Abstract Algebra

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Contents

Chapter 1

1.1 Introductory Notes

1.1.1 Things to Remember

Note:

- Definitions will usually be stated as "if" even though they mean "if and only if".
- Any form of proof is valid. Avoid proofs by contradiction because of disbelief in the law of excluded middle.
- When you define an object, you can *only* utilize its definition to prove anything about it.

1.1.2 Set Review

Definition 1.1.1: Set

In mathematics, a set is an undefined term. Basically, "everyone knows what it is." A few examples of sets are:

- The empty set is the set with no elements. It is denoted by ϕ or $\emptyset$.
- $\mathbb{N}$ is the set of natural numbers.
- $\mathbb{Z}$ is the set of integers.
- $\mathbb{Q}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{C}$ is the set of complex numbers.

Note:

- A set is a well-defined collection of objects. The objects in a set are called elements of the set.
- A set is generally defined as a capital letter.
- $(A = B) \iff (\forall x : x \in A \iff x \in B)$
- $(A \subset B) \iff (\forall x \in A : x \in B)$
- A is a proper subset of B if $A \subset B$ and $A \neq B$.

Theorem 1.1.1

$A = B \iff A \subset B \wedge B \subset A$

Note:

- $A \cup B = x : x \in A \vee x \in B$
- $A \cap B = x : x \in A \wedge x \in B$
- $A \backslash B = x : x \in A \wedge x \notin B$
- $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$

1.1.3 Cartesian Products and Functions

Note:

- $A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Example 1.1.1 (Cartesian Product of two sets)

Let $A = \{1, 2, \Delta\}$ and $B = \{0, \pi\}$

- $(1, 0)$
- $(2, 0)$
- $(\Delta, 0)$
- $(1, \pi)$
- $(2, \pi)$
- (Δ, π)

Note:

Relations are subsets of Cartesian Products. For example, we can say that $<$ is a relation on the subset of $\mathbb{R} \times \mathbb{R}$ consisting of all ordered pairs of real numbers such that the first element is less than the second.

Definition 1.1.2: Function

A function f from a set A to a set B is a subset of $A \times B$ such that for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.

Note:

Let R be a relation from A to B.

- A is the domain
- B is the codomain
- $\{b : aRb\}$ is the image
- R is injective (one-to-one) if $a_1Rb \wedge a_2Rb \implies a_1 = a_2$
- R is surjective (onto) if $\forall b \in B : \exists a \in A : aRb$. Basically if the image is the entire codomain.
- R is bijective if it is injective and surjective

Note:

$A \xrightarrow{\text{R}} B$
$B \xrightarrow{\text{S}} C$
Define the composition as $S \circ R = \{(a, c) : \text{there is some } b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$

Theorem 1.1.2

Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then

- $h \circ (g \circ f) = (h \circ g) \circ f$
- If f and g are injective, so is $g \circ f$
- If f and g are surjective, so is $g \circ f$
- If f and g are bijective, so is $g \circ f$

1.1.4 Equivalence Relations

Definition 1.1.3: Equivalence Relation

An equivalence relation is a relation that has the following special properties:

- Reflexivity: aRa for all $a \in A$
- Symmetry: $aRb \implies bRa$
- Transitivity: $aRb \wedge bRc \implies aRc$

Definition 1.1.4: Partition

Given a set S, a partition of S is a collection of subsets of S such that their union is S.

Note:

Equivalence relations go hand in hand with partitions.

Note:

If $\sim$ is an equivalence relation $a \sim b$, then $\sim$ partiations a set X into chunks. $X/\sim$ is the set of chunks.
Addition is *well-defined* as an operation on $\mathbb{Z}/x\mathbb{Z}$ for $x \in \mathbb{Z}$.

1.1.5 Complex Numbers and Matrices

Definition 1.1.5: Complex Number

A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. $i^2 = -1$.

Note:

Complex numbers generally take the from $z = a + bi$.
$\bar{z} = a - bi$ is the complex conjugate of z.
Divide complex numbers by multiplying by the complex conjugate of the denominator

Definition 1.1.6: Matrix

A matrix is a rectangular array of numbers. A $m \times n$ matrix is an array of m rows and n columns. Define the group of $m \times n$ matrices over a field $\mathbb{F}$ as $\mathbb{F}^{m \times n}$.

