21-235 Math Studies Analysis I

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Contents

Chapter 1

1.1 Ordered Fields (Review)

Definition 1.1.1: Order

Let E be a set. An *order* on E is a relation $<$ on E such that for all $x, y, z \in E$:

1. (Trichotomy) Exactly one of the following holds: $x < y$, $x = y$, or $x > y$.
2. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Example 1.1.1 (Examples of Ordered Sets)

1. This definition develops orders on basic number systems: e.g. $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
2. Define $\lesssim$ on $\mathbb{Z}$ as follows: We say that $m \lesssim n$ for $m, n \in \mathbb{Z}$ if:
   (a) m is even and n is odd
   (b) m, n are even and $m < n$
   (c) m, n are odd and $m < n$.

Key Concepts:

– upper/lower bounds of sets

– bounded sets

– max/min

– supremum/infimum

– supremum/infimum property: An ordered set E satisfies such a property if every nonempty set $A \subseteq E$ that's bounded above/below has a supremum/infimum in E.

– Fact: sup prop $\implies$ inf prop

Definition 1.1.2: Ordered Field

Let $\mathbb{F}$ be a field with order $<$. We say that $\mathbb{F}$ is an *ordered field* provided that:

1. For all $x, y, z \in \mathbb{F}$, if $x < y$, then $x + z < y + z$.
2. For all $x, y \in \mathbb{F}$, if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Example 1.1.2
$\mathbb{Q}$ is a field.

Facts of any ordered field:

1. $0 < 1$
2. $\nexists x \in \mathbb{F}$ such that $x^2 = -1$.

Definition 1.1.3: Ordered Subfield, Homomorphism, Isomorphism

Let $\mathbb{F}$ be an ordered field.

1. A set $\mathbb{K} \subseteq \mathbb{F}$ is called an *ordered subfield* if $mathbbK$ is an algeraic subfield and $\mathbb{K}$ is an ordered field equipped with $<$ from $\mathbb{F}$.
2. Let $\mathbb{G}$ be an ordered field and let $f : \mathbb{F} \to \mathbb{G}$. We say that f is an *ordered field homomorphism* if it's a field homomorphism and $f(x) < f(y)$ whenever $x < y$.
3. f is an ordered field isomorphism if f is an ordered field homomorphism and f is bijective.

Note:

1. If $f : \mathbb{F} \to \mathbb{G}$ is an ordered field homomorphism, $f(\mathbb{F})$ is an ordered subfield of $\mathbb{G}$.
2. OF property $\implies$ f is injective.
3. $\therefore$ every ordered field homomorphism $f : \mathbb{F} \to \mathbb{G}$ is such that f induces a bijection $f : \mathbb{F} \to f(\mathbb{F}) \subseteq \mathbb{G}$.

Theorem 1.1.1 $\mathbb{Q}$ is the smallest ordered field. More precisely, if $\mathbb{F}$ is an ordered field, then there exists a canonical ordered field homomorphism $f : \mathbb{Q} \to \mathbb{F}$.

Upshot/notation abuse: We identify $f(\mathbb{Q}) = \mathbb{Q}$ to view $\mathbb{Q} \subseteq \mathbb{F}$. In turn, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{F}$.

1.2 Types of Ordered Fields

Definition 1.2.1: Archimedean, Dedekind complete

Let $\mathbb{F}$ be an ordered field.

1. We say that $\mathbb{F}$ is Archimedean if $\forall 0 < x \in \mathbb{F}$, $\exists n \in \mathbb{N}$ such that $n > x$.
2. We say that $\mathbb{F}$ is Dedekind complete if it satisfies the supremum property.

Facts:

1. $\mathbb{Q}$ is Archimedean.
2. If $\mathbb{F}$ is Dedekind complete, then $\forall 0 < x \in \mathbb{F}$ and $\forall 0 < n \in \mathbb{N}$, $\exists!\ 0 < y \in \mathbb{F}$ such that $y^n = x$.
3. $\mathbb{Q}$ is not Dedekind complete. ($\sqrt{2}$ is a counterexample.)

Theorem 1.2.1
Suppose $\mathbb{F}$ is a Dedekind complete ordered field. Then $\mathbb{F}$ is Archimedean.

Proof. If not, then $\mathbb{N} \subset \mathbb{F}$ is bounded above, and so the supremum property provides $x \in \mathbb{F}$ such that $x = \sup \mathbb{N}$. But then $x - 1$ is an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that $x - 1 < n$. Hence $x < n + 1$, which contradicts the definition of x as an upper bound. Therefore, $\mathbb{F}$ is Archimedean. 😄

1.3 Dedekind Completion

Throughout this section, let $\mathbb{F}$ be an Archimedean ordered field.

Definition 1.3.1: Dedekind cut

We say a set $C \subseteq \mathbb{F}$ is *Dedekind cut* if:

1. $C \neq \emptyset$ and $C \neq \mathbb{F}$.
2. If $p \in C$ and $q \in \mathbb{F}$ such that $q < p$, then $q \in C$.
3. If $p \in C$, then $\exists r \in C$ such that $p < r$.

