Question: 2

Prove that $\mathbb{C}^*$ is isomorphic to the subgroup of $GL_2(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Solution:

We define our isomorphism as $\phi(z) = \phi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

We start by showing that this function is one-to-one. So, if we have two complex numbers, $a+bi, c+di$, then we have to show that $\phi(a+bi) = \phi(c+di)$ implies $a+bi = c+di$. We have that $\phi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and that $\phi(c+di) = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$. Since matrices are equal if and only if their entries are equal, we have that $a = c$ from the entries $1,1$ and $2,2$, and we have that $b = d$ from the entires $1,2$ and $2,1$. Therefore, we have that $a+bi = c+di$ and that ϕ is one-to-one.

This function is clearly onto as well. For any $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, the corresponding value in $\mathbb{C}^*$ is $a+bi$.

Therefore, since ϕ is one-to-one and onto, it is an isomorphism. 😃

Question: 13

Let $\omega = \text{cis}(2\pi/n)$ be the primitive nth root of unity. Prove that matrices

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

generate a multiplicative group isomorphic to D_n.

Solution: We start by realizing that $A^n = B^2 = I_2$. We also see that $(BA)^2 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}\right)^2 = I_2$.

Therefore, we can create the group presentation $\langle A, B | A^n = B^2 = (BA)^2 = I_2 \rangle$, which is a definition for D_n. 😃

Question: 18

Prove that the subgroup of $\mathbb{Q}^*$ consisting of elements of the form $2^m 3^n$ for $m, n \in \mathbb{Z}$ is an internal direct product isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Solution: Let's first define our groups:

$$H = \{2^m : m \in \mathbb{Z}\}$$
$$K = \{3^n : n \in \mathbb{Z}\}$$
$$S = \{2^m 3^n : m, n \in \mathbb{Z}\} = HK$$

The line above shows one step of showing that S is the internal direct product of H and K. The next step is to show that $2^m \neq 3^n$ for any $m, n \neq 0$. This is true because all numbers have a unique prime factorization by the fundamental theorem of arithmetic and therefore $2^m \neq 3^n$ for any $m, n \neq 0$ since they have different prime factorization. However, when $m = n = 0$, we have that $2^0 = 1 = 3^0$, which is the identity. Therefore, $H \cap K = \{1\} = \{e\}$.

To show commutativity, we have that $2^m 3^n = 3^n 2^m$ for any $m, n \in \mathbb{Z}$ because integers are abelian.

Therefore, S is an internal direct product of H and K.

Now for isomorphism to $\mathbb{Z} \times \mathbb{Z}$. We start by defining our function as $\phi(2^m 3^n) = (m, n)$. We have to show that this function is one-to-one and onto.

To show that this is one-to-one, assume we have $\phi(2^{m_1} 3^{n_1}) = \phi(2^{m_2} 3^{n_2})$. We have that $(m_1, n_1) = (m_2, n_2)$, which means $m_1 = m_2$ and $n_1 = n_2$. Therefore, we have that $2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2}$, and as such, this function is one-to-one.

ϕ is onto iff for all (m, n), there exists a $2^m 3^n$ such that $\phi(2^m 3^n) = (m, n)$. By construction of ϕ, it is clearly onto.

As such, $S \cong \mathbb{Z} \times \mathbb{Z}$. 😊

Question: 23

Prove or disprove the following assertion. Let $G, H,$ and K be groups. If $G \times K \cong H \times K$ then $G \cong H$.

Solution: Counterexample: $K = \prod_{i=0}^{\infty} \mathbb{Z}$, $G = \mathbb{Z}$, $H = \{e\}$. $G \times K = K = H \times K$, but $G \ncong H$. 😊

Question: 29

Show that S_n is isomorphic to a subgroup of A_{n+2}.

Solution: Define $\tau = (n{+}1\ n{+}2) \in S_{n+2}$. Then, we define our ϕ as $\phi : S_n \longrightarrow A_{n+2}$ as $\phi(\sigma) = \sigma\tau$ if n is odd and $\phi(\sigma) = \sigma$ if n is even. This is obviously injective and satisfies $\phi(\sigma\tau) = \phi(\sigma_1)\phi(\sigma_2)$ for all $\sigma_1, \sigma_2 \in S_n$. Now we use the fact that τ commutes with all of S_n and that $\tau^2 = e$ to show that

$$\phi(\sigma_1)\phi(\sigma_2) = \begin{cases} \sigma_1\sigma_2 & \text{if both are even or both are odd} \\ \sigma_1\sigma_2\tau & \text{if only one is even} \end{cases}$$

Showing this tells us that ϕ is an injective homomorphism. Therefore, ϕ is an isomorphism from S_n with an image that is a subgroup of A_{n+2}. 😊

Question: 36

Prove that $A \mapsto B^{-1}AB$ is an automorphism of $SL_2(\mathbb{R})$ for all B in $GL_2(\mathbb{R})$.

Solution: We define our function $\phi : SL_2(\mathbb{R}) \longrightarrow SL_2(\mathbb{R})$ as $\phi(A) = B^{-1}AB$.

$$\phi(AC) = B^{-1}ACB = B^{-1}AICB = B^{-1}ABB^{-1}CB = (B^{-1}AB)(B^{-1}CB) = \phi(A)\phi(C)$$

So ϕ is a homomorphism.

To show that ϕ is one-to-one, we say that if $\phi(A) = \phi(C)$, then:

$$B^{-1}AB = B^{-1}CB \iff B(B^{-1}AB)B^{-1} = B(B^{-1}CB)B^{-1} \iff A = C,$$

meaning ϕ is one-to-one.

Now see that

$$\det(\phi(A)) = \det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(B)\det(A)\frac{1}{\det(B)} = \det(A) = 1,$$

so ϕ is onto because every $\phi(A)$ maps to an A in $SL_2(\mathbb{R})$.

Therefore, since ϕ is a bijective homomorphism from a group to itself, it is an automorphism. 😄