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Vector Analysis

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Contents

Chapter 1

1.1 The Real Numbers

Definition 1.1.1: Partial Order

Let X be a set with a binary relation $\leqslant$. $\leqslant$ is a *partial order* if:

1. $x \leqslant x$ for all $x \in X$ (reflexivity)
2. $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$ for all $x, y, z \in X$ (transitivity)
3. $x \leqslant y$ and $y \leqslant x$ implies $x = y$ for all $x, y \in X$ (antisymmetry)

Definition 1.1.2: Partially Ordered Set (poset)

A set X with a partial order $\leqslant$ is called a *partially ordered set* or *poset*. It is notated as $(X, \leqslant)$.

Definition 1.1.3: Total Order

A partial order $\leqslant$ is a *total order* if for all $x, y \in X$, we have $x \leqslant y$ or $y \leqslant x$.

Example 1.1.1 (poset)

Let Y be a set. Define $X = \{\text{all subsets of } Y\} = \mathcal{P}(Y)$. Let $E, F \in Y$, we say that $E \leqslant F$ if $E \subseteq F$. Then $(X, \leqslant)$ is a poset. This is not a total order.

Definition 1.1.4: Upper Bound, Bounded Above, Supremum, Maximum

Let $(X, \leqslant)$ be a poset. Let $E \subseteq X$.

1. $y \in X$ is an *upper bound* of E if $x \leqslant y$ for all $x \in E$.
2. E is *bounded above* if it has at least one upper bound.
3. If E is nonempty and bounded above, then the *supremum*, if it exists, of E, denoted $\sup E$, is the least upper bound of E.
4. E has a *maximum* if there is $y \in E$ such that $x \leqslant y$ for all $x \in E$.

Properties worth mentioning:

1. If E has a maximum, then $\sup E$ exists and is equal to the maximum.

   Proof. Let y be the maximum of E. If $z \in X$, is an upper bound of E, then $z \geqslant y$ because $y \in E$. Since z was arbitrary, this is true for any upper bound. Thus, y is the least upper bound of E. 😃

Example 1.1.2

Let Y be a nonempty set, $(\mathcal{P}(Y), \leqslant)$ poset.
Fix nonempty $Z \subseteq Y$.

$$E = \{W \subseteq Y : W \subset Z\}$$

Trivially, Z is an upper bound of E. Realize that any superset of Z is an upper bound as well. We can postulate that the supremum of E is Z. We will now prove it:

Proof. Need to show that if F is an upper bound of E, then $F \supseteq Z$. If $x \in Z$, then $\{x\} \in E$ by definition of E, so $F \supseteq x$ for all $x \in Z$. Thus, $F \supseteq Z$. ☺

Note that there is no maximum of E.

Definition 1.1.5: Lower Bound, Bounded Below, Infimum, Minimum

Let $(X, \leqslant)$ be a poset. Let $E \subseteq X$.

1. $y \in X$ is a *lower bound* of E if $y \leqslant x$ for all $x \in E$.
2. E is *bounded below* if it has at least one lower bound.
3. If E is nonempty and bounded below, then the *infimum*, if it exists, of E, denoted $\inf E$, is the greatest lower bound of E.
4. E has a *minimum* if there is $y \in E$ such that $y \leqslant x$ for all $x \in E$.

Going back to example 1.1.2, we can see that E is bounded below by $\emptyset$. The infimum of E is $\emptyset$. The minimum of E is also $\emptyset$.

Definition 1.1.6: Complete

Let $(X, \leqslant)$ poset. X is *complete* if every nonempty subset of X that is bounded above has a supremum.

Example 1.1.3 ($\mathbb{Q}$)

$(\mathbb{Q}, \leqslant)$ is not complete.

Claim 1.1.1 $\mathbb{R}$

There is a complete ordered field $(\mathbb{R}, +, \cdot, \leqslant)$. Its elements are called real numbers.

1.2 First Recitation, 1/18

Exercise 1.2.1 Function Example

Let X be the set of all functions $f : D_f \to Z$ with $D_f \subseteq Y$. We say that $f \leqslant g$ if $D_f \subseteq D_g$ and $f(x) = g(x)$ for all $x \in D_f$. Is $(X, \leqslant)$ a poset? Is it complete?

Proof. To show that $(X, \leqslant)$ is complete, we need to show that every nonempty subset of X that is bounded above has a supremum. Let $E \subseteq X$ be nonempty and bounded above. Let $G = \bigcup_{f \in E} D_f$. G is the union of all the domains of the functions in E. G is bounded above by the union of the upper bounds of the domains of the functions in E. Let $H = \bigcup_{f \in E} f(D_f)$. H is bounded above by the union of the upper bounds of the ranges of the functions in E. Let $F : G \to H$ be defined as $F(x) = f(x)$ for all $x \in D_f$. F is the supremum of E. ☺

1.3 Natural Numbers

Exercise 1.3.1

Take $(X, +, \cdot, \leqslant)$ ordered field. Prove:

1. If $0 \leqslant x$, then $-x \leqslant 0$.
2. If $x \leqslant y$, and $0 \leqslant z \neq 0$, then $xz \leqslant yz$.
3. For all $x \in X$, $0 \leqslant x^2$.
4. Prove $0 < 1$.

Proof. Fields have the following important properties:

- If $a \leqslant b$, then $a + c \leqslant b + c$.
- If $a, b \geqslant 0$, then $ab \geqslant 0$.

1. Take the first property with $a = 0$, $b = x$, and $c = -x$. Then $0 \leqslant x \implies 0 + (-x) \leqslant x + (-x) \implies -x \leqslant 0$.
2. If $x \leqslant y$, then $0 \leqslant y + (-x)$. By the second property, $0 \leqslant z \cdot (y + (-x)) = zy + (-zx)$. Then $0 \leqslant zy + (-zx) \implies zx \leqslant zy$.
3. We split into the three trichotomy cases:
   - If $x = 0$, then $0 \leqslant 0^2$.
   - If $x < 0$ with $x \neq 0$, then $0 \leqslant -x$. By the second property, $0 \leqslant (-x)^2 = (-x)(-x) = x^2$.
   - If $x > 0$, then $0 \leqslant x$. By the second property, $0 \leqslant x^2$.
4. FSOC, assume $0 > 1$ and multiply both sides by 1. Then we get $0 \cdot 1 > 1 \cdot 1 \Rightarrow 0 > (1)^2$, which is a contradiction to the third property we proved.

😊

Definition 1.3.1: Inductive

Take $E \subseteq \mathbb{R}$. E is *inductive* if $1 \in E$ and $x \in E$ implies $x + 1 \in E$.

Example 1.3.1 (Inductive Sets)

- $\mathbb{R}$ is inductive.
- $\{x \in \mathbb{R} : 0 \leqslant x\}$

  Proof. $1 \in E$ because $1 \geqslant 0$. If $x \in E$, then $x + 1 \geqslant 0$, so $x + 1 \in E$. 😊

Definition 1.3.2: Natural Numbers

The intersection of all inductive sets is denoted $\mathbb{N}$. The elements of $\mathbb{N}$ are called *natural numbers*.

Properties of $\mathbb{N}$:

- $\mathbb{N} \neq \emptyset$. Since $1 \in$ every inductive set, $1 \in \mathbb{N}$.
- $\mathbb{N}$ is an inductive set.

Theorem 1.3.1 Induction

For every $n \in \mathbb{N}$, let $P(n)$ be a proposition such that:

1. $P(1)$ is true.
2. If $P(n)$, then $P(n+1)$.

Then $P(n)$ is true for every $n \in \mathbb{N}$

Proof. $E = \{n \in \mathbb{N} \ : \ P(n)\}$ is inductive by 1. and 2. So, $\mathbb{N} \subseteq E$, but $E \subseteq \mathbb{N}$ by definition of $\mathbb{N}$. Thus, $E = \mathbb{N}$. ☺

Theorem 1.3.2 Archimedean Property

Let $a, b \in \mathbb{R}$ with $a > 0$. Then there is $n \in \mathbb{N}$ such that $na > b$.

Proof. If $b \leqslant 0$, then we take $n = 1$. Assume $b > 0$. For sake of contradiction, assume there does not exist n such that $na > b$. Then $E = \{na : n \in \mathbb{N}\}$ is bounded above by b. Let $c = \sup E$. $c - a \leqslant c$, so $c - a$ is not an upper bound of E. Thus, there is $n \in \mathbb{N}$ such that $c - a \leqslant na$. Then $c \leqslant (n+1)a$. But c is an upper bound of E, so $c \geqslant (n+1)a$. Thus, $c = (n+1)a$. But $c \in E$, so $c = na$ for some $n \in \mathbb{N}$. Thus, $na = (n+1)a$, so $n = n+1$, which is a contradiction. ☺

Definition 1.3.3: Integers

$\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$

Theorem 1.3.3 Integer Part

For every $x \in \mathbb{R}$, there is a unique $k \in \mathbb{Z}$ such that $k \leqslant x < k+1$.

Definition 1.3.4: Integer Part

The k that satisfies the above theorem is called the *integer part* of x, denoted $\lfloor x \rfloor$.

