21603 Model Theory I

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Contents

Chapter 1

1.1 random info

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1. Set Theory
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1973 book by Chang and Keisler - Model Theory - Highly recommended for elementary model theory.
What is model thoery? Model Theory = logic + universal algebra
1984 - W. Hodges - Shorter Model Theory
model theory = algebraic geometry - field theory
Algebraic structures:

1. groups
2. rings
3. vector spaces
4. fields
5. graphs - (V, E)
6. ordered structures

Around 1870, mathematicians started to layout the foundations for mathematics. One of the ideas was axiomatization. One example was Euclidean axioms for plane geometry.

1.2 Structures and Languages

Definition 1.2.1: Language

L is a language if $L = F \cup R \cup C$ are parameter disjoint.

Definition 1.2.2: L-structure

Let L be a language (similarity type/signature). Then $\mathcal{M}$ is an L-structure provided:

$$\mathcal{M} = (U, \{f^{\mathcal{M}} \mid f \in F\}, \{r^{\mathcal{M}} \mid r \in R\}, \{c^{\mathcal{M}} \mid c \in C\})$$

where U is a nonempty set. U is also called the universe of $\mathcal{M}$.
For any $f \in F$ there is $U(f)$ natural number such that $f^{\mathcal{M}} : U^{n(F)} \to U$, $R^{\mathcal{M}} \subseteq U^{n(R)}$, $C^{\mathcal{M}} \subseteq U$, $\forall c \in C$.

Notation: $|\mathcal{M}| = U$. The cardinal of $\mathcal{M}$ is $|U|$. $\|\mathcal{M}\|$ denotes the cardinality of $\mathcal{M}$.

Definition 1.2.3: Theory

Let L be a language. A theory T is a set of sentences in L. A sentence is a finite set of symbols from L.

Example 1.2.1 (Sentences)
$L_{\text{gr}} = \{e, \cdot\}$. $e \in C$, $\cdot \in F$. $T_{\text{gr}} = \{\forall x \forall y \forall z(x\cdot(y\cdot z) = (x\cdot y)\cdot z), \forall x(x\cdot e = x, e\cdot x = x), \forall x \exists y(x\cdot y = e, y\cdot x = e)\}$. These are the group axioms (associativity, identity, existence of inverse).

Definition 1.2.4: Term

Let L be a language. A term is:

1. c is a term for any $c \in C$.
2. x when x is a variable.
3. $\tau_1, \ldots, \tau_k$ terms, $f \in F$, $n(f) = k$, then $f(\tau_1, \ldots, \tau_k)$ is a term.

Definition 1.2.5: Term

Term(L) is a minimal set of finite strings of symbols from $L \cup \{(,)\} \cup X$ that contains $C \cup x$ and closed under the following rule:

$$\tau_1, \ldots, \tau_k \in \text{Term}(L), f\, k-\text{place function symbol, then} f(\tau_1, \ldots, \tau_k) \in \text{Term}(L)$$

Example 1.2.2 (L_r)
$L_r = \{0, 1, +, -\}$. $\text{Term}(L_r) \supseteq \{\sum a_j x_1^{n_j} \mid a_j \in \mathbb{Z}, n_j \in \mathbb{N}\}$.

Example 1.2.3 (L_{gr})
$\text{Term}(L_{\text{gr}}) \supseteq \{x_1 \cdot x_n \cdots x_n \mid x_i \in X, n \in \omega\}$.

Definition 1.2.6: AFml

Let L be a language. The set of atomic fomrulas denotes by AFml(L) is the smallest set of formulas in L that contains $L \cup \{(,), =\} \cup X$ such that:

1. If $\tau_1, \tau_2 \in \text{Term}(L)$, then $\tau_1 = \tau_2 \in \text{AFml}(L)$.
2. Given $R(x_1, \ldots, x_n)$ relation symbol and $\tau_1, \ldots, \tau_n \in \text{Term}(L)$, then $R(\tau_1, \ldots, \tau_n) \in \text{AFml}(L)$.

Definition 1.2.7: Fml

$\mathrm{Fml}(L)$ is the set of (first order) formulas in L. Which is the mininmal set of finite strings of symbols from $L \cup \{(,),=,\neg,\vee,\wedge,\implies,\impliedby,\iff,\forall,\exists\} \cup X$ such that:

1. $\mathrm{Fml}(L) \supseteq \mathrm{AFml}(L)$.
2. If φ is a formula, then $\neg\varphi$ is a formula.
3. If $x \in \{\wedge,\vee,\implies,\iff\}$ and $\varphi,\psi \in \mathrm{Fml}(L)$, then $(\varphi x \psi) \in \mathrm{Fml}(L)$.
4. If $\varphi \in \mathrm{Fml}(L)$, $Q \in \{\forall,\exists\}$, and $x \in X$, then $Qx\varphi \in \mathrm{Fml}(L)$.
5. If $\varphi \in \mathrm{Fml}(L)$, $\mathrm{FV}(\varphi)$ is the set of free variables in φ defined by induction on the structure of φ.
   Case 1: $\varphi \in \mathrm{AFml}(L)$.
   (a) φ is $\tau_1 = \tau_2$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \mathrm{FV}(\tau_2)$.
   (b) φ is $R(\tau_1,\ldots,\tau_n)$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \ldots \cup \mathrm{FV}(\tau_n)$.

   Case 2:
   (a) if φ is $\neg\psi$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$.
   (b) if $\varphi = \psi_1 * \psi_2$ for $* \in \{\wedge,\vee,\implies,\iff\}$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi_1) \cup \mathrm{FV}(\psi_2)$.

