21-235 Math Studies Analysis I

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Contents

Chapter 1

1.1 Ordered Fields (Review)

Definition 1.1.1: Order

Let E be a set. An *order* on E is a relation $<$ on E such that for all $x, y, z \in E$:

1. (Trichotomy) Exactly one of the following holds: $x < y$, $x = y$, or $x > y$.
2. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Example 1.1.1 (Examples of Ordered Sets)

1. This definition develops orders on basic number systems: e.g. $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
2. Define $\lesssim$ on $\mathbb{Z}$ as follows: We say that $m \lesssim n$ for $m, n \in \mathbb{Z}$ if:
   (a) m is even and n is odd
   (b) m, n are even and $m < n$
   (c) m, n are odd and $m < n$.

Key Concepts:

- upper/lower bounds of sets
- bounded sets
- max/min
- supremum/infimum
- supremum/infimum property: An ordered set E satisfies such a property if every nonempty set $A \subseteq E$ that's bounded above/below has a supremum/infimum in E.
- Fact: sup prop $\implies$ inf prop

Definition 1.1.2: Ordered Field

Let $\mathbb{F}$ be a field with order $<$. We say that $\mathbb{F}$ is an *ordered field* provided that:

1. For all $x, y, z \in \mathbb{F}$, if $x < y$, then $x + z < y + z$.
2. For all $x, y \in \mathbb{F}$, if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Example 1.1.2
$\mathbb{Q}$ is a field.

Facts of any ordered field:

1. $0 < 1$
2. $\nexists x \in \mathbb{F}$ such that $x^2 = -1$.

Definition 1.1.3: Ordered Subfield, Homomorphism, Isomorphism

Let $\mathbb{F}$ be an ordered field.

1. A set $\mathbb{K} \subseteq \mathbb{F}$ is called an *ordered subfield* if $mathbbK$ is an algeraic subfield and $\mathbb{K}$ is an ordered field equipped with $<$ from $\mathbb{F}$.
2. Let $\mathbb{G}$ be an ordered field and let $f : \mathbb{F} \to \mathbb{G}$. We say that f is an *ordered field homomorphism* if it's a field homomorphism and $f(x) < f(y)$ whenever $x < y$.
3. f is an ordered field isomorphism if f is an ordered field homomorphism and f is bijective.

Note:

1. If $f : \mathbb{F} \to \mathbb{G}$ is an ordered field homomorphism, $f(\mathbb{F})$ is an ordered subfield of $\mathbb{G}$.
2. OF property $\implies$ f is injective.
3. $\therefore$ every ordered field homomorphism $f : \mathbb{F} \to \mathbb{G}$ is such that f induces a bijection $f : \mathbb{F} \to f(\mathbb{F}) \subseteq \mathbb{G}$.

Theorem 1.1.1 $\mathbb{Q}$ is the smallest ordered field. More precisely, if $\mathbb{F}$ is an ordered field, then there exists a canonical ordered field homomorphism $f : \mathbb{Q} \to \mathbb{F}$.

Upshot/notation abuse: We identify $f(\mathbb{Q}) = \mathbb{Q}$ to view $\mathbb{Q} \subseteq \mathbb{F}$. In turn, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{F}$.

1.2 Types of Ordered Fields

Definition 1.2.1: Archimedean, Dedekind complete

Let $\mathbb{F}$ be an ordered field.

1. We say that $\mathbb{F}$ is Archimedean if $\forall 0 < x \in \mathbb{F}$, $\exists n \in \mathbb{N}$ such that $n > x$.
2. We say that $\mathbb{F}$ is Dedekind complete if it satisfies the supremum property.

Facts:

1. $\mathbb{Q}$ is Archimedean.
2. If $\mathbb{F}$ is Dedekind complete, then $\forall 0 < x \in \mathbb{F}$ and $\forall 0 < n \in \mathbb{N}$, $\exists!\ 0 < y \in \mathbb{F}$ such that $y^n = x$.
3. $\mathbb{Q}$ is not Dedekind complete. ($\sqrt{2}$ is a counterexample.)

Theorem 1.2.1
Suppose $\mathbb{F}$ is a Dedekind complete ordered field. Then $\mathbb{F}$ is Archimedean.

Proof. If not, then $\mathbb{N} \subset \mathbb{F}$ is bounded above, and so the supremum property provides $x \in \mathbb{F}$ such that $x = \sup \mathbb{N}$. But then $x - 1$ is an upper bound for $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that $x - 1 < n$. Hence $x < n + 1$, which contradicts the definition of x as an upper bound. Therefore, $\mathbb{F}$ is Archimedean. 😄

1.3 Dedekind Completion

Throughout this section, let $\mathbb{F}$ be an Archimedean ordered field.

Definition 1.3.1: Dedekind cut

We say a set $C \subseteq \mathbb{F}$ is *Dedekind cut* if:

1. $C \neq \emptyset$ and $C \neq \mathbb{F}$.
2. If $p \in C$ and $q \in \mathbb{F}$ such that $q < p$, then $q \in C$.
3. If $p \in C$, then $\exists r \in C$ such that $p < r$.

We will write $\mathbb{F}^*$ for the set of all Dedekind cuts in $\mathbb{F}$. It is called the *Dedekind completion* of $\mathbb{F}$.

Note:

Let $C \subseteq \mathbb{F}$ be a cut. Then:

1. If $p \in C$, then $q \notin C$, then $p < q$.
2. If $r \notin C$, and $r < s \in \mathbb{F}$, then $s \notin C$.

Example 1.3.1 (Cut examples)

1. Let $q \in \mathbb{F}$ and define $C_q = \{p \in \mathbb{F} \mid p < q\}$. Then C_q is a cut.

   Proof. (a) $q - 1 < q \implies q - 1 \in C_q$. $q \not< q \implies q \notin C_q \implies C_q \neq \mathbb{F}$.

   (b) Let $p \in C_q$. Suppose $s \in \mathbb{F}$ such that $s < p$. Then $s < q \implies s \in C_q$.

   (c) Let $p \in C_q$. Then $p < \frac{p+q}{2} < q \implies \frac{p+q}{2} \in C_q$.

   😊

2. Suppose $\mathbb{F}$ is such that $\nexists x \in \mathbb{F}$ such that $x^2 = 2$. Let $C = \{p \in \mathbb{F} \mid p \leqslant 0 \text{ or } 0 < p^2 < 2\}$. Then C is a cut.

   Proof. (a) $1 \in C$ and $1^2 = 1 < 2$. $2 \notin C$ and $2^2 = 4 > 2$.

   (b) Let $p \in C$ and $q \in \mathbb{F}$ such that $q < p$. If $q \leqslant 0$, then $q \in C$ trivially. Suppose $0 < q < p$. Then $0 < q^2 < p^2 < 2$, so $q \in C$.

   (c) Let $p \in C$. If $p \leqslant 0$, then $1 \in C$ and $p < 1$, so we're done. Suppose $0 < p^2 < 2$. Note, $0 < 2 - p^2$, so $\frac{2p+1}{2-p^2} > 0$. Then we can define $r = 1 + \frac{2p+1}{2-p^2} \geqslant \max(1, \frac{2p+1}{2-p^2})$. Then $(p + 1/r)^2 = p^2 + \frac{2p}{r} + \frac{1}{r^2}$. We have:

   $$\begin{aligned} p^2 + \frac{2p}{r} + \frac{1}{r^2} &< p^2 + \frac{2p}{r} + \frac{1}{r} \\ &= p^2 + \frac{2p+1}{r} \\ &\leqslant p^2 + 2 - p^2 \\ &= 2. \end{aligned}$$

   So, $p < p + 1/r < 2$ and $p + 1/r \in C$.

   😊

1.3.1 Ordering $\mathbb{F}^*$

Lenma 1.3.1

The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then exactly one holds:
   - $\mathcal{A} \subset \mathcal{B}$
   - $\mathcal{A} = \mathcal{B}$
   - $\mathcal{B} \subset \mathcal{A}$
2. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$.

Proof. Proof of 2 is trivial, as well as the equality part for 1.

- If $\mathcal{A} = \mathcal{B}$, we're done.
- Suppose $\exists b \in \mathcal{B} \setminus \mathcal{A}$. If $a \in \mathcal{A}$, then $a < b$, but $\mathcal{B}$ is a cut so $a \in \mathcal{B}$, so $\mathcal{A} \subset \mathcal{B}$.
- Suppose $\exists a \in \mathcal{A} \setminus \mathcal{B}$. Then $a < b$ for all $b \in \mathcal{B}$, so $a \in \mathcal{B}$, so $\mathcal{B} \subset \mathcal{A}$.

☺

Definition 1.3.2: Order on cuts

Given $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we say that $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$. The lemma above shows that this is infact an order.

Lenma 1.3.2

Let $E \subseteq \mathbb{F}^*$ be nonempty and bounded above. Then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A}$ is a cut.

Proof.

1. Since $E \neq \emptyset$, there exists $\mathcal{A} \in E$. So $\mathcal{A} \neq \emptyset$, hence $\mathcal{B} \neq \emptyset$.

   Since E is bounded above, there exists $\mathcal{C} \in \mathbb{F}^*$ such that $\mathcal{A} \subset \mathcal{C}$ for all $\mathcal{A} \in E$. Since $\mathcal{C}$ is a cut, there is $q \in \mathbb{F}$ such that $q \notin \mathcal{C}$. Then $q \notin \mathcal{A}$ for all $\mathcal{A} \in E$, so $q \notin \mathcal{B}$.
2. Let $p \in \mathcal{B}$ and $q \in \mathbb{F}$ such that $q < p$. Since $\mathcal{B}$ is a union of cuts, it follows that $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, $q \in \mathcal{A} \subseteq \mathcal{B}$.
3. Let $p \in \mathcal{B}$. Then $p \in \mathcal{A}$ for some $\mathcal{A} \in E$. Since $\mathcal{A}$ is a cut, there exists $r \in \mathcal{A}$ such that $p < r$. Since $\mathcal{A} \subset \mathcal{B}$, we have $r \in \mathcal{B}$.

☺

Theorem 1.3.1

$\mathbb{F}^*$ equipped with the order $<$ satisfies the supremum property.

Proof. Let $E \subseteq \mathbb{F}$ be a nonempty set that is bounded above. From last time, we know that $\mathcal{B} = \cup_{\mathcal{A} \in E} \mathcal{A}$ is a cut. We claim that $\mathcal{B} = \sup E$.

If $\mathcal{A} \in E$, then $\mathcal{A} \subseteq \mathcal{B}$. And so $\mathcal{A} \leqslant \mathcal{B}$, so $\mathcal{B}$ is an upper bound for E.

Next, suppose that $\mathcal{C} \in \mathbb{F}^*$ is an upper bound of E. This means that $\mathcal{A} \leqslant \mathcal{C}$ for every $\mathcal{A} \in E$, meaning $\mathcal{A} \subseteq \mathcal{C} \forall \mathcal{A} \in E$. So $\mathcal{B} \subseteq \mathcal{C}$. As such, $\mathcal{B} \leqslant \mathcal{C}$, so $\mathcal{B} = \sup E$. ☺

Remark: In none of the results leading up to this theorem did we use that $\mathbb{F}$ is anything other than an ordered set. This shows that the cut construction of Dedekind works in general for ordered sets and yields $\mathbb{F}^*$ that satisfies the supremum property. Also, $\{\mathcal{C}_p \mid p \in \mathbb{F}\} \subseteq \mathbb{F}^*$.

1.3.2 Addition

Idea: $\mathbb{F} \cong \{\mathcal{C}_p \mid p \in \mathbb{F}\}$.

Lenma 1.3.3

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. Then $\mathcal{C} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is a cut.

Proof. Claim: $\mathcal{A}, \mathcal{B} \neq \emptyset \implies \mathcal{C} \neq \emptyset$.

$\mathcal{A}, \mathcal{B}$ are cuts, so $\exists M_1, M_2 \in \mathbb{F}$ such that $a < M_1$ for all $a \in \mathcal{A}$ and $b < M_2$ for all $b \in \mathcal{B}$. Then $a + b < M_1 + M_2$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, so $a + b < M_1 + M_2$, meaning $M_1 + M_2 \notin \mathcal{C}$.

Also, let $c = a+b \in \mathcal{C}$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Let $q < c \implies q - a < b \implies q - a \in \mathcal{B}$. Hence, $q = a + (q-a) \in \mathcal{C}$.

Thirdly, let $c = a + b \in \mathcal{C}$ for $a \in \mathcal{A}, b \in \mathcal{B}$. Since $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, $\exists r_a, r_b$ such that $a < r_a \in \mathcal{A}$, $b < r_b \in \mathcal{B}$. Then $c = a + b < r_a + r_b$, so $r_a + r_b \in \mathcal{C}$.

As such, $\mathcal{C}$ is a cut. ☺

Before we define addition, we need to define the negative of a cut.

Heuristic: What we want is that $-\mathcal{C}_1 = \mathcal{C}_{-1}$. The way we do this is by defining $\mathcal{C}_{-p} = \{q \in \mathbb{F} \mid \exists p > q : p \in -\mathcal{C}_p^C\}$. This is the same as $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}_p\}$.

Now we study $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}\}$.

Lenma 1.3.4

Let $\mathcal{C} \in \mathbb{F}^*$. Then $\{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{C}\}$ is a cut.

Definition 1.3.3: Addition

For $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, we define $\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and $-\mathcal{A} = \{q \in \mathbb{F} \mid \exists p > q : -p \notin \mathcal{A}\}$.

Theorem 1.3.2

Define $0 = \mathcal{C}_0 \in \mathbb{F}^*$. The following hold:

1. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} \in \mathbb{F}^*$.
2. $\mathcal{A}, \mathcal{B} \in \mathbb{F}^* \implies \mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$.
3. $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^* \implies (\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$.
4. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + 0 = \mathcal{A}$.
5. $\mathcal{A} \in \mathbb{F}^* \implies \mathcal{A} + (-\mathcal{A}) = 0$.

Proof. Easy proof, too lazy to write out. ☺

Also: $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{F}^*$ and $\mathcal{A} < \mathcal{B} \implies \mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$.