Proof. Let $E = \{k \in \mathbb{Z} : k \leqslant x\}$. First we show that E is nonempty.

- If $x \geqslant 0$, then $0 \in E$, so E is nonempty.
- If $x < 0$, then $-x > 0$. By the Archimedean property, there is $n \in \mathbb{N}$ such that $n > -x$. Thus, $-n < x$. So, $-n \in E$, so E is nonempty.

Now we show that E is bounded from above. Very clearly, x is an upper bound. By supremum property, there is $L = \sup(E)$ and $L \in \mathbb{R}$. $L - 1$ is not an upper bound, which means that there is an element $k \in E$ such that $L - 1 < k$. But since L is the supremum, $L \geqslant k$. Thus, $L - 1 < k \leqslant L$. So, $L < k + 1$ so $k + 1 \notin E$. Now, $k \leqslant x$ since $k \in E$. Now we show that k is unique. Assume there is $m \in \mathbb{Z}$ such that $m \leqslant x < m+1$. Then $m \in E$, so $m \leqslant L$. But L is the supremum, so $L \geqslant m$. Thus, $L = m$. So, $k = m$. ☺

Definition 1.3.5: $\mathbb{Q}$

If $p \in \mathbb{Z}$ with $p \neq 0$, then $\exists p^{-1} \in \mathbb{R}$. Define $\mathbb{Q} = \{pq^{-1} : p, q \in \mathbb{Z}, p \neq 0\}$.

1.4 Density of Rationals

Theorem 1.4.1 Density of the Rationals

Let $a, b \in \mathbb{R}$ with $a < b$. Then there is $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. We have $a < b \implies 0 = a + (-a) < b - a \implies 0 < \frac{1}{b-a}$. By the integer part theorem, there is $q \in \mathbb{Z}$ such that $\frac{1}{b-a} < q$. So now, $\frac{1}{q} < b - a \implies a < a + \frac{1}{q} < b$. Multiply both sides by $q > 0$ to get $aq < a + 1 < bq$. By the integer part theorem, there is $p \in \mathbb{Z}$ such that $p \leqslant qa < p + 1$ (i.e. $p = \lfloor qa \rfloor$). Since $qa < p + 1 \leqslant qa + 1 < qb$. Getting rid of unnecessary stuff, we have $qa < p + 1 < qb$. Thus, $a < \frac{p+1}{q} < b$. Let $r = \frac{p+1}{q}$. Then $r \in \mathbb{Q}$ and $a < r < b$. 😊

Definition 1.4.1: Irrational Numbers

$\mathbb{R} \setminus \mathbb{Q}$ is the set of *irrational numbers*.

Exercise 1.4.1 TODO in Recitation 1/23

- Prove that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$.
- Prove that "$\sqrt{2}$" exists in $\mathbb{R}$. (prove that there is at least one irrational number)
  - Have to play with the set $E = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$.

Theorem 1.4.2 Density of Irrationals

Let $a, b \in \mathbb{R}$ with $a < b$. Then there is $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Proof. $a < b \implies a\sqrt{2} < b\sqrt{2}$. By the density of rationals, there is $r \in \mathbb{Q}$ such that $a\sqrt{2} < r < b\sqrt{2}$. Then $a < \frac{r}{\sqrt{2}} < b$. Let $x = \frac{r}{\sqrt{2}}$. If $r = 0$, then $a\sqrt{2} < 0 < b\sqrt{2}$. By previous theorem, we can find $q \in \mathbb{Q}$ such that $a\sqrt{2} < q < 0 < b\sqrt{2}$. Then $a < \frac{q}{\sqrt{2}} < b$. Let $x = \frac{q}{\sqrt{2}}$. Then $x \in \mathbb{R} \setminus \mathbb{Q}$ and $a < x < b$. 😊

Note:

Take $x \in \mathbb{R}$, $E = \{r \in \mathbb{Q} : r < x\}$. x is the upper bound of E. This set is nonempty because we can take $x - 1 < r < x$. Now we prove that $x = \sup E$.

Proof. Assume $\exists L$ upper bound of E such that $L < x$. Then $L < x \implies$ there exists some $r \in \mathbb{Q}$ such that $L < r < x$, but $r \in E$, so L is not an upper bound of E. Thus, L cannot be an upper bound of E and x is the least upper bound of E. 😊

Since now we know that $\sqrt{2} = \sup\{r \in \mathbb{Q} : r < \sqrt{2}\}$, we can also define $3^{\sqrt{2}} = \sup\{3^r : r \in \mathbb{Q}, r < \sqrt{2}\}$.

Definition 1.4.2: x^0

Let $0 \neq x \in \mathbb{R}$. We define $x^0 = 1$.

Definition 1.4.3: x^n

Let $x \in \mathbb{R}$, $n \in \mathbb{N}$. We start with $x^1 := x$. Then assume x^m has been defined. Then we say $x^{m+1} := x^m \cdot x$.

Definition 1.4.4: $x^{p/m}$

Let $x \in \mathbb{R}$, $p \in \mathbb{Z}$, $m \in \mathbb{N}$. We say $x^{p/m} = \sqrt[m]{x^p}$.

Exercise 1.4.2 Properties of Exponenets

Let $x \in \mathbb{R}$, $r, q \in \mathbb{Q}$, and $x, r, q > 0$. Prove the following:

- $x^r \cdot x^q = x^{r+q}$
- $(x^r)^q = (x^q)^r = x^{rq}$

Proof. ☺

Definition 1.4.5: Negative Exponent

Take $x > 0$, $r = -\frac{p}{m}$ for $p, m \in \mathbb{N}$. First, we have that $x^{-r} := (x^{-1})^{p/m}$.

Exercise 1.4.3 More Properties of Exponents

Take $x \in \mathbb{R}, x > 0, r, q \in \mathbb{Q}$. Prove the following:

- If $r > 0$, prove that $x^r > 1$.
- If $r < q$, prove that $x^r < x^q$.

1.5 1/23 - Recitation - Proving Irrationality of $\sqrt{2}$

Existence of $\sqrt{2}$:

1. Let $E = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$. Prove that E is non-empty and that E is bounded above.

   Proof. We know that $0 < 1$ and from that we get $1^2 = 1 < 2$, which can be checked by subtracting 1 from both sides. As such E is nonempty.

   Now we show that E is bounded above. We know that $2^2 = 4 > 2 > a^2 \in E$, so $2^2 > a^2 \Rightarrow 2 > a$, so 2 is an upper bound of E. ☺

2. By the completeness of $(\mathbb{R}, \leqslant)$, E has a supremum, L. Prove that $L > 0$ and that $L^2 = 2$.

   Proof. Since L is the least upper bound, it has to be greater than 1 which is in the set E. Therefore, $L > 1 > 0 \implies L > 0$.

   Now we show that $L^2 \geqslant 2$. For sake of contradiction, assume $L^2 < 2$. Since $L > 0$, this means that $L \in E$. By the density of rationals, there exists $r \in \mathbb{Q}$ such that $L < r < \sqrt{2}$. Since L is an upper bound of E, $r \notin E$. But $r \in \mathbb{Q}$, so $r^2 \neq 2$. Thus, $r^2 > 2$. Since $r > 0$, $r^2 > 2 \implies r > \sqrt{2}$. But $r < \sqrt{2}$, so we have a contradiction. Thus, $L^2 \geqslant 2$. ☺

3. Prove that if $y \in \mathbb{R} \setminus E$ and $y > 0$, then y is an upper bound of E.

   Proof. Assume $y \in \mathbb{R} \setminus E$ and $y > 0$. We need to show that y is an upper bound of E. Assume for sake of contradiction that y is not an upper bound of E. Then there exists $x \in E$ such that $x > y$. But $x \in E \implies x^2 < 2$. Since $y > 0$, $x^2 < 2 \implies y^2 < 2$. But $y \notin E$, so $y^2 \geqslant 2$. But this would mean that $y \in E$. Contradiction. Thus, y is an upper bound of E. ☺

4. Prove that $L^2 = 2$.

Proof. We know that $L^2 \geqslant 2$ from part 2. Now we show that $L^2 \leqslant 2$. Assume for sake of contradiction that $L^2 > 2$.

How small does $\epsilon > 0$ need to be such that $(L - \epsilon)^2 > 2$ as well.

Start with $(L - \epsilon)^2 = L^2 - 2L\epsilon + \epsilon^2$, which is greater than $L^2 - 2L\epsilon$ since $\epsilon > 0$. So now, how small does ϵ need to be such that $L^2 > 2 \implies L^2 - 2L\epsilon > 2$ too.

$$
\begin{aligned}
2L\epsilon &< 2 - L^2 \\
\epsilon &< \frac{2 - L^2}{2L}
\end{aligned}
$$

Since $L^2 > 2$, this means that an ϵ can be found. This means that L is not the least upper bound. Contradiction. Thus, $L^2 \leqslant 2$. 😄

1.6 Exponents

Definition 1.6.1: $\sqrt{2}$

$$\sqrt{2} := \sup\{x \in \mathbb{R} : x > 0, x^2 < 2\}$$

Exercise 1.6.1

For $n \in \mathbb{N}, n \geqslant 2$. Fix $x > 0$.

$$E = \{y \in \mathbb{R} : y > 0, y^n < x\}.$$

Prove that $l = \sup E$ satisfies $l^n = x$.

