Question: 12

If $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, show that the left cosets are identical to the right cosets. That is, show $gH = Hg$ for all $g \in G$.

Solution: If we take any value $x \in gH$, then $x = gh$ for some $g \in G$ and $h \in H$. Then we can look at the value $xg^{-1} = ghg^{-1} \in H$. Therefore, $x = xg^{-1}g \in Hg$. So, we have shown $gH \subseteq Hg$.

Now if we take any $x \in Hg$, then $x = hg^{-1}$ for some $g \in G$ and $h \in H$. We can use g^{-1} instead of g because G is a group and therefore has inverse closure. Then we can look at $gx = ghg^{-1} \in H$. Therefore, $x = g^{-1}gx \in gH$. So, we have shown $Hg \subseteq gH$.

From the above, we can conclude that $gH = Hg$ for all $g \in G$.

Question: 17

Suppose that $[G : H] = 2$. If a and b are not in H, show that $ab \in H$.

Solution: We have that $[G : H] = 2$ and that $a, b \in G$ but $a, b \notin H$. If $a \notin H$, then $a^{-1} \notin H$ as well. Since these values aren't in H, we can conclude that $a^{-1}H \neq H$ and that $bH \neq H$. But we also know that there are only two cosets of H in G, and one of these cosets is H itself. This means that the two left cosets, $a^{-1}H$ and bH, are the equal.

Now we can take any element from $a^{-1}H$, let's just call it $a^{-1}h$. Since the cosets are equal, we know that $a^{-1}h = bh'$ for some $h' \in H$. Pre-multiplying both sides by a and post multiplying by h'^{-1} shows that $ab = hh'^{-1} \in H$, thereby completing the proof. 😄

Question: 19

Let H and K be subgroups of group G. Prove that $gH \cap gK$ is a coset of $H \cap K$ in G.

Solution: We can prove this by showing that $gH \cap gK = g(H \cap K)$.

If we take any $x \in gH \cap gK$, then $x = gh = gk$ for some $g \in G, h \in H$, and $k \in K$. Since $gh = gk$, we can conclude that $h = k$ by pre-multiplying by g^{-1}. Therefore, $h, k \in H \cap K$. Then, $x = gh = gk \in g(H \cap K)$. Therefore, $gH \cap gK \subseteq g(H \cap K)$.

In the opposite direction, if we take any $x \in g(H \cap K)$, then $x = gy$ for some $y \in H \cap K$. Since $y \in H$, then $x = gy \in gH$. And since $y \in K$, then $x = gk \in gK$. Therefore, $x \in gH \cap gK$ and as such, $g(H \cap K) \subseteq gH \cap gK$.

From the above, we can conclude that $gH \cap gK = g(H \cap K)$ and therefore, $gH \cap gK$ is a coset of $H \cap K$ in G. 😄

Question: 20

Let H and K be subgroups of group G. Define a relation $\sim$ on G by $a \sim b$ if there exists an $h \in H$ and a $k \in K$ such that $hak = b$. Show that this relation is an equivalence relation. The corresponding equivalence classes are called ***double cosets***. Compute the double cosets of $H = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$ in A_4.

Solution: First I will show that this is an equivalence relation:

- **Reflexivity**: $a \sim a$ for all $a \in G$. This is true because if we take $a \in G$, then we can choose $h = e$ and $k = e$ and then $hak = eae = a$.

- **Symmetry**: We know that $a \sim b \iff b \sim a$ because if $hak = b$, then we can show that $h^{-1}bk^{-1} = a$ by left and right multiplying by h^{-1} and k^{-1} respectively. We also know that $h^{-1} \in H$ and that $k^{-1} \in K$ because they are groups and therefore have inverse closure.

- **Transitivity**: If $a \sim b \wedge b \sim c$, then we know that $h_1ak_1 = b$ and that $h_2bk_2 = c$. Then we can do a substitution to see that $h_2h_1ak_1k_2 = c$. Since $h_2h_1 \in H$ and $k_1k_2 \in K$ by group closure, we can conclude that $a \sim c$.

A_4 itself is described by the elements:

$$A_4 = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4),$$

$$(1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}.$$

Then, for $a \in A_4$,

$$HaH = \{hak : h, k \in H\}.$$

With this information, we can start listing the double cosets of H in A_4.

$$H(1)H = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$$

$$H(1\ 2)(3\ 4)H = \{(1)(1\ 2)(3\ 4)(1), (1)(1\ 2)(3\ 4)(1\ 2\ 3), (1)(1\ 2)(3\ 4)(1\ 3\ 2),$$

$$(1\ 2\ 3)(1\ 2)(3\ 4)(1), (1\ 2\ 3)(1\ 2)(3\ 4)(1\ 2\ 3), (1\ 2\ 3)(1\ 2)(3\ 4)(1\ 3\ 2),$$

$$(1\ 3\ 2)(1\ 2)(3\ 4)(1), (1\ 3\ 2)(1\ 2)(3\ 4)(1\ 2\ 3), (1\ 3\ 2)(1\ 2)(3\ 4)(1\ 3\ 2)\}$$

$$= \{(1\ 2)(3\ 4), (2\ 4\ 3), (1\ 4\ 3), (1\ 3\ 4), (1\ 2\ 4), (1\ 4)(2\ 3), (2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 2)\}$$

Since these two equivalence classes, $[(1)]$ and $[(1\ 2)(3\ 4)]$, contain every element in A_4, they are the double cosets of H in A_4.