Important Remark: The Archimedean property is actually needed for the above theorem in orer to prove the 5th condition.

1.3.3 Multiplication

Lenma 1.3.5
Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ such that $\mathcal{A}, \mathcal{B} > 0$. Then $C = \{p \in \mathbb{F} \mid p \leqslant 0\} \cup \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\}$ is a cut.

Lenma 1.3.6
Let $\mathcal{A} \in \mathbb{F}^*$ be such that $\mathcal{A} > 0$. Then $C = \{p \in \mathbb{F}^* \mid p \leqslant 0\} \cup \{0 < q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$ is a cut.

Definition 1.3.4: Multiplication

Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$. We define multiplication as:

1. If $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = \{ab \mid 0 < a \in \mathcal{A}, 0 < b \in \mathcal{B}\}$.
2. If $\mathcal{A} = 0$ or $\mathcal{B} = 0$, then $\mathcal{A} \cdot \mathcal{B} = 0$.
3. If $\mathcal{A} > 0$ and $\mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B}))$.
4. If $\mathcal{A} < 0$ and $\mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} = -((-\mathcal{A}) \cdot \mathcal{B})$.
5. If $\mathcal{A}, \mathcal{B} < 0$, then $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B})$.

We define multiplication inversion via:

1. If $\mathcal{A} > 0$, then $\mathcal{A}^{-1} = \{q \in \mathbb{F} \mid \exists p > q : p^{-1} \notin \mathcal{A}\}$.
2. If $\mathcal{A} < 0$, then $\mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$.

Theorem 1.3.3
Set $1 = C_1$. The following hold:

1. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} \in \mathbb{F}^*$.
2. If $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$.
3. If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $(\mathcal{A} \cdot \mathcal{B}) \cdot C = \mathcal{A} \cdot (\mathcal{B} \cdot C)$.
4. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot 1 = \mathcal{A}$.
5. If $\mathcal{A} \in \mathbb{F}^*$, then $\mathcal{A} \cdot \mathcal{A}^{-1} = 1$.

Also if $\mathcal{A}, \mathcal{B} \in \mathbb{F}^*$ and $\mathcal{A}, \mathcal{B} > 0$, then $\mathcal{A} \cdot \mathcal{B} > 0$.

Theorem 1.3.4
If $\mathcal{A}, \mathcal{B}, C \in \mathbb{F}^*$, then $\mathcal{A} \cdot (\mathcal{B} + C) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot C$.

We now know that $\mathbb{F}^*$ is an ordered field.

1.4 Robert Reci

Theorem 1.4.1
$\mathbb{Q}$ is the smallest ordered field.

Proof. Let $\mathbb{F}$ be any ordered field. Let $1 \in \mathbb{F}$. Let $\iota : \mathbb{N} \to \mathbb{F}$, $n \mapsto 1 + \cdots + 1$ n times. Then $\iota(-n) = -\iota(n)$ for $n \in \mathbb{N}_0$ and $-n \in \mathbb{Z}^-$.

Then we say $\iota(p/q) = \iota(p)\iota(q)^{-1}$ for $p/q \in \mathbb{Q}$. ☺

Corollary 1.4.1 Every ordered field is infinite

$\iota[\mathbb{Q}] \subseteq \mathbb{F}$ is infinite.

Roots

Let $\mathbb{F}$ be a Dedekind complete ordered field, $0 < x \in \mathbb{F}$, $n \in \mathbb{N}$. Then $\exists! y \in \mathbb{F}$ such that $y > 0$ and $y^n = x$.

Proof. $n = 1$ is silly. Assume $n \geqslant 2$. Let $E = \{z \in \mathbb{F} \mid z > 0 \text{ and } z^n < x\}$. Then E is nonempty and bounded above by x. Let $y = \sup E$. We claim that $y^n = x$.

We want to show that $y^n \not> > x$ and $y^n \not< < x$.

Lemma 1.4.1

In any commutative ring R, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.

And hence for $0 < a < b$ in $\mathbb{F}$, we have $0 < b^n - a^n = (b-a)nb^{n-1}$.

Suppose $y^n < x$, so $x - y^n > 0$. We define $h = \frac{1}{2}\min\left(1, \frac{x-y^n}{n(y+1)^{n-1}}\right)$. $0 < h < 1$, also $0 < h < \frac{x-y^n}{n(y+1)^{n-1}}$.

Then, by the inequality below the lemma, we have

$$\begin{aligned} 0 &< (y+h)^n - y^n \\ &< hn(y+h)^{n-1} \\ &< hn(y+1)^{n-1} \\ &< x - y^n, \end{aligned}$$

so $(y+h)^n < x$, which contradicts the definition of y as the supremum. ☺

Definition 1.4.1: Ring*

A ring is a field where actually we don't care about inverses anymore.

Definition 1.4.2: Domain

R is a domain when $xy = 0 \implies x = 0 \wedge y = 0$.

Let R be a ring. For $(r,s) \in R \times R \setminus \{0\}$, we say $(r,s) \sim (r',s')$ if $rs' = r's$.

The field of fractions, $\mathrm{Frac}(R)$ is the set of equivalence classes of $R \times R \setminus \{0\}$ under $\sim$ equipped with the operations $[(r,s)] + [(r',s')] = [(rs' + r's, ss')]$ and $[(r,s)] \cdot [(r',s')] = (rr', ss')$.

We check that $[(r,s)] \cdot [(s,r)] = [(rs, sr)] = [(1,1)]$.

Let $\mathbb{F}$ a field, $\mathbb{F}^x$ its polynomial ring. Let $\mathbb{F}(x)$ be the field of fractions of $\mathbb{F}^x$. Then $\mathbb{F}(x) := \mathrm{Frac}(\mathbb{F}^x)$ is the field of rational functions in x with coefficients in $\mathbb{F}$.

Given $p, q \in \mathbb{F}^x$, say $p/q > 0$ if p and q have the same sign. Say $f, g \in \mathbb{F}(x)$, that $f > g$ when $f - g > 0$.

Theorem 1.4.2

$\mathbb{F}(x)$ is never Archimedean.

Proof. x is an upper bound for all $n \in \mathbb{N}$. ☺

Note:

If $\mathbb{F}$ is Archimedean, $|\mathbb{F}| \leqslant 2^{\aleph_0}$.

Theorem 1.4.3
Let λ be an infinite cardinal. Then there is an ordered field of cardinality λ.

Corollary 1.4.2
The Archimedean property is not a first-order property.

1.5 Completeness

Lemma 1.5.1
Suppose $\mathbb{F}$ is an ordered field that is not Dedekind complete. Then $\exists$ and infinite $E \subsetneq \mathbb{F}$ such that:

1. E bounded above, $\emptyset \neq U(E)$ is open, $\emptyset \neq U(E)^C$ is open.
2. $a \in U(E)^C$, $b \in U(E) \implies a < b$.
3. $f : \mathbb{F} \to \mathbb{F}$ with $f(x) = \begin{cases} 1 & x \in U(E) \\ 0 & x \in U(E)^C \end{cases}$ is differentiable with $f' = 0$.

Theorem 1.5.1 Characteristics of Dedekind Completeness
Let $\mathbb{F}$ be an ordered field. The following are equivalent:

1. $\mathbb{F}$ is Dedekind complete.
2. $\mathbb{F}$ has the intermediate value property: If $f : [a,b] \to \mathbb{F}$ is continuous and $\min(f(a), f(b)) < c < \max(f(a), f(b))$, then $\exists x \in [a,b]$ such that $f(x) = c$.
3. $\mathbb{F}$ satisfies the mean value property: If $f : [a,b] \to \mathbb{F}$ is continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.
4. $\mathbb{F}$ satisfies Cauchy mean value property: If $f, g : [a,b] \to \mathbb{F}$ are both continuous and differentiable on (a,b), then $\exists x \in (a,b)$ such that $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
5. $\mathbb{F}$ satisfies the extreme value property: If $f : [a,b] \to \mathbb{F}$ is continuous, then f attains a maximum and minimum on $[a,b]$.

Proof. $1 \implies 2$: Let $f : [a,b] \to \mathbb{F}$ and continuous. WLOG, assume $f(a) < c < f(b)$. Define $E = \{x \in [a,b] \mid f(x) < c\}$. E is nonempty and bounded above by b. Let $x = \sup E$. We claim that $f(x) = c$. Since f is continuous, $\exists \kappa > 0$ such that $f(t) < c \, \forall t \in [a, a+\kappa]$ and $f(t) > c \, \forall t \in [b-\kappa, b]$. So, $a + \frac{\kappa}{2} < x < b - \frac{\kappa}{2}$.

Suppose BWOC $f(x) < c$. Again by continuity, $\exists \delta > 0$ such that $f(t) < c$ for all $t \in B(x, \delta) \subsetneq [a,b]$. Then $x + \frac{\delta}{2} \in E$, contradiction.

Then suppose BWOC $f(x) > c$. Again, $\exists \delta > 0$ such that $f(t) > c$ for all $t \in B(x, \delta) \subsetneq [a,b]$. Then $\exists z \in E$ such that $x - \frac{\delta}{2} < z \leqslant x$ and $f(z) < c$. But then $c < f(z) < c$, contradiction.

So $f(x) = c$ by trichotomy.

$2 \implies 1$: We'll show $\neg 1 \implies \neg 2$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the strange function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then f is continuous on $[a,b]$, $f(a) - < 1 = f(b)$, but there is not $x \in [a,b]$ with $f(x) = \frac{1}{2}$, by construction.

$1 \implies 5$: First we claim that if $\mathbb{F}$ is Dedekind and $f : [a,b] \to \mathbb{F}$ is continuous, then $f([a,b]) \subseteq \mathbb{F}$ is a bounded set. We prove the claim.

Consider $E = \{x \in [a,b] \mid f([a,x]) \text{ is bounded}\}$. $a \in E$ and E is bounded, so we can let $s = \sup E$. Next note that by continuity, if $[c,d] \subseteq [a,b]$ such that $f([c,d])$ is bounded, then $\exists \delta > 0$ such that $f([a,b] \cap [c-\delta, d+\delta])$ is bounded. Using this, deduce in turn that $a < s$, $s = \max E$, and $s = b$.

So now suppose $\mathbb{F}$ is Dedekind complete and let $f : [a,b] \to \mathbb{F}$ be continuous. The claim establishes that $f([a,b]) \subseteq \mathbb{F}$ is a bounded set, so we can let $\begin{cases} \mu = \inf f([a,b]) \\ \lambda = \sup f([a,b]) \end{cases}$. Suppose BWOC that $f(x) < \lambda$ for all $x \in [a,b]$. Then teh function $g : [a,b] \to \mathbb{F}$ defined by $g(x) = \frac{1}{\lambda - f(x)}$ is continuous and positive. So by the claim, there is $k > 0$ such that $g(x) \leqslant k$ for all $x \in [a,b]$. But then

$$\frac{1}{\lambda - f(x)} \leqslant k \implies \frac{1}{k} \leqslant \lambda - f(x) \implies f(x) \leqslant \lambda - \frac{1}{k},$$

for all $x \in [a,b]$. But this contradicts the definition of λ, as we just found a better upper bound.

Therefore, there does exists $M \in [a,b]$ such that $f(M) = \lambda$, which is $\max f([a,b])$.

The min follows from a similar argument.

$5 \implies 4$: Let $f, g : [a,b] \to \mathbb{F}$ be continuous and differentiable on (a,b). Let $h : [a,b] \to \mathbb{F}$ via $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. It suffices to show $\exists x \in (a,b)$ such that $h'(x) = 0$.

By construction, $h(a) = h(b)$. If $h(x) = h(a)$ for all $x \in [a,b]$, then $h' = 0$ and we're done. Suppose then that h is not constant. Then EVT shows that f attains its maximal/minimum values, and at least one must occur at the point $x \in (a,b)$, therefore $h'(x) = 0$.

$4 \implies 3$: Let $g(x) = x$. Done.

$3 \implies 1$. We'll show $\neg 1 \implies \neg 3$. Suppose $\mathbb{F}$ is not Dedekind complete. Then we can let $f : \mathbb{F} \to \mathbb{F}$ be the function from the lemma, and we can pick $a < b$ with $a \in U(E)^C$ and $b \in U(E)$. Then consider the restriction $f : [a,b] \to \mathbb{F}$. Then $1 = 1 - 0 = f(b) - f(a)$. Then, $f'(x)(b-a) = 0 \cdot (b-a) = 0$ for all $x \in \mathbb{F}$. $0 \neq 1$ so $\neg 3$ as desired. 😄

Chapter 2

$\mathbb{R}, \mathbb{C}, \bar{\mathbb{R}}$

Theorem 2.0.1

$\mathbb{R}$ is uncountable.

Proof. $\mathbb{Q} \subseteq \mathbb{R}$, so $\mathbb{R}$ is definitely infinite. Suppose BWOC that there was a bijection $f : \mathbb{N} \to \mathbb{R}$. Set $I_0 = [f(0)+1, f(0)+2]$ and not that $f(0) \notin I_0$. Suppose we are given closed, nested, non-singleton intervals $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_0$ such that $f(k) \notin I_k$ for $0 \leqslant k \leqslant n$. If $f(n+1) \notin I_n$, then set $I_{n+1} = I_n$. Otherwise, set I_{n+1} to some non-singleton closed interval contained in I_n such that $f(n+1) \notin I_{n+1}$.

Since $\mathbb{R}$ is Dedekind complete, we have that $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$. So, there is an x such that $x \in I_n$ for all $n \in \mathbb{N}$. But then $x \neq f(n)$ for all $n \in \mathbb{N}$, contradiction since f is a bijection. 😊

Note:

Upshot: Most of $\mathbb{R}$ is transcendental over $\mathbb{Q}$.

2.1 Extended Reals: $\bar{\mathbb{R}}$

Definition 2.1.1: Extended Reals

$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We endow $\bar{\mathbb{R}}$ with the following order: We write $x < y$ for $x, y \in \bar{\mathbb{R}}$ if:

1. $x, y \in \mathbb{R}$ and $x < y$.
2. $x = -\infty$ and $y \in \bar{\mathbb{R}} \setminus \{-\infty\}$.
3. $x \in \bar{\mathbb{R}} \setminus \{\infty\}$ and $y = \infty$.

