21-235 Math Studies Analysis I

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Contents

Chapter 1

1.1 Ordered Fields (Review)

Definition 1.1.1: Order

Let E be a set. An *order* on E is a relation $<$ on E such that for all $x, y, z \in E$:

1. (Trichotomy) Exactly one of the following holds: $x < y$, $x = y$, or $x > y$.
2. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Example 1.1.1 (Examples of Ordered Sets)

1. This definition develops orders on basic number systems: e.g. $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
2. Define $\lesssim$ on $\mathbb{Z}$ as follows: We say that $m \lesssim n$ for $m, n \in \mathbb{Z}$ if:
   (a) m is even and n is odd
   (b) m, n are even and $m < n$
   (c) m, n are odd and $m < n$.

Key Concepts:

- upper/lower bounds of sets
- bounded sets
- max/min
- supremum/infimum
- supremum/infimum property: An ordered set E satisfies such a property if every nonempty set $A \subseteq E$ that's bounded above/below has a supremum/infimum in E.
- Fact: sup prop $\implies$ inf prop

Definition 1.1.2: Ordered Field

Let $\mathbb{F}$ be a field with order $<$. We say that $\mathbb{F}$ is an *ordered field* provided that:

1. For all $x, y, z \in \mathbb{F}$, if $x < y$, then $x + z < y + z$.
2. For all $x, y \in \mathbb{F}$, if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Example 1.1.2

$\mathbb{Q}$ is a field.

Facts of any ordered field:

1. $0 < 1$
2. $\nexists x \in \mathbb{F}$ such that $x^2 = -1$.

Definition 1.1.3: Ordered Subfield, Homomorphism, Isomorphism

Let $\mathbb{F}$ be an ordered field.

1. A set $\mathbb{K} \subseteq \mathbb{F}$ is called an *ordered subfield* if $mathbbK$ is an algeraic subfield and $\mathbb{K}$ is an ordered field equipped with $<$ from $\mathbb{F}$.
2. Let $\mathbb{G}$ be an ordered field and let $f : \mathbb{F} \to \mathbb{G}$. We say that f is an *ordered field homomorphism* if it's a field homomorphism and $f(x) < f(y)$ whenever $x < y$.
3. f is an ordered field isomorphism if f is an ordered field homomorphism and f is bijective.

Note:

1. If $f : \mathbb{F} \to \mathbb{G}$ is an ordered field homomorphism, $f(\mathbb{F})$ is an ordered subfield of $\mathbb{G}$.
2. OF property $\implies$ f is injective.
3. $\therefore$ every ordered field homomorphism $f : \mathbb{F} \to \mathbb{G}$ is such that f induces a bijection $f : \mathbb{F} \to f(\mathbb{F}) \subseteq \mathbb{G}$.

Theorem 1.1.1 $\mathbb{Q}$ is the smallest ordered field. More precisely, if $\mathbb{F}$ is an ordered field, then there exists a canonical ordered field homomorphism $f : \mathbb{Q} \to \mathbb{F}$.

Upshot/notation abuse: We identify $f(\mathbb{Q}) = \mathbb{Q}$ to view $\mathbb{Q} \subseteq \mathbb{F}$. In turn, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{F}$.

1.2 Types of Ordered Fields

Definition 1.2.1: Archimedean, Dedekind complete

Let $\mathbb{F}$ be an ordered field.

1. We say that $\mathbb{F}$ is Archimedean if $\forall 0 < x \in \mathbb{F}$, $\exists n \in \mathbb{N}$ such that $n > x$.
2. We say that $\mathbb{F}$ is Dedekind complete if it satisfies the supremum property.

Facts:

1. $\mathbb{Q}$ is Archimedean.
2. If $\mathbb{F}$ is Dedekind complete, then $\forall 0 < x \in \mathbb{F}$ and $\forall 0 < n \in \mathbb{N}$, $\exists!\ 0 < y \in \mathbb{F}$ such that $y^n = x$.
3. $\mathbb{Q}$ is not Dedekind complete. ($\sqrt{2}$ is a counterexample.)

Theorem 1.2.1

Suppose $\mathbb{F}$ is a Dedekind complete ordered field. Then $\mathbb{F}$ is Archimedean.

Proof. If not, then $\mathbb{N} \subset \mathbb{F}$ is bounded above, and so the supremum property provides $x \in \mathbb{F}$ such that $x = \sup \mathbb{N}$. But then $x - 1$ is an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that $x - 1 < n$. Hence $x < n + 1$, which contradicts the definition of x as an upper bound. Therefore, $\mathbb{F}$ is Archimedean. 😊

1.3 Dedekind Completion

Throughout this section, let $\mathbb{F}$ be an Archimedean ordered field.

Definition 1.3.1: Dedekind cut

We say a set $C \subseteq \mathbb{F}$ is *Dedekind cut* if:

1. $C \neq \emptyset$ and $C \neq \mathbb{F}$.
2. If $p \in C$ and $q \in \mathbb{F}$ such that $q < p$, then $q \in C$.
3. If $p \in C$, then $\exists r \in C$ such that $p < r$.

We will write $\mathbb{F}^*$ for the set of all Dedekind cuts in $\mathbb{F}$. It is called the *Dedekind completion* of $\mathbb{F}$.

Note:

Let $C \subseteq \mathbb{F}$ be a cut. Then:

1. If $p \in C$, then $q \notin C$, then $p < q$.
2. If $r \notin C$, and $r < s \in \mathbb{F}$, then $s \notin C$.

Example 1.3.1 (Cut examples)

1. Let $q \in \mathbb{F}$ and define $C_q = \{p \in \mathbb{F} \mid p < q\}$. Then C_q is a cut.

   Proof. (a) $q - 1 < q \implies q - 1 \in C_q$. $q \not< q \implies q \notin C_q \implies C_q \neq \mathbb{F}$.

