21-235 Math Studies Analysis I

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Contents

Chapter 1

1.1 Ordered Fields (Review)

Definition 1.1.1: Order

Let E be a set. An *order* on E is a relation $<$ on E such that for all $x, y, z \in E$:

1. (Trichotomy) Exactly one of the following holds: $x < y$, $x = y$, or $x > y$.
2. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Example 1.1.1 (Examples of Ordered Sets)

1. This definition develops orders on basic number systems: e.g. $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
2. Define $\lesssim$ on $\mathbb{Z}$ as follows: We say that $m \lesssim n$ for $m, n \in \mathbb{Z}$ if:
   (a) m is even and n is odd
   (b) m, n are even and $m < n$
   (c) m, n are odd and $m < n$.

Key Concepts:

- upper/lower bounds of sets
- bounded sets
- max/min
- supremum/infimum
- supremum/infimum property: An ordered set E satisfies such a property if every nonempty set $A \subseteq E$ that's bounded above/below has a supremum/infimum in E.
- Fact: sup prop $\implies$ inf prop

Definition 1.1.2: Ordered Field

Let $\mathbb{F}$ be a field with order $<$. We say that $\mathbb{F}$ is an *ordered field* provided that:

1. For all $x, y, z \in \mathbb{F}$, if $x < y$, then $x + z < y + z$.
2. For all $x, y \in \mathbb{F}$, if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Example 1.1.2
$\mathbb{Q}$ is a field.

Facts of any ordered field:

1. $0 < 1$
2. $\nexists x \in \mathbb{F}$ such that $x^2 = -1$.

Definition 1.1.3: Ordered Subfield, Homomorphism, Isomorphism

Let $\mathbb{F}$ be an ordered field.

1. A set $\mathbb{K} \subseteq \mathbb{F}$ is called an *ordered subfield* if $mathbbK$ is an algeraic subfield and $\mathbb{K}$ is an ordered field equipped with $<$ from $\mathbb{F}$.
2. Let $\mathbb{G}$ be an ordered field and let $f : \mathbb{F} \to \mathbb{G}$. We say that f is an *ordered field homomorphism* if it's a field homomorphism and $f(x) < f(y)$ whenever $x < y$.
3. f is an ordered field isomorphism if f is an ordered field homomorphism and f is bijective.

Note:

1. If $f : \mathbb{F} \to \mathbb{G}$ is an ordered field homomorphism, $f(\mathbb{F})$ is an ordered subfield of $\mathbb{G}$.
2. OF property $\implies$ f is injective.
3. $\therefore$ every ordered field homomorphism $f : \mathbb{F} \to \mathbb{G}$ is such that f induces a bijection $f : \mathbb{F} \to f(\mathbb{F}) \subseteq \mathbb{G}$.

Theorem 1.1.1 $\mathbb{Q}$ is the smallest ordered field. More precisely, if $\mathbb{F}$ is an ordered field, then there exists a canonical ordered field homomorphism $f : \mathbb{Q} \to \mathbb{F}$.

Upshot/notation abuse: We identify $f(\mathbb{Q}) = \mathbb{Q}$ to view $\mathbb{Q} \subseteq \mathbb{F}$. In turn, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{F}$.

1.2 Types of Ordered Fields

Definition 1.2.1: Archimedean, Dedekind complete

Let $\mathbb{F}$ be an ordered field.

1. We say that $\mathbb{F}$ is Archimedean if $\forall 0 < x \in \mathbb{F}$, $\exists n \in \mathbb{N}$ such that $n > x$.
2. We say that $\mathbb{F}$ is Dedekind complete if it satisfies the supremum property.

Facts:

1. $\mathbb{Q}$ is Archimedean.
2. If $\mathbb{F}$ is Dedekind complete, then $\forall 0 < x \in \mathbb{F}$ and $\forall 0 < n \in \mathbb{N}$, $\exists!\ 0 < y \in \mathbb{F}$ such that $y^n = x$.
3. $\mathbb{Q}$ is not Dedekind complete. ($\sqrt{2}$ is a counterexample.)

Theorem 1.2.1
Suppose $\mathbb{F}$ is a Dedekind complete ordered field. Then $\mathbb{F}$ is Archimedean.

Proof. If not, then $\mathbb{N} \subset \mathbb{F}$ is bounded above, and so the supremum property provides $x \in \mathbb{F}$ such that $x = \sup \mathbb{N}$. But then $x - 1$ is an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that $x - 1 < n$. Hence $x < n + 1$, which contradicts the definition of x as an upper bound. Therefore, $\mathbb{F}$ is Archimedean. 😊

1.3 Dedekind Completion

Throughout this section, let $\mathbb{F}$ be an Archimedean ordered field.

Definition 1.3.1: Dedekind cut

We say a set $C \subseteq \mathbb{F}$ is *Dedekind cut* if:

1. $C \neq \emptyset$ and $C \neq \mathbb{F}$.
2. If $p \in C$ and $q \in \mathbb{F}$ such that $q < p$, then $q \in C$.
3. If $p \in C$, then $\exists r \in C$ such that $p < r$.

We will write $\mathbb{F}^*$ for the set of all Dedekind cuts in $\mathbb{F}$. It is called the *Dedekind completion* of $\mathbb{F}$.

Note:

Let $C \subseteq \mathbb{F}$ be a cut. Then:

1. If $p \in C$, then $q \notin C$, then $p < q$.
2. If $r \notin C$, and $r < s \in \mathbb{F}$, then $s \notin C$.

Example 1.3.1 (Cut examples)

1. Let $q \in \mathbb{F}$ and define $C_q = \{p \in \mathbb{F} \mid p < q\}$. Then C_q is a cut.

   Proof. (a) $q - 1 < q \implies q - 1 \in C_q$. $q \not< q \implies q \notin C_q \implies C_q \neq \mathbb{F}$.

   (b) Let $p \in C_q$. Suppose $s \in \mathbb{F}$ such that $s < p$. Then $s < q \implies s \in C_q$.

   (c) Let $p \in C_q$. Then $p < \frac{p+q}{2} < q \implies \frac{p+q}{2} \in C_q$.

   😃

2. Suppose $\mathbb{F}$ is such that $\nexists x \in \mathbb{F}$ such that $x^2 = 2$. Let $C = \{p \in \mathbb{F} \mid p \leqslant 0 \text{ or } 0 < p^2 < 2\}$. Then C is a cut.

   Proof. (a) $1 \in C$ and $1^2 = 1 < 2$. $2 \notin C$ and $2^2 = 4 > 2$.

   (b) Let $p \in C$ and $q \in \mathbb{F}$ such that $q < p$. If $q \leqslant 0$, then $q \in C$ trivially. Suppose $0 < q < p$. Then $0 < q^2 < p^2 < 2$, so $q \in C$.

   (c) Let $p \in C$. If $p \leqslant 0$, then $1 \in C$ and $p < 1$, so we're done. Suppose $0 < p^2 < 2$. Note, $0 < 2 - p^2$, so $\frac{2p+1}{2-p^2} > 0$. Then we can define $r = 1 + \frac{2p+1}{2-p^2} \geqslant \max(1, \frac{2p+1}{2-p^2})$. Then $(p + 1/r)^2 = p^2 + \frac{2p}{r} + \frac{1}{r^2}$. We have:

   $$\begin{aligned} p^2 + \frac{2p}{r} + \frac{1}{r^2} &< p^2 + \frac{2p}{r} + \frac{1}{r} \\ &= p^2 + \frac{2p+1}{r} \\ &\leqslant p^2 + 2 - p^2 \\ &= 2. \end{aligned}$$

   So, $p < p + 1/r < 2$ and $p + 1/r \in C$.

   😃

1.3.1 Ordering $\mathbb{F}^*$

Lenma 1.3.1

The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then exactly one holds:
   - $\mathcal{A} \subset \mathcal{B}$
   - $\mathcal{A} = \mathcal{B}$
   - $\mathcal{B} \subset \mathcal{A}$
2. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$.

Proof. Proof of 2 is trivial, as well as the equality part for 1.

- If $\mathcal{A} = \mathcal{B}$, we're done.
- Suppose $\exists b \in \mathcal{B} \setminus \mathcal{A}$. If $a \in \mathcal{A}$, then $a < b$, but $\mathcal{B}$ is a cut so $a \in \mathcal{B}$, so $\mathcal{A} \subset \mathcal{B}$.
- Suppose $\exists a \in \mathcal{A} \setminus \mathcal{B}$. Then $a < b$ for all $b \in \mathcal{B}$, so $a \in \mathcal{B}$, so $\mathcal{B} \subset \mathcal{A}$.

☺

Definition 1.3.2: Order on cuts

Given $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we say that $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$. The lemma above shows that this is infact an order.

Lenma 1.3.2

Let $E \subseteq \mathbb{F}^*$ be nonempty and bounded above. Then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$ is a cut.

Proof.

1. Since $E \neq \emptyset$, there exists $\mathcal{A} \in E$. So $\mathcal{A} \neq \emptyset$, hence $\mathcal{B} \neq \emptyset$.

   Since E is bounded above, there exists $\mathcal{C} \in \mathbb{F}^*$ such that $\mathcal{A} \subset \mathcal{C}$ for all $\mathcal{A} \in E$. Since $\mathcal{C}$ is a cut, there is $q \in \mathbb{F}$ such that $q \notin \mathcal{C}$. Then $q \notin \mathcal{A}$ for all $\mathcal{A} \in E$, so $q \notin \mathcal{B}$.
2. Let $p \in \mathcal{B}$ and $q \in \mathbb{F}$ such that $q < p$. Since $\mathcal{B}$ is a union of cuts, it follows that $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, $q \in \mathcal{A} \subseteq \mathcal{B}$.
3. Let $p \in \mathcal{B}$. Then $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, there exists $r \in \mathcal{A}$ such that $p < r$. Since $\mathcal{A} \subset \mathcal{B}$, we have $r \in \mathcal{B}$.

☺

Theorem 1.3.1

$\mathbb{F}^*$ equipped with the order $<$ satisfies the supremum property.

Proof. Let $E \subseteq \mathbb{F}$ be a nonempty set that is bounded above. From last time, we know that $\mathcal{B} = \cup_{\mathcal{A} \in E} \mathcal{A}$ is a cut. We claim that $\mathcal{B} = \sup E$.

If $\mathcal{A} \in E$, then $\mathcal{A} \subseteq \mathcal{B}$. And so $\mathcal{A} \leqslant \mathcal{B}$, so $\mathcal{B}$ is an upper bound for E.

Next, suppose that $\mathcal{C} \in \mathbb{F}^*$ is an upper bound of E. This means that $\mathcal{A} \leqslant \mathcal{C}$ for every $\mathcal{A} \in E$, meaning $\mathcal{A} \subseteq \mathcal{C} \forall \mathcal{A} \in E$. So $\mathcal{B} \subseteq \mathcal{C}$. As such, $\mathcal{B} \leqslant \mathcal{C}$, so $\mathcal{B} = \sup E$. ☺

Remark: In none of the results leading up to this theorem did we use that $\mathbb{F}$ is anything other than an ordered set. This shows that the cut construction of Dedekind works in general for ordered sets and yields $\mathbb{F}^*$ that satisfies the supremum property. Also, $\{\mathcal{C}_p \mid p \in \mathbb{F}\} \subseteq \mathbb{F}^*$.

1.3.2 Addition

Idea: $\mathbb{F} \cong \{C_p \mid p \in \mathbb{F}\}$.

Lemma 1.3.3

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. Then $C = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is a cut.

Proof. Claim: $\mathcal{A}, \mathcal{B} \neq \emptyset \implies C \neq \emptyset$.

$\mathcal{A}, \mathcal{B}$ are cuts, so $\exists M_1, M_2 \in \mathbb{F}$ such that $a < M_1$ for all $a \in \mathcal{A}$ and $b < M_2$ for all $b \in \mathcal{B}$. Then $a + b < M_1 + M_2$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, so $a + b < M_1 + M_2$, meaning $M_1 + M_2 \notin C$.

Also, let $c = a+b \in C$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Let $q < c \implies q-a < b \implies q-a \in \mathcal{B}$. Hence, $q = a+(q-a) \in C$.

Thirdly, let $c = a + b \in C$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Since $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, $\exists r_a, r_b$ such that $a < r_a \in \mathcal{A}$, $b < r_b \in \mathcal{B}$. Then $c = a + b < r_a + r_b$, so $r_a + r_b \in C$.

As such, C is a cut. 😃

Before we define addition, we need to define the negative of a cut.

Heuristic: What we want is that $-C_1 = C_{-1}$. The way we do this is by defining $C_{-p} = \{q \in \mathbb{F} \mid \exists p > q : p \in -C_p^C\}$. This is the same as $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C_p\}$.

Now we study $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C\}$.

Lemma 1.3.4

Let $C \in \mathbb{F}^*$. Then $\{q \in \mathbb{F} \mid \exists p > q : -p \notin C\}$ is a cut.

Definition 1.3.3: Addition

For $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we define $\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and $-\mathcal{A} = \{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{A}\}$.

Theorem 1.3.2

Define $0 = C_0 \in \mathbb{F}^*$. The following hold:

1. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} \in \mathbb{F}^*$.
2. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$.
3. $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^* \implies (\mathcal{A} + \mathcal{B}) + C = \mathcal{A} + (\mathcal{B} + C)$.
4. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + 0 = \mathcal{A}$.
5. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + (-\mathcal{A}) = 0$.

Proof. Easy proof, too lazy to write out. 😃

Also: $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$ and $\mathcal{A} < \mathcal{B} \implies \mathcal{A} + C < \mathcal{B} + C$.

Important Remark: The Archimedean property is actually needed for the above theorem in orer to prove the 5th condition.

1.3.3 Multiplication

Lenma 1.3.5
Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ such that $\mathcal{A}, \mathcal{B} > 0$. Then $C = \{p \in \mathbb{F} \mid p \leqslant 0\} \cup \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\}$ is a cut.

Lenma 1.3.6
Let $\mathcal{A} \in \mathbb{F}^*$ be such that $\mathcal{A} > 0$. Then $C = \{p \in \mathbb{F}^* \mid p \leqslant 0\} \cup \{0 < q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$ is a cut.

Definition 1.3.4: Multiplication

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. We define multiplication as:

1. If $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = \{ab \mid 0 < a \in \mathcal{A}, 0 < b \in \mathcal{B}\}$.
2. If $\mathcal{A} = 0$ or $\mathcal{B} = 0$, then $\mathcal{A} \cdot \mathcal{B} = 0$.
3. If $\mathcal{A} > 0$ and $\mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B}))$.
4. If $\mathcal{A} < 0$ and $\mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = -((-\mathcal{A}) \cdot \mathcal{B})$.
5. If $\mathcal{A}, \mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B})$.