Facts:

- $(\bar{\mathbb{R}}, <)$ is an ordered set that satisfies the supremum property.
- All sets in $\bar{\mathbb{R}}$ are bounded above.
- All sets in $\bar{\mathbb{R}}$ admit a sup/inf, i.e.
  - $\sup : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.
  - $\inf : \mathcal{P}(\bar{\mathbb{R}}) \to \bar{\mathbb{R}}$.

  Note: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Also, $A \subseteq B \subseteq \bar{\mathbb{R}}$ implies $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$. And if $E \neq \emptyset$, then $\inf E \leqslant \sup E$.

Note:

$\bar{\mathbb{R}}$ isn't an OF because if it were, then it would be Dedekind complete and then there would exists an ordered field isomorphism $f : \mathbb{R} \to \bar{\mathbb{R}}$ such that $f(x) = \infty$ for some $x \in \mathbb{R}$. but then $f(x+1) = f(x)+f(1) = \infty+1 = \infty$, which is not a true statement.

Definition 2.1.2

We endow $\bar{\mathbb{R}}$ with the following "algebra."

1. If $x \in \mathbb{R}$, we set $x + \infty = \infty + x = \infty$.
2. If $x \in \mathbb{R}$, we set $x + (-\infty) = (-\infty) + x = -\infty$.
3. $\infty + \infty = \infty$.
4. $-\infty + (-\infty) = -\infty$.
5. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = \infty$.
6. If $0 < x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$.
7. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot \infty = \infty \cdot x = -\infty$.
8. If $0 > x \in \bar{\mathbb{R}}$, we set $x \cdot (-\infty) = (-\infty) \cdot x = \infty$.
9. If $x \in \mathbb{R}$, we set $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.
10. $\infty^{-1} = 0 = (-\infty)^{-1}$.
11. If $0 < x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = \infty$.
12. If $0 > x \in \bar{\mathbb{R}}$, we set $\frac{x}{0} = -\infty$.

Forbidden/undefined: $\infty + (-\infty)$, $\infty \cdot 0$, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{\mp\infty}$.

2.1.1 Sequences in $\bar{\mathbb{R}}$

Definition 2.1.3: Sequence

A sequence in $\bar{\mathbb{R}}$ is $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$ for $\ell \in \mathbb{Z}$.

In turn, we define new sequences $\{a_N\}_{N=\ell}^{\infty}, \{b_N\}_{N=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$:

- $a_N = \inf\{x_n \mid n \geqslant N\}$.
- $b_N = \sup\{x_n \mid n \geqslant N\}$.

We then set $\liminf_{n\to\infty} x_n = \sup_{N\geqslant\ell} \inf_{n\geqslant N} x_n = \sup_{N\geqslant\ell} a_N$ and $\limsup_{n\to\infty} x_n = \inf_{N\geqslant\ell} \sup_{n\geqslant N} x_n = \inf_{N\geqslant\ell} b_N$.

Example 2.1.1

Let $x_n = \begin{cases} (-1)^n & n \equiv 0 \mod 2 \\ n & n \equiv 1 \mod 2 \end{cases}$. Then, $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 1$.

Proposition 2.1.1

Let $\{x_n\}_{n=\ell}^{\infty} \subseteq \bar{\mathbb{R}}$. Then $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$.

Proof. Let $M, N \geqslant \ell$ and $K = \max(M, N)$. Then, $\inf_{n>N} x_n \leqslant \inf_{n>K} x_n \leqslant \sup_{n\geqslant K} x_n \leqslant \sup_{n\geqslant M} x_n$.

Thus, $\liminf_{n\to\infty} x_n = \sup_{N\geqslant \ell}\inf_{n\geqslant N} x_n \leqslant \sup_{n\geqslant M} x_n$ for all $M \geqslant \ell$. So, $\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n$. 😄

Proposition 2.1.2

Let $a_n, b_n \in \bar{\mathbb{R}}$ and suppose $\exists K \geqslant \ell$ such that $a_n \leqslant b_n$ for all $n \geqslant K$. Then, $\liminf_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} b_n$ and $\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} b_n$.

Proof. We can claim that if $k \geqslant K$, then

$$\inf\{a_n \mid n \geqslant k\} \leqslant \inf\{b_n \mid n \geqslant k\}$$
$$\sup\{b_n \mid n \geqslant k\} \leqslant \sup\{a_n \mid n \geqslant k\}.$$

Indeed, if $\exists k \geqslant K$ such that $\inf\{a_n \mid n \geqslant k\} > \inf\{b_n \mid n \geqslant k\}$, then $\exists m \geqslant k$ such that $b_m < \inf\{a_n \mid n \geqslant k\} \leqslant a_m \leqslant b_m$, contradiction. Ditto for sup.

Now define for $N \geqslant \ell$, $C_N = \inf_{n\geqslant N} a_n$, $D_N = \inf_{n\geqslant N} b_N$, $E_N = \sup_{n\geqslant N} a_n$, and $F_N = \sup_{n\geqslant N} b_n$.

The above claims show that $N \geqslant K$ then $C_N \leqslant D_N$ and $E_N \leqslant F_N$. Then we iterate to learn:

$$\liminf_{n\to\infty} a_n = \sup_{N\geqslant\ell} C_N \leqslant \sup_{N\geqslant\ell} D_N = \liminf_{n\to\infty} b_n$$
$$\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell} E_N \leqslant \inf_{N\geqslant\ell} F_N = \limsup_{n\to\infty} b_n.$$

😄

Theorem 2.1.1

Suppose $a_n, b_n \in \bar{\mathbb{R}}$. The following hold:

1. If $\limsup_{n\to\infty} a_n < x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n < x$ for all $n \geqslant N$.
2. If $\liminf_{n\to\infty} a_n > x \in \bar{\mathbb{R}}$, then $\exists N \geqslant \ell$ such that $a_n > x$ for all $n \geqslant N$.
3. $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} -a_n$.
4. $\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty} -a_n$.
5. $\limsup_{n\to\infty} a_n + b_n \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided that all arithmetic operations are well-defined.
6. $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leqslant \liminf_{n\to\infty} a_n + b_n$, provided that all arithmetic operations are well-defined.

Proof. 1. Suppose $\limsup_{n\to\infty} a_n = \inf_{N\geqslant\ell}\sup_{n\geqslant N} a_n < x$. This implies that $\exists N \geqslant \ell$ such that $\sup_{n\geqslant N} a_n < x$, meaning $a_n < x$ for all $n \geqslant N$.

2. Similar as above.
3. For any $\emptyset \neq X \subseteq \mathbb{F}$, we have that $-\sup(-X) = \inf X$ and $-\inf(-X) = \sup X$. So the result follows.
4. Same as above.
5. We break into cases:
   (a) $\limsup a_n = \infty$ or $\limsup b_n = \infty$. Then $\limsup a_n + b_n = \infty \geqslant \limsup a_n + \limsup b_n$.
   (b) Suppose either $\limsup a_n = -\infty$ or $\limsup b_n = -\infty$. WLOG consider the first option. Since $\limsup b_n < \infty$, then there eixsts $N_1 \geqslant \ell$ and $K \geqslant \mathbb{R}$ such that $b_n < K$ for $n \geqslant N_1$. Now let $m \in \mathbb{N}$ and note that $-\infty < -m - K$. We can use the first result of the theorem to pick $N_2 \geqslant \ell$ such that $n \geqslant N_2 \implies a_n < -m - K$. Then, if $n \geqslant \max(N_1, N_2)$, we have $a_n + b_n < -m$, so $\limsup a_n + b_n = -\infty \leqslant \limsup a_n + \limsup b_n$.

(c) $\limsup a_n, \limsup b_n \in \mathbb{R}$. Let $\epsilon > 0$, then $\exists N_1, N_2 \geqslant \ell$ such that $n \geqslant N_1 \implies a_n < \limsup a_n + \frac{\epsilon}{2}$ and $n \geqslant N_2 \implies b_n < \limsup b_n + \frac{\epsilon}{2}$. Then, $n \geqslant \max(N_1, N_2) \implies a_n + b_n < \limsup a_n + \limsup b_n + \epsilon$, so $\limsup a_n + b_n \leqslant \limsup a_n + \limsup b_n + \epsilon$ for all ϵ.

6. Same as above.

😁

Lemma 2.1.1

Let $x_n \subseteq \mathbb{R}$. The following are equivalent for $x \in \mathbb{R}$:

1. $x_n \to x$ as $n \to \infty$.
2. $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Proof. Let $\epsilon > 0$. Then $\exists N \geqslant \ell$ such that $n \geqslant N \implies x - \epsilon < x_n < x + \epsilon$. Thus, $x - \epsilon \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant x + \epsilon$ for all $\epsilon > 0$. This implies that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

Now let $\epsilon > 0$. Then by the previous theorem, there exists $N_1, N_2 \geqslant \ell$ such that $\begin{cases} x - \epsilon < x_n & n \geqslant N_1 \\ x_n < x + \epsilon & n \geqslant N_2 \end{cases}$.

Thus, $n \geqslant \max(N_1, N_2) \implies x - \epsilon < x_n < x + \epsilon$, so $x_n \to x$ as $n \to \infty$. 😁

Definition 2.1.4

Let $x_n \in \bar{\mathbb{R}}$ and $x \in \bar{\mathbb{R}}$. We say that $x_n \to x$ as $n \to \infty$ if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x$.

Remarks:

1. The lemma shows this extends the notion of convergence in $\mathbb{R}$.
2. Limits are unique, when they exist.

Example 2.1.2

1. $\lim_{n\to\infty} n = \infty$ ($n \to \infty$ as $n \to \infty$).
2. Version of squeeze lemma
3. TFAE:
   - $x_n \to \infty$ as $n \to \infty$.
   - $\liminf_{n\to\infty} x_n = \infty$.
   - $\forall M \in \mathbb{N}$, there exists $N \geqslant \ell$ such that $n \geqslant N \implies M \leqslant x_N$.

Chapter 3

Metric Spaces

Definition 3.0.1: Metric

Let X be a nonempty set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x, y) \geqslant 0$ for all $x, y \in X$, and $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 3.0.2

A metric space is (X, d) for $X \neq \emptyset$ and d a metric on X.

Example 3.0.1

1. $\mathbb{R}$ with $d(x, y) = |x - y|$.
2. $\mathbb{C}$ with $d(x, y) = |x - y|$.
3. (Discrete Metric) Let $X \neq \emptyset$ be any set. Then $d : X \times X \to \{0, 1\}$ defined by $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ is a metric on X.
4. Let V be a normed metric space with norm $\|\cdot\|$. Then $d(x, y) = \|x - y\|$ is a metric on V.
5. Suppose (Y, d) is a metric space and suppose $f : X \to Y$ is an injection where $X \neq \emptyset$ is a set. Then $\sigma : X \times X \to \mathbb{R}$ defined by $\sigma(x, y) = d(f(x), f(y))$ is a metric on X.

   Proof. We need to show that σ satisfies the three properties of a metric.

   (a) $\sigma(x, y) \geqslant 0$ because $d \geqslant 0$ and $\sigma(x, y) = 0 \iff d(f(x), f(y)) = 0 \iff f(x) = f(y) \iff x = y$.

   (b) The other two are very trivial.

   😄

6. Let Y be a metric space and $\emptyset \neq X \subseteq Y$. Then $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = d_Y(x, y)$ is a metric on X.

7. Consider $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to \mathbb{R}$ with $f(x) = \log x$ and $g(x) = \frac{1}{x}$. Then $d_f(x,y) = \left|\log \frac{x}{y}\right|$ and $d_g(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|x||y|}$ are metrics on $(0,\infty)$.

8. Let V, W be finite dimensional vector spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(V,W) = \{T : V \to W : T \text{ linear}\}$. Then define $\operatorname{rk}(T) = \dim \operatorname{ran} T$ for $T \in L(V,W)$. Note that $\operatorname{ran}(T+S) = \{Tx + Sx \mid x \in \mathbb{F}\} \subseteq \{Tx + Sy \mid x, y \in \mathbb{F}\} = \operatorname{ran} T + \operatorname{ran} S$. Then, $\operatorname{rk}(T+S) \leqslant \operatorname{rk}(T) + \operatorname{rk}(S)$.

   Define $d(T,S) = \operatorname{rk}(T-S) \in \mathbb{N} \subseteq [0,\infty]$.

   - $d(T,S) = 0 \iff \operatorname{rk}(T-S) = 0 \iff T - S = 0$.
   - Has symmetry.
   - Triangle inequality: $d(T-S) = \operatorname{rk}(T - R + R - S) \leqslant \operatorname{rk}(T-R) + \operatorname{rk}(R-S) = d(T,R) + d(R,S)$.

9. Let $f : \bar{R}R \to [-1,1]$ via $f(x) = \begin{cases} 1 & x = \infty \\ -1 & x = -\infty \\ \frac{x}{\sqrt{1+x^2}} & x \in \mathbb{R} \end{cases}$. Then $d(x,y) = |f(x) - f(y)|$ is a metric on $\bar{\mathbb{R}}$.

Definition 3.0.3

Let X be a metric space.

1. For $x \in X$ and $r \geqslant 0$, we define $B(x,r) = \{y \in X \mid d(x,y) < r\}$. And $B[x,r] = \{y \in X \mid d(x,y) \leqslant r\}$.
2. A set $E \subseteq XX$ is bounded if $\exists (R \geqslant 0)$ such that $E \subseteq B(x,R)$ for some $x \in X$.
3. Let Y be any set and $f : Y \to X$. We say f is a bounded function if $f(Y) \subseteq X$ is bounded. We write $\mathcal{B}(Y;X) = \{g : Y \to X \mid g \text{ is bounded}\}$.

Example 3.0.2

1. $f : \mathbb{R} \to \mathbb{C}$ via $f(t) = e^{it} \implies f(t) = 1 \implies f(\mathbb{R}) \subseteq B[0,1]$ is bounded. So, $f \in \mathcal{B}(\mathbb{R};\mathbb{C})$.