   (b) Let $p \in C_q$. Suppose $s \in \mathbb{F}$ such that $s < p$. Then $s < q \implies s \in C_q$.

   (c) Let $p \in C_q$. Then $p < \frac{p+q}{2} < q \implies \frac{p+q}{2} \in C_q$.

   😃

2. Suppose $\mathbb{F}$ is such that $\nexists x \in \mathbb{F}$ such that $x^2 = 2$. Let $C = \{p \in \mathbb{F} \mid p \leqslant 0 \text{ or } 0 < p^2 < 2\}$. Then C is a cut.

   Proof. (a) $1 \in C$ and $1^2 = 1 < 2$. $2 \notin C$ and $2^2 = 4 > 2$.

   (b) Let $p \in C$ and $q \in \mathbb{F}$ such that $q < p$. If $q \leqslant 0$, then $q \in C$ trivially. Suppose $0 < q < p$. Then $0 < q^2 < p^2 < 2$, so $q \in C$.

   (c) Let $p \in C$. If $p \leqslant 0$, then $1 \in C$ and $p < 1$, so we're done. Suppose $0 < p^2 < 2$. Note, $0 < 2 - p^2$, so $\frac{2p+1}{2-p^2} > 0$. Then we can define $r = 1 + \frac{2p+1}{2-p^2} \geqslant \max(1, \frac{2p+1}{2-p^2})$. Then $(p + 1/r)^2 = p^2 + \frac{2p}{r} + \frac{1}{r^2}$. We have:

   $$\begin{aligned} p^2 + \frac{2p}{r} + \frac{1}{r^2} &< p^2 + \frac{2p}{r} + \frac{1}{r} \\ &= p^2 + \frac{2p+1}{r} \\ &\leqslant p^2 + 2 - p^2 \\ &= 2. \end{aligned}$$

   So, $p < p + 1/r < 2$ and $p + 1/r \in C$.

   😃

1.3.1 Ordering $\mathbb{F}^*$

Lenma 1.3.1

The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then exactly one holds:
   - $\mathcal{A} \subset \mathcal{B}$
   - $\mathcal{A} = \mathcal{B}$
   - $\mathcal{B} \subset \mathcal{A}$
2. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$.

Proof. Proof of 2 is trivial, as well as the equality part for 1.

- If $\mathcal{A} = \mathcal{B}$, we're done.
- Suppose $\exists b \in \mathcal{B} \setminus \mathcal{A}$. If $a \in \mathcal{A}$, then $a < b$, but $\mathcal{B}$ is a cut so $a \in \mathcal{B}$, so $\mathcal{A} \subset \mathcal{B}$.
- Suppose $\exists a \in \mathcal{A} \setminus \mathcal{B}$. Then $a < b$ for all $b \in \mathcal{B}$, so $a \in \mathcal{B}$, so $\mathcal{B} \subset \mathcal{A}$.

☺

Definition 1.3.2: Order on cuts

Given $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we say that $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$. The lemma above shows that this is infact an order.

Lenma 1.3.2

Let $E \subseteq \mathbb{F}^*$ be nonempty and bounded above. Then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$ is a cut.

Proof.

1. Since $E \neq \emptyset$, there exists $\mathcal{A} \in E$. So $\mathcal{A} \neq \emptyset$, hence $\mathcal{B} \neq \emptyset$.

   Since E is bounded above, there exists $\mathcal{C} \in \mathbb{F}^*$ such that $\mathcal{A} \subset \mathcal{C}$ for all $\mathcal{A} \in E$. Since $\mathcal{C}$ is a cut, there is $q \in \mathbb{F}$ such that $q \notin \mathcal{C}$. Then $q \notin \mathcal{A}$ for all $\mathcal{A} \in E$, so $q \notin \mathcal{B}$.
2. Let $p \in \mathcal{B}$ and $q \in \mathbb{F}$ such that $q < p$. Since $\mathcal{B}$ is a union of cuts, it follows that $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, $q \in \mathcal{A} \subseteq \mathcal{B}$.
3. Let $p \in \mathcal{B}$. Then $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, there exists $r \in \mathcal{A}$ such that $p < r$. Since $\mathcal{A} \subset \mathcal{B}$, we have $r \in \mathcal{B}$.

☺

Theorem 1.3.1

$\mathbb{F}^*$ equipped with the order $<$ satisfies the supremum property.

Proof. Let $E \subseteq \mathbb{F}$ be a nonempty set that is bounded above. From last time, we know that $\mathcal{B} = \cup_{\mathcal{A} \in E} \mathcal{A}$ is a cut. We claim that $\mathcal{B} = \sup E$.

If $\mathcal{A} \in E$, then $\mathcal{A} \subseteq \mathcal{B}$. And so $\mathcal{A} \leqslant \mathcal{B}$, so $\mathcal{B}$ is an upper bound for E.

Next, suppose that $\mathcal{C} \in \mathbb{F}^*$ is an upper bound of E. This means that $\mathcal{A} \leqslant \mathcal{C}$ for every $\mathcal{A} \in E$, meaning $\mathcal{A} \subseteq \mathcal{C} \forall \mathcal{A} \in E$. So $\mathcal{B} \subseteq \mathcal{C}$. As such, $\mathcal{B} \leqslant \mathcal{C}$, so $\mathcal{B} = \sup E$. ☺

Remark: In none of the results leading up to this theorem did we use that $\mathbb{F}$ is anything other than an ordered set. This shows that the cut construction of Dedekind works in general for ordered sets and yields $\mathbb{F}^*$ that satisfies the supremum property. Also, $\{\mathcal{C}_p \mid p \in \mathbb{F}\} \subseteq \mathbb{F}^*$.

1.3.2 Addition

Idea: $\mathbb{F} \cong \{\mathcal{C}_p \mid p \in \mathbb{F}\}$.

