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Question: 1

For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

a. $G = S_4$ and $H = A_4$

b. $G = A_5$ and $H = \{(1), (1\,2\,3), (1\,3\,2)\}$

c. $G = S_4$ and $H = D_4$

d. $G = Q_8$ and $H = \{1, -1, I, -I\}$

e. $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution:

a. D

b. D

c. D

d. D

e. H is normal in G because $G = \mathbb{Z}$ is abelian and all subgroups of abelian groups are normal.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Question: 4

Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is, in the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$.

a. Show that U is a subgroup of T.

b. Prove that U is abelian.

c. Prove that U is normal in T.

d. Show that T/U is abelian.

e. Is T normal in $GL_2(\mathbb{R})$?

Solution:

Question: 5

Show that the intersection of two normal subgroups is a normal subgroup.

Solution:

Question: 11

If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Solution:

Question: 13

Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

a. Calculate the center of S_3.

b. Calculate the center of $GL_2(\mathbb{R})$.

c. Show that the center of any group G is a normal subgroup of G.

d. If $G/Z(G)$ is cyclic, show that G is abelian.

Solution:

a. $Z(S_3) = \{(e)\}$

b. If we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, we can multiply out AB and BA to see the following equality:

$$ae + bg = ae + cf \Rightarrow bg = cf$$

However, this equality needs to hold true for all choices of g, f because our B was arbitrary and not related to A. This means that $b = c = 0$. This means that the equation

$$af + bh = be + df$$

reduces to $af = df$ or $a = d$. This means that we can say

$$Z(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \backslash \{0\} \right\}.$$

c. By definition, for any $z \in Z(G)$, the following equation will hold true, $zG = Gz$. By the definition of a normal subgroup, $Z(G)$ is normal in G.

d.

Question: 14

Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$; that is, G' is the subgroup of all finite products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the **commutator subgroup** of G.

a. Show that G' is a normal subgroup of G.

b. Let N be a normal subgroup of G. Prove that G/N is abelian iff N contains the commutator subgroup of G.

Solution:

a.