21-269
Vector Analysis

Rohan Jain

Contents

Chapter 1

1.1 The Real Numbers

Definition 1.1.1: Partial Order

Let X be a set with a binary relation $\leqslant$. $\leqslant$ is a *partial order* if:

1. $x \leqslant x$ for all $x \in X$ (reflexivity)
2. $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$ for all $x, y, z \in X$ (transitivity)
3. $x \leqslant y$ and $y \leqslant x$ implies $x = y$ for all $x, y \in X$ (antisymmetry)

Definition 1.1.2: Partially Ordered Set (poset)

A set X with a partial order $\leqslant$ is called a *partially ordered set* or *poset*. It is notated as $(X, \leqslant)$.

Definition 1.1.3: Total Order

A partial order $\leqslant$ is a *total order* if for all $x, y \in X$, we have $x \leqslant y$ or $y \leqslant x$.

Example 1.1.1 (poset)

Let Y be a set. Define $X = \{\text{all subsets of } Y\} = \mathcal{P}(Y)$. Let $E, F \in Y$, we say that $E \leqslant F$ if $E \subseteq F$. Then $(X, \leqslant)$ is a poset. This is not a total order.

Definition 1.1.4: Upper Bound, Bounded Above, Supremum, Maximum

Let $(X, \leqslant)$ be a poset. Let $E \subseteq X$.

1. $y \in X$ is an *upper bound* of E if $x \leqslant y$ for all $x \in E$.
2. E is *bounded above* if it has at least one upper bound.
3. If E is nonempty and bounded above, then the *supremum*, if it exists, of E, denoted $\sup E$, is the least upper bound of E.
4. E has a *maximum* if there is $y \in E$ such that $x \leqslant y$ for all $x \in E$.

Properties worth mentioning:

1. If E has a maximum, then $\sup E$ exists and is equal to the maximum.

   Proof. Let y be the maximum of E. If $z \in X$, is an upper bound of E, then $z \geqslant y$ because $y \in E$. Since z was arbitrary, this is true for any upper bound. Thus, y is the least upper bound of E. 😃

Example 1.1.2

Let Y be a nonempty set, $(\mathcal{P}(Y), \leqslant)$ poset.
Fix nonempty $Z \subseteq Y$.

$$E = \{W \subseteq Y : W \subset Z\}$$

Trivially, Z is an upper bound of E. Realize that any superset of Z is an upper bound as well. We can postulate that the supremum of E is Z. We will now prove it:

Proof. Need to show that if F is an upper bound of E, then $F \supseteq Z$. If $x \in Z$, then $\{x\} \in E$ by definition of E, so $F \supseteq x$ for all $x \in Z$. Thus, $F \supseteq Z$. ☺

Note that there is no maximum of E.

Definition 1.1.5: Lower Bound, Bounded Below, Infimum, Minimum

Let $(X, \leqslant)$ be a poset. Let $E \subseteq X$.

1. $y \in X$ is a *lower bound* of E if $y \leqslant x$ for all $x \in E$.
2. E is *bounded below* if it has at least one lower bound.
3. If E is nonempty and bounded below, then the *infimum*, if it exists, of E, denoted $\inf E$, is the greatest lower bound of E.
4. E has a *minimum* if there is $y \in E$ such that $y \leqslant x$ for all $x \in E$.

Going back to example 1.1.2, we can see that E is bounded below by $\emptyset$. The infimum of E is $\emptyset$. The minimum of E is also $\emptyset$.

Definition 1.1.6: Complete

Let $(X, \leqslant)$ poset. X is *complete* if every nonempty subset of X that is bounded above has a supremum.

Example 1.1.3 ($\mathbb{Q}$)

$(\mathbb{Q}, \leqslant)$ is not complete.

Claim 1.1.1 $\mathbb{R}$

There is a complete ordered field $(\mathbb{R}, +, \cdot, \leqslant)$. Its elements are called real numbers.

1.2 First Recitation, 1/18

Exercise 1.2.1 Function Example

Let X be the set of all functions $f : D_f \to Z$ with $D_f \subseteq Y$. We say that $f \leqslant g$ if $D_f \subseteq D_g$ and $f(x) = g(x)$ for all $x \in D_f$. Is $(X, \leqslant)$ a poset? Is it complete?