Lenma 1.3.3

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. Then $\mathcal{C} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is a cut.

Proof. Claim: $\mathcal{A}, \mathcal{B} \neq \emptyset \implies \mathcal{C} \neq \emptyset$.

$\mathcal{A}, \mathcal{B}$ are cuts, so $\exists M_1, M_2 \in \mathbb{F}$ such that $a < M_1$ for all $a \in \mathcal{A}$ and $b < M_2$ for all $b \in \mathcal{B}$. Then $a + b < M_1 + M_2$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, so $a + b < M_1 + M_2$, meaning $M_1 + M_2 \notin \mathcal{C}$.

Also, let $c = a+b \in \mathcal{C}$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Let $q < c \implies q-a < b \implies q-a \in \mathcal{B}$. Hence, $q = a+(q-a) \in \mathcal{C}$.

Thirdly, let $c = a + b \in \mathcal{C}$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Since $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, $\exists r_a, r_b$ such that $a < r_a \in \mathcal{A}$, $b < r_b \in \mathcal{B}$. Then $c = a + b < r_a + r_b$, so $r_a + r_b \in \mathcal{C}$.

As such, $\mathcal{C}$ is a cut. ☺

Before we define addition, we need to define the negative of a cut.

Heuristic: What we want is that $-\mathcal{C}_1 = \mathcal{C}_{-1}$. The way we do this is by defining $\mathcal{C}_{-p} = \{q \in \mathbb{F} \mid \exists p > q : p \in -\mathcal{C}_p^C\}$. This is the same as $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}_p\}$.

Now we study $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}\}$.

Lenma 1.3.4

Let $\mathcal{C} \in \mathbb{F}^*$. Then $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}\}$ is a cut.

Definition 1.3.3: Addition

For $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we define $\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and $-\mathcal{A} = \{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{A}\}$.

Theorem 1.3.2

Define $0 = \mathcal{C}_0 \in \mathbb{F}^*$. The following hold:

1. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} \in \mathbb{F}^*$.
2. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$.
3. $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^* \implies (\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$.
4. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + 0 = \mathcal{A}$.
5. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + (-\mathcal{A}) = 0$.

Proof. Easy proof, too lazy to write out. ☺

Also: $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} < \mathcal{B} \implies \mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$.

Important Remark: The Archimedean property is actually needed for the above theorem in orer to prove the 5th condition.

1.3.3 Multiplication

Lenma 1.3.5

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ such that $\mathcal{A}, \mathcal{B} > 0$. Then $C = \{p \in \mathbb{F} \mid p \leqslant 0\} \cup \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\}$ is a cut.

Lenma 1.3.6

Let $\mathcal{A} \in \mathbb{F}^*$ be such that $\mathcal{A} > 0$. Then $C = \{p \in \mathbb{F}^* \mid p \leqslant 0\} \cup \{0 < q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$ is a cut.

Definition 1.3.4: Multiplication

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. We define multiplication as:

1. If $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = \{ab \mid 0 < a \in \mathcal{A}, 0 < b \in \mathcal{B}\}$.
2. If $\mathcal{A} = 0$ or $\mathcal{B} = 0$, then $\mathcal{A} \cdot \mathcal{B} = 0$.
3. If $\mathcal{A} > 0$ and $\mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B}))$.
4. If $\mathcal{A} < 0$ and $\mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = -((-\mathcal{A}) \cdot \mathcal{B})$.
5. If $\mathcal{A}, \mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B})$.

We define multiplication inversion via:

1. If $\mathcal{A} > 0$, then $\mathcal{A}^{-1} = \{q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$.
2. If $\mathcal{A} < 0$, then $\mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$.

Theorem 1.3.3

Set $1 = C_1$. The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} \in \mathbb{F}^*$.
2. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$.
3. If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $(\mathcal{A} \cdot \mathcal{B}) \cdot C = \mathcal{A} \cdot (\mathcal{B} \cdot C)$.
4. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot 1 = \mathcal{A}$.
5. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{A}^{-1} = 1$.

Also if $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ and $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} > 0$.

Theorem 1.3.4

If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $\mathcal{A} \cdot (\mathcal{B} + C) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot C$.

We now know that $\mathbb{F}^*$ is an ordered field.

1.4 Robert Reci

Theorem 1.4.1

$\mathbb{Q}$ is the smallest ordered field.

Proof. Let $\mathbb{F}$ be any ordered field. Let $1 \in \mathbb{F}$. Let $\iota : \mathbb{N} \to \mathbb{F}$, $n \mapsto 1 + \cdots + 1$ n times. Then $\iota(-n) = -\iota(n)$ for $n \in \mathbb{N}_0$ and $-n \in \mathbb{Z}^-$.

Then we say $\iota(p/q) = \iota(p)\iota(q)^{-1}$ for $p/q \in \mathbb{Q}$. ☺

Corollary 1.4.1 Every ordered field is infinite

$\iota[\mathbb{Q}] \subseteq \mathbb{F}$ is infinite.

Roots

Let $\mathbb{F}$ be a Dedekind complete ordered field, $0 < x \in \mathbb{F}$, $n \in \mathbb{N}$. Then $\exists! y \in \mathbb{F}$ such that $y > 0$ and $y^n = x$.

Proof. $n = 1$ is silly. Assume $n \geqslant 2$. Let $E = \{z \in \mathbb{F} \mid z > 0 \text{ and } z^n < x\}$. Then E is nonempty and bounded above by x. Let $y = \sup E$. We claim that $y^n = x$.

We want to show that $y^n \not> x$ and $y^n \not< x$.

Lemma 1.4.1

In any commutative ring R, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.

And hence for $0 < a < b$ in $\mathbb{F}$, we have $0 < b^n - a^n = (b-a)nb^{n-1}$.