Proof. We first need to show that $\sup E$ exists. Let $y = x/(1+x)$. Then, $0 \leqslant y < 1$, so $y^n \leqslant y < x$. Thus, $y \in E$. So, E is nonempty. E is also bounded from above because x is an upper bound of E. Thus, $\sup E$ exists by the completeness of $\mathbb{R}$. Let $l = \sup E$. We now show that $l^n = x$.

First we show that $l^n \leqslant x$. FSOC, assume $l^n > x$. If you choose an $\epsilon > 0$ that is small enough, then $(l-\epsilon)^n > x$ as well. We can't do this because $y > l - \epsilon$ for some $y \in E$ since l is the supremum of E. As such, we arrive at a contradiction which means that $l^n \leqslant x$.

To show that $l^n \geqslant x$, assume FSOC that $l^n < x$. Then we can choose an ϵ such that $(l+\epsilon)^n < x$, meaning we have an element $(l+\epsilon)$ which is in E but bigger than the supremum, which is a contradiction.

Thus, $l^n \geqslant x$. 😆

Definition 1.6.2: $\sqrt[m]{x}$

$$\sqrt[m]{x} := \sup\{y \in \mathbb{R} : y > 0, y^m < x\}$$

Definition 1.6.3: $x^{p/q}$

$$x^{p/q} := \left(\sqrt[q]{x}\right)^p$$

Definition 1.6.4: x^q

For $q \in \mathbb{R}$, $q > 0$, and $x > 1$.

$$x^q := \sup\{x^r : r \in \mathbb{Q}, 0 < r < q\}$$

Example 1.6.1

$$\sqrt{2} = \sup\{r \in \mathbb{Q} : r > 0, r < \sqrt{2}\}$$

Theorem 1.6.1

Take $a, b \in \mathbb{R}$, $a, b > 0$ and $x \in \mathbb{R} > 1$. Then $x^a \cdot x^b = x^{a+b}$.

Proof. Let $E_i = \{x^r : r \in \mathbb{Q}, r > 0, r < i\}$. Consider E_a, E_b, E_{a+b}. Then let $l_i = \sup(E_i)$. Consider l_a, l_b, l_{a+b}. We want to show that $l_a \cdot l_b = l_{a+b}$ by showing that both $l_a \cdot l_b \leqslant l_{a+b}$ and $l_a \cdot l_b \geqslant l_{a+b}$.

Let $r \in \mathbb{Q}$ with $0 < r < a$. Let $s \in \mathbb{Q}$ with $0 < s < b$. Then we have that $x^r \cdot x^s = x^{r+s}$ (from the exercise two days ago and since $r, s \in \mathbb{Q}$.) we know that $0 < r + s < a + b$ and is rational. Thus, $x^{r+s} \in E_{a+b}$. Thus, $x^r \cdot x^s \leqslant l_{a+b}$.

We want to divide both sides by x^s while fixing r. So, we have that $x^r \leqslant \frac{l_{a+b}}{x^s}$, which is true for all $r \in \mathbb{Q}$, such that $0 < r < a$. Thus, $\frac{l_{a+b}}{x^s}$ is an upper bound for E_a. Thus, $l_a \leqslant \frac{l_{a+b}}{x^s}$. Thus, $x^s \leqslant \frac{l_{a+b}}{l_a}$, meaning that $\frac{l_{a+b}}{l_a}$ is an upper bound for E_b. Thus, $l_b \leqslant \frac{l_{a+b}}{l_a}$. Thus, $l_a \cdot l_b \leqslant l_{a+b}$.

Now we show that $l_a \cdot l_b \geqslant l_{a+b}$. Let $t \in \mathbb{Q}$ with $0 < t < a + b$. We need $0 < r \in \mathbb{Q} < a$ and $0 < s \in \mathbb{Q} < b$ with $t = r + s$. We start by looking at $t - a < b$. By the density of $\mathbb{Q}$, find $s \in \mathbb{Q}$ such that $t - a < s < b$. Take $s > 0$ beacuse $b > 0$. So $t - s < a$. By the density of $\mathbb{Q}$, find $0 < p \in \mathbb{Q}$ such that $t - s < p < a$. So $t < s + p$. So, $x^t < x^{s+p} = x^s x^p \leqslant l_a l_b$ since $x^s \in E_b$ and $x^p \in E_a$. We know that $l_a l_b$ is an upper bound of E_{a+b}, so $l_{a+b} \leqslant l_a l_b$.

Therefore $l_a \cdot l_b = l_{a+b}$. 😄

Definition 1.6.5: Negative Exponents

Let $x > 1$, $a < 0$. Then:
$$x^a := (x^{-a})^{-1}$$

Definition 1.6.6: Exponents between 0 and 1

Let $x \in \mathbb{R}$ with $0 < x < 1$ and $a > 0$. Then:
$$x^a := \left(\frac{1}{x}\right)^{-a}$$

An important note is that if we have $E \subseteq (0, \infty)$ with a bounded E. Then if we define $F = \{\frac{1}{x} : x \in E\}$, then we have the following:

$$\sup E = \frac{1}{\inf F}$$
$$\inf E = \frac{1}{\sup F}$$

1.7 1/25 - Recitation - Sequences of Set

Definition 1.7.1: Sequence of a Set

Given a set X, a sequence on X is a function $f : \mathbb{N} \to X$. We denote $f(n)$ as x_n. We can also denote the sequence as $\{x_n\}_{n=1}^{\infty}$.

Definition 1.7.2

Let $(X, \leqslant)$ be a poset and $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Then $E = \{x_n : n \in \mathbb{N}\}$ is a subset of X. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded from above is the set E is bounded from above. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded from below is the set E is bounded from below. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded if it is bounded from above and below.

Definition 1.7.3: Limit Superior

Let $(X, \leqslant)$ be a poset. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Suppose $\{x_n\}_n$ is bounded from above. Then, we define the *limit superior* of x_n as $n \to \infty$ as:

$$\limsup_{n \to \infty} x_n = \inf_{n \in \mathbb{N}} \sup_{k \geqslant n} x_k$$

Definition 1.7.4: Limit Inferior

Let $(X, \leqslant)$ be a poset. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Suppose $\{x_n\}_n$ is bounded from below. Then, we define the *limit inferior* of x_n as $n \to \infty$ as:

$$\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} x_k$$

Exercise 1.7.1

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on $\mathbb{R}$ bounded above. Prove that $L \in \mathbb{R}$ is the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$ iff for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:

   (a) $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$.

   (b) $L - \epsilon < x_n$ for infinitely many n.

Proof. Let $L \in \mathbb{R}$ be the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$. Let $\epsilon > 0$. L being the lim sup means that $L = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} x_k$. Thus, $L \leqslant \sup_{k\geqslant n} x_k$ for all $n \in \mathbb{N}$. Thus, $L - \epsilon < \sup_{k\geqslant n} x_k$ for all $n \in \mathbb{N}$. Then $L - \epsilon$ is not an upper bound of $\{x_n\}_{n=1}^{\infty}$. Thus, there is $n_\epsilon \in \mathbb{N}$ such that $L - \epsilon < x_{n_\epsilon}$. Thus, $L - \epsilon < x_n$ for infinitely many n. Now we show that $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$. Assume for sake of contradiction that there is $n \geqslant n_\epsilon$ such that $x_n \geqslant L + \epsilon$. Then $L + \epsilon$ is an upper bound of $\{x_n\}_{n=1}^{\infty}$. But L is the $\limsup$, so $L \geqslant L + \epsilon$. Contradiction. Thus, $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$.

Now we show the other direction. Assume that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$ and $L - \epsilon < x_n$ for infinitely many n. We want to show that L is the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$. We know that L is an upper bound of $\{x_n\}_{n=1}^{\infty}$. We need to show that L is the least upper bound. Assume for sake of contradiction that L is not the least upper bound. Then there is $L' < L$ such that L' is an upper bound of $\{x_n\}_{n=1}^{\infty}$. Let $\epsilon = L - L'$. Then $L' < L - \epsilon$. But $L - \epsilon < x_n$ for infinitely many n. But $L' < L - \epsilon$, so L' is not an upper bound of $\{x_n\}_{n=1}^{\infty}$. Contradiction. 😃

1.8 Vector Spaces

Example 1.8.1 (Vector Spaces)

- Euclidean Space $\subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ is a vector. $x = (x_1, \dots, x_n)$.
- Polynomial Space from $\mathbb{R} \to \mathbb{R}$. $x \in \mathbb{R}^x$. $x = a_0 + a_1x + \cdots + a_nx^n$.
- $f : [a, b] \to \mathbb{R}$ continuous functions.

Definition 1.8.1: Boundedness of Functions

Let E be a set and $f : E \to \mathbb{R}$.

1. f is bounded from above if the set $f(E) = \{y \in \mathbb{R} : y = f(x), x \in E\}$ is bounded from above.
2. f is bounded from below if the set $f(E) = \{y \in \mathbb{R} : y = f(x), x \in E\}$ is bounded from below.
3. f is bounded if $f(E)$ is bounded.

