

# 15756 Randomized Algorithms

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# Chapter 1

## 100 Prisoners Problem

This is a problem based on a riddle:

### Question

There is a jail and a warden. However, the warden is very lazy and says if they win a game, they're all set free, otherwise they all die. There is 100 boxes in a room and each box has a hidden number from 1 to 100. They are assorted in a random permutation. Each prisoner is also given a number from 1 to 100. Each prisoner is allowed to enter the room and will be allowed to open 50 boxes and if they find their number, they win. They will only live if everyone finds their number. The prisoners can't communicate with each other. What strategy should they use to maximize their chances of winning?

Naive strategy: Each prisoner opens 50 boxes at random. The probability of winning is  $\frac{1}{2^{100}}$ .

### Theorem 1.0.1

There exists a strategy such that the probability of winning is  $\geq \frac{30}{100}$ .

*Proof.* The algorithm is as follows:

1. Each prisoner opens the box with their number.
2. If they find their number, they open the box with the number they found.
3. They continue this process until they find their number or they open 50 boxes.

Cycle notation: Consider the following arrangement, where the ordinal is the box number:

1. 7
2. 6
3. 4
4. 3
5. 8
6. 1
7. 2
8. 0
9. 5
10. 10

This can be represented as 4 individual cycles:  $(1, 7, 2, 6)$ ,  $(3, 4)$ ,  $(5, 8, 9)$ ,  $(10)$ .

A critical observation to make is that prisoner  $i$  will win if and only if the cycle containing  $i$  is of length  $\leq 50$ . That is, everyone wins iff there are no cycles of length  $> 50$ .

We warm up by considering opening 75 boxes instead of 50. The observation is that if anyone loses, at least 75 people will lose. Let  $X$  be a random variable representing the number of people who lose. Then,  $E[X] = 25$  as each player has a  $\frac{1}{4}$  chance of losing. Recall Markov's inequality:

### Theorem 1.0.2

Let  $X$  be a non-negative random variable. Then, for any  $t > 0$ ,  $\Pr[X \geq t] \leq \frac{E[X]}{t}$ .

Applying Markov's inequality, we have  $\Pr[X \geq 75] \leq \frac{25}{75} = \frac{1}{3}$ . Thus, the probability of winning is  $\geq \frac{2}{3}$ . Moving back to the original problem, we can apply the same logic.

$$\Pr(\text{anyone loses}) = \Pr(\text{at least one cycle of length} > 50).$$

Recall how we can count the number of cycles of length  $\geq 50$ . For a cycle to be exactly length  $n > \frac{100}{n} = 50$ , we need to choose  $n$  boxes and then permute them. Thus, the number of cycles of length  $n$  is  $\frac{100!}{n \cdot (100-n)!}$ . Now we have to worry about the number of arrangements of the other  $100 - n$  boxes, which is  $(100 - n)!$ . Multiplying these two quantities, we get:

$$\frac{100!}{n \cdot (100 - n)!} \cdot (100 - n)! = \frac{100!}{n}.$$

As there are  $100!$  total permutations, we realize that exactly  $\frac{1}{n}$  of the permutations contain a cycle of length  $n$ . Thus, the probability of losing is:

$$\Pr(\text{anyone loses}) = \sum_{n=51}^{100} \frac{1}{n} = \sum_{n=1}^{100} \frac{1}{n} - \sum_{n=1}^{50} \frac{1}{n} = H_{100} - H_{50} \approx \ln 100 - \ln 50 = \ln 2 \approx 0.693.$$

Thus, the probability of winning is  $\geq 1 - 0.693 \approx 0.307$  as desired.  $\odot$

We move on to proving that this algorithm is actually optimal:

### Theorem 1.0.3

The algorithm described above is optimal.

*Proof.* We consider a second version of the game, where any box that a previous prisoner has opened stays open. So if prisoner  $i$  walks in to see that  $i$  has been revealed, he just leaves. Otherwise, he opens boxes until he finds  $i$  or has opened 50 boxes. Boxes are never closed.

### Lemma 1.0.1

Cycle algorithm is no better at version 2 than version 1.

*Proof.* Option 1: Someone from my cycle in the past lost.

Option 2: I am the first person in my cycle to enter.

In both versions, the boxes in my cycle are closed. This means that in both versions, I win iff the cycle is of length  $\leq 50$ .

Option 3: Someone from my cycle has already entered, and they won. This means the cycle is of length  $\leq 50$ , so I win in both versions.  $\odot$

### Lemma 1.0.2

All algorithms are equally good in version 2.

*Proof.* By symmetry: If I'm opening a box, I have to pick out of the remaining boxes, and they all have the same probability of containing my number. ☺

Together, these lemmas show that the cycle algorithm is optimal in version 1. ☺

# Chapter 2

## Graph algorithms

\*Note: all graphs are going to be undirected and unweighted.

### 2.1 Graph coloring

Suppose we are given a graph  $G = (V, E)$ . Suppose  $G$  is 3-colorable. Recall that a 3-coloring is a function  $c : V \rightarrow \{1, 2, 3\}$  such that  $c(u) \neq c(v)$  for all  $(u, v) \in E$ . Simply put, you can color the vertices such that no two adjacent vertices have the same color.

#### Question

Can we efficiently find a 3-coloring of  $G$ ?

**Answer:** No, this is NP-hard.

**Observation:** It's possible to partition the vertices in the graph into two parts that are each  $\Delta$ -free. This is easy because we can create the partition of one set being the odd vertices (1 and 3) and the other set being the even vertices (2).

#### Theorem 2.1.1 McDiarmid (1993)

Given a 3-colorable graph, you can construct a triangle-free partition of the vertices in  $\text{poly}(n)$  expected time.

*Proof.* Consider the following algorithm: in each round, find a triangle in some side, pick a random vertex from the triangle, and move it to the other side. Repeat this process until there are no more triangles.

Now we analyze the algorithm. Fix a 3-coloring of  $G$ . Let  $X$  be the number of vertices of color 1 on the left side. Let  $Y$  be the number of vertices of color 2 on the right side. Now we analyze what happens to  $X + Y$  during a round:

- If we move a vertex from the left to the right,  $X$  can decrease by 1 with probability  $\frac{1}{3}$ ,  $Y$  can increase by 1 with probability  $\frac{1}{3}$ , or both can stay the same with probability  $\frac{1}{3}$ .
- If we move a vertex from the right to the left,  $Y$  can decrease by 1,  $X$  can increase by 1, or both can stay the same all with probability  $\frac{1}{3}$ .

So overall,  $X + Y$  increases by 1 with probability  $\frac{1}{3}$ , decreases by 1 with probability  $\frac{1}{3}$ , and stays the same with probability  $\frac{1}{3}$ . This is just a random walk.

The random walk starts at an unknown value, but if it ever leaves  $[0, n]$ , the algorithm terminates. It can look something like  $+1, -1, 0, 0, -1, +1, +1, +1$ , and at some point  $t_1$  the algorithm finishes. For sake of discussion, assume this hypothetical random walk keeps going. At some point, the walk may leave  $[0, n]$  at time  $t_2$ . Note that  $t_2$  can exist only if  $t_1$  exists, and  $t_1 < t_2$ . So  $E[t_2] \geq E[t_1]$ .

Goal: prove that  $E[t_2] \in O(n^2)$ . This will imply that  $E[t_1] \in O(n^2)$ , which will imply that the algorithm runs in  $\text{poly}(n)$  time.

We step away from the graph problem to analyze random walks further. Consider a random walk on  $\mathbb{Z}$  that starts at 0. Let  $T$  be the first time the walk arrives at  $n$  or  $-n$ . We want to show that  $E[T] \in O(n^2)$ .

Fact: after  $t$  steps, the middle 90% of possible positions would be  $O(\sqrt{t})$  from the origin. Additionally, within this 90%, the walk is about uniformly distributed.

Using the fact above, we want  $t = n^2$  to be the time it takes to reach  $n$  or  $-n$ .

Now we return to the original problem.

### Lemma 2.1.1

For any power of 2, the expected time to get from the origin to  $n$  or  $-n$  is  $n^2$ .

*Proof.* If  $n = 1$ , the expected time is 1. Define  $T_n$  as the expected time to get from the origin to  $n$  or  $-n$ . Pretend we split the number line into  $-n, -n/2, 0, n/2, n$ . Then we can write  $T_n$  as:

$$T(n) = T(n/2) + T(n/2) + \frac{1}{2}T(n).$$

Solving, we get  $T(n) = 4T(n/2) \implies T(n) = n^2$ . ⊖

This means that our algorithm is  $O(n^2)$  rounds, meaning the algorithm runs in  $\text{poly}(n)$  time. ⊖

## 2.2 Shortest paths

Given  $G = (V, E)$ , for each  $x, y \in V$ , we want  $d(x, y) \pm O(1)$  in time  $O(n^{2.5} \log n)$ .

Naive algorithm:  $O(|V| \cdot |E|) = O(n^3)$ .

Now consider a shortest path from  $x$  to  $y$ .

- Option 1: Every vertex on the path has degree  $< \sqrt{n}$ .

Then for each vertex, we can BFS, but only on low degree vertices. This takes  $O(n^{2.5})$  time. The result is that we get exact distances for all vertices on the path.

- Option 2: There is a vertex on the path with degree  $\geq \sqrt{n}$ .

Now we sample  $100\sqrt{n} \log n$  iid uniformly random vertices and call this set  $S$ . We claim that with high probability, every high degree vertex  $w$  will be incident to a vertex  $s \in S$ .

If we take this claim for granted, observe that for  $x, y$  in option 2,  $d(x, y) = d(x, s) + d(s, y) \pm 2$  for some  $s \in S$ . So for each  $s \in S$ , do a BFS which takes  $O(n^{2.5} \log n)$  time. For each pair  $(x, y)$ , compute the minimum  $d(x, s) + d(s, y)$  over all  $s \in S$ . This takes  $O(n^{2.5} \log n)$  time.

Now we prove the claim made in option 2.

*Proof.* Let  $w$  be a high degree vertex. We want to find  $P(\text{none of } w\text{'s neighbors are sampled})$ .

$$\begin{aligned} P(\text{none of } w\text{'s neighbors are sampled}) &\leq P(\text{given sample misses})^{|S|} \\ &= \left(1 - \frac{1}{\sqrt{n}}\right)^{100\sqrt{n} \log n}. \end{aligned}$$

Now we use the fact that  $(1 - \frac{1}{n})^n \leq \frac{1}{e}$ . So now,

$$\left(1 - \frac{1}{\sqrt{n}}\right)^{100\sqrt{n} \log n} \leq \frac{1}{e^{100 \log n}} = \frac{1}{n^{100}}.$$

This is the probability of  $w$  screwing up. The probability of any high degree vertex screwing up is  $\leq \frac{n}{n^{100}} = \frac{1}{n^{99}}$  ⊖



## 2.3 Global Min-Cut

We start by stating the question we are trying to solve.

### Question: Global Min-Cut Problem

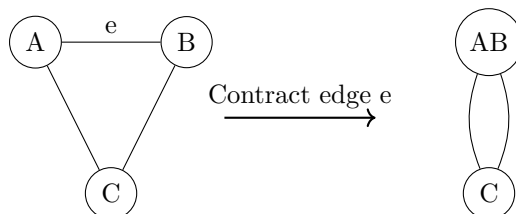
We are given an unweighted and undirected graph  $G = (V, E)$ . A “cut” of the graph is a partition of the vertices into two sets such that each set is non-empty. The “cost” of the cut (also called the cut “size”) is the number of edges that have one endpoint in each set. The goal is to find the cut with the smallest cost.

Let  $n := |V|$ . We consider two randomized algorithms:

- Karger’s Algorithm (1995) -  $\text{poly}(n)$ -time.
- Karger-Stein Algorithm (1996) -  $\tilde{O}(n^2)$  time.

### 2.3.1 Edge Contraction

The key idea in both algorithms is edge contraction. Given an edge  $(u, v)$ , we can contract it by removing the edge and merging the two vertices into a single vertex. The new vertex will have all the edges that  $u$  and  $v$  had. For example:



Each contraction reduces the number of vertices by 1, and we want to stop when we have two vertices left. This leaves us with two vertices and many edges between them.

#### Note:

We allow multi-edges.

**Question:** Should we keep self-edges?

**Answer:** No. Delete them.

The observation we make is that we win iff edges at the end are a min-cut.