We define multiplication inversion via:

1. If $\mathcal{A} > 0$, then $\mathcal{A}^{-1} = \{q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$.
2. If $\mathcal{A} < 0$, then $\mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$.

Theorem 1.3.3
Set $1 = C_1$. The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} \in \mathbb{F}^*$.
2. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$.
3. If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $(\mathcal{A} \cdot \mathcal{B}) \cdot C = \mathcal{A} \cdot (\mathcal{B} \cdot C)$.
4. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot 1 = \mathcal{A}$.
5. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{A}^{-1} = 1$.

Also if $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ and $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} > 0$.

Theorem 1.3.4
If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $\mathcal{A} \cdot (\mathcal{B} + C) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot C$.

We now know that $\mathbb{F}^*$ is an ordered field.

1.4 Robert Reci

Theorem 1.4.1
$\mathbb{Q}$ is the smallest ordered field.

Proof. Let $\mathbb{F}$ be any ordered field. Let $1 \in \mathbb{F}$. Let $\iota : \mathbb{N} \to \mathbb{F}$, $n \mapsto 1 + \cdots + 1$ n times. Then $\iota(-n) = -\iota(n)$ for $n \in \mathbb{N}_0$ and $-n \in \mathbb{Z}^-$.

Then we say $\iota(p/q) = \iota(p)\iota(q)^{-1}$ for $p/q \in \mathbb{Q}$. ☺

Corollary 1.4.1 Every ordered field is infinite

$\iota[\mathbb{Q}] \subseteq \mathbb{F}$ is infinite.

Roots

Let $\mathbb{F}$ be a Dedekind complete ordered field, $0 < x \in \mathbb{F}$, $n \in \mathbb{N}$. Then $\exists! y \in \mathbb{F}$ such that $y > 0$ and $y^n = x$.

Proof. $n = 1$ is silly. Assume $n \geqslant 2$. Let $E = \{z \in \mathbb{F} \mid z > 0 \text{ and } z^n < x\}$. Then E is nonempty and bounded above by x. Let $y = \sup E$. We claim that $y^n = x$.

We want to show that $y^n \not>> x$ and $y^n \not<< x$.

Lenma 1.4.1

In any commutative ring R, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.

And hence for $0 < a < b$ in $\mathbb{F}$, we have $0 < b^n - a^n = (b-a)nb^{n-1}$.

Suppose $y^n < x$, so $x - y^n > 0$. We define $h = \frac{1}{2}\min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. $0 < h < 1$, also $0 < h < \frac{x-y^n}{n(y+1)^{n-1}}$.

Then, by the inequality below the lemma, we have

$$\begin{aligned} 0 &< (y+h)^n - y^n \\ &< hn(y+h)^{n-1} \\ &< hn(y+1)^{n-1} \\ &< x - y^n, \end{aligned}$$

so $(y+h)^n < x$, which contradicts the definition of y as the supremum. ☺

Definition 1.4.1: Ring*

A ring is a field where actually we don't care about inverses anymore.

Definition 1.4.2: Domain

R is a domain when $xy = 0 \implies x = 0 \wedge y = 0$.

Let R be a ring. For $(r,s) \in R \times R \setminus \{0\}$, we say $(r,s) \sim (r',s')$ if $rs' = r's$.

The field of fractions, $\mathrm{Frac}(R)$ is the set of equivalence classes of $R \times R \setminus \{0\}$ under $\sim$ equipped with the operations $[(r,s)] + [(r',s')] = [(rs' + r's, ss')]$ and $[(r,s)] \cdot [(r',s')] = (rr', ss')$.

We check that $[(r,s)] \cdot [(s,r)] = [(rs, sr)] = [(1,1)]$.

Let $\mathbb{F}$ a field, $\mathbb{F}^x$ its polynomial ring. Let $\mathbb{F}(x)$ be the field of fractions of $\mathbb{F}^x$. Then $\mathbb{F}(x) := \mathrm{Frac}(\mathbb{F}^x)$ is the field of rational functions in x with coefficients in $\mathbb{F}$.

Given $p, q \in \mathbb{F}^x$, say $p/q > 0$ if p and q have the same sign. Say $f, g \in \mathbb{F}(x)$, that $f > g$ when $f - g > 0$.

Theorem 1.4.2

$\mathbb{F}(x)$ is never Archimedean.

Proof. x is an upper bound for all $n \in \mathbb{N}$. ☺

Note:

If $\mathbb{F}$ is Archimedean, $|\mathbb{F}| \leqslant 2^{\aleph_0}$.

Theorem 1.4.3

Let λ be an infinite cardinal. Then there is an ordered field of cardinality λ.

Corollary 1.4.2

The Archimedean property is not a first-order property.

1.5 Completeness

Lemma 1.5.1

Suppose $\mathbb{F}$ is an ordered field that is not Dedekind complete. Then $\exists$ and infinite $E \subseteq \mathbb{F}$ such that:

1. E bounded above, $\emptyset \neq U(E)$ is open, $\emptyset \neq U(E)^C$ is open.
2. $a \in U(E)^C$, $b \in U(E) \implies a < b$.
3. $f : \mathbb{F} \to \mathbb{F}$ with $f(x) = \begin{cases} 1 & x \in U(E) \\ 0 & x \in U(E)^C \end{cases}$ is differentiable with $f' = 0$.

Theorem 1.5.1 Characteristics of Dedekind Completeness

Let $\mathbb{F}$ be an ordered field. The following are equivalent:

1. $\mathbb{F}$ is Dedekind complete.
2. $\mathbb{F}$ has the intermediate value property: If $f : [a,b] \to \mathbb{F}$ is continuous and $\min(f(a), f(b)) < c < \max(f(a), f(b))$, then $\exists x \in [a,b]$ such that $f(x) = c$.
3. $\mathbb{F}$ satisfies the mean value property: If $f : [a,b] \to \mathbb{F}$ is continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.
4. $\mathbb{F}$ satisfies Cauchy mean value property: If $f, g : [a,b] \to \mathbb{F}$ are both continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
5. $\mathbb{F}$ satisfies the extreme value property: If $f : [a,b] \to \mathbb{F}$ is continuous, then f attains a maximum and minimum on $[a,b]$.

Proof. $1 \implies 2$: Let $f : [a,b] \to \mathbb{F}$ and continuous. WLOG, assume $f(a) < c < f(b)$. Define $E = \{x \in [a,b] \mid f(x) < c\}$. E is nonempty and bounded above by b. Let $x = \sup E$. We claim that $f(x) = c$. Since f is continuous, $\exists \kappa > 0$ such that $f(t) < c \, \forall t \in [a, a+\kappa]$ and $f(t) > c \, \forall t \in [b-\kappa, b]$. So, $a + \frac{\kappa}{2} < x < b - \frac{\kappa}{2}$.

Suppose BWOC $f(x) < c$. Again by continuity, $\exists \delta > 0$ such that $f(t) < c$ for all $t \in B(x, \delta) \subseteq [a,b]$. Then $x + \frac{\delta}{2} \in E$, contradiction.

Then suppose BWOC $f(x) > c$. Again, $\exists \delta > 0$ such that $f(t) > c$ for all $t \in B(x,\delta) \subseteq [a,b]$. Then $\exists z \in E$ such that $x - \frac{\delta}{2} < z \leqslant x$ and $f(z) < c$. But then $c < f(z) < c$, contradiction.

So $f(x) = c$ by trichotomy.

$2 \implies 1$: We'll show $\neg 1 \implies \neg 2$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the strange function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then f is continuous on $[a,b]$, $f(a) - < 1 = f(b)$, but there is not $x \in [a,b]$ with $f(x) = \frac{1}{2}$, by construction.

$1 \implies 5$: First we claim that if $\mathbb{F}$ is Dedekind and $f : [a,b] \to \mathbb{F}$ is continuous, then $f([a,b]) \subseteq \mathbb{F}$ is a bounded set. We prove the claim.

Consider $E = \{x \in [a,b] \mid f([a,x]) \text{ is bounded}\}$. $a \in E$ and E is bounded, so we can let $s = \sup E$. Next note that by continuity, if $[c,d] \subseteq [a,b]$ such that $f([c,d])$ is bounded, then $\exists \delta > 0$ such that $f([a,b] \cap [c-\delta, d+\delta])$ is bounded. Using this, deduce in turn that $a < s$, $s = \max E$, and $s = b$.

So now suppose $\mathbb{F}$ is Dedekind complete and let $f : [a,b] \to \mathbb{F}$ be continuous. The claim establishes that $f([a,b]) \subseteq \mathbb{F}$ is a bounded set, so we can let $\begin{cases} \mu = \inf f([a,b]) \\ \lambda = \sup f([a,b]) \end{cases}$. Suppose BWOC that $f(x) < \lambda$ for all $x \in [a,b]$. Then teh function $g : [a,b] \to \mathbb{F}$ defined by $g(x) = \frac{1}{\lambda - f(x)}$ is continuous and positive. So by the claim, there is $k > 0$ such that $g(x) \leqslant k$ for all $x \in [a,b]$. But then

$$\frac{1}{\lambda - f(x)} \leqslant k \implies \frac{1}{k} \leqslant \lambda - f(x) \implies f(x) \leqslant \lambda - \frac{1}{k},$$

for all $x \in [a,b]$. But this contradicts the definition of λ, as we just found a better upper bound.

Therefore, there does exists $M \in [a,b]$ such that $f(M) = \lambda$, which is $\max f([a,b])$.

The min follows from a similar argument.

5 $\implies$ 4: Let $f, g : [a,b] \to \mathbb{F}$ be continuous and differentiable on (a,b). Let $h : [a,b] \to \mathbb{F}$ via $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. It suffices to show $\exists x \in (a,b)$ such that $h'(x) = 0$.

By construction, $h(a) = h(b)$. If $h(x) = h(a)$ for all $x \in [a,b]$, then $h' = 0$ and we're done. Suppose then that h is not constant. Then EVT shows that f attains its maximal/minimum values, and at least one must occur at the point $x \in (a,b)$, therefore $h'(x) = 0$.

4 $\implies$ 3: Let $g(x) = x$. Done.

3 $\implies$ 1. We'll show $\neg 1 \implies \neg 3$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then consider the restriction $f : [a,b] \to \mathbb{F}$. Then $1 = 1 - 0 = f(b) - f(a)$. Then, $f'(x)(b-a) = 0 \cdot (b-a) = 0$ for all $x \in \mathbb{F}$. $0 \neq 1$ so $\neg 3$ as desired. 😄

Chapter 2

$\mathbb{R}, \mathbb{C}, \bar{\mathbb{R}}$

Theorem 2.0.1

$\mathbb{R}$ is uncountable.

Proof. $\mathbb{Q} \subseteq \mathbb{R}$, so $\mathbb{R}$ is definitely infinite. Suppose BWOC that there was a bijection $f : \mathbb{N} \to \mathbb{R}$. Set $I_0 = [f(0)+1, f(0)+2]$ and not that $f(0) \notin I_0$. Suppose we are given closed, nested, non-singleton intervals $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_0$ such that $f(k) \notin I_k$ for $0 \leqslant k \leqslant n$. If $f(n+1) \notin I_n$, then set $I_{n+1} = I_n$. Otherwise, set I_{n+1} to some non-singleton closed interval contained in I_n such that $f(n+1) \notin I_{n+1}$.

Since $\mathbb{R}$ is Dedekind complete, we have that $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$. So, there is an x such that $x \in I_n$ for all $n \in \mathbb{N}$. But then $x \neq f(n)$ for all $n \in \mathbb{N}$, contradiction since f is a bijection. 😊

Note:

Upshot: Most of $\mathbb{R}$ is transcendental over $\mathbb{Q}$.

2.1 Extended Reals: $\bar{\mathbb{R}}$

Definition 2.1.1: Extended Reals

$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We endow $\bar{\mathbb{R}}$ with the following order: We write $x < y$ for $x, y \in \bar{\mathbb{R}}$ if:

1. $x, y \in \mathbb{R}$ and $x < y$.
2. $x = -\infty$ and $y \in \bar{\mathbb{R}} \setminus \{-\infty\}$.
3. $x \in \bar{\mathbb{R}} \setminus \{\infty\}$ and $y = \infty$.

Facts:

- $(\bar{\mathbb{R}}, <)$ is an ordered set that satisfies the supremum property.
- All sets in $\bar{\mathbb{R}}$ are bounded above.
- All sets in $\bar{\mathbb{R}}$ admit a sup/inf, i.e.
  - $\sup : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.
  - $\inf : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.

  Note: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Also, $A \subseteq B \subseteq \bar{\mathbb{R}}$ implies $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$. And if $E \neq \emptyset$, then $\inf E \leqslant \sup E$.

Note:

$\bar{\mathbb{R}}$ isn't an OF because if it were, then it would be Dedekind complete and then there would exists an ordered field isomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \infty$ for some $x \in \mathbb{R}$. but then $f(x+1) = f(x)+f(1) = \infty+1 = \infty$, which is not a true statement.

Definition 2.1.2

We endow $\bar{\mathbb{R}}$ with the following "algebra."

1. If $x \in \mathbb{R}$, we set $x + \infty = \infty + x = \infty$.
2. If $x \in \mathbb{R}$, we set $x + (-\infty) = (-\infty) + x = -\infty$.
3. $\infty + \infty = \infty$.
4. $-\infty + (-\infty) = -\infty$.
5. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = \infty$.
6. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$.
7. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = -\infty$.
8. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = \infty$.
9. If $x \in \mathbb{R}$, we set $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.
10. $\infty^{-1} = 0 = (-\infty)^{-1}$.
11. If $0 < x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = \infty$.
12. If $0 > x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = -\infty$.

Forbidden/undefined: $\infty + (-\infty)$, $\infty \cdot 0$, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{\mp\infty}$.

2.1.1 Sequences in $\bar{\mathbb{R}}$

Definition 2.1.3: Sequence

A sequence in $\bar{\mathbb{R}}$ is $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$ for $\ell \in \mathbb{Z}$.

In turn, we define new sequences $\{a_N\}_{N=\ell}^{\infty}, \{b_N\}_{N=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$:

- $a_N = \inf\{x_n \mid n \geqslant N\}$.
- $b_N = \sup\{x_n \mid n \geqslant N\}$.