We will write $\mathbb{F}^*$ for the set of all Dedekind cuts in $\mathbb{F}$. It is called the *Dedekind completion* of $\mathbb{F}$.

Note:

Let $C \subseteq \mathbb{F}$ be a cut. Then:

1. If $p \in C$, then $q \notin C$, then $p < q$.
2. If $r \notin C$, and $r < s \in \mathbb{F}$, then $s \notin C$.

Example 1.3.1 (Cut examples)

1. Let $q \in \mathbb{F}$ and define $C_q = \{p \in \mathbb{F} \mid p < q\}$. Then C_q is a cut.

   Proof. (a) $q - 1 < q \implies q - 1 \in C_q$. $q \not< q \implies q \notin C_q \implies C_q \neq \mathbb{F}$.

   (b) Let $p \in C_q$. Suppose $s \in \mathbb{F}$ such that $s < p$. Then $s < q \implies s \in C_q$.

   (c) Let $p \in C_q$. Then $p < \frac{p+q}{2} < q \implies \frac{p+q}{2} \in C_q$.

   😄

2. Suppose $\mathbb{F}$ is such that $\nexists x \in \mathbb{F}$ such that $x^2 = 2$. Let $C = \{p \in \mathbb{F} \mid p \leqslant 0 \text{ or } 0 < p^2 < 2\}$. Then C is a cut.

   Proof. (a) $1 \in C$ and $1^2 = 1 < 2$. $2 \notin C$ and $2^2 = 4 > 2$.

   (b) Let $p \in C$ and $q \in \mathbb{F}$ such that $q < p$. If $q \leqslant 0$, then $q \in C$ trivially. Suppose $0 < q < p$. Then $0 < q^2 < p^2 < 2$, so $q \in C$.

   (c) Let $p \in C$. If $p \leqslant 0$, then $1 \in C$ and $p < 1$, so we're done. Suppose $0 < p^2 < 2$. Note, $0 < 2 - p^2$, so $\frac{2p+1}{2-p^2} > 0$. Then we can define $r = 1 + \frac{2p+1}{2-p^2} \geqslant \max(1, \frac{2p+1}{2-p^2})$. Then $(p + 1/r)^2 = p^2 + \frac{2p}{r} + \frac{1}{r^2}$. We have:

   $$\begin{aligned} p^2 + \frac{2p}{r} + \frac{1}{r^2} &< p^2 + \frac{2p}{r} + \frac{1}{r} \\ &= p^2 + \frac{2p+1}{r} \\ &\leqslant p^2 + 2 - p^2 \\ &= 2. \end{aligned}$$

   So, $p < p + 1/r < 2$ and $p + 1/r \in C$.

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1.3.1 Ordering $\mathbb{F}^*$

Lemma 1.3.1

The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then exactly one holds:
   - $\mathcal{A} \subset \mathcal{B}$
   - $\mathcal{A} = \mathcal{B}$
   - $\mathcal{B} \subset \mathcal{A}$
2. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$.

Proof. Proof of 2 is trivial, as well as the equality part for 1.

- If $\mathcal{A} = \mathcal{B}$, we're done.
- Suppose $\exists b \in \mathcal{B} \setminus \mathcal{A}$. If $a \in \mathcal{A}$, then $a < b$, but $\mathcal{B}$ is a cut so $a \in \mathcal{B}$, so $\mathcal{A} \subset \mathcal{B}$.
- Suppose $\exists a \in \mathcal{A} \setminus \mathcal{B}$. Then $a < b$ for all $b \in \mathcal{B}$, so $a \in \mathcal{B}$, so $\mathcal{B} \subset \mathcal{A}$.

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Definition 1.3.2: Order on cuts

Given $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we say that $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$. The lemma above shows that this is infact an order.

Lemma 1.3.2

Let $E \subseteq \mathbb{F}^*$ be nonempty and bounded above. Then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$ is a cut.

Proof.

1. Since $E \neq \emptyset$, there exists $\mathcal{A} \in E$. So $\mathcal{A} \neq \emptyset$, hence $\mathcal{B} \neq \emptyset$.

   Since E is bounded above, there exists $\mathcal{C} \in \mathbb{F}^*$ such that $\mathcal{A} \subset \mathcal{C}$ for all $\mathcal{A} \in E$. Since $\mathcal{C}$ is a cut, there is $q \in \mathbb{F}$ such that $q \notin \mathcal{C}$. Then $q \notin \mathcal{A}$ for all $\mathcal{A} \in E$, so $q \notin \mathcal{B}$.
2. Let $p \in \mathcal{B}$ and $q \in \mathbb{F}$ such that $q < p$. Since $\mathcal{B}$ is a union of cuts, it follows that $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, $q \in \mathcal{A} \subseteq \mathcal{B}$.
3. Let $p \in \mathcal{B}$. Then $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, there exists $r \in \mathcal{A}$ such that $p < r$. Since $\mathcal{A} \subset \mathcal{B}$, we have $r \in \mathcal{B}$.

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Theorem 1.3.1

$\mathbb{F}^*$ equipped with the order $<$ satisfies the supremum property.