Suppose $y^n < x$, so $x - y^n > 0$. We define $h = \frac{1}{2}\min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. $0 < h < 1$, also $0 < h < \frac{x-y^n}{n(y+1)^{n-1}}$.

Then, by the inequality below the lemma, we have

$$\begin{aligned} 0 &< (y+h)^n - y^n \\ &< hn(y+h)^{n-1} \\ &< hn(y+1)^{n-1} \\ &< x - y^n, \end{aligned}$$

so $(y+h)^n < x$, which contradicts the definition of y as the supremum. ☺

Definition 1.4.1: Ring*

A ring is a field where actually we don't care about inverses anymore.

Definition 1.4.2: Domain

R is a domain when $xy = 0 \implies x = 0 \wedge y = 0$.

Let R be a ring. For $(r,s) \in R \times R \setminus \{0\}$, we say $(r,s) \sim (r',s')$ if $rs' = r's$.

The field of fractions, $\mathrm{Frac}(R)$ is the set of equivalence classes of $R \times R \setminus \{0\}$ under $\sim$ equipped with the operations $[(r,s)] + [(r',s')] = [(rs' + r's, ss')]$ and $[(r,s)] \cdot [(r',s')] = (rr', ss')$.

We check that $[(r,s)] \cdot [(s,r)] = [(rs, sr)] = [(1,1)]$.

Let $\mathbb{F}$ a field, $\mathbb{F}^x$ its polynomial ring. Let $\mathbb{F}(x)$ be the field of fractions of $\mathbb{F}^x$. Then $\mathbb{F}(x) := \mathrm{Frac}(\mathbb{F}^x)$ is the field of rational functions in x with coefficients in $\mathbb{F}$.

Given $p, q \in \mathbb{F}^x$, say $p/q > 0$ if p and q have the same sign. Say $f, g \in \mathbb{F}(x)$, that $f > g$ when $f - g > 0$.

Theorem 1.4.2

$\mathbb{F}(x)$ is never Archimedean.

Proof. x is an upper bound for all $n \in \mathbb{N}$. ☺

Note:

If $\mathbb{F}$ is Archimedean, $|\mathbb{F}| \leqslant 2^{\aleph_0}$.

Theorem 1.4.3

Let λ be an infinite cardinal. Then there is an ordered field of cardinality λ.

Corollary 1.4.2

The Archimedean property is not a first-order property.

1.5 Completeness

Lenma 1.5.1

Suppose $\mathbb{F}$ is an ordered field that is not Dedekind complete. Then $\exists$ and infinite $E \subsetneq \mathbb{F}$ such that:

1. E bounded above, $\emptyset \neq U(E)$ is open, $\emptyset \neq U(E)^C$ is open.
2. $a \in U(E)^C$, $b \in U(E) \implies a < b$.
3. $f : \mathbb{F} \to \mathbb{F}$ with $f(x) = \begin{cases} 1 & x \in U(E) \\ 0 & x \in U(E)^C \end{cases}$ is differentiable with $f' = 0$.

Theorem 1.5.1 Characteristics of Dedekind Completeness

Let $\mathbb{F}$ be an ordered field. The following are equivalent:

1. $\mathbb{F}$ is Dedekind complete.
2. $\mathbb{F}$ has the intermediate value property: If $f : [a,b] \to \mathbb{F}$ is continuous and $\min(f(a), f(b)) < c < \max(f(a), f(b))$, then $\exists x \in [a,b]$ such that $f(x) = c$.
3. $\mathbb{F}$ satisfies the mean value property: If $f : [a,b] \to \mathbb{F}$ is continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.
4. $\mathbb{F}$ satisfies Cauchy mean value property: If $f, g : [a,b] \to \mathbb{F}$ are both continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
5. $\mathbb{F}$ satisfies the extreme value property: If $f : [a,b] \to \mathbb{F}$ is continuous, then f attains a maximum and minimum on $[a,b]$.

Proof. $1 \implies 2$: Let $f : [a,b] \to \mathbb{F}$ and continuous. WLOG, assume $f(a) < c < f(b)$. Define $E = \{x \in [a,b] \mid f(x) < c\}$. E is nonempty and bounded above by b. Let $x = \sup E$. We claim that $f(x) = c$. Since f is continuous, $\exists \kappa > 0$ such that $f(t) < c \; \forall t \in [a, a+\kappa]$ and $f(t) > c \; \forall t \in [b-\kappa, b]$. So, $a + \frac{\kappa}{2} < x < b - \frac{\kappa}{2}$.

Suppose BWOC $f(x) < c$. Again by continuity, $\exists \delta > 0$ such that $f(t) < c$ for all $t \in B(x,\delta) \subseteq [a,b]$. Then $x + \frac{\delta}{2} \in E$, contradiction.

Then suppose BWOC $f(x) > c$. Again, $\exists \delta > 0$ such that $f(t) > c$ for all $t \in B(x,\delta) \subseteq [a,b]$. Then $\exists z \in E$ such that $x - \frac{\delta}{2} < z \leqslant x$ and $f(z) < c$. But then $c < f(z) < c$, contradiction.

So $f(x) = c$ by trichotomy.

$2 \implies 1$: We'll show $\neg 1 \implies \neg 2$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the strange function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then f is continuous on $[a,b]$, $f(a) - < 1 = f(b)$, but there is not $x \in [a,b]$ with $f(x) = \frac{1}{2}$, by construction.

$1 \implies 5$: First we claim that if $\mathbb{F}$ is Dedekind and $f : [a,b] \to \mathbb{F}$ is continuous, then $f([a,b]) \subseteq \mathbb{F}$ is a bounded set. We prove the claim.

Consider $E = \{x \in [a,b] \mid f([a,x]) \text{ is bounded}\}$. $a \in E$ and E is bounded, so we can let $s = \sup E$. Next note that by continuity, if $[c,d] \subseteq [a,b]$ such that $f([c,d])$ is bounded, then $\exists \delta > 0$ such that $f([a,b] \cap [c-\delta, d+\delta])$ is bounded. Using this, deduce in turn that $a < s$, $s = \max E$, and $s = b$.

