Math 21-269, Vector Analysis I, Spring 2024 Assignment 1

The due date for this assignment is Friday, January 26.

1. Let $G \subseteq \mathbb{R}$ be an additive group. Prove that the following two conditions are equivalent:

   (a) for every $0 < x < y$ there exists $g \in G$ such that $x < g < y$,

   (b) $\inf\{g \in G : g > 0\} = 0$.

2. $$f(x) = \arcsin \frac{1 - x^2}{1 + x^2},$$

   (a) find the domain D of f,

   (b) find its derivative and the sets in which f is increasing,

   (c) sketch the graph of f,

   (d) find the supremum and the infimum of the set

   $$E = \left\{ y \in \mathbb{R} : y = \arcsin \frac{1 - x^2}{1 + x^2}, \, x \in D \right\}.$$

JUSTIFY ALL YOUR ANSWERS

Solutions

1. **Proof.** We start by showing $(a) \implies (b)$. We need to prove that for all $\epsilon > 0$, there exists $g \in G$ such that $0 < g < \epsilon$. This is saying that there is no positive $g < \epsilon$ that is a lower bound, and therefore the infimum is 0. Fix $x = \dfrac{\epsilon}{2}$ and $y = \epsilon$ for $\epsilon > 0$ (which means $\dfrac{\epsilon}{2} > 0$). By condition (a), there exists $g \in G$ such that $\dfrac{\epsilon}{2} < g < \epsilon$. This yields that $0 < g < \epsilon$ as desired. Thus, we have shown that (a) $\implies$ (b).

   Now, we show that (b) $\implies$ (a). We need to prove that for all $0 < x < y$, there exists $g \in G$ such that $x < g < y$. Given any $0 < x < y$, choose ϵ such that $0 < \epsilon < y - x$. We know that there is an $\epsilon \in \{g \in G : g > 0\}$ that suffices this inequality because if there weren't, that would mean $y - x$ would be a lower bound of $\{g \in G : g > 0\}$, which contradicts the fact that the infimum is 0.
   Consider the element $\epsilon \left\lfloor \dfrac{x}{e} \right\rfloor + \epsilon$, which we know is in G because it is an additive group (multiplication is repeated addition). Then, we have:

   $$x < \epsilon \left\lfloor \frac{x}{e} \right\rfloor + \epsilon \leq x + \epsilon < x + (y - x) = y$$

   Thus, we have shown that (b) $\implies$ (a).

   Therefore, we have shown that (a) and (b) are equivalent. ∎

2. (a) The argument of arcsin needs to be in $[-1,1]$. So, we are limited to points where $\dfrac{1-x^2}{1+x^2} \in [-1,1]$. I claim that this is true for all $x \in \mathbb{R}$.

Proof. If $\dfrac{1-x^2}{1+x^2} < -1$, we have the following:

$$\begin{aligned} \frac{1-x^2}{1+x^2} &< -1 \\ 1-x^2 &< -1-x^2 \\ 1 &< -1 \end{aligned}$$

The above is a contradiction. If we have $\dfrac{1-x^2}{1+x^2} > 1$, we have the following:

$$\begin{aligned} \frac{1-x^2}{1+x^2} &> 1 \\ 1-x^2 &> 1+x^2 \\ 1 &> 2 \end{aligned}$$

The above is another contradiction.

Thus, we have that $\dfrac{1-x^2}{1+x^2} \in [-1,1]$ for all $x \in \mathbb{R}$. Thus, the domain of f is $\mathbb{R}$. ∎

(b) Finding the derivative:

$$\begin{aligned} f(x) &= \arcsin \frac{1-x^2}{1+x^2} \\ f'(x) &= \frac{d}{dx} \arcsin \frac{1-x^2}{1+x^2} \\ &= \frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \\ &= \frac{\dfrac{\frac{d}{dx}(1-x^2)(1+x^2)-(1-x^2)\cdot\frac{d}{dx}(1+x^2)}{(x^2+1)^2}}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \\ &= \frac{(-2x)(1+x^2)-(1-x^2)(2x)}{(x^2+1)^2\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \\ &= \boxed{\frac{-4x}{(x^2+1)^2\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}} \end{aligned}$$

