Question: 6

Evaluate
$$\int_1^2 \frac{\log x}{2-2x+x^2}dx.$$

Solution: We start by u substitution of $x = 2u$.

$$I = \int_1^2 \frac{\log x}{2-2x+x^2}dx = \int_{1/2}^1 \frac{\log 2u}{2-4u+4u^2}2du = \int_{1/2}^1 \frac{\log u}{1-2u+2u^2}du + \int_{1/2}^1 \frac{\log 2}{1-2u+2u^2}du$$

Now we do another stupid, unmotivated substitution of $u = 1/v$ into the first integral.

$$\int_{1/2}^1 \frac{\log u}{1-2u+2u^2}du = \int_2^1 \frac{-\log v}{1-\frac{2}{v}+\frac{2}{v^2}}\left(-\frac{1}{v^2}dv\right) = -\int_1^2 \frac{\log v}{2-2v+v^2}dv = -I$$

As such, we get:

$$2I = (\log 2)\int_{1/2}^1 \frac{1}{1-2u+2u^2}du = (\log 2)\left[\arctan(2u-1)\right]_{1/2}^1 = \log 2 \cdot \frac{\pi}{4}$$
$$\implies I = \frac{\pi}{8}\log 2$$

Question: 7

Evaluate
$$\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x}dx.$$

Solution: This is made trivial by Frullani integral, which gives the following result:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = [f(\infty)-f(0)]\log\left(\frac{a}{b}\right)$$

Quickly plugging in values, we get
$$\frac{\pi}{2}\cdot\ln(\pi)$$
as the result.

Since this was short, here's a short proof of the Frullani integral.

$$\begin{aligned} f'(xt) &= \frac{\partial}{\partial t}\left(\frac{f(xt)}{x}\right) \\ \frac{f(ax)-f(bx)}{x} &= \left[\frac{f(xt)}{x}\right]_{t=b}^{t=a} \\ &= \int_b^a f'(xt)dt \end{aligned}$$

From here, we get that

$$\begin{aligned} \int_0^\infty \int_b^a f'(xt)dtdx &= \int_b^a \int_0^\infty f'(xt)dxdt && \text{(by fubini's theorem)} \\ &= \int_b^a \left.\frac{f(xt)}{x}\right|_0^\infty dt \\ &= \int_b^a \frac{f(\infty)-f(0)}{x}dt \\ &= [f(\infty)-f(0)]\log\left(\frac{a}{b}\right) \quad ☺ \end{aligned}$$

, which is our intended result.