   Case 3: φ is $Qx\psi$, $Q \in \{\forall,\exists\}$. Then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \setminus \{x\}$.
6. $\mathrm{Sent}(L)$ are the sentences in L. $\mathrm{Sent}(L) = \{\varphi \in \mathrm{Fml}(L) \mid \mathrm{FV}(\varphi) = \emptyset\}$.

Example 1.2.4

If $L_f = \{+,\cdot,0,1\}$, then $T_f = \{$

- $\forall x\forall y\forall z(x\cdot(y\cdot z) = (x\cdot y)\cdot z)$,
- $\forall x\forall y\forall z(x+(y+z) = (x+y)+z)$,
- $\forall x\forall y(x+y = y+x)$,
- $\forall x\forall y(x\cdot y = y\cdot x)$,
- $\forall x(x\cdot 1 = x, 1\cdot x = x)$,
- $\forall x(x+0 = x, 0+x = x)$,
- $\forall x\exists y(x\cdot y = 1, y\cdot x = 1)$,
- $\forall x\exists y(x+y = 0, y+x = 0)$,
- $\forall x\forall y\forall z(x\cdot(y+z) = (x\cdot y)+(x\cdot z))$

$\}$.

Definition 1.2.8: L-theory

T is an L-theory if $T \subseteq \mathrm{Sent}(L)$.

The example above is "field theory".

Definition 1.2.9

Let M be an L-structure. $\tau(\bar{x})$ is a term, $\bar{a} \in |M|^{\ell(n)}$. T
Case 1: $\tau(\bar{x}) = c$ for some constant symbol. Then $\tau^M(\bar{a}) = c^M$.
Case 2: $\tau(\bar{x}) = x_i$. Then $\tau^M(\bar{a}) = a_i$.
Case 3: $\tau(\bar{x}) = f(\tau_1, \dots, \tau_k)$. Then $\tau^M(\bar{a}) = f^M(\tau_1^M(\bar{a}), \dots, \tau_k^M(\bar{a}))$.

Definition 1.2.10: $\models$

Let L be a language, $\varphi \in \mathrm{Fml}(L)$, M and L-structure, $n = \ell(\bar{x})$, $\bar{a} \in |M|^n$. Define $M \models \varphi(\bar{a})$ at $\bar{a}$ by induction on the structure of φ:

- If φ is atomic,
  - when $\varphi(x)$ is $\tau_1 = \tau_2$, then $M \models \varphi(\bar{a})$ iff $\tau_1(\bar{a}) = \tau_2(\bar{a})$.
  - when $\varphi(x)$ is $R(\tau_1, \dots, \tau_k)$, then $M \models \varphi(\bar{a})$ iff $(\tau_1(\bar{a}), \dots, \tau_k(\bar{a})) \in R^M$.
- If φ is not atomic, then:
  - if φ is $\neg\psi$, then $M \models \varphi(\bar{a})$ iff $M \models \psi(\bar{a})$ is false.
  - if φ is $\psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $M \models \varphi(\bar{a})$ iff $M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a})$.
  - if φ is $\exists y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff there is $b \in |M|$ such that $M \models \psi(b, \bar{a})$.
  - if φ is $\forall y \psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff for all $b \in |M|$, $M \models \psi(b, \bar{a})$.

Definition 1.2.11

Let M be an L-structure and T an L-theory. $M \models T$ iff for every $\varphi \in T$, $M \models \varphi$. We say T "satisfies" M.

Example 1.2.5 (Models)
$M \models T_f \iff (|M|, +^M, \cdot^M, 0^M, 1^M)$ is a field.

Definition 1.2.12: Mod

$\mathrm{Mod}(T) = \{M\ L\text{-structure} \mid M \models T\}$.

Example 1.2.6
$\mathrm{Mod}(T_f)$ is the class of all fields and $\mathrm{Mod}(T_{\mathrm{gr}})$ is the class of all groups.

Definition 1.2.13: Structure Isomorphism

Let M, N both be L-structures. f is an isomorphism from M onto N if $f : |M| \to |N|$ is a bijection such that:

- $f(c^M) = c^N$ for all $c \in C$.
- $G(x_1, \dots, x_k)$ function symbol. $a_1, \dots, a_k \in |M|$, then $f(G^M(a_1, \dots, a_k)) = G^N(f(a_1), \dots, f(a_k))$.
- $R(x_1, \dots, x_k)$ predicate symbol. $a_1, \dots, a_k \in |M|$, then $(a_1, \dots, a_k) \in R^M$ iff $(f(a_1), \dots, f(a_k)) \in R^N$.

We write $f : M \cong N$. Also $M \cong N \iff \exists f : M \cong N$.

Definition 1.2.14

Let $\lambda \geqslant \aleph_0$, T an L-theory. T is λ-categorical if for all $M, N \models T$ of cardinality λ, $M \cong N$.

Theorem 1.2.1 Los Conjecture (1954)

Let L be a language, T a first order L-theory, in an at most countable language. If $\exists \lambda > \aleph_0$ such that T is λ-categorical, then for all $\mu > \aleph_0$, T is μ-categorical.

Somewhere around 1961-1965, Morley proved this conjecture.

Chapter 2

Basic Concepts

Lenma 2.0.1

1. $M \cong N \implies N \cong M$.
2. $M \cong M$, $f = \mathrm{id}_{|M|}$.
3. Let M_1, M_2, M_3 be all L-structures. Then $f_1 : M_1 \cong M_2$ and $f_2 : M_2 \cong M_3 \implies f_2 \circ f_1 : M_1 \cong M_3$.

In other words, $\cong$ is an equivalence relation on $\mathrm{Struct}(L)$.

$M/\cong = \{N \text{ is an } L(M)\text{-structure} \mid N \cong M\}$.

