Math 21-269, Vector Analysis I, Spring 2024
Assignment 2

The due date for this assignment is Friday, February 9.

Given N bounded intervals $I_1, \ldots, I_N \subset \mathbb{R}$, a *rectangle* in $\mathbb{R}^N$ is a set of the form

$$R := I_1 \times \cdots \times I_N.$$

The elementary measure of a rectangle is given by

$$\operatorname{meas} R := (b_1 - a_1) \cdot (b_2 - a_2) \cdot \cdots \cdot (b_n - a_n)$$

where $a_n \leq b_n$ are the endpoints of I_n.

Given a rectangle R, by a *partition* $\mathcal{P}$ of R we mean a finite set of rectangles $R_1, \ldots, R_n$ such that $R_i \cap R_j = \emptyset$ if $i \neq j$ and

$$R = \bigcup_{i=1}^{n} R_i.$$

Let's take for granted that if $R \subset \mathbb{R}^N$ is a rectangle and $\mathcal{P} = \{R_1, \ldots, R_n\}$ is a partition of R, then[1]

$$\operatorname{meas} R = \sum_{i=1}^{n} \operatorname{meas} R_i.$$

Given a rectangle $R \subset \mathbb{R}^N$ and two partitions $\mathcal{P}$ and $\mathcal{Q}$ of R, we say that $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, if each rectangle of $\mathcal{Q}$ is contained in some rectangle of $\mathcal{P}$.

Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to \mathbb{R}$ be a bounded function. Given a partition $\mathcal{P} = \{R_1, \ldots, R_n\}$ of R, we define the *lower* and *upper sums* of f for the partition $\mathcal{P}$ respectively by

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\boldsymbol{x} \in R_i} f(\boldsymbol{x}),$$

$$U(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \sup_{\boldsymbol{x} \in R_i} f(\boldsymbol{x}).$$

Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the *upper Riemann integral* $\overline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x}$ of f and the *lower Riemann integral* $\underline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x}$ of f are defined as

$$\overline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x} = \inf \{U(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\},$$

$$\underline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x} = \sup \{L(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\}.$$

If a nonempty set $E \subseteq \mathbb{R}$ is not bounded from above, we define $\sup E := \infty$. Similarly, if E is not bounded from below, we set $\inf E := -\infty$.

[1] This is a good exercise, in case you are bored.

1. Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ be bounded functions.

   (a) Prove that if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of R and $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then

   $$U(f, \mathcal{Q}) \le U(f, \mathcal{P}), \quad L(f, \mathcal{P}) \le L(f, \mathcal{Q}).$$

   (b) Prove that

   $$\underline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x} \le \overline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

   (c) Prove that if $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of R and $\mathcal{S}$ is a refinement $\mathcal{S}$ of both $\mathcal{P}$ and $\mathcal{Q}$, then

   $$\begin{aligned} L(f, \mathcal{P}) + L(g, \mathcal{Q}) &\le L(f + g, \mathcal{S}), \\ U(f + g, \mathcal{S}) &\le U(f, \mathcal{P}) + U(g, \mathcal{Q}). \end{aligned}$$

   (d) Prove that

   $$\begin{aligned} \underline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x} + \underline{\int_R} g(\boldsymbol{x})\, d\boldsymbol{x} &\le \underline{\int_R} (f(\boldsymbol{x}) + g(\boldsymbol{x}))\, d\boldsymbol{x}, \\ \overline{\int_R} (f(\boldsymbol{x}) + g(\boldsymbol{x}))\, d\boldsymbol{x} &\le \overline{\int_R} f(\boldsymbol{x})\, d\boldsymbol{x} + \overline{\int_R} g(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}$$

2. For every $\boldsymbol{x} \in \mathbb{R}^N$, let

   $$\|\boldsymbol{x}\|_1 := |x_1| + \cdots + |x_N|, \quad \|\boldsymbol{x}\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.$$

   (a) Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms.

   (b) For every $\boldsymbol{x} \in \mathbb{R}^N$, let

   $$\|\boldsymbol{x}\| := \inf\{\|\boldsymbol{y}\|_1 + \|\boldsymbol{z}\|_\infty : \boldsymbol{y} \in \mathbb{R}^N, \boldsymbol{z} \in \mathbb{R}^N, \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}\}.$$

   Prove that $\|\cdot\|$ is a norm.

3. Compute the liminf and the limsup of the following sequences

   (a) $x_n = \frac{1+(-1)^n}{n}$,

   (b) $x_n = \sin \frac{n\pi}{2}$,

   (c) $x_n = (-1)^n n \sin \frac{1}{n}$,

   (d) $x_n = x^n$, where $x \in \mathbb{R}$.