2. $f : (0,\infty) \to \mathbb{R}$ via $f(t) = \frac{\log t}{\sqrt{1+(\log t)^2}}$. So, $f \in \mathcal{B}((0,\infty);\mathbb{R})$.

3. Let X be a metric space and Y a nonempty set. Consider $\mathcal{B}(X;Y)$. If $f \in \mathcal{B}(X;Y)$, then $\exists y \in Y$ and $R \geqslant 0$ such that $d(f(x),y) \leqslant R$ for all x. Thus, $\sup_{x\in X} d(f(x),y) := \sup\{d(f(x),y) \mid x \in X\} \in [0,R]$.

   Similarly, if $f, g \in \mathcal{B}(X;Y)$, then exists $R \geqslant 0$ and $y_1, y_2 \in Y$ such that $d(f(x),y_1) \leqslant R$ and $d(g(x),y_2) \leqslant R$ for all $x \in X$. Then, $d(f(x),g(x)) \leqslant d(f(x),y_1) + d(y_1,y_2) + d(y_2,g(x)) \leqslant 2R + d(y_1,y_2) < \infty$ for all $x \in X$. So, $\sup_{x\in X} d(f(x),g(x)) < \infty$. We now define

$$\begin{aligned} d : \mathcal{B}(X;Y) \times \mathcal{B}(X;Y) &\to [0,\infty) \\ (f,g) &\mapsto \sup_{x\in X} d(f(x),g(x)). \end{aligned}$$

   Proof. Consider the properties of a metric:

   - $d(f,g) = 0 \iff \sup_{x\in X} d(f(x),g(x)) = 0 \iff d(f(x),g(x)) = 0 \iff f(x) = g(x)$ for all $x \in X \iff f = g$.
   - Symmetry is trivial.
   - Let $f, g, h \in \mathcal{B}(X;Y)$. Then, $d(f,h) = \sup_{x\in X} d(f(x),h(x)) \leqslant \sup_{x\in X} d(f(x),g(x)) + d(g(x),h(x)) \leqslant d(f,g) + d(g,h)$.

   ☺

Definition 3.0.4

Let X and Y be metric spaces:

1. A map $f : X \to Y$ is an isometric embedding if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Note, such an f is injective.
2. f is an isometry if it's an isometric embedding and surjective.
3. X and Y are isometric if there exists an isometry $f : X \to Y$.

Example 3.0.3

1. Consider $\mathbb{R}^n$ with $|\cdot| = \|\cdot\|_2$, that is, 2-norm.
2. Recall $O(n) = \{\mathcal{M} \in \mathbb{R}^{n\times n} \mid \mathcal{M}^T\mathcal{M} = I\}$ and $R \in O(n) \implies |Rx| = |x|$.
   Let $a \in \mathbb{R}^n$, $R \in O(n)$, and set $f : \mathbb{R}^n \to \mathbb{R}^n$ via $f(x) = a + Rx$. Then,
   $$|f(x) - f(y)| = |a + Rx - (a + Ry)| = |Rx - Ry| = |R(x - y)|.$$
   Also, $y = f(x) = a + Rx \iff y - a = Rx$. So, f is an isometry.
3. Consider $x \mapsto ix \in \mathbb{C}$ for $x \in \mathbb{R}$. This is an isometric embedding but obviously not an isometry for it is not surjective.

The next example is so important that we call it a theorem. Recall $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{R})$ for $X \neq \emptyset$ is a set. Note that if V is a normed vector space, then $\mathcal{B}(X;V)$ is too: $\|f\|_{\mathcal{B}} = \sup_{x\in X}\|f(x)\|_V$ is a norm (exercise) and $d_{\mathcal{B}}(f, g) = \|f - g\|_{\mathcal{B}}$.

Theorem 3.0.1

Let X be a metric space and fix an arbitrary element $a \in X$. For $x \in X$, we'll define $\varphi_x : X \to \mathbb{R}$ via $\varphi_x(y) = d(x, y) - d(y, a)$. The following hold:

1. $\varphi_x \in \mathcal{B}(X)$ for all $x \in X$.
2. Define $\Phi : X \to \mathcal{B}(X)$ via $\Phi(x) = \varphi_x$. Then, Φ is an isometric embedding.

Proof. First note, $|\varphi_x(y)| = |d(x, y) - d(y, a)| \leqslant d(x, a)$ by the triangle inequality. So, $\|\varphi_X\|_{\mathcal{B}} = \sup_{y\in X}|\varphi_x(y)| \leqslant d(x, a) < \infty$. This shows the first result.

Next, fix $x, z \in X$ and consider $\varphi_x(y) - \varphi_z(y) = d(x, y) - d(y, a) - d(z, y) + d(y, a)$. So,

$$|\varphi_x(y) - \varphi_z(y)| = |d(x, y) - d(y, z)| \leqslant d(x, z).$$

Thus, $d_{\mathcal{B}}(\varphi_x, \varphi_y) = \|\varphi_x - \varphi_y\|_{\mathcal{B}} = \sup_{y\in X}|\varphi_x(y) - \varphi_z(y)| \leqslant d(x, z)$.

On the other hand, $|\varphi_x(z) - \varphi_z(z)| = |d(x, z) - \cancel{d(z, z)}^{0}| = d(x, z)$. So, $d_{\mathcal{B}}(\varphi_x, \varphi_z) = d(x, z)$. ☺

Chapter 4

Basic Metric Space Topology

FILL IN LATER

Proposition 4.0.1

Let $Y_1, \dots, Y_n$ be metric spaces and consider $Y = \prod_{i=1}^n Y_i$, endowed with a p-metric from Homework 3. That is,

$$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^n d_{Y_i}^p(x_i,y_i)\right)^{1/p} & 1 \leqslant p < \infty \\ \max_{1\leqslant i \leqslant n} d_{Y_i}^p(x_i,y_i) & p = \infty \end{cases}.$$

Suppose $\{y_k\}_{k=\ell}^\infty \subseteq Y$ is given by $y_k = (y_{k,1}, \dots, y_{k,n})$. The following hold:

1. Let $y = (y_1, \dots, y_n) \in Y$. Then $y_k \longrightarrow y$ in Y as $n \to \infty \iff y_{k,i} \longrightarrow y_i$ in Y_i as $k \to \infty$ for all $1 \leqslant i \leqslant n$.

2. $\{y_k\}_{k=\ell}^\infty$ is Cauchy in Y if and only if $\{y_{k,i}\}_{k=\ell}^\infty$ is Cauchy in Y_i for all $1 \leqslant i \leqslant n$.

Proof. We'll only prove 1. as 2. is very similar. Suppose $y_k \longrightarrow y$ as $k \longrightarrow \infty$. Note that for $1 \leqslant i \leqslant n$, $d_i(y_{k,i}, y_i) \leqslant d_Y(y_k, y)$. Thus, for $\epsilon > 0$, we pick $K \geqslant \ell$ such that if $k \geqslant K$, then $d_Y(y_k, y) \leqslant \epsilon$. But then $k \geqslant K \implies d_i(y_{k,i}) \leqslant d_Y(y_k, y) \leqslant \epsilon$ for all $1 \leqslant i \leqslant n$, meaning $y_{k,i} \longrightarrow y_i$ as $k \longrightarrow \infty$ for $1 \leqslant i \leqslant n$.

Now suppose $y_{k,i} \longrightarrow y_i$ as $k \longrightarrow \infty$ for all $1 \leqslant i \leqslant n$. Let $\epsilon > 0$ and pick $K_i \geqslant \ell$ such that $k \geqslant K_i \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$. Let $K = \max K_i \geqslant \ell$, and note $k \geqslant K \implies d_i(y_{k,i}, y_i) < \frac{\epsilon}{n^{1/p}}$ for all $1 \leqslant i \leqslant n$. This means

$$\begin{cases} \left(\sum_{i=1}^n d_i^p(y_{k,i}, y_i)\right)^{1/p} \leqslant \left(\sum_{i=1}^n \frac{\epsilon^p}{n}\right)^{1/p} = \epsilon & 1 \leqslant p < \infty \\ \max_i d_i(y_{k,i}, y_i) < \epsilon & p = \infty \end{cases}$$

So, $y_k \longrightarrow y$ as $k \longrightarrow \infty$. 😆

Definition 4.0.1

Let $X \neq \emptyset$ be a set and d_1, d_2 be metrics on X. We say d_1 and d_2 are equivalent if $\exists c_1, c_2 > 0$ such that $c_1 d_1(x,y) \leqslant d_2(x,y) \leqslant c_2 d_1(x,y)$ for all $x, y \in X$.

The point is that equivalent metrics give the same notions of convergence, Cauchyness, and boundedness.

Example 4.0.1 (Equivalent Norms)

1. All norms on $\mathbb{F}^n$ are equivalent.
2. From recitation, $\|\cdot\|_p$ are all equivalent on $\mathbb{F}^n$ for $1 \leqslant p \leqslant \infty$.
3. Let $Y_1, \ldots, Y_n$ be metric spaces and form $Y = \prod_{i=1}^n Y_i$. Then

$$d_p(x,y) = \|(d_1(x,y), \ldots, d_n(x,y))\|_p \asymp \|(d_1(x,y), \ldots, d_n(x,y))\|_q = d_q(x,y)$$

   Therefore, $d_p \asymp d_q$ in Y.

   Note: This does not mean all metrics on Y are equivalent.

Example 4.0.2

Let $V_1, \ldots, V_n, W$ be normed vector sapces over $\mathbb{F}$. We define $\mathcal{L}(V_1, \ldots, V_n; W)$ is the set of $\{T \in L(V_1, \ldots, V_n; W) \mid \|T\|_{\mathcal{L}} < \infty\}$ where $\|T\|_{\mathcal{L}} := \sup\{\|T(v_1, \ldots, v_n)\|_W \mid v_i \in V_i : \|v_i\|_{V_i} < 1\} \in [0, \infty]$. Facts:

1. This is indeed a norm.
2. $T \in \mathcal{L} \iff \|T(v_1, \ldots, v_n)\|_W \leqslant c \prod_{i=1}^n \|v_i\|_{V_i}$ for all $v_i \in V_i$ for some $0 \leqslant c < \infty$. $c = \|T\|_{\mathcal{L}}$ is the best constant.

Theorem 4.0.1 Algebra of Sequences

Let $V_1, \ldots, V_n, W$ be normed vector spaces over a common field $\mathbb{F}$. The following hold:

1. Let $\{v_{k,i}\}_{k=\ell}^\infty \subseteq V_i$ for $1 \leqslant i \leqslant n$ be such that $v_{k,i} \to v_i$ in V_i as $k \to \infty$. Let $\{T_k\}_{k=\ell}^\infty \subseteq \mathcal{L}(V_1, \ldots, V_n; W)$ be such that $T_k \to T$ as $k \to \infty$. Then $T_k(v_{k,1}, \ldots, v_{k,n}) \to T(v_1, \ldots, v_n)$ in W as $k \to \infty$.
2. If $\{u_k\}, \{v_k\} \subseteq V_1$ are such that $u_k \to u$, $v_k \to v$ then $u_k + v_k \to u + v$ as $k \to \infty$.

Proof. We'll only do 1 because 2 is easy. We start with $n = 2$ for simplicity. Suppose $\{x_k\} \subseteq V_1$, $\{y_k\} \subseteq V_w$ such taht $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Then let $\sup_{k \geqslant \ell} \max\{\|x_k\|_{V_1}, \|y_k\|_{V_2}, \|T_k\|_{\mathcal{L}}\} = M < \infty$. Then,

$$T_k(x_k, y_k) - T(x,y) = T_k(x_k, y_k - y) + T_k(x_k, y) - T(x,y)$$
$$T_k(x_k, y_k + y) + T(x_k - x, y) + T_k(x,y) - T(x,y).$$

This shows that

$$\|T_k(x_k, y_k) - T(x,y)\|_W \leqslant \|T_k\|_{\mathcal{L}}\|x_k\|_{V_1}\|y - y_k\|_{V_2} + \|T_k\|_{\mathcal{L}}\|x - x_k\|_{V_1}\|y_k\|_{V_2} + \|T - T_k\|_{\mathcal{L}}\|x_k\|_{V_1}\|$$
$$\leqslant M^2\|y - y_k\|_{V_2} + M^2\|x - x_k\|_{V_1} + M^2\|T - T_k\|_{\mathcal{L}} \to 0$$

as $k \to \infty$. 😄

Definition 4.0.2

1. We say a metric space X is complete if every Cauchy sequence in X is convergent in X.
2. We say a normed vector space is Banach if it's complete.
3. We say an inner product space is a Hilbert space if it's Banach.

Example 4.0.3

1. $(\mathbb{R}, |\cdot|)$ is complete.
2. $X = \prod_{i=1}^n X_i$ with p-metric is complete if and only if each X_i is complete. In particular, $(\mathbb{R}^n, \|\cdot\|)$ is complete.
3. $\mathbb{F}^n$ is complete with any more.
4. $\mathbb{R} \setminus \{0\}$ is not complete with $|\cdot|$ as the metric.
5. $\mathbb{Q}^n$ with $|\cdot|$ is not complete.

Example 4.0.4

1. V is a finite dimensional normed vector spaces. $\varphi : \mathbb{F}^n \to V$ isomorphism. Then $\mathbb{F}^n \ni x \mapsto \|\varphi(x)\|_V \in [0,\infty)$ defines a norm on $\mathbb{F}^N$, which we call $|||x|||$. Then $(\mathbb{F}^n, |||\cdot|||)$ is isometric to $(V, \|\cdot\|_V)$, and hence V is complete.
2. Let $\emptyset \neq X$ be a set endowed with the discrete metric. Suppose $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy and pick $N \geqslant \ell$ such that $n, m \geqslant N \implies d(x_n, x_m) < 1$. Then $x_n = x_m = x_N$. So $x_n \to x_N$ as $n \to \infty$. Therefore X is complete.

Note that $Y = \prod Y_i$ is complete iff each individual Y_i is complete.

Theorem 4.0.2

Let $V_1, \ldots, V_k, W$ be normed vector spaces over $\mathbb{F}$. If W is Banach, then so is $\mathcal{L}(V_1, \ldots, V_k)$.

