ is an equivalence relation on $\mathrm{Struct}(L)$.

$M/{\cong} = \{N \text{ is an } L(M)\text{-structure} \mid N \cong M\}$.

> **Definition 2.0.1: Spectrum function of T**
>
> Let T be a first order theory $(T \subseteq \mathrm{Sent}(L))$ of cardinality λ. Then $I(\lambda, T)$ is the number of pairwise nonisomorphic models of T of cardinality λ. We have
>
> $$I(\lambda, T) = |M/{\cong}|$$
>
> where $M \models T$ and $\|M\| = \lambda$.

Consider $\lambda \mapsto I(\lambda, T)$, $\lambda \in \mathrm{Card}$ (the class of cardinal numbers). But what is the shape of $\lambda \mapsto I(\lambda, T)$. Is it weakly monotone? That is, $\mu > \lambda \implies I(\mu, T) \geqslant I(\lambda, T)$?

> **Theorem 2.0.1** Morley's Conjecture (~1965)
>
> Suppose T is first order and $|L(T)| \leqslant \aleph_0$. Then $\mu > \lambda > \aleph_0 \implies I(\mu, T) \geqslant I(\lambda, T)$.

The basic problem is that given M and N both of cardinality λ, $M \not\cong N$, find M', N' both of cardinality μ such that $M' \cong N'$. In 1990, Shelah solved Morley's Conjecture. However, this is an open question for uncountable T.

> **Theorem 2.0.2** Morley's Category Theorem
>
> Let T be a first order theory for $|L(T)| \leqslant \aleph_0$. Then $\exists \lambda > \aleph_0, I(\lambda, T) = 1$ then $\forall \mu > \aleph_0$, $I(\mu, T) = 1$.

Shelah listed all possible functions $\lambda \mapsto I(\lambda, T)$ and, by hand, verified that they were weakly monotone.

> **Example 2.0.1**
>
> 1. $I(\lambda, T) = 1$ for all $\lambda > \aleph_0$.
>
> 2. $I(\lambda, T) = 2^\lambda$ for all $\lambda > \aleph_0$.

Hart, Hrushovski, and Laskowski found all the 13 functions.

Definition 2.0.2: Submodel

Let M, N be L-structures. M is a submodel of N if:

1. $|M| \leqslant |N|$

2. $\forall a_1, \ldots, a_n \in |M|$ and $F(x_1, \ldots, x_n)$, $F^M(a_1, \ldots, a_n) = F^N(a_1, \ldots, a_n)$.

3. $c^M = c^N$ for all constant symbols c.

4. $R^M = R^N \cap (|M| \times \cdots \times |M|)$.

Definition 2.0.3: Elementarily Equivalent

Let M, N be L-structure. M is elementarily equivalent to N denoted by $M \equiv N$ provided $M \models \varphi \iff N \models \varphi$ for any $\varphi \in \text{Sent}(L)$.

Definition 2.0.4

Let M be an L-structure. The theory of M is denoted $(M) = \{\text{Th}(M)\varphi \in \text{Sent}(L) \mid M \models \varphi\}$.

Let $N := (\omega, +, \cdot, 0, 1)$. Then TA $= \text{Th}(N)$ "True Arithmetic". For example the twin primes conjecture is $\{p \mid p \text{ and } p + 2 \text{ are both primes}\}$ is infinite. If it is true, then it is a member of TA.

Theorem 2.0.3

Let M, N be L-stuructres. If $M \cong N$, then $M \equiv N$.

Theorem 2.0.4

Let M, N be L-structures. Suppose $f : M \cong N$. Then for any $\bar{a} \in |M|$ and any $\varphi(\bar{x}) \in \text{Fml}(L)$ with $\ell(\bar{x}) = \ell(\bar{a})$, $M \models \varphi[\bar{a}] \iff N \models \varphi[f(\bar{a})]$.

Proof. Suppose $\varphi(\bar{x})$ is atomic.

Lenma 2.0.2

Suppose $f : M \cong N$ and $\tau(\bar{x})$ sequence of terms. $\bar{a} \in |M|$, $\ell(\bar{x}) = \ell(\bar{a})$. Then $f(\tau(\bar{a})) = \tau(f(\bar{a}))$.

Proof. By induction on the length of τ.

Case 1: $\tau(\bar{x})$ is x. Then $f(\tau(\bar{a})) = f(a) = \tau(f(\bar{a}))$.

Case 2: $\tau(\bar{x}) = c$. then $f(c^M) = c^N$ by definition of isomorphism.