So now suppose $\mathbb{F}$ is Dedekind complete and let $f : [a,b] \to \mathbb{F}$ be continuous. The claim establishes that $f([a,b]) \subseteq \mathbb{F}$ is a bounded set, so we can let $\begin{cases} \mu = \inf f([a,b]) \\ \lambda = \sup f([a,b]) \end{cases}$. Suppose BWOC that $f(x) < \lambda$ for all $x \in [a,b]$. Then teh function $g : [a,b] \to \mathbb{F}$ defined by $g(x) = \frac{1}{\lambda - f(x)}$ is continuous and positive. So by the claim, there is $k > 0$ such that $g(x) \leqslant k$ for all $x \in [a,b]$. But then

$$\frac{1}{\lambda - f(x)} \leqslant k \implies \frac{1}{k} \leqslant \lambda - f(x) \implies f(x) \leqslant \lambda - \frac{1}{k},$$

for all $x \in [a,b]$. But this contradicts the definition of λ, as we just found a better upper bound.

Therefore, there does exists $M \in [a,b]$ such that $f(M) = \lambda$, which is $\max f([a,b])$.

The min follows from a similar argument.

$5 \implies 4$: Let $f, g : [a,b] \to \mathbb{F}$ be continuous and differentiable on (a,b). Let $h : [a,b] \to \mathbb{F}$ via $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. It suffices to show $\exists x \in (a,b)$ such that $h'(x) = 0$.

By construction, $h(a) = h(b)$. If $h(x) = h(a)$ for all $x \in [a,b]$, then $h' = 0$ and we're done. Suppose then that h is not constant. Then EVT shows that f attains its maximal/minimum values, and at least one must occur at the point $x \in (a,b)$, therefore $h'(x) = 0$.

$4 \implies 3$: Let $g(x) = x$. Done.

$3 \implies 1$. We'll show $\neg 1 \implies \neg 3$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then consider the restriction $f : [a,b] \to \mathbb{F}$. Then $1 = 1 - 0 = f(b) - f(a)$. Then, $f'(x)(b-a) = 0 \cdot (b-a) = 0$ for all $x \in \mathbb{F}$. $0 \neq 1$ so $\neg 3$ as desired. 😄

Chapter 2

$\mathbb{R}, \mathbb{C}, \bar{\mathbb{R}}$

Theorem 2.0.1

$\mathbb{R}$ is uncountable.

Proof. $\mathbb{Q} \subseteq \mathbb{R}$, so $\mathbb{R}$ is definitely infinite. Suppose BWOC that there was a bijection $f : \mathbb{N} \to \mathbb{R}$. Set $I_0 = [f(0)+1, f(0)+2]$ and not that $f(0) \notin I_0$. Suppose we are given closed, nested, non-singleton intervals $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_0$ such that $f(k) \notin I_k$ for $0 \leqslant k \leqslant n$. If $f(n+1) \notin I_n$, then set $I_{n+1} = I_n$. Otherwise, set I_{n+1} to some non-singleton closed interval contained in I_n such that $f(n+1) \notin I_{n+1}$.

Since $\mathbb{R}$ is Dedekind complete, we have that $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$. So, there is an x such that $x \in I_n$ for all $n \in \mathbb{N}$. But then $x \neq f(n)$ for all $n \in \mathbb{N}$, contradiction since f is a bijection. 😄

Note:

Upshot: Most of $\mathbb{R}$ is transcendental over $\mathbb{Q}$.

2.1 Extended Reals: $\bar{\mathbb{R}}$

Definition 2.1.1: Extended Reals

$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We endow $\bar{\mathbb{R}}$ with the following order: We write $x < y$ for $x, y \in \bar{\mathbb{R}}$ if:

1. $x, y \in \mathbb{R}$ and $x < y$.
2. $x = -\infty$ and $y \in \bar{\mathbb{R}} \setminus \{-\infty\}$.
3. $x \in \bar{\mathbb{R}} \setminus \{\infty\}$ and $y = \infty$.

Facts:

- $(\bar{\mathbb{R}}, <)$ is an ordered set that satisfies the supremum property.
- All sets in $\bar{\mathbb{R}}$ are bounded above.
- All sets in $\bar{\mathbb{R}}$ admit a sup/inf, i.e.
  - $\sup : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.
  - $\inf : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.

  Note: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Also, $A \subseteq B \subseteq \bar{\mathbb{R}}$ implies $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$. And if $E \neq \emptyset$, then $\inf E \leqslant \sup E$.

Note:

$\bar{\mathbb{R}}$ isn't an OF because if it were, then it would be Dedekind complete and then there would exists an ordered field isomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \infty$ for some $x \in \mathbb{R}$. but then $f(x+1) = f(x)+f(1) = \infty+1 = \infty$, which is not a true statement.

Definition 2.1.2

We endow $\bar{\mathbb{R}}$ with the following "algebra."

1. If $x \in \mathbb{R}$, we set $x + \infty = \infty + x = \infty$.
2. If $x \in \mathbb{R}$, we set $x + (-\infty) = (-\infty) + x = -\infty$.
3. $\infty + \infty = \infty$.
4. $-\infty + (-\infty) = -\infty$.
5. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = \infty$.
6. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$.
7. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = -\infty$.
8. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = \infty$.
9. If $x \in \mathbb{R}$, we set $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.
10. $\infty^{-1} = 0 = (-\infty)^{-1}$.
11. If $0 < x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = \infty$.
12. If $0 > x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = -\infty$.

Forbidden/undefined: $\infty + (-\infty)$, $\infty \cdot 0$, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{\mp\infty}$.