Proof. Suppose $\{T_n\}_{n=\ell}^\infty \subseteq \mathcal{L}(V_1, \ldots, V_k; W$ is Cauchy. For fixed $v_1, \ldots, v_k) \in \prod_{i=1}^k V_i$, we bound

$$\|T_n(v_1, \ldots, v_k) - T_m(v_1, \ldots, v_k)\|_W \leqslant \|T_n - T_m\|_{\mathcal{L}} \prod_{i=1}^k \|v_i\|_{V_i}.$$

Therefore, $\{T_n(v_1, \ldots, v_k)\}_{n=\ell}^\infty \subseteq W$ is Cauchy and hence convergent. We may thus define $T : V_1 \times \cdots \times V_k \to W$ via $T(v_1, \ldots, v_k) = \lim_{n\to\infty} T_n(v_1, \ldots, v_k)$.

1. $T \in L(V_1, \ldots, V_k; W)$:

$$T_n(\alpha x + \beta y, v_2, \ldots, v_k) = \alpha T_n(x, v_2, \ldots, v_k) + \beta T_n(y, v_2, \ldots, v_k)$$

   As $n \to \infty$, we get:

$$T(\alpha x + \beta y, v_2, \ldots, v_k) = \alpha T(x, v_2, \ldots, v_k) + \beta T(y, v_2, \ldots, v_k).$$

   Repeat in other slots if $k \geqslant 2$. As such, it is multilinear.

2. $T \in \mathcal{L}(V_1, \ldots, V_k; W)$: Fix $v_i \in V_i$ with $\|v_i\|_{V_i} \leqslant 1$. Then

$$\begin{aligned}\|T(v_1, \ldots, v_k)\|_W &= \lim_{n\to\infty} \|T_n(v_1, \ldots, v_k)\|_W \\ &\leqslant \left(\limsup_{n\to\infty} \|T_n\|_{\mathcal{L}}\right) \prod_{n=1}^\infty \|v_i\|_{V_i} \leqslant \limsup_{n\to\infty} \|T_n\|_{\mathcal{L}} < \infty.\end{aligned}$$

3. $T_n \to T$ in $\mathcal{L}$ as $n \to \infty$: Let $\epsilon > 0$ and pick $N \geqslant \ell$ such that $n, m \geqslant N \implies \|T_n - T_m\|_{\mathcal{L}} < \frac{\epsilon}{2}$. Then let $v_i \in V_i$ with $\|v_i\|_{V_i} \leqslant 1$. Then,

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W = \lim_{m\to\infty} \|T_m(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leqslant \lim_{m\to\infty} \|T_m - T_n\|_{\mathcal{L}} < \frac{\epsilon}{2}.$$

But this implies

$$\|T(v_1, \dots, v_k) - T_n(v_1, \dots, v_k)\|_W \leqslant \frac{\epsilon}{2}.$$

By taking the supremum, we get that $\|T - T_n\|_{\mathcal{L}} \leqslant \frac{\epsilon}{2} < \epsilon$.

😊

Corollary 4.0.1

$V^* = \mathcal{L}(V; \mathbb{F})$ is always Banach.

Definition 4.0.3

Let X be a metric space, $E \subseteq X$.

1. $x \in E$ is an interior point if $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq E$. $E^\circ = \{x \in E \mid x \text{ is an interior point}\}$. E is open iff $E = E^\circ$. E is closed iff E^c is open.
2. $x \in X$ is a boundary point of E if $\forall \epsilon > 0$, $B(x, \epsilon) \cap E \neq \emptyset$ and $B(x, \epsilon) \cap E^c = \emptyset$. We write $\partial E = \{x \in X \mid x \text{ is a boundary point of } E\}$. $\bar{E} = E^\circ \cup \partial E$.
3. We say $x \in X$ is a limit point (accumulation point) of E if $\forall \epsilon > 0$ $(B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$. We write $E' = \{x \in X \mid x \text{ is a limit point of } E\}$. If $x \in E \setminus E'$, then x is an isolated point.

Example 4.0.5

Let $(X, disc)$ be given. Claim: all subsets of X are both open and closed.

Proof. $B(x, 1) = \{x\} \implies E \subseteq X$ can be written as

$$E = \cup_{x\in E} B(x, 1),$$

which is open. Therefore $E = (E^c)^c$ is also closed. 😊

Any metric space in which all sets are open and closed is called a discrete space.

Theorem 4.0.3

Let X be a metric space and $C \subseteq X$. The following are equivalent:

1. C is closed.
2. C is sequentially closed; If $\{x_n\}_{n=\ell}^\infty \subseteq C$ is such that $x_n \to x$ in X as $n \to \infty$, then $x \in C$.

Proof. $1 \to 2$. Let $\{x_n\} \subseteq C$ be such that $x_n \to x \in X$. Suppose BWOC that $x \in C^c$, which is open. Then $\exists N \geqslant \ell$ such that $n \geqslant N \implies x_n \in C^c \cup C$, which is a contradiction.

$2 \to 1$. BWOC, suppose that C is not closed, which emans C^c is not open. Then $\exists x \in C^c$ such that we can pick $\{x_n\}_{n=0}^\infty \subseteq C$ such that $x_n \in B(x, 2^{-n}) \cap C$. This means that $\{x_n\}_{n=0}^\infty \subseteq C$ and $x_n \to x$ as $n \to \infty$. But $x \notin C$, so we have a contradiction. 😊

Corollary 4.0.2

Let X be a complete metric space, and $\emptyset \neq C \subseteq X$. Then C is closed in X iff C is a complete metric space with the metric from X.

Proof. $\implies$: Let $\{x_n\}_{n=\ell}^{\infty} \subseteq C$ be Cauchy. Then $x_n \to x \in X$ as $n \to \infty$ because X is complete. By since C is closed, $x \in C$.

$\impliedby$: Let $\{x_n\} \subseteq C$ be such that $x_n \to x$ in X as $n \to \infty$. Then $\{x_n\}$ is cauchy in C, meaning it's convergent in C, so $x \in C$, so C is sequentially closed. ☺

Definition 4.0.4

Let X be a metric space and $A \subseteq B \subseteq X$. We say A is dense in B if $\forall b \in B$, $\exists \{a_n\} \subseteq A$ such that $a_n \to b$ as $n \to \infty$.

Example 4.0.6

1. $\mathbb{Q}$ is dense in $\mathbb{R}$. $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. $(\mathbb{Q}^n + i\mathbb{Q}^n) \subseteq \mathbb{C}^n$ is dense.
2. $B(x, r) \subseteq \mathbb{R}^n$ is dense in $B[x, r]$.
3. Let X be given the discrete metric. $B(x, 1) = \{x\}$, but $B[x, 1] = X$, so as long as $X \neq \{x\}$, we do not have $B(x, 1) \subseteq B[x, 1]$ is dense.

Proposition 4.0.2

Let X be a metrid space, $A \subseteq B \subseteq X$. The following are equivalent:

1. A is dense in B.
2. $B \subseteq \bar{A}$.
3. $\forall x \in B$ and $\epsilon > 0$, $\exists a \in A$ such that $d(x, a) < \epsilon$.
4. $\forall x \in B$ and $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$.

Proof. Recall $\bar{A} = A \cup A'$.

$1 \implies 2$. Let $b \in B$. If $b \in A$, we're done. Otherwise $b \notin A$, but by density $\exists \{a_n\}_{n=\ell}^{\infty} \subseteq A \setminus \{b\}$ such that $a_n \to b$ as $n \to \infty$. Thus, $b \in A'$.

$2 \implies 1$. Suppose $B \subseteq A \cup A' = \bar{A}$. Let $b \in B$. If $b \in A$, let $\{a\}_{n=\ell}^{\infty} = b$ then we're done.

So suppose $b \in A' \setminus A$. By definition of limit point, we can pick a sequence $\{a_n\}$ such that $a_n \to b$ as $n \to \infty$. So A is dense in B.

$3 \iff 4$ is trivial.

$2 \iff 3$. Again, use $\bar{A} = A \cup A'$. ☺

Corollary 4.0.3

Let X be a metric space and $A \subseteq B \subseteq X$. If A is dense in B, then A is also dense in $\bar{B}$.

Proof. $A \subseteq B$ is dense $\implies A \subseteq B \subseteq \bar{A}$. So $\bar{B} \subseteq \bar{A}$, meaning A is dense is $\bar{B}$ as desired. ☺

Definition 4.0.5

Let X be a metric space. We say X is separable if X has a countable dense subset.

Example 4.0.7 (Separable Vector Spaces)

1. $\mathbb{R}^n$ is separable, ditto for $\mathbb{C}^n$.
2. Let V be a finite dimensional normed vector space. Let $\varphi : \mathbb{F}^n \to V$ be an isomorphism. Endow $\mathbb{F}^n$ with norm $|||x||| = \|\varphi(x)\|_V$, which is equivalent to $|\cdot|$ on $\mathbb{F}^n$. Then V is separable with $\varphi(\mathbb{Q}^n)$ as a countable dense subset.
3. $\ell^\infty(\mathbb{N};\mathbb{F})$ is not separable, but $\ell^p(\mathbb{N};\mathbb{F})$ is for $1 \leqslant p < \infty$.

Definition 4.0.6

Let X, X^* be metric spaces. We say that X^* completes X if:

1. X^* is complete.
2. $\exists f : X \to X^*$ an isometric embedding.
3. $f(x) \subseteq X^*$ is dense.

Theorem 4.0.4 Uniqueness of completions

Let X, Y, Z be metric spaces. Suppose Y and Z both complete X. Then Y and Z are isometric.

Proof. Let $g : X \to Y$ and $h : X \to Z$ be isometric embeddings. We will construct an isometric $f : Y \to Z$. Let $y \in Y$. Since $g(X) \subseteq Y$ is dense, $\exists \{y_n\}_{n=\ell}^\infty \subseteq g(X)$ such that $y_n \to y$ as $n \to \infty$.

Then $\exists! \{x_n\}_{n=\ell}^\infty \subseteq X$ such that $g(x_n) = y_n$ for all $n \geqslant \ell$. Then upon setting $z_n = h(x_n) = h \circ g^{-1}(y_n)$, we have

$$d_Z(z_n, z_m) = d_X(x_n, x_m) = d_Y(y_n, y_m).$$

This means $\{z_n\}$ is Cauchy, and therefore convergent as Z is complete.

Suppose $\{y_n'\}_{n=\ell}^\infty$ is another sequence such that $y_n' \to y$ as $n \to \infty$. Note

$$d_Y(y_n, y_n') = d_X(g^{-1}(y_n), g^{-1}(y_n')) = d_Z(h(g^{-1}(y_n)), h(g^{-1}(y_n'))) = d_Z(z_n, z_n').$$

Therefore, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n'$. So, we can define $f : Y \to Z$ as $f(y) = \lim_{n\to\infty} h(g^{-1}(y_n))$ for any sequence $\{y_n\} \subseteq g(X)$ such that $y_n \to y$ as $n \to \infty$.

We claim that f is an isometric embedding. Let $y, y' \in Y$ and pick $\{y_n\}_{n=\ell}^\infty$ and $\{y_n'\}_{n=\ell}^\infty$ such that $y_n \to y$ and $y_n' \to y'$ as $n \to \infty$. Then,

$$d_Y(y_n, y_n') = d_X(g^{-1}(y_n), g^{-1}(y_n')) = d_Z(h(g^{-1}(y_n)), h(g^{-1}(y_n'))) \to d_Z(f(y), f(y')) = d_Y(y, y'),$$

so f is an isometric embedding.

We claim that f is surjective. Let $z \in Z$ and pick $\{x_n\}_{n=\ell}^\infty$ such that $h(x_n) = z_n \to z$ as $n \to \infty$. Then let $y_n = g(x_n)$. Then $\{y_n\}_{n=\ell}^\infty \subseteq Y$ are Cauchy and hence convergent to $y \in Y$. Then $f(y) = \lim_{n\to\infty} h \circ g^{-1}(y_n) = \lim_{n\to\infty} z_n = z$. So $f : Y \to Z$ is an isometry! 😊

Note:

This is analogous to the uniqueness of Dedekind complete ordered fields. In principal, there can be different techniques for finding /constructing completions of a given metric space, but in the end they're isometric.

Theorem 4.0.5

Let $X \neq \emptyset$ be a set and Y be a metric space. Then $\mathcal{B}(X;Y)$ is complete if and only if Y is complete.

Proof. HW5 ☺

Corollary 4.0.4

Let $X \neq \emptyset$ be a set. Then $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{R})$ is a Banach space.

Proof. $\mathbb{R}$ is complete. ☺

Theorem 4.0.6

Let X be a metric space. Then X has a completion.

Proof. Let $\Phi : X \longrightarrow \mathcal{B}(X)$ be the isometric embedding we previously constructed. Let $X^* = \overline{\Phi(X)}$, which is closed in (B) and hence a complete metric space. By construction, $\Phi(X)$ is dense in X^*. So, X^* is complete. ☺

Remarks:

1. Why not just set $\mathbb{R} = \bar{\mathbb{Q}}$? It's cyclic!
2. $\exists$ another construction of X^* which is more "direct" and proceeds through Cauchy(X) from HW4. This idea has room to play. It can be hacked to yield an alternate construction of $\bar{\mathbb{R}}$ from $\mathbb{Q}$ or any other Archimedean ordered field.

4.1 Limits and Continuity

Definition 4.1.1

Let X, Y be metric spaces, $E \subseteq X$, $z \in E'$, $f : E \longrightarrow Y$. We say that f has limit $y \in Y$ as $x \longrightarrow z$, written as $f(x) \longrightarrow y$ as $x \longrightarrow z$ or $\lim_{x\to z} f(x) = y$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that $x \in E$ and $0 < d_X(x,z) < \delta \implies d_Y(f(x), y) < \epsilon$.

Remarks:

1. limits are unique when they exist
2. the definition only requires $z \in E'$, not $z \in E$. that is, $f(z)$ doesn't need to be defined and even if it is, the defintion doesn't care what it is.

Theorem 4.1.1 Sequential characterization of limits

Let X, Y be metric spaces, $E \subseteq X$, $f : E \to Y$, $z \in E'$, $y \in Y$. The following are equivalent:

1. $f(x) \to y$ as $x \to z$.