We then set $\liminf_{n\to\infty} x_n = \sup_{N\geqslant \ell} \inf_{n\geqslant N} x_n = \sup_{N\geqslant\ell} a_N$ and $\limsup_{n\to\infty} x_n = \inf_{N\geqslant\ell} \sup_{n\geqslant N} x_n = \inf_{N\geqslant\ell} b_N$.

Example 2.1.1

Let $x_n = \begin{cases} (-1)^n & n \equiv 0 \mod 2 \\ n & n \equiv 1 \mod 2 \end{cases}$. Then, $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 1$.

Proposition 2.1.1

Let $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$. Then $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$.

Proof. Let $M, N \geqslant \ell$ and $K = \max(M, N)$. Then, $\inf_{n>N} x_n \leqslant \inf_{n>K} x_n \leqslant \sup_{n\geqslant K} x_n \leqslant \sup_{n\geqslant M} x_n$.

Thus, $\liminf_{n\to\infty} x_n = \sup_{N\geqslant \ell} \inf_{n\geqslant N} x_n \leqslant \sup_{n\geqslant M} x_n$ for all $M \geqslant \ell$. So, $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$. 😊

Proposition 2.1.2

Let $a_n, b_n \in \bar{\mathbb{R}}$ and suppose $\exists K \geqslant \ell$ such that $a_n \leqslant b_n$ for all $n \geqslant K$. Then, $\liminf_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} b_n$ and $\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} b_n$.

Proof. We can claim that if $k \geqslant K$, then

$$\inf\{a_n \mid n \geqslant k\} \leqslant \inf\{b_n \mid n \geqslant k\}$$
$$\sup\{b_n \mid n \geqslant k\} \leqslant \sup\{a_n \mid n \geqslant k\}.$$

Indeed, if $\exists k \geqslant K$ such that $\inf\{a_n \mid n \geqslant k\} > \inf\{b_n \mid n \geqslant k\}$, then $\exists m \geqslant k$ such that $b_m < \inf\{a_n \mid n \geqslant k\} \leqslant a_m \leqslant b_m$, contradiction. Ditto for sup.

Now define for $N \geqslant \ell$, $C_N = \inf_{n\geqslant N} a_n$, $D_N = \inf_{n\geqslant N} b_N$, $E_N = \sup_{n\geqslant N} a_n$, and $F_N = \sup_{n\geqslant N} b_n$.

The above claims show that $N \geqslant K$ then $C_N \leqslant D_N$ and $E_N \leqslant F_N$. Then we iterate to learn:

$$\liminf_{n\to\infty} a_n = \sup_{N\geqslant\ell} C_N \leqslant \sup_{N\geqslant\ell} D_N = \liminf_{n\to\infty} b_n$$
$$\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell} E_N \leqslant \inf_{N\geqslant\ell} F_N = \limsup_{n\to\infty} b_n.$$

😊

Theorem 2.1.1

Suppose $a_n, b_n \in \bar{\mathbb{R}}$. The following hold:

1. If $\limsup_{n\to\infty} a_n < x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n < x$ for all $n \geqslant N$.
2. If $\liminf_{n\to\infty} a_n > x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n > x$ for all $n \geqslant N$.
3. $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} -a_n$.
4. $\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty} -a_n$.
5. $\limsup_{n\to\infty} a_n + b_n \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided that all arithmetic operations are well-defined.
6. $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leqslant \liminf_{n\to\infty} a_n + b_n$, provided that all arithmetic operations are well-defined.

Proof.

1. Suppose $\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell} \sup_{n\geqslant N} a_n < x$. This implies that $\exists N \geqslant \ell$ such that $\sup_{n\geqslant N} a_n < x$, meaning $a_n < x$ for all $n \geqslant N$.
2. Similar as above.
3. For any $\emptyset \neq X \subseteq \mathbb{F}$, we have that $-\sup(-X) = \inf X$ and $-\inf(-X) = \sup X$. So the result follows.
4. Same as above.
5. We break into cases:
   (a) $\limsup a_n = \infty$ or $\limsup b_n = \infty$. Then $\limsup a_n + b_n = \infty \geqslant \limsup a_n + \limsup b_n$.
   (b) Suppose either $\limsup a_n = -\infty$ or $\limsup b_n = -\infty$. WLOG consider the first option. Since $\limsup b_n < \infty$, then there eixsts $N_1 \geqslant \ell$ and $K \geqslant \mathbb{R}$ such that $b_n < K$ for $n \geqslant N_1$. Now let $m \in \mathbb{N}$ and note that $-\infty < -m - K$. We can use the first result of the theorem to pick $N_2 \geqslant \ell$ such that $n \geqslant N_2 \implies a_n < -m - K$. Then, if $n \geqslant \max(N_1, N_2)$, we have $a_n + b_n < -m$, so $\limsup a_n + b_n = -\infty \leqslant \limsup a_n + \limsup b_n$.

(c) $\limsup a_n, \limsup b_n \in \mathbb{R}$. Let $\epsilon > 0$, then $\exists N_1, N_2 \geqslant \ell$ such that $n \geqslant N_1 \implies a_n < \limsup a_n + \frac{\epsilon}{2}$ and $n \geqslant N_2 \implies b_n < \limsup b_n + \frac{\epsilon}{2}$. Then, $n \geqslant \max(N_1, N_2) \implies a_n + b_n < \limsup a_n + \limsup b_n + \epsilon$, so $\limsup a_n + b_n \leqslant \limsup a_n + \limsup b_n + \epsilon$ for all ϵ.

6. Same as above.

☺

Lemma 2.1.1

Let $x_n \subseteq \mathbb{R}$. The following are equivalent for $x \in \mathbb{R}$:

1. $x_n \to x$ as $n \to \infty$.

2. $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Proof. Let $\epsilon > 0$. Then $\exists N \geqslant \ell$ such that $n \geqslant N \implies x - \epsilon < x_n < x + \epsilon$. Thus, $x - \epsilon \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant x + \epsilon$ for all $\epsilon > 0$. This implies that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Now let $\epsilon > 0$. Then by the previous theorem, there exists $N_1, N_2 \geqslant \ell$ such that $\begin{cases} x - \epsilon < x_n & n \geqslant N_1 \\ x_n < x + \epsilon & n \geqslant N_2 \end{cases}$.
Thus, $n \geqslant \max(N_1, N_2) \implies x - \epsilon < x_n < x + \epsilon$, so $x_n \to x$ as $n \to \infty$. ☺

Definition 2.1.4

Let $x_n \in \bar{\mathbb{R}}$ and $x \in \bar{\mathbb{R}}$. We say that $x_n \to x$ as $n \to \infty$ if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x$.

Remarks:

1. The lemma shows this extends the notion of convergence in $\mathbb{R}$.

2. Limits are unique, when they exist.

Example 2.1.2

1. $\lim_{n\to\infty} n = \infty$ ($n \to \infty$ as $n \to \infty$).

2. Version of squeeze lemma

3. TFAE:

   - $x_n \to \infty$ as $n \to \infty$.
   - $\liminf_{n\to\infty} x_n = \infty$.
   - $\forall M \in \mathbb{N}$, there exists $N \geqslant \ell$ such that $n \geqslant N \implies M \leqslant x_N$.

Chapter 3

Metric Spaces

Definition 3.0.1: Metric

Let X be a nonempty set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x, y) \geqslant 0$ for all $x, y \in X$, and $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 3.0.2

A metric space is (X, d) for $X \neq \emptyset$ and d a metric on X.

Example 3.0.1

1. $\mathbb{R}$ with $d(x, y) = |x - y|$.
2. $\mathbb{C}$ with $d(x, y) = |x - y|$.
3. (Discrete Metric) Let $X \neq \emptyset$ be any set. Then $d : X \times X \to \{0, 1\}$ defined by $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ is a metric on X.
4. Let V be a normed metric space with norm $\|\cdot\|$. Then $d(x, y) = \|x - y\|$ is a metric on V.
5. Suppose (Y, d) is a metric space and suppose $f : X \to Y$ is an injection where $X \neq \emptyset$ is a set. Then $\sigma : X \times X \to \mathbb{R}$ defined by $\sigma(x, y) = d(f(x), f(y))$ is a metric on X.

   Proof. We need to show that σ satisfies the three properties of a metric.

   (a) $\sigma(x, y) \geqslant 0$ because $d \geqslant 0$ and $\sigma(x, y) = 0 \iff d(f(x), f(y)) = 0 \iff f(x) = f(y) \iff x = y$.

   (b) The other two are very trivial.

   😊

6. Let Y be a metric space and $\emptyset \neq X \subseteq Y$. Then $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = d_Y(x, y)$ is a metric on X.

7. Consider $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to \mathbb{R}$ with $f(x) = \log x$ and $g(x) = \frac{1}{x}$. Then $d_f(x, y) = \left|\log \frac{x}{y}\right|$ and $d_g(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|x||y|}$ are metrics on $(0, \infty)$.

8. Let V, W be finite dimensional vector spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(V, W) = \{T : V \to W : T \text{ linear}\}$. Then define $\operatorname{rk}(T) = \dim \operatorname{ran} T$ for $T \in L(V, W)$. Note that $\operatorname{ran}(T+S) = \{Tx+Sx \mid x \in \mathbb{F}\} \subseteq \{Tx+Sy \mid x, y \in \mathbb{F}\} = \operatorname{ran} T + \operatorname{ran} S$. Then, $\operatorname{rk}(T + S) \leqslant \operatorname{rk}(T) + \operatorname{rk}(S)$.

   Define $d(T, S) = \operatorname{rk}(T - S) \in \mathbb{N} \subseteq [0, \infty]$.

   - $d(T, S) = 0 \iff \operatorname{rk}(T - S) = 0 \iff T - S = 0$.
   - Has symmetry.
   - Triangle inequality: $d(T - S) = \operatorname{rk}(T - R + R - S) \leqslant \operatorname{rk}(T - R) + \operatorname{rk}(R - S) = d(T, R) + d(R, S)$.

9. Let $f : \bar{R}R \to [-1, 1]$ via $f(x) = \begin{cases} 1 & x = \infty \\ -1 & x = -\infty \\ \frac{x}{\sqrt{1+x^2}} & x \in \mathbb{R} \end{cases}$. Then $d(x, y) = |f(x) - f(y)|$ is a metric on $\bar{\mathbb{R}}$.

Definition 3.0.3

Let X be a metric space.

1. For $x \in X$ and $r \geqslant 0$, we define $B(x, r) = \{y \in X \mid d(x, y) < r\}$. And $B[x, r] = \{y \in X \mid d(x, y) \leqslant r\}$.
2. A set $E \subseteq XX$ is bounded if $\exists (R \geqslant 0)$ such that $E \subseteq B(x, R)$ for some $x \in X$.
3. Let Y be any set and $f : Y \to X$. We say f is a bounded function if $f(Y) \subseteq X$ is bounded. We write $\mathcal{B}(Y; X) = \{g : Y \to X \mid g \text{ is bounded}\}$.

Example 3.0.2

1. $f : \mathbb{R} \to \mathbb{C}$ via $f(t) = e^{it} \implies f(t) = 1 \implies f(\mathbb{R}) \subseteq B[0, 1]$ is bounded. So, $f \in \mathcal{B}(\mathbb{R}; \mathbb{C})$.
2. $f : (0, \infty) \to \mathbb{R}$ via $f(t) = \frac{\log t}{\sqrt{1+(\log t)^2}}$. So, $f \in \mathcal{B}((0, \infty); \mathbb{R})$.
3. Let X be a metric space and Y a nonempty set. Consider $\mathcal{B}(X; Y)$. If $f \in \mathcal{B}(X; Y)$, then $\exists y \in Y$ and $R \geqslant 0$ such that $d(f(x), y) \leqslant R$ for all x. Thus, $\sup_{x \in X} d(f(x), y) := \sup\{d(f(x), y) \mid x \in X\} \in [0, R]$.

   Similarly, if $f, g \in \mathcal{B}(X; Y)$, then exists $R \geqslant 0$ and $y_1, y_2 \in Y$ such that $d(f(x), y_1) \leqslant R$ and $d(g(x), y_2) \leqslant R$ for all $x \in X$. Then, $d(f(x), g(x)) \leqslant d(f(x), y_1) + d(y_1, y_2) + d(y_2, g(x)) \leqslant 2R + d(y_1, y_2) < \infty$ for all $x \in X$. So, $\sup_{x \in X} d(f(x), g(x)) < \infty$. We now define

$$\begin{aligned} d : \mathcal{B}(X; Y) \times \mathcal{B}(X; Y) &\to [0, \infty) \\ (f, g) &\mapsto \sup_{x \in X} d(f(x), g(x)). \end{aligned}$$

   Proof. Consider the properties of a metric:

   - $d(f, g) = 0 \iff \sup_{x \in X} d(f(x), g(x)) = 0 \iff d(f(x), g(x)) = 0 \iff f(x) = g(x)$ for all $x \in X \iff f = g$.
   - Symmetry is trivial.
   - Let $f, g, h \in \mathcal{B}(X; Y)$. Then, $d(f, h) = \sup_{x \in X} d(f(x), h(x)) \leqslant \sup_{x \in X} d(f(x), g(x)) + d(g(x), h(x)) \leqslant d(f, g) + d(g, h)$.

   😊

Definition 3.0.4

Let X and Y be metric spaces:

1. A map $f : X \to Y$ is an isometric embedding if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Note, such an f is injective.
2. f is an isometry if it's an isometric embedding and surjective.
3. X and Y are isometric if there exists an isometry $f : X \to Y$.

Example 3.0.3

1. Consider $\mathbb{R}^n$ with $|\cdot| = \|\cdot\|_2$, that is, 2-norm.
2. Recall $O(n) = \{\mathcal{M} \in \mathbb{R}^{n\times n} \mid \mathcal{M}^T\mathcal{M} = I\}$ and $R \in O(n) \implies |Rx| = |x|$.
   Let $a \in \mathbb{R}^n$, $R \in O(n)$, and set $f : \mathbb{R}^n \to \mathbb{R}^n$ via $f(x) = a + Rx$. Then,
   $$|f(x) - f(y)| = |a + Rx - (a + Ry)| = |Rx - Ry| = |R(x - y)|.$$
   Also, $y = f(x) = a + Rx \iff y - a = Rx$. So, f is an isometry.
3. Consider $x \mapsto ix \in \mathbb{C}$ for $x \in \mathbb{R}$. This is an isometric embedding but obviously not an isometry for it is not surjective.