### 2.3.2 Karger’s Algorithm

The algorithm is as follows:

- Randomly pick an edge  $(u, v)$ .
- Contract  $(u, v)$ .
- Repeat until 2 vertices are left.

#### Lemma 2.3.1

For any min-cut  $C$ , the probability that Karger’s algorithm finds  $C$  is  $\geq \frac{1}{\text{poly}(n)}$ .

*Proof.* Let  $e_1, e_2, \dots$  be the edges we contract. Let’s start by considering  $P(e_1 \in C)$ .

We first have  $P(e_1 \in C) = \frac{|C|}{|E|}$ , but there’s something better to notice. Let  $d$  be the minimum degree of any vertex in  $G$ . Then  $|C| \leq d$  and  $|E| \geq \frac{dn}{2}$ . So  $P(e_1 \in C) \leq \frac{2}{n}$  and  $P(e_1 \notin C) \geq 1 - \frac{2}{n}$ .

Now consider  $P(e_2 \notin C \mid e_1 \notin C)$ . By the same analysis, we have  $P(e_2 \notin C \mid e_1 \notin C) \geq 1 - \frac{2}{n-1}$ . This is because after we contract  $e_1$ , we have  $n-1$  vertices left. So,  $C$  is still a min-cut with respect to the remaining vertices. Set  $d$  as the new minimum degree, and we have  $|C| \leq d$  and  $|\text{remaining edges}| \geq \frac{d(n-1)}{2}$ . Then:

$$P(e_2 \in C \mid e_1 \notin C) = \frac{|C|}{|\text{remaining edges}|} \leq \frac{d}{\frac{d(n-1)}{2}} = \frac{2}{n-1}.$$

Now:

$$\begin{aligned} P(e_3 \notin C \mid e_1, e_2 \notin C) &\geq 1 - \frac{2}{n-2} \\ &\vdots \\ P(e_k \notin C \mid e_1, \dots, e_{k-1} \notin C) &\geq 1 - \frac{2}{n-k+1}. \end{aligned}$$

So,

$$\begin{aligned} P(\text{we find } C) &= P(e_1, \dots, e_{n-2} \notin C) \geq \left(\frac{n-2}{n}\right) \left(\frac{n-3}{n-1}\right) \dots \left(\frac{1}{3}\right) \\ &\approx \frac{2}{n(n-1)}. \end{aligned}$$

⊕

**Conclusion:** The algorithm succeeds with probability  $\geq \frac{2}{n(n-1)}$ .

**New Goal:** Find min-cut with  $\geq 1 - \frac{1}{n^2}$  probability.

**Idea:** Repeat  $t$  times and return the best cut we find. (We will determine the value for  $t$  below.)

The probability that all  $t$  trials fail is  $\leq \left(1 - \frac{2}{n(n-1)}\right)^t$ . We want this to be  $\leq \frac{1}{n^2}$ . We utilize the inequality below:

$$\left(1 - \frac{1}{k}\right)^k \leq \frac{1}{e} \leq \left(1 - \frac{1}{k}\right)^{k-1}.$$

So,

$$\begin{aligned} \left(1 - \frac{2}{n(n-1)}\right)^t &\leq \frac{1}{e^{\frac{2t}{n(n-1)}}} \implies t = 2 \binom{n}{2} \log n \\ &\leq \frac{1}{e^{2 \log n}} = \frac{1}{n^2}, \end{aligned}$$

as desired.

### 2.3.3 Karger-Stein Algorithm

We start with the following motivation. We have:

$$P(\text{first } n/2 \text{ contractions succeed}) \geq \frac{n-2}{n} \frac{n-3}{n-1} \dots \frac{n/2-1}{n/2+1} \approx \frac{1}{4}.$$

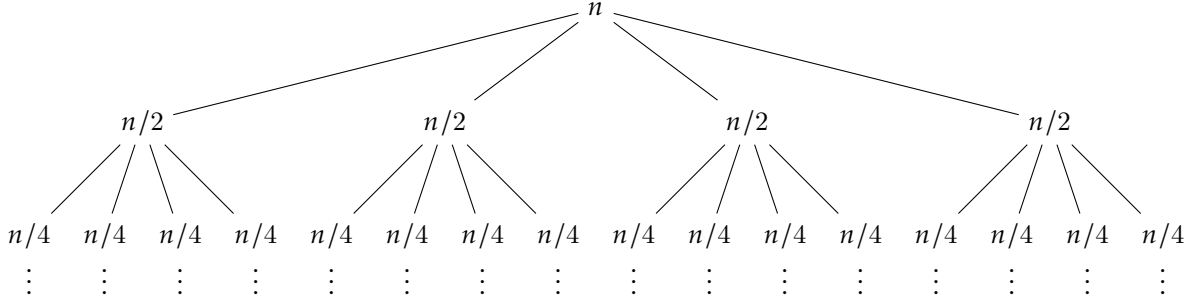
This is because after we telescope, the bottom two terms are both near  $n$  while the top two terms are near  $n/2$ . So now, if the first  $n/2$  contractions succeed, what's the probability that the next  $n/4$  succeed?

$$P(\text{next } n/4 \text{ contractions succeed}) \geq \frac{n/2-2}{n/2} \frac{n/2-3}{n/2-1} \dots \frac{n/4-1}{n/4+1} \approx \frac{1}{4}.$$

And this pattern continues.

We now go over the Karger-Stein algorithm:

- Represent the problem as a tree.
- For the first  $n/2$  contractions, run Karger's algorithm 4 times.
- For the next  $n/4$  contractions, run Karger's algorithm 16 times.
- So on, until the base case of the algorithm which is when there are two vertices left.



**Time Analysis:** The first level takes  $\tilde{O}(n^2)$  time. The second level takes  $\tilde{O}\left(\frac{n^2}{2}\right)$  time. The third level takes  $\tilde{O}\left(\frac{n^2}{4}\right)$  time. So the total time is  $\tilde{O}(n^2)$ .

So, at level  $i$ :

- There are  $4^i$  edges.
- The work per edge  $\leq (|\text{remaining vertices}|)^2 = \tilde{O}\left(\frac{n}{2^i}\right)^2 = \tilde{O}\left(\frac{n^2}{4^i}\right)$ .

This implies that the cost of the  $i$ th level is less than  $\tilde{O}(n^2)$ . Summing, we get a total cost  $\tilde{O}(n^2)$ .

Now we consider the probability that our algorithm succeeds. Again, we have a 4-ary tree with a depth of  $t = O(\log n)$ . Each edge flips a coin where  $P(\text{heads}) = \frac{1}{4}$ . So what we want is a  $t$  length chain of all heads. We can show the following result:

$$P(\text{good path}) \geq \frac{1}{t} \geq \Omega\left(\frac{1}{\log n}\right).$$

The goal is an algorithm that succeeds with probability greater than  $1 - \frac{1}{n^2}$ .

**Exercise:**  $O(\log^2 n)$  repetitions suffice.

## 2.4 Galton Walton Tree

Building on the previous problem, we introduce the concept of a Galton Walton tree. This is a 4-ary tree with  $t$  levels where each edge flips a biased coin. The probability of heads is  $1/4$ . A root-to-leaf path in this tree is “successful” if all edges are heads.

**Claim:**  $P(\text{successful path}) \geq \frac{1}{t}$ .

*Proof.* Pretend the paths are independent. Then we have that

$$P(\text{No successful path}) = \left(1 - \frac{1}{4^t}\right)^{4^t} \leq \frac{1}{e}.$$

This is not a very interesting result. It is worth noting that the paths are extremely dependent on each other. Consider a path of length  $t - 2$  and creating two separate paths by diverging at the last edge. This should illustrate the dependence of the paths.

Now let  $f_t$  the probability of tailing in a tree with  $t$  levels. We can define this as a recurrence on  $f_{t-1}$ . We have:

$$f_t = \left( \frac{1}{4}f_{t-1} + \frac{3}{4} \right)^4.$$

To proceed with induction on  $t$ , we first prove the following identity:

$$1 - 1/k < \frac{1}{e^{1/k}} < 1 - 1/(k+1).$$

This is a useful identity to have in our back pocket. We have:

$$\begin{aligned} (1 - 1/k)^k &< 1/e < (1 - 1/k)^{k-1} \\ \implies 1 - 1/k &< 1/e^{1/k} \\ 1/e^{1/(k-1)} &< 1 - 1/k \implies 1/e^{1/k} < 1 - 1/(k+1). \end{aligned}$$

Putting it all together, we get the desired result. We can now proceed with induction.

Our hypothesis:  $f_{t-1} \leq 1 - \frac{1}{t-1}$ . What we want to show is that  $f_t \leq 1 - \frac{1}{t}$ .

We have:

$$\begin{aligned} f_t &\leq \left( \frac{3}{4} + \frac{1}{4} \left( 1 - \frac{1}{t-1} \right) \right)^4 \\ &= \left( 1 - \frac{1}{4(t-1)} \right)^4 \\ &\leq \left( \frac{1}{e^{\frac{1}{4(t-1)}}} \right)^4 \\ &= \frac{1}{e^{1/(t-1)}} && \text{(by the above identity)} \\ &< 1 - \frac{1}{t}. && \text{(by the above identity)} \end{aligned}$$

This completes the induction, as well as the entire proof. ☺

# Chapter 3

## Lovasz Local Lemma

### Lemma 3.0.1 Lovasz Local Lemma

Let  $A_1, A_2, \dots, A_n$  be “bad events.” Suppose:

- $P(A_i) < p$  for some  $p$ .
- Each  $A_i$  has a “dependency set”  $D_i \subseteq \{A_1, A_2, \dots, A_n\}$  such that  $A_i$  is independent of  $\{A_1, A_2, \dots, A_n\} \setminus D_i$  and  $|D_i| \leq d$ .

If  $p < \frac{1}{ed}$ , then there exists an outcome where none of the  $A_i$  occur. That is,  $P(\text{no bad events}) \geq \frac{1}{d^n}$ .

**Application: KSAT.** In KSAT, we have a bunch of boolean variables  $x_1, x_2, \dots$ . A K-SAT formula is a conjunction of  $n$  clauses, where each clause selects  $k$  unique variables that cannot take a specific form. For example, if we have  $k = 3$ , we might have a clause that says  $(x_1, x_2, x_3) \neq (0, 1, 0)$ . The question is whether we can find a satisfying assignment of these variables.

### Theorem 3.0.1

Suppose each clause overlaps in variables  $\leq d - 1$  other clauses. Also suppose  $\frac{1}{2^k} < \frac{1}{ed}$ . Then, there exists a satisfying assignment.

*Proof.*  $x_1, x_2, \dots$  are random 0, 1 variables. Let  $A_i$  be the event that clause  $i$  is not satisfied. Then  $P(A_i) \leq \frac{1}{2^k}$ . By the LLL, there exists a satisfying assignment. ☺

## 3.1 Algorithmic Lovasz Local Lemma

### 3.1.1 Algorithmic Setting

We have random variables  $X_1, X_2, \dots, X_n$  and events  $A_i$  that are determined by some subset of the  $X_i$ s. We can think about this as a dependency graph where all the  $X_i$  and  $A_i$  are nodes, and edges are drawn between  $X_i$  and  $A_j$  if  $X_i$  determines  $A_j$ .

### Lemma 3.1.1 Algorithmic Lovasz Local Lemma

Suppose each  $A_i$  overlaps with at most  $d - 1$  other  $A_j$ s on which  $X_i$ s they use. Suppose  $P(A_i) < \frac{1}{4d}$ . Then there exists a polynomial time algorithm\* that finds an outcome of  $X_i$ s such that none of the  $A_i$ s occur.

### 3.1.2 Moser's Fix-it Algorithm

**Algorithm:**

- Sample  $X_1, X_2, \dots, X_m$ .
- While  $\exists$  bad event  $A_i$ :
  - Pick  $A_i$  arbitrarily.
  - Resample the  $X_i$ 's that determine  $A_i$ .

#### Theorem 3.1.1 Moser's Theorem (KSAT)

Consider KSAT:

- $m$  vars,  $X_1, \dots, X_m$ .
- $n$  clauses  $C_1, \dots, C_n$ .
- Each  $C_i$  depends on  $\leq d$  other  $C_i$ 's.