Definition 2.0.1: Spectrum function of T

Let T be a first order theory ($T \subseteq \mathrm{Sent}(L)$) of cardinality λ. Then $I(\lambda, T)$ is the number of pairwise nonisomorphic models of T of cardinality λ. We have

$$I(\lambda, T) = |M/\cong|$$

where $M \models T$ and $\|M\| = \lambda$.

Consider $\lambda \mapsto I(\lambda, T)$, $\lambda \in \mathrm{Card}$ (the class of cardinal numbers). But what is the shape of $\lambda \mapsto I(\lambda, T)$. Is it weakly monotone? That is, $\mu > \lambda \implies I(\mu, T) \geqslant I(\lambda, T)$?

Theorem 2.0.1 Morley's Conjecture (~1965)
Suppose T is first order and $|L(T)| \leqslant \aleph_0$. Then $\mu > \lambda > \aleph_0 \implies I(\mu, T) \geqslant I(\lambda, T)$.

The basic problem is that given M and N both of cardinality λ, $M \ncong N$, find M', N' both of cardinality μ such that $M' \cong N'$. In 1990, Shelah solved Morley's Conjecture. However, this is an open question for uncountable T.

Theorem 2.0.2 Morley's Category Theorem
Let T be a first order theory for $|L(T)| \leqslant \aleph_0$. Then $\exists \lambda > \aleph_0, I(\lambda, T) = 1$ then $\forall \mu > \aleph_0$, $I(\mu, T) = 1$.

Shelah listed all possible functions $\lambda \mapsto I(\lambda, T)$ and, by hand, verified that they were weakly monotone.

Example 2.0.1

1. $I(\lambda, T) = 1$ for all $\lambda > \aleph_0$.
2. $I(\lambda, T) = 2^\lambda$ for all $\lambda > \aleph_0$.

Hart, Hrushovski, and Laskowski found all the 13 functions.

Definition 2.0.2: Submodel

Let M, N be L-structures. M is a submodel of N if:

1. $|M| \leqslant |N|$
2. $\forall a_1, \ldots, a_n \in |M|$ and $F(x_1, \ldots, x_n)$, $F^M(a_1, \ldots, a_n) = F^N(a_1, \ldots, a_n)$.
3. $c^M = c^N$ for all constant symbols c.
4. $R^M = R^N \cap (|M| \times \cdots \times |M|)$.

Definition 2.0.3: Elementarily Equivalent

Let M, N be L-structure. M is elementarily equivalent to N denoted by $M \equiv N$ provided $M \models \varphi \iff N \models \varphi$ for any $\varphi \in \mathrm{Sent}(L)$.

Definition 2.0.4

Let M be an L-structure. The theory of M is denoted $(M) = \{\mathrm{Th}(M)\varphi \in \mathrm{Sent}(L) \mid M \models \varphi\}$.

Let $N := (\omega, +, \cdot, 0, 1)$. Then $\mathrm{TA} = \mathrm{Th}(N)$ "True Arithmetic". For example the twin primes conjecture is $\{p \mid p$ and $p+2$ are both primes$\}$ is infinite. If it is true, then it is a member of TA.

Theorem 2.0.3

Let M, N be L-stuructres. If $M \cong N$, then $M \equiv N$.

Theorem 2.0.4

Let M, N be L-structures. Suppose $f : M \cong N$. Then for any $\bar{a} \in |M|$ and any $\varphi(\bar{x}) \in \mathrm{Fml}(L)$ with $\ell(\bar{x}) = \ell(\bar{a})$, $M \models \varphi[\bar{a}] \iff N \models \varphi[f(\bar{a})]$.

Proof. Suppose $\varphi(\bar{x})$ is atomic.

Lenma 2.0.2

Suppose $f : M \cong N$ and $\tau(\bar{x})$ sequence of terms. $\bar{a} \in |M|$, $\ell(\bar{x}) = \ell(\bar{a})$. Then $f(\tau(\bar{a})) = \tau(f(\bar{a}))$.

Proof. By induction on the length of τ.

Case 1: $\tau(\bar{x})$ is x. Then $f(\tau(\bar{a})) = f(a) = \tau(f(\bar{a}))$.

Case 2: $\tau(\bar{x}) = c$. then $f(c^M) = c^N$ by definition of isomorphism.

Case 3: $\tau(\bar{x}) = G(y_1, \ldots, y_n)$ function symbol. Then $\tau_1(\bar{x}), \ldots, \tau_n(\bar{x})$ are terms. By induction, $f(\tau(\bar{a})) = f(G^M(\tau_1(\bar{a}), \ldots, \tau_n(\bar{a}))) = G^N(f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) = \tau^N(f(\bar{a}))$. 😊

Now returning to the proof:

Case 1: $\varphi(\bar{x})$ is $\tau_1(\bar{x}) = \tau_2(\bar{x})$. Then, we have $M \models \varphi(\bar{a}) \iff \tau_1(\bar{a}) = \tau_2(\bar{a}) \iff f(\tau_1(\bar{a})) = f(\tau_2(\bar{a})) \iff \tau_1^N(f(\bar{a})) = \tau_2^N(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Case 2: $\varphi(\bar{x})$ is $R(\tau_1(\bar{x}), \ldots, \tau_n(\bar{x}))$. When $R(y_1, \ldots, y_n)$ is a relation symbol and $\tau_i(\bar{x})$ are terms. Then $M \models \varphi(\bar{a}) \iff (\tau_1(\bar{a}), \ldots, \tau_n(\bar{a})) \in R^M \iff (f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) \in R^N \iff (\tau_1(f(\bar{a})), \ldots, \tau_n(f(\bar{a}))) \iff N \models \varphi(f(\bar{a}))$.