Definition 1.8.2: Inner Product

A function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is an *inner product* if it satisfies the following properties:

- $(x, x) \geqslant 0$ for all $x \in X$.
- $(x, x) = 0$ iff $x = 0$.
- $(x, y) = (y, x)$ for all $x, y \in X$.
- $(sx + ty, z) = s(x, z) + t(y, z)$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}$.

Example 1.8.2 (Examples of Inner Products)

- $\mathbb{R}^n$ with dot products.
- $f : [a,b] \to \mathbb{R}$ with $(f,g) = \int_a^b f(x)g(x)dx$. This is is not an inner product because we can define:

$$f = \begin{cases} 1 & x = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

which has an integral of 0. But $f \neq 0$. If we add that f is continuous, then it is an inner product.

Definition 1.8.3: Norm

Let V be a vector space with an inner product $(\cdot,\cdot)$. Then the *norm* of $x \in X$ is defined as $||\cdot|| : X \to [0,\infty)$ such that:

1. $||x|| = 0 \iff x = 0$
2. $||tx|| = |t|||x||$ for all $x \in X$
3. $||x + y|| \leqslant ||x|| + ||y||$ for all $x, y \in X$

Example 1.8.3 (Examples of Norms)

- $||x|| = \sqrt{(x,x)}$ for $x \in \mathbb{R}^n$
- $X = \{f : E \to \mathbb{R}, f \text{ bounded}\}$. $||f|| = \sup_{x\in E} |f(x)|$.
  - First property is obviously true.
  - For the second property, we use the fact that

$$\sup(tF) = \begin{cases} t\sup(F) & \text{if } t \geqslant 0 \\ t\inf(F) & \text{if } t < 0 \end{cases}$$

  - For the third property, we use the triangle inequality:

$$\sup|f+g| \leqslant \sup|f| + \sup|g|$$
$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant \sup|f| + \sup|g|$$

Note:
Space of bounded functions denoted as $\ell^\infty(E) = \{f : E \to \mathbb{R} : f \text{ bounded}\}$.

Theorem 1.8.1 Cauchy Schwarz Inequality

Let X be a vector space with an inner product $(\cdot,\cdot)$. Then for all $x, y \in X$, we have that $|(x,y)| \leqslant \sqrt{(x,x)} \cdot \sqrt{(y,y)}$.

Proof. Let $y \neq 0$. Consider $(x+ty, x+ty) = (x, x+ty) + t(y, x+ty) = (x,x) + t(x,y) + t(y,x) + t^2(y,y)$. We can combine the middle terms to get $t^2(y,y)+2(x,y)+(x,x)$, which is quadratic in t. Take $t=-\frac{(x,y)}{(y,y)}$.

$$\begin{aligned}0 &\leqslant (x,x)-2\frac{(x,x)^2}{(y,y)}+\frac{(x,y)^2}{(y,y)}\\ 0 &\leqslant (x,x)(y,y)-2(x,y)^2+(x,y)^2\\ 0 &\leqslant (x,x)(y,y)-(x,y)^2\\ (x,y)^2 &\leqslant (x,x)(y,y)\\ |(x,y)| &\leqslant \sqrt{(x,x)}\cdot\sqrt{(y,y)}\end{aligned}$$

😄

1.9 Inner Products, Norms, and Metric Spaces

Theorem 1.9.1

Let X be a vector space with an inner product $(\cdot,\cdot)$. Then $||x|| := \sqrt{(x,x)}$ is a norm.

Proof. We check the properties of norms:

1. $||x|| = 0 \iff \sqrt{(x,x)} = 0 \iff (x,x) = 0 \iff x = 0$.
2. $||tx|| = \sqrt{(tx,tx)} = \sqrt{t^2(x,x)} = |t|\sqrt{(x,x)} = |t|||x||$.
3. $||x+y||^2 = (x+y,x+y) = (x,x)+2(x,y)+(y,y) = ||x||^2+2(x,y)+||y||^2 \leqslant ||x||^2+2|(x,y)|+||y||^2 \leqslant ||x||^2+2||x||\cdot||y||+||y||^2 = (||x||+||y||)^2$.

😄

Corollary 1.9.1 Parallelogram Identity

Let X be a vector space with inner product $(\cdot,\cdot)$. Then for all $x,y\in X$, we have that

$$||x+y||^2+||x-y||^2 = 2||x||^2+2||y||^2$$

Proof.

$$\begin{aligned}||x+y||^2+||x-y||^2 &= (x+y,x+y)+(x-y,x-y)\\ &= (x,x)+2(x,y)+(y,y)+(x,x)-2(x,y)+(y,y)\\ &= 2(x,x)+2(y,y)\\ &= 2||x||^2+2||y||^2\end{aligned}$$

😄

If we subtract them instead, we get

$$\frac{||x+y||^2-||x-y||^2}{4} = (x,y) \tag{*}$$

So, if $||\cdot||$ is a norm, then if i want to define an inner product, I can use *.

Exercise 1.9.1

Let $||\cdot||$ be a norm. Then $(x,y) := \frac{1}{4}(||x+y||^2-||x-y||^2)$ is an inner product iif the parallelogram identity holds.

Linearity of inner products is the hard part to prove because we have to consider:

- $t \in \mathbb{N}$
- $t = \frac{1}{2}$
- $t \in \mathbb{Q}$
- $t \in \mathbb{R}$ (density of $\mathbb{Q}$)

Note:

For recitation:

1. $X = \{f : E \to \mathbb{R} \text{ bounded}\}, ||f|| = \sup_E |f|$, does not satisfy the parallelogram identity.
2. $x \in \mathbb{R}^N$, $||x||_1 = |x_1| + |x_2| + \cdots + |x_N|$ does not satisfy the parallelogram identity.

Definition 1.9.1: Metric

Let X be a set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$

Definition 1.9.2: Metric Space

A set X with a metric d is called a *metric space* and is denoted as (X, d).

Example 1.9.1 (Metrics)

Let X be a set. Then the following is a metric on X:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Theorem 1.9.2 If X is a vector space with $||\cdot||$ as a norm. Then

$$d(x, y) := ||x - y||$$

is a metric on X.

Proof. We check all the properties of metrics.

- $d(x, y) = 0 = ||x - y|| \Rightarrow 0 = x - y \iff x = y$.
- $d(x, y) = ||x - y|| = ||y - x|| = d(y, x)$.
- $d(x, y) = ||x - y|| = ||x - z + z - y|| \leqslant ||x - z|| + ||z - y|| = d(x, z) + d(z, y)$.

😄

Example 1.9.2
Let's define

$$d(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$$

as a metric on $\mathbb{R}$. However, this is not a norm because $d(tx,ty) \neq td(x,y)$.

Definition 1.9.3: Ball

Let (X,d) be a metric space. Let $x \in X$ and $r > 0$. Then the *ball* of radius r centered at x is defined as $B_r(x) = \{y \in X : d(x,y) < r\}$.

Example 1.9.3

- Take $X = \mathbb{R}^2$ with $(x,y) \in \mathbb{R}$. Then defin $||(x,y)||_\infty = \max(|x|,|y|)$ is a norm. Take $B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : ||(x,y)-(0,0)||_\infty < 1\}$. This is a square with vertices $(1,1),(-1,1),(-1,-1),(1,-1)$.
- If we have $||(x,y)||_1 = |x|+|y|$, then $B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : ||(x,y)-(0,0)||_1 < 1\}$. This is a square with vertices $(1,0),(0,1),(-1,0),(0,-1)$.

Definition 1.9.4: Interior

Let (X,d) be a metric space and $E \subseteq X$. $x \in E$ is called an *interior point* of E if there is $B(x,r) \subseteq E$. The set of all interior points of E is called the *interior* of E and is denoted as E°.

Definition 1.9.5: Open Set

E is *open* if $E = E^\circ$.

1.10 Open Sets

Example 1.10.1 (Balls)
$B(x,r)$ is open.

Proof. Let $y \in B(x,r)$ and take $B(y, r-d(x,y))$. Let $z \in B(y,r-d(x,y))$. Then $d(x,z) \leqslant d(x,y)+d(y,z) < d(x,y)+r-d(x,y) = r$. Thus, $z \in B(x,r)$. Thus, $B(y,r-d(x,y)) \subseteq B(x,r)$. Thus, $B(x,r)$ is open. 😊

Example 1.10.2 ($\mathbb{R}$)

1. $E = (0,1) \cap \mathbb{Q}$ is not open. Because the irrationals are dense, we can always find a rational number in any ball. Thus, $E^\circ = \emptyset$.
2. $E = (3,4)$ is open. Let $x \in E$. Take $B(x, \min(x-3, 4-x))$. Then $B(x,\min(x-3,4-x)) \subseteq E$. Thus, E is open.
3. $E = [3,4)$ is not open. $E^\circ = (3,4)$.
4. $E = \{x \in \mathbb{R} : x^3 - 3x + 4 > 0\}$. This is open and we'll be able to use continuity to prove this easily later.
5. $l^\infty([0,1]) = \{f : [0,1] \to \mathbb{R} \text{ bounded}\}$. $||f||_\infty = \sup_{[0,1]} |f|$. $d(f,g) = ||f-g||_\infty$. $E = \{f \in l^\infty([0,1]) : f(x) > 0 \ \forall x \in [0,1]\}$ is open? (finish in recitation)

Properties of open sets (X, d):

- $\emptyset$ is open. X is open.
- Infinite intersections of open sets are not necessarily open. For example, we have $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is not open.
- Finite intersections of open sets are open. Consider $U_1, \ldots U_n$. Let $x \in \bigcap_{i=1}^{n} U_i$. Then $x \in U_i$ for all i. Since U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Let $r = \min(r_1, \ldots, r_n)$. Then $B(x, r) \subseteq U_i$ for all i. Thus, $B(x, r) \subseteq \bigcap_{i=1}^{n} U_i$.
- Unions of open sets are open because if a point in the union is contained in one of the open sets, then there is a ball in that set that is contained in the union.