If  $8d < 2^k$ , then Moser's algorithm performs  $t = O(n)$  (really  $O(\frac{n}{d})$ ) fix-its in expectation.

**Proof Idea:** Use random bits, for example  $R = 0010010100101\dots$ . Construct a “transcript”  $T$  such that if you tell me  $T$ , I can recover all the random bits you used. The interesting property of  $T$  is that:

$$|T| \leq (\text{number of fix-its}) \cdot (2 \lg d) + O(n).$$

The number of random bits we use is less than the number of fix-its times  $k$ . We know that  $k > (2 + \lg d) \cdot 100$ .

If Moser's algorithm runs for a long time, say  $> 100n$  fix-its, then

$$|T| \leq 100n \cdot (2 \lg d) + O(n).$$

Then,

$$\begin{aligned} \text{number of random bits} &= 100n \cdot k \\ &> 100n \cdot (2 + \lg d) \cdot 100. \end{aligned}$$

This means  $|T| \ll$  number of random bits. So if we can show that such a  $T$  exists, then we can show that the expected number of fix-its to be pretty small.

### 3.1.3 Proof of Moser's Theorem

*Proof.* We start by defining variables we are going to use:

- $R$  = random bits used by the algorithm.
- $M_1$  = final values of  $X_i$ 's.
- $M_2$  = sequence of which clauses we fix.

#### Lemma 3.1.2

$M_1$  and  $M_2$  fully determine  $R$ .

*Proof.* Work backwards! Consider the last  $C_i$  that was fixed. We know its variables after the fix and the variables before the fix. So we can “undo” the fix. We can keep doing this until we get to the beginning. ☺

**Lemma 3.1.3**

$M_2$  can be compressed to  $n + t(2 + \lg d)$  bits.

We'll proceed by assuming the lemma first. So now we have  $|M_1| = m$  and  $|R| = m + tk$ . Also since  $8d < 2^k$  we have  $3 + \lg d \leq k$ . So,  $|R| \geq m + t(3 + \lg d)$ .

Now,

$$\begin{aligned} |M_1| + |M_2| &= n + m + t(2 + \lg d) \\ &\leq |R| + n - t. \end{aligned}$$

If  $E[t] > n$ , then  $E[|M_1 \circ M_2|] < E[|R|]$ . This is a contradiction by the incompressibility of random bits. So,  $E[t] \leq n$  as desired.  $\odot$

Now we return to proving the second lemma.

*Proof.* We start by defining processes.

Define  $\text{process}(C_i)$ :

- Resample  $C_i$ 's variables.
- While  $\exists$  another  $C_j$  that is broken and depends on  $C_i$ :
  - Select such a  $C_j$ .
  - $\text{process}(C_j)$ .

Define Algorithm:

- For  $i = 1, \dots, n$ :
  - If  $C_i$  is broken,  $\text{process}(C_i)$ .

**Claim:** When the algorithm finishes, all clauses holds.

**Claim:** Algorithm terminates with probability 1.

We now look towards constructing  $M_2$ :

- $A = \text{Part 1: } n \text{ bits, which } C_i\text{'s get fixed in for loop.}$

Now recall the process algorithm. Notice here that for each  $C_j$ , there are only  $d + 1$  options. This means that we can communicate  $C_j$  with  $\lg(d + 1)$  bits, which we'll call  $q$ . We now rewrite process considering a string  $B$ .

Define  $\text{process}(C_i)$ :

- Resample  $C_i$ 's variables.
- While  $\exists$  another  $C_j$  that is broken and depends on  $C_i$ :
  - Select such a  $C_j$ .
  - $B = B + \text{"1"} + q$ .
  - $\text{process}(C_j)$ .
- $B = B + \text{"0"}.$

So given  $M_2$ ,  $A$  lets us recover which clauses the for loop fixes.  $B$  lets us figure out which clauses are fixed within each process subroutine. This means we can recover the entire sequence of clause fixes that the algorithm performs.

So,  $|A| = n$  and  $|B| = t(2 + \lg d)$ . This means  $|A \circ B| = n + t(2 + \lg d)$ , which completes the proof.  $\odot$

## Chapter 4

# Chernoff Bounds

We start with an example. Suppose I flip 100 fair coins. What is the probability that the number of heads is greater than or equal to 75? Turns out,

$$P(\text{heads} \geq 75) \approx \frac{1}{3.5 \times 10^6}.$$

### Theorem 4.0.1

Consider  $n$  fair coin flips,  $X_1, X_2, \dots, X_n$  iid with  $X_i = \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$ . Let  $X = \sum X_i$ . Then,

$$P\left(X > \frac{n}{2} + K\sqrt{n}\right) \leq e^{-\Omega(K^2)}.$$

### Corollary 4.0.1

$$P\left(X \geq \frac{3}{4}n\right) \leq e^{-n/32}$$

You don't actually need  $X_1, \dots, X_n$  to be independent. It suffices to have that, for any  $i$  and any outcomes of  $X_1, \dots, X_{i-1}$ ,

$$P(X_i = 1 \mid X_1, \dots, X_{i-1}) \leq 1/2.$$

#### Note:

“Azuma’s Inequality” is a citable Chernoff bound. Alternatives include “multiplicative version of Azuma’s inequality” and “Freedman’s inequality.”

We turn to the proof of the theorem.

*Proof.* Idea: suppose we have a function  $f$  such that:

- $f$  is non-negative.
- $f$  is increasing on  $[n/2, n]$ .

Then,

$$P\left(x \geq \frac{n}{2} + K\sqrt{n}\right) \leq P\left(f(x) \geq f\left(\frac{n}{2} + K\sqrt{n}\right)\right) \leq \frac{E[f(x)]}{f\left(\frac{n}{2} + K\sqrt{n}\right)}.$$



**Example 4.0.1** (Trying  $f$ 's)

$f(x) = \left(x - \frac{n}{2}\right)^2$ . This clearly satisfies the requirements. This tells us that:

$$\begin{aligned} P\left(x > \frac{n}{2} + K\sqrt{n}\right) &\leq \frac{E[f(x)]}{K^2 n} \\ &= \frac{\text{Var}(X)}{K^2 n}. \end{aligned}$$

We use the linearity of variance. We have  $\text{Var}(X_i) = \frac{1}{4}$ . So,  $\text{Var}(X) = \frac{n}{4}$ . Thus,

$$P\left(x > \frac{n}{2} + K\sqrt{n}\right) \leq \frac{1}{4K^2}.$$

This is exactly Chebyshev's inequality.

We now find an  $f$  that proves our result. Try  $f(x) = e^{\lambda x}$  for some  $\lambda \in (0, 1)$ . This is approximately  $(1 + \lambda)^x$ . We first find the expectation:

$$\begin{aligned} E[f(x)] &= E[e^{\lambda \sum_i X_i}] = E\left[\prod_i e^{\lambda X_i}\right] \\ &= \prod_i E[e^{\lambda X_i}] && \text{(by independence)} \\ &= \left(\frac{1 + e^\lambda}{2}\right)^n. \end{aligned}$$

**Fact:**

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \quad (4.1)$$

$$\leq 1 + \lambda + \lambda^2 \cdot \left(\frac{1}{2!} + \frac{1}{3!} + \cdots\right) \quad (4.2)$$

$$\leq 1 + \lambda + \lambda^2. \quad (4.3)$$

It follows that

$$E[e^{\lambda X_i}] \leq 1 + \frac{1}{2}\lambda + \frac{1}{2}\lambda^2.$$

Now we make use of the other fact that  $1 + x \leq e^x$ . So

$$\begin{aligned} E[e^{\lambda X_i}] &\leq 1 + \frac{1}{2}\lambda + \frac{1}{2}\lambda^2 \\ &\leq e^{\lambda/2 + \lambda^2/2}. \end{aligned}$$

Thus,

$$E[f(x)] \leq e^{n(\lambda/2 + \lambda^2/2)}.$$

So now,

$$\begin{aligned} P\left(X > \frac{n}{2} + K\sqrt{n}\right) &\leq \frac{E[f(x)]}{f\left(\frac{n}{2} + K\sqrt{n}\right)} \\ &\leq \frac{e^{n(\lambda/2 + \lambda^2/2)}}{e^{n\lambda/2 + \lambda K\sqrt{n}}} \\ &= e^{n\lambda^2/2 - \lambda K\sqrt{n}}. \end{aligned}$$

Now we try to find an appropriate value for  $\lambda$ . We try:

$$\begin{aligned}\lambda K \sqrt{n} &= n \lambda^2 \\ \frac{K}{\sqrt{n}} &= \lambda.\end{aligned}$$

This yields:

$$e^{n\lambda^2/2 - \lambda K \sqrt{n}} = e^{-\lambda K \sqrt{n}/2} = e^{-k^2/2},$$

as desired. ☺

## 4.1 Quicksort Algorithm (1961)

We review the algorithm for quicksort:

- Pick random pivot  $P$ .
- Partition elements into 2 pieces: elements smaller/larger than  $P$ .
- Recurse on the pieces.

### Theorem 4.1.1

The running time of quicksort satisfies  $T = O(n \lg n)$  with probability at least  $1 - \frac{1}{n^2}$ .

*Proof.* Note that the time is  $\leq \# \text{levels} \cdot O(n)$ . So what we have to prove is that with high probability, the depth of the algorithm doesn't exceed  $\lg n$ . So,

$$\begin{aligned}P(\text{max depth} \geq 1000 \lg n) &\leq P(\exists \text{element with max depth} \geq 1000 \lg n) \\ &\leq n \cdot P(\text{a given } v \text{ has depth} \geq 1000 \lg n).\end{aligned}$$

We want to show that the probability on the last line is  $\leq \frac{1}{n^3}$ . We have that

$$P(\text{next subproblem} \leq 3/4 \text{current size}) \geq \frac{1}{2}.$$

Let  $P_1, P_2, \dots$  be subproblems in  $v$ 's path. Let

$$X_i = \begin{cases} 1 & P_{i+1} \text{ exists and } |P_{i+1}| > \frac{3}{4}|P_i| \\ 0 & \text{otherwise} \end{cases}$$

We observe that  $P(X_i = 1 \mid X_1, \dots, X_{i-1}) \leq 1/2$ . So by Chernoff bound, if we look at  $X = X_1 + X_2 + \dots + X_{1000 \lg n}$ , we get:

$$P\left(X \geq \frac{3}{4}m\right) \leq e^{-m/32} \ll \frac{1}{n^3}.$$

If  $X \leq \frac{3}{4}m$ , either  $P_m$  doesn't exist or  $|P_m| \leq n \cdot \left(\frac{3}{4}\right)^{m/4} = n \cdot \left(\frac{3}{4}\right)^{250 \lg n} \ll 1$ , which is a contradiction. This means  $P_m$  in fact does not exist. As such, we are able to conclude that we have a running time of  $O(n \lg n)$  with the desired probability. ☺

## 4.2 Revisiting Theorem 4.0.1

We start with warm-ups to give an alternative proof to Theorem 4.0.1.

### 4.2.1 Poor Man's Chernoff Bound

#### Theorem 4.2.1

$$P(X \geq 2K\sqrt{n}) \leq \frac{1}{2^K}.$$

*Proof.* We make use of the core facts below:

**Core fact 1:**  $P(X \geq 2\sqrt{n}) \leq \frac{1}{4}$ , by Chebyshev's inequality.

**Core fact 1':**  $P(\text{any prefix has sum} \geq 2\sqrt{n}) \leq \frac{1}{2}$ . This is a direct result of **Core fact 1**, and the implication will be shown in the homework.

We can graph the running sum as a function of time and track the times that we hit the values  $2\sqrt{n}$ ,  $4\sqrt{n}$ , and  $6\sqrt{n}$  and call them  $t_1, t_2, t_3$ , et cetera. What the core fact is saying is that  $P(t_1 \text{ exists}) \leq \frac{1}{2}$ . Then, we can build the the following chain:

$$\begin{aligned} P(t_1 \text{ exists}) &\leq \frac{1}{2} \\ P(t_2 \text{ exists} \mid t_1 \text{ exists}) &\leq \frac{1}{2} \\ &\vdots \\ P(t_i \text{ exists} \mid t_{i-1} \text{ exists}) &\leq \frac{1}{2}. \end{aligned}$$

So,

$$P(X \geq 2K\sqrt{n}) \leq P(t_K \text{ exists}) \leq \frac{1}{2^K},$$

as desired. ☺

### 4.2.2 Chernoff Bound for Geometric Random Variables

#### Theorem 4.2.2

Let  $X_1, \dots, X_n$  be non-negative independent random variables such that for all  $j \in \mathbb{N}$ ,

$$P(X_i \geq j) \leq p^j.$$

Then  $X = \sum X_i$  satisfies

$$P(X \geq 2n) \leq (4p)^n.$$

*Proof.*  $X \geq 2n \implies \sum \lfloor X_i \rfloor \geq n \implies \exists \langle Y_1, \dots, Y_n \rangle =: \vec{Y}$  such that

- $\sum Y_i = n$ .
- $X_i \geq Y_i \forall i$ .