Suppose φ is $\neg\psi$. Then $M \models \varphi(\bar{a}) \iff M \not\models \psi(\bar{a}) \iff N \not\models \psi(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Suppose φ is $\psi_1 \wedge \psi_2$. Then $M \models \varphi(\bar{a}) \iff M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a}) \iff N \models \psi_1(f(\bar{a}))$ and $N \models \psi_2(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.

Suppose $\varphi(\bar{x})$ is $\exists y \psi(y, \bar{x})$. Then $M \models \varphi(\bar{a}) \iff$ there is $b \in |M|$ such that $M \models \psi(b, \bar{a}) \iff$ there is $c \in |N|$ such that $N \models \psi(c, f(\bar{a})) \iff N \models \exists \psi(y, f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$. 😊

General Remark:

$$M \models \exists y \varphi(y, \bar{a}) \iff M \models \neg \forall y \neg \varphi(y, \bar{a})$$
$$M \models \neg \exists y \varphi(y, \bar{x}) \iff \forall y \neg \varphi(y, \bar{a}).$$

Example 2.0.2

$L_{gr} = \{\cdot, 1\}$. $(\mathbb{Q}, +, 0), (\mathbb{R}, +, 0)$ are not isomorphic because diff cardinality. $(\mathbb{Q}, +, 0), (\mathbb{Z}, +, 0)$ are not isomorphic because:

$$(\mathbb{Q}, +, 0) \models \forall x \exists y (x = y + y).$$

This sentence is not true for $\mathbb{Z}$ under addition.

$N = (\omega, +, \cdot, 0, 1)$ is called the standard model of arithmetic. TA = Th(N), true arithmetic.

Question

Given $M_1, M_2 \models$ TA both countable. Are they isomorphic?

Question

What is $I(\aleph_0, \text{TA})$? Voted on $2^{\aleph_0}$.... and it is.

Let T be a theory and $\varphi \in \text{Sent}(L)$. We say T proves φ (denoted $T \vdash \varphi$) if there exists a finite set of sequences from L, $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for all i, $\varphi_i \in T$ or φ_i is a tautology or there are $j, k < i$ where $\varphi_j = (\varphi_k \implies \varphi_i)$.

1. $Q \to P$: the rule of inference. "modus ponens".
2.

Other rules (possible members of φ):

- $x = y$, $y = z$ then $x = z$.
- If $\varphi_i = \forall x \varphi(x)$, then also $\forall y \varphi(y)$ in the sequence.
- If $\forall x \varphi(x)$ also $\varphi(\tau(\bar{c}))$.

Definition 2.0.5

A set of sentences is a consistent theory if there is no φ such that φ and $\neg\varphi$ are both in the theory. T is inconsistent if it is not consistent.

Theorem 2.0.5 Godel's Completeness Theorem

Let T be some set of sentences in L. Then T is consistent iff T has a model.

Godel only proved it for when $|L| \leqslant \aleph_0$.

Theorem 2.0.6 Compactness Theorem

Let $T \subseteq \text{Sent}(L)$. If for any finite $T_0 \subseteq T$, T_0 has a model, then T has a model.

Proof. Enough to show by completeness that is consistent. If T inconsistent, $T \vdash \varphi$ and $T \vdash \neg\varphi$, there are $T_1, T_2 \subset T$ finite such that $T_1 \vdash \varphi$ and $T_2 \vdash \neg\varphi$. Then $T_1 \cup T_2 \vdash \varphi \wedge \neg\varphi$. By assumption on T, $\exists M_0 \models T_1 \cup T_2$. Then $M_0 \models \varphi \wedge \neg\varphi$ which is a contradiction. ☺

ZF cannot prove the compactness theorem.

Let G be a group, $A \subseteq G$. Then subgroup generated by A is denoted:

$$\langle A \rangle := \bigcap \{H \mid H \leqslant G, H \supseteq A\}.$$

Proposition 2.0.1

$\langle A \rangle = \{a_1^{\epsilon_1} \cdot a_2^{\epsilon_2}, \dots, a_n^{\epsilon_n} \mid n \in \mathbb{N}, a_i \in A, \epsilon_i \in \{1, -1\}\}$.

Theorem 2.0.7 Submodel Theorem

Let M be an L-structure. Denote by $\lambda := |L| + \aleph_0$. For any $A \subseteq |M|$, there is $N \subseteq M$ such that $|N| \supseteq A$ and $\|N\| \leqslant |A| + \lambda$.

Remark: When $|L| \leqslant \aleph_0$, then $\lambda = \aleph_0$. For infinite A we have $|A| \geqslant \|N\| \geqslant |A|$, so by CB, $\|N\| = |A|$.

Proof. By induction on $n < \omega$, define $\{A_n \subseteq |M| \mid n < \omega\}$ such that $A_0 = A \cup \{c^M \mid c \text{ constant symbols}\}$. For $n+1$, let $A_{n+1} = A_n \cup \{f^M(a_1, \dots, a_k) \mid f \text{ function symbol}\}$. Take $B = \bigcup_{n<\omega} A_n$. Now let $N = (B, F^M, R^M, c^M)_{F,R,c \in L}$.

We claim that N is as required.

$|N| \supseteq A$: $B = \bigcup_{n<\omega} A_n \supseteq A_0 \supseteq A$.

$N \subseteq M$: Enough to show $F(x_1, \dots, x_k)$ is a function symbol for $a_1, \dots, a_k \in B$.

$F(a_1, \dots, a_k) \in B$: Given $a_1, \dots, a_k \in B$, forall $1 \leqslant n \leqslant k$, $\exists n_i < \omega$, $a_i \in A_{n_i}$. Let $\mu = \max\{n_1, \dots, n_k\}$. By $A_{n+1} \supseteq A_n$ for all n, we have $A_\mu \supseteq A_{n_i}$ for all $i \leqslant k$. So $a_1, \dots, a_k \in A_\mu$. By definition of $A_{\mu+1}$, $F(a_1, \dots, a_k) \in A_{\mu+1} \subseteq B = |N|$.