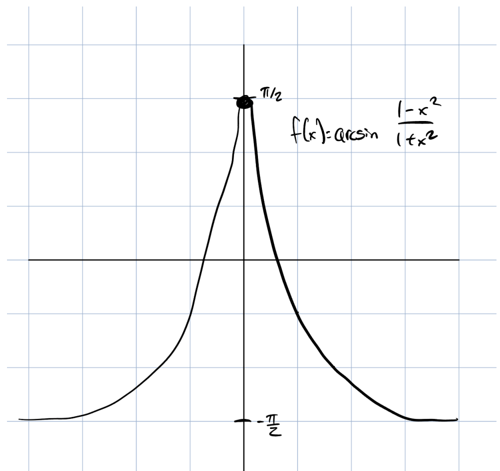
We can see that $f'(x) < 0$ when $x < 0$. This is because the denominator is always positive (a squared value multiplied by a square rooted value, both of which are for sure positive). Though this is not the case when $x = 0$, which makes the square root value 0 and therefore f is indifferentiable at $x = 0$. However, we still only need to analyze the sign of $-4x$, which is positive when x is negative and vice versa. Therefore, f is increasing on $(-\infty, 0)$.

(c) Here are some key facts we need to realize before sketching:

i. f is defined for all $x \in \mathbb{R}$.

ii. f is increasing on $(-\infty, 0)$ and vice versa.

iii. f is limited by the range of arcsin, which is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since the argument of arcsin has an asymptote at $y = -1$, we can say that f has an asymptote at $y = -\frac{\pi}{2}$.



(d) I claim that the supremum is $\frac{\pi}{2}$ and the infimum is $-\frac{\pi}{2}$.

Supremum: $\arcsin(z)$ as a function is bounded above by $\frac{\pi}{2}$, which occurs when $z = 1$. We can see that $z = \frac{1-x^2}{1+x^2}$, which equals 1 when $x = 0$. Thus, the supremum is $\frac{\pi}{2}$.

Infimum: $\arcsin(z)$ as a function is bounded below by $-\frac{\pi}{2}$, which occurs when $z = -1$. Since the horizontal asymptote of $z = \frac{1-x^2}{1+x^2}$ is $y = -1$, we can say that z approaches, but never reaches -1. This means that for f, we approach, but never cross $-\frac{\pi}{2}$. Thus, the infimum is $-\frac{\pi}{2}$.

If we want to be more rigorous, let $y = -\frac{\pi}{2} + \epsilon$ for $\pi \geq \epsilon \geq 0$.

$$\begin{aligned}
-\frac{\pi}{2} + \epsilon &= \arcsin \frac{1-x^2}{1+x^2} \\
\sin\left(-\frac{\pi}{2} + \epsilon\right) &= \frac{1-x^2}{1+x^2} \\
-\cos(\epsilon) &= \frac{1-x^2}{1+x^2} \\
-\cos(\epsilon)(1+x^2) &= 1-x^2 \\
-\cos(\epsilon) - x^2\cos(\epsilon) &= 1-x^2 \\
x^2\cos(\epsilon) - x^2 &= 1+\cos(\epsilon) \\
x^2(\cos(\epsilon)-1) &= 1+\cos(\epsilon) \\
x^2 &= \frac{1+\cos(\epsilon)}{\cos(\epsilon)-1} \\
x &= \pm\sqrt{\frac{1+\cos(\epsilon)}{\cos(\epsilon)-1}}
\end{aligned}$$

The only time this value will be undefined is when the denominator of the square root is equal to 0. This happens when $\cos(\epsilon) = 1 \Rightarrow \epsilon = 0$. Thus, we have that $x = \pm\sqrt{\frac{1+\cos(\epsilon)}{\cos(\epsilon)-1}}$ is defined for all $\pi \geq \epsilon > 0$. Thus, we have that $-\frac{\pi}{2} + \epsilon$ is in the range of f for all $\pi \geq \epsilon > 0$. Thus, the infimum is $-\frac{\pi}{2}$.