For a given option  $\langle Y_1, \dots, Y_n \rangle$  for  $\vec{Y}$ ,

$$\begin{aligned} P(\vec{Y} \text{ occurs}) &= P(X_i \geq Y_i \forall i) \\ &\leq \prod_i P(X_i \geq Y_i) \\ &\leq \prod_i p^{Y_i} \\ &= p^{\sum_i Y_i} \\ &= p^n. \end{aligned}$$

Now we bound the number of options for  $\vec{Y}$ . We can express  $\vec{Y}$  as a binary string, for example:

$$\underbrace{000}_{Y_1=3} \underbrace{1}_{Y_2=1} \underbrace{0}_{Y_3=0} \underbrace{1}_{Y_4=1} \dots$$

So the number of 0's is  $\sum Y_i = n$  and the number of 1's is  $n$ . This means that the number of options satisfying  $\sum Y_i = n$  is less than the number of binary strings of length  $2n$  which is  $4^n$ .

So,

$$\begin{aligned} P(X \geq 2n) &\leq P(\text{some } \vec{Y} \text{ occurs}) \\ &\leq (\# \text{ of options for } \vec{Y}) \cdot P(\vec{Y} \text{ occurs}) \\ &\leq 4^n \cdot p^n \\ &= (4p)^n. \end{aligned}$$

⊕

### 4.2.3 Anti-(Chernoff Bound)

#### Theorem 4.2.3

$$P\left(X \geq \frac{K}{4}\sqrt{n}\right) \geq \frac{1}{4^{K^2}}$$

*Proof.* Break the coins into  $K^2$  groups. So each group has  $m = \frac{n}{K^2}$  coins.

**Core fact:**  $P\left(X \geq \frac{1}{4}\sqrt{n}\right) \geq \frac{1}{4}$ .

Let  $C_i$  be the sum of the  $i$ -th group. So by the core fact,

$$P\left(C_i \geq \frac{1}{4}\sqrt{m}\right) \geq \frac{1}{4}.$$

So,

$$P\left(\text{every } C_i \geq \frac{1}{4}\sqrt{m}\right) \geq \frac{1}{4^{K^2}}$$

If this occurs, then

$$X = \sum C_i \geq K^2 \cdot \left(\frac{1}{4}\sqrt{m}\right) = \frac{K}{4}\sqrt{n}.$$

⊕

#### 4.2.4 Real Chernoff Bound

##### Theorem 4.2.4

$$P(X \geq 16k\sqrt{n}) \leq \frac{1}{2^{k^2}}.$$

*Proof.* We start by defining  $Y_i = \max\left(0, \frac{C_i}{8\sqrt{m}}\right)$ . We can apply the Poor Man's Chernoff bound to each  $C_i$  to get

$$\begin{aligned} P(C_i \geq 2\ell\sqrt{m}) &\leq \frac{1}{2^\ell} \\ \Rightarrow P\left(Y_i \geq \frac{2\ell\sqrt{m}}{8\sqrt{m}}\right) &\leq \frac{1}{2^\ell} \\ \Rightarrow P(Y_i \geq \ell) &\leq \frac{1}{2^{4\ell}} \leq \frac{1}{16^\ell}. \end{aligned}$$

This is saying that  $Y_i$  are geometric random variables. So by the geometric random variable bound,

$$P\left(\sum Y_i \geq 2K^2\right) \leq (4 \cdot 1/16)^{K^2} = \left(\frac{1}{4}\right)^{K^2}.$$

But also

$$\begin{aligned} X \geq 16K\sqrt{n} &\Rightarrow \sum C_i \geq 16K\sqrt{n} \\ &\Rightarrow \sum Y_i \geq \frac{16K\sqrt{n}}{8\sqrt{m}} = 2K^2. \end{aligned}$$

☺

##### Theorem 4.2.5

Let  $X_1, X_2, \dots, X_n$  be independent  $\{0, 1\}$  coin flips with  $P(X_i = 1) = p$  for some  $p \leq \frac{1}{2}$ . Define  $X = \sum X_i$ ,  $\mu = E[X] = np$ .

Small-deviation regime: For  $K \in \{1, \dots, \sqrt{\mu}\}$ ,

$$P(X \geq \mu + K\sqrt{\mu}) \leq e^{-\Theta(K^2)}.$$

Large-deviation regime: For  $J \geq 2$  such that  $J\mu \leq n$ ,

$$P(X \geq J\mu) \leq \frac{1}{J^{\Theta(J\mu)}}.$$

*Proof.* (large-deviation case)

##### Warmup 1

**Theorem 1** (Poor Man's Chernoff Bound) If  $\mu \leq 1$ , then

$$P(X \geq K) \leq \mu^K.$$

*Proof.* **Core fact**  $P(X \geq 1) \leq \mu$  by Markov's inequality.

Now repeat the ideas with  $t_1, t_2, \dots$  to say that  $P(X \text{ reaches } i) = \mu^i$ .

☺

**Warmup 2** This was already done prior as it is the exact same.

### Warmup 3

#### Theorem 3

$$P(X \geq J) \geq \frac{1}{J^{O(J\mu)}}$$

*Proof.* We'll have  $J\mu$  groups where each group has mean  $\frac{1}{J}$ .

**Core fact:** If  $\mu \leq 1$ , then  $P(X \geq 1) \geq \Omega(\mu)$ .

By the core fact, each  $C_i$  is  $\geq 1$  with probability  $\geq \Omega\left(\frac{1}{J}\right)$ . So,

$$P(\text{every } C_i \geq 1) = P(X \geq J\mu) \geq \Omega\left(\frac{1}{J}\right)^{J\mu}.$$

“Squiggly square to signify almost proof done.” ☹

So the Poor Man's Chernoff Bound implies  $P(C_i \geq K) \leq \frac{1}{J^K}$ . Then by Chernoff for geometric random variables,

$$P\left(\sum C_i \geq 2J\mu\right) \leq \left(\frac{4}{J}\right)^{J\mu}$$

☺

## 4.3 More Chernoff Bounds

### Theorem 4.3.1 Chernoff Bound

Let  $X_1, X_2, \dots, X_n \in [0, 1]$  be independent random variables. Let  $X = \sum X_i$  and  $\mu = E[X]$ . Then, for  $k \in O(\sqrt{\mu})$ ,

$$P(|X - \mu| \geq k\sqrt{\mu}) \leq e^{-\Omega(k^2)}.$$

And for  $J \geq 2$ ,

$$P(X \geq J\mu) \leq \left(\frac{1}{J}\right)^{\Omega(J\mu)}.$$

Small-deviation regime:

- There are  $k^2$  chunks.
- So each has expected value  $\approx \frac{\mu}{k^2}$ .
- If each exceeds expected value by  $\sqrt{\frac{\mu}{k^2}}$ , then the total sum is  $\geq k^2 \cdot \sqrt{\frac{\mu}{k^2}} = k\sqrt{\mu}$ .

Large-deviation regime:

- $J\mu$  chunks.
- Each chunk has expected value  $\approx \frac{1}{J}$ .
- If each chunk is at least 1, then the total sum is at least  $J\mu$ .

**Theorem 4.3.2** Bennett's Inequality, Bernstein's Bound

Same bound, but let  $v = \text{Var}(X)$ .

$$P(X - E[X] \geq k\sqrt{v}) \leq e^{-\Theta(k)} \text{ for } k \in \{1, 2, \dots, \sqrt{v}\}$$

$$P(X - E[X] \geq Jv) \leq \left(\frac{1}{J}\right)^{\Omega(Jv)} \text{ for } J \geq 2.$$

**4.3.1 Adaptive Version****Theorem 4.3.3** Azuma's Inequality

Let  $X_1, X_2, \dots, X_n \in [-1, 1]$ . Suppose  $E[X_i | X_1, \dots, X_{i-1}] = 0$ . Then,  $X = \sum X_i$  satisfies

$$P(X \geq K\sqrt{n}) \leq e^{-\Omega(K^2)}.$$

Fancier versions:

Have random variables  $X_1, X_2, \dots, X_n \in [0, 1]$  which are revealed one after the other. After  $X_1, \dots, X_{i-1}$  are revealed, Alice gets to select the distribution  $P_i$  for  $X_i$  and then  $X_i$  is revealed from  $P_i$ .

Let  $\mu_i = E[X_i | P_i]$ ,  $v_i = \text{Var}(X_i | P_i)$ ,  $X = \sum X_i$ ,  $\mu = \sum \mu_i$  and  $v = \sum v_i$ . As a corollary to Azuma's inequality,

$$P(X \geq \mu + k\sqrt{n}) \leq e^{-\Omega(k^2)}.$$

Now if  $\mu$  is deterministically at most  $\bar{\mu}$ , then

$$P(X \geq K\sqrt{\bar{\mu}} + \bar{\mu}) \leq e^{-\Omega(K^2)}$$

$$P(X \geq J\bar{\mu}) \leq \left(\frac{1}{J}\right)^{\Omega(J\bar{\mu})}.$$

**Theorem 4.3.4** Freedman's Inequality

If  $v$  is deterministically at most  $\bar{v}$ , then

$$P(X \geq K\sqrt{\bar{v}} + \mu) \leq e^{-\Omega(K^2)} \text{ for } K \in \{1, 2, \dots, \sqrt{\bar{v}}\}$$

$$P(X \geq \mu + J\bar{v}) \leq \left(\frac{1}{J}\right)^{\Omega(J\bar{v})} \text{ for } J \geq 2.$$

**Note:**

Bill has used the following bound in  $\sim 90\%$  of his papers.

"Sometimes when you don't need this bound you want to find a way to use it anyway."

**Theorem 4.3.5** McDiarmid's Inequality

Let  $X_1, X_2, \dots, X_n$  be independent. Let  $F : X_1, \dots, X_n \rightarrow \mathbb{R}$ . Suppose: If I change some  $X_i$  to a new value  $\bar{X}_i$ ,

$$|F(X_1, \dots, X_i, \dots, X_n) - F(X_1, \dots, \bar{X}_i, \dots, X_n)| \leq 1.$$

Then,

$$P(|F - E[F]| \geq K\sqrt{n}) \leq e^{-\Omega(K^2)}.$$

**Example 4.3.1**

Consider a random graph:

- $n$  vertices.
- each  $(i, j)$  is an edge with probability  $\frac{1}{2}$ .

Consider the chromatic number  $\chi(G)$ . There's a way to show that

$$P(|\chi(G) - E[\chi(G)]| \geq k\sqrt{n}) \leq e^{-\Omega(k^2)}.$$

We cannot define the random variables  $X_i$  on whether an edge is included or not. This is because we end up getting  $n^2$  random variables, which makes the inequality flop. We proceed by defining  $X_1, X_2, \dots, X_n$  where  $X_i$  is the number of edges from vertex  $i$  to vertices  $> i$ . Then the result follows directly from McDiarmid's inequality.

*Proof Idea:* Let

$$\begin{aligned} Y_0 &= E[F] \\ Y_1 &= E[F \mid X_1] \\ Y_2 &= E[F \mid X_1, X_2] \\ &\vdots \\ Y_n &= E[F \mid X_1, \dots, X_n] = F. \end{aligned}$$

Then let

$$\begin{aligned} Z_1 &= Y_1 - Y_0 \\ Z_2 &= Y_2 - Y_1 \\ &\vdots \\ Z_n &= Y_n - Y_{n-1}. \end{aligned}$$

Claim 1:  $E[Z_i \mid Z_1, \dots, Z_{i-1}] = 0$ .

Claim 2:  $|Z_i| \leq 1$ .

