Question: 3

Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$ for each of the following pairs of polynomials.

a. $a(x) = 5x^3 + 6x^2 - 3x + 4$ and $b(x) = x - 2$ in $\mathbb{Z}_7[x]$

b. $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1$ and $b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$

c. $a(x) = 4x^5 - x^3 + x^2 + 4$ and $b(x) = x^3 - 2$ in $\mathbb{Z}_5[x]$

d. $a(x) = x^5 + x^3 - x^2 - x$ and $b(x) = x^3 + x$ in $\mathbb{Z}_2[x]$

Solution:

a. $5x^3 + 6x^2 - 3x + 4 = (5x^2 + 4x + 6)(x - 2) + (2x + 5) \pmod 7$.

b. $6x^4-2x^3+x^2-3x+1 = (6x^2-8x+21)(x^2+x-2)+(-40x+43) = (6x^2-x)(x^2+x-2)+(2x+1) \pmod 7$.

c. $4x^5 - x^3 + x^2 + 4 = (4x^2 + 3x + 1)(x^3 - 2) + (3x^2 + 3x + 2) \pmod 5$.

d. $x^5 + x^3 - x^2 - x =$

Question: 4

Find the greatest common divisor of each of the following pairs $p(x)$ and $q(x)$ of polynomials. If $d(x) = \gcd(p(x), q(x))$, find two polynomials $a(x)$ and $b(x)$ such that $d(x) = a(x)p(x) + b(x)q(x)$.

a. $p(x) = x^3 - 6x^2 + 14x - 15$ and $q(x) = x^3 - 8x^2 + 21x - 18$, where $p(x), q(x) \in \mathbb{Q}^x$.

b. $p(x) = x^3 + x^2 - x + 1$ and $q(x) = x^3 + x - 1$, where $p(x), q(x) \in \mathbb{Z}_2[x]$.

c. $p(x) = x^3 + x^2 - 4x + 4$ and $q(x) = x^3 + 3x - 2$, where $p(x), q(x) \in \mathbb{Z}_5[x]$.

d. $p(x) = x^3 - 2x + 4$ and $q(x) = 4x^3 + x + 3$, where $p(x), q(x) \in \mathbb{Q}^x$.

Solution:

a. d

b. d

c.

Question: 5

Find all of the zeros for each of the following polynomials.

a. $5x^3 + 4x^2 - x + 9$ in $\mathbb{Z}_{12}[x]$

b. $3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5[x]$

c. $5x^4 + 2x^2 - 3$ in $\mathbb{Z}_7[x]$

d. $x^3 + x + 1$ in $\mathbb{Z}_2[x]$

Solution: You just have to plug all the numbers from 0 to $n-1$ in $\mathbb{Z}_n$ to see if there are any zeros $\pmod{n}$.

a. There are no zeroes.

b. $x \cong 2 \pmod{5}$.

c. $x \cong 3, 4 \pmod{7}$.

d. There are no zeroes.

Question: 6

Find all of the units in $\mathbb{Z}^x$.

Solution:

Question: 7

Find a unit $p(x)$ in $\mathbb{Z}_4[x]$ such that $\deg p(x) > 1$.

Solution: If we look at $p(x) = (2x+1)^2 = 4x^2 + 4x + 1 = 1 \pmod{4}$, we have found at degree 2 unit.

Question: 10

Give two different factorizations of $x^2 + x + 8$ in $\mathbb{Z}_{10}[x]$.

Solution: Testing all values of x from 0 to 9, we see that the zeroes $x \cong 1, 3, 6, 8$. Pairing these numbers to add up to -1, we see that they can't. Therefore, we have to make two of these values negative. So, we can say that $x \cong 1, 3, -4, -2$ and pair as follows:

$$\begin{aligned} x^2 + x + 8 &= (x-1)(x+2) \\ &= (x-3)(x+4). \end{aligned}$$

Question: 25

Let F be a field and $f(x) = a_0 + a_1 x + \cdots + a^n x^n$ be in $F[x]$. Define $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$ to be the ***derivative*** of $f(x)$.

a. Prove that
$$(f+g)'(x) = f'(x) + g'(x)$$
Conclude that we can define a homomorphism of abelian groups $D : F[x] \to F[x]$ by $D(f(x)) = f'(x)$.

b. Calculate the kernel of D if $\operatorname{char} F = 0$.

c. Calculate the kernel of D if $\operatorname{char} F = p$.

d. Prove that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

e. Suppose that we can factor a polynomial $f(x) \in F[x]$ into linear factors, say

$$f(x) = a(x - a_1)(x - a_2) \cdots (x - a_n).$$

Prove that $f(x)$ has no repeated factors if and only if $f(x)$ and $f'(x)$ are relatively prime.

Solution:

a.