2.1.1 Sequences in $\bar{\mathbb{R}}$

Definition 2.1.3: Sequence

A sequence in $\bar{\mathbb{R}}$ is $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$ for $\ell \in \mathbb{Z}$.

In turn, we define new sequences $\{a_N\}_{N=\ell}^{\infty}, \{b_N\}_{N=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$:

- $a_N = \inf\{x_n \mid n \geqslant N\}$.
- $b_N = \sup\{x_n \mid n \geqslant N\}$.

We then set $\liminf_{n\to\infty} x_n = \sup_{N\geqslant\ell} \inf_{n\geqslant N} x_n = \sup_{N\geqslant\ell} a_N$ and $\limsup_{n\to\infty} x_n = \inf_{N\geqslant\ell} \sup_{n\geqslant N} x_n = \inf_{N\geqslant\ell} b_N$.

Example 2.1.1

Let $x_n = \begin{cases} (-1)^n & n \equiv 0 \mod 2 \\ n & n \equiv 1 \mod 2 \end{cases}$. Then, $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 1$.

Proposition 2.1.1

Let $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$. Then $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$.

Proof. Let $M, N \geqslant \ell$ and $K = \max(M, N)$. Then, $\inf_{n>N} x_n \leqslant \inf_{n>K} x_n \leqslant \sup_{n\geqslant K} x_n \leqslant \sup_{n\geqslant M} x_n$.

Thus, $\liminf_{n\to\infty} x_n = \sup_{N\geqslant \ell} \inf_{n\geqslant N} x_n \leqslant \sup_{n\geqslant M} x_n$ for all $M \geqslant \ell$. So, $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$. 😊

Proposition 2.1.2

Let $a_n, b_n \in \bar{\mathbb{R}}$ and suppose $\exists K \geqslant \ell$ such that $a_n \leqslant b_n$ for all $n \geqslant K$. Then, $\liminf_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} b_n$ and $\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} b_n$.

Proof. We can claim that if $k \geqslant K$, then

$$\inf\{a_n \mid n \geqslant k\} \leqslant \inf\{b_n \mid n \geqslant k\}$$
$$\sup\{b_n \mid n \geqslant k\} \leqslant \sup\{a_n \mid n \geqslant k\}.$$

Indeed, if $\exists k \geqslant K$ such that $\inf\{a_n \mid n \geqslant k\} > \inf\{b_n \mid n \geqslant k\}$, then $\exists m \geqslant k$ such that $b_m < \inf\{a_n \mid n \geqslant k\} \leqslant a_m \leqslant b_m$, contradiction. Ditto for sup.

Now define for $N \geqslant \ell$, $C_N = \inf_{n\geqslant N} a_n$, $D_N = \inf_{n\geqslant N} b_N$, $E_N = \sup_{n\geqslant N} a_n$, and $F_N = \sup_{n\geqslant N} b_n$.

The above claims show that $N \geqslant K$ then $C_N \leqslant D_N$ and $E_N \leqslant F_N$. Then we iterate to learn:

$$\liminf_{n\to\infty} a_n = \sup_{N\geqslant\ell} C_N \leqslant \sup_{N\geqslant\ell} D_N = \liminf_{n\to\infty} b_n$$
$$\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell} E_N \leqslant \inf_{N\geqslant\ell} F_N = \limsup_{n\to\infty} b_n.$$

😊

Theorem 2.1.1

Suppose $a_n, b_n \in \bar{\mathbb{R}}$. The following hold:

1. If $\limsup_{n\to\infty} a_n < x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n < x$ for all $n \geqslant N$.
2. If $\liminf_{n\to\infty} a_n > x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n > x$ for all $n \geqslant N$.
3. $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} -a_n$.
4. $\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty} -a_n$.
5. $\limsup_{n\to\infty} a_n + b_n \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided that all arithmetic operations are well-defined.
6. $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leqslant \liminf_{n\to\infty} a_n + b_n$, provided that all arithmetic operations are well-defined.

Proof.

1. Suppose $\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell} \sup_{n\geqslant N} a_n < x$. This implies that $\exists N \geqslant \ell$ such that $\sup_{n\geqslant N} a_n < x$, meaning $a_n < x$ for all $n \geqslant N$.
2. Similar as above.
3. For any $\emptyset \neq X \subseteq \mathbb{F}$, we have that $-\sup(-X) = \inf X$ and $-\inf(-X) = \sup X$. So the result follows.
4. Same as above.
5. We break into cases:
   (a) $\limsup a_n = \infty$ or $\limsup b_n = \infty$. Then $\limsup a_n + b_n = \infty \geqslant \limsup a_n + \limsup b_n$.
   (b) Suppose either $\limsup a_n = -\infty$ or $\limsup b_n = -\infty$. WLOG consider the first option. Since $\limsup b_n < \infty$, then there eixsts $N_1 \geqslant \ell$ and $K \geqslant \mathbb{R}$ such that $b_n < K$ for $n \geqslant N_1$. Now let $m \in \mathbb{N}$ and note that $-\infty < -m - K$. We can use the first result of the theorem to pick $N_2 \geqslant \ell$ such that $n \geqslant N_2 \implies a_n < -m - K$. Then, if $n \geqslant \max(N_1, N_2)$, we have $a_n + b_n < -m$, so $\limsup a_n + b_n = -\infty \leqslant \limsup a_n + \limsup b_n$.

(c) $\limsup a_n, \limsup b_n \in \mathbb{R}$. Let $\epsilon > 0$, then $\exists N_1, N_2 \geqslant \ell$ such that $n \geqslant N_1 \implies a_n < \limsup a_n + \frac{\epsilon}{2}$ and $n \geqslant N_2 \implies b_n < \limsup b_n + \frac{\epsilon}{2}$. Then, $n \geqslant \max(N_1, N_2) \implies a_n + b_n < \limsup a_n + \limsup b_n + \epsilon$, so $\limsup a_n + b_n \leqslant \limsup a_n + \limsup b_n + \epsilon$ for all ϵ.

