Question: 34

Find all the subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Use this information to show that $\mathbb{Z}_3 \times \mathbb{Z}_3 \ncong \mathbb{Z}_9$.

Solution: The subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are:

- $\mathbb{Z}_3 \times \mathbb{Z}_3$
- $\{(0,0)\}$
- $\{(0,0),(1,0),(2,0)\}$
- $\{(0,0),(0,1),(0,2)\}$
- $\{(0,0),(1,1),(2,2)\}$
- $\{(0,0),(1,2),(2,1)\}$

The subgroups of $\mathbb{Z}_9$ are:

- $\mathbb{Z}_9$
- $\{0\}$
- $\{0,3,6\}$

Since these groups have different sets of subgroups, they are not isomorphic. In other words, $\mathbb{Z}_3 \times \mathbb{Z}_3 \ncong \mathbb{Z}_9$.

Question: 35

Find all the subgroups of the symmetry group of an equilateral triangle.

Solution: The subgroups of D_3 are:

- D_3
- $\{e\}$
- $\{e, f_A\}$
- $\{e, f_B\}$
- $\{e, f_C\}$
- $\{e, \rho_{120}, \rho_{240}\}$

The solutions to questions 34 and 35 somewhat demonstrate an important theorem, Lagrange's Theorem, which says that the order of the subgroups of G should divide $|G|$.

Question: 41

Prove that

$$G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \wedge a, b \neq 0\}$$

is a subgroup of $\mathbb{R}^*$ under the group operation of multiplication.

Solution: There are 4 things we have to check for: closed under the operation, identity, and inverse:

Closure: If we have $a+b\sqrt{2} \in G$ and $c+d\sqrt{2} \in G$, then their product is $(ac+2bd)+(ad+bc)\sqrt{2}$. We know that this is in G because if $a,b,c,d \in \mathbb{Q}$, then we know that $(ac+2bd),(ad+bc) \in \mathbb{Q}$ too because of closure of rationals. Therefore G is closed under the operation of multiplication.

Identity: The identity for G is 1. We know that 1 is in G because $0, 1 \in \mathbb{Q}$ and if $a,b \in \mathbb{Q}$, then $a+b\sqrt{2}$ is in G.

Inverse: If $a+b\sqrt{2} \in G$, then $a-b\sqrt{2} \in G$ because if $a,b \in \mathbb{Q}$, then $a-b\sqrt{2} \in \mathbb{Q}$. So, we can calculate the inverse of $a+b\sqrt{2}$.

$$\begin{aligned} \frac{1}{a+b\sqrt{2}} &= \frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} \\ &= \frac{a-b\sqrt{2}}{a^2-2b^2} \\ &= \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2} \end{aligned} \tag{1}$$

This is an element of G because both quantities, $\frac{a}{a^2-2b^2}$ and $\frac{b}{a^2-2b^2}$ are in $\mathbb{Q}$ if $a,b \in \mathbb{Q}$. From the above, we can conclude that $G < \mathbb{R}^*$. 😄

Question: 45

Prove that the intersection of two subgroups of a group G is also a subgroup of G.

Solution: Let's call the two subgroups H and K, so the intersection is $H \cap K$.

1. **Closure**: If $a,b \in H \cap K$, then $a,b \in H$ and $a,b \in K$. So $ab \in H$ and $ab \in K$. Therefore $ab \in H \cap K$.
2. **Identity**: The identity of G is the identity of H and the identity of K, since subgroups contain the identity of their parent group. Since this identity is in H and K, it is also in $H \cap K$.
3. **Inverse**: If $a \in H \cap K$, then $a \in H \Rightarrow a^{-1} \in H$ and $a \in K \Rightarrow a^{-1} \in K$. So $a^{-1} \in H \cap K$.

From the above, we can conclude that $H \cap K$ is a subgroup of G. 😄

Question: 46

Prove or disprove: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

Solution: FSOC, assume $H \not\subset K$ and $K \not\subset H$ and that $H \cup K$ is a group. Since the sets don't contain themselves, there exists an element $h \in H$ and $k \in K$ such that $h \notin K$ and $k \notin H$. However, since we are assuming that $H \cup K$ is a group, $hk \in H \cup K$. Therefore, either $hk \in H$ or $hk \in K$.