The next example is so important that we call it a theorem. Recall $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{R})$ for $X \neq \emptyset$ is a set. Note that if V is a normed vector space, then $\mathcal{B}(X; V)$ is too: $\|f\|_{\mathcal{B}} = \sup_{x\in X} \|f(x)\|_V$ is a norm (exercise) and $d_{\mathcal{B}}(f, g) = \|f - g\|_{\mathcal{B}}$.

Theorem 3.0.1

Let X be a metric space and fix an arbitrary element $a \in X$. For $x \in X$, we'll define $\varphi_x : X \to \mathbb{R}$ via $\varphi_x(y) = d(x, y) - d(y, a)$. The following hold:

1. $\varphi_x \in \mathcal{B}(X)$ for all $x \in X$.
2. Define $\Phi : X \to \mathcal{B}(X)$ via $\Phi(x) = \varphi_x$. Then, Φ is an isometric embedding.

Proof. First note, $|\varphi_x(y)| = |d(x, y) - d(y, a)| \leqslant d(x, a)$ by the triangle inequality. So, $\|\varphi_X\|_{\mathcal{B}} = \sup_{y\in X} |\varphi_x(y)| \leqslant d(x, a) < \infty$. This shows the first result.

Next, fix $x, z \in X$ and consider $\varphi_x(y) - \varphi_z(y) = d(x, y) - d(y, a) - d(z, y) + d(y, a)$. So,

$$|\varphi_x(y) - \varphi_z(y)| = |d(x, y) - d(y, z)| \leqslant d(x, z).$$

Thus, $d_{\mathcal{B}}(\varphi_x, \varphi_y) = \|\varphi_x - \varphi_y\|_{\mathcal{B}} = \sup_{y\in X} |\varphi_x(y) - \varphi_z(y)| \leqslant d(x, z)$.

On the other hand, $|\varphi_x(z) - \varphi_z(z)| = |d(x, z) - \cancelto{0}{d(z, z)}| = d(x, z)$. So, $d_{\mathcal{B}}(\varphi_x, \varphi_z) = d(x, z)$. ☺

Chapter 4

Basic Metric Space Topology

FILL IN LATER

Proposition 4.0.1

Let $Y_1, \dots, Y_n$ be metric spaces and consider $Y = \prod_{i=1}^n Y_i$, endowed with a p-metric from Homework 3. That is,

$$d_p(x, y) = \begin{cases} \left(\sum_{i=1}^n d_{Y_i}^p(x_i, y_i)\right)^{1/p} & 1 \leqslant p < \infty \\ \max_{1 \leqslant i \leqslant n} d_{Y_i}^p(x_i, y_i) & p = \infty \end{cases}.$$

Suppose $\{y_k\}_{k=\ell}^\infty \subseteq Y$ is given by $y_k = (y_{k,1}, \dots, y_{k,n})$. The following hold:

1. Let $y = (y_1, \dots, y_n) \in Y$. Then $y_k \to y$ in Y as $n \to \infty \iff y_{k,i} \to y_i$ in Y_i as $k \to \infty$ for all $1 \leqslant i \leqslant n$.
2. $\{y_k\}_{k=\ell}^\infty$ is Cauchy in Y if and only if $\{y_{k,i}\}_{k=\ell}^\infty$ is Cauchy in Y_i for all $1 \leqslant i \leqslant n$.

Proof. We'll only prove 1. as 2. is very similar. Suppose $y_k \to y$ as $k \to \infty$. Note that for $1 \leqslant i \leqslant n$, $d_i(y_{k,i}, y_i) \leqslant d_Y(y_k, y)$. Thus, for $\epsilon > 0$, we pick $K \geqslant \ell$ such that if $k \geqslant K$, then $d_Y(y_k, y) \leqslant \epsilon$. But then $k \geqslant K \implies d_i(y_{k,i}) \leqslant d_Y(y_k, y) \leqslant \epsilon$ for all $1 \leqslant i \leqslant n$, meaning $y_{k,i} \to y_i$ as $k \to \infty$ for $1 \leqslant i \leqslant n$.

Now suppose $y_{k,i} \to y_i$ as $k \to \infty$ for all $1 \leqslant i \leqslant n$. Let $\epsilon > 0$ and pick $K_i \geqslant \ell$ such that $k \geqslant K_i \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$. Let $K = \max K_i \geqslant \ell$, and note $k \geqslant K \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$ for all $1 \leqslant i \leqslant n$. This means

$$\begin{cases} \left(\sum_{i=1}^n d_i^p(y_{k,i}, y_i)\right)^{1/p} \leqslant \left(\sum_{i=1}^n \frac{\epsilon^p}{n}\right)^{1/p} = \epsilon & 1 \leqslant p < \infty \\ \max_i d_i(y_{k,i}, y_i) < \epsilon & p = \infty \end{cases}$$

So, $y_k \to y$ as $k \to \infty$. 😄

Definition 4.0.1

Let $X \neq \emptyset$ be a set and d_1, d_2 be metrics on X. We say d_1 and d_2 are equivalent if $\exists c_1, c_2 > 0$ such that $c_1 d_1(x, y) \leqslant d_2(x, y) \leqslant c_2 d_1(x, y)$ for all $x, y \in X$.

The point is that equivalent metrics give the same notions of convergence, Cauchyness, and boundedness.

Example 4.0.1 (Equivalent Norms)

1. All norms on $\mathbb{F}^n$ are equivalent.
2. From recitation, $\|\cdot\|_p$ are all equivalent on $\mathbb{F}^n$ for $1 \leqslant p \leqslant \infty$.
3. Let $Y_1, \ldots, Y_n$ be metric spaces and form $Y = \prod_{i=1}^n Y_i$. Then

$$d_p(x,y) = \|(d_1(x,y), \ldots, d_n(x,y))\|_p \asymp \|(d_1(x,y), \ldots, d_n(x,y))\|_q = d_q(x,y)$$

Therefore, $d_p \asymp d_q$ in Y.

Note: This does not mean all metrics on Y are equivalent.

Example 4.0.2

Let $V_1, \ldots, V_n, W$ be normed vector sapces over $\mathbb{F}$. We define $\mathcal{L}(V_1, \ldots, V_n; W)$ is the set of $\{T \in L(V_1, \ldots, V_n; W) \mid \|T\|_{\mathcal{L}} < \infty\}$ where $\|T\|_{\mathcal{L}} := \sup\{\|T(v_1, \ldots, v_n)\|_W \mid v_i \in V_i : \|v_i\|_{V_i} < 1\} \in [0, \infty]$.
Facts:

1. This is indeed a norm.
2. $T \in \mathcal{L} \iff \|T(v_1, \ldots, v_n)\|_W \leqslant c \prod_{i=1}^n \|v_i\|_{V_i}$ for all $v_i \in V_i$ for some $0 \leqslant c < \infty$. $c = \|T\|_{\mathcal{L}}$ is the best constant.

Theorem 4.0.1 Algebra of Sequences

Let $V_1, \ldots, V_n, W$ be normed vector spaces over a common field $\mathbb{F}$. The following hold:

1. Let $\{v_{k,i}\}_{k=\ell}^{\infty} \subseteq V_i$ for $1 \leqslant i \leqslant n$ be such that $v_{k,i} \to v_i$ in V_i as $k \to \infty$. Let $\{T_k\}_{k=\ell}^{\infty} \subseteq \mathcal{L}(V_1, \ldots, V_n; W)$ be such that $T_k \to T$ as $k \to \infty$. Then $T_k(v_{k,1}, \ldots, v_{k,n}) \to T(v_1, \ldots, v_n)$ in W as $k \to \infty$.
2. If $\{u_k\}, \{v_k\} \subseteq V_1$ are such that $u_k \to u$, $v_k \to v$ then $u_k + v_k \to u + v$ as $k \to \infty$.

Proof. We'll only do 1 because 2 is easy. We start with $n = 2$ for simplicity. Suppose $\{x_k\} \subseteq V_1$, $\{y_k\} \subseteq V_w$ such taht $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Then let $\sup_{k \geqslant \ell} \max\{\|x_k\|_{V_1}, \|y_k\|_{V_2}, \|T_k\|_{\mathcal{L}}\} = M < \infty$. Then,

$$T_k(x_k, y_k) - T(x,y) = T_k(x_k, y_k - y) + T_k(x_k, y) - T(x,y)$$
$$T_k(x_k, y_k + y) + T(x_k - x, y) + T_k(x,y) - T(x,y).$$

This shows that

$$\|T_k(x_k, y_k) - T(x,y)\|_W \leqslant \|T_k\|_{\mathcal{L}} \|x_k\|_{V_1} \|y - y_k\|_{V_2} + \|T_k\|_{\mathcal{L}} \|x - x_k\|_{V_1} \|y_k\|_{V_2} + \|T - T_k\|_{\mathcal{L}} \|x_k\|_{V_1} \|$$
$$\leqslant M^2 \|y - y_k\|_{V_2} + M^2 \|x - x_k\|_{V_1} + M^2 \|T - T_k\|_{\mathcal{L}} \to 0$$

as $k \to \infty$. 😄

Definition 4.0.2

1. We say a metric space X is complete if every Cauchy sequence in X is convergent in X.
2. We say a normed vector space is Banach if it's complete.
3. We say an inner product space is a Hilbert space if it's Banach.

Example 4.0.3

1. $(\mathbb{R}, |\cdot|)$ is complete.
2. $X = \prod_{i=1}^n X_i$ with p-metric is complete if and only if each X_i is complete. In particular, $(\mathbb{R}^n, \|\cdot\|)$ is complete.
3. $\mathbb{F}^n$ is complete with any more.
4. $\mathbb{R} \setminus \{0\}$ is not complete with $|\cdot|$ as the metric.
5. $\mathbb{Q}^n$ with $|\cdot|$ is not complete.

Example 4.0.4

1. V is a finite dimensional normed vector spaces. $\varphi : \mathbb{F}^n \to V$ isomorphism. Then $\mathbb{F}^n \ni x \mapsto \|\varphi(x)\|_V \in [0, \infty)$ defines a norm on $\mathbb{F}^N$, which we call $|||x|||$. Then $(\mathbb{F}^n, |||\cdot|||)$ is isometric to $(V, \|\cdot\|_V)$, and hence V is complete.
2. Let $\emptyset \neq X$ be a set endowed with the discrete metric. Suppose $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy and pick $N \geqslant \ell$ such that $n, m \geqslant N \implies d(x_n, x_m) < 1$. Then $x_n = x_m = x_N$. So $x_n \to x_N$ as $n \to \infty$. Therefore X is complete.

Note that $Y = \prod Y_i$ is complete iff each individual Y_i is complete.

Theorem 4.0.2

Let $V_1, \ldots, V_k, W$ be normed vector spaces over $\mathbb{F}$. If W is Banach, then so is $\mathcal{L}(V_1, \ldots, V_k)$.

Proof. Suppose $\{T_n\}_{n=\ell}^\infty \subseteq \mathcal{L}(V_1, \ldots, V_k; W$ is Cauchy. For fixed $v_1, \ldots, v_k) \in \prod_{i=1}^k V_i$, we bound

$$\|T_n(v_1, \ldots, v_k) - T_m(v_1, \ldots, v_k)\|_W \leqslant \|T_n - T_m\|_{\mathcal{L}} \prod_{i=1}^k \|v_i\|_{V_i}.$$

Therefore, $\{T_n(v_1, \ldots, v_k)\}_{n=\ell}^\infty \subseteq W$ is Cauchy and hence convergent. We may thus define $T : V_1 \times \cdots \times V_k \to W$ via $T(v_1, \ldots, v_k) = \lim_{n\to\infty} T_n(v_1, \ldots, v_k)$.

1. $T \in L(V_1, \ldots, V_k; W)$:

$$T_n(\alpha x + \beta y, v_2, \ldots, v_k) = \alpha T_n(x, v_2, \ldots, v_k) + \beta T_n(y, v_2, \ldots, v_k)$$

   As $n \to \infty$, we get:

$$T(\alpha x + \beta y, v_2, \ldots, v_k) = \alpha T(x, v_2, \ldots, v_k) + \beta T(y, v_2, \ldots, v_k).$$

   Repeat in other slots if $k \geqslant 2$. As such, it is multilinear.

2. $T \in \mathcal{L}(V_1, \ldots, V_k; W)$: Fix $v_i \in V_i$ with $\|v_i\|_{V_i} \leqslant 1$. Then

$$\begin{aligned}\|T(v_1, \ldots, v_k)\|_W &= \lim_{n\to\infty} \|T_n(v_1, \ldots, v_k)\|_W \\ &\leqslant \left(\limsup_{n\to\infty} \|T_n\|_{\mathcal{L}}\right) \prod_{n=1}^\infty \|v_i\|_{V_i} \leqslant \limsup_{n\to\infty} \|T_n\|_{\mathcal{L}} < \infty.\end{aligned}$$

3. $T_n \to T$ in $\mathcal{L}$ as $n \to \infty$: Let $\epsilon > 0$ and pick $N \geqslant \ell$ such that $n, m \geqslant N \implies \|T_n - T_m\|_{\mathcal{L}} < \frac{\epsilon}{2}$. Then let $v_i \in V_i$ with $\|v_i\|_{V_i} \leqslant 1$. Then,

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W = \lim_{m\to\infty} \|T_m(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leqslant \lim_{m\to\infty} \|T_m - T_n\|_{\mathcal{L}} < \frac{\epsilon}{2}.$$

But this implies

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leqslant \frac{\epsilon}{2}.$$

By taking the supremum, we get that $\|T - T_n\|_{\mathcal{L}} \leqslant \frac{\epsilon}{2} < \epsilon$.

☺

Corollary 4.0.1

$V^* = \mathcal{L}(V; \mathbb{F})$ is always Banach.

Definition 4.0.3

Let X be a metric space, $E \subseteq X$.

1. $x \in E$ is an interior point if $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq E$. $E^\circ = \{x \in E \mid x \text{ is an interior point}\}$. E is open iff $E = E^\circ$. E is closed iff E^c is open.
2. $x \in X$ is a boundary point of E if $\forall \epsilon > 0$, $B(x, \epsilon) \cap E \neq \emptyset$ and $B(x, \epsilon) \cap E^c = \emptyset$. We write $\partial E = \{x \in X \mid x \text{ is a boundary point of } E\}$. $\bar{E} = E^\circ \cup \partial E$.
3. We say $x \in X$ is a limit point (accumulation point) of E if $\forall \epsilon > 0$ $(B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$. We write $E' = \{x \in X \mid x \text{ is a limit point of } E\}$. If $x \in E \setminus E'$, then x is an isolated point.

Example 4.0.5

Let $(X, disc)$ be given. Claim: all subsets of X are both open and closed.