$\|N\| \leqslant |A| + \mu$: We proceed with induction on $n < \omega$.

$|A_n| \leqslant \lambda + |A|$: $n = 0$. By definition of A_0, $|A_0| \leqslant |A| + |L| \leqslant |A| + \lambda \implies |L| \leqslant \lambda$.

So $|A_{n+1}| \leqslant |A_n| + |L| + \sum_{k \leqslant \omega} |A_n|^k \leqslant \mu + \sum_{k<\omega} \mu^k = \sum_{k<\omega} \mu = \mu + \aleph_0 \mu = \mu = |A| + \lambda$. ☺

Definition 2.0.6

Let M be an L structure, $L_0 \subseteq L$, $M \restriction L_0 := \langle |M|, F^M, R^M, c^M \rangle_{F,R,c \in L_0}$. We can also say M is an expansion of $M \restriction L_0$.

Example 2.0.3

Suppose you have a field $(F, +, \cdot, 0, 1)$, so $L = (+, \cdot, 0, 1)$. Then let $L_0 = \{+, 0\}$. Then $F \restriction L_0$ is the additive group of F.

Theorem 2.0.8

Let T be a first order theory with $\lambda \geqslant |L(T)| + \aleph_0$. If T has an infinite model, then $\exists N \models T$ such that $\|N\| \geqslant \lambda$.

Remark: This is a very simple version of the Upward Lowenheim-Skolem Theorem.

Proof. Let $\{c_i \mid i < \lambda\}$ be a set of constant symbols not in $L(T)$. Let $T_1 = T \cup \{c_i \neq c_j \mid i \neq j, i, j < \lambda\}$. We claim that if $N_1 \models T_1$, then $N := N_1 \restriction L(T)$ is as required.

As $N_1 \models T_1$ and $T \subseteq T_1$, $N_1 \models T$, so $N \models T$.

Let $a_i := c_i$ for all $i < \lambda$. Let $i < j < \lambda$. Since $N_1 \models c_i \neq c_j$, by definition of $\models$, $a_i \neq a_j$. But $\{a_i \mid i < \lambda\} \subseteq |N_1| = |N|$.

So by claim, it is enought o show that there exists $N_1 \models T_1$. We apply the compactness theorem to T_1. Let $T_0 \subseteq T_1$ be finite.

Let $i_1, \dots i_n < \lambda$ such that $T_0 \subseteq T \cup \{c_{i_\ell} \neq c_{i_k} \mid \ell \neq k, \ell, k \leqslant n\}$. As T has an infinite model M, pick $\{a_1, \dots, a_n\} \subseteq |M|$. Let $M_0 = \langle |M|, R^M, F^M, c^M, a_1, \dots, a_n \rangle_{R,F,c \in L(T)}$. $c_i^{M_0} := a_i$. Clearly $M_0 \models T_0$. ☺

CT - the statement of the compactness theorem. Facts:

- Tychonov's theorem. Product of compact topological spaces is compact. This is known to be equivalent to the axiom of choice.

- Ty H $\iff$ Tychonov's theorem for Hausdorff spaces.
- BPI: Boolean Prime Ideal Theorem.

Theorem 2.0.9
BPI $\iff$ CT $\iff$ Ty H.

This was proved by ZF. More facts:

- ZFC $\vdash$ CT.
- $\exists M \models (\text{ZF} + \neg\text{BPI})$. So, ZF $\nvdash$ CT.

Remark: $M \subseteq N \nRightarrow M \equiv N$.

Example 2.0.4
$N = \langle \omega, +, 0\rangle$. Then $M = \langle \mathbb{Z}, +, 0\rangle$. $N \subseteq M$, but M is a group and N isn't.

Example 2.0.5
Suppose $\{M_k \mid k < \omega\}$. We have that $M_k \subseteq M_{k+1}$ for all k. Then $N := \bigcup_{k<\omega} M_k$. Then,

- $F^N := \bigcup_{k<\omega} F_k^M$.
- $R^N := \bigcup_{k<\omega} R_k^M$.

More generally, let $(I, <)$ be a linearly ordered set. Suppose $\{M_i \mid i \in I\}$ is such that $i \leqslant j \implies M_i \subseteq M_j$. Then $N := \bigcup_{i\in I} M_i$.
Question: Does $N \equiv M_k$ for any k?
Answer: No.

Suppose we know in addition that for all k, $M_k \equiv M_{k+1}$. The answer remains no.

Now let $L = \{<\}$. For $k \in \omega$, $M_k = [-k-1, k+1]$. Clearly $M_k \cong M_{k+1} \models \exists x \forall y[y > x \vee y = x]$. But $N \equiv \mathbb{R} \models \forall x \neg\exists y[y > x \vee y = x]$.

Definition 2.0.7

Let M and N be L-structures. M is an elementary submodel of N (denoted by $M \prec N$) if:

1. $M \subseteq N$.
2. for every $\varphi(x) \in \text{Fml}(L)$ and every $\bar{a} \in |M|$ with $\ell(x) = \ell(\bar{a})$¡ we know $M \models \varphi[\bar{a}] \iff N \models \varphi[\bar{a}]$.

Theorem 2.0.10 Tarski Vaught Chain Theorem (1956)
If $(I, <)$ linearly ordered set and $\{M_i \mid i \in I\}$ elementary chain and $M := \bigcup_{i\in I} M_i$. Then $M_i \prec N$ for all $i \in I$.