So by Azuma,  $P(\sum Z_i > K\sqrt{n}) \leq e^{-\Omega(K^2)}$ . But  $\sum Z_i = Y_n - Y_0 = F - E[F]$ .



# Chapter 5

## Oblivious Routing

We start with an  $n$ -node hypercube:

- $n$  vertices:  $1, \dots, n$ .
- edges: For each pair  $i, j$  if  $i, j$  differ in exactly one bit, then we have edges  $(i, j)$  and  $(j, i)$ .

This graph contains  $n \log n$  edges. Each vertex  $i$  wants to send a message to another vertex  $\pi(i)$ , where  $\pi$  is a permutation. Each edge can only send 1 message per time step (may have to wait in queue before going down edge).

Goal: Complete all the message sending within time  $O(\log n)$ .

Oblivious Routing: Each message decides its path at the beginning of time.

### Theorem 5.0.1 Brebner, Valiant (Leslie) (1981)

With probability  $\geq 1 - \frac{1}{n^{1000}}$ , we can achieve  $O(\log n)$  total time.

*Proof.* Algorithm:

- To send  $i \rightarrow \pi(i)$ ;
  - Pick random  $r_i \in \{1, \dots, n\}$ .
  - Phase 1: Send  $i \rightarrow r_i$ .
  - Phase 2: Send  $r_i \rightarrow \pi(i)$ .

We wait until a preplanned time to begin phase 2. To send  $i \rightarrow r_i$ , use bit-fixing where we fix bits from left to right. Let  $u_i :=$  message going from  $i \rightarrow r_i$ ,  $P_i =$  path that  $u_i$  takes. Focus on a fixed  $i$ .

Claim: With high probability,  $u_i$  completes  $p_i$  within  $O(\log n)$  time.

### Lemma 5.0.1 Lemma 1

If  $i \neq j$ , then edges in  $P_i \cap P_j$  form a contiguous subpath.

*Proof.* Consider an edge in  $P_i$ . For example if we have

010111011001  
010111111001

we can say that the 011001 agrees with  $i$  and  $j$ , while the 0101111 agrees with  $r_i$  and  $r_j$ .

The values of  $k$  where paths agree,

$$\{k \mid r_i, r_j \text{ agree first } k \text{ bits}\} \cap \{k \mid i, j \text{ agree on final } n - k + 1 \text{ bits}\}.$$

Note that the first set is a prefix of  $1, \dots, n$  and the second set is a suffix of  $1, \dots, n$ . Therefore, their intersection is a contiguous interval. ☺

**Lemma 5.0.2** Lemma 2

The number of time steps that  $u_i$  spends in queues is at most  $|S_i|$ , where  $S_i = \{j \neq i \mid P_i \cap P_j \neq \emptyset\}$ .

*Proof.* Say that a message  $w$  that is currently on  $P_i$  has “delay”  $d$  if

- $w$  is on the  $t$ -th step of  $P_i$  for some  $t$ .
- We are in the  $t + d$ -th time step of the algorithm.

If  $u_i$  waits in a queue and its delay goes from  $d$  to  $d + 1$ , give a note with the number  $d$  on it to whoever used the edge that  $u_i$  wanted to use. Later on, if a message has a note “ $d$ ”, and  $w$  waits in a queue:

- if edge  $\notin P_i$ , keep the note
- otherwise, pass the note to message that’s using the edge we’re waiting on.

If note  $d$  is currently held by a message  $w$  and if  $w$  is still on  $P_i$ , then  $w$ ’s delay is exactly  $d$ . This means that no two notes will be held by the same message. This implies that the number of notes is equal to the number of messages hold notes, which is then less or equal to  $|S_i|$ .  $\odot$

Now let  $e$  be an edge. Define  $m_e :=$  the number of messages that use  $e$ .

**Lemma 5.0.3** Lemma 3

$E[m_e] \leq 1$ .

*Proof.* What does it mean to use edge  $e$ ? The first  $k$  bits are free variables for  $j$  while the last  $n - k + 1$  agree with  $j$ . So the number of options for  $j = 2^{k-1}$ . For each such  $j$ ,

$$P(r_j \text{ has correct first } k \text{ bits}) = \frac{1}{2^k}.$$

So,

$$E[\# j \text{ that use } e] \leq \sum_{\text{valid } j} P(j \text{ uses } e) \leq 2^{k-1} \frac{1}{2^k} = \frac{1}{2}.$$

$\odot$

**Lemma 5.0.4** Lemma 4

With high probability,  $|S_i| \leq O(\log n)$ .

*Proof.* First, fix  $P_i$ .

$$|S_i| = \sum_{j \neq i} \mathbb{I}[P_i \cap P_j \neq \emptyset],$$

where  $\mathbb{I}$  is the indicator function. Note that these are all independent random variables determined by  $r_j$ . This is exactly the type of thing we want to apply Chernoff bounds to.

Define  $\mu := E[|S_i|] \leq \sum_{e \in P_i} E[\# \text{ messages } j \neq i \text{ that use } e] \leq \frac{\log n}{2}$ .  $\odot$

So let’s say  $\mu = \frac{\log n}{2}$ . So,

$$P(|S_i| \geq J\mu) \leq \left(\frac{1}{J}\right)^{\Omega(J\mu)} \leq \frac{1}{2} \leq \frac{1}{\text{poly} n}.$$

$\odot$

Suppose I flip a fair coin repeatedly:

Question 1: How many flips do I need to get  $\geq 1$  heads with high probability.

Answer:  $O(\log n) \implies P(\text{all tails}) \leq \frac{1}{2^{\Theta(\log n)}}$ .

Question 2: How many flips do I need to get  $\geq \log n$  heads with high probability?

Answer:  $O(\log n)$ . If I flip  $1000 \log n$  coins,  $P(< \log n \text{ heads}) = P(> 999 \log n \text{ tails})$ . This means I exceeded the mean greatly as  $E[\# \text{ of tails}] = 500 \log n$ . If  $\mu = E[T]$ , then

$$\begin{aligned} T &> 1.9\mu \\ &> \mu + 0.9\sqrt{\mu}(\sqrt{\mu}). \end{aligned}$$

And the probability of this is  $\leq e^{-\Omega(\log n)}$ .

# Chapter 6

## Metric Embeddings

### Definition 6.0.1: Metric Space

A metric space  $X$  is a set equipped with a function  $d : X \times X \rightarrow \mathbb{R}$  such that

1.  $d(x, y) \geq 0$  for all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$
3.  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$

### Example 6.0.1

- $\ell^p$ -distance. For  $p \geq 1$ ,  $X = \mathbb{R}^k$ . Then

$$d_{\ell^p}(x, y) = \left( \sum_{i=1}^k |x_i - y_i|^p \right)^{1/p}.$$

- Graph-distance. Take any undirected graph with positive weights, then you can talk about the graph distance between two vertices where

$$d(v, u) = \text{graph distance}.$$

### Definition 6.0.2: Metric Embeddings

Given two metric spaces,  $(X_1, d_1), (X_2, d_2)$ , a map  $\phi : X_1 \rightarrow X_2$  is a metric embedding with distortion at most  $\alpha$  if  $\exists \beta$  scaling parameter such that  $\forall x, y \in X_1$ ,  $d_1(x, y)$  is within a factor of  $\alpha$  of  $\beta d_2(\phi(x), \phi(y))$ . That is,

$$\beta d_2(\phi(x), \phi(y)) \leq d_1(x, y) \leq \alpha \beta d_2(\phi(x), \phi(y)).$$

**Theorem 6.0.1** Bourgain's Theorem

Let  $(X, d)$  be an  $n$ -point metric. Then there exists a function  $\phi : (X; d) \rightarrow (\mathbb{R}^{O(\log^2 n)}, d_{\ell^1})$  such that the distortion is  $O(\log n)$ .

*Proof.* We review the algorithm. Let  $c$  be a large constant.

- For  $i = 1, \dots, \log n$ ,  
     For  $j = 1, \dots, c \log n$ ,  
      $S_{i,j}$  = sample each element  $x \in X$  with probability  $2^{-i}$ .

⊕

**Lemma 6.0.1** Key Lemma

For  $i \in \{1, \dots, t\}$  and  $j \in \{1, \dots, c \log n\}$ , we have with probability  $\Omega(1)$  that

$$|d(x, S_{i,j}) - d(y, S_{i,j})| \geq r_{i+1} - r_i.$$

I LEFT LECTURE HERE, FILLIN LATER

## 6.1 Day 2

### Theorem 6.1.1

Let  $(X, d)$  be an  $n$ -point metric space. Suppose that  $\forall x \neq y \in X, d(x, y) \in [1, n^2]$ . We will embed  $X$  into a tree metric  $(T, d_T)$  with the following properties:

- depth of tree is  $O(\log n)$ .
- For any root-leaf path, edge weights are decreasing powers of 2.
- Points in  $X$  get mapped to leaves of the tree.

There exists a randomized embedding  $\phi : X \rightarrow (T, d_T)$ , where  $T$  is also a random variable, such that

- $d_T(\phi(x), \phi(y)) \geq d(x, y)$ .
- $E[d_T(\phi(x), \phi(y))] \leq O(\log n)d(x, y)$ .

*Proof.* The algorithm:

1. Pick a random permutation  $\pi = \pi_1, \dots, \pi_n$  of the elements of  $X$ .
2. Pick uniformly random  $\beta \in [1, 2]$ . Define  $\beta_i := 2^{i-1} \cdot \beta$ .

For each  $x \in X$  and  $i \in \{0, \dots, 2 \log n\}$ , define

$$\text{center}_i(x) = \text{first } \pi_j \text{ in permutation to satisfy } d(x, \pi_j) \leq \beta_i.$$

3. Construct  $\phi$  as follows:

Each node represents a set of elements. Root at level  $2 \log n + 1 = \text{all of } X$ . For any node  $w$  at level  $i + 1$ , if node  $> 1$  element.

- Group  $w$ 's elements  $z$  by  $\text{center}_i(z) \rightarrow \text{child nodes}$ .

Two elements  $a, b \in w$  are in the same child iff  $\text{center}_i(a) = \text{center}_i(b)$ .

4. Finally, edge from level  $i + 1$  to level  $i$  has weight  $2^i$ .

Analysis:

We first claim that for  $x, y \in X$ ,  $d_T(\phi(x), \phi(y)) \geq d(x, y)$ .

*Proof.* Observe that at level  $i$ , if  $x, y$  in the same node, then  $d(x, y) \leq 2\beta_i \leq 2^{i+1}$ . So let  $i$  be the highest level the path goes through, this implies that  $d(x, y) \leq 2^{i+1}$ . But also  $d_T(\phi(x), \phi(y)) \geq 2 \cdot 2^i \geq d(x, y)$  as desired. ☺

### Proposition 6.1.1

$$E[d_T(\phi(x), \phi(y))] \leq O(\log n)d(x, y).$$

*Proof.*

$$\begin{aligned} d_T(\phi(x), \phi(y)) &\leq O(\text{largest edge weight along the path}) \\ &\leq \sum_{i=0}^{2 \log n} \mathbb{I}(\text{center}_i(x) \neq \text{center}_i(y)) \cdot 2^{i+1}. \end{aligned}$$

So we have

$$\begin{aligned}
E[d_T(\phi(x), \phi(y))] &\leq \sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} P(B_i = r) \cdot P(\text{center}_i(x) \neq \text{center}_i(y) \mid \beta_i = r) 2^{i+1} dr \\
&= \sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} \frac{1}{2^{i-1}} \cdot P(\text{center}_i(x) \neq \text{center}_i(y) \mid \beta_i = r) 2^{i+1} dr \\
&= \sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} P(\text{center}_i(x) \neq \text{center}_i(y) \mid \beta_i = r) dr.
\end{aligned}$$

If we look through  $\pi_1, \pi_2, \dots, \pi_n$ , the first of  $\{\text{center}_i(x), \text{center}_i(y)\}$  to appear is  $w$  implies that  $w$  “cuts”  $(x, y)$  (and the centers are non-equal).