Proof. $B(x, 1) = \{x\} \implies E \subseteq X$ can be written as

$$E = \cup_{x\in E} B(x, 1),$$

which is open. Therefore $E = (E^c)^c$ is also closed. ☺

Any metric space in which all sets are open and closed is called a discrete space.

Theorem 4.0.3

Let X be a metric space and $C \subseteq X$. The following are equivalent:

1. C is closed.
2. C is sequentially closed; If $\{x_n\}_{n=\ell}^\infty \subseteq C$ is such that $x_n \to x$ in X as $n \to \infty$, then $x \in C$.

Proof. $1 \to 2$. Let $\{x_n\} \subseteq C$ be such that $x_n \to x \in X$. Suppose BWOC that $x \in C^c$, which is open. Then $\exists N \geqslant \ell$ such that $n \geqslant N \implies x_n \in C^c \cup C$, which is a contradiction.

$2 \to 1$. BWOC, suppose that C is not closed, which emans C^c is not open. Then $\exists x \in C^c$ such that we can pick $\{x_n\}_{n=0}^\infty \subseteq C$ such that $x_n \in B(x, 2^{-n}) \cap C$. This means that $\{x_n\}_{n=0}^\infty \subseteq C$ and $x_n \to x$ as $n \to \infty$. But $x \notin C$, so we have a contradiction. ☺

Corollary 4.0.2

Let X be a complete metric space, and $\emptyset \neq C \subseteq X$. Then C is closed in X iff C is a complete metric space with the metric from X.

Proof. $\implies$: Let $\{x_n\}_{n=\ell}^{\infty} \subseteq C$ be Cauchy. Then $x_n \to x \in X$ as $n \to \infty$ because X is complete. By since C is closed, $x \in C$.

$\impliedby$: Let $\{x_n\} \subseteq C$ be such that $x_n \to x$ in X as $n \to \infty$. Then $\{x_n\}$ is cauchy in C, meaning it's convergent in C, so $x \in C$, so C is sequentially closed. 😊

Definition 4.0.4

Let X be a metric space and $A \subseteq B \subseteq X$. We say A is dense in B if $\forall b \in B$, $\exists \{a_n\} \subseteq A$ such that $a_n \longrightarrow b$ as $n \longrightarrow \infty$.

Example 4.0.6

1. $\mathbb{Q}$ is dense in $\mathbb{R}$. $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. $(\mathbb{Q}^n + i\mathbb{Q}^n) \subseteq \mathbb{C}^n$ is dense.
2. $B(x, r) \subseteq \mathbb{R}^n$ is dense in $B[x, r]$.
3. Let X be given the discrete metric. $B(x, 1) = \{x\}$, but $B[x, 1] = X$, so as long as $X \neq \{x\}$, we do not have $B(x, 1) \subseteq B[x, 1]$ is dense.

Proposition 4.0.2

Let X be a metrid space, $A \subseteq B \subseteq X$. The following are equivalent:

1. A is dense in B.
2. $B \subseteq \bar{A}$.
3. $\forall x \in B$ and $\epsilon > 0$, $\exists a \in A$ such that $d(x, a) < \epsilon$.
4. $\forall x \in B$ and $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$.

Proof. Recall $\bar{A} = A \cup A'$.

$1 \implies 2$. Let $b \in B$. If $b \in A$, we're done. Otherwise $b \notin A$, but by density $\exists \{a_n\}_{n=\ell}^{\infty} \subseteq A \setminus \{b\}$ such that $a_n \longrightarrow b$ as $n \longrightarrow \infty$. Thus, $b \in A'$.

$2 \implies 1$. Suppose $B \subseteq A \cup A' = \bar{A}$. Let $b \in B$. If $b \in A$, let $\{a\}_{n=\ell}^{\infty} = b$ then we're done.

So suppose $b \in A' \setminus A$. By definition of limit point, we can pick a sequence $\{a_n\}$ such that $a_n \longrightarrow b$ as $n \longrightarrow \infty$. So A is dense in B.

$3 \iff 4$ is trivial.

$2 \iff 3$. Again, use $\bar{A} = A \cup A'$. 😊

Corollary 4.0.3

Let X be a metric space and $A \subseteq B \subseteq X$. If A is dense in B, then A is also dense in $\bar{B}$.

Proof. $A \subseteq B$ is dense $\implies A \subseteq B \subseteq \bar{A}$. So $\bar{B} \subseteq \bar{A}$, meaning A is dense is $\bar{B}$ as desired. 😊

Definition 4.0.5

Let X be a metric space. We say X is separable if X has a countable dense subset.

Example 4.0.7 (Separable Vector Spaces)

1. $\mathbb{R}^n$ is separable, ditto for $\mathbb{C}^n$.
2. Let V be a finite dimensional normed vector space. Let $\varphi : \mathbb{F}^n \to V$ be an isomorphism. Endow $\mathbb{F}^n$ with norm $|||x||| = \|\varphi(x)\|_V$, which is equivalent to $|\cdot|$ on $\mathbb{F}^n$. Then V is separable with $\varphi(\mathbb{Q}^n)$ as a countable dense subset.
3. $\ell^\infty(\mathbb{N}; \mathbb{F})$ is not separable, but $\ell^p(\mathbb{N}; \mathbb{F})$ is for $1 \leqslant p < \infty$.

Definition 4.0.6

Let X, X^* be metric spaces. We say that X^* completes X if:

1. X^* is complete.
2. $\exists f : X \to X^*$ an isometric embedding.
3. $f(x) \subseteq X^*$ is dense.

Theorem 4.0.4 Uniqueness of completions

Let X, Y, Z be metric spaces. Suppose Y and Z both complete X. Then Y and Z are isometric.

Proof. Let $g : X \to Y$ and $h : X \to Z$ be isometric embeddings. We will construct an isometric $f : Y \to Z$. Let $y \in Y$. Since $g(X) \subseteq Y$ is dense, $\exists \{y_n\}_{n=\ell}^\infty \subseteq g(X)$ such that $y_n \to y$ as $n \to \infty$.

Then $\exists! \{x_n\}_{n=\ell}^\infty \subseteq X$ such that $g(x_n) = y_n$ for all $n \geqslant \ell$. Then upon setting $z_n = h(x_n) = h \circ g^{-1}(y_n)$, we have

$$d_Z(z_n, z_m) = d_X(x_n, x_m) = d_Y(y_n, y_m).$$

This means $\{z_n\}$ is Cauchy, and therefore convergent as Z is complete.

Suppose $\{y_n'\}_{n=\ell}^\infty$ is another sequence such that $y_n' \to y$ as $n \to \infty$. Note

$$d_Y(y_n, y_n') = d_X(g^{-1}(y_n), g^{-1}(y_n')) = d_Z(h(g^{-1}(y_n)), h(g^{-1}(y_n'))) = d_Z(z_n, z_n').$$

Therefore, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n'$. So, we can define $f : Y \to Z$ as $f(y) = \lim_{n\to\infty} h(g^{-1}(y_n))$ for any sequence $\{y_n\} \subseteq g(X)$ such that $y_n \to y$ as $n \to \infty$.

We claim that f is an isometric embedding. Let $y, y' \in Y$ and pick $\{y_n\}_{n=\ell}^\infty$ and $\{y_n'\}_{n=\ell}^\infty$ such that $y_n \to y$ and $y_n' \to y'$ as $n \to \infty$. Then,

$$d_Y(y_n, y_n') = d_X(g^{-1}(y_n), g^{-1}(y_n')) = d_Z(h(g^{-1}(y_n)), h(g^{-1}(y_n'))) \to d_Z(f(y), f(y')) = d_Y(y, y'),$$

so f is an isometric embedding.

We claim that f is surjective. Let $z \in Z$ and pick $\{x_n\}_{n=\ell}^\infty$ such that $h(x_n) = z_n \to z$ as $n \to \infty$. Then let $y_n = g(x_n)$. Then $\{y_n\}_{n=\ell}^\infty \subseteq Y$ are Cauchy and hence convergent to $y \in Y$. Then $f(y) = \lim_{n\to\infty} h \circ g^{-1}(y_n) = \lim_{n\to\infty} z_n = z$. So $f : Y \to Z$ is an isometry! 😊

Note:

This is analogous to the uniqueness of Dedekind complete ordered fields. In principal, there can be different techniques for finding /constructing completions of a given metric space, but in the end they're isometric.

Theorem 4.0.5
Let $X \neq \emptyset$ be a set and Y be a metric space. Then $\mathcal{B}(X;Y)$ is complete if and only if Y is complete.

Proof. HW5 ☺

Corollary 4.0.4
Let $X \neq \emptyset$ be a set. Then $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{R})$ is a Banach space.

Proof. $\mathbb{R}$ is complete. ☺

Theorem 4.0.6
Let X be a metric space. Then X has a completion.

Proof. Let $\Phi : X \to \mathcal{B}(X)$ be the isometric embedding we previously constructed. Let $X^* = \overline{\Phi(X)}$, which is closed in (B) and hence a complete metric space. By construction, $\Phi(X)$ is dense in X^*. So, X^* is complete. ☺

Remarks:

1. Why not just set $\mathbb{R} = \bar{\mathbb{Q}}$? It's cyclic!
2. $\exists$ another construction of X^* which is more "direct" and proceeds through Cauchy(X) from HW4. This idea has room to play. It can be hacked to yield an alternate construction of $\bar{\mathbb{R}}$ from $\mathbb{Q}$ or any other Archimedean ordered field.

4.1 Limits and Continuity

Definition 4.1.1

Let X, Y be metric spaces, $E \subseteq X$, $z \in E'$, $f : E \to Y$. We say that f has limit $y \in Y$ as $x \to z$, written as $f(x) \to y$ as $x \to z$ or $\lim_{x \to z} f(x) = y$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that $x \in E$ and $0 < d_X(x,z) < \delta \implies d_Y(f(x),y) < \epsilon$.

Remarks:

1. limits are unique when they exist
2. the definition only requires $z \in E'$, not $z \in E$. that is, $f(z)$ doesn't need to be defined and even if it is, the defintion doesn't care what it is.

Theorem 4.1.1 Sequential characterization of limits

Let X, Y be metric spaces, $E \subseteq X$, $f : E \to Y$, $z \in E'$, $y \in Y$. The following are equivalent:

1. $f(x) \to y$ as $x \to z$.
2. $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(B(z, \delta) \setminus \{z\}) \subseteq B_Y(y, \epsilon)$.
3. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{x\}$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to y$ as $n \to \infty$.

Proof. $1 \iff 2$ is a triviality. Now we show $1 \implies 3$. Let $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ be such that $x_n \to z$ as $n \to \infty$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $0 < d_X(x, z) < \delta \implies d_Y(f(x), y) < \epsilon$. Pick $N \geqslant \ell$ such that $n \geqslant N$ implies $0 < d_X(x_n, z) < \delta$. So, $d_Y(f(x_n), y) < \epsilon$. Therefore, $f(x_n) \to y$ as $n \to \infty$.

Now for $3 \implies 1$. Suppose BWOC $\neg 1$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in E$ such that $0 < d(x, z) < \delta$, $d(f(x), y) \geqslant \epsilon$.

For $\delta = 2^{-n}$, $n \in \mathbb{N}$, we then get $\{x_n\}_{n=0}^{\infty} \subseteq E \setminus \{z\}$ such that $d(x_n, z) < 2^{-n}$, but $d(f(x_n), y) \geqslant \epsilon$. Now we use 3: $x_n \to z$ as $n \to \infty$, so $f(x_n) \to y$ as $n \to \infty$. In particular, $\exists N \geqslant 0$ such that $n \geqslant N \implies d(f(x_n), y) < \epsilon$. This is a contradiction. 😄

Theorem 4.1.2 Limits and components

Let $X, Y_1, \ldots, Y_n$ be metric spaces, and let $Y = \prod Y_i$ endowed with usual p-metric. Let $E \subseteq X$, $z \in E'$, $f : E \to Y$. Write $f = (f_1, \ldots, f_n)$ where $f_i : E \to Y_i$. The following are equivalent for $y = (y_1, \ldots, y_n) \in Y$:

1. $f(x) \to y$ as $x \to z$.
2. $f_i(x) \to y_i$ as $x \to z$ for $1 \leqslant i \leqslant n$.

Proof. This follows from the sequential characterization of limits combined with the characterization of limits of sequences in the product space Y. 😄

Theorem 4.1.3 Algebra of limits

Let X be a metric space, $E \subseteq ZX$, $z \in E'$. The following hold:

1. Let V be a normed vector space and suppose $f, g : E \to V$, $\alpha : E \to \mathbb{F}$ are such that $f(x) \to v_1$, $g(x) \to v_2$, and $\alpha(x) \to \beta$ as $x \to z$. Then:
   (a) $f(x) + g(x) \to v_1 + v_2$ as $x \to z$.
   (b) $\alpha(x)f(x) \to \beta v_1$ as $x \to z$.
2. Let $V_1, \ldots, V_k, W$ be normed vector spaces over $\mathbb{F}$. Suppose $f_i : E \to V_i$ and $T : E \to \mathcal{L}(V_1, \ldots, V_k; W)$ are such that $f_i(x) \to v_i$ as $x \to z$ and $T(x) \to M$ as $x \to z$. Then,
$$E \ni x \mapsto T(x)(f_1(x), \ldots, f_k(x)) \in W$$
   satisfies $T(x)(f_1(x), \ldots, f_k(x)) \to M(v_1, \ldots, v_k)$ as $x \to z$.

Proof. Use characterization of limits via sequences together with algebra of sequential limits. 😄

Definition 4.1.2

Let X, Y be metric spaces, $E \subseteq X$, $z \in E$, and $f : E \to Y$. We say f is continuous at z if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in E$ and $d(x, z) < \delta \implies d(f(x), f(z)) < \epsilon$. We say f is continuous on E if f is continuous at every point of E.

Remarks:

1. If z is isolated, i.e. $z \in E \setminus E'$, then the definition of continuity is true vacuously and so f is continuous as z.
2. Unlike when computing limits, we need $f(z)$ defined, and $x = z$ is allowed.
3. We can think of $f : E \to Y$ with E a metric space on its own with $d_E = d_X$.

Theorem 4.1.4 Characterizations of continuity

Let X, Y be metric spaces and $z \in E \subseteq X$ and $f : E \to Y$. The following are equivalent:

1. f is continuous at z.
2. $\forall \epsilon > 0$, $\exists \delta > 0$ such $f(E \cap B(z, \delta)) \subseteq B(f(z), \epsilon)$.
3. If $z \in E'$, then $f(x) \to f(z)$ as $x \to z$.
4. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to f(z)$ as $n \to \infty$.
5. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to f(z)$ as $n \to \infty$.
6. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ is such that $x_n \to z$ as $n \to \infty$, then $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is convergent.