Proof. Clearly $M_i \subseteq N$ for all $i \in I$. So now we proceed by induction on $\varphi(x) \in \text{Fml}(L)$.

For $\varphi \in \text{Fml}(L)$, $\bar{a} \in |M|$, then $M_i \models \varphi[\bar{a}] \iff N \models \varphi[\bar{a}]$ for every $i \in I$ and $\bar{a} \in |M_i|$.

Lenma 2.0.3
$M \subseteq N \implies M \models (\varphi[\bar{a}] \iff N \models \varphi[\bar{a}] \forall \bar{a} \in |M|$ and every atomic formula $\varphi \in \text{Fml}(L))$.

We check that when $\varphi(\bar{x}) \in \text{AFml}(L)$:

(a) $\varphi(\bar{x})$ is $\tau_1(\bar{x}) = \tau_2(\bar{x})$.
$M \models \tau_1(\bar{a}) = \tau_2(\bar{a})$, so $\tau_1^M[\bar{a}] = \tau_2^M[\bar{a}] \iff \tau_1^N[\bar{a}] = \tau_2^N[\bar{a}]$. As such, $M \models \varphi(\bar{a})$.

(b) $\varphi(\bar{x})$ is $R(\tau_1, \ldots, \tau_k)$.

Then $M \models \varphi[\bar{a}] \implies \langle \tau_1^M[\bar{a}], \ldots, \tau_k^M[\bar{a}] \rangle \in R^M \implies \langle \tau_1^N[\bar{a}], \ldots, \tau_k^N[\bar{a}] \rangle \in R^N \implies N \models \varphi[\bar{a}]$.

So if φ is atomic, we have $(*)$
If $\varphi(\bar{x}) = \neg\psi(\bar{x})$, then $M_i \models \varphi[\bar{a}] \iff$ not true $M_i \models \psi(\bar{a}) \iff$ not true $N \models \psi[\bar{a}] \iff N \models \neg\psi[\bar{a}]$.
Then if $\varphi(\bar{x})$ is $\psi_1(\bar{x}) \wedge \psi_2(\bar{x})$,

$$M_i \models \varphi[\bar{a}] \iff M_i \models \psi_1[\bar{a}] \wedge M_i \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \wedge N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}].$$

If $\varphi(\bar{x})$ is $\exists y \psi(y, \bar{x})$,

$$M_i \models \varphi[\bar{a}] \iff \text{there is} b \in |M_i| \text{such that} M_i \models \psi[b, \bar{a}].$$

So by the inductive hypothesis,

$$N \models \psi[b, \bar{a}] \implies b \in |N|.$$

As such, $N \models \exists y \psi(y, \bar{x}) \implies N \models \varphi[\bar{x}]$.

Suppose $N \models \varphi[\bar{x}] \implies \exists b \in |N| \psi[b, \bar{a}]$. As $N = \bigcup M_i$, $\exists j \in I$ such that $b \in M_j$. So let $k = \max(i, j)$. Since $i_1 < i_2 \implies M_{i_1} \subseteq M_{i_2}$, we have $M_i \subseteq M_k$ and $M_j \subseteq M_k$. So $\bar{a} \in |M_i|, b \in |M_j| \implies \bar{a}, b \in |M_k|$.

Apply $(*)$ to $i \leftarrow k$, so $M_k \models \exists y \psi(y, \bar{a})$. Since $M_i \subseteq M_k$ and $\bar{a} \in M_i$, we have $M_i \models \exists y \psi(y, \bar{a}) \implies M_k \models \exists y \psi(y, \bar{a})$.

😊

Theorem 2.0.11 Tarski Vaught Test

Suppose $M \subseteq N$. TFAE:

1. $M \prec N$
2. For every $\varphi(y, \bar{x}) \in \mathrm{Fml}(L)$, $\forall \bar{a} \in |M|$, if $N \models \exists y \varphi(y, \bar{a})$, then there is $b \in |M|$ such that $N \models \varphi[b, \bar{a}]$.

Proof. Obviously $1 \implies 2$. For the other direction, we know that for atomic $\varphi(\bar{x})$, $M \subseteq N \implies (M \models \varphi[\bar{a}] \iff N \models \varphi(\bar{a}))$ for every $\bar{a} \in |M|$.

We proceed by induction on φ.
Suppose φ is $\neg\psi$. $M \models \varphi[\bar{a}] \iff M \not\models \psi[\bar{a}] \iff N \not\models \psi[\bar{a}] \iff N \models \varphi[\bar{a}]$.
Suppose φ is $\psi_1 \wedge \psi_2$.

$$M \models \varphi[\bar{a}] \iff M \models \psi_1[\bar{a}] \wedge M \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \wedge N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}]$$

.

Suppose φ is $\psi_1 \vee \psi_2$.

$$M \models \varphi[\bar{a}] \iff M \models \psi_1[\bar{a}] \vee M \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \vee N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}].$$

Suppose φ is $\exists y \psi(y, \bar{x})$. Then,

$$M \models \varphi(\bar{a}) \iff \exists b \in |M|, M \models \psi(b, \bar{a}) \implies \exists b \in |M|, N \models \psi(b, \bar{a}) \implies b \in |N|, N \models \psi[b, \bar{a}] \implies N \models \varphi[\bar{a}].$$

Suppose $N \models \varphi[\bar{a}]$,

$$N \models \exists y \psi(y, \bar{a}).$$

By the assumption, there is $b \in |M|$ such that $N \models \psi[b, \bar{a}]$. As $b, \bar{a} \in |M|$, there is $b \in |M|$ such that $M \models \psi[b, \bar{a}] \implies M \models \exists y \psi(y, \bar{a}) \implies M \models \varphi[\bar{a}]$. 😊

Theorem 2.0.12 Downward Lowenheim Skolem-Tarski

Let M be L-structure, $A \subseteq |M|$. Then there is $N \prec M$, $|N| \supseteq A$, and $\|M\| = |A| + \lambda$.

