

Question: 1

For each of the following groups G , determine whether H is a normal subgroup of G . If H is a normal subgroup, write out a Cayley table for the factor group G/H .

- $G = S_4$ and $H = A_4$
- $G = A_5$ and $H = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$
- $G = S_4$ and $H = D_4$
- $G = Q_8$ and $H = \{1, -1, i, -i\}$
- $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution:

- $|G/H| = 2$, so the only group that G/H could be is \mathbb{Z}_2 . Here is the Cayley table:

+	0	1
0	0	1
1	1	0

- Counterexample: $(1\ 5)(2\ 3)(1\ 2\ 3) = (1\ 3\ 5) \neq (1\ 2\ 3)(1\ 5)(2\ 3) = (1\ 5\ 2)$.

- $|G/H| = 15$ and there is only one group of order 15 which is \mathbb{Z}_{15} . Here is the Cayley table:

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

- Same as part a.
- H is normal in G because $G = \mathbb{Z}$ is abelian and all subgroups of abelian groups are normal.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Question: 4

Let T be the group of nonsingular upper triangular 2×2 matrices with entries in \mathbb{R} ; that is, in the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$.

- Show that U is a subgroup of T .
- Prove that U is abelian.
- Prove that U is normal in T .
- Show that T/U is abelian.
- Is T normal in $GL_2(\mathbb{R})$?

Solution:

- To prove that U is a subgroup of T , we have to prove the following criteria: associativity, identity, inverse, and closure.

(a) Associativity: Let $a, b, c \in \mathbb{R}$. Then $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. Then

$$A(BC) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = (AB)C. \text{ Therefore, } U \text{ is associative.}$$

(b) Identity: Consider $u \in U$ where $x = 0$. Then $u = I_2$ which is the identity for all 2×2 matrices.

(c) Inverse: The inverse of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. This can be checked by multiplying the two matrices together and getting I_2 .

(d) Closure: Let $u, v \in U$. Then $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Then $uv = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \in U$. Therefore, U is closed.

Therefore, U is a subgroup of T .

b. U is abelian because if we have $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, then $uv = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix} = vu. \quad \ominus$

c. U is normal in T because T is abelian and all subgroups of abelian groups are normal.

d. Consider the following:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}.$$

From this, we know that every coset in T/U has a representative diagonal matrix. We know that diagonal matrices commute, so we know that T/U is abelian.

e. Counterexample:

$$\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \begin{pmatrix} n & n \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} n^3 & 0 \\ n^3 & n^3 \end{pmatrix}.$$

Question: 5

Show that the intersection of two normal subgroups is a normal subgroup.

Solution: Let H and K be two normal subgroups of G . Then, for $h \in H$ and $k \in K$ and $g \in G$, $ghg^{-1} \in H$ and $gkg^{-1} \in K$. Now, let $T = H \cap K$.

Let $t \in T \Rightarrow t \in H$ and $t \in K$

$$\Rightarrow gtg^{-1} \in H \text{ and } gtg^{-1} \in K$$

$$\Rightarrow gtg^{-1} \in H \cap K$$

$$\Rightarrow gtg^{-1} \in T$$

\therefore for all $g \in G, t \in T, gtg^{-1} \in T$. Therefore, T is a normal subgroup of G .

Question: 11

If a group G has exactly one subgroup H of order k , prove that H is normal in G .

Solution: For $g \in G$, consider the conjugate subgroup $gHg^{-1} \leq G$. We also know that the order of gHg^{-1} is the same as the order of H , which we called k . But, since H is the only subgroup of order k , any subgroup that has order k must be H . Therefore, $gHg^{-1} = H$. Therefore, H is normal in G . \ominus

Question: 13

Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

- Calculate the center of S_3 .
- Calculate the center of $GL_2(\mathbb{R})$.
- Show that the center of any group G is a normal subgroup of G .
- If $G/Z(G)$ is cyclic, show that G is abelian.

Solution:

a. $Z(S_3) = \{e\}$

- b. If we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, we can multiply out AB and BA to see the following equality:

$$ae + bg = ae + cf \Rightarrow bg = cf$$

However, this equality needs to hold true for all choices of g, f because our B was arbitrary and not related to A . This means that $b = c = 0$. This means that the equation

$$af + bh = be + df$$

reduces to $af = df$ or $a = d$. This means that we can say

$$Z(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}.$$

- c. By definition, for any $z \in Z(G)$, the following equation will hold true, $zG = Gz$. By the definition of a normal subgroup, $Z(G)$ is normal in G .
- d. By definition $G/Z(G) = \langle xZ(G) \rangle$ for some $xZ(G) \in G/Z(G)$ where x is the representative for the coset $xZ(G)$.

If we let $a \in G$, then we know that $aZ(G) = (xZ(G))^m$ for some m . We can also rewrite $(xZ(G))^m = x^m Z(G)$.

If we take another $b \in G$, then we know that $bZ(G) = (xZ(G))^n$ for some n . We can then rewrite $(xZ(G))^n = x^n Z(G)$.

From these two equations, we get that $ax^{-m}, bx^{-n} \in Z(G)$. For shorthand purposes, we can say $p = ax^{-m}$ and $q = bx^{-n}$. Then, we get that $a = px^m$ and $b = qx^n$. Multiplying both gives us $ab = (px^m)(qx^n) = pqx^{m+n}$. The last step was done because we know that $Z(G)$ is abelian.

If we multiply the other way, we know see that $ba = (qx^n)(px^m) = pqx^{m+n}$. This means that $ab = ba$. Therefore, $G/Z(G)$ is abelian. \ominus

Question: 14

Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$; that is, G' is the subgroup of all finite products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the **commutator subgroup** of G .

- Show that G' is a normal subgroup of G .
- Let N be a normal subgroup of G . Prove that G/N is abelian iff N contains the commutator subgroup of G .

Solution:

- Let $s = aba^{-1}b^{-1}$ be the generator of G' . We can say that for any $g \in G$, $gsg^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$. By this structure, we can see that $gsg^{-1} \in G'$. Effectively, conjugating by g is a homomorphism G' is normal in G .
- If $a, b \in G$ and we assume G/N is abelian, then we have $(aN)(bN) = (bN)(aN) \Leftrightarrow Nab = Nba \Leftrightarrow Naba^{-1}b^{-1} = N \Leftrightarrow aba^{-1}b^{-1} \in N$.

Now, if we assume that $aba^{-1}b^{-1} \in N$, this is the same as $ab(ba)^{-1} \in N$. This means that $Nab = Nba$, or as we showed before, $(aN)(bN) = (bN)(aN)$. Therefore, G/N is abelian. \ominus .