If $hk \in H$, then $h^{-1}(hk) = k \in H$, $\rightarrow\leftarrow$. A similar case can be made for $hk \in K$. Therefore, $H \cup K$ is not always a subgroup of G. 😄

Addendum: If $H \cong K$, then $H \cup K = H = K < G$.

Question: 47

Prove or disprove: If H and K are subgroups of a group G, then $HK = \{hk : h \in H \wedge k \in K\}$ is a subgroup of G. What if G is abelian?

Solution: Two cases: non-abelian and abelian.

- **non-abelian**: If G is not abelian, then HK is not a subgroup of G. Let $h_1k_1, h_2k_2 \in HK$. Then $(h_1k_1)(h_2k_2) = h_1k_1h_2k_2$ is not guaranteed to be in HK. 😄

- **abelian**: If G is abelian, then HK is a subgroup of G.

  - **Closure**: Let $h_1k_1, h_2k_2 \in HK$. Then $(h_1k_1)(h_2k_2) = h_1k_1h_2k_2 = h_1h_2k_1k_1 = (h_1h_2)(k_1k_2) \in HK$. This works because H and K are abelian since G is abelian.
  - **Identity**: Since $e \in H$ and $e \in K$, $ee = e \in HK$.
  - **Inverse**: Choose any $h \in H$ and $k \in K$. Then say $h^{-1} = x \in H$ and $k^{-1} = y \in K$. Then $hk, xy \in HK$. $(hk)^{-1} = yx$, but since G is abelian, $yx = xy$. Therefore, $hk^{-1} = xy \in HK$. Therefore, inverses exist in HK.

  Therefore HK is a subgroup of G if G is abelian. 😄

Question: 48

Let G be a group and $g \in G$. Show that

$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$$

is a subgroup of G. This subgroup is called the **center** of G.

Solution:

- **Closure**: If $x, y \in Z(G)$, then $\exists g \in G : (xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy) \in Z(G)$. Therefore, $Z(G)$ is closed.

- **Identity**: The identity of G, call it e, obviously satifies $eg = ge$ for all $g \in G$. Therefore, $e \in Z(G)$.

- **Inverse**: If $x \in Z(G)$, then $\exists g \in G : gx = xg \forall g \in G \Rightarrow gx^{-1} = x^{-1}g \forall g \in G$. Therefore, $x^{-1} \in Z(G)$.

Therefore, $Z(G)$ is a subgroup of G. 😄

Question: 53

Let H be a subgroup of G and

$$C(H) = \{g \in G : gh = hg \text{ for all } h \in H\}.$$

Prove that $C(H)$ is a subgroup of G. This group is called the **centralizer** of H in G.

Solution:

- **Closure**: If $x, y \in C(H)$, then $\exists h \in H : (xy)h = x(yh) = x(hy) = (xh)y = (hx)y = h(xy) \in C(H)$. Therefore, $C(H)$ is closed.
- **Identity**: Have e be the identity of G. Then $e \in H$ so we have $eh = he \forall h \in H$. Therefore, $e \in C(H)$.
- **Inverse**: Let $c \in C(H) \Rightarrow ch = hc \forall h \in H$. If we pre and post multiply by c^{-1}, we get $c^{-1}chc^{-1} = c^{-1}hcc^{-1} = hc^{-1} = c^{-1}h \forall h \in H$. Therefore, $c^{-1} \in C(H)$.

  As such, $C(H)$ is a subgroup of G. 😄

Question: 54

Let H be a subgroup of G. If $g \in G$, show that $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is also a subgroup of G.

Solution:

1. **Closure**: If we let $h_1, h_2 \in H$, then $gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$, demonstrating closure.
2. **Identity**: Since subgroups have their parent group's identity, it is in H. As such, $geg^{-1} = e \in gHg^{-1}$.
3. **Inverse**: If $h \in H$, then we can use the fact that H is a subgroup and contains inverses to show that ghg^{-1} has an inverse. $(ghg^{-1})^{-1} = (g^{-1^{-1}})h^{-1}g^1 = gh^1g^{-1} \in gHg^{-1}$.

   Therefore, gHg^{-1} is a subgroup of G. 😄