Question: 1

What is the largest positive integer that is a factor of $P(1) - 2P(7) + P(13)$, for every polynomial P with integer coefficients?

Solution: It is a result that for any polynomial P with integer coefficients, $a - b$ divides $P(a) - P(b)$. So, we rewrite this as $P(13) - P(7) + P(1) - P(7)$. Now, we know that $13 - 7 = 6$ and $1 - 7 = -6$. So, we can factor out a 6 from the expression, meaning that this is the largest integer we can guarantee to be a factor of the expression.

Question: 3

Prove that for every prime number p, the polynomial

$$P(x) = \sum_{i=0}^{p-1} x^i$$

cannot be expressed as the product of two non-constant polynomials with integer coefficients.

Solution: This is equivalent to showing that it is irreducible. We will use the idea that $f(x)$ is irreducible iff $f(x+1)$ is irreducible. So,

$$f(x+1) = \frac{(x+1)^p - 1}{(x+1) - 1} = x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + p$$

This clearly fails Eisenstein's for p. So, $f(x+1)$ is irreducible, and thus $f(x)$ is irreducible.