Let  $w_1, w_2, \dots, w_n$  be elements of  $x$  sorted by distance from  $w_i$  to  $\{x, y\}$ .

$$\sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} P(\text{center}_i(x) \neq \text{center}_i(y) \mid \beta_i = r) dr \leq \sum_{j=1}^n \sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} P(w \text{ cuts } (x, y) \mid \beta_i = r) dr.$$

Call the integrand  $\Xi$ . We consider what must occur for  $w_j$  to cut  $(x, y)$ . Suppose WLOG  $d(w_j, x) \leq d(w_j, y)$ . We need

- $d(w_j, x) \leq \beta_i \leq d(w_j, y)$ .
- $w_j$  must appear before  $w_1, \dots, w_{j-1}$  in  $\pi$ , which has probability  $\leq \frac{1}{j}$ .

So we get

$$\begin{aligned}
\sum_{j=1}^n \sum_{i=0}^{2 \log n} \int_{r=2^{i-1}}^{2^i} P(w \text{ cuts } (x, y) \mid \beta_i = r) dr &\leq \sum_{j=1}^n \int_{z=d(w_j, x)}^{d(w_j, y)} \frac{1}{j} dz \\
&= \sum_{j=1}^n \frac{1}{j} |d(w_j, y) - d(w_j, x)| \\
&\leq d(x, y) \sum_{j=1}^n \frac{1}{j} = d(x, y) O(\log n).
\end{aligned}$$

☺

☺

## 6.2 Day 3

### Theorem 6.2.1 Johnson-Lindenstrauss Lemma (1994)

Let  $\epsilon \in (0, 1)$ . Let  $S \subseteq \mathbb{R}^d$  be a set of  $n$  points. There exists a map  $\phi : S \rightarrow \mathbb{R}^{O(\log n) \cdot \epsilon^{-2}}$  which satisfies:

$$\forall a, b \in S, (1 \pm \epsilon) \|a - b\|_2 = \|\phi(a) - \phi(b)\|_2.$$

Our goal is to construct a linear  $\phi$  such that for any  $a \in \mathbb{R}^d$ , with high probability in  $n$ ,

$$\|\phi(a)\|_2 = (1 \pm \epsilon) \|a\|_2.$$

If we can do this, then  $\forall a, b \in S$ , we have with high probability that

$$\begin{aligned} \|\phi(a) - \phi(b)\|_2 &= \|\phi(a - b)\|_2 \\ &= (1 \pm \epsilon) \|a - b\|_2. \end{aligned}$$

Then by applying the union bound over all  $\binom{n}{2}$  pairs, can complete the proof.

We start with a warmup task. We want to construct a linear function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $E[f(a)^2] = \|a\|_2^2$ . If we can find a function like this that also suffices:

$$P(f(a)^2 > K \|a\|_2^2) \leq e^{-\Omega(K)},$$

then we can finish the theorem off.

**Review of normal random variables:** Let  $v$  = variance,  $\sigma = \sqrt{v}$  (standard-deviation). Then  $N(0, v)$  represents a normal random variable with mean 0 and standard deviation  $\sigma$ .

**Fact 1:** Let  $X \sim N(0, \sigma^2)$ . Then

$$P(|X| \geq K\sigma) < 2e^{-K^2/3}.$$

**Fact 2:** If  $X \sim (0, v_1)$  and  $Y \sim (0, v_2)$ , and  $X \perp Y$ , then  $X + Y \sim (0, v_1 + v_2)$ .

**Fact 3:** Let  $X \sim N(0, v)$ ,  $c \in \mathbb{R}$ . Then

$$cX \sim N(0, c^2v).$$

For vector  $\vec{a} = (a_1, \dots, a_d)$ , define

$$f(a) = \sum_{i=1}^d X_i \cdot a_i$$

where  $X_i$  is an independent random variable such that  $X_i \sim N(0, 1)$ . That means  $f(a) \sim N(0, \|a\|_2^2)$ .

### Corollary 6.2.1 1

$$E[f(a)^2] = \|a\|_2^2.$$

### Corollary 6.2.2 2

$$P(f(a)^2 > K \|a\|_2^2) \leq e^{-\Omega(K)}.$$

*Proof.* By the previous Chernoff bound,

$$\begin{aligned} P(|f(a)| > J \|a\|_2) &\leq e^{-J^2/3} \\ P(f(a)^2 > J^2 \|a\|_2^2) &\leq 2e^{-J^2/3}. \end{aligned}$$

By setting  $K = J^2$ , we get the corollary. ⊗



We now turn to the proof of the Johnson-Lindenstrauss lemma.

*Proof.* Define  $\phi(\vec{a})$  as

$$\phi(\vec{a}) = \langle \phi_1(\vec{a}), \dots, \phi_r(\vec{a}) \rangle$$

where  $r = c\epsilon^{-2} \log n$  where  $c$  is a large constant. Now we define  $\phi_i$ . We take the same argument as in the previous proof:

$$\phi_i(a) = \frac{1}{\sqrt{r}} \sum_{j=1}^d X_{i,j} \cdot a_j$$

where all the  $X_{i,j} \sim N(0, 1)$  and are independent. We want to show:

$$\begin{aligned} \|\phi(\vec{a})\|_2^2 &= (1 \pm \epsilon) \|a\|_2^2 \\ \iff \sum \phi_i(\vec{a})^2 &= (1 \pm \epsilon) \|a\|_2^2. \end{aligned}$$

We now go over a Chernoff bound for geometric random variables. Let  $X_1, \dots, X_n$  be independent. Let  $\mu = E[\sum X_i]$ . Suppose each  $X_i$  satisfies

$$P(|X_i| \geq K) \leq e^{-\Omega(k)}.$$

Let  $X = \sum X_i$ . In the small-deviation regime ( $K \in [1, \sqrt{\mu}]$ ), we have

$$P(|X - E[X]| \geq K\sqrt{\mu}) \leq e^{-\Omega(k^2)}.$$

Then for  $J \geq 2$ , we have

$$P(X \geq J\mu) \leq \left(\frac{1}{2}\right)^{\Omega(J\mu)}.$$

Note that the large-deviation regime bound we're used to does not apply because it is not true in the  $n = 1$  case. Now for each  $\phi_i$ , we have

$$r \cdot \phi_i(\vec{a}) \sim N(0, \|a\|_2^2).$$

If we wanted to normalize this, we would get

$$Y_i := \left( \frac{\sqrt{r} \cdot \phi_i(\vec{a})}{\|a\|_2} \right)^2 \sim N(0, 1)^2.$$

$Y_i$  has mean 1, and  $|Y_i|$  is a geometric random variable. This means

$$P \left[ \left| \sum_{i=1}^r Y_i - r \right| \geq K\sqrt{r} \right] \leq e^{-\Omega(k^2)}.$$

Let's set  $K\sqrt{r} = \epsilon r$ , or that  $K = \epsilon\sqrt{r}$ . So,

$$P \left[ \sum_{i=1}^r Y_i \neq (1 \pm \epsilon)r \right] = e^{-\Omega(\epsilon^2 r)}.$$

But  $-\Omega(\epsilon^2 r) = \frac{1}{\text{poly}(n)}$ . What follows is:

$$\begin{aligned} \sum \frac{r \phi_i(\vec{a})^2}{\|a\|_2^2} &= (1 \pm \epsilon)r \\ \implies \sum \phi_i(\vec{a})^2 &= (1 \pm \epsilon) \|a\|_2^2. \end{aligned}$$

This is exactly what we wanted to prove. ⊕

We now prove the facts about normal random variables.

**Corollary 6.2.3 from Central Limit Theorem**

Suppose you define  $X^{(n)} := \sum_{i=1}^n X_i$  where  $X_i$  takes on 1 or  $-1$  with probability  $1/2$ . Then as  $n \rightarrow \infty$ ,  $\frac{\sqrt{v}}{\sqrt{n}}X^{(n)}$  converges pointwise to a single distribution. This is the definition of  $N(0, v)$ .

As a shorthand,

$$\lim_{n \rightarrow \infty} \frac{\sqrt{v}}{n} X^{(n)} \sim N(0, v).$$

We skip the first fact because it's an easy Chernoff bound. Fact 3 is also straight-forward from the definition of a random variable. So we will just prove that if we have  $A, B \sim N(0, v_1), N(0, v_2)$ , then  $A + B \sim N(0, v_1 + v_2)$ . As an easy example, take  $v_1 = v_2 = 1$ . So then the distribution of  $A + B$  would be

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}}(X^{(n)} + Y^{(n)}) &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n + Y_1 + \cdots + Y_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} X^{(2n)} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{2}}{\sqrt{m}} X^m && (\text{where } m = 2n) \\ &= \lim_{m \rightarrow \infty} \frac{\sqrt{2}}{\sqrt{m}} \cdot X^{(m)} \sim N(0, 2). \end{aligned}$$

## Chapter 7

# The Train Track Problem

While very sleep deprived and on a train ride, Bill crossed a bridge and got very confused because he thought there were no tracks below him. So he thought of a problem regarding the number of wheels on the left and right quarter of the train, and how much track is actually necessary for a train to go forever.

### Question

Consider a set  $W$  of wheels where  $W \subseteq \{0, \dots, \ell - 1\}$  and  $|W| = n$ . Then consider the track  $T \subseteq \{1, \dots, 2\ell - 1\}$ . A track  $T$  is valid if

$$\forall i \in \{1, \dots, \ell\}, (i + W) \cap T \neq \emptyset.$$

Our goal is to use as little track as possible.

A clear observation we can make is that if  $W$  is an arithmetic progression (or rather, evenly spread out), we can say that there exists a valid track  $T$  of size  $O\left(\frac{\ell}{n}\right)$  (just put a track at every length of the quarter train).



There are two theorems we want to prove:

### Theorem 7.0.1

For any  $W$ , there exists a valid  $T$  such that  $|T| \leq O\left(\frac{\ell}{n} \cdot \log n\right)$ .

### Theorem 7.0.2

$\exists W$  such that all valid  $T$  satisfy  $|T| \geq \Omega\left(\frac{\ell}{n} \cdot \log n\right)$ .

*Proof of Theorem 7.0.1.* Given  $W$ , WLOG,  $0 \in W$ .

**Step 1:** Let

$$T_1 = \text{each } i \in \{1, \dots, 2\ell - 1\} \text{ with probability } \frac{\ln n}{n}.$$

**Step 2:** Let

$$F = \{i \in \{1, \dots, \ell\} \mid (i + W) \cap T_1 = \emptyset\}.$$

That is, “ $W$  falls through  $T$ , at point  $i$ ”.

**Step 3:** Define  $T = T_1 \cup F$ .

**Note:**

Because  $0 \in W$ ,  $T$  is valid.

We move to analyzing  $T$ . First, we have

$$\begin{aligned} E[|T|] &= E[|T_1|] + E[|F|] \\ &\leq \frac{2\ell \ln n}{n} + ?. \end{aligned}$$

So,

$$\begin{aligned} E[|F|] &= \sum_{i=1}^{\ell} P[(i + W) \cap T = \emptyset] \\ &= \sum_{i=1}^{\ell} \prod_{j \in (i+W)} P(j \notin T_1) \\ &= \sum_{i=1}^{\ell} \left(1 - \frac{\ln n}{n}\right)^n \\ &\approx \sum_{i=1}^{\ell} \frac{1}{n} \\ &= \frac{\ell}{n}. \end{aligned}$$

So

$$\begin{aligned} E[|T|] &\leq \frac{2\ell \ln n}{n} + \frac{\ell}{n} \\ &= \ell \left( \frac{2 \ln n + 1}{n} \right) \\ &= O\left(\frac{\ell \ln n}{n}\right). \end{aligned}$$

From this, it follows that there exists a valid  $T$  such that  $|T| \leq O\left(\frac{\ell}{n} \cdot \ln n\right)$  that is an outcome of this randomized algorithm.

☺

*Proof of Theorem 7.0.2.* Without loss of generality, we can set  $\ell = 2n$  for the rest of this section.

**Proposition 7.0.1**

Let  $W =$  include each  $i \in \{0, \dots, 2n-1\}$  independently with probability  $1/2$ . Let  $T \subseteq \{1, \dots, 4n\}$  satisfy

$$|T| \leq \frac{\ln n}{100}.$$

Then,  $P(T \text{ valid}) < \frac{1}{n^{\Omega(\sqrt{n})}}$ .