Proof. 1 $\iff$ 2 is obvious as well as 3 $\iff$ 4 since we proved it in the sequential chracterization of limits.

We'll prove 1 $\implies$ 5 $\implies$ 6 $\implies$ 4 and 3 $\implies$ 1.

3 $\implies$ 1: If $z \in E \setminus E'$, we're done because of earlier remark. So let $z \in E \setminus E'$. Then 3 is in play: $f(x) \to f(z)$ as $x \to z$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $0 < d(x, z) < \delta \implies d(f(x), f(z)) < \epsilon$.

Note, $x = z \iff d(x, z) = 0$, in which case $d(f(x), f(z)) = 0 < \epsilon$. So f is continuous at z.

1 $\implies$ 5: Suppose f is continuous at z and let $\{x_n\} \subseteq E$ be such that $x_n \to z$ as $n \to \infty$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $d(x, z) < \delta \implies d(f(x_n), f(z)) < \epsilon$. Pick $N \geqslant \ell$ such that $n \geqslant N$ implies $d(x_n, z) < \delta \implies d(f(x_n), f(z)) < \epsilon$. So $f(x_n) \to f(z)$ as $n \to \infty$.

5 $\implies$ 6. Trivial

6 $\implies$ 4. Let $\{x_n\} \subseteq E \setminus \{z\}$ be such that $x_n \to z$ as $n \to \infty$. Define $\{y_n\} \subseteq E$ via

$$y_n = \begin{cases} x_n & n = \ell + 2k \\ z & n = \ell + 2k + 1 \end{cases}.$$

Then $y_n \to z$ as $n \to \infty$. 6 implies that $f(y_n)$ converges. So we can pick a subsequence to show that is converges to $f(z)$. 😆

Corollary 4.1.1 Corallary 1

Let X, Y be metric spaces, $f : X \to Y$. f is continuous if and only if if $\{x_n\} \subseteq X$ is convergent, then $\{f(x_n)\} \subseteq Y$ is convergent.

Corollary 4.1.2 Corallary 2

Let X, Y be metric spaces with X separable. Let $f : X \to Y$ be continuous. Then $f(X) \subseteq Y$ is separable.

Theorem 4.1.5 Continuity and products

Let $X, Y_1, \ldots, Y_k$ be metric spaces and let $f : X \to Y := \prod Y_i$. Let $z \in X$. Write $f = (f_1, \ldots, f_k)$ where $f_i : X \to Y_i$. The following are equivalent:

1. f is continuous at z.
2. Each f_i is continuous at z.

Proof. Proof direct from limit chracterization. ☺

Theorem 4.1.6 Algebra of continuity

Sum, product, and multilinear functions of continuous functions are continuous.

Theorem 4.1.7 Composition

Let X, Y, Z be metric spaces and $f : X \to Y$ and $g : Y \to Z$. Suppose f is continuous at $z \in X$ and g is continuous at $f(z)$. Then $g \circ f : X \to Z$ is continuous at z.

Theorem 4.1.8

Let X, Y be metric spaces and $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. $f^{-1}(U)$ is open $\forall U \subseteq Y$ open.
3. $f^{-1}(U)$ is closed $\forall U \subseteq Y$ closed.

Theorem 4.1.9 Multilinearity and continuity

Let $V_1, \ldots, V_k$ and W be normed vector spaces over $\mathbb{F}$, and let $T \in L(V_1, \ldots, V_k; W)$. The following are equivalent:

1. $T \in \mathcal{L}(V_1, \ldots, V_k; W)$ i.e. T is a bounded multilinear map.
2. T is continuous.
3. T is continuous at $0 \in \prod V_i$.

Proof. $1 \implies 2$: Let $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$ be two vectors in $V_1 \times \ldots \times V_k$. We write

$$T(v_1, \ldots, v_k) - T(u_1, \ldots, u_k) = T(v_1 - u_1, v_2, \ldots, v_k) + T(u_1, v_2 - u_2, \ldots, v_k) + \ldots + T(u_1, \ldots, u_{k-1} v_k - u_k).$$

This implies that $\|T(u) - T(v)\| \leqslant \|T\|_{\mathcal{L}} \Big[\|v_1 - u_1\|_{V_1} \prod_{i=2}^k \|v_i\|_{V_i} + \|u_1\|_{V_1} \|u_2 - v_2\|_{V_2} \prod_{i=3}^k \|u_i\|_{V_i} + \cdots + \prod_{i=1}^{k-1} \|u_1\|_{V_i} \|u_k - v_k\|_V$

From this est, it's easy to conclude that T is continuous at u.

$2 \implies 3$: trivial

$3 \implies 1$: Suppose T is continuous at 0. Let $\epsilon = 1$ and let $\delta > 0$ such that $\|u\|_p = \begin{cases} \left(\sum \|u_i\|_{V_i}^p \right)^{1/p} & p < \infty \\ \max & p = \infty \end{cases} < \delta$. This implies that $\|T(u)\|_W = \|T(u) - T(0)\|_W < \epsilon = 1$.

Let $u_i \in V_i$ be such that $\|u_i\|_{V_i} = 1$. Then

$$\| \left(\frac{\delta}{2k^{1/p}} u_1, \ldots, \frac{\delta u_k}{2k^{1/p}} \right) = \frac{\delta}{2} < \delta.$$

So, T applied to that value is less than 1. But this means

$$\left(\frac{\delta}{2k^{1/p}}\right)\|T(u)\|_W < 1$$
$$\|T(u)\|_W \leqslant \left(\frac{2k^{1/p}}{\delta}\right)^k.$$

as well. By taking the supremum, we get that T is bounded and in $\mathcal{L}$. 😊

Definition 4.1.3

Let V, W be normed vector spaces over $\mathbb{F}$.

1. Recall $L^k(V;W) = L(V_1, \ldots, V_k; W)$ and similarly for $\mathcal{L}$. Given $T \in L^k(V;W)$ and $v \in V$, we write $Tv^{\otimes k} = T(v^{\otimes k}) = T(v, \ldots, v)$.
2. A polynomial is a map $p : V \to W$ given by $p(v) = \sum_{k=0}^{d} T_k v^{\otimes k}$ for $T_k \in L^k(V;W)$. We write $d =$ degree of p given that $T_d \neq 0$. Note: by the continuity of $T_k \in \mathcal{L}^k(V;W)$ and algebra of continuity, all polynomials are continuous.

Definition 4.1.4

Let X, Y be metric spaces and $f : X \to Y$.

1. We say f is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x, y \in X$ and $d_X(x,y) < \epsilon$ then $d_Y(f(x), f(y)) < \epsilon$.
2. f is Lipschitz if $\exists c \geqslant 0$ such that $d(f(x), f(y)) \leqslant c d(x,y)$ for all $x, y \in X$.

Facts:

1. Lipschitz $\implies$ uniformly continuous $\implies$ continuous.
2. Compositions of uniformly continuous functions are uniformly continuous.
3. Compositions of Lipschitz functions are Lipschitz.
4. Suppose $f, g : X \to V$ for V a normed vector space. If f, g are uniformly continuous or Lipschitz, then $\alpha f + \beta g$ are too $\forall \alpha, \beta \in \mathbb{F}$.

Lenma 4.1.1

Suppose X, Y are metric spaces, $f : X \to Y$ is uniformly continuous if $\{x_n\}_{n=\ell}^{\infty} \subseteq X$ is Cauchy, then $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is Cauchy.

Proof. Let $\epsilon > 0$, then there exists $\delta > 0$ such that

$$x, y \in X \wedge d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Pick $N \geqslant \ell$ such that $m, n \geqslant N \implies d(x_n, x_m) < \delta$. This means that $d(f(x_n), f(x_m)) < \epsilon$. Therefore $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is Cauchy. 😊

Example 4.1.1

1. Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$.
2. Let V, W be normed vector spaces, $a \in W$, $T \in \mathcal{L}(V, W)$. Then $f : V \to W$ via $f(x) = a + Tx$ is Lipschitz:
$$\begin{aligned} \|f(x) - f(y)\|_W &= \|Tx - Ty\|_W && (4.1) \\ &\leqslant \|T\|_{\mathcal{L}} \|x - y\|_V. && (4.2) \end{aligned}$$
So, f is Lipschitz.
But moving back to the first example, f is not uniformly continuous. However, f maps Cauchy sequences to Cauchy sequences. Indeed, suppose $\{x_n\}_n$ is Cauchy and bounded by M. Then,
$$|f(x_n) - f(x_m)| = |x_n^2 - x_m^2| = |x_n + x_m||x_n - x_m| \leqslant 2M|x_n - x_m|.$$
So f is not uniformly continuous. Suppose not, then $\exists \delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < 1$.
Let $x = n \in \mathbb{N}$ and $y = n + \frac{\delta}{2}$. Then
$$|x - y| = \frac{\delta}{2} < \delta,$$
so $1 > |f(y) - f(x)| = (n + \delta/2)^2 - n^2 = \delta n + \frac{\delta^2}{4}$. This is a contradiction.
3. Let X be a metric space. Let $a \in X$ and define $f : X \to \mathbb{R}$ via $f(x) = d(x, a)$. f is Lipschitz as $|f(x) - f(y)| = |d(x, a) - d(y, a)| \leqslant d(x, y)$. This can be generalized.
4. Consider $\sin : \mathbb{R} \to \mathbb{R}$.
$$|\sin(x) - \sin(y)| = |\cos(w)(x - y)| \leqslant |x - y|$$
for some w below x and y. Therefore sin is Lipschitz. Ditto for cos.
5. Let X be a metric space, $V_1, \ldots, V_k, W$ be normed vector spaces. And suppose $f_i : X \to V_i$ is uniformly continuous xor Lipschitz. Further suppose $T : X \to \mathcal{L}(V_1, \ldots, V_k; W)$ is uniformly continous xor Lipschitz. If $T, f_1, \ldots, f_k$ are all also bounded, then $X \ni x \mapsto T(x)(f_1(x), \ldots, f_k(x)) \in W$ is uniformly continuous xor Lipschitz.

Proof. Recall
$$T(u_1, \ldots, u_k) - T(v_1, \ldots, v_k) = \\ T(u_1 - v_1, u_2, \ldots, u_k) + \cdots + T(v_1, \ldots, v_{k-1}, u_k - v_k).$$
Now use this with $u_i = f_i(x)$ and $v_i = f_i(y)$.

😊

Definition 4.1.5

Let $f : X \to Y$ for X, Y metric spaces. We define $K(f) \in [0, \infty]$ to be
$$K(f) = \begin{cases} 0 & |X| = 1 \\ \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} & \text{otherwise} \end{cases}.$$
$K(f)$ is called the Lipschitz constant for f.

Facts:

1. $K(f) = 0 \iff f$ is constant. $K(f)$ is finite $\iff f$ is Lipschitz. Also, $d(f(x), f(y)) \leqslant K(f)d(x,y)$.
2. Suppose $g : Y \to Z$, Z is a metric space. Then $K(g \circ f) \leqslant K(g)K(f)$.

   Proof. $d_Z(g \circ f(x), g \circ f(y)) \leqslant K(g)d_Y(f(x), f(y)) \leqslant K(g)K(f)d(x,y)$. This yields the result. 😄

3. If $Y = X$, i.e. $f : X \to X$, then $K(f^{(n)}) \leqslant K(f)^n$.

$$f^{(0)} = I_X$$
$$f^{(n)} = f \circ f^{(n-1)}.$$

Definition 4.1.6

Let X, Y be metric spaces and $f : X \to Y$.

1. We say f is expansive if $\infty > K(f) > 1$.
2. We say f is non-expansion if $K(f) \leqslant 1$.
3. We say f is contractive if $K(f) < 1$.
4. Suppose $Y = X$. We say f is eventually contractive if $\exists 1 \leqslant n \in \mathbb{N}$ such that $f^{(n)}$ is contractive.

Example 4.1.2

Let $\alpha \in [0,1]$, $\beta \in (0,1)$, $\gamma \in [0,\infty)$ and $x \in (-\infty, 0]$. Set $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} \beta & x \in (-\infty, 0] \\ \beta + (1-\beta)\left(\frac{x}{\beta}\right)^\alpha & x \in [0,\beta] \\ 1 & x \in [\beta, 1] \\ 1 + \gamma(x-1) & x \in (1,\infty) \end{cases}.$$

Exercise:

$$K(f) = \begin{cases} \infty, & \alpha \in (0,1) \\ \max\left(\gamma, \frac{1}{\beta} - 1\right) & \alpha = 1 \end{cases}.$$

Also,

$$f^{(2)}(x) = \begin{cases} 1 & x \leqslant 1 \\ 1 + \gamma^2(x-1) & x > 1 \end{cases}.$$

So $K(f^{(2)}) = \gamma^2 \implies f^{(2)}$ is Lipschitz. If $\gamma < 1$ then f is eventually contractive.

Theorem 4.1.10 Banach Fixed Point Theorem

Let X be a complete metric space and $f : X \to X$ be eventually contractive. Then there exists a unique fixed point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Suppose intitially that f is contractive, i.e. $K(f) = \gamma \in [0,1)$. Let $x_0 \in X$ arbitrarily. Inductively define $\{x_n\}_{n=0}^\infty \subseteq X$ via $x_{n+1} = f(x_n)$, i.e. $x_n = f^{(n)}(x_0)$. For $n > m \geqslant 0$, we bound

$$d(x_n, x_m) \leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leqslant \cdots \leqslant \sum_{i=m}^{n-1} d(x_i, x_{i+1}) = \sum_{i=m}^{n-1} d(f^{(i)}(x_0), f^{(i)}(x_1)) \leqslant \sum_{i=1}^{n-1} \gamma^i d(x_0, x_1) = d(x_0, x_1) \sum_{i=m}^{n-1} \gamma^i.$$

Since $\gamma < 1$, the infinite sum of γ^i converges. That is, $\left\{\sum_{i=0}^{k} \gamma^i\right\}_{k=0}^{\infty}$ is Cauchy. This and the bound implies that $\{x_n\}_{n=0}^{\infty}$ is Cauchy.

Now note that $x_{n+1} = f(x_n)$, and this converges to $x = f(x)$ because f is continuous. Therefore x is a fixed point.

Suppose $y \in X$ is such that $f(y) = y$. Then, $d(x,y) = d(f(x), f(y)) \leqslant \gamma d(x,y) \implies (1-\gamma)d(x,y) \leqslant 0 \implies x = y$.