Proof. To show that $(X, \leqslant)$ is complete, we need to show that every nonempty subset of X that is bounded above has a supremum. Let $E \subseteq X$ be nonempty and bounded above. Let $G = \bigcup_{f \in E} D_f$. G is the union of all the domains of the functions in E. G is bounded above by the union of the upper bounds of the domains of the functions in E. Let $H = \bigcup_{f \in E} f(D_f)$. H is bounded above by the union of the upper bounds of the ranges of the functions in E. Let $F : G \to H$ be defined as $F(x) = f(x)$ for all $x \in D_f$. F is the supremum of E. ☺

1.3 Natural Numbers

Exercise 1.3.1

Take $(X, +, \cdot, \leqslant)$ ordered field. Prove:

1. If $0 \leqslant x$, then $-x \leqslant 0$.
2. If $x \leqslant y$, and $0 \leqslant z \neq 0$, then $xz \leqslant yz$.
3. For all $x \in X$, $0 \leqslant x^2$.
4. Prove $0 < 1$.

Proof. Fields have the following important properties:

- If $a \leqslant b$, then $a + c \leqslant b + c$.
- If $a, b \geqslant 0$, then $ab \geqslant 0$.

1. Take the first property with $a = 0$, $b = x$, and $c = -x$. Then $0 \leqslant x \implies 0 + (-x) \leqslant x + (-x) \implies -x \leqslant 0$.
2. If $x \leqslant y$, then $0 \leqslant y + (-x)$. By the second property, $0 \leqslant z \cdot (y + (-x)) = zy + (-zx)$. Then $0 \leqslant zy + (-zx) \implies zx \leqslant zy$.
3. We split into the three trichotomy cases:
   - If $x = 0$, then $0 \leqslant 0^2$.
   - If $x \leqslant 0$ with $x \neq 0$, then $0 \leqslant -x$. By the second property, $0 \leqslant (-x)^2 = (-x)(-x) = x^2$.
   - If $x > 0$, then $0 \leqslant x$. By the second property, $0 \leqslant x^2$.
4. FSOC, assume $0 > 1$ and multiply both sides by 1. Then we get $0 \cdot 1 > 1 \cdot 1 \Rightarrow 0 > (1)^2$, which is a contradiction to the third property we proved.

😄

Definition 1.3.1: Inductive

Take $E \subseteq \mathbb{R}$. E is *inductive* if $1 \in E$ and $x \in E$ implies $x + 1 \in E$.

Example 1.3.1 (Inductive Sets)

- $\mathbb{R}$ is inductive.
- $\{x \in \mathbb{R} : 0 \leqslant x\}$

  Proof. $1 \in E$ because $1 \geqslant 0$. If $x \in E$, then $x + 1 \geqslant 0$, so $x + 1 \in E$. 😄

Definition 1.3.2: Natural Numbers

The intersection of all inductive sets is denoted $\mathbb{N}$. The elements of $\mathbb{N}$ are called *natural numbers*.

Properties of $\mathbb{N}$:

- $\mathbb{N} \neq \emptyset$. Since $1 \in$ every inductive set, $1 \in \mathbb{N}$.
- $\mathbb{N}$ is an inductive set.

Theorem 1.3.1 Induction

For every $n \in \mathbb{N}$, let $P(n)$ be a proposition such that:

1. $P(1)$ is true.
2. If $P(n)$, then $P(n+1)$.

Then $P(n)$ is true for every $n \in \mathbb{N}$

Proof. $E = \{n \in \mathbb{N} \ : \ P(n)\}$ is inductive by 1. and 2. So, $\mathbb{N} \subseteq E$, but $E \subseteq \mathbb{N}$ by definition of $\mathbb{N}$. Thus, $E = \mathbb{N}$. ☺

Theorem 1.3.2 Archimedean Property

Let $a, b \in \mathbb{R}$ with $a > 0$. Then there is $n \in \mathbb{N}$ such that $na > b$.

Proof. If $b \leqslant 0$, then we take $n = 1$. Assume $b > 0$. For sake of contradiction, assume there does not exist n such that $na > b$. Then $E = \{na : n \in \mathbb{N}\}$ is bounded above by b. Let $c = \sup E$. $c - a \leqslant c$, so $c - a$ is not an upper bound of E. Thus, there is $n \in \mathbb{N}$ such that $c - a \leqslant na$. Then $c \leqslant (n+1)a$. But c is an upper bound of E, so $c \geqslant (n+1)a$. Thus, $c = (n+1)a$. But $c \in E$, so $c = na$ for some $n \in \mathbb{N}$. Thus, $na = (n+1)a$, so $n = n+1$, which is a contradiction. ☺

Definition 1.3.3: Integers

$\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$

Theorem 1.3.3 Integer Part

For every $x \in \mathbb{R}$, there is a unique $k \in \mathbb{Z}$ such that $k \leqslant x < k+1$.