Case 3: $\tau(\bar{x}) = G(y_1, \ldots, y_n)$ function symbol. Then $\tau_1(\bar{x}), \ldots, \tau_n(\bar{x})$ are terms. By induction, $f(\tau(\bar{a})) = f(G^M(\tau_1(\bar{a}), \ldots, \tau_n(\bar{a}))) = G^N(f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) = \tau^N(f(\bar{a}))$. ☺

Now returning to the proof:

Case 1: $\varphi(\bar{x})$ is $\tau_1(\bar{x}) = \tau_2(\bar{x})$. Then, we have $M \models \varphi(\bar{a}) \iff \tau_1(\bar{a}) = \tau_2(\bar{a}) \iff f(\tau_1(\bar{a})) = f(\tau_2(\bar{a})) \iff \tau_1^N(f(\bar{a})) = \tau_2^N(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Case 2: $\varphi(\bar{x})$ is $R(\tau_1(\bar{x}), \ldots, \tau_n(\bar{x}))$. When $R(y_1, \ldots, y_n)$ is a relation symbol and $\tau_i(\bar{x})$ are terms. Then $M \models \varphi(\bar{a}) \iff (\tau_1(\bar{a}), \ldots, \tau_n(\bar{a})) \in R^M \iff (f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) \in R^N \iff (\tau_1(f(\bar{a})), \ldots, \tau_n(f(\bar{a}))) \iff N \models \varphi(f(\bar{a}))$.

Suppose φ is $\neg\psi$. Then $M \models \varphi(\bar{a}) \iff M \not\models \psi(\bar{a}) \iff N \not\models \psi(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Suppose φ is $\psi_1 \wedge \psi_2$. Then $M \models \varphi(\bar{a}) \iff M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a}) \iff N \models \psi_1(f(\bar{a}))$ and $N \models \psi_2(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Suppose $\varphi(\bar{x})$ is $\exists y \psi(y, \bar{x})$. Then $M \models \varphi(\bar{a}) \iff$ there is $b \in |M|$ such that $M \models \psi(b, \bar{a}) \iff$ there is $c \in |N|$ such that $N \models \psi(c, f(\bar{a})) \iff N \models \exists \psi(y, f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$. ☺

General Remark:

$$M \models \exists y \varphi(y, \bar{a}) \iff M \models \neg \forall y \neg \varphi(y, \bar{a})$$
$$M \models \neg \exists y \varphi(y, \bar{x}) \iff \forall y \neg \varphi(y, \bar{a}).$$

Example 2.0.2

$L_{gr} = \{\cdot, 1\}$. $(\mathbb{Q}, +, 0), (\mathbb{R}, +, 0)$ are not isomorphic because diff cardinality. $(\mathbb{Q}, +, 0), (\mathbb{Z}, +, 0)$ are not isomorphic because:

$$(\mathbb{Q}, +, 0) \models \forall x \exists y (x = y + y).$$

This sentence is not true for $\mathbb{Z}$ under addition.

$N = (\omega, +, \cdot, 0, 1)$ is called the standard model of arithmetic. TA = Th(N), true arithmetic.

Question

Given $M_1, M_2 \models$ TA both countable. Are they isomorphic?

Question

What is $I(\aleph_0, \text{TA})$? Voted on $2^{\aleph_0}$.... and it is.

Let T be a theory and $\varphi \in \text{Sent}(L)$. We say T proves φ (denoted $T \vdash \varphi$) if there exists a finite set of sequences from L, $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for all i, $\varphi_i \in T$ or φ_i is a tautology or there are $j, k < i$ where $\varphi_j = (\varphi_k \implies \varphi_i)$.

1. $Q \to P$: the rule of inference. "modus ponens".

2.

Other rules (possible members of φ):

- $x = y$, $y = z$ then $x = z$.

- If $\varphi_i = \forall x \varphi(x)$, then also $\forall y \varphi(y)$ in the sequence.

- If $\forall x \varphi(x)$ also $\varphi(\tau(\bar{c}))$.

Definition 2.0.5

A set of sentences is a consistent theory if there is no φ such that φ and $\neg \varphi$ are both in the theory. T is inconsistent if it is not consistent.

Theorem 2.0.5 Godel's Completeness Theorem

Let T be some set of sentences in L. Then T is consistent iff T has a model.

Godel only proved it for when $|L| \leqslant \aleph_0$.

Theorem 2.0.6 Compactness Theorem

Let $T \subseteq \text{Sent}(L)$. If for any finite $T_0 \subseteq T$, T_0 has a model, then T has a model.

Proof. Enough to show by completeness that is consistent. If T inconsistent, $T \vdash \varphi$ and $T \vdash \neg \varphi$, there are $T_1, T_2 \subset T$ finite such that $T_1 \vdash \varphi$ and $T_2 \vdash \neg \varphi$. Then $T_1 \cup T_2 \vdash \varphi \wedge \neg \varphi$. By assumption on T, $\exists M_0 \models T_1 \cup T_2$. Then $M_0 \models \varphi \wedge \neg \varphi$ which is a contradiction. ☺

ZF cannot prove the compactness theorem.