

Question: 2

Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

a. $\phi : \mathbb{R}^* \rightarrow GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

b. $\phi : \mathbb{R} \rightarrow GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

c. $\phi : GL_2(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\phi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d.$$

d. $\phi : GL_2(\mathbb{R}) \rightarrow \mathbb{R}^*$ defined by

$$\phi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = ad - bc.$$

e. $\phi : M_2(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\phi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = b.$$

Solution:

- a. $\phi(a)\phi(b) = \phi(ab)$. $\text{Ker}(\phi) = \phi^{-1}(I_2) = 1$
- b. $\phi(a)\phi(b) = \phi(a + b)$. $\text{Ker}(\phi) = \phi^{-1}(I_2) = 0$
- c. Trace is known to not be a group homomorphism.
- d. $\phi(a)\phi(b) = \phi(ab)$. $\text{Ker}(\phi) = SL_2(\mathbb{R})$.
- e. $\phi(a) + \phi(b) = \phi(a + b)$. $\text{Ker}(\phi) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$

Question: 9

If $\phi : G \rightarrow H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is also abelian.

Solution: If $x, y \in \phi(G)$, then there exist $g, h \in G$ with $x = \phi(g)$ and $y = \phi(h)$. So, $xy = \phi(g)\phi(h) = \phi(gh) = \phi(hg) = \phi(h)\phi(g) = yx$, so $\phi(G)$ is abelian. ☺

Question: 11

Show that a homomorphism defined on a cyclic group is completely determined by its action on the generator of the group.

Solution: Let $G = \langle g \rangle$ for some $g \in G$, a cyclic group. Now, let ϕ be a homomorphism on G . $\phi(\langle g \rangle)$ is a cyclic group and we need to show that $\phi(\langle g \rangle) = \langle \phi(g) \rangle$. If $|G| = n$, then we have the following:

$$\begin{aligned}\phi(\langle g \rangle) &= \phi\{e, g, \dots, g^{n-1}\} \\ &= \{\phi(e), \phi(g), \dots, \phi(g^{n-1})\} \\ &= \{\phi(e), \phi(g), \dots, \phi(g)^{n-1}\} \\ &= \langle \phi(g) \rangle\end{aligned}$$

Therefore, our claim is true.

Question: 14

Let G be a finite group and N a normal subgroup of G . If H is a subgroup of G/N , prove that $\phi^{-1}(H)$ is a subgroup of G of order $|H| \cdot |N|$, where $\phi : G \rightarrow G/N$ is the canonical homomorphism.

Solution: Let $a, b \in \phi^{-1}(H)$, then $\phi(a) = aN$ is an element of H and $a^{-1}N$ is also an element of H . Then we also have that $\phi(a)\phi(b) = aNbN = abN = \phi(ab)$ since N is normal. Hence, $\phi(ab)$ is an element of $H \Rightarrow ab \in \phi^{-1}(H)$, so $\phi^{-1}(H)$ is a subgroup.

To determine the order of this group, we analyze the cosets. For each $h \in H$, we can have hN . Since there are $|N|$ elements in each coset, $|H|$ cosets, and cosets are disjoint, we have that $|H| \cdot |N|$ elements exist in $\phi^{-1}(H)$. \ominus

Question: P1

Let T be the circle group as defined in the text: the set of all complex numbers of modulus one, with operation being multiplication. Let $\psi : T \rightarrow T$ be the map defined by $\psi(z) = z^2$.

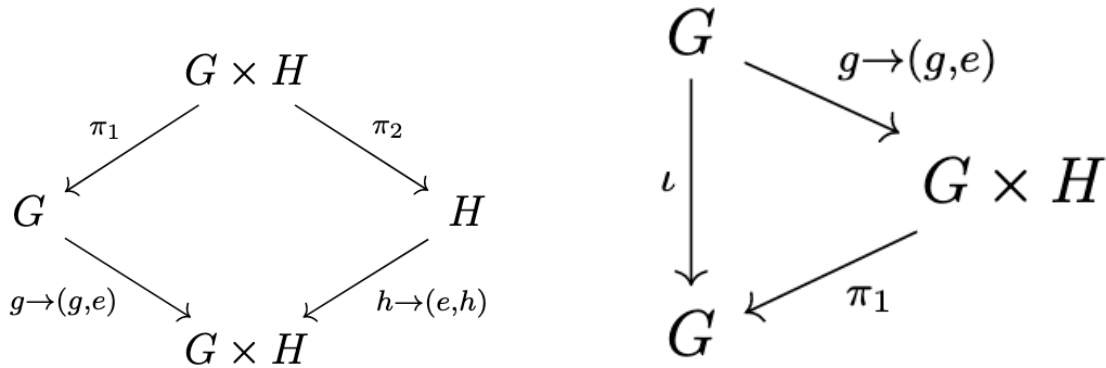
- Prove that ψ is a homomorphism of T .
- Determine the kernel K of ψ .
- Show that $T/K \cong T$.

Solution:

- This is a homomorphism because $\psi(xy) = \psi(x)\psi(y)$.
- The kernel is the set of all complex numbers z such that $z^2 = 1$. This is the set of all complex numbers z such that $z = \pm 1$.
- Trivial by First Isomorphism Theorem.

Question: P2

Determine which, if either, of the diagrams below are correct commutative diagrams. Explain. The map $\iota : G \rightarrow G \times H$ represents the identity map where $\iota(x) = x$ and π_1 and π_2 are projection maps.



Solution: A diagram is commutative if all paths lead to the same place. So, for the first diagram, if we start at (g, h) , we see that the first path will yield us (g, e) while the second path yields (e, h) . Therefore, this diagram is not commutative.

For the second diagram, we see that the path from $G \rightarrow G$ for any $g \in G$ will yield g . Additionally, the path from $G \rightarrow G \times H \rightarrow G$ for any $g \in G$ will also yield g . Therefore, this diagram is commutative.