Question: 5

List the subgroups of S_4. Find each of the following sets:

a. $\{\sigma \in S_4 : \sigma(1) = 3\}$

b. $\{\sigma \in S_4 : \sigma(2) = 2\}$

c. $\{\sigma \in S_4 : \sigma(1) = 3 \text{ and } \sigma(2) = 2\}$

Are any of these sets subgroups of S_4?

Solution: The subgroups of S_4 are:

- $\langle e \rangle$
- $\langle (1\ 2) \rangle$
- $\langle (1\ 3) \rangle$
- $\langle (1\ 4) \rangle$
- $\langle (2\ 3) \rangle$
- $\langle (2\ 4) \rangle$
- $\langle (3\ 4) \rangle$
- $\langle (1\ 2)(3\ 4) \rangle$
- $\langle (1\ 3)(2\ 4) \rangle$
- $\langle (1\ 4)(2\ 3) \rangle$
- $\langle (1\ 2\ 3) \rangle$
- $\langle (1\ 2\ 4) \rangle$
- $\langle (1\ 3\ 4) \rangle$
- $\langle (2\ 3\ 4) \rangle$
- $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$
- $\langle (1\ 2), (3\ 4) \rangle$
- $\langle (1\ 3), (2\ 4) \rangle$
- $\langle (1\ 4), (2\ 3) \rangle$
- $\langle (1\ 2\ 3\ 4) \rangle$
- $\langle (1\ 2\ 4\ 3) \rangle$
- $\langle (1\ 3\ 2\ 4) \rangle$

- $\langle(1\ 2\ 3), (1\ 2)\rangle$
- $\langle(1\ 2\ 4), (1\ 2)\rangle$
- $\langle(1\ 3\ 4), (1\ 3)\rangle$
- $\langle(2\ 3\ 4), (2\ 3)\rangle$
- $\langle(1\ 2\ 3\ 4), (1\ 3)\rangle$
- $\langle(1\ 2\ 4\ 3), (1\ 4)\rangle$
- $\langle(1\ 3\ 2\ 4), (1\ 2)\rangle$
- A_4
- S_4

a. $\{(1\ 3), (1\ 3)(2\ 4), (1\ 3\ 4), (1\ 3\ 2), (1\ 3\ 4\ 2), (1\ 3\ 2\ 4)\}$

b. $\{e, (1\ 3), (1\ 4), (3\ 4), (1\ 3\ 4), (1\ 4\ 3)\}$

c. $\{(1\ 3), (1\ 3\ 4)\}$

The set given by part b. is a subgroup of S_4. It is the only one you would even consider because it is the only one with the identity in it. It is isomorphic to the subgroup $\langle(1\ 3\ 4), (1\ 3)\rangle$.

Question: 6

Find all of the subgroups in A_4. What is the order of each subgroup?

Solution:

- A_4, order of 12
- $\langle(1)\rangle = \{(1)\}$, order of 1
- $\langle(1\ 2)(3\ 4)\rangle = \{(1), (1\ 2), (3\ 4)\}$, order of 3
- $\langle(1\ 3)(2\ 4)\rangle = \{(1), (1\ 3), (2\ 4)\}$, order of 3
- $\langle(1\ 4)(2\ 3)\rangle = \{(1), (1\ 4), (2\ 3)\}$, order of 3
- $\langle(1\ 2\ 3)\rangle = \langle(1\ 3\ 2)\rangle = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$, order of 3
- $\langle(1\ 2\ 4)\rangle = \langle(1\ 4\ 2)\rangle = \{(1), (1\ 2\ 4), (1\ 4\ 2)\}$, order of 3
- $\langle(1\ 3\ 4)\rangle = \langle(1\ 4\ 3)\rangle = \{(1), (1\ 3\ 4), (1\ 4\ 3)\}$, order of 3
- $\langle(2\ 3\ 4)\rangle = \langle(2\ 4\ 3)\rangle = \{(1), (2\ 3\ 4), (2\ 4\ 3)\}$, order of 3
- $\{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, order of 4

Question: 7

Find all possible orders of elements in S_7 and A_7.

Solution: We start by proving that the order of permutations is defined by the least common multiple of lengths of disjoint cycles. For S_7, this corresponds to partitions of 7. This lemma is not hard to prove by splitting any permutation into disjoint cycles and then raising them to the power of the least common multiple of the lengths of the cycles.

Listing them all out shows that elements in S_7 can have order of $1, 2, 3, 4, 5, 6, 7, 10, 12$. A_7 has the same list except any elements with odd parity are disregarded, so the order of the elements would be $1, 2, 3, 4, 5, 6, 7$.

Question: 23

If σ is a cycle of odd length, prove that σ^2 is also a cycle.

Solution: Let's say that σ has length k and that it can be written in cycle notation as $(a_1, a_2, \ldots, a_k)$. Then σ^2 is the permutation $(a_1, a_3, \ldots, a_k, a_2, a_4, \ldots a_{k-1})$. The term a_{k-1} would be sent to a_1 which would complete the cycle if k is odd since $k-1$ would then be even ☺.

Question: 25

Prove that in A_n for $n \geqslant 3$, any permutation is a product of cycles of length 3.

Solution: First we need to prove that $|A_n|$ is even. This is true because $|A_n| = n!/2$ and that value will be even for $n \geqslant 3$.

Now, consider $\sigma = \underbrace{(a_1\ a_2) \ldots (a_{4n-1}\ a_{4n})}_{2n}$.

If we consider any pair of transpositions, $(a_\alpha\ a_\beta)(a_\gamma\ a_\delta)$, we can do the following:

$$(a_\alpha\ a_\beta)(a_\beta\ a_\gamma)(a_\beta\ a_\gamma)(a_\gamma\ a_\delta)$$
$$(a_\alpha\ a_\beta)[(a_\beta\ a_\gamma)(a_\beta\ a_\gamma)](a_\gamma\ a_\delta)$$
$$(a_\alpha\ a_\beta\ a_\gamma)(a_\beta\ a_\gamma\ a_\delta)$$

Therefore, in A_n, any permutation is a product of cycles of length 3. ☺

Question: 29

Recall that the the ***center*** of a group G is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

Find the center of D_8. What about the center of D_{10}? What is the center of D_n?

Solution: It is important to start by noting that rotations in D_n commute. If we define r as an action of D_n that rotations the n-gon by $2\pi/n$ radians, then the set of rotations in D_n are r^i for $i \in [1, n]$ where $r^n = e$. If we also define a flip f that flips the n-gon over the vertical axis. Then the set of flips in D_n is fr^j for $j \in [0, n-1]$.

If we take any i, j, then we can see that $r^i(sr^j) = r^i(sr^jss) = r^i(srs)^js = r^i(r^{-1})^js = r^ir^{-j}s = r^{i-j}s$. If we reverse the order, then we can see that $(sr^j)r^i = sr^{j+i}ss = r^{-i-j}s$. Therefore, we can conclude that $r^i(sr^j) = (sr^j)r^i$ only when $i - j \equiv -i - j \pmod{n} \iff 2i \equiv 0 \pmod{n}$. As such, the center of D_n, $Z(D_n) = \{e, r^{n/2}\}$. This means that $Z(D_8) = \{e, r^4\}$ and $Z(D_{10}) = \{e, r^5\}$.