We'll defer the proof of the proposition to later. But assuming we have proved it, then

$$\begin{aligned} P\left(\exists \text{ any } T \text{ of size } \frac{\ln n}{100} \text{ that is valid}\right) &\leq \sum_{T \subseteq \{1, \dots, 4n-1\}, |T| \leq \frac{\ln n}{100}} P(T \text{ valid}) \\ &\leq (4n)^{\frac{\ln n}{100}} \cdot \frac{1}{n^{\Omega(\sqrt{n})}} = \frac{1}{n^{\Omega(\sqrt{n})}}. \end{aligned}$$

Notice that with probability at least  $1/2$ ,  $|W| \geq n$ . So, there exists  $W$  such that  $|W| \geq n$  such that there is no valid  $T$  of size  $\leq \frac{\ln n}{100}$ . It follows that there is  $W$  of exactly size  $n$  such that this is true.

So given Proposition 7.0.1, we will have proved Theorem 7.0.2. ☺

*Proof of Proposition 7.0.1.* Let

$$F = \{i \in \{1, \dots, 2n\} \mid (i + W) \cap T = \emptyset\}.$$

We have

$$\begin{aligned} E[|F|] &= \sum_{i=1}^{2n} P[(i + W) \cap T = \emptyset] \\ &= \sum_{i=1}^{2n} \prod_{j \in T} \underbrace{P(j \notin W + i)}_{\geq 1/2} \\ &\geq 2n \cdot \left(\frac{1}{2}\right)^T \\ &\gg n^{0.9}. \end{aligned}$$

If  $f := |F|$ , then  $f$  is a function  $f(X_1, \dots, X_n)$  where  $X_i = \mathbb{I}[i \in W]$ . And if you toggle  $X_i$ ,  $f$  changes by at most  $|T| \leq \frac{\ln n}{100}$ . Now we are in a perfect spot to apply McDiarmid's inequality:

$$\begin{aligned} \underbrace{P(f = 0)}_{T \text{ is valid}} &\leq P(f < E[f] - n^{0.9}) \\ &\leq n^{-\Omega(\sqrt{n})}. \end{aligned} \quad (k = \frac{n^{0.4}}{\Theta(\lg n)})$$

☺

## Chapter 8

# Poissonization

### Definition 8.0.1

A random variable  $X$  is Poisson distributed,  $X \sim \text{Poisson}(\lambda)$  if

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

for  $\lambda, k \geq 0$ .

### Note:

Another definition is Binomial  $\left(n, \frac{\lambda}{n}\right)$  as  $n \rightarrow \infty$ .

The above note is true because we can write

$$\begin{aligned} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} &\approx \left(\frac{en}{k}\right)^k \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &\approx \left(\frac{e}{k}\right)^k \lambda^k e^{-\lambda(1-k/n)} \\ &\approx \left(\frac{1}{k!}\right)^k \lambda^k e^{-\lambda} \\ &= \frac{e^{-\lambda} \lambda^k}{k!} \end{aligned}$$

as desired.

Now we review some properties of the Poisson distribution:

- If  $X \sim \text{Poisson}(\lambda)$ , then  $E[X] = \text{Var}(X) = \lambda$ .
- If  $X \sim \text{Poisson}(\lambda)$  and  $Y \sim \text{Poisson}(\mu)$ , then  $X + Y \sim \text{Poisson}(\lambda + \mu)$ .
- If  $X \sim \text{Poisson}(\lambda)$ , then for any  $c > 0$ ,

$$P(|X - \lambda| \geq c) \leq 2e^{\frac{-c^2}{2(c+\lambda)}}$$

In the old balls and bins problems, we would drop  $n$  balls into  $m$  bins. This was awkward because the number of balls in different bins aren't independent random variables. So consider a different variant of the problem:

### Question: New balls and bins

Suppose  $k \sim \text{Poisson}(n)$ . We toss  $k$  balls uniformly at random into  $m$  bins.

**Claim 8.0.1**

The number of balls in bin  $i$  is distributed  $\text{Poisson}(\frac{n}{m})$  for all  $i$ . The number of balls in bin 1, bin 2, ..., bin  $m$  are independent random variables.

*Proof.* We take  $m = 2$  as this is easily generalizable. Let  $X_1, X_2$  be the number of balls in bins 1 and 2 respectively. We claim that

$$P(X_1 = i, X_2 = j) = P(\text{Poisson}(n/2) = i) \cdot P(\text{Poisson}(n/2) = j).$$

We have

$$\begin{aligned} P(X_1 = i, X_2 = j) &= P(k = i + j) \cdot P(\text{Binomial}(i + j, 1/2) = i) \\ &= \frac{e^{-n} n^{i+j}}{(i+j)!} \cdot \binom{i+j}{i} \left(\frac{1}{2}\right)^{i+j} \\ &= e^{-n/2} \cdot e^{-n/2} \cdot n^i \cdot n^j \cdot \left(\frac{1}{i!}\right) \left(\frac{1}{j!}\right) \cdot \left(\frac{1}{2}\right)^i \cdot \left(\frac{1}{2}\right)^j \\ &= P(\text{Poisson}(n/2) = i) \cdot P(\text{Poisson}(n/2) = j) \end{aligned}$$

as desired. This proves the first claim. Now,

$$\begin{aligned} P(X_1 = i) &= \sum_j P(X_1 = i, X_2 = j) \\ &= \sum_j P(\text{Poisson}(n/2) = i) \cdot P(\text{Poisson}(n/2) = j) \\ &= P(\text{Poisson}(n/2) = i). \end{aligned}$$

Together, this gives us the second claim that  $X_1 \perp X_2$ . ⊕

To learn Poissonization, we'll review the coupon collector. So let  $X$  be the number of steps until we get all  $n$  coupons.

**Claim 8.0.2**

For any constant  $c$ ,

$$\lim_{n \rightarrow \infty} P(X \geq n \log n + cn) = 1 - e^{-e^{-c}}.$$

*Proof.* To prove this claim, we use Poissonization and view it as a balls and bins problems. So instead of  $n$  balls and  $m$  bins, we have  $n \log n + cn$  steps and  $n$  coupons. Similarly, we'll have  $k \sim \text{Poisson}(n \log n + cn)$  steps and  $n$  coupons. Here are the steps we take:

1. We want to show

$$\lim_{n \rightarrow \infty} P(\text{see every coupon after } k) = 1 - e^{-e^{-c}}.$$

2. Then we show

$$P(\text{see every type of coupon after } n \log n + cn) = P(\text{see every type of coupon after } k) \pm o(1).$$

Together, this will show the desired result. So suppose we do  $k \sim \text{Poisson}(n \log n + cn)$  steps. Then the number of coupons of each type is independently distributed as  $\text{Poisson}(\log n + c)$ . So,

$$\begin{aligned} P(\text{we see all the coupons}) &= (1 - P(\text{Poisson}(\log n + c) = 0))^n \\ &= (1 - e^{-\log n + c})^n \\ &= (1 - e^{-c}/n)^n \\ &\approx 1 - e^{-e^{-c}} \end{aligned}$$

as desired.

So now we de-Poissonify.

(a) We want to show

$$P(\text{see all coupons after } b \text{ steps}) = P(\text{see all coupons after } b \pm n^{0.9} \text{ steps}) \pm o(1).$$

So let  $X$  be the number of boxes needed to get all the coupons. The two events we consider are  $X \geq b - n^{0.9}$  and  $X \geq b + n^{0.9}$ . So,

$$\begin{aligned} P(X \geq b + n^{0.9}) &= P(X \geq b - n^{0.9}) + P(\text{find the last coupon in boxes } b - n^{0.9} + 1, \dots, b + n^{0.9}) \\ &\leq P(X \geq b - n^{0.9}) + \frac{2n^{0.9}}{n} \\ &= P(X \geq b - n^{0.9}) + o(1). \end{aligned}$$

This sort of proves the claim.

(b) So now we can have that

$$P(|k - (n \log n + cn)| \geq n^{0.9}) = o(1).$$

This follows directly from the tail-bound that we mentioned earlier.

This completes the overall result.

☺



# Chapter 9

## Hash Tables

### 9.1 Random-Probing Hash table

For this section, a hash table has two important operations:

1. `insert(key)` (adding)
2. `query(key)` (exists)

Also for this section, we'll support up to  $(1 - \epsilon)n$  total insertions. The hash table will be an array with  $n$  slots.



So to insert a key  $k$ , we consider a sequence of random hashes  $h_1(k), h_2(k), \dots$  and each  $h_i(k)$  is uniformly random in  $1, \dots, n$  and  $h_i$ 's are independent. We place the key in the first available position out of the sequence  $h_1, h_2, h_3, \dots$

To query  $k$ , we try  $h_1(k), h_2(k), \dots$  until

- we find  $k$ , in which case we return true.
- or we find a free slot, in which case we return false.

Observations:

1. If the hash table is  $1 - \delta$  full, then

$$E[\text{time for next insertion}] = \delta^{-1}.$$

2. Suppose I fill  $1 - \epsilon$  full. Then,

$$\begin{aligned} E[\text{time to query random elements of those present}] &\approx \frac{n}{2} \cdot O(2) + \frac{n}{4} \cdot O(4) + \frac{n}{8} \cdot O(8) + \dots + \epsilon n \cdot O(\epsilon^{-1}) \\ &\approx \underbrace{1 + 1 + \dots + 1}_{\lg \epsilon^{-1} \text{ terms}} \\ &= O(\lg \epsilon^{-1}). \end{aligned}$$

“Open-addressed hash tables”: Any hash table that is “like random-probing” but that possibly uses a different distribution for its probe sequence

$$h_1(k), h_2(k), \dots$$

Some examples include linear probing and double hashing.

**Claim 9.1.1** Ullman '72

For any open address hash table,

$$E[\text{time to query random elements of those present}] \geq \Omega(\lg \epsilon^{-1}).$$

This was proven by Andrew Yao in 1985. The title of his paper was “Uniform Hashing is Optimal.” He ended the paper with the following conjecture:

**Claim 9.1.2** Yao '85

For any open address hash table, if you fill it to  $(1 - \epsilon)$  full, the expected time of the next insertion is at least  $\Omega(\epsilon^{-1})$ .

**Theorem 9.1.1**

Possible to achieve

$$E[\text{insertion time}] = O(\lg^2 \epsilon^{-1}).$$

We now turn attention towards the coupon collector problem:

**Question: Coupon Collector**

We have a hat with  $n$  coupons in it. We repeatedly draw random coupons from it, with replacement. The classical question is how many draws in expectation do we need to collect all  $n$  coupons? The answer to this is  $O(n \log n)$ .

**Theorem 9.1.2**

Let  $\delta \in (0, 1)$  be such that  $\delta \geq \frac{1}{n^{1/10}}$ . Let  $X$  be the number of draws until we have collected a  $(1 - \delta)$  fraction of all the coupons. Then, with high probability,  $X \leq 2n \ln \delta^{-1}$ .

*Proof.* Do  $2n \ln \delta^{-1}$  draws. Let  $Y$  be the number of distinct coupons not sampled. Then,

$$\begin{aligned} E[Y] &= \sum_{i=1}^n P(\text{coupon } i \text{ is not sampled}) \\ &= n \cdot \left(1 - \frac{1}{n}\right)^{2n \ln \delta^{-1}} \\ &\approx n\delta^2. \end{aligned}$$

Applying McDiarmid's to  $Y$ ,

- $Y$  is a function of  $m = 2n \ln \delta^{-1}$  random variables.
- Each of these random variables can only affect  $Y$  by  $\pm 1$ .

So by McDiarmid's, we have with high probability that

$$\begin{aligned} Y &\leq E[Y] + \tilde{O}(\sqrt{m}) \\ \implies Y &\leq \delta^2 n + \tilde{O}(\sqrt{m}) \\ \implies \text{number of sampled coupons} &\geq n - \underbrace{(\delta^2 n + \tilde{O}(\sqrt{m}))}_{\ll \delta n} \\ &\geq (1 - \delta)n. \end{aligned}$$

We now learn about a new hash table design. Break up the table as follows:

- $p_1$  is the first  $\frac{n}{2}$  slots.
- $p_2$  is the next  $\frac{n}{4}$  slots.
- $\vdots$
- Do this until there is a  $\epsilon n/100$  size part.

Probe sequence:

- Try  $100 \lg \epsilon^{-1}$  probes in  $p_1$ .
- Try  $100 \lg \epsilon^{-1}$  probes in  $p_2$ .
- $\vdots$

If this fails for all parts,

- Do  $100 \log n$  probes in the final part.
- Loop through the array until there is a free spot in the worst case.

The most recent step happens with very very low probability, but is required for correctness of the data structure.