JUSTIFY ALL YOUR ANSWERS

1. (a) $\mathcal{Q}$ being a refinement of $\mathcal{P}$ means that for every $P \in \mathcal{P}$, there exists $\{Q_i\}_{1\le i\le r}$ such that $\bigcup_{1\le i\le p} Q_i = P$. So, we have:

$$U(f,\mathcal{Q}) = \sum_{Q\in\mathcal{Q}} \operatorname{meas} Q \sup_{x\in Q} f(x)$$

So, let $\{Q_i\}_{1\le i\le r} = S$. We argue that:

$$\sum_{Q\in S} \operatorname{meas} Q \sup_{x\in Q} f(x) \le \sum_{Q\in S} \operatorname{meas} Q \sup_{x\in P} f(x) = \operatorname{meas} P \sup_{x\in P} f(x)$$

This is true because the supremum of the union of some sets is very obviously greater than or equal to the supremum of each individual set. We then proceed by realizing

$$\sum_{Q\in\mathcal{Q}} \operatorname{meas} Q \sup_{x\in Q} f(x) = \sum_{P\in\mathcal{P}} \sum_{Q\in S} \operatorname{meas} Q \sup_{x\in Q} f(x)$$

since the partitions are still included exactly one time. Combining everything, we get :

$$\begin{aligned} U(f,\mathcal{Q}) &= \sum_{Q\in\mathcal{Q}} \operatorname{meas} Q \sup_{x\in Q} f(x) \\ &= \sum_{P\in\mathcal{P}} \sum_{Q\in S} \operatorname{meas} Q \sup_{x\in Q} f(x) \\ &\le \sum_{P\in\mathcal{P}} \operatorname{meas} P \sup_{x\in P} f(x) \\ &= U(f,\mathcal{P}) \end{aligned}$$

For the L case, we can use the same argument to show that $L(f,\mathcal{P}) \le L(f,\mathcal{Q})$.

$$\begin{aligned} L(f,\mathcal{Q}) &= \sum_{Q\in\mathcal{Q}} \operatorname{meas} Q \inf_{x\in Q} f(x) \\ &= \sum_{P\in\mathcal{P}} \sum_{Q\in S} \operatorname{meas} Q \inf_{x\in Q} f(x) \\ &\ge \sum_{P\in\mathcal{P}} \operatorname{meas} P \inf_{x\in P} f(x) \\ &= L(f,\mathcal{P}) \end{aligned}$$

(b) We know $L(f,\mathcal{P}) \le U(f,\mathcal{P})$ for all partitions $\mathcal{P}$ of R. This can be derived from the definitions of L and U and the fact that $\inf_{x\in R} f(x) \le \sup_{x\in R} f(x)$. Because of this, we essentially want to show that $\sup\{L(f,\mathcal{P})\} \le \inf\{U(f,\mathcal{P})\}$. We basically have that for all $x \in \sup\{L(f,\mathcal{P})\}$ and $y \in \inf\{U(f,\mathcal{P})\}$, $x \le y$. We also have that $\sup\{L(f,\mathcal{P})\}$ is bounded from above and $\inf\{U(f,\mathcal{P})\}$ is bounded from below. As such, we know that all y is an upper bound for all x. This means that $\sup\{L(f,\mathcal{P})\}$ is a lower bound for $\inf\{U(f,\mathcal{P})\}$. As such, we have that $\sup\{L(f,\mathcal{P})\} \leq \inf\{U(f,\mathcal{P})\}$.

(c) (a) gave us the following results:

- $L(f,\mathcal{P}) + L(g,\mathcal{Q}) \leq L(f,\mathcal{S}) + L(g,\mathcal{S})$
- $U(f,\mathcal{S}) + U(g,\mathcal{S}) \leq U(f,\mathcal{P}) + U(g,\mathcal{Q})$

We now show that $\inf\{f(x)\} + \inf\{g(x)\} \leq \inf\{f(x) + g(x)\}$. We have:

$$f(x) + g(x) \geq \inf\{f(x)\} + \inf\{g(x)\},$$

which means that we have a lower bound for $\{f(x)+g(x)\}$. As such:

$$\inf\{f(x)\} + \inf\{g(x)\} \leq \inf\{f(x) + g(x)\}.$$

So if we apply this given inequality to each element of $\mathcal{S}$, we get our desired result:

$$\begin{aligned} L(f,\mathcal{P}) + L(g,\mathcal{Q}) &\leq L(f,\mathcal{S}) + L(g,\mathcal{S}) \\ &\leq L(f+g,\mathcal{S}) \end{aligned}$$