Proof. For every $\psi(y, \bar{x}) \in \mathrm{Fml}(L)$, let G_ψ be a new function symbol, $L_1 = L \cup \{G_\psi \mid \psi \in \mathrm{Fml}(L)\}$. M_1 is an expansion of M to L_1.

$G_\psi : |M|^{\ell(\bar{x})} \to |M|$. Fix $<$ a well-ordering of $|M|$. For $\bar{a} \in |M|^{\ell(\bar{x})}$,

$$G_\psi^{M_1}(\bar{a}) := \begin{cases} b_0 & M \not\models \exists y \psi(y, \bar{a}) \\ \min\{b \in |M| \mid M \models \varphi[b, \bar{a}]\} & \end{cases}.$$

Apply the submodel theorem to find $N_1 \subseteq M_1$ containing A of cardinality $\lambda + |A|$. Take $N := N_1 \cap L_1$.

Verify $N \prec M$. Follows for $N_1 \prec M_1$. Clearly $N_1 \subseteq M_1$. Verify the Tarski-Vaught test (condition 2). Given $\varphi(y, \bar{x}), \bar{a} \in |N|$,

$$\begin{aligned} M_1 &\models \exists y \psi(y, \bar{a}) \text{ by definition of } G_\varphi \\ M_1 &\models \varphi(G_\varphi(\bar{a}), \bar{a}) \text{ since } N_1 \subseteq M_1 \\ G_\varphi^{N_1}(\bar{a}) &= G_\varphi^{M_1}(\bar{a}). \end{aligned}$$

☺

Corollary 2.0.1 Upward Lowenheim Skolem Tarski

Given L, M an infinite L-structure. For every $\lambda \geqslant |L| + \|M\|$, there exists $N \succ M$ of cardinality λ.

Corollary 2.0.2 Downward Lowenheim Skolem Tarski (stronger)

For all M L-structure, $\lambda \leqslant \|M\|$, $\lambda \geqslant |L| + \aleph_0$, there exists $N \prec M$ of cardinality λ.

Question Suppose T is first order with $|L(T)| \leqslant \aleph_0$. If $I(\lambda, T) \neq 0$ for some $\lambda \geqslant \aleph_0$, then $I(\mu, T) \neq 0$ for all $\mu \geqslant \aleph_0$.

Answer By the two above corralaries, yes.

Proof. Given M and λ, pick $A \subseteq |M|$, $|A| = \lambda$. Apply the last theorem to find $N \prec M$, $|N| \supseteq A$ of cardinality λ. ☺

Fact $\exists f : \omega \times \omega \to \omega$. So $M = \langle omega, f \rangle \models f$ is a bijection.

By compactness and Downward Lowenheim Skolem, T is first order such that $|L(T)| \leqslant \aleph_0$. Then if T has an inifnite model, then for all infinite A, $\exists M \models T$, $\|M\| = |A|$.

Given A infinite, apply the above to M to find $N \models \mathrm{Th}(M)$, $\|N\| = |A|$ such that f^N induces a bijection for $A \times A \to A$.

Definition 2.0.8

$\lambda \geqslant \aleph_0$ is regular provided for every A, $|A| = \lambda$ and every $\mu < \lambda$, B, $|B| = \mu$ and every $f : A \to B$, there exists $b \in B$ such that $|f^{-1}(b)| = \lambda$.

Above, A are the pigeons and B are the pigeon holes.

Definition 2.0.9

$\lambda \geqslant \aleph_0$, $\lambda^+ = \min\{\mu \text{ cardinality} \mid \mu > \lambda\}$. For examples, $\aleph_0 = |\mathbb{N}|$ and $\aleph_0^+ = \aleph_1$.

Kurepa Theorem (1930)

Theorem 2.0.13 Erdos-Rado Theorem (1952)
For all $\lambda \geqslant \aleph_0$, λ^+ is regular.

Proof. Assume λ^+ is not regular. Then there is $f : \lambda^+ \to \lambda$ such that for all $\alpha in \lambda$, $|f^{-1}(\alpha)| < \lambda^+$ or that $|f^{-1}(\alpha)| \leqslant \lambda$. Since the domain of f is λ^+, So,

$$\lambda^+ = \operatorname{dom} f = \cup_{\alpha<\lambda} f^{-1}(\alpha).$$

So, $\lambda^+ = |\lambda^+| \leqslant \sup_{\alpha \leqslant \lambda} |f^{-1}(\alpha)| = \sup_{\alpha<\lambda} \lambda = \lambda$. This is a contradiction. ☺

Definition 2.0.10

λ, μ, κ cardinals, $n < \omega$. $\lambda \to (\mu)^n_\kappa$ is true or false. For all $F : [\lambda]^n \to \kappa$, $\exists A \subseteq \lambda, |A| = \mu$ such that for all $\bar{a} \in [A]^n, F(\bar{a}) = \beta$. This would mean A is monochromatic in β.

Theorem 2.0.14 Infinite Ramsey
$\aleph_0 \to (\aleph_0)^2_2$.

Theorem 2.0.15 Sierpinski
$\mathrm{ZFC} \vdash \aleph_1 \not\to (\aleph_1)^2_2$.

Proof. By monotonicity, it is enough to show $2^{\aleph_0} \not\to (\aleph_1)^2_2$. Fix $<^*$ a well order on $\mathbb{R}$.