Now consider the general case. $\exists z \leqslant n$ such that $f^{(n)}$ is contractive. By the previous analysis, there exists a unique $x \in x$ such that $f^{(n)} = x$. Thus, $f(x) = f^{(n+1)}(x) = f^{(n)}(f(x)) \implies f(x)$ is a fixed point of $f^{(n)}$. So this means $f(x) = x$. Suppose now $y = f(y)$, this means $y = f^{(n)}(y) \implies y = x$. 😊

Note:

Say f is contractive for simplicity. What we knew was that if $n > m \geqslant 0$, then

$$d(x_n, x_m) \leqslant d(x_0, x_1) \sum_{i=m}^{n-1} \gamma^i \leqslant d(x_0, x_1) \sum_{i=m}^{\infty} \gamma^i = d(x_0, x_1) \frac{\gamma^m}{1-\gamma}.$$

So,

$$d(x, x_m) = \lim_{n\to\infty} d(x_n, x_m) \leqslant \frac{d(x_0, x_1)\gamma^m}{1-\gamma}.$$

Example 4.1.3 (putnam(?))

Let $X \neq \emptyset$ be a set, $g : X \to \mathbb{R}$ be bounded and $h : X \to \mathbb{R}$ be arbitrary. Let $0 < \gamma < 1$.
Claim: $\exists! f \in \mathcal{B}(X;\mathbb{R})$ such that $f(x) = g(x) + \gamma\cos(h(x) + f(x))$.

Proof. Define $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ as $\Phi(f) = g + \gamma\cos(h+f) \in \mathcal{B}(X)$.
Fact 1: $\mathcal{B}(X)$ is complete.
Fact 2: $\Phi(f_1) - \Phi(f_2) = \gamma[\cos(h+f_1) - \cos(h+f_2)]$. That is,

$$|\Phi(f_1)(x) - \Phi(f_2)(x)| \leqslant \gamma|f_1(x) - f_2(x)|$$
$$\|\Phi(f_1) - \Phi(f_2)\|_{\mathcal{B}(X)} \leqslant \gamma\|f_1 - f_2\|_{\mathcal{B}(X)}.$$

Therefore, Φ is a contaction, meaning there is a unique $f \in \mathcal{B}(X)$ such that $f = \Phi(f) = g + \gamma\cos(h+f)$. 😊

Example 2: Solving the quadratic equation. Claim: suppose V, W are Banach spaces over $\mathbb{F}$, $A \in \mathcal{L}^2(V;W)$, $B \in \mathcal{L}(V;W), c \in W$. Suppose $A \neq 0$ and B is invertible with $B^{-1} \in \mathcal{L}(W;V)$. We claim that there eixsts $x \in V$ such that $A(x,x) + Bx + c = 0$ provided that $4\|B^{-1}\|^2_{\mathcal{L}(W;V)}\|A\|_{\mathcal{L}^2(V;W)}\|c\|_W < 1$.

Proof. If $c = 0$, then $x = 0$ does the job, so suppose $c \neq 0$. It suffices to prove that when $W = V$ and $B = I_V$. Indeed, suppose we proved this. Then,

$$A(x,x) + Bx + c = 0 \text{ in } W \iff B^{-1}A(x,x) + x + B^{-1}c = 0 \text{ in } V.$$

But $B^{-1} \circ A \in \mathcal{L}^2(V;V)$, $\|B^{-1} \circ A\|_{\mathcal{L}^2} \leqslant \|B^{-1}\|_{\mathcal{L}}\|A\|_{\mathcal{L}^2}$.
$B^{-1}c \in V$, $\|B^{-1}c\|_V \leqslant \|B^{-1}\|_{\mathcal{L}}\|c\|_W$.
Then, this means that $4\|B^{-1}A\|_{\mathcal{L}^2}\|B^{-1}c\|_V < 1 \implies \exists x \in V$ such that we are done.

Proof. We prove the special case. We want to thsow that $A(x,x) + x + c = 0$ for $A \in \mathcal{L}^2(V;V), c \in V \setminus \{0\}$.
Note,

$$A(x,x) + x + c = 0 \iff x = -c - A(x,x) \iff x \text{ is a fixed point of } f : V \to V, f(x) = -c - A(x,x).$$

The idea is that if $A = 0$, then $x = -c$ is a solution. The strategy is to try to find $R \geqslant 0$ such that

1. $f : B[-c,R] \to B[-c,R]$,
2. f is eventually contractive on $B[-c,R]$.

IF we can prove this, then $\exists! x \in B[-c,R]$ such that $x = f(x) = -c - A(x,x) \implies A(x,x) + x + c = 0$. 😊

😊

Example 4.1.4

$\exists x \in V$ such that $A(x,x) + Bx + c = 0$. Reduce to case $V + W$, $B = I$.
Claim: $\exists x \in V$ such that $A(x,x) + x + c = 0$ where $A \in \mathcal{L}^2(V,V)$, $c \in V \setminus \{0\}$.
Let $f : V \to V$ such that $f(x) = -c - A(x,x)$. Let $R \geqslant 0$ (TBD) and consider $x \in B[-c,R]$.

$$\begin{aligned} f(x) = c = -A(x,x) = -[A(x+c-c, x+c-c)] &= -[A(x+c,x+c) - A(x+c,c) - A(c,x+c) + A(c,c)] \\ &\implies \|f(x)+c\| \leqslant \|A\|[\|x+c\|^2 + 2\|c\|\|x+c\| + \|c\|^2] \\ &\leqslant \|A\|[R^2 + 2\|c\|R + \|c\|^2] \leqslant R. \end{aligned}$$

$\|A\|R^2 + (2\|c\|\|R\| - 1)R + \|A\|\|c\|^2 \leqslant 0 \iff (2\|A\|\|c\| - 1)^2 - 4\|A\|^2\|c\|^2 \geqslant 0$. This just gives

$$1 - 4\|A\|\|c\| \geqslant 0 \implies 1 \geqslant 4\|A\|\|c\|.$$

That is, if this holds, then $R \in [R_-, R_+]$. So,

$$R_\pm = \frac{1 - 2\|A\|\|C\| \pm \sqrt{1 - 4\|A\|\|c\|}}{2\|A\|}$$

and $R_- > 0$. We now konw that $4\|A\|\|c\| \leqslant 1 \implies$ for $R \in [R_-, R_+]$, $f : B[-c,R] \to B[-c,R]$.
Next, for $x, y \in B[-c,R]$,

$$\begin{aligned} \|f(x) - f(y)\| = \|A(y,y) - A(x,x)\| &= \|A(y-x,x) + A(y,y-x)\| \\ &= \|A(y-x,y+c) - A(y-x,c) + A(x+c,y-x) - A(c,y-x)\| \\ &\leqslant \|A\|[\|x-y\|R + \|x-y\|\|c\| + \|x-y\|R - \|x-y\|\|c\|] \\ &= 2\|A\|[R + \|c\|]\|x-y\|. \end{aligned}$$

We win if this quantity is strictly less than 1. Note:

$$2\|A\|R_\pm + 2\|A\|\|c\| = 1 \pm \sqrt{1 - 4\|A\|\|c\|}.$$

This is why we choose R_-. So

$$4\|A\|\|c\| - 1 \implies f : B[-c,R_-] \to B[-c,R_-] \text{ is a contraction.}$$

By Banach Fixed Point Theorem, $\exists! x \in B[-c,R_-]$ such that $f(x) = x$.

Note:

$f : B[-c,R_-]\circlearrowleft$

4.2 Homeomorphisms

Definition 4.2.1

Let X, Y be metric spaces and $f : X \to Y$ be a bijection.

1. f is a homeomorphism if f, f^{-1} are continuous. We write $X \simeq_{hom} Y$ in this case.
2. f is a uniform homeomorphism if f, f^{-1} are uniformly continuous. We write $X \simeq_{uni} Y$ in this case.
3. f is a bi-Lipschitz homeomorphism if f, f^{-1} are Lipschitz continuous. We write $X \simeq_{bi-L} Y$ in this case.

Facts:

- $\simeq_*$ are equivalence relations (also $X \simeq_{iso} Y \iff \exists$ an iso.). $[X]_{iso} \subseteq [X]_{bi-L} \subseteq [X]_{uni} \subseteq [X]_{hom}$.
- $f : X \to Y$ is a homeomorphism iff

$$\begin{cases} f \text{ is bijective} \\ f^{-1}(U) \text{ is open } \ \forall U \subseteq Y \text{ open.} \\ f(V) \text{ is open } \ \forall V \subseteq X \text{ open.} \end{cases}$$

$f : X \to Y$ is bi-Lipschitz iff f is surjective and $\exists c_0, c_1 > 0$ such that $c_0 d(x, y) \leqslant d(f(x), f(y)) \leqslant c_1 d(x, y)$ for all $x, y \in X$.

Example 4.2.1

1. Suppose X is a finite metric space, $f : X \to Y$ a bijection. We claim that f continuous implies f Lipschitz. If the cardinality of X is 1, then this is pointless (not really??). Suppose $|X| \geqslant 2$.

   Define $K(f) = \max\left\{\frac{d(f(x),f(y))}{d(x,y)} \mid x \neq y\right\} < \infty$.
2. Consider $f : (0, 1) \to (1, \infty)$ via $f(x) = \frac{1}{x}$. This is a bijection and a homeomorphism. It's not a uniform homeomorphism. This shows that $[(0, 1)]_{uni} \subset [(0, 1)]_{hom}$.
3. Let V, W be normed vector spaces. We can add "linear" to any of our homeomorphism notions. In this case, linear bi-Lipschitz $\iff$ linear homeomorphism.

   Indeed, suppose that $T : V \to W$ is a linear map. This means that $T \in \mathcal{L}(V; W)$. We can use the same logic for T^{-1}. Thus,

$$\|Tx - Ty\|_W \leqslant \|T\|_{\mathcal{L}} \|x - y\|_V.$$

   Also,

$$\begin{aligned} \|x - y\|_V &= \|T^{-1}Tx - T^{-1}Ty\|_V \\ &\leqslant \|T^{-1}\|_{\mathcal{L}} \|Tx - Tv\|_V \\ \implies \frac{1}{\|T\|_{\mathcal{L}}} \|x - y\|_V &\leqslant \|Tx - Ty\|_W \leqslant \|T\|_{\mathcal{L}} \|x - y\|_V. \end{aligned}$$

   Therefore, T is bi-Lipschitz.

Question: In general, is it the case that $[V]_{hom} = [V]_{bi-L}$ for $V \neq \{0\}$ a normed vector space?
Answer: No. In fact, $[V]_{bi-L} \subset [V]_{uni}$. Remember, we only care about metrics, not necessarily norms.

Proof. Fix $(V, \|\cdot\|)$ a normed vector space. Claim, $d : V \times V \to \mathbb{R}$ given by $d(x, y) = \sqrt{\|x-y\|}$ is a metric on V. This is obviously symmetric and positive, so we just check the triangle inequality:

$$d(x,y) = \sqrt{\|x-y\|} \leqslant \sqrt{\|x-z\| + \|z-y\|} \leqslant \sqrt{\|x-z\|} + \sqrt{\|y-z\|} = d(x,z) + d(z,y).$$

The last inequality is true by squaring. We'll now show that $(V, \|\cdot\|) \simeq_{uni} (V, d)$ with the identity map. That is $I : (V, \|\cdot\|) \leftrightarrow (V, d)$ is uniformly continuous in both directions.

Let $\epsilon > 0$. Then $d(x, y) < \delta \iff \|x - y\| < \delta^2$. So taking $\delta = \sqrt{\epsilon}$ shows that I is uniformly continuous from (V, d) to $(V, \|\cdot\|)$.

Similarly, $\|x - y\| < \delta \iff d(x, y) < \sqrt{\delta}$. So take $\delta = \epsilon^2$ and we see that I is uniformly continuous from $(V, \|\cdot\|)$ to (V, d).

Next, we claim that (V, d) and $(V, \|\cdot\|)$ are not bi-Lipschitz homeomorphic. So suppose BWOC there exists an $f : (V, \|\cdot\|) \to (V, d)$ that is bi-Lipschitz homeomorphic. In particular, then there exists $c_0, c_1 > 0$ such that

$$c_0\|x-y\| \leqslant d(f(x), f(y)) = \sqrt{\|f(x) - f(y)\|} \leqslant c_1\|x-y\|.$$

for all $x, y \in V$. In particular, if $c := c_1^2$, then $\|f(x) - f(y)\| \leqslant c\|x - y\|^2$ for $x, y \in V$. Let $x \neq y$ in V and $1 \leqslant n \in \mathbb{N}$. Let $x_i = x + \frac{i}{n}(y - x)$ for $0 \leqslant i \leqslant n$. This yields

$$\|x_{i+1}x_n\| = \frac{1}{n}\|x-y\|.$$

Thus,

$$\|f(y) - f(x)\| \leqslant \sum_{i=0}^{n-1} \|f(x_{i+1}) - f(x_i)\| \leqslant \sum_{i=0}^{n-1} c\|x_{i+1} - x_i\|^2 = c\sum_{i=0}^{n-1} \frac{\|x-y\|^2}{n^2} = \frac{c\|x-y\|^2}{n}.$$

Send $n \to \infty$ to get that $f(x) = f(y)$, contradiction! Therefore, $[V]_{bi-L} \subset [V]_{uni}$. ☺

Question: Are there any non-linear bi-Lipschitz homeomorphisms on $V \neq \{0\}$ a normed vector space.
Answer: Yes, there are lots, at least if V is complete. Suppose V is a banach space and suppose $g : V \to V$ is a contraction. We claim that $f = I + g$ is a bi-Lipschitz homeomorphism.

Proof. We follow the following steps:

1. f is a bijection. We'll show $\forall y \in V$, $\exists x \in V$ such that $x + g(x) = y$. Fix y, and define $h : V \to V$ via $h(x) = y - g(x)$. Then we're done if we can show that $\exists! x \in X$ such that $h(x) = x$. But, $K(h) = K(g)$, so h is a contraction and therefore the Banach fixed point theorem applies.

