

Question: 11

Let $\mathbb{Z}^{\sqrt{2}} = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

- Prove that $\mathbb{Z}[\sqrt{2}]$ is an integral domain.
- Find all the units of $\mathbb{Z}[\sqrt{2}]$.
- Determine the field of fractions of $\mathbb{Z}[\sqrt{2}]$.
- Prove that $\mathbb{Z}[\sqrt{2}i]$ is a Euclidean domain under the Euclidean valuation $v(a + b\sqrt{2}i) = a^2 + 2b^2$.

Solution:

- Consider $(a+b\sqrt{2})(c+d\sqrt{2}) = 0$. This means that $(a+b\sqrt{2})(a-b\sqrt{2})(c+d\sqrt{2})(c-d\sqrt{2}) = (a^2-2b^2)(c^2-2d^2) = 0$. The irrationality of $\sqrt{2}$ and that fact that a, b, c, d are all integers tells us that either $a = b = 0$ or $c = d = 0$. ☺

- If u is a unit, then so are $-u$, $1/u$, and $-1/u$. At least one of these has to be greater than 1 if $u \neq 0, 1$. As such, it is enough to show that if $u < 1$, then u is a power of $1+\sqrt{2}$. So, we can write that $(1+\sqrt{2})^k < u < (1+\sqrt{2})^{k+1}$. Dividing by $(1+\sqrt{2})^k$ gives us $1 < u(1+\sqrt{2})^{-k} < 1+\sqrt{2}$. Since $1+\sqrt{2}$ is the smallest unit not equal to 0, $u(1+\sqrt{2})^{-k} = 1 \Rightarrow u = (1+\sqrt{2})^k$. Since norm is multiplicative, we have that all powers of $1+\sqrt{2}$ are units.

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$$Q = \left\{ \frac{a+b\sqrt{2}}{c+d\sqrt{2}} : a, b, c, d \in \mathbb{Z}, c+d\sqrt{2} \neq 0 \right\}$$

- We first need to show that $v(xy) = v(x)v(y)$ for $x, y \in \mathbb{Z}[\sqrt{2}i]$.

$$\begin{aligned} v((a+b\sqrt{2}i)(c+d\sqrt{2}i)) &= v((ac-2bd) + (ad+bc)\sqrt{2}i) \\ &= (ac-2bd)^2 + 2(ad+bc)^2 \\ &= a^2c^2 + 2a^2d^2 + 2b^2c^2 + 4b^2d^2 \\ v(a+b\sqrt{2}i)v(c+d\sqrt{2}i) &= (a^2+2b^2)(c^2+2d^2) \\ &= a^2c^2 + 2a^2d^2 + 2b^2c^2 + 4b^2d^2 \end{aligned}$$

For nonzero values, we have that $v(x) \geq 1$ and it follows that $v(x) \leq v(xy)$.

Next, let $a+b\sqrt{2}i, c+d\sqrt{2}i \in \mathbb{Z}[\sqrt{2}i]$ with nonzero $c+d\sqrt{2}i$. Now define q_1 to be the closest integer to $\frac{ac}{c^2}$ and q_2 as the closest integer to $\frac{bc}{c^2}$. Define s_1, s_2, r_1, r_2 as follows:

$$\begin{aligned} s_1 + s_2\sqrt{2}i &= \left(\frac{ac+2bd}{c^2+d^2} + \frac{bc-ad}{c^2+2d^2}\sqrt{2}i \right) - (q_1 + q_2\sqrt{2}i) \\ r_1 + r_2\sqrt{2}i &= (a+b\sqrt{2}i) - (q_1 + q_2\sqrt{2}i)(c+d\sqrt{2}i) \end{aligned}$$

From this, we need to show that $r_1^2 + 2r_2^2 < c^2 + 2d^2$. Start by noting that $|s_1|, |s_2| \leq \frac{1}{2}$ by definition of q_1 and q_2 . So,

$$(s_1 + s_2\sqrt{2}i)(c + d\sqrt{2}i) = (a + b\sqrt{2}i) - (q_1 + q_2\sqrt{2}i)(c + d\sqrt{2}i) = r_1 + r_2\sqrt{2}i$$