Definition 1.3.4: Integer Part

The k that satisfies the above theorem is called the *integer part* of x, denoted $\lfloor x \rfloor$.

Proof. Let $E = \{k \in \mathbb{Z} : k \leqslant x\}$. First we show that E is nonempty.

- If $x \geqslant 0$, then $0 \in E$, so E is nonempty.
- If $x < 0$, then $-x > 0$. By the Archimedean property, there is $n \in \mathbb{N}$ such that $n > -x$. Thus, $-n < x$. So, $-n \in E$, so E is nonempty.

Now we show that E is bounded from above. Very clearly, x is an upper bound. By supremum property, there is $L = \sup(E)$ and $L \in \mathbb{R}$. $L - 1$ is not an upper bound, which means that there is an element $k \in E$ such that $L - 1 < k$. But since L is the supremum, $L \geqslant k$. Thus, $L - 1 < k \leqslant L$. So, $L < k + 1$ so $k + 1 \notin E$. Now, $k \leqslant x$ since $k \in E$. Now we show that k is unique. Assume there is $m \in \mathbb{Z}$ such that $m \leqslant x < m + 1$. Then $m \in E$, so $m \leqslant L$. But L is the supremum, so $L \geqslant m$. Thus, $L = m$. So, $k = m$. ☺

Definition 1.3.5: $\mathbb{Q}$

If $p \in \mathbb{Z}$ with $p \neq 0$, then $\exists p^{-1} \in \mathbb{R}$. Define $\mathbb{Q} = \{pq^{-1} : p, q \in \mathbb{Z}, p \neq 0\}$.

1.4 Density of Rationals

Theorem 1.4.1 Density of the Rationals

Let $a, b \in \mathbb{R}$ with $a < b$. Then there is $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. We have $a < b \implies 0 = a + (-a) < b - a \implies 0 < \frac{1}{b-a}$. By the integer part theorem, there is $q \in \mathbb{Z}$ such that $\frac{1}{b-a} < q$. So now, $\frac{1}{q} < b - a \implies a < a + \frac{1}{q} < b$. Multiply both sides by $q > 0$ to get $aq < a + 1 < bq$. By the integer part theorem, there is $p \in \mathbb{Z}$ such that $p \leqslant qa < p + 1$ (i.e. $p = \lfloor qa \rfloor$). Since $qa < p + 1 \leqslant qa + 1 < qb$. Getting rid of unnecessary stuff, we have $qa < p + 1 < qb$. Thus, $a < \frac{p+1}{q} < b$. Let $r = \frac{p+1}{q}$. Then $r \in \mathbb{Q}$ and $a < r < b$. 😊

Definition 1.4.1: Irrational Numbers

$\mathbb{R} \setminus \mathbb{Q}$ is the set of *irrational numbers*.

Exercise 1.4.1 TODO in Recitation 1/23

- Prove that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$.
- Prove that "$\sqrt{2}$" exists in $\mathbb{R}$. (prove that there is at least one irrational number)
  - Have to play with the set $E = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$.

Theorem 1.4.2 Density of Irrationals

Let $a, b \in \mathbb{R}$ with $a < b$. Then there is $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Proof. $a < b \implies a\sqrt{2} < b\sqrt{2}$. By the density of rationals, there is $r \in \mathbb{Q}$ such that $a\sqrt{2} < r < b\sqrt{2}$. Then $a < \frac{r}{\sqrt{2}} < b$. Let $x = \frac{r}{\sqrt{2}}$. If $r = 0$, then $a\sqrt{2} < 0 < b\sqrt{2}$. By previous theorem, we can find $q \in \mathbb{Q}$ such that $a\sqrt{2} < q < 0 < b\sqrt{2}$. Then $a < \frac{q}{\sqrt{2}} < b$. Let $x = \frac{q}{\sqrt{2}}$. Then $x \in \mathbb{R} \setminus \mathbb{Q}$ and $a < x < b$. 😊

Note:

Take $x \in \mathbb{R}$, $E = \{r \in \mathbb{Q} : r < x\}$. x is the upper bound of E. This set is nonempty because we can take $x - 1 < r < x$. Now we prove that $x = \sup E$.

