Question: 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = f(x^2)$. Prove that f is constant.

Solution: We will start by contradiction. That is, assume f is non constant and that $\exists a, b > 1 : f(a) \neq f(b)$. Then take a very large amount of square roots of a, which tends to 1. Then by definition of the limit, the limit of $f(x)$ as x tends to 1 is $f(a)$. But by definition of continuity, it's also $f(1)$. We can apply this to any positive $a \geqslant 1$

If $0 \leqslant a < 1$, then the case is the same except everything tends to 0.

Negatives follow trivially. 😄

Question: 11

Prove that there is no function f from the set of non negative integers to itself such that $f(f(n)) = n + 1987$ for every n.

Solution: First, we analyze the fact that f is injective. This is because, if $f(m) = f(n)$, then $f(f(n)) = f(f(m)) = n + 1987 = m + 1987 \implies n = m$.

Consider the set of natural numbers that are not in the image of f. Namely, $A = \mathbb{N} \setminus f(\mathbb{N})$. Now consider $f(A)$. It is not hard to see that $f(A) = f(\mathbb{N}) \setminus f(f(\mathbb{N}))$. This is because $f(\mathbb{N}) \in f(A)$ and if any value x is in $f(\mathbb{N})$, then it can't be in $f(f(\mathbb{N}))$ because of injectivity.

Since A subtracts $f(\mathbb{N})$, $A \cap f(A) = \emptyset$. On the other hand, $A \cup f(A) = \mathbb{N} \setminus f(f(\mathbb{N}))$, which is the set of integers from 0 to 1986 inclusive. But f is injective so it must have the same number of elements, which doesn't work because the previous union has an odd number of elements. 😄