Definition 1.10.1: Topological Space

Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset, X \in \mathcal{T}$.
2. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$. (finite intersections)
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$. (arbitrary unions)

Elements of $\mathcal{T}$ are called open sets.

Definition 1.10.2: Closed

Let (X, d) be a metric space. We say $C \subseteq X$ is *closed* if $X \setminus C$ is open.

Note that X and $\emptyset$ are both open and closed.

Example 1.10.3 (Open and Closed Sets)

- $[0, 1)$ is not open or closed.
- $[0, 1]$ is closed.

Properties of closed sets:

- $\emptyset$ and X are closed.
- Infinite intersections of closed sets are closed. (De Morgan's Law)
- Finite unions of closed sets are closed. For example, if we have $\bigcup_{m=1}^{\infty}(-\infty, -\frac{1}{m}) = (-\infty, 0)$ which is closed.

1.11 2/1 - Rectitation

Recall:

1. Let $\{x_n\}$ be a sequence bounded above in $\mathbb{R}$. Then $L \in \mathbb{R}$ is the limit superior of $\{x_n\}$ if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
   (a) $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$.
   (b) $x_n > L - \epsilon$ for infinitely many n.
2. Let $\{x_n\}$ be a sequence bounded below in $\mathbb{R}$. Then $L \in \mathbb{R}$ is the limit inferior of $\{x_n\}$ if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:
   (a) $x_n < L + \epsilon$ for infinitely many n.
   (b) $x_n > L - \epsilon$ for all $n \geqslant n_\epsilon$.

Now consider the following sequence:

$$x_n = (-1)^n \frac{2n}{n+1} \in \mathbb{R}$$

Prove that $\limsup_{n\to\infty} x_n = 2$.

Proof. We need to show that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:

1. $x_n < 2 + \epsilon$ for all $n \geqslant n_\epsilon$.
2. $2 - \epsilon < x_n$ for infinitely many n.

Let $\epsilon > 0$. We need to find $n_\epsilon \in \mathbb{N}$ such that $x_n < 2 + \epsilon$ for all $n \geqslant n_\epsilon$ and $2 - \epsilon < x_n$ for infinitely many n. We can find $n_\epsilon \in \mathbb{N}$ such that $2 - \epsilon < x_n$ for all $n \geqslant n_\epsilon$. Then $x_n < 2 + \epsilon$ for all $n \geqslant n_\epsilon$. Thus, $\limsup_{n\to\infty} x_n = 2$. 😄

Now prove that for any $\{x_n\}$ in $\mathbb{R}$, prove that $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$.

Proof. Comes quickly from properties of limits and that the inf is less than the sup. 😄

Now prove that $\liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n$ and that $\limsup_{n\to\infty} -x_n = -\liminf_{n\to\infty} x_n$.

Proof. We start by using the property that $\inf(-E) = -\sup(E)$. Then we use the property that $\sup(-E) = -\inf(E)$. So,

$$\begin{aligned}
\liminf_{n\to\infty} -x_n &= \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} -x_k \\
&= \sup_{n\in\mathbb{N}} -\sup_{k\geqslant n} x_k \\
&= -\inf_{n\in\mathbb{N}} \sup_{k\geqslant n} x_k \\
&= -\limsup_{n\to\infty} x_n
\end{aligned}$$

😄

1.12 Closure

Definition 1.12.1: Closure

Let (X,d) be a metric space with $A \subset X$. Then the *closure* of A is defined as $\overline{A}$, the intersection of all sets that contain E.

Definition 1.12.2: Boundary Point

Let (X,d) be a metric space with $E \subseteq X$. Then $x \in X$ is a *boundary point* of E if for every $r > 0$, $B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X \setminus E) \neq \emptyset$. The set of all boundary points is denoted as ∂E.

Theorem 1.12.1

Let (X,d) be a metric space and $E \subseteq X$. Then $\overline{E} = E \cup \partial E$.

Proof. Let $x \in \overline{E}$. FSOC, assume $x \notin E \cup \partial E$. Since $x \notin \partial E$, there exists $r > 0$ such that $B(x,r)$ that doesn't intersect with either E or complement of E. But since $x \notin E$, only the second option can occur. So there exists r such that $B(x,r) \cap E = \emptyset$. Because of that and the fact that $B(x,r)$ is open, it follows that $X \setminus B(x,r)$ is closed and contains E. By the definition of $\overline{E}$, we have that $\overline{E} \subseteq X \setminus B(x,r)$. But this is a contradiction because $x \in \overline{E}$.

Conversely, let $x \in E \cup \partial E$ and assume $x \notin \overline{E}$. Since $\overline{E}$ is closed, $X \setminus \overline{E}$ is open. Using the fact that $x \in E \cup \partial E$, we have that we can find a $B(x,r) \in X \setminus \overline{E}$. But this is a contradiction because $B(x,r)$ is open and contains E. Thus, $E \cup \partial E \subseteq \overline{E}$. 😄

Definition 1.12.3: Accumulation Point

Let (X,d) be a metric space with $E \subseteq X$. Then $x \in X$ is an *accumulation point* of E if for every $r > 0$, there exists $y \in E$ such that $y \neq x$ and $d(x,y) < r$.

Definition 1.12.4: Interval

$I \subseteq \mathbb{R}$ is an *interval* if we have that $z \in I$ for all $x < z < y$.

Definition 1.12.5: Rectangle

$R \subseteq \mathbb{R}^N$ is a *rectangle* if $R = I_1 \times \cdots \times I_N$ where $I_1, \ldots, I_N$ are intervals in $\mathbb{R}$.

Definition 1.12.6: Sequence

Let X be a set. A *sequence* is a function $f : \mathbb{N} \to X$. We denote $f(n)$ as x_n.

Definition 1.12.7: Convergent Sequence

Let (X,d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ is *convergent* if there exists $x \in X$ such that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $d(x,x_n) < \epsilon$ for all $n \geqslant n_\epsilon$. We write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

1.13 Bolzano-Weierstrass

Theorem 1.13.1 Bolzano-Weierstrauss

If $E \subset \mathbb{R}^N$ is bounded and contains infinitely many distinct points, then E has an accumulation point

Proof.

Lenma 1.13.1 1

If $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for all n, then $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

Proof. For all a_n and b_n, we have:

$$a_1 \leqslant a_2 \leqslant \cdots$$
$$b_1 \geqslant b_2 \geqslant \cdots$$

Let

$$A := \{a_1, a_2, \ldots\}.$$

We have that $a_n \leqslant b_n \leqslant b_1$ for all n. So A is bounded above, so by the supremum property, there exists $x = \sup A \in \mathbb{R}$ and $a_n \leqslant x$ for all $n \in \mathbb{N}$. We claim that $x \leqslant b_n$ as well. If not, then there exists $m \in \mathbb{N}$ such that $b_m < x$. Since x is an upper bound of A, we'll have that there's an $n \in \mathbb{N}$ such that $b_m < a_n \leqslant x$. Find $k \geqslant m, n$, then we have $b_m < a_n \leqslant a_k \leqslant b_k \leqslant b_m$, which is a contradiction. This proves the claim. Hence, $x \in [a_n, b_n]$ for all n. Thus, $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$. 😊

Lenma 1.13.2 2

Let R_n be a closed and bounded rectangle. Assume that $R_1 \supseteq R_2 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} R_n \neq \emptyset$.

Proof. We know that

$$R_n = [a_{1,n}, b_{1,n}] \times \cdots \times [a_{N,n}, b_{N,n}]$$
$$R_{n+1} = [a_{1,n+1}, b_{1,n+1}] \times \cdots \times [a_{N,n+1}, b_{N,n+1}]$$

We can apply lemma 1 N times (for each of the components of R_n) to find that $x_1, x_2, \ldots, x_N \in \mathbb{R}$ such that $a_{i,n} \leqslant x_i \leqslant b_{i,n}$ for all $1 \leqslant i \leqslant N$. Then, if you take $x = (x_1, \ldots, x_N)$, then $x \in R_m$ for all n. Thus, $x \in \bigcap_{n=1}^{\infty} R_n$. 😊

Lenma 1.13.3 3

Let (X, d) be a metric space with $E \subseteq X$. Then $x \in X$ is an accumulation point of E if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in E such that $x_n \to x$ as $n \to \infty$.