6. Same as above.

😊

Lemma 2.1.1

Let $x_n \subseteq \mathbb{R}$. The following are equivalent for $x \in \mathbb{R}$:

1. $x_n \to x$ as $n \to \infty$.
2. $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Proof. Let $\epsilon > 0$. Then $\exists N \geqslant \ell$ such that $n \geqslant N \implies x - \epsilon < x_n < x + \epsilon$. Thus, $x - \epsilon \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant x + \epsilon$ for all $\epsilon > 0$. This implies that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Now let $\epsilon > 0$. Then by the previous theorem, there exists $N_1, N_2 \geqslant \ell$ such that $\begin{cases} x - \epsilon < x_n & n \geqslant N_1 \\ x_n < x + \epsilon & n \geqslant N_2 \end{cases}$.
Thus, $n \geqslant \max(N_1, N_2) \implies x - \epsilon < x_n < x + \epsilon$, so $x_n \to x$ as $n \to \infty$. 😊

Definition 2.1.4

Let $x_n \in \bar{\mathbb{R}}$ and $x \in \bar{\mathbb{R}}$. We say that $x_n \to x$ as $n \to \infty$ if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x$.

Remarks:

1. The lemma shows this extends the notion of convergence in $\mathbb{R}$.
2. Limits are unique, when they exist.

Example 2.1.2

1. $\lim_{n\to\infty} n = \infty$ ($n \to \infty$ as $n \to \infty$).
2. Version of squeeze lemma
3. TFAE:
   - $x_n \to \infty$ as $n \to \infty$.
   - $\liminf_{n\to\infty} x_n = \infty$.
   - $\forall M \in \mathbb{N}$, there exists $N \geqslant \ell$ such that $n \geqslant N \implies M \leqslant x_N$.

Chapter 3

Metric Spaces

Definition 3.0.1: Metric

Let X be a nonempty set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x,y) \geqslant 0$ for all $x, y \in X$, and $d(x,y) = 0 \iff x = y$.
2. $d(x,y) = d(y,x)$ for all $x, y \in X$.
3. $d(x,y) \leqslant d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Definition 3.0.2

A metric space is (X,d) for $X \neq \emptyset$ and d a metric on X.

Example 3.0.1

1. $\mathbb{R}$ with $d(x,y) = |x-y|$.
2. $\mathbb{C}$ with $d(x,y) = |x-y|$.
3. (Discrete Metric) Let $X \neq \emptyset$ be any set. Then $d : X \times X \to \{0,1\}$ defined by $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ is a metric on X.
4. Let V be a normed metric space with norm $\|\cdot\|$. Then $d(x,y) = \|x-y\|$ is a metric on V.
5. Suppose (Y,d) is a metric space and suppose $f : X \to Y$ is an injection where $X \neq \emptyset$ is a set. Then $\sigma : X \times X \to \mathbb{R}$ defined by $\sigma(x,y) = d(f(x), f(y))$ is a metric on X.

   Proof. We need to show that σ satisfies the three properties of a metric.

   (a) $\sigma(x,y) \geqslant 0$ because $d \geqslant 0$ and $\sigma(x,y) = 0 \iff d(f(x), f(y)) = 0 \iff f(x) = f(y) \iff x = y$.

   (b) The other two are very trivial.

   😊

6. Let Y be a metric space and $\emptyset \neq X \subseteq Y$. Then $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = d_Y(x,y)$ is a metric on X.

7. Consider $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to \mathbb{R}$ with $f(x) = \log x$ and $g(x) = \frac{1}{x}$. Then $d_f(x,y) = \left|\log \frac{x}{y}\right|$ and $d_g(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|x||y|}$ are metrics on $(0,\infty)$.

8. Let V, W be finite dimensional vector spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(V,W) = \{T : V \to W : T \text{ linear}\}$. Then define $\operatorname{rk}(T) = \dim \operatorname{ran} T$ for $T \in L(V,W)$. Note that $\operatorname{ran}(T+S) = \{Tx+Sx \mid x \in \mathbb{F}\} \subseteq \{Tx+Sy \mid x,y \in \mathbb{F}\} = \operatorname{ran} T + \operatorname{ran} S$. Then, $\operatorname{rk}(T+S) \leqslant \operatorname{rk}(T) + \operatorname{rk}(S)$.

   Define $d(T,S) = \operatorname{rk}(T-S) \in \mathbb{N} \subseteq [0,\infty]$.

   - $d(T,S) = 0 \iff \operatorname{rk}(T-S) = 0 \iff T - S = 0$.
   - Has symmetry.
   - Triangle inequality: $d(T-S) = \operatorname{rk}(T-R+R-S) \leqslant \operatorname{rk}(T-R) + \operatorname{rk}(R-S) = d(T,R) + d(R,S)$.

9. Let $f : \bar{R}R \to [-1,1]$ via $f(x) = \begin{cases} 1 & x = \infty \\ -1 & x = -\infty \\ \frac{x}{\sqrt{1+x^2}} & x \in \mathbb{R} \end{cases}$. Then $d(x,y) = |f(x) - f(y)|$ is a metric on $\bar{\mathbb{R}}$.

Definition 3.0.3

Let X be a metric space.

1. For $x \in X$ and $r \geqslant 0$, we define $B(x,r) = \{y \in X \mid d(x,y) < r\}$. And $B[x,r] = \{y \in X \mid d(x,y) \leqslant r\}$.
2. A set $E \subseteq XX$ is bounded if $\exists (R \geqslant 0)$ such that $E \subseteq B(x,R)$ for some $x \in X$.
3. Let Y be any set and $f : Y \to X$. We say f is a bounded function if $f(Y) \subseteq X$ is bounded. We write $\mathcal{B}(Y;X) = \{g : Y \to X \mid g \text{ is bounded}\}$.