We can do the same thing for the upper sums to get the other inequality. We proceed by showing that $\sup\{f(x) + g(x)\} \leq \sup\{f(x)\} + \sup\{g(x)\}$. We have:

$$f(x) + g(x) \leq \sup\{f(x)\} + \sup\{g(x)\},$$

which means that we have an upper bound for $\{f(x) + g(x)\}$. As such:

$$\sup\{f(x) + g(x)\} \leq \sup\{f(x)\} + \sup\{g(x)\}.$$

So if we apply this given inequality to each element of $\mathcal{S}$, we get our desired result:

$$\begin{aligned} U(f+g,\mathcal{S}) &\leq U(f,\mathcal{P}) + U(g,\mathcal{Q}) \\ &\leq U(f,\mathcal{S}) + U(g,\mathcal{S}) \end{aligned}$$

(d) We have that for all partitions $\mathcal{P}_\circ$ of R, we have:

$\inf\{U(f,\mathcal{P})|\mathcal{P} \text{ partitions } R\} + \inf\{U(g,\mathcal{P})|\mathcal{P} \text{ partitions } R\} \leq U(f,\mathcal{P}_\circ) + U(g,\mathcal{P}_\circ)$

This means that the LHS is a lower bound for the following set:

$$\{U(f+g,\mathcal{P})|\mathcal{P} \text{ partitions } R\}$$

So the infimum of this set is greater than or equal to the aforementioned LHS, leading to:

$$\underline{\int_R} f(\boldsymbol{x})\ d\boldsymbol{x} + \underline{\int_R} g(\boldsymbol{x})\ d\boldsymbol{x} \leq \underline{\int_R} (f(\boldsymbol{x}) + g(\boldsymbol{x}))\ d\boldsymbol{x}$$

We repeat this process for upper integrals with L instead of U and sups instead of infs. So, for any partition $\mathcal{P}_\circ$ of R, we have:

$\sup\{L(f,\mathcal{P})|\mathcal{P} \text{ partitions } R\} + \sup\{L(g,\mathcal{P})|\mathcal{P} \text{ partitions } R\} \geq L(f,\mathcal{P}_\circ) + L(g,\mathcal{P}_\circ)$

This means that the LHS is an upper bound for the following set:

$$\{L(f+g,\mathcal{P})|\mathcal{P} \text{ partitions } R\}$$

So the supremum of this set is less than or equal to the aforementioned LHS, leading to:

$$\overline{\int_R} (f(\boldsymbol{x}) + g(\boldsymbol{x}))\ d\boldsymbol{x} \leq \overline{\int_R} f(\boldsymbol{x})\ d\boldsymbol{x} + \overline{\int_R} g(\boldsymbol{x})\ d\boldsymbol{x}.$$

As such, we have shown both inequalities.

2. Recall the definition of a norm:

Let V be a vector space with an inner product $(\cdot,\cdot)$. Then the *norm* of $x \in X$ is defined as $||\cdot|| : X \to [0,\infty)$ such that:

- $||x|| = 0 \iff x = 0$
- $||tx|| = |t|||x||$ for all $x \in X$
- $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in X$

(a) First we show that $\|\cdot\|_1$ is a norm.

- If we look at:

$$\|x\|_1 = |x_1| + \cdots + |x_N|,$$

we see that the RHS is strictly positive if $x_i \neq 0$, so by contrapositive, if $\|x\|_1 = 0$, then $x_i = 0$ for all $i \Rightarrow x = 0$. This satisfies the first property of a norm.

- If we look at:

$$tx = t(x_1, \dots, x_N) = (tx_1, \dots, tx_N)$$
$$\|tx\|_1 = |tx_1| + \cdots + |tx_N| = |t||x_1| + \cdots + |t||x_N| = |t|(|x_1| + \cdots + |x_N|) = |t|\|x\|_1$$

- We know that $|x_i + y_i| \leq |x_i| + |y_i|$ for all i, so we can apply this to the sum of the absolute values of the components of $x + y$ to get:

$$\begin{aligned} \|x + y\|_1 &= |x_1 + y_1| + \cdots + |x_N + y_N| \\ &\leq |x_1| + |y_1| + \cdots + |x_N| + |y_N| \\ &= \|x\|_1 + \|y\|_1 \end{aligned}$$

Now we move to $\|\cdot\|_\infty$.

- If we look at:

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_N|\},$$

we see that the RHS is strictly positive if $x_i \neq 0$, so by contrapositive, if $\|x\|_\infty = 0$, then $x_i = 0$ for all $i \Rightarrow x = 0$. This satisfies the first property of a norm.

