Question: 3

Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$ for each of the following pairs of polynomials.

a. $a(x) = 5x^3 + 6x^2 - 3x + 4$ and $b(x) = x - 2$ in $\mathbb{Z}_7[x]$

b. $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1$ and $b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$

c. $a(x) = 4x^5 - x^3 + x^2 + 4$ and $b(x) = x^3 - 2$ in $\mathbb{Z}_5[x]$

d. $a(x) = x^5 + x^3 - x^2 - x$ and $b(x) = x^3 + x$ in $\mathbb{Z}_2[x]$

Solution:

Question: 4

Find the greatest common divisor of each of the following pairs $p(x)$ and $q(x)$ of polynomials. If $d(x) = \gcd(p(x), q(x))$, find two polynomials $a(x)$ and $b(x)$ such that $d(x) = a(x)p(x) + b(x)q(x)$.

a. $p(x) = x^3 - 6x^2 + 14x - 15$ and $q(x) = x^3 - 8x^2 + 21x - 18$, where $p(x), q(x) \in \mathbb{Q}^x$.

b. $p(x) = x^3 + x^2 - x + 1$ and $q(x) = x^3 + x - 1$, where $p(x), q(x) \in \mathbb{Z}_2[x]$.

c. $p(x) = x^3 + x^2 - 4x + 4$ and $q(x) = x^3 + 3x - 2$, where $p(x), q(x) \in \mathbb{Z}_5[x]$.

d. $p(x) = x^3 - 2x + 4$ and $q(x) = 4x^3 + x + 3$, where $p(x), q(x) \in \mathbb{Q}^x$.

Solution:

Question: 5

Find all of the zeros for each of the following polynomials.

a. $5x^3 + 4x^2 - x + 9$ in $\mathbb{Z}_{12}[x]$

b. $3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5[x]$

c. $5x^4 + 2x^2 - 3$ in $\mathbb{Z}_7[x]$

d. $x^3 + x + 1$ in $\mathbb{Z}_2[x]$

Solution:

Question: 6

Find all of the units in $\mathbb{Z}^x$.

Solution:

Question: 7

Find a unit $p(x)$ in $\mathbb{Z}_4[x]$ such that $\deg p(x) > 1$.

Solution:

Question: 10

Give two different factorizations of $x^2 + x + 8$ in $\mathbb{Z}_{10}[x]$.

Solution:

Question: 25

Let F be a field and $f(x) = a_0 + a_1x + \cdots + a^n x^n$ be in $F[x]$. Define $f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$ to be the ***derivative*** of $f(x)$.

a. Prove that
$$(f + g)'(x) = f'(x) + g'(x)$$
Conclude that we can define a homomorphism of abelian groups $D : F[x] \to F[x]$ by $D(f(x)) = f'(x)$.

b. Calculate the kernel of D if $\operatorname{char} F = 0$.

c. Calculate the kernel of D if $\operatorname{char} F = p$.

d. Prove that
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

e. Suppose that we can factor a polynomial $f(x) \in F[x]$ into linear factors, say
$$f(x) = a(x - a_1)(x - a_2)\cdots(x - a_n).$$
Prove that $f(x)$ has no repeated factors if and only if $f(x)$ and $f'(x)$ are relatively prime.

Solution: