Question: 2

Does the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2(n)}$$

converge or diverge?

Solution: We recall Cauchy's condensation test, which yields the following result:

For non-increasing, non-negative $f(x)$, $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n f(2^n)$ converges.

So, we get the following:

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2(n)} \leqslant \sum_{n=2}^{\infty} \frac{2^n}{2^n \log^2(2^n)} = \sum_{n=2}^{\infty} \frac{1}{n^2 \log^2(2)}$$

By the p-test, our condensed series converges and therefore so does the original series.

Question: 8

Evaluate

$$\lim_{x\to\infty} (2x)^{1+1/2x} - (x)^{1+1/x} - x$$

Solution: We start by factoring out an x and realizing that $x^{1/x} = e^{\ln(x)/x}$. Doing this, we get:

$$\lim_{x\to\infty} x(2(2x)^{1/2x} - (x)^{1/x} - 1)$$
$$\lim_{x\to\infty} x(2e^{\ln(2x)/2x} - e^{\ln(x)/x} - 1)$$
$$\lim_{x\to\infty} x\left(2 + \frac{2\ln(2x)}{2x} + O\left(\frac{\ln(x)^2}{x^2}\right) - \left(1 + \frac{\ln(x)}{x} + O\left(\frac{\ln(x)^2}{x^2}\right)\right) - 1\right)$$
$$\lim_{x\to\infty} x\left((2-1-1) + \left(\frac{\ln(2x)}{x} - \frac{\ln(x)}{x}\right) + \left(O\left(\frac{\ln(x)^2}{x^2}\right) - O\left(\frac{\ln(x)^2}{x^2}\right)\right)\right)$$
$$\lim_{x\to\infty} x\left(\frac{\ln(2)}{x}\right) = \ln(2)$$