- We look at:

$$|t|\|x\|_\infty = t\max\{|x_1|, \dots, |x_N|\} = \max\{|tx_1|, \dots, |tx_N|\} = \|tx\|_\infty$$

- We have:

$$\begin{aligned} \|x + y\|_\infty &= \max\{|x_1 + y_1|, \dots, |x_N + y_N|\} \\ &\leq \max\{|x_1| + |y_1|, \dots, |x_N| + |y_N|\} \\ &\leq \max\{|x_1|, \dots, |x_N|\} + \max\{|y_1|, \dots, |y_N|\} \\ &= \|x\|_\infty + \|y\|_\infty \end{aligned}$$

(b) Turns out $\|\cdot\| = \|\cdot\|_\infty$. We can see this by the following:

$$\begin{aligned} \|y\|_1 + \|z\|_\infty &\geq \|y\|_\infty + \|z\|_\infty \\ &\geq \|y + z\|_\infty \\ &\geq \|x\|_\infty \end{aligned}$$

So $\|x\|_\infty$ is a lower bound of our set. But we can see that it is also in the set, because we can take $y = 0$ and $z = x$. Since a lower bound that is in the set is the infimum, we have that $\|x\| = \|x\|_\infty$. As such, $\|\cdot\|$ is a norm.

3. Recall the definitions of liminf and limsup:

$$\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} x_k \right)$$
$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k \right)$$

Additionally, we showed during recitation that liminf and limsup have the following properties:

- Let $\{x_n\}$ be a sequence bounded above in $\mathbb{R}$. Then $L \in \mathbb{R}$ is the limit superior of $\{x_n\}$ if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon$ for all $n \geq n_\epsilon$.
  - $x_n > L - \epsilon$ for infinitely many n.
- Let $\{x_n\}$ be a sequence bounded below in $\mathbb{R}$. Then $L \in \mathbb{R}$ is the limit inferior of $\{x_n\}$ if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon$ for infinitely many n.
  - $x_n > L - \epsilon$ for all $n \geq n_\epsilon$.

(a) I claim that the liminf and limsup of the sequence $x_n = \frac{1+(-1)^n}{n}$ are both 0. I will prove this by using the conditions above:

- **limsup**: Let $L = 0$. Then for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon = \epsilon$ for all $n \geq n_\epsilon$.
  - $x_n > L - \epsilon = -\epsilon$ for infinitely many n.

  For the first condition, we are essentially asking if there exists $n_\epsilon \in \mathbb{N}$ such that $\dfrac{2}{n_\epsilon} < \epsilon$ for any positive ϵ.

  $$\frac{2}{n_\epsilon} < \epsilon$$
  $$n_\epsilon > \frac{2}{\epsilon}$$

  By the Archimedean property, we know that this n_ϵ exists for any $\epsilon > 0$. As such, this condition is sufficed.
  For the second property, we are asking if $x_n > -\epsilon$ for infinitely many n. Since x_n is always positive and $\epsilon < 0$, this condition is also sufficed.
- **liminf**: Let $L = 0$. Then for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon = \epsilon$ for infinitely many n.
  - $x_n > L - \epsilon = -\epsilon$ for all $n \geq n_\epsilon$.

  For the first condition, we are asking if there exists $n_\epsilon \in \mathbb{N}$ such that $\dfrac{2}{n_\epsilon} < \epsilon$ for infinitely many n. This is clearly true since in the limsup case, we showed that $x_n < \epsilon$ for all $n \geq n_\epsilon$ for any $\epsilon > 0$, which is an infinite number.
  For the second property, we are asking if $x_n > -\epsilon$ for all $n \geq n_\epsilon$. Since x_n is always positive and $\epsilon < 0$, this condition is also sufficed if we even choose $n_\epsilon = 1$.

(b) I claim the limsup is 1 and the liminf is -1.

- **limsup**: Let $L = 1$. Then for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon = 1 + \epsilon$ for all $n \geq n_\epsilon$.
  - $x_n > L - \epsilon = 1 - \epsilon$ for infinitely many n.

  If we analyze x_n, we see the following:

  $$x_n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ -1 & n \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

  If we take any $\epsilon > 1$, we see that $x_n < 1 + \epsilon$ for all $n \geq 1$, so we can take $n_\epsilon = 1$.
  Additionally, we know there are infinite many n such that $x_n > 1 - \epsilon$ since $x_n = 1$ for all $n \equiv 0 \pmod 4$.
- **liminf**: Let $L = -1$. Then for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
  - $x_n < L + \epsilon = -1 + \epsilon$ for infinitely many n.
  - $x_n > L - \epsilon = -1 - \epsilon$ for all $n \geq n_\epsilon$.