For $a < b \in \mathbb{R}$, define $f(a,b) = \begin{cases} 0 & \text{if } a <^* b \\ 1 & \text{if } a >^* b \end{cases}$. FSOC, suppose $A \subseteq \mathbb{R}$ is uncountable monochromatic for f. WMA $A = \{a_\alpha \mid \alpha < \omega\}$ increasing in $<^*$. As A is monochromatic for f, $\exists \ell \in \{0,1\}$ such that $| forall x < \omega, f(a_\alpha, a_{\alpha+1}) = \ell ¿$

Case 1: suppose $\ell = 1$. Namely, $\forall \alpha$, $\mathbb{R} \models a_\alpha < a_{\alpha+1}$. Remember that $\mathbb{Q}$ is dense in $\mathbb{R}$. So $\forall \alpha < \omega$, pick $q_\alpha \in \mathbb{Q} \cup (a_\alpha, a_{\alpha+1})$. As $\{a_\alpha\}$ increasing, $\alpha < \beta \implies (a_\alpha, a_{\alpha+1}) \cap (a_\beta, a_{\beta+1}) = \emptyset$. We found $\alpha < \beta \implies q_\alpha \neq q_\beta$. So $\{q_\alpha \mid \alpha < \omega\}$ is uncountable subset of $\mathbb{Q}$, a contradiction.

Case 2: suppose $\ell = 0$. Then $\forall \alpha < \omega$, $a_{\alpha+1} < a_\alpha$. Similarly, pick $q_\alpha \in (a_{\alpha+1}, a_\alpha) \cup \mathbb{Q}$. $\alpha \neq \beta \implies q_\alpha \neq q_\beta$. So $\{q_\alpha \mid \alpha < \omega\}$ is uncountable subset of $\mathbb{Q}$, a contradiction. ☺

Question: Is there a cardinal $\lambda > \aleph_0$ such that $\lambda \to (\lambda)^2_2$.

Theorem 2.0.16
$(2^{\aleph_0})^+ \to (\aleph_1)^2_2$.

Theorem 2.0.17 ER
$\forall \lambda \geqslant \aleph_0$, $\forall n < \omega$,

$$\beth_{n+1}(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda.$$

Definition 2.0.11

Let $\lambda \geqslant \aleph_0$, $\alpha \in Or$.

$$\beth_\alpha(\lambda) = \begin{cases} \lambda & \alpha = 0 \\ 2^{\beth_\beta(\lambda)} & \alpha = \beta + 1 \\ \sup_{\beta<\alpha} \beth_\beta(\lambda) & \alpha \text{ limit} \end{cases}.$$

Theorem 2.0.18 Cantor's Continuum Hypothesis
$2^{\aleph_0} = \aleph_1$.

ZF $\nvdash$ CH and ZF $\nvdash$ $\neg$CH.

Theorem 2.0.19 Generalized Continuum Hypothesis
$\forall \lambda \geqslant \aleph_0,\ 2^\lambda = \lambda^+ \iff \forall \alpha 2^{\aleph_\alpha} = \aleph_{\alpha+1} \iff \forall \alpha,\ \aleph_\alpha = \beth_\alpha$.

Theorem 2.0.20
ZF $\vdash$ [GCH $\to$ AC]

Definition 2.0.12

Let M be an L-structure, $A \subseteq |M|$, $\bar{b} \in |M|$, $\ell(\bar{b}) < \omega$. The type of $\bar{b}$ over A in M, denoted by $\mathrm{tp}(\bar{b}/A, M)$ is

$$\{\varphi(\bar{x}, \bar{a}) \mid \varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}(L), \bar{a} \in A, \ell(\bar{y}) = \ell(\bar{a}), M \models \varphi[\bar{b}, \bar{a}]\}.$$

Remark: If M is "nice" and $A \subseteq |M|$ is "small", $\bar{b}_1, \bar{b}_2 \in |M|$, then

$$\mathrm{tp}(\bar{b}_1/A, M) = \mathrm{tp}(\bar{b}_2/A, M) \iff \exists f \in \mathrm{Aut}_A(M), f(\bar{b}_1) = \bar{b}_2$$

Proof. (ER)
By induction on n. For $n = 0$¡ $\beth_0(\lambda)^+ = \lambda^+$. ER claims $\lambda^+ \to (\lambda)^1_\lambda \impliedby \lambda^+$ is regular.
For $n+1$, the inductive assumption is that $\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda$¿ We want to show that $\left(2^{\beth_n(\lambda)}\right)^+ \to (\lambda)^{n+2}_\lambda$.
Let $\mu = \beth_n(\lambda)$. We are assuming $\mu^+ \to (\lambda^+)^{n+1}_\lambda$. We want to show $(2^\mu)^+ \to (\lambda^+)^{n+2}_\lambda$.
Suppose $F : [(2^m u)^+]^{n+2} \to \lambda$. We want to find $A \subseteq (2^\mu)^+$ of cardinality λ^+ that is monochromatic for F.
Define $M = \langle (2^\mu)^+, F, \epsilon, i \rangle_{i<\lambda}$.
So, $L(M) = \{F, \epsilon, C_i \mid i < \lambda\}$, $c_i^M = i$.
Note that for $n < \omega$ and $A \subseteq |M|$, $S^n(A, M) = \{\mathrm{tp}(\bar{b}/A, M) \mid \bar{b} \in |M|\}$.
The idea is to find $\{M_i \prec M \mid i < \lambda^+\}$ where $\|M_i\| = 2^\mu$ for all $i < \lambda^+$ such that

$$\forall i < \lambda^+, \forall A \subseteq |M_i|, |A| \leqslant \mu, \forall p \in S^1(A, M), \exists \bar{b} \in |M_{i+1}|, \mathrm{tp}(\bar{b}/A, M_i) = p.$$

😆