2. $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(B(z, \delta) \setminus \{z\}) \subseteq B_Y(y, \epsilon)$.

3. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{x\}$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to y$ as $n \to \infty$.

Proof. 1 $\iff$ 2 is a triviality. Now we show 1 $\implies$ 3. Let $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ be such that $x_n \to z$ as $n \to \infty$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $0 < d_X(x, z) < \delta \implies d_Y(f(x), y) < \epsilon$. Pick $N \geqslant \ell$ such that $n \geqslant N$ implies $0 < d_X(x_n, z) < \delta$. So, $d_Y(f(x_n), y) < \epsilon$. Therefore, $f(x_n) \to y$ as $n \to \infty$.

Now for 3 $\implies$ 1. Suppose BWOC $\neg 1$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in E$ such that $0 < d(x, z) < \delta$, $d(f(x), y) \geqslant \epsilon$.

For $\delta = 2^{-n}$, $n \in \mathbb{N}$, we then get $\{x_n\}_{n=0}^{\infty} \subseteq E \setminus \{z\}$ such that $d(x_n, z) < 2^{-n}$, but $d(f(x_n), y) \geqslant \epsilon$. Now we use 3: $x_n \to z$ as $n \to \infty$, so $f(x_n) \to y$ as $n \to \infty$. In particular, $\exists N \geqslant 0$ such that $n \geqslant N \implies d(f(x_n), y) < \epsilon$. This is a contradiction. 😄

Theorem 4.1.2 Limits and components

Let $X, Y_1, \ldots, Y_n$ be metric spaces, and let $Y = \prod Y_i$ endowed with usual p-metric. Let $E \subseteq X$, $z \in E'$, $f : E \to Y$. Write $f = (f_1, \ldots, f_n)$ where $f_i : E \to Y_i$. The following are equivalent for $y = (y_1, \ldots, y_n) \in Y$:

1. $f(x) \to y$ as $x \to z$.

2. $f_i(x) \to y_i$ as $x \to z$ for $1 \leqslant i \leqslant n$.

Proof. This follows from the sequential characterization of limits combined with the characterization of limits of sequences in the product space Y. 😄

Theorem 4.1.3 Algebra of limits

Let X be a metric space, $E \subseteq ZX$, $z \in E'$. The following hold:

1. Let V be a normed vector space and suppose $f, g : E \to V$, $\alpha : E \to \mathbb{F}$ are such that $f(x) \to v_1$, $g(x) \to v_2$, and $\alpha(x) \to \beta$ as $x \to z$. Then:

   (a) $f(x) + g(x) \to v_1 + v_2$ as $x \to z$.

   (b) $\alpha(x) f(x) \to \beta v_1$ as $x \to z$.

2. Let $V_1, \ldots, V_k, W$ be normed vector spaces over $\mathbb{F}$. Suppose $f_i : E \to V_i$ and $T : E \to \mathcal{L}(V_1, \ldots, V_k; W)$ are such that $f_i(x) \to v_i$ as $x \to z$ and $T(x) \to M$ as $x \to z$. Then,

$$E \ni x \mapsto T(x)(f_1(x), \ldots, f_k(x)) \in W$$

   satisfies $T(x)(f_1(x), \ldots, f_k(x)) \to M(v_1, \ldots, v_k)$ as $x \to z$.

Proof. Use characterization of limits via sequences together with algebra of sequential limits. 😄

Definition 4.1.2

Let X, Y be metric spaces, $E \subseteq X$, $z \in E$, and $f : E \to Y$. We say f is continuous at z if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in E$ and $d(x, z) < \delta \implies d(f(x), f(z)) < \epsilon$. We say f is continuous on E if f is continuous at every point of E.

Remarks:

1. If z is isolated, i.e. $z \in E \setminus E'$, then the definition of continuity is true vacuously and so f is continuous as z.
2. Unlike when computing limits, we need $f(z)$ defined, and $x = z$ is allowed.
3. We can think of $f : E \to Y$ with E a metric space on its own with $d_E = d_X$.

Theorem 4.1.4 Characterizations of continuity

Let X, Y be metric spaces and $z \in E \subseteq X$ and $f : E \to Y$. The following are equivalent:

1. f is continuous at z.
2. $\forall \epsilon > 0$, $\exists \delta > 0$ such $f(E \cap B(z, \delta)) \subseteq B(f(z), \epsilon)$.
3. If $z \in E'$, then $f(x) \to f(z)$ as $x \to z$.
4. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to f(z)$ as $n \to \infty$.
5. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ is such that $x_n \to z$ as $n \to \infty$, then $f(x_n) \to f(z)$ as $n \to \infty$.
6. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ is such that $x_n \to z$ as $n \to \infty$, then $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is convergent.

Proof. 1 $\iff$ 2 is obvious as well as 3 $\iff$ 4 since we proved it in the sequential chracterization of limits.

We'll prove 1 $\implies$ 5 $\implies$ 6 $\implies$ 4 and 3 $\implies$ 1.

3 $\implies$ 1: If $z \in E \setminus E'$, we're done because of earlier remark. So let $z \in E \setminus E'$. Then 3 is in play: $f(x) \to f(z)$ as $x \to z$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $0 < d(x, z) < \delta \implies d(f(x), f(z)) < \epsilon$.

Note, $x = z \iff d(x, z) = 0$, in which case $d(f(x), f(z)) = 0 < \epsilon$. So f is continuous at z.

1 $\implies$ 5: Suppose f is continuous at z and let $\{x_n\} \subseteq E$ be such that $x_n \to z$ as $n \to \infty$. Let $\epsilon > 0$ and pick $\delta > 0$ such that $x \in E$ and $d(x, z) < \delta \implies d(f(x_n), f(z)) < \epsilon$. Pick $N \geqslant \ell$ such that $n \geqslant N$ implies $d(x_n, z) < \delta \implies d(f(x_n), f(z)) < \epsilon$. So $f(x_n) \to f(z)$ as $n \to \infty$.

5 $\implies$ 6. Trivial

6 $\implies$ 4. Let $\{x_n\} \subseteq E \setminus \{z\}$ be such that $x_n \to z$ as $n \to \infty$. Define $\{y_n\} \subseteq E$ via

$$y_n = \begin{cases} x_n & n = \ell + 2k \\ z & n = \ell + 2k + 1 \end{cases}.$$

Then $y_n \to z$ as $n \to \infty$. 6 implies that $f(y_n)$ converges. So we can pick a subsequence to show that is converges to $f(z)$. 😆

Corollary 4.1.1 Corallary 1

Let X, Y be metric spaces, $f : X \to Y$. f is continuous if and only if if $\{x_n\} \subseteq X$ is convergent, then $\{f(x_n)\} \subseteq Y$ is convergent.

Corollary 4.1.2 Corallary 2

Let X, Y be metric spaces with X separable. Let $f : X \to Y$ be continuous. Then $f(X) \subseteq Y$ is separable.

Theorem 4.1.5 Continuity and products

Let $X, Y_1, \ldots, Y_k$ be metric spaces and let $f : X \to Y := \prod Y_i$. Let $z \in X$. Write $f = (f_1, \ldots, f_k)$ where $f_i : X \to Y_i$. The following are equivalent:

1. f is continuous at z.
2. Each f_i is continuous at z.

Proof. Proof direct from limit chracterization. ☺

Theorem 4.1.6 Algebra of continuity

Sum, product, and multilinear functions of continuous functions are continuous.

Theorem 4.1.7 Composition

Let X, Y, Z be metric spaces and $f : X \to Y$ and $g : Y \to Z$. Suppose f is continuous at $z \in X$ and g is continuous at $f(z)$. Then $g \circ f : X \to Z$ is continuous at z.

Theorem 4.1.8

Let X, Y be metric spaces and $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. $f^{-1}(U)$ is open $\forall U \subseteq Y$ open.
3. $f^{-1}(U)$ is closed $\forall U \subseteq Y$ closed.

Theorem 4.1.9 Multilinearity and continuity

Let $V_1, \ldots, V_k$ and W be normed vector spaces over $\mathbb{F}$, and let $T \in L(V_1, \ldots, V_k; W)$. The following are equivalent:

1. $T \in \mathcal{L}(V_1, \ldots, V_k; W)$ i.e. T is a bounded multilinear map.
2. T is continuous.
3. T is continuous at $0 \in \prod V_i$.

Proof. $1 \implies 2$: Let $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$ be two vectors in $V_1 \times \ldots \times V_k$. We write

$$T(v_1, \ldots, v_k) - T(u_1, \ldots, u_k) = T(v_1 - u_1, v_2, \ldots, v_k) + T(u_1, v_2 - u_2, \ldots, v_k) + \ldots + T(u_1, \ldots, u_{k-1} v_k - u_k).$$

This implies that $\|T(u) - T(v)\| \leqslant \|T\|_{\mathcal{L}} \left[\|v_1 - u_1\|_{V_1} \prod_{i=2}^k \|v_i\|_{V_i} + \|u_1\|_{V_1} \|u_2 - v_2\|_{V_2} \prod_{i=3}^k \|u_i\|_{V_i} + \cdots + \prod_{i=1}^{k-1} \|u_1\|_{V_i} \|u_k - v_k\|_{V}\right.$

From this est, it's easy to conclude that T is continuous at u.

$2 \implies 3$: trivial

$3 \implies 1$: Suppose T is continuous at 0. Let $\epsilon = 1$ and let $\delta > 0$ such that $\|u\|_p = \begin{cases} \left(\sum \|u_i\|_{V_i}^p\right)^{1/p} & p < \infty \\ \max & p = \infty \end{cases} < \delta$. This implies that $\|T(u)\|_W = \|T(u) - T(0)\|_W < \epsilon = 1$.

Let $u_i \in V_i$ be such that $\|u_i\|_{V_i} = 1$. Then

$$\| \left(\frac{\delta}{2k^{1/p}} u_1, \ldots, \frac{\delta u_k}{2k^{1/p}} \right) = \frac{\delta}{2} < \delta.$$

So, T applied to that value is less than 1. But this means

$$\left(\frac{\delta}{2k^{1/p}}\right)\|T(u)\|_W < 1$$
$$\|T(u)\|_W \leqslant \left(\frac{2k^{1/p}}{\delta}\right)^k.$$

as well. By taking the supremum, we get that T is bounded and in $\mathcal{L}$. 😄

Definition 4.1.3

Let V, W be normed vector spaces over $\mathbb{F}$.

1. Recall $L^k(V;W) = L(V_1, \ldots, V_k; W)$ and similarly for $\mathcal{L}$. Given $T \in L^k(V;W)$ and $v \in V$, we write $Tv^{\otimes k} = T(v^{\otimes k}) = T(v, \ldots, v)$.
2. A polynomial is a map $p : V \to W$ given by $p(v) = \sum_{k=0}^{d} T_k v^{\otimes k}$ for $T_k \in L^k(V;W)$. We write $d =$ degree of p given that $T_d \neq 0$. Note: by the continuity of $T_k \in \mathcal{L}^k(V;W)$ and algebra of continuity, all polynomials are continuous.

Definition 4.1.4

Let X, Y be metric spaces and $f : X \to Y$.

1. We say f is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x, y \in X$ and $d_X(x,y) < \epsilon$ then $d_Y(f(x), f(y)) < \epsilon$.
2. f is Lipschitz if $\exists c \geqslant 0$ such that $d(f(x), f(y)) \leqslant c d(x,y)$ for all $x, y \in X$.

Facts:

1. Lipschitz $\implies$ uniformly continuous $\implies$ continuous.
2. Compositions of uniformly continuous functions are uniformly continuous.
3. Compositions of Lipschitz functions are Lipschitz.
4. Suppose $f, g : X \to V$ for V a normed vector space. If f, g are uniformly continuous or Lipschitz, then $\alpha f + \beta g$ are too $\forall \alpha, \beta \in \mathbb{F}$.

Lenma 4.1.1

Suppose X, Y are metric spaces, $f : X \to Y$ is uniformly continuous if $\{x_n\}_{n=\ell}^{\infty} \subseteq X$ is Cauchy, then $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is Cauchy.

Proof. Let $\epsilon > 0$, then there exists $\delta > 0$ such that

$$x, y \in X \wedge d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Pick $N \geqslant \ell$ such that $m, n \geqslant N \implies d(x_n, x_m) < \delta$. This means that $d(f(x_n), f(x_m)) < \epsilon$. Therefore $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is Cauchy. 😄

Example 4.1.1

1. Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$.
2. Let V, W be normed vector spaces, $a \in W$, $T \in \mathcal{L}(V, W)$. Then $f : V \to W$ via $f(x) = a + Tx$ is Lipschitz:

$$\begin{aligned} \|f(x) - f(y)\|_W &= \|Tx - Ty\|_W && (4.1) \\ &\leqslant \|T\|_{\mathcal{L}} \|x - y\|_V. && (4.2) \end{aligned}$$

   So, f is Lipschitz.

   But moving back to the first example, f is not uniformly continuous. However, f maps Cauchy sequences to Cauchy sequences. Indeed, suppose $\{x_n\}_n$ is Cauchy and bounded by M. Then,

$$|f(x_n) - f(x_m)| = |x_n^2 - x_m^2| = |x_n + x_m||x_n - x_m| \leqslant 2M|x_n - x_m|.$$

   So f is not uniformly continuous. Suppose not, then $\exists \delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < 1$.

   Let $x = n \in \mathbb{N}$ and $y = n + \frac{\delta}{2}$. Then

$$|x - y| = \frac{\delta}{2} < \delta,$$

   so $1 > |f(y) - f(x)| = (n + \delta/2)^2 - n^2 = \delta n + \frac{\delta^2}{4}$. This is a contradiction.
3. Let X be a metric space. Let $a \in X$ and define $f : X \to \mathbb{R}$ via $f(x) = d(x, a)$. f is Lipschitz as $|f(x) - f(y)| = |d(x, a) - d(y, a)| \leqslant d(x, y)$. This can be generalized.
4. Consider $\sin : \mathbb{R} \to \mathbb{R}$.

$$|\sin(x) - \sin(y)| = |\cos(w)(x - y)| \leqslant |x - y|$$

   for some w below x and y. Therefore sin is Lipschitz. Ditto for cos.
5. Let X be a metric space, $V_1, \ldots, V_k, W$ be normed vector spaces. And suppose $f_i : X \to V_i$ is uniformly continuous xor Lipschitz. Further suppose $T : X \to \mathcal{L}(V_1, \ldots, V_k; W)$ is uniformly continous xor Lipschitz. If $T, f_1, \ldots, f_k$ are all also bounded, then $X \ni x \mapsto T(x)(f_1(x), \ldots, f_k(x)) \in W$ is uniformly continuous xor Lipschitz.