Proof. Let $x \in X$ be an accumulation point of E. Take $r = \dfrac{1}{n}$. Find $x_n \in B\left(x, \dfrac{1}{n}\right) \cap E$ with $x_n \neq x$. We claim $x_n \to x$. Given $\epsilon > 0$, find $n_\epsilon \geqslant \dfrac{1}{\epsilon}$. Then $d(x, x_n) < \dfrac{1}{n} \leqslant \dfrac{1}{n_\epsilon}$ for all $n \geqslant n_\epsilon$. Thus, $x_n \to x$ as $n \to \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in E such that $x_n \to x$ as $n \to \infty$. We claim that $x \in \operatorname{acc}(E)$. Let $r > 0$ and take $\epsilon = r$. Then there exists $n_\epsilon \in \mathbb{N}$ such that $d(x, x_n) < \epsilon = r$ for all $n \geqslant n_\epsilon$. Thus, $x_n \in B(x, r) \cap E$ for all $n \geqslant n_\epsilon$. Thus, $x \in \operatorname{acc}(E)$. 😊

Now we prove the actual theorem. Let $E \subseteq \mathbb{R}^N$ be bounded. $E \subseteq B(0, r)$ for some r. Let Q_1 be the closed cube centered at 0 with sidelength $2r$. Pick some point $x_1 \in E \subseteq Q_1$. Subdivide Q_1 into 2^N closed cubes of sidelength $\dfrac{2r}{2}$. Let Q_2 be the closed cube containing x_1. Pick some point $x_2 \in E \cap Q_2$ with $x_2 \neq x_1$. Inductively, assume $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n$ have been chosen. Then Q_n is a closed cube of sidelength $\dfrac{2r}{2^{n-1}}$ containing x_n. Each

Q_n contains infinitely many elements of E. Assume also that $x_1, x_2, \cdots x_n \in E$ have been chosen with $x_i \in Q_i$ and $x_i \neq x_j$ for $i \neq j$.

Now we can subdivide Q_n to get Q_{n+1} and continue this process infinitely.

By Lemma 2, we know that $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} Q_n$. Now we need to show there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in E such that $x_n \to x$ as $n \to \infty$ but $x_i \neq x$ for any i because then the rest of the points won't converge to x. If $x = x_i$ for some i, we can just pick another point.

So WLOG, assume $x_n \neq x$ for any n. So we claim $x_n \to x$ as $n \to \infty$. We know that in Q_n, the difference between any two points in this cube is given by:

$$||x_n - x|| = \sqrt{(x_{n,1} - x_1)^2 + (x_{n,2} - x_2)^2 + \cdots + (x_{n,N} - x_N)^2} \leqslant \sqrt{\frac{2r}{2^{n-1}} + \frac{2r}{2^{n-1}} + \cdots + \frac{2r}{2^{n-1}}} = \sqrt{N}\frac{2r}{2^{n-1}}$$

This value is less than ϵ for all large n, so this concludes the proof. 😊

1.14 2/6 - Recitation - Spaces

Let $X = \{f : [0,1] \to \mathbb{R} \text{ bounded}\}$. Define $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Prove that $(X, \|\cdot\|)$ does not suffice parallelogram identity. That is, show a counterexample to the parallelogram identity, which is

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

Proof. Counterexample: Let $f(x) = x$ and $g(x) = 1$. 😊

Now given a normed space which satisifies the parallelogram identity, can we define an inner product?

Proof. Yes. We can define $(f, g) = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2)$. We can prove that this is an inner product.

Linearity of products because the other properties are easy to prove. We need to show that $(x + y, z) = (x, z) + (y, z)$. I'm so lazy so I won't tbh.

We now show that $(tx, y) = t(x, y) \forall t \in \mathbb{Z}$. We proceed with induction for $t \in \mathbb{Z}^+$

Our two base cases are $t = 0, 1$. For $t = 0$, we have that $(0x, y) = (0, y) = 0 = 0(0, y)$. For $t = 1$, we have that $(x, y) = (x, y) = 1(x, y)$.

Now we assume that $(tx, y) = t(x, y)$ for some $t \in \mathbb{Z}^+$. Then we have that $(t + 1)x = tx + x$. Then we have that $(t + 1)x, y = (tx + x, y) = (tx, y) + (x, y) = t(x, y) + (x, y) = (t + 1)(x, y)$. Thus, we have that $(tx, y) = t(x, y)$ for all $t \in \mathbb{Z}^+$.

Now we have to deal with $t \in \mathbb{Z}^-$. We have that $(tx, y) = -t(-x, y) = -t(x, y) = t(x, y)$. Thus, we have that $(tx, y) = t(x, y)$ for all $t \in \mathbb{Z}$.

To proceed, we deal with $t \in \mathbb{Q}$. We have that $t = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. Then we have that $n(tx, y) = (ntx, y) = (mx, y) = m(x, y) = t(mx, y) = t(n(x, y))$. Thus, we have that $n(tx, y) = t(n(x, y))$. Thus, we have that $(tx, y) = t(x, y)$ for all $t \in \mathbb{Q}$. 😊

1.15 Compactness

Definition 1.15.1: Subsequence

Let X be a set and $f : \mathbb{N} \to X$ a sequence. Let $g : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then $f \circ g : \mathbb{N} \to X$ is a *subsequence* of f. We denote m_k as $g(k)$, so $f(g(k)) = f(m_k) = x_{m_k}$. So we denote the whole sequence as $\{x_{m_k}\}_k$.

Definition 1.15.2: Sequentially Compact

Let (X, d) be a metric space. $K \subseteq X$ is *sequentially compact* if every sequence $\{x_n\}_n$ in K and there exists a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in K$.

Example 1.15.1 ($\mathbb{R}$)

1. $(0, 1]$ is not sequentially compact. Consider the sequence $x_n = \dfrac{1}{n}$. This sequence has no convergent subsequence that tends to 0 since 0 is not in the set. The issue is that it's not closed.
2. $[0, \infty)$ is not sequentially compact. Consider the sequence $x_n = n$. This sequence has no convergent subsequence that tends to ∞ since ∞ is not in the set. So, $[0, \infty)$ is not sequentially compact. The issue is that it's not bounded.

Theorem 1.15.1

Let (X, d) be a metric space. If $K \subseteq X$ is sequentially compact, then K is closed and bounded.

Proof. Claim: K is closed. We want $X \setminus K$ to be open. Let $x \in X \setminus K$. We want $B(x, r) \in X \setminus K$ for some $r > 0$. By contradiction, for all $r > 0$, assume $\exists y \in B(x, r) \cap K$. Take $r = \dfrac{1}{m} \Rightarrow y_m \in B(x, \dfrac{1}{m}) \cap K$. $d(y_m, x) < \dfrac{1}{m} \to 0$, so $y_m \to x$. But $x \notin K$ even though $y_m \in K$. This is a contradiction, so K is closed.

Claim: K is bounded. By contradiction, assume K is not bounded. Let $x_0 \in X$. Then $K \nsubseteq B(x_0, r)$ for any $r > 0$. Take $r = n$. Then $\exists x_n \in K$ such that $d(x_n, x_0) \geqslant n$. So $\{x_n\}_n \in K$. K is sequentially compact, so there exists a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in K$. But $n_k \leqslant d(x_{n_k}, x_0) \leqslant d(x_{n_k}, x) + d(x, x_0)$. But $d(x_{n_k}, x) \to 0$ as $k \to \infty$, so $n_k \to \infty < d(x_{m_k}, x_0) \leqslant d(x, x_0)$ which is a fixed number, so we have a contradiction. As such, K is bounded. 😊

Theorem 1.15.2

Let $K \subseteq \mathbb{R}^N$. Then K is sequentially compact if and only if K is closed and bounded.

Proof. We just showed the first direction. So, we need to show that if K is closed and bounded, then K is sequentially compact.

So now, assume K is closed and bounded. Let $\{x_n\}_n$ be a sequence in K. We want to show that there exists a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in K$.

Consider the set $E = \{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}_N$. We now case on whether E has infinitely many distinct points or not.

If E doesn't have infinitely many distinct points, there exists $x \in K$ such that $x_n = x$ for einfinitely many n. Then $x_{n_k} = x$ for all k, so $x_{n_k} \to x$ as $k \to \infty$.

Now we consider the case where Bolzano-Weierstrass applies. By B-W, E has an accumulation point $x \in \mathbb{R}^N$. So we can find a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to x$ as $k \to \infty$. But $x \in K$ because K is closed. Thus, K is sequentially compact. 😊

Note:

Let $(X, \|\cdot\|)$ be a normed space. If every closed and bounded set is sequentially compact, then X has finite dimension.

Exercise 1.15.1

Recall $l^\infty([0,1]) = \{f : [0,1] \to \mathbb{R} \text{ bounded}\}$. Define $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$. $B(0,1) = \{g \in l^\infty([0,1]) : \|g\|_\infty < 1\}$. Prove that $B(0,1) = \{g \in l^\infty([0,1]) : |g(x)| < 1 \ \forall x \in [0,1]\}$. Also prove that this not sequentially compact.

1.16 2/8 - Recitation

Let $n \in \mathbb{N}$, $x, y \in \mathbb{R}$.