Example 3.0.2

1. $f : \mathbb{R} \to \mathbb{C}$ via $f(t) = e^{it} \implies f(t) = 1 \implies f(\mathbb{R}) \subseteq B[0,1]$ is bounded. So, $f \in \mathcal{B}(\mathbb{R};\mathbb{C})$.
2. $f : (0,\infty) \to \mathbb{R}$ via $f(t) = \frac{\log t}{\sqrt{1+(\log t)^2}}$. So, $f \in \mathcal{B}((0,\infty);\mathbb{R})$.
3. Let X be a metric space and Y a nonempty set. Consider $\mathcal{B}(X;Y)$. If $f \in \mathcal{B}(X;Y)$, then $\exists y \in Y$ and $R \geqslant 0$ such that $d(f(x),y) \leqslant R$ for all x. Thus, $\sup_{x\in X} d(f(x),y) := \sup\{d(f(x),y) \mid x \in X\} \in [0,R]$.

   Similarly, if $f,g \in \mathcal{B}(X;Y)$, then exists $R \geqslant 0$ and $y_1, y_2 \in Y$ such that $d(f(x),y_1) \leqslant R$ and $d(g(x),y_2) \leqslant R$ for all $x \in X$. Then, $d(f(x),g(x)) \leqslant d(f(x),y_1) + d(y_1,y_2) + d(y_2,g(x)) \leqslant 2R + d(y_1,y_2) < \infty$ for all $x \in X$. So, $\sup_{x\in X} d(f(x),g(x)) < \infty$. We now define

$$\begin{aligned} d : \mathcal{B}(X;Y) \times \mathcal{B}(X;Y) &\to [0,\infty) \\ (f,g) &\mapsto \sup_{x\in X} d(f(x),g(x)). \end{aligned}$$

   Proof. Consider the properties of a metric:

   - $d(f,g) = 0 \iff \sup_{x\in X} d(f(x),g(x)) = 0 \iff d(f(x),g(x)) = 0 \iff f(x) = g(x)$ for all $x \in X \iff f = g$.
   - Symmetry is trivial.
   - Let $f,g,h \in \mathcal{B}(X;Y)$. Then, $d(f,h) = \sup_{x\in X} d(f(x),h(x)) \leqslant \sup_{x\in X} d(f(x),g(x)) + d(g(x),h(x)) \leqslant d(f,g) + d(g,h)$.

   😊

Definition 3.0.4

Let X and Y be metric spaces:

1. A map $f : X \to Y$ is an isometric embedding if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Note, such an f is injective.
2. f is an isometry if it's an isometric embedding and surjective.
3. X and Y are isometric if there exists an isometry $f : X \to Y$.

Example 3.0.3

1. Consider $\mathbb{R}^n$ with $|\cdot| = \|\cdot\|_2$, that is, 2-norm.
2. Recall $O(n) = \{\mathcal{M} \in \mathbb{R}^{n\times n} \mid \mathcal{M}^T\mathcal{M} = I\}$ and $R \in O(n) \implies |Rx| = |x|$.
   Let $a \in \mathbb{R}^n$, $R \in O(n)$, and set $f : \mathbb{R}^n \to \mathbb{R}^n$ via $f(x) = a + Rx$. Then,
   $$|f(x) - f(y)| = |a + Rx - (a + Ry)| = |Rx - Ry| = |R(x - y)|.$$
   Also, $y = f(x) = a + Rx \iff y - a = Rx$. So, f is an isometry.
3. Consider $x \mapsto ix \in \mathbb{C}$ for $x \in \mathbb{R}$. This is an isometric embedding but obviously not an isometry for it is not surjective.

The next example is so important that we call it a theorem. Recall $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{R})$ for $X \neq \emptyset$ is a set. Note that if V is a normed vector space, then $\mathcal{B}(X;V)$ is too: $\|f\|_{\mathcal{B}} = \sup_{x\in X}\|f(x)\|_V$ is a norm (exercise) and $d_{\mathcal{B}}(f,g) = \|f - g\|_{\mathcal{B}}$.

Theorem 3.0.1

Let X be a metric space and fix an arbitrary element $a \in X$. For $x \in X$, we'll define $\varphi_x : X \to \mathbb{R}$ via $\varphi_x(y) = d(x,y) - d(y,a)$. The following hold:

1. $\varphi_x \in \mathcal{B}(X)$ for all $x \in X$.
2. Define $\Phi : X \to \mathcal{B}(X)$ via $\Phi(x) = \varphi_x$. Then, Φ is an isometric embedding.

Proof. First note, $|\varphi_x(y)| = |d(x,y) - d(y,a)| \leqslant d(x,a)$ by the triangle inequality. So, $\|\varphi_X\|_{\mathcal{B}} = \sup_{y\in X}|\varphi_x(y)| \leqslant d(x,a) < \infty$. This shows the first result.

Next, fix $x, z \in X$ and consider $\varphi_x(y) - \varphi_z(y) = d(x,y) - d(y,a) - d(z,y) + d(y,a)$. So,
$$|\varphi_x(y) - \varphi_z(y)| = |d(x,y) - d(y,z)| \leqslant d(x,z).$$
Thus, $d_{\mathcal{B}}(\varphi_x, \varphi_y) = \|\varphi_x - \varphi_y\|_{\mathcal{B}} = \sup_{y\in X}|\varphi_x(y) - \varphi_z(y)| \leqslant d(x,z)$.

On the other hand, $|\varphi_x(z) - \varphi_z(z)| = |d(x,z) - \cancelto{0}{d(z,z)}| = d(x,z)$. So, $d_{\mathcal{B}}(\varphi_x, \varphi_z) = d(x,z)$. ☺

Chapter 4

Basic Metric Space Topology