Linear Algebra — HW # 2 $\boxed{\textit{deeznuts}}$

1. Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix denoted A^T, whose columns are formed from the corresponding rows of A. For example:

$$A = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$$

Show that $(AB)^T = B^T A^T$. *Hint: Consider the j^{th} column of $(AB)^T$.*

Solution 1: Say that A is an $m \times n$ matrix and that B is an $n \times p$ matrix.

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

Then we look at the RHS.

$$(B^T A^T)_{ij} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj} = \sum_{k=1}^{n} B_{ki} A_{jk} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

.

Therefore, $(AB)^T = B^T A^T$. $\square$

2. Use elimination to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution 2:

Steps to get rref:

$R_2 \Rightarrow R_2 - R_1$

$R_3 \Rightarrow R_3 - R_1$

$R_3 \Rightarrow R_3 - R_2$

$R_2 \Rightarrow R_2 - R_3$

$R_1 \Rightarrow R_1 - R_2 - R_3$

Resultant matrix: $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

3. Find three numbers q for which the matrix $A = \begin{bmatrix} 2 & q & q \\ q & q & q \\ 8 & 7 & q \end{bmatrix}$ is singular (i.e., non-invertible). Briefly explain why each of these numbers makes this true.

Solution 3: A matrix is singular if its determinant is 0, so, we calculate the determinant of this matrix first and set it equal to 0.

$$2(q^2 - 7q) - q(q^2 - 8q) + q(7q - 8q) = 2q^2 - 14q - q^3 + 8q^2 - q^2 = -q^3 + 9q^2 - 14q = 0$$

This polynomial has solutions of $\boxed{0, 2, \text{ and } 7}$, which are also the values which make A singular.

4. As might be expected, repeated multiplication of a matrix is denoted with an exponent. For example: $A^2 = AA$ and $A^4 = AAAA$.

   (a) Find a real non-zero 2×2 matrix A such that $A^2 = -I_2$.

   Solution 4a: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & d^2+bc \end{bmatrix}$

   From here, we know that A^2_{12} and A^2_{21} equal 0, so we set $a + d = 0$. Then we arbitrarily choose $a = 1$ and $d = -1$. We also know that $bc = -2$ in order to make the first and fourth entires equal to -1, so we arbitrarily choose $b = 1$ and $c = -2$.

   Therefore, the matrix we want is $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$.

   (b) Find a real non-zero matrix B such that $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   Solution 4b: Same concept as before. So, we set $a+d = 0$ arbitrarily. One option is $a = -d = 1$. From there, we get that $bc = -1$, so we arbitrarily choose $b = -c = 1$ to get the matrix $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

5. Suppose $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Find A.

   Solution 5:

$$A * B = AB$$
$$A * B * B^{-1} = A = (AB)B^{-1}$$

   So now, find inverse of B.

$$\det(B) = 7 - 6 = 1$$
$$B^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

   Therefore

$$A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

6. A matrix is **tridiagonal** if it has zero entries everywhere except the main diagonal and the two adjacent diagonals. Find the LU factorization of the following tridiagonal matrix:

$$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Solution 6:

We first get T into row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The upper matrix on the RHS is our U. To get our L, we take the product of the inverses of all the matrices on the LHS from left to right.

So, then we get

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} U$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU$$