Thus,

$$v(r_1 + r_2\sqrt{2}i) = v(s_1 + s_2\sqrt{2}i)v(c + d\sqrt{2}i) \leq \left(\frac{1}{4} + 2 \cdot \frac{1}{4} \right) v(c + d\sqrt{2}i) < v(c + d\sqrt{2}i)$$

Therefore, the original statement was proved and also we proved that $\mathbb{Z}[\sqrt{2}i]$ is a Euclidean domain. ☺

Question: 17

Prove or disprove: Every subdomain of a UFD is also a UFD.

Solution: $\mathbb{Z}[3i] \subseteq \mathbb{C}$ is a subdomain of a UFD, but is not a UFD. ☹

Question: 18

An ideal of a commutative ring R is said to be **finitely generated** if there exist elements a_1, \dots, a_n in R such that every element r in the ideal can be written as $a_1r_1 + \dots + a_nr_n$ for some r_1, \dots, r_n in R . Prove that R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.

Solution: We start by proving that if R satisfies ACC, then its ideals are finitely generated. Let I be a nonzero ideal and a_1 a nonzero value of I . If $I = \langle a_1 \rangle$, then I is finitely generated. If not, then $I_1 = \langle a_1 \rangle$ is a subset of I . Now consider $a_2 \in I \setminus I_1$. Let, $I_2 = \langle a_1, a_2 \rangle$. If $I = I_2$, we are done. If not, we have an $a_3 \in I \setminus I_2$ and we continue the process to have that $I_1 \subseteq I_2 \subseteq I_3 \dots$. By ACC, there exists an N such that $I_n = I_N$ for all $n \geq N$. But if $I_{N+1} = I_N$, then there are no elements in I that aren't in I_N . Therefore, $I = \langle a_1, \dots, a_N \rangle$ and I is finitely generated.

For the converse, we supposed the ideals of R are finitely generated. We have that $I = \bigcup_{n=1}^{\infty} I_n$ is an ideal. But since every ideal is finitely generated, we have that $I = \langle a_1, \dots, a_k \rangle$ for some k . But then for $n = 1, 2, 3, \dots, k$, $a_i \in I_{b_i}$ for some integer b_i . Let $N = \max(b_1, \dots, b_k)$. Then $a_i \in I_{b_i} \subseteq I_N$. Therefore, $I \subseteq I_N$. So, $I_n = I_N$ for all $n \geq N$ and R satisfies ACC. ☺

Question: 19

Let D be an integral domain with a descending chain of ideals $I_1 \supset I_2 \supset I_3 \supset \dots$. Suppose that there exists N such that $I_k = I_N$ for all $k \geq N$. A ring satisfying this condition is said to satisfy the **descending chain condition**, or DCC. Rings satisfying the DCC are called **Artinian rings**, after Emil Artin. Show that if D satisfies the descending chain condition, it must satisfy the ascending chain condition.

Solution: Consider $I_i = \langle a^i \rangle$ for some $a \in D$. Since $a^{n+1}d = a^n(ad)$, we have that $a^{n+1} \subseteq a^nD$. So, we have a descending chain of ideals as follows:

$$aD \supseteq a^2D \supseteq \dots \supseteq a^nD \supseteq a^{n+1}D \supseteq \dots$$

which stabilizes since D is Artinian. So, we can say that

$$a^{m+1}D = a^mD$$

for some positive integer m . Since $a^m = a^m 1_D \in D$, there exists b such that $a^{m+1} = a^m b$, or that $a^m(1_D - ab) = 0$. This yields that $ab = 1_D$ as $a, a^m \neq 0$ since D is an integral domain. We have shown that $a \neq 0 \in D$ has an inverse and as such, D is a field which satisfies the ascending chain condition. ☺