2. f is bi-Lipschitz. We have

$$\begin{aligned}\|f(x) - f(y)\| = \|x - y + g(x) - g(y)\| &\leqslant \|x-y\| + \|g(x) - g(y)\| \\ &\leqslant \|x-y\| + K(g)\|x-y\| = (1 + K(g))\|x-y\|.\end{aligned}$$

   This means that $K(f) \leqslant 1 + K(g)$. On the other hand,

$$\begin{aligned}\|x-y\| &\leqslant \|x + g(x) - y - g(y)\| + \|g(x) - g(y)\| \\ &\leqslant \|f(x) - f(y)\| + K(g)\|x-y\|.\end{aligned}$$

   This implies $(1 - K(g))\|x - y\| \leqslant \|f(x) - f(y)\| \leqslant (1 + K(g))\|x - y\|$. Therefore, f is bi-Lipschitz.

☺

Next: Let $f : V \to V$ be a bi-Lipschitz homeomorphism. Let $g : V \to V$ be a bi-Lipschitz homeomorphism with $K(g)K(f^{-1}) < 1$. We claim $h = f + g$ is a bi-Lipschitz homeomorphism.

Proof. Indeed, $h = f + g = f + g \circ f^{-1} \circ f = (I + g \circ f^{-1}) \circ f$. We just need to show that $g \circ f^{-1}$ is bi-Lipschitz because then h will be.

But, $K(g \circ f^{-1}) \leqslant K(g)K(f^{-1}) < 1$ by assumption. Therefore, $g \circ f^{-1}$ is a bi-Lipschitz homeomorphism, and so is h. 😀

Definition 4.2.2

Let X be a metric space

1. We say a property of X is a topological property if it is common to $[X]_{hom}$. That is, it's true in X if and only if it's true in Y for all $Y \simeq_{hom} X$.
2. We say a property of X is a uniform property if it is common to $[X]_{uni}$.
3. We say a property of X is a strong property if it is common to $[X]_{bi-L}$.

Example 4.2.2

1. $(0, 1) \simeq_{hom} \mathbb{R}$. Let $f : (0, 1) \to \mathbb{R}$ be defined as

$$f(x) = \log\left(\frac{1}{x} - 1\right)$$
$$f^{-1}(y) = \frac{1}{1 + e^y}.$$

$(0, 1)$ is not complete and bounded, but $\mathbb{R}$ is copmlete and unbounded. This is a strong example of how homeomorphisms do not maintain all properties.

Proposition 4.2.1

Let $X \simeq_{bi-L} Y$. Then X is bounded iff Y is bounded.

Proof. $\exists c_0, c_1 > 0$ and a bijection $f : X \to Y$ such that $c_0 d(x, y) \leqslant d(f(x), f(y)) \leqslant c_1 d(x, y)$ for all $x, y \in X$. We'll show that Y bounded implies X bounded. The other direction is free.

So if Y is bounded, then $Y \subseteq B(z, r)$ for some $z \in Y$. But, $z = f(x)$ for some $x \in X$. Therefore, $d(x, y) \leqslant \frac{1}{c_0} d(f(x), f(y))$ for all $y \in X$. But that distance is less than r, so we also have that $d(x, y) < \frac{r}{c_0}$. This means that $X \subseteq B(x, r/c_0)$. So X is bounded. 😀

Corollary 4.2.1

Boundedness is a strong property.

Example 4.2.3

Let (X, d) be a metric space that is not bounded. We've seen that $\sigma = \frac{d}{1+d}$ is also a metric on X. (X, σ) is bounded because $X = B_\sigma[x, 1]$ for all $x \in X$. These are not bi-Lipschitz homeomorphic by the previous proposition, but they can be uniformly homeomorphic. We prove that the identity map does the job.
To see this, let $\epsilon > 0$ and note

$$d < \epsilon \implies \frac{d}{1+d} \leqslant d < \epsilon.$$

This means $I : (X, d) \to (X, \sigma)$ is uniformly continuous. On the other hand, assume $\delta < 1$ and observe that

$$\frac{d}{d+1} < \delta \implies d(1-\delta) < \delta \iff d < \frac{\delta}{1-\delta}.$$

We set $\epsilon := \frac{\delta}{1-\delta}$. So then we hae $\delta = \frac{\epsilon}{1+\epsilon}$, so $I : (X, \sigma) \to (X, d)$ is uniformly continuous.

Corollary 4.2.2

Boundedness is a strong property (preserved by bi-Lipschitz, not by uniform).

Proposition 4.2.2

Let X, Y be metric spaces, $f : X \to Y$ be a uniform homeomorphism. The following hold:

1. $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy if and onlyl if $\{f(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy.
2. X complete if and only if Y complete.

Proof. We prove this in parts.

1. We know $g : X \to Y$ uniformly continuous implies that $\{g(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy when $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy. Now apply this to $g = f$ and $g = f^{-1}$ to get the result.
2. Suppose Y is complete. Let $\{x_n\}_{n=\ell}^\infty \subseteq X$ be Cauchy. Then we know $\{f(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy and hence convergent to some $y \in Y$. Thus $x_n = f^{-1}(f(x_n)) \to f^{-1}(y)$ in X as $n \to \infty$. Therefore X is complete. The converse holds by symmetry.

😃

Corollary 4.2.3

Completeness is a uniform property.

Proof. The above proposition and $(0, 1) \simeq_{hom} \mathbb{R}$. 😃

4.3 More Metric Space Topology

Definition 4.3.1

Let X be a metric space, $E \subseteq X$. We say E is totally bounded if $\forall \epsilon > 0$, there exist $x_1, \dots x_n \in X$ such that $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$.

Facts:

1. $E \subseteq X$ totally bounded $\implies$ E is bounded.
2. If $A \subseteq E \subseteq X$ and E is totally bounded, then A is totally bounded.
3. We don't have to use balls. $E \subseteq X$ is totall bounded if and only if $\forall \epsilon > 0$, $\exists A_1, \dots, A_n \subseteq X$ such that $\operatorname{diam}(A_i) < \epsilon$ for $i = 1, \dots, n$ and $E \subseteq \bigcup_{i=1}^n A_i$.

Proof. We prove the third item.

First realize that $\operatorname{diam}(B(x, \epsilon)) = 2\epsilon$. So if we know E is totally bounded, we can pick $E \subseteq \bigcup_{i=1}^n B\left(x_i, \frac{\epsilon}{3}\right)$. Then the diameter of each ball is $\frac{2\epsilon}{3} < \epsilon$, so let $A_i := B\left(x_i, \frac{\epsilon}{3}\right)$ for $i = 1, \dots, n$.

Now suppose that $E \subseteq \bigcup_{i=1}^n A_i$ with diam $< \frac{\epsilon}{3}$. Then let $x_i \in A_i$ be arbitrary and note $A_i \subseteq B(x_i, \epsilon)$. This means $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$. 😄

Example 4.3.1

1. Suppose $E \subseteq X$ is finite. Then E is totally bounded (just take the individual points).
2. $a, b \in \mathbb{R}$, $a < b$. Then (a, b) and $[a, b]$ are totally bounded, but $\mathbb{R}$ itself is not totally bounded.
3. Let X be an infinite set with the discrete metric. Then $B(x, 1)$ is a singleton set, so X itself is not totally bounded.
4. Let $E \subseteq \{\chi_A \in \ell^\infty(\mathbb{N}; \mathbb{R}) \mid A \subseteq \mathbb{N}\}$ where
$$\chi_a(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}.$$
E is bounded because $\|\chi_A\|_\infty \leqslant 1$. Also, if $A \neq B$, then $\|\chi_A - \chi_B\|_\infty = 1$.
Then $B(\chi_A, 1) \cap E = \{\chi_A\}$. Therefore E is bounded but not totally bounded.
5. Let V be a finite dimensional normed vector space. $B(x, r)$ and $B[x, r]$ are totally bounded.

Proposition 4.3.1

Let X be a metric space with $E \subseteq X$ totally bounded. Then $\overline{E}$ is totally bounded.

Proof. Recall $\overline{E} = E \cup E'$. Let $\epsilon > 0$ and pick $X_1, \dots, X_n \in X$ such that $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon/2)$.

Let $x \in E'$, which means that $\emptyset \neq (B(x, \epsilon/2) \cap E) \setminus \{x\}$, so pick any y in this set. In particular, $d(x, y) < \epsilon/2$. Since $y \in E$, there is an x_i such that $d(x_i, y) < \epsilon/2$. By the triangle inequality,
$$d(x, x_i) \leqslant d(x, y) + d(y, x_i) < \frac{2\epsilon}{2} = \epsilon.$$
That is, $E' \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$. Therefore, E itself will be contained in the same union of balls as well. 😄

Theorem 4.3.1

Let X, Y be metric spaces.

1. If $f : X \to Y$ is uniformly continuous and $E \subseteq X$ is totally bounded, then $f(E) \subseteq Y$ is totally bounded.
2. If $X \simeq_{uni} Y$, then X is totally bounded if and only if Y is totally bounded. In particular, TB is a uniform property.

Proof. If suffices to just prove the first item.

Let $\epsilon > 0$ and pick $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$ for all $x \in X$. Since E is totally bounded, there exist $x_1, \ldots, x_n$ in X such that $E \subseteq \bigcup_{i=1}^n B(x_i,\delta)$. Thus, $f(E) \subseteq \bigcup_{i=1}^n f(B(x_i,\delta)) \subseteq \bigcup_{i=1}^n B(f(x_i),\epsilon)$. ☺

Proposition 4.3.2

Let X be a totally bounded metric space. Then X is separable.

Proof. Since X is totally bounded, for every $n \in \mathbb{N}$ there is a finite set $\emptyset \neq E_n \subseteq X$ such that $X = \bigcup_{x \in E_n} B(x, 2^{-n})$. No we can define the countable set $E = \bigcup_{n \in \mathbb{N}} E_n \subseteq X$.

Now given any $x \in X$, we know there exists $x_n \in E_n$ such that $d(x, x_n) < 2^{-n}$. ☺

Example 4.3.2

$\mathcal{B}(X;Y)$ is not separable when X is an infinite set and $|Y| \geqslant 2$.

Theorem 4.3.2

Let X be a metric space and $E \subseteq X$. The following are equivalent:

1. E is totally bounded.
2. If $\{x_n\}_{n=\ell}^\infty \subseteq E$. Then there exists a Cauchy subsequence $\{x_{n_k}\}_{k=\ell}^\infty$.

Proof. Suppose $E \subseteq X$ is totally bounded. We will use this notation for JUST this proof:
Given two sequences $x = \{x_n\}_{n=\ell}^\infty, y = \{y_n\}_{n=\ell}^\infty \subseteq E$. We will write $x\sigma y$ to mean x is a subsequence of y. Note, if $x\sigma y$ and $y\sigma z$, then $x\sigma z$.

Set $x^\ell = x$, some given sequence $\{x_n\}_{n=\ell}^\infty \subseteq E$. E is totally bounded, so $E \subseteq \bigcup_{y \in F_\ell} B(y, 2^{-\ell-1})$ for F_ℓ finite. By the pigeonhole principle, $\exists y \in F_\ell$ such that $x_n^\ell \in B(y, 2^{-\ell-1})$ for infinitely many n. We may thus select $x^{\ell+1}\sigma x^\ell$ such that $x_n^{\ell+1} \in B(y, 2^{-\ell-1})$ for all n.

In particular, $d(x_n^{\ell+1}, x_m^{\ell+1}) < 2^{-\ell}$ for all $n, m \geqslant \ell$. Iterate this argument. This produces a sequence of subsequences. That is,

$$\cdots \sigma x^{m+1} \sigma x^m \sigma \cdots \sigma = x$$

such that $d(x_k^m, x_n^m) < 2^{-m+1}$ for all $m \geqslant \ell$, $n, k \geqslant \ell$.

Now let $\{y_n\}_{n=\ell}^\infty \subseteq E$ be given by $y_n = x_n^n$. Thus, for $m, n \geqslant N$, we know by construction that $d(y_n, y_m) = d(x_n^n, x_m^m) < 2^{-N+1}$. This easily shows that $\{y_n\}_{n=\ell}^\infty \subseteq E$ is Cauchy.

For the backward direction, we'll show that $\neg 1 \implies \neg 2$. Suppose E is not totally bounded. Then there exists an $\epsilon > 0$ such that E cannot be covered by finitely many ϵ-balls. Let $x_0 \in E$ be arbitrary. $E \not\subseteq B(x_0, \epsilon)$. So there exists $x \in E$ such that $d(x_0, x) \geqslant \epsilon$.

Now suppose we have $x_0, \ldots, x_n \in E$ such that $d(x_i, x_j) \geqslant \epsilon$ for $i \neq j \leqslant n$. Note that $E \not\subseteq \bigcup_{i=0}^n B(x, \epsilon)$, so we can pick an $x_{n+1} \in E$ that has a minimal distance of ϵ to all the previous points.

Proceeding via induction, we find that there exists $\{x_n\}_{n=0}^\infty$ such that $d(x_n, x_m) \geqslant \epsilon$ for all $m \neq n$ which cannot have a Cauchy subsequence. ☺

Corollary 4.3.1

If X is totally bounded and complete, then it is sequentially compact, meaning all sequences in X have convergent subsequences.

Definition 4.3.2

Let X be a metric space.

1. We say $\{U_\alpha\}_{\alpha \in A}$ (A is any index set) is an open cover of E if $E \subseteq \bigcup_{\alpha \in A} U_\alpha$ and each U_α is open.
2. An open subcover of the open cover $\{U_\alpha\}_{\alpha \in A}$ is a collection $\{U_\beta\}_{\beta \in B}$ for any $B \subseteq A$ such that it remains an open cover of E. We say an open subcover $\{U_\beta\}_{\beta \in B}$ is finite if B is finite.
3. We say E is compact if each open cover of E admits a finite open subcover.

Example 4.3.3

1. Let $E \subseteq X$ be finite. Then E is compact.
2. $(0,1) \subset \mathbb{R}$ is not compact. Take $\left\{\left(0, \frac{n}{n+1}\right)\right\}_{n=1}^{\infty}$. Suppose $B \subseteq \mathbb{N} \setminus \{0\}$ is a finite set such that $\left\{\left(0, \frac{n}{n+1}\right)\right\}_{n=\in B}$ is an open subcover. Then let $N = \max B$. This implies that

$$(0,1) \subseteq \left(0, \frac{N}{N+1}\right)$$

   which is a contradiction.
3. Let $E = \{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Pick $\alpha_0 \in A$ such that $0 \in U_{\alpha_0}$. Since $2^{-n} \longrightarrow 0$ as $n \longrightarrow \infty$, in fact U_{α_0} contains all but finitely many of the 2^{-n} terms. That is, there exists N such that $2^{-n \in U + \alpha_0}$ for all $n \geqslant N$. Now pick a finite subcover of $\{2^{-n} \mid 0 \leqslant n \leqslant N\}$ and we're done. E ic ompact, though E is infinite.