Note:

Multiplication by an $m \times n$ matrix is a function from $\mathbb{F}^n$ to $\mathbb{F}^m$. It is associative because all functions are associative.

Example 1.1.2 (2 × 2 matrix exercise)

Consider $\mathbb{Z}^{2\times 2}$. Define a relation $A \sim B$ if there is an integer matrix P whose determinant is one and $B = P^{-1}AP$. Note that if an integer matrix has a determinant 1 it is invertible and its inverse is also an integer matrix with determinant 1.

1. Show that this is an equivalence relation.
2. Show that two matrices with different determinants cannot be similar.
3. Determine whether $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.
4. Determine whether $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

Solution:

1. Reflexive: $A = P^{-1}AP$ for $P = I_2$.
   Symmetric: $P^{-1}AP = P^{-1}BP$ for some P with determinant 1.
   Transitive: $B = P_1^{-1}AP_1 \wedge C = P_2^{-1}BP_2 \Rightarrow C = P_2^{-1}P_1^{-1}AP_1P_2$
2. Determinants are a multiplicative property. If $B = P^{-1}AP$ and $\det(B) \neq \det(A)$, then $\det(B) \neq 1 * \det(A) * 1$.
3. No, different JCF.
4. Yes, same JCF.

1.1.6 Number Theory

Note:

Know induction, division algorithm, GCD and Bezout's lemma, and Primes and the Fundmental Theorem of Arithmetic.

Example 1.1.3 (Weak Induction)

Prove that $5 | n^5 - n$ for all n.

Proof: Proof by induction.

1. $n = 1$ is true, $5|0$.
2. If it is true then $n = k$, show that it is true when $n = k + 1$.
   $(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^3 + 5k + 1 - (k+1) = (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)$.
   Both quantities are divisible by 5.

Therefore, $5 | n^5 - n$ for all n. ☺

Example 1.1.4 (Strong Induction)

Prove that every integer n can be written as $n = d_1 1! + d_2 2! + \cdots + d_k k!$ for some $d_1, \ldots, d_k \leqslant k \in \mathbb{Z}$ and $k \geqslant 1$.

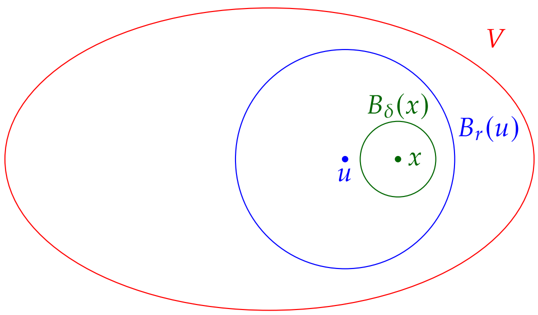
Proof: Strong induction. Given n, chose s s.t. $s! \leqslant n < (s+1)!$. Then we can write $n = q \cdot s! + r$.

1. $q \leqslant s$ (if $q \geqslant s + 1$, then $n \geqslant (s+1)!$, which goes against our claim)
2. $r < s!$

Assume that this is true for any $k < n$. Then we can write $n = q \cdot s! + r$ for some $r < s!$. Then we can write r in the same format since it is true for all $k < n$. 😄

1.2 Random Examples

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

😄

Corollary 1.2.1

By the result of the proof, we can then show...

Lenma 1.2.1

Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.2.1

$1 + 1 = 2$.

1.3 Random

Definition 1.3.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|$ $V \to \mathbb{R}_{\geqslant 0}$ satisfying

1. $\|x\| = 0 \iff x = 0 \; \forall \; x \in V$
2. $\|\lambda x\| = |\lambda| \|x\| \; \forall \; \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\; x \in V$
3. $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \; x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where 2 becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \; \lambda \in \mathbb{C}$, $x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise
Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2 \\
\implies &\sum_i (x_i^2 + 2x_iy_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2 \\
\implies &\left[\sum_i x_iy_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle\cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i}y_i$$

Then we still have $\langle x, x\rangle \geqslant 0$

1.4 Algorithms

Algorithm 1: what

```
Input: This is some input
Output: This is some output
/* This is a comment */

some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                    // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```