Proof. Assume $\exists L$ upper bound of E such that $L < x$. Then $L < x \implies$ there exists some $r \in \mathbb{Q}$ such that $L < r < x$, but $r \in E$, so L is not an upper bound of E. Thus, L cannot be an upper bound of E and x is the least upper bound of E. 😊

Since now we know that $\sqrt{2} = \sup\{r \in \mathbb{Q} : r < \sqrt{2}\}$, we can also define $3^{\sqrt{2}} = \sup\{3^r : r \in \mathbb{Q}, r < \sqrt{2}\}$.

Definition 1.4.2: x^0

Let $0 \neq x \in \mathbb{R}$. We define $x^0 = 1$.

Definition 1.4.3: x^n

Let $x \in \mathbb{R}$, $n \in \mathbb{N}$. We start with $x^1 := x$. Then assume x^m has been defined. Then we say $x^{m+1} := x^m \cdot x$.

Definition 1.4.4: $x^{p/m}$

Let $x \in \mathbb{R}$, $p \in \mathbb{Z}$, $m \in \mathbb{N}$. We say $x^{p/m} = \sqrt[m]{x^p}$.

Exercise 1.4.2 Properties of Exponenets

Let $x \in \mathbb{R}$, $r, q \in \mathbb{Q}$, and $x, r, q > 0$. Prove the following:

- $x^r \cdot x^q = x^{r+q}$
- $(x^r)^q = (x^q)^r = x^{rq}$

Proof. ☺

Definition 1.4.5: Negative Exponent

Take $x > 0$, $r = -\frac{p}{m}$ for $p, m \in \mathbb{N}$. First, we have that $x^{-r} := (x^{-1})^{p/m}$.

Exercise 1.4.3 More Properties of Exponents

Take $x \in \mathbb{R}, x > 0, r, q \in \mathbb{Q}$. Prove the following:

- If $r > 0$, prove that $x^r > 1$.
- If $r < q$, prove that $x^r < x^q$.

1.4.1 1/23 - Recitation - Proving Irrationality of $\sqrt{2}$

Existence of $\sqrt{2}$:

1. Let $E = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$. Prove that E is non-empty and that E is bounded above.

   Proof. We know that $0 < 1$ and from that we get $1^2 = 1 < 2$, which can be checked by subtracting 1 from both sides. As such E is nonempty.

   Now we show that E is bounded above. We know that $2^2 = 4 > 2 > a^2 \in E$, so $2^2 > a^2 \Rightarrow 2 > a$, so 2 is an upper bound of E. ☺

2. By the completeness of $(\mathbb{R}, \leqslant)$, E has a supremum, L. Prove that $L > 0$ and that $L^2 = 2$.

   Proof. Since L is the least upper bound, it has to be greater than 1 which is in the set E. Therefore, $L > 1 > 0 \implies L > 0$.

   Now we show that $L^2 \geqslant 2$. For sake of contradiction, assume $L^2 < 2$. Since $L > 0$, this means that $L \in E$. By the density of rationals, there exists $r \in \mathbb{Q}$ such that $L < r < \sqrt{2}$. Since L is an upper bound of E, $r \notin E$. But $r \in \mathbb{Q}$, so $r^2 \neq 2$. Thus, $r^2 > 2$. Since $r > 0$, $r^2 > 2 \implies r > \sqrt{2}$. But $r < \sqrt{2}$, so we have a contradiction. Thus, $L^2 \geqslant 2$. ☺

3. Prove that if $y \in \mathbb{R} \setminus E$ and $y > 0$, then y is an upper bound of E.

   Proof. Assume $y \in \mathbb{R} \setminus E$ and $y > 0$. We need to show that y is an upper bound of E. Assume for sake of contradiction that y is not an upper bound of E. Then there exists $x \in E$ such that $x > y$. But $x \in E \implies x^2 < 2$. Since $y > 0$, $x^2 < 2 \implies y^2 < 2$. But $y \notin E$, so $y^2 \geqslant 2$. But this would mean that $y \in E$. Contradiction. Thus, y is an upper bound of E. ☺

4. Prove that $L^2 = 2$.

   Proof. We know that $L^2 \geqslant 2$ from part 2. Now we show that $L^2 \leqslant 2$. Assume for sake of contradiction that $L^2 > 2$.