  Since $x_n = -1$ for infinitely many n, we know that $x_n < -1 + \epsilon$ for infinitely many n. Additionally, we know that $x_n > -1 - \epsilon$ for all $n \geq 1$.

(c) For small-angle approximation, we have that:

$$\sin(\theta) \approx \theta \qquad \text{(for small } \theta\text{)}$$

As such, we have that for large n,

$$x_n = (-1)^n n \sin\left(\frac{1}{n}\right) \approx (-1)^n n \left(\frac{1}{n}\right) = (-1)^n$$

We then have:

$$\lim_{m\to\infty} x_{2m} = 1$$
$$\lim_{m\to\infty} x_{2m+1} = -1$$

As such, we have that $\liminf_{n\to\infty} x_n = -1$ and $\limsup_{n\to\infty} x_n = 1$ because the liminf is the infimum of the set of all subsequential limits and the limsup is the supremum of the set of all subsequential limits.

(d) There are 7 cases to consider here depending on the value of x.

- If $x = 0$, then $x_n = 0$ for all n and so $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = 0$.
- If $x = 1$, then $x_n = 1$ for all n and so $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = 1$.

- If $x = -1$, then $x_n = (-1)^n$ and so $\liminf_{n\to\infty} x_n = -1$ and $\limsup_{n\to\infty} x_n = 1$. We know this because it's basically the same as problem (b) because is has an infinite number of -1s and 1s, and our solution in (b) did not rely at all on the fact that $x_n = 0$ for some values of n.
- If $-1 > x > 0$, then the limit superior is 0 and the limit inferior is 0. For the limit superior, we know that there exists $x_n < \epsilon$ for all $n \geq n_\epsilon$. This is because the subsequence x_n for even n is strictly decreasing and positive, it will eventually be less than ϵ. Additionally, we have that $x_n > -\epsilon$ for infinitely many n. This is because the subsequence x_n for odd n is strictly increasing and negative, it will eventually be greater than $-\epsilon$, and the even n will be positive, so obviously greater than $-\epsilon$.
  For the limit inferior, we said that $x_n < L + \epsilon$ for any $n > n_\epsilon$, which is an infinite number of n. Now again since x_n is positive, we have that $x_n > L - \epsilon$ for all $n \geq 1$. This is because the subsequence x_n for odd n is strictly increasing and negative, it will eventually be greater than $-\epsilon$, and the even n will be positive, so obviously greater than $-\epsilon$.
- If $0 < x < 1$ then the limit superior is 0 and the limit inferior is 0. x_n is always positive and decreasing since a^x is decreasing for $0 < a < 1$. As such, we have that $x_n < \epsilon$ for all $n \geq n_\epsilon$ is the first number such that $x^{n_\epsilon} < \epsilon$. Since x^n is positive, we also have $x_n > -\epsilon$ for infinitely many n.
  For the limit inferior, we said that $x_n < L + \epsilon$ for any $n > n_\epsilon$, which is an infinite number of n. Now again since x_n is positive, we have that $x_n > L - \epsilon$ for all $n \geq 1$.
- It's worth noting the following:
  - If x_n isn't bounded from above, then

    $$\limsup_{n\to\infty} x_n = \infty$$

    Proof. Suppose $\{x_n\}$ is not bounded from above. Then for any $k \in \mathbb{N}$, we also have that $\{x_n : n \geq k\}$ is also not bounded from above. Thus $s_k = \sup\{x_n : n \geq k\} = \infty$ for any k. As such, $\limsup_{n\to\infty} x_n = \infty$. ■
  - If x_n isn't bounded from below, then

    $$\liminf_{n\to\infty} x_n = -\infty$$

    Proof. Suppose $\{x_n\}$ is not bounded from below. Then for any $k \in \mathbb{N}$, we also have that $\{x_n : n \geq k\}$ is also not bounded from below. Thus $s_k = \inf\{x_n : n \geq k\} = -\infty$ for any k. As such, $\liminf_{n\to\infty} x_n = -\infty$. ■

  If $x > 1$, we have an unbounded strictly increasing sequence. In other words, $x_n \to \infty$ as $n \to \infty$. The above proof shows that $\limsup_{n\to\infty} x_n = \infty$.

- If $x < -1$, then we have that x_n isn't bounded from top or bottom. As such, we have that $\liminf_{n\to\infty} x_n = -\infty$ and $\limsup_{n\to\infty} x_n = \infty$.