1. Prove that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

   Proof. Base case: $n = 1$ is trivial.

   Now assume that for any $n \in \mathbb{N}$, $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})$. We want to show that this is true for $n+1$. We have that $x^{n+1}-y^{n+1} = x(x^n-y^n)+y^n(x-y) = x(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n(x-y)$. Then we get $(x-y)(x^n+x^{n-1}y+\cdots+xy^{n-1}+y^n) = (x-y)(x^n+x^{n-1}y+\cdots+xy^{n-1}+y^n)$. ☺

2. Prove that when $|x-y| \leqslant 1$, then $|x^n - y^n| \leqslant n(1+|x|)^{n-1}|x-y|$.

   Proof. Let $|x-y| \leqslant 1$. Then we have that $|x^n - y^n| = |(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})| \leqslant |x-y|(|x^{n-1}| + |x^{n-2}y| + \cdots + |xy^{n-2}| + |y^{n-1}|) \leqslant |x-y|(|x^{n-1}| + |x^{n-2}||y| + \cdots + |x||y^{n-2}| + |y^{n-1}|) \leqslant |x-y|(|x^{n-1}|+|x^{n-2}||y|+\cdots+|x||y^{n-2}|+|y^{n-1}|) \leqslant |x-y|(|x^{n-1}|+|x^{n-2}|+\cdots+|x|+1) \leqslant n(1+|x|)^{n-1}|x-y|$. ☺

3. Let $E = \{x \in \mathbb{R} : x^n > 3\}$ for a fixed n. Prove that E is open.

   Proof. Let $x \in E$. We want to show that there is an $r > 0$ such that $B(x,r) \subseteq E$. Take $r = \dfrac{x^n - 3}{n(1+|x|)^{n-1}}$ and take $y \in B(x,r)$. Then $|x-y| < r \Rightarrow |x^n|-|y^n| \leqslant |x^n - y^n| \leqslant n(1+|x|)^{n-1}|x-y| < n(1+|x|)^{n-1}r < x^n - 3$. Then $y^n \geqslant x^n - n(1+|x|)^{n-1}r > 3$. Thus, $y \in E$. Thus, $B(x,r) \subseteq E$. Thus, E is open. ☺

4. Consider the space $l^\infty([0,1]) = \{f : [0,1] \to \mathbb{R} \text{ bounded}\}$. Define $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.
   Let $E = \{f \in l^\infty([0,1]) : f(x) > 0 \ \forall x \in [0,1]\}$. Prove that E is not open.

   Proof. Consider
   $$f(x) = \begin{cases} x & x \in [0,1) \\ 1 & x = 1 \end{cases}$$
   Then let $r > 0$ and consider $g(x) = f(x) \cdot \dfrac{r}{2}$. Then $g(x) \in B(f,r)$. But $g(x) \notin E$ because $g(1) = \dfrac{r}{2}$. Thus, $B(f,r) \nsubseteq E$. Thus, E is not open. ☺

1.17 Limits

Definition 1.17.1: Limits

Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $f : E \to Y$. Let $x_0 \in \operatorname{acc} E$.
Take $l \in Y$. l is the *limit* of f as $x \to x_0$. We write $\lim_{x\to x_0} f(x) = l$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < d_X(x,x_0) < \delta \Rightarrow d_Y(f(x),l) < \epsilon$. We can also write it as $f(x) \to l$ as $x \to x_0$.

Note:

Even if $x_0 \in E$, you don't take in the definition for the limit.

Theorem 1.17.1

Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $f : E \to Y$, and $x_0 \in \operatorname{acc} E$. If $\lim_{x\to x_0} f(x)$ exists, then it is unique.

Proof. Assume that $\lim_{x\to x_0} f(x) = l$ and $\lim_{x\to x_0} f(x) = m$. Take $\epsilon = \frac{d_Y(l,m)}{2} > 0$. Then there exists $\delta_1 > 0$ such that $0 < d_X(x,x_0) < \delta_1 \Rightarrow d_Y(f(x),l) < \epsilon$. There also exists $\delta_2 > 0$ such that $0 < d_X(x,x_0) < \delta_2 \Rightarrow d_Y(f(x),m) < \epsilon$. Take $\delta = \min(\delta_1,\delta_2)$. Then $0 < d_X(x,x_0) < \delta \Rightarrow d_Y(f(x),l) < \epsilon$ and $d_Y(f(x),m) < \epsilon$. Then $d_Y(l,m) \leqslant d_Y(l,f(x)) + d_Y(f(x),m) < 2\epsilon = d_Y(l,m)$. This is a contradiction, so $l = m$. ☺

Example 1.17.1 ($\mathbb{R}^2$)

Take $(x_0, y_0) \in \mathbb{R}^2$ and $y_0 \neq 0$. Compute

$$\lim_{(x,y)\to(x_0,y_0)} \frac{x}{y}$$

We want to show that this is $\dfrac{x_0}{y_0}$. We have the set $E = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$. We also know that $(x_0, y_0) \in \operatorname{acc} E$. What we know if that $(x,y) \to (x_0, y_0)$: $|x - x_0|$ and $|y - y_0|$ are going to be small. Then

$$\begin{aligned}
\left| f(x,y) - \frac{x_0}{y_0} \right| &= \left| \frac{x}{y} - \frac{x_0}{y_0} \right| \\
&= \left| \frac{xy_0 - x_0y_0}{yy_0} \right| \\
&= \left| \frac{xy_0 - x_0y_0 + x_0y_0 - x_0y}{yy_0} \right| \\
&= \left| \frac{y_0(x - x_0) + x_0(y_0 - y)}{yy_0} \right| \\
&\leqslant \frac{|y_0||x - x_0| + |x_0||y_0 - y|}{|y||y_0|} \\
&= \frac{|x - x_0|}{|y|} + \frac{|x_0||y_0 - y|}{|y||y_0|}
\end{aligned}$$

Then we have $\delta < \frac{|y_0|}{2}$. If $|y - y_0| < \delta < \frac{y_0}{2}$, then we get $|y| \geqslant \frac{|y_0|}{2} \Rightarrow \frac{1}{|y|} \leqslant \frac{2}{|y_0|}$.

$$\frac{|x - x_0|}{|y|} + \frac{|x_0||y_0 - y|}{|y||y_0|} \leqslant \frac{2|x - x_0|}{|y_0|} + \frac{2|x_0||y_0 - y|}{|y_0|^2}$$

Take $\delta = \min\left\{\epsilon, \frac{|y_0|}{2}\right\} > 0$. Then $0 < \|(x,y) - (x_0,y_0)\| < \delta$.

$$\begin{aligned}
|x - x_0| &= \sqrt{(x - x_0)^2} \leqslant \sqrt{(x - x_0)^2 + (y - y_0)^2} \\
|y - y_0| &\leqslant \delta
\end{aligned}$$

So,

$$\left| f(x,y) - \frac{x_0}{y_0} \right| < \epsilon \left(\frac{2}{|y_0} + \frac{2}{|y_0|^2} \right)$$

Say you can prove that for every $\epsilon > 0$, $\exists \delta > 0$ such that

$$d(f(x), l) < \epsilon |\log(\epsilon)| \text{ for all } x \in E \text{ such that } 0 < d(x, x_0) < \delta$$

For every $\eta > 0$ ("my epsilon"), since $\lim_{\epsilon\to 0^+} \epsilon|\log(\epsilon)| = 0$, $\exists \delta_1 > 0$ such that $\epsilon|\log(\epsilon)| < \eta$ for all $0 < \epsilon < \delta_1$.

So given $\eta > 0$, take $0 < \epsilon < \delta_1$. Find η from $d(f(x), l) < \epsilon|\log(\epsilon)| < \eta$ for all $x \in E$ such that $0 < d(x, x_0) < \delta$. This means that

$$d_Y(f(x), l) < \epsilon|\log(\epsilon)| < \eta$$

for all $x \in E$, $0 < d(x, x_0) < \delta$. Thus, $\lim_{x\to x_0} f(x) = l$.

1.18 Limits Continued

Definition 1.18.1: Restriction

Assume that $\lim_{x \to x_0} f(x) = l$ exists. Let $F \subseteq E$ such that $x_0 \in \operatorname{acc} F$. The function $f : F \to Y$ is called the *restriction* of f to F. It is denoted as $f|_F$.

Note:

If $\lim_{x \to x_0} f(x) = l$, then $\lim_{x \to x_0} f|_F(x) = l$.

So to prove that the limit does not exist, you can conjure up two restrictions of the function and show that the limits are different.

Example 1.18.1 (Limits that don't exist)

Consider $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$. We can find two restrictions of the function and show that the limits are different.

- $\frac{1}{x} = 2\pi n + \frac{\pi}{2}$
- $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$

So we have:

- $\sin x_n = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$
- $\sin x_n = \sin\left(\frac{1}{2\pi n}\right) = 0$

Thus, the limit does not exist.

Exercise 1.18.1 TODO in Recitation

- $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$ (no)
- $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2+y^2}$ (yes, 0)
- $\lim_{(x,y) \to (0,0)} \frac{x^{10000000000} y}{y - \sin(x)}$ (no)

Now we talk about the composition of limits.