   Proof. Recall

$$T(u_1, \ldots, u_k) - T(v_1, \ldots, v_k) = \\ T(u_1 - v_1, u_2, \ldots, u_k) + \cdots + T(v_1, \ldots, v_{k-1}, u_k - v_k).$$

   Now use this with $u_i = f_i(x)$ and $v_i = f_i(y)$.

😊

Definition 4.1.5

Let $f : X \to Y$ for X, Y metric spaces. We define $K(f) \in [0, \infty]$ to be

$$K(f) = \begin{cases} 0 & |X| = 1 \\ \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} & \text{otherwise} \end{cases}.$$

$K(f)$ is called the Lipschitz constant for f.

Facts:

1. $K(f) = 0 \iff f$ is constant. $K(f)$ is finite $\iff f$ is Lipschitz. Also, $d(f(x), f(y)) \leqslant K(f)d(x, y)$.
2. Suppose $g : Y \to Z$, Z is a metric space. Then $K(g \circ f) \leqslant K(g)K(f)$.

   Proof. $d_Z(g \circ f(x), g \circ f(y)) \leqslant K(g)d_Y(f(x), f(y)) \leqslant K(g)K(f)d(x, y)$. This yields the result. 😊

3. If $Y = X$, i.e. $f : X \to X$, then $K(f^{(n)}) \leqslant K(f)^n$.

$$f^{(0)} = I_X$$
$$f^{(n)} = f \circ f^{(n-1)}.$$

Definition 4.1.6

Let X, Y be metric spaces and $f : X \to Y$.

1. We say f is expansive if $\infty > K(f) > 1$.
2. We say f is non-expansion if $K(f) \leqslant 1$.
3. We say f is contractive if $K(f) < 1$.
4. Suppose $Y = X$. We say f is eventually contractive if $\exists 1 \leqslant n \in \mathbb{N}$ such that $f^{(n)}$ is contractive.

Example 4.1.2

Let $\alpha \in [0, 1]$, $\beta \in (0, 1)$, $\gamma \in [0, \infty)$ and $x \in (-\infty, 0]$. Set $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} \beta & x \in (-\infty, 0] \\ \beta + (1 - \beta)\left(\frac{x}{\beta}\right)^\alpha & x \in [0, \beta] \\ 1 & x \in [\beta, 1] \\ 1 + \gamma(x - 1) & x \in (1, \infty) \end{cases}.$$

Exercise:

$$K(f) = \begin{cases} \infty, & \alpha \in (0, 1) \\ \max\left(\gamma, \frac{1}{\beta} - 1\right) & \alpha = 1 \end{cases}.$$

Also,

$$f^{(2)}(x) = \begin{cases} 1 & x \leqslant 1 \\ 1 + \gamma^2(x - 1) & x > 1 \end{cases}.$$

So $K(f^{(2)}) = \gamma^2 \implies f^{(2)}$ is Lipschitz. If $\gamma < 1$ then f is eventually contractive.

Theorem 4.1.10 Banach Fixed Point Theorem

Let X be a complete metric space and $f : X \to X$ be eventually contractive. Then there exists a unique fixed point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Suppose intitially that f is contractive, i.e. $K(f) = \gamma \in [0, 1)$. Let $x_0 \in X$ arbitrarily. Inductively define $\{x_n\}_{n=0}^\infty \subseteq X$ via $x_{n+1} = f(x_n)$, i.e. $x_n = f^{(n)}(x_0)$. For $n > m \geqslant 0$, we bound

$$d(x_n, x_m) \leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leqslant \cdots \leqslant \sum_{i=m}^{n-1} d(x_i, x_{i+1}) = \sum_{i=m}^{n-1} d(f^{(i)}(x_0), f^{(i)}(x_1)) \leqslant \sum_{i=1}^{n-1} \gamma^i d(x_0, x_1) = d(x_0, x_1) \sum_{i=m}^{n-1} \gamma^i.$$

Since $\gamma < 1$, the infinite sum of γ^i converges. That is, $\left\{\sum_{i=0}^{k} \gamma^i\right\}_{k=0}^{\infty}$ is Cauchy. This and the bound implies that $\{x_n\}_{n=0}^{\infty}$ is Cauchy.

Now note that $x_{n+1} = f(x_n)$, and this converges to $x = f(x)$ because f is continuous. Therefore x is a fixed point.

Suppose $y \in X$ is such that $f(y) = y$. Then, $d(x, y) = d(f(x), f(y)) \leqslant \gamma d(x, y) \implies (1-\gamma)d(x, y) \leqslant 0 \implies x = y$.

Now consider the general case. $\exists z \leqslant n$ such that $f^{(n)}$ is contractive. By the previous analysis, there exists a unique $x \in x$ such that $f^{(n)} = x$. Thus, $f(x) = f^{(n+1)}(x) = f^{(n)}(f(x)) \implies f(x)$ is a fixed point of $f^{(n)}$. So this means $f(x) = x$. Suppose now $y = f(y)$, this means $y = f^{(n)}(y) \implies y = x$. 😊

Note:

Say f is contractive for simplicity. What we knew was that if $n > m \geqslant 0$, then

$$d(x_n, x_m) \leqslant d(x_0, x_1) \sum_{i=m}^{n-1} \gamma^i \leqslant d(x_0, x_1) \sum_{i=m}^{\infty} \gamma^i = d(x_0, x_1) \frac{\gamma^m}{1-\gamma}.$$

So,

$$d(x, x_m) = \lim_{n\to\infty} d(x_n, x_m) \leqslant \frac{d(x_0, x_1)\gamma^m}{1-\gamma}.$$

Example 4.1.3 (putnam(?))

Let $X \neq \emptyset$ be a set, $g : X \to \mathbb{R}$ be bounded and $h : X \to \mathbb{R}$ be arbitrary. Let $0 < \gamma < 1$.
Claim: $\exists! f \in \mathcal{B}(X; \mathbb{R})$ such that $f(x) = g(x) + \gamma \cos(h(x) + f(x))$.

Proof. Define $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ as $\Phi(f) = g + \gamma \cos(h + f) \in \mathcal{B}(X)$.
Fact 1: $\mathcal{B}(X)$ is complete.
Fact 2: $\Phi(f_1) - \Phi(f_2) = \gamma[\cos(h + f_1) - \cos(h + f_2)]$. That is,

$$|\Phi(f_1)(x) - \Phi(f_2)(x)| \leqslant \gamma|f_1(x) - f_2(x)|$$
$$\|\Phi(f_1) - \Phi(f_2)\|_{\mathcal{B}(X)} \leqslant \gamma\|f_1 - f_2\|_{\mathcal{B}(X)}.$$

Therefore, Φ is a contaction, meaning there is a unique $f \in \mathcal{B}(X)$ such that $f = \Phi(f) = g + \gamma \cos(h + f)$. 😊

Example 2: Solving the quadratic equation. Claim: suppose V, W are Banach spaces over $\mathbb{F}$, $A \in \mathcal{L}^2(V; W)$, $B \in \mathcal{L}(V; W), c \in W$. Suppose $A \neq 0$ and B is invertible with $B^{-1} \in \mathcal{L}(W; V)$. We claim that there eixsts $x \in V$ such that $A(x, x) + Bx + c = 0$ provided that $4\|B^{-1}\|^2_{\mathcal{L}(W;V)}\|A\|_{\mathcal{L}^2(V;W)}\|c\|_W < 1$.

Proof. If $c = 0$, then $x = 0$ does the job, so suppose $c \neq 0$. It suffices to prove that when $W = V$ and $B = I_V$. Indeed, suppose we proved this. Then,

$$A(x, x) + Bx + c = 0 \text{ in } W \iff B^{-1}A(x, x) + x + B^{-1}c = 0 \text{ in } V.$$

But $B^{-1} \circ A \in \mathcal{L}^2(V; V)$, $\|B^{-1} \circ A\|_{\mathcal{L}^2} \leqslant \|B^{-1}\|_{\mathcal{L}}\|A\|_{\mathcal{L}^2}$.
$B^{-1}c \in V$, $\|B^{-1}c\|_V \leqslant \|B^{-1}\|_{\mathcal{L}}\|c\|_W$.
Then, this means that $4\|B^{-1}A\|_{\mathcal{L}^2}\|B^{-1}c\|_V < 1 \implies \exists x \in V$ such that we are done.

Proof. We prove the special case. We want to thsow that $A(x, x) + x + c = 0$ for $A \in \mathcal{L}^2(V; V), c \in V \setminus \{0\}$.
Note,

$$A(x, x) + x + c = 0 \iff x = -c - A(x, x) \iff x \text{ is a fixed point of } f : V \to V, f(x) = -c - A(x, x).$$

The idea is that if $A = 0$, then $x = -c$ is a solution. The strategy is to try to find $R \geqslant 0$ such that

1. $f : B[-c, R] \to B[-c, R]$,
2. f is eventually contractive on $B[-c, R]$.

IF we can prove this, then $\exists! x \in B[-c, R]$ such that $x = f(x) = -c - A(x, x) \implies A(x, x) + x + c = 0$. 😊

😊

Example 4.1.4

$\exists x \in V$ such that $A(x,x) + Bx + c = 0$. Reduce to case $V + W$, $B = I$.
Claim: $\exists x \in V$ such that $A(x,x) + x + c = 0$ where $A \in \mathcal{L}^2(V,V)$, $c \in V \setminus \{0\}$.
Let $f : V \to V$ such that $f(x) = -c - A(x,x)$. Let $R \geqslant 0$ (TBD) and consider $x \in B[-c, R]$.

$$\begin{aligned} f(x) = c = -A(x,x) = -[A(x+c-c, x+c-c)] = -[A(x+c, x+c) - A(x+c, c) - A(c, x+c) + A(c,c)] \\ \implies \|f(x) + c\| \leqslant \|A\|[\|x+c\|^2 + 2\|c\|\|x+c\| + \|c\|^2] \\ \leqslant \|A\|[R^2 + 2\|c\|R + \|c\|^2] \leqslant R. \end{aligned}$$

$\|A\|R^2 + (2\|c\|\|R\| - 1)R + \|A\|\|c\|^2 \leqslant 0 \iff (2\|A\|\|c\| - 1)^2 - 4\|A\|^2\|c\|^2 \geqslant 0$. This just gives

$$1 - 4\|A\|\|c\| \geqslant 0 \implies 1 \geqslant 4\|A\|\|c\|.$$

That is, if this holds, then $R \in [R_-, R_+]$. So,

$$R_\pm = \frac{1 - 2\|A\|\|C\| \pm \sqrt{1 - 4\|A\|\|c\|}}{2\|A\|}$$

and $R_- > 0$. We now konw that $4\|A\|\|c\| \leqslant 1 \implies$ for $R \in [R_-, R_+]$, $f : B[-c, R] \to B[-c, R]$.
Next, for $x, y \in B[-c, R]$,

$$\begin{aligned} \|f(x) - f(y)\| = \|A(y,y) - A(x,x)\| &= \|A(y-x, x) + A(y, y-x)\| \\ &= \|A(y-x, y+c) - A(y-x, c) + A(x+c, y-x) - A(c, y-x)\| \\ &\leqslant \|A\|[\|x-y\|R + \|x-y\|\|c\| + \|x-y\|R - \|x-y\|\|c\|] \\ &= 2\|A\|[R + \|c\|]\|x-y\|. \end{aligned}$$

We win if this quantity is strictly less than 1. Note:

$$2\|A\|R_\pm + 2\|A\|\|c\| = 1 \pm \sqrt{1 - 4\|A\|\|c\|}.$$

This is why we choose R_-. So

$$4\|A\|\|c\| - 1 \implies f : B[-c, R_-] \to B[-c, R_-] \text{ is a contraction.}$$

By Banach Fixed Point Theorem, $\exists! x \in B[-c, R_-]$ such that $f(x) = x$.

Note:

$f : B[-c, R_-]\circlearrowleft$

4.2 Homeomorphisms

Definition 4.2.1

Let X, Y be metric spaces and $f : X \to Y$ be a bijection.

1. f is a homeomorphism if f, f^{-1} are continuous. We write $X \simeq_{hom} Y$ in this case.
2. f is a uniform homeomorphism if f, f^{-1} are uniformly continuous. We write $X \simeq_{uni} Y$ in this case.
3. f is a bi-Lipschitz homeomorphism if f, f^{-1} are Lipschitz continuous. We write $X \simeq_{bi-L} Y$ in this case.

Facts:

- $\simeq_*$ are equivalence relations (also $X \simeq_{iso} Y \iff \exists$ an iso.). $[X]_{iso} \subseteq [X]_{bi-L} \subseteq [X]_{uni} \subseteq [X]_{hom}$.
- $f : X \to Y$ is a homeomorphism iff

$$\begin{cases} f \text{ is bijective} \\ f^{-1}(U) \text{ is open } \ \forall U \subseteq Y \text{ open.} \\ f(V) \text{ is open } \ \forall V \subseteq X \text{ open.} \end{cases}$$

  $f : X \to Y$ is bi-Lipschitz iff f is surjective and $\exists c_0, c_1 > 0$ such that $c_0 d(x, y) \leqslant d(f(x), f(y)) \leqslant c_1 d(x, y)$ for all $x, y \in X$.

Example 4.2.1

1. Suppose X is a finite metric space, $f : X \to Y$ a bijection. We claim that f continuous implies f Lipschitz. If the cardinality of X is 1, then this is pointless (not really??). Suppose $|X| \geqslant 2$.

   Define $K(f) = \max\left\{\frac{d(f(x),f(y))}{d(x,y)} \mid x \neq y\right\} < \infty$.