   How small does $\epsilon > 0$ need to be such that $(L - \epsilon)^2 > 2$ as well.

Start with $(L-\epsilon)^2 = L^2 - 2L\epsilon + \epsilon^2$, which is greater than $L^2 - 2L\epsilon$ since $\epsilon > 0$. So now, how small does ϵ need to be such that $L^2 > 2 \implies L^2 - 2L\epsilon > 2$ too.

$$\begin{aligned} 2L\epsilon &< 2 - L^2 \\ \epsilon &< \frac{2 - L^2}{2L} \end{aligned}$$

Since $L^2 > 2$, this means that an ϵ can be found. This means that L is not the least upper bound. Contradiction. Thus, $L^2 \leqslant 2$. 😄

1.5

Definition 1.5.1: $\sqrt{2}$

$$\sqrt{2} := \sup\{x \in \mathbb{R} : x > 0, x^2 < 2\}$$

Exercise 1.5.1

For $n \in \mathbb{N}, n \geqslant 2$. Fix $x > 0$.

$$E = \{y \in \mathbb{R} : y > 0, y^m < x\}.$$

Prove that $l = \sup E$ satisfies $l^m = x$.

Definition 1.5.2: $\sqrt[m]{x}$

$$\sqrt[m]{x} := \sup\{y \in \mathbb{R} : y > 0, y^m < x\}$$

Definition 1.5.3: $x^{p/q}$

$$x^{p/q} := \left(\sqrt[q]{x}\right)^p$$

Definition 1.5.4: x^q

For $q \in \mathbb{R}$, $q > 0$, and $x > 1$.

$$x^q := \sup\{x^r : r \in \mathbb{Q}, 0 < r < q\}$$

Example 1.5.1

$$\sqrt{2} = \sup\{r \in \mathbb{Q} : r > 0, r < \sqrt{2}\}$$

Theorem 1.5.1

Take $a, b \in \mathbb{R}$, $a, b > 0$ and $x \in \mathbb{R} > 1$. Then $x^a \cdot x^b = x^{a+b}$.

Proof. Let $E_i = \{x^r : r \in \mathbb{Q}, r > 0, r < i\}$. Consider E_a, E_b, E_{a+b}. Then let $l_i = \sup(E_i)$. Consider l_a, l_b, l_{a+b}. We want to show that $l_a \cdot l_b = l_{a+b}$ by showing that both $l_a \cdot l_b \leqslant l_{a+b}$ and $l_a \cdot l_b \geqslant l_{a+b}$.

Let $r \in \mathbb{Q}$ with $0 < r < a$. Let $s \in \mathbb{Q}$ with $0 < s < b$. Then we have that $x^r \cdot x^s = x^{r+s}$ (from the exercise two days ago and since $r, s \in \mathbb{Q}$.) we know that $0 < r + s < a + b$ and is rational. Thus, $x^{r+s} \in E_{a+b}$. Thus, $x^r \cdot x^s \leqslant l_{a+b}$.

We want to divide both sides by x^s while fixing r. So, we have that $x^r \leqslant \dfrac{l_{a+b}}{x^s}$, which is true for all $r \in \mathbb{Q}$, such that $0 < r < a$. Thus, $\dfrac{l_{a+b}}{x^s}$ is an upper bound for E_a. Thus, $l_a \leqslant \dfrac{l_{a+b}}{x^s}$. Thus, $x^s \leqslant \dfrac{l_{a+b}}{l_a}$, meaning that $\frac{l_{a+b}}{l_a}$ is an upper bound for E_b. Thus, $l_b \leqslant \dfrac{l_{a+b}}{l_a}$. Thus, $l_a \cdot l_b \leqslant l_{a+b}$.