Example 1.18.2

Consider

$$g(y) = \begin{cases} 1 & y \neq 0 \\ 2 & y = 0 \end{cases}.$$

The limit of $g(y)$ as $y \to 0$ is 1. Now consider $f(x) = 0$. The limit of $f(x)$ as $x \to x_0$ is 0. Now consider $g(f(x))$. The limit of $g(f(x))$ as $x \to x_0$ is 2.

Theorem 1.18.1 Composition of Limits

Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces, $E \subseteq X$, $F \subseteq Y$, $f : E \to F$, $g : F \to Z$, and $x_0 \in \operatorname{acc} E$. Assume there exists $\lim_{x \to x_0} f(x) = l \in Y$. Assume $l \in \operatorname{acc} F$ and that there is $\lim_{y \to l} g(y) = L \in Z$. Assume that either $f(x) \neq l$ for all $x \in E$ or $l \in F$ and $g(l) = L$. Then there is $\lim_{x \to x_0} g(f(x)) = L$.

Proof. Since $\lim_{y \to l} g(y) = L$, there for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Z(g(y), L) < \epsilon$ for all $y \in F$ with $0 < d_Y(y, l) < \delta$. We would like to take $y = f(x)$. Use δ as "my epsilon" for the definition of the limit of $f(x)$. Then to find $\eta > 0$ such that $d_X(f(x), l) < \delta$ for all $x \in E$ with $0 < d_X(x, x_0) < \eta$. Now we split into cases:

- Assume $f(x) \neq l$ for all $x \in E$. Then $0 < d_Y(f(x), l)$ so we can take $y = f(x)$ to get $d_z(g(f(x)), L) < \epsilon$ for all $x \in E$ with $0 < d_X(x, x_0) < \eta$. This means that there exists $\lim_{x \to x_0} g(f(x)) = L$.
- Assume $l \in F$ and $g(l) = L$. If $f(x) = l$, then $d_Z(g(f(x)), L) = d_Z(g(l), L) = 0$ for all $x \in E$ with $0 < d_X(x, x_0) < \eta$. If $f(x) \neq l$, then take $y = f(x)$ to get $0 < d_Y(f(x), l)$ so we can take $y = f(x)$ to get $d_z(g(f(x)), L) < \epsilon$ for all $x \in E$ with $0 < d_X(x, x_0) < \eta$. This means that there exists $\lim_{x \to x_0} g(f(x)) = L$. This means that there exists $\lim_{x \to x_0} g(f(x)) = L$.

😊

Corollary 1.18.1 Limits of the Sum/Products/Quotients

Let (X, d) be a metric space and $E \in X$. Then take $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$. Let $x_0 \in X$ and $x_0 \in \operatorname{acc} E$. Assume $\lim_{x \to x_0} f(x) = l$ and $\lim_{x \to x_0} g(x) = m$. Then we have the following results:

- $\lim_{x \to x_0} (f + g)(x) = l + m$.
- $\lim_{x \to x_0} (f \cdot g)(x) = l \cdot m$.
- $\lim_{x \to x_0} \frac{f}{g}(x) = \frac{l}{m}$.

Proof. We can use the composition of limits to prove this. We'll just proceed with the quotient case. Consider $x \to (f(x), g(x))$. Then consider the function that takes $(s, t) \to \frac{s}{t}$ and call it h. THen we have $\frac{f(x)}{g(x)} = h(f(x), g(x))$. We then have $\lim_{x \to x_0} (f(x), g(x)) = (l, m)$ and $\lim_{(s,t) \to (l,m)} h(s, t) = \frac{l}{m} = h(l, m)$. So now we can use the composition of limits to get $\lim_{x \to x_0} \frac{f}{g}(x) = \frac{l}{m}$.

The other two cases are similar. For products, you need to show that the limit as $(x, y) \to (x_0, y_0)$ of xy is $x_0 y_0$ and similarly for sum. 😊

1.19 Squeeze Theorem

Theorem 1.19.1 Squeeze Theorem

Let (X, d_X) be a metric space, $E \subseteq X$, $f : E \to \mathbb{R}$, $g : E \to \mathbb{R}$, and $h : E \to \mathbb{R}$. Let $x_0 \in \operatorname{acc} E$ and have $f \leqslant g \leqslant h$. Assume that $\lim_{x \to x_0} f(x) = l = \lim_{x \to x_0} h(x)$. Then $\lim_{x \to x_0} g(x) = l$.

Proof. Assume $\lim_{x \to x_0} f(x) = l = \lim_{x \to x_0} h(x)$. Then for every $\epsilon > 0$, there exists $\delta_1 > 0$ such that $0 < d_X(x, x_0) < \delta_1 \implies |f(x) - l| < \epsilon$ and $0 < d_X(x, x_0) < \delta_2 \implies |h(x) - l| < \epsilon$. Take $\delta = \min(\delta_1, \delta_2)$ and $x \in E$ with $0 < d(x, x_0) < \delta$. Then $l - \epsilon < f(x) < g(x) < h(x) < l + \epsilon$. Then $|g(x) - l| < \epsilon$ for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. Thus, $\lim_{x \to x_0} g(x) = l$. 😊

Example 1.19.1

$$\lim_{x \to 0} |x|^a \sin \frac{1}{x} = 0$$

for all $a > 0$.
Let $Q > 0$. Then $O \leqslant ||x|^Q \sin \frac{1}{x}| \leqslant |x|^Q$. Since both sides tend to 0 as $x \to 0$, then the middle does as well.

Definition 1.19.1: Increasing

$f : E \to \mathbb{R}$ is *increasing* if $f(x) \leqslant f(y)$ for all $x \leqslant y$. It is strictly increasing if $f(x) < f(y)$ for all $x < y$.

Definition 1.19.2: Decreasing

$f : E \to \mathbb{R}$ is *decreasing* if $f(x) \geqslant f(y)$ for all $x \leqslant y$. It is strictly decreasing if $f(x) > f(y)$ for all $x < y$.

Definition 1.19.3: Divergent

Let (X, d_x) be a metric space with $E \subseteq X$, $x_0 \in \operatorname{acc} E$, and $f : E \to \mathbb{R}$. We say that f diverges to $+\infty$ as $x \to x_0$ if for every $M > 0 \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) > M$ for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. We say that f diverges to $-\infty$ as $x \to x_0$ if for every $M < 0 \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) < M$ for all $x \in E$ with $0 < d_X(x, x_0) < \delta$.

Theorem 1.19.2

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be increasing. Let $x_0 \in \mathbb{R}$. Assume x_0 is an accumulation point of $E \cap (-\infty, x_0)$. Then there is

$$\lim_{x \to x_0^-} f(x) = \sup_{E \cap (-\infty, x_0)} f(x)$$

Now if x_0 is an accumulation point of $E \cap (x_0, \infty)$, then there is

$$\lim_{x \to x_0^+} f(x) = \inf_{E \cap (x_0, \infty)} f(x)$$

Proof.

- Case 1: Assume f is bounded form above on $E \cap (-\infty, x_0)$. Let $l = \sup_{E \cap (-\infty, x_0)} f(x)$. Then for every $\epsilon > 0$, there exists $x_1 \in E \cap (-\infty, x_0)$ such that $l - \epsilon < f(x_1) \leqslant l$. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that $l - \epsilon < f(x_1) \leqslant l$. Take $\delta = x_0 - x_1 > 0$. Let $x \in E$ with $x_0 - \delta < x < x_0$. Since f is increasing, we have $l - \epsilon < f(x_1) \leqslant f(x) \leqslant l < l + \epsilon$. Thus, $\lim_{x \to x_0^-} f(x) = l$.
- Case 2: If f is not bounded from above, then for every $M > 0$, there exists $x_1 \in E \cap (-\infty, x_0)$ such that $f(x_1) > M$. Let $\delta = x_0 - x_1 > 0$. Then for every $x \in E$ with $x_1 = x_0 - \delta < x < x_0$, we have $f(x) \geqslant f(x_1) > M$. Thus, $\lim_{x \to x_0^-} f(x) = +\infty$.

The other case is similar. ☺

Definition 1.19.4: Infinite Sum

Let X be a set and take $f : X \to [0, \infty]$. The *infinite sum* is defined as:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq X \text{ finite} \right\}.$$

Lenma 1.19.1

Let X be nonempty with $f : X \to [0, \infty]$. Assume that $\sum_{x \in X} f(x) < \infty$. Then $\{x \in X : f(x) > 0\}$ is countable.

Proof. Take $n \in \mathbb{N}$ and define $X_n = \{x \in X : f(x) \geqslant \frac{1}{n}\}$. Let $E \subseteq X_n$ be finite. Then $\frac{1}{n}|E| < \sum_{x \in E} f(x) \leqslant M$. Then $|E| < nM$. Thus, X_n is countable. Then $\bigcup_{n \in \mathbb{N}} X_n = \{x \in X : f(x) > 0\}$ is countable. ☺

Theorem 1.19.3

Take $I \subseteq \mathbb{R}$ to be an interval with $f : I \to \mathbb{R}$ increasing. Then for all but countably many $x_0 \in I$, there is $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$.

Proof. Let $x_0 \in I$. 😄