2. Consider $f : (0, 1) \to (1, \infty)$ via $f(x) = \frac{1}{x}$. This is a bijection and a homeomorphism. It's not a uniform homeomorphism. This shows that $[(0, 1)]_{uni} \subset [(0, 1)]_{hom}$.

3. Let V, W be normed vector spaces. We can add "linear" to any of our homeomorphism notions. In this case, linear bi-Lipschitz $\iff$ linear homeomorphism.

   Indeed, suppose that $T : V \to W$ is a linear map. This means that $T \in \mathcal{L}(V; W)$. We can use the same logic for T^{-1}. Thus,

$$\|Tx - Ty\|_W \leqslant \|T\|_{\mathcal{L}} \|x - y\|_V.$$

   Also,

$$\begin{aligned} \|x - y\|_V &= \|T^{-1}Tx - T^{-1}Ty\|_V \\ &\leqslant \|T^{-1}\|_{\mathcal{L}} \|Tx - Tv\|_V \\ &\implies \frac{1}{\|T\|_{\mathcal{L}}} \|x - y\|_V \leqslant \|Tx - Ty\|_W \leqslant \|T\|_{\mathcal{L}} \|x - y\|_V. \end{aligned}$$

   Therefore, T is bi-Lipschitz.

Question: In general, is it the case that $[V]_{hom} = [V]_{bi-L}$ for $V \neq \{0\}$ a normed vector space?
Answer: No. In fact, $[V]_{bi-L} \subset [V]_{uni}$. Remember, we only care about metrics, not necessarily norms.

Proof. Fix $(V, \|\cdot\|)$ a normed vector space. Claim, $d : V \times V \to \mathbb{R}$ given by $d(x, y) = \sqrt{\|x - y\|}$ is a metric on V. This is obviously symmetric and positive, so we just check the triangle inequality:

$$d(x, y) = \sqrt{\|x - y\|} \leqslant \sqrt{\|x - z\| + \|z - y\|} \leqslant \sqrt{\|x - z\|} + \sqrt{\|y - z\|} = d(x, z) + d(z, y).$$

The last inequality is true by squaring. We'll now show that $(V, \|\cdot\|) \simeq_{uni} (V, d)$ with the identity map. That is $I : (V, \|\cdot\|) \leftrightarrow (V, d)$ is uniformly continuous in both directions.

Let $\epsilon > 0$. Then $d(x, y) < \delta \iff \|x - y\| < \delta^2$. So taking $\delta = \sqrt{\epsilon}$ shows that I is uniformly continuous from (V, d) to $(V, \|\cdot\|)$.

Similarly, $\|x - y\| < \delta \iff d(x, y) < \sqrt{\delta}$. So take $\delta = \epsilon^2$ and we see that I is uniformly continuous from $(V, \|\cdot\|)$ to (V, d).

Next, we claim that (V, d) and $(V, \|\cdot\|)$ are not bi-Lipschitz homeomorphic. So suppose BWOC there exists an $f : (V, \|\cdot\|) \to (V, d)$ that is bi-Lipschitz homeomorphic. In particular, then there exists $c_0, c_1 > 0$ such that

$$c_0\|x - y\| \leqslant d(f(x), f(y)) = \sqrt{\|f(x) - f(y)\|} \leqslant c_1\|x - y\|.$$

for all $x, y \in V$. In particular, if $c := c_1^2$, then $\|f(x) - f(y)\| \leqslant c\|x - y\|^2$ for $x, y \in V$. Let $x \neq y$ in V and $1 \leqslant n \in \mathbb{N}$. Let $x_i = x + \frac{i}{n}(y - x)$ for $0 \leqslant i \leqslant n$. This yields

$$\|x_{i+1}x_n\| = \frac{1}{n}\|x - y\|.$$

Thus,

$$\|f(y) - f(x)\| \leqslant \sum_{i=0}^{n-1} \|f(x_{i+1}) - f(x_i)\| \leqslant \sum_{i=0}^{n-1} c\|x_{i+1} - x_i\|^2 = c\sum_{i=0}^{n-1} \frac{\|x - y\|^2}{n^2} = \frac{c\|x - y\|^2}{n}.$$

Send $n \to \infty$ to get that $f(x) = f(y)$, contradiction! Therefore, $[V]_{bi-L} \subset [V]_{uni}$. ☺

Question: Are there any non-linear bi-Lipschitz homeomorphisms on $V \neq \{0\}$ a normed vector space.
Answer: Yes, there are lots, at least if V is complete. Suppose V is a banach space and suppose $g : V \to V$ is a contraction. We claim that $f = I + g$ is a bi-Lipschitz homeomorphism.

Proof. We follow the following steps:

1. f is a bijection. We'll show $\forall y \in V$, $\exists x \in V$ such that $x + g(x) = y$. Fix y, and define $h : V \to V$ via $h(x) = y - g(x)$. Then we're done if we can show that $\exists! x \in X$ such that $h(x) = x$. But, $K(h) = K(g)$, so h is a contraction and therefore the Banach fixed point theorem applies.

2. f is bi-Lipschitz. We have

$$\begin{aligned}\|f(x) - f(y)\| = \|x - y + g(x) - g(y)\| &\leqslant \|x - y\| + \|g(x) - g(y)\| \\ &\leqslant \|x - y\| + K(g)\|x - y\| = (1 + K(g))\|x - y\|.\end{aligned}$$

   This means that $K(f) \leqslant 1 + K(g)$. On the other hand,

$$\begin{aligned}\|x - y\| &\leqslant \|x + g(x) - y - g(y)\| + \|g(x) - g(y)\| \\ &\leqslant \|f(x) - f(y)\| + K(g)\|x - y\|.\end{aligned}$$

   This implies $(1 - K(g))\|x - y\| \leqslant \|f(x) - f(y)\| \leqslant (1 + K(g))\|x - y\|$. Therefore, f is bi-Lipschitz.

☺

Next: Let $f : V \to V$ be a bi-Lipschitz homeomorphism. Let $g : V \to V$ be a bi-Lipschitz homeomorphism with $K(g)K(f^{-1}) < 1$. We claim $h = f + g$ is a bi-Lipschitz homeomorphism.

Proof. Indeed, $h = f + g = f + g \circ f^{-1} \circ f = (I + g \circ f^{-1}) \circ f$. We just need to show that $g \circ f^{-1}$ is bi-Lipschitz because then h will be.

But, $K(g \circ f^{-1}) \leqslant K(g)K(f^{-1}) < 1$ by assumption. Therefore, $g \circ f^{-1}$ is a bi-Lipschitz homeomorphism, and so is h. 😊

Definition 4.2.2

Let X be a metric space

1. We say a property of X is a topological property if it is common to $[X]_{hom}$. That is, it's true in X if and only if it's true in Y for all $Y \simeq_{hom} X$.
2. We say a property of X is a uniform property if it is common to $[X]_{uni}$.
3. We say a property of X is a strong property if it is common to $[X]_{bi-L}$.

Example 4.2.2

1. $(0, 1) \simeq_{hom} \mathbb{R}$. Let $f : (0, 1) \to \mathbb{R}$ be defined as

$$f(x) = \log\left(\frac{1}{x} - 1\right)$$

$$f^{-1}(y) = \frac{1}{1 + e^y}.$$

   $(0, 1)$ is not complete and bounded, but $\mathbb{R}$ is copmlete and unbounded. This is a strong example of how homeomorphisms do not maintain all properties.

Proposition 4.2.1

Let $X \simeq_{bi-L} Y$. Then X is bounded iff Y is bounded.

Proof. $\exists c_0, c_1 > 0$ and a bijection $f : X \to Y$ such that $c_0 d(x, y) \leqslant d(f(x), f(y)) \leqslant c_1 d(x, y)$ for all $x, y \in X$. We'll show that Y bounded implies X bounded. The other direction is free.

So if Y is bounded, then $Y \subseteq B(z, r)$ for some $z \in Y$. But, $z = f(x)$ for some $x \in X$. Therefore, $d(x, y) \leqslant \frac{1}{c_0} d(f(x), f(y))$ for all $y \in X$. But that distance is less than r, so we also have that $d(x, y) < \frac{r}{c_0}$. This means that $X \subseteq B(x, r/c_0)$. So X is bounded. 😊

Corollary 4.2.1

Boundedness is a strong property.

Example 4.2.3

Let (X, d) be a metric space that is not bounded. We've seen that $\sigma = \frac{d}{1+d}$ is also a metric on X. (X, σ) is bounded because $X = B_\sigma[x, 1]$ for all $x \in X$. These are not bi-Lipschitz homeomorphic by the previous proposition, but they can be uniformly homeomorphic. We prove that the identity map does the job.
To see this, let $\epsilon > 0$ and note

$$d < \epsilon \implies \frac{d}{1+d} \leqslant d < \epsilon.$$

This means $I : (X, d) \to (X, \sigma)$ is uniformly continuous. On the other hand, assume $\delta < 1$ and observe that

$$\frac{d}{d+1} < \delta \implies d(1-\delta) < \delta \iff d < \frac{\delta}{1-\delta}.$$

We set $\epsilon := \frac{\delta}{1-\delta}$. So then we hae $\delta = \frac{\epsilon}{1+\epsilon}$, so $I : (X, \sigma) \to (X, d)$ is uniformly continuous.

Corollary 4.2.2

Boundedness is a strong property (preserved by bi-Lipschitz, not by uniform).

Proposition 4.2.2

Let X, Y be metric spaces, $f : X \to Y$ be a uniform homeomorphism. The following hold:

1. $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy if and onlyl if $\{f(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy.
2. X complete if and only if Y complete.

Proof. We prove this in parts.

1. We know $g : X \to Y$ uniformly continuous implies that $\{g(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy when $\{x_n\}_{n=\ell}^\infty \subseteq X$ is Cauchy. Now apply this to $g = f$ and $g = f^{-1}$ to get the result.
2. Suppose Y is complete. Let $\{x_n\}_{n=\ell}^\infty \subseteq X$ be Cauchy. Then we know $\{f(x_n)\}_{n=\ell}^\infty \subseteq Y$ is Cauchy and hence convergent to some $y \in Y$. Thus $x_n = f^{-1}(f(x_n)) \to f^{-1}(y)$ in X as $n \to \infty$. Therefore X is complete. The converse holds by symmetry.

😃

Corollary 4.2.3

Completeness is a uniform property.

Proof. The above proposition and $(0, 1) \simeq_{hom} \mathbb{R}$. 😃

4.3 More Metric Space Topology

Definition 4.3.1

Let X be a metric space, $E \subseteq X$. We say E is totally bounded if $\forall \epsilon > 0$, there exist $x_1, \ldots x_n \in X$ such that $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$.

Facts:

1. $E \subseteq X$ totally bounded $\implies$ E is bounded.
2. If $A \subseteq E \subseteq X$ and E is totally bounded, then A is totally bounded.
3. We don't have to use balls. $E \subseteq X$ is totall bounded if and only if $\forall \epsilon > 0$, $\exists A_1, \ldots, A_n \subseteq X$ such that $\operatorname{diam}(A_i) < \epsilon$ for $i = 1, \ldots, n$ and $E \subseteq \bigcup_{i=1}^n A_i$.

Proof. We prove the third item.

First realize that $\operatorname{diam}(B(x, \epsilon)) = 2\epsilon$. So if we know E is totally bounded, we can pick $E \subseteq \bigcup_{i=1}^n B\left(x_i, \frac{\epsilon}{3}\right)$. Then the diameter of each ball is $\frac{2\epsilon}{3} < \epsilon$, so let $A_i := B\left(x_i, \frac{\epsilon}{3}\right)$ for $i = 1, \ldots, n$.

Now suppose that $E \subseteq \bigcup_{i=1}^n A_i$ with diam $< \frac{\epsilon}{3}$. Then let $x_i \in A_i$ be arbitrary and note $A_i \subseteq B(x_i, \epsilon)$. This means $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$. 😊

Example 4.3.1

1. Suppose $E \subseteq X$ is finite. Then E is totally bounded (just take the individual points).
2. $a, b \in \mathbb{R}$, $a < b$. Then (a, b) and $[a, b]$ are totally bounded, but $\mathbb{R}$ itself is not totally bounded.
3. Let X be an infinite set with the discrete metric. Then $B(x, 1)$ is a singleton set, so X itself is not totally bounded.
4. Let $E \subseteq \{\chi_A \in \ell^\infty(\mathbb{N}; \mathbb{R}) \mid A \subseteq \mathbb{N}\}$ where

$$\chi_a(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}.$$

   E is bounded because $\|\chi_A\|_\infty \leqslant 1$. Also, if $A \neq B$, then $\|\chi_A - \chi_B\|_\infty = 1$.

   Then $B(\chi_A, 1) \cap E = \{\chi_A\}$. Therefore E is bounded but not totally bounded.
5. Let V be a finite dimensional normed vector space. $B(x, r)$ and $B[x, r]$ are totally bounded.

Proposition 4.3.1

Let X be a metric space with $E \subseteq X$ totally bounded. Then $\overline{E}$ is totally bounded.

Proof. Recall $\overline{E} = E \cup E'$. Let $\epsilon > 0$ and pick $X_1, \ldots, X_n \in X$ such that $E \subseteq \bigcup_{i=1}^n B(x_i, \epsilon/2)$.

Let $x \in E'$, which means that $\emptyset \neq (B(x, \epsilon/2) \cap E) \setminus \{x\}$, so pick any y in this set. In particular, $d(x, y) < \epsilon/2$. Since $y \in E$, there is an x_i such that $d(x_i, y) < \epsilon/2$. By the triangle inequality,

$$d(x, x_i) \leqslant d(x, y) + d(y, x_i) < \frac{2\epsilon}{2} = \epsilon.$$

That is, $E' \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$. Therefore, E itself will be contained in the same union of balls as well. 😊

Theorem 4.3.1

Let X, Y be metric spaces.

1. If $f : X \longrightarrow Y$ is uniformly continuous and $E \subseteq X$ is totally bounded, then $f(E) \subseteq Y$ is totally bounded.

2. If $X \simeq_{uni} Y$, then X is totally bounded if and only if Y is totally bounded. In particular, TB is a uniform property.

Proof. If suffices to just prove the first