Now we show that $l_a \cdot l_b \geqslant l_{a+b}$. Let $t \in \mathbb{Q}$ with $0 < t < a + b$. We need $0 < r \in \mathbb{Q} < a$ and $0 < s \in \mathbb{Q} < b$ with $t = r + s$. We start by looking at $t - a < b$. By the density of $\mathbb{Q}$, find $s \in \mathbb{Q}$ such that $t - a < s < b$. Take $s > 0$ beacuse $b > 0$. So $t - s < a$. By the density of $\mathbb{Q}$, find $0 < p \in \mathbb{Q}$ such that $t - s < p < a$. So $t < s + p$. So, $x^t < x^{s+p} = x^s x^p \leqslant l_a l_b$ since $x^s \in E_b$ and $x^p \in E_a$. We know that $l_a l_b$ is an upper bound of E_{a+b}, so $l_{a+b} \leqslant l_a l_b$.

Therefore $l_a \cdot l_b = l_{a+b}$. 😆

Definition 1.5.5: Negative Exponents

Let $x > 1$, $a < 0$. Then:

$$x^a := (x^{-a})^{-1}$$

Definition 1.5.6: Exponents between 0 and 1

Let $x \in \mathbb{R}$ with $0 < x < 1$ and $a > 0$. Then:

$$x^a := \left(\frac{1}{x}\right)^{-a}$$

An important note is that if we have $E \subseteq (0, \infty)$ with a bounded E. Then if we define $F = \{\frac{1}{x} : x \in E\}$, then we have the following:

$$\sup E = \frac{1}{\inf F}$$
$$\inf E = \frac{1}{\sup F}$$

1.6 1/25 - Recitation - Sequences of Set

Definition 1.6.1: Sequence of a Set

Given a set X, a sequence on X is a function $f : \mathbb{N} \to X$. We denote $f(n)$ as x_n. We can also denote the sequence as $\{x_n\}_{n=1}^{\infty}$.

Definition 1.6.2

Let $(X, \leqslant)$ be a poset and $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Then $E = \{x_n : n \in \mathbb{N}\}$ is a subset of X. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded from above is the set E is bounded from above. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded from below is the set E is bounded from below. We say that $\{x_n\}_{n=1}^{\infty}$ is bounded if it is bounded from above and below.

Definition 1.6.3: Limit Superior

Let $(X, \leqslant)$ be a poset. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Suppose $\{x_n\}_n$ is bounded from above. Then, we define the *limit superior* of x_n as $n \to \infty$ as:

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} x_k$$

Definition 1.6.4: Limit Inferior

Let $(X, \leqslant)$ be a poset. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on X. Suppose $\{x_n\}_n$ is bounded from below. Then, we define the *limit inferior* of x_n as $n \to \infty$ as:

$$\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} x_k$$

Exercise 1.6.1

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on $\mathbb{R}$ bounded above. Prove that $L \in \mathbb{R}$ is the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$ iff for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that:

   (a) $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$.

   (b) $L - \epsilon < x_n$ for infinitely many n.

Proof. Let $L \in \mathbb{R}$ be the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$. Let $\epsilon > 0$. L being the lim sup means that $L = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} x_k$. Thus, $L \leqslant \sup_{k\geqslant n} x_k$ for all $n \in \mathbb{N}$. Thus, $L - \epsilon < \sup_{k\geqslant n} x_k$ for all $n \in \mathbb{N}$. Then $L - \epsilon$ is not an upper bound of $\{x_n\}_{n=1}^{\infty}$. Thus, there is $n_\epsilon \in \mathbb{N}$ such that $L - \epsilon < x_{n_\epsilon}$. Thus, $L - \epsilon < x_n$ for infinitely many n. Now we show that $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$. Assume for sake of contradiction that there is $n \geqslant n_\epsilon$ such that $x_n \geqslant L + \epsilon$. Then $L + \epsilon$ is an upper bound of $\{x_n\}_{n=1}^{\infty}$. But L is the $\limsup$, so $L \geqslant L + \epsilon$. Contradiction. Thus, $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$.

Now we show the other direction. Assume that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $x_n < L + \epsilon$ for all $n \geqslant n_\epsilon$ and $L - \epsilon < x_n$ for infinitely many n. We want to show that L is the $\limsup$ of $\{x_n\}_{n=1}^{\infty}$. We know that L is an upper bound of $\{x_n\}_{n=1}^{\infty}$. We need to show that L is the least upper bound. Assume for sake of contradiction that L is not the least upper bound. Then there is $L' < L$ such that L' is an upper bound of $\{x_n\}_{n=1}^{\infty}$. Let $\epsilon = L - L'$. Then $L' < L - \epsilon$. But $L - \epsilon < x_n$ for infinitely many n. But $L' < L - \epsilon$, so L' is not an upper bound of $\{x_n\}_{n=1}^{\infty}$. Contradiction. 😊

1.7 Vector Spaces

Example 1.7.1 (Vector Spaces)

- Euclidean Space $\subseteq \mathbb{R}^n$. $x \in \mathbb{R}^n$ is a vector. $x = (x_1, \ldots, x_n)$.
- Polynomial Space from $\mathbb{R} \to \mathbb{R}$. $x \in \mathbb{R}^x$. $x = a_0 + a_1 x + \cdots + a_n x^n$.
- $f : [a, b] \to \mathbb{R}$ continuous functions.