Now suppose  $p_i$  gets  $\geq 50\%$  full for some  $i$ . This means that  $p_{i-1}$  must have gotten at least  $\frac{|p_i|}{2} \cdot 100 \lg \epsilon^{-1}$  probes. So, with high probability, if  $p_i$  gets 50% full, we have that

$$\frac{|p_i|}{2} \cdot 100 \lg \epsilon^{-1} = |p_{i-1}| \cdot 25 \lg \epsilon^{-1},$$

which means that  $p_{i-1}$  is  $1 - \epsilon^{12.5}$  full. Then  $p_{i-2}$  is also at least  $1 - \epsilon^{12.5}$  full, and all  $p_{i-1}, p_{i-2}, \dots, p_1$  are all at least  $1 - \epsilon^{12.5}$  full. We claim that the final part is  $\leq \frac{1}{2}$  full. This is because if the final part were  $\geq \frac{1}{2}$  full, then all the earlier parts would be  $\geq 1 - \epsilon^{12.5}$  full, meaning that the overall number of elements is

$$\begin{aligned} &\geq (1 - \epsilon/10)(|p_1| + \dots + |p_k|) \\ &= n \cdot \left(1 - \frac{\epsilon}{50}\right) \\ &> (1 - \epsilon)n. \end{aligned}$$

But this yields a contradiction, meaning the final part never reaches 50% full. As such, we have disproved Yao's claim. So,

- Once we get to the final part, the expected additional time is  $O(1)$ .
- So, the total expected time is

$$\begin{aligned} &O(1) + \text{number of probes before final part} \\ &= O(1) + 100 \lg \epsilon^{-1} \cdot O(\lg \epsilon^{-1}) = O(\lg^2 \epsilon^{-1}). \end{aligned}$$

## 9.2 Linear Probing

To insert an element  $v$  into a hash table that incorporates linear probing, we just

- place  $v$  in the first free slot out of  $h(v), h(v) + 1, \dots$  (modulo the table size).

Searching for an element  $v$  uses the same process. At relatively low capacity, linear probing is the most efficient hash table known to man. But now suppose we perform  $(1 - 1/x) \cdot n$  insertions and perform one more insertion.

### Claim 9.2.1 Peterson 1957

The expected insertion time is  $O(x)$ .

### Theorem 9.2.1 Knuth 1962

The expected insertion time is  $\Theta(x^2)$ .

Intuition for clustering:

- The longer your run gets, the more likely you are to hash into the run. “Winner keeps winning.”
- “Globbing effect.” This is when an insertion connects two clusters.

Intuition #2:

- Consider an interval  $I$  with size  $x^2$ . Let  $R$  be the number of items whose hashes are in  $I$ . We consider properties of  $R$ . So first,

$$E[R] = (1 - 1/x) \cdot x^2 = x^2 - x$$

$$\text{std}(R) \approx \sqrt{E[R]} \approx x.$$

Moral: intervals of size  $\leq x$  often have more elements hashing to them than can fit.

### Lemma 9.2.1

Throw  $(1 - 1/x) \cdot n$  balls into  $n$  bins. Let  $k > 0$ . Define  $P_i$  to be the number of balls in the first  $i$  bins. Then

$$P(\exists \text{ any } i \geq kx^2 \text{ such that } P_i \geq i) \leq e^{-\Omega(k)}.$$

Call the event above  $*$ .

Warmup: What is  $P(P_{kx^2} \geq kx^2)$ ? (call this event  $\dagger$ )

$$\mu = E[P_{kx^2}] = (1 - 1/x) \cdot kx^2 = kx^2 - kx.$$

So  $\sqrt{\mu} \leq \sqrt{kx^2} = \sqrt{k}x$ . So,

$$P(P_{kx^2} \geq kx^2) \approx P(P_{kx^2} \geq \mu + \sqrt{k}\sqrt{\mu}) = e^{-\Omega(k)}.$$

Now we make a critical claim. If we condition on any prefix being problematic, then

$$P(\dagger \mid *) = \Omega(1).$$

This would mean that  $P(*) = \Theta(P(\dagger))$ , so by the warmup, we'd be done.

*Proof.* Assume event  $*$ . Let  $i$  be the largest  $i$  such that  $P_i \geq i$ . Note that  $kx^2 \leq i$  because we assumed  $*$ . If you tell me  $i, P_i$ , what is the conditional distribution for  $P_{kx^2}$ . Then,

$P_{kx^2} \sim$  binomial random variable given by sum of  $P_i$  balls,  
each of which has probability  $kx^2/i$  of landing in the first  $kx^2$  bins.

Therefore,  $P_{kx}$  is now a binomial random variable with mean  $P_i \cdot \frac{kx^2}{i} \geq kx^2$ . Any binomial random variable with mean  $\mu \geq 1$  has  $\Omega(1)$  probability of meeting or exceeding its mean. So it follows that  $P(\dagger \mid *) \geq \Omega(1)$ . This completes the proof of the lemma.

Returning to Knuth's theorem, consider insertion of an element  $u$  at  $(1 - 1/x)$  full. Then we want to show that

$$P(\text{insertion time} \geq kx^2) \leq e^{-\Omega(k)}.$$

Now consider a table where the range  $[h(v) - r, h(v) + kx^2]$  is saturated. Then, there exists a suffix of array slots ending at slot  $h(v) + kx^2 - 1$  such that

- $|\text{suffix}| \geq kx^2$
- the number of elements that hash to suffix  $\geq |\text{suffix}|$ .

This is where we apply the lemma. This directly gives us that

$$P(\text{insertion time} \geq kx^2) \leq e^{-\Omega(k)}$$

which completes the proof. ☺

### 9.3 Ordered Linear Probing

#### Theorem 9.3.1 Knuth 1963

Suppose you fill a linear-probing hash table to  $(1 - 1/x)$  full. Then, the expected insertion time is  $\Theta(x^2)$ .

Amble and Knuth had the following idea: modify the hash table such that in each “run”, maintain an invariant that the elements in the run are in sorted order by hash.

Aside: if elements have the same hash, imagine we break ties with another hash function.

#### Note:

Also note that this is a history-independent data structure, meaning that the final state of the hash table is independent of the order of the elements added.

#### Theorem 9.3.2 Amble and Knuth 1973

$E[\text{time to query an element}] = O(x)$ .

*Proof.* Just to simplify things, we will assume positive queries, meaning we only query elements that exist in the hash table.

Let  $S = \{\text{elements in the hash table}\}$ . For  $s \in S$ ,  $Q_s$  is the query time for  $s$ . Also define  $I_s$  as the insertion time for  $s$  when it was inserted.

**Observation 1:**  $E[Q_s]$  is the same  $\forall s \in S$ .

**Proof:** By symmetry and because the hash table is history-independent. ☺

**Observation 2:**

$$\sum_{s \in S} Q_s = \sum_{s \in S} I_s.$$

**Proof:** Just think about it lowkey. Basically if the insertion time is  $k$ , then we increase the sum of the query times by  $k$ . ☺

**Observation 3:**

$$E \left[ \sum_{s \in S} I_s \right] = \Theta(n \cdot x).$$

**Proof:** Well we have

$$\begin{aligned} E \left[ \sum_{s \in S} I_s \right] &= \Theta \left( \frac{n}{2} \cdot 2^2 + \frac{n}{4} \cdot 4^2 + \frac{n}{8} \cdot 8^2 + \cdots + \frac{n}{x} \cdot x^2 \right) \\ &= \Theta(n \cdot x). \end{aligned} \quad (\text{dominated by last summand})$$

This completes the proof of the observation. ☺  
Moving to the proof of the theorem, we have:

$$\begin{aligned}
E[Q_s] &= \frac{1}{|S|} \cdot \sum_{s \in S} E[Q_s] \\
&= \frac{1}{|S|} \cdot \sum_{s \in S} E[I_s] \\
&= \frac{1}{\Theta(n)} \cdot \Theta(n \cdot x) \\
&= \Theta(x).
\end{aligned}$$

☺

### 9.3.1 Deletions

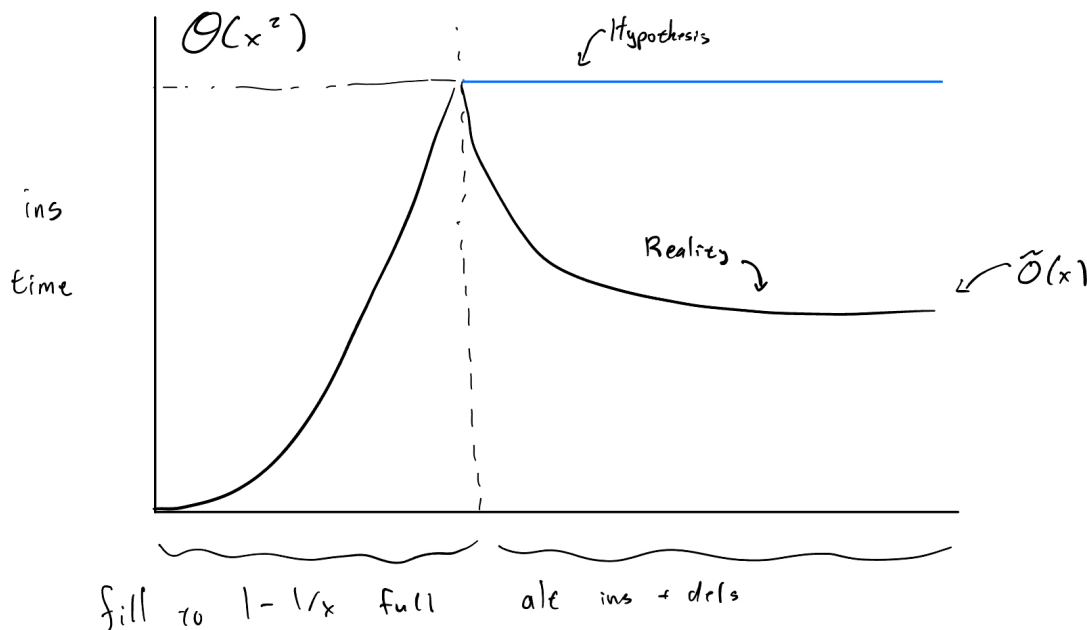
Idea: use “tombstones.” That is, rather than entirely deleting the element, we just mark the index as “deleted.” What that means is that when we query that index, all we know is that there used to be an element at that spot. Also when trying to insert an element, the tombstone spot is just as good as any other open spot to insert the element.

However, queries cannot treat tombstones as empty. This is because the element we are searching for could still be to the right of a tombstone. So queries skip over tombstones.

Every so often, we should pause and rebuild the hash table, but with the tombstones removed. How often? The classical measure is every  $R = \frac{n}{2x}$  operations.

**Question:** What if we have a table with insertions and deletions?

Experiment: initially fill our hash table to  $1 - 1/x$  full. Then, we alternate between insertions and deletions. We get this graph:



**Theorem 9.3.3** Kuszmaul (2022)

Consider any workload such that the hash table is never  $> 1 - 1/x$  full. Then, the amortized expected time per operation is  $\tilde{O}(x)$ . The actual optimal bound is  $\Theta(x \lg^{1.5} x)$ .

**Big Idea 1:** Consider all the operations in a given rebuild window. Then we draw a random line in the hash table and ask how many insertions crossed the line. If we can answer this for every dotted line, we can sum to get the sum of the insertion times.

**Big Idea 2:** Before rebuilding, consider the state of the hash table. Draw an X in every empty slot before the random line. Then for every deletion that occurs in the rebuild window, we draw an X at the time of deletion and at the position of the deleted element. Similarly for insertions, but use  $\checkmark$ . Now we draw a path from the bottom left to the top right of the graph (nondecreasing). It turns out that

the number of insertions that cross the line =  $\max_{\text{all possible paths } p} (\text{number of } \checkmark \text{ under path } - \text{number of X under path}).$

**Big Idea 3:** With enough work, we get a simpler problem which we call the path surplus problem. That is, we have an  $x \times x$  grid where each entry is randomly  $\pm 1$ . Consider every monotonic path to the right. For each path  $p$ , we have

$$\text{surplus}(p) = \sum \text{entries under } p.$$

The question is what is the expected max surplus over all path?

**Theorem 9.3.4**

The expected maximum surplus is  $\Theta(x \lg^{0.75} x)$ .