Definition 1.7.1: Boundedness of Functions

Let E be a set and $f : E \to \mathbb{R}$.

1. f is bounded from above if the set $f(E) = \{y \in \mathbb{R} : y = f(x), x \in E\}$ is bounded from above.
2. f is bounded from below if the set $f(E) = \{y \in \mathbb{R} : y = f(x), x \in E\}$ is bounded from below.
3. f is bounded if $f(E)$ is bounded.

Definition 1.7.2: Inner Product

A function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is an *inner product* if it satisfies the following properties:

- $(x, x) \geqslant 0$ for all $x \in X$.
- $(x, x) = 0$ iff $x = 0$.
- $(x, y) = (y, x)$ for all $x, y \in X$.
- $(sx + ty, z) = s(x, z) + t(y, z)$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}$.

Example 1.7.2 (Examples of Inner Products)

- $\mathbb{R}^n$ with dot products.
- $f : [a, b] \to \mathbb{R}$ with $(f, g) = \int_a^b f(x)g(x)dx$. This is is not an inner product because we can define:

$$f = \begin{cases} 1 & x = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

  which has an integral of 0. But $f \neq 0$. If we add that f is continuous, then it is an inner product.

Definition 1.7.3: Norm

Let V be a vector space with an inner product $(\cdot, \cdot)$. Then the *norm* of $x \in X$ is defined as $||\cdot|| : X \to [0, \infty)$ such that:

1. $||x|| = 0 \iff x = 0$
2. $||tx|| = |t|||x||$ for all $x \in X$
3. $||x + y|| \leqslant ||x|| + ||y||$ for all $x, y \in X$

Example 1.7.3 (Examples of Norms)

- $||x|| = \sqrt{(x, x)}$ for $x \in \mathbb{R}^n$
- $X = \{f : E \to \mathbb{R}, f \text{ bounded}\}$. $||f|| = \sup_{x \in E} |f(x)|$.
  - First property is obviously true.
  - For the second property, we use the fact that

$$\sup(tF) = \begin{cases} t\sup(F) & \text{if } t \geqslant 0 \\ t\inf(F) & \text{if } t < 0 \end{cases}$$

  - For the third property, we use the triangle inequality:

$$\sup|f + g| \leqslant \sup|f| + \sup|g|$$
$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant \sup|f| + \sup|g|$$

Note:

Space of bounded functions denoted as $\ell^\infty(E) = \{f : E \to \mathbb{R} : f \text{ bounded}\}$.

Theorem 1.7.1 Cauchy Schwarz Inequality

Let X be a vector space with an inner product $(\cdot, \cdot)$. Then for all $x, y \in X$, we have that $|(x, y)| \leqslant \sqrt{(x, x)} \cdot \sqrt{(y, y)}$.

Proof. Let $y \neq 0$. Consider $(x + ty, x + ty) = (x, x + ty) + t(y, x + ty) = (x, x) + t(x, y) + t(y, x) + t^2(y, y)$. We can combine the middle terms to get $t^2(y,y)+2(x,y)+(x,x)$, which is quadratic in t. Take $t=-\dfrac{(x,y)}{(y,y)}$.

$$
\begin{aligned}
0 &\leqslant (x,x) - 2\frac{(x,x)^2}{(y,y)} + \frac{(x,y)^2}{(y,y)} \\
0 &\leqslant (x,x)(y,y) - 2(x,y)^2 + (x,y)^2 \\
0 &\leqslant (x,x)(y,y) - (x,y)^2 \\
(x,y)^2 &\leqslant (x,x)(y,y) \\
|(x,y)| &\leqslant \sqrt{(x,x)} \cdot \sqrt{(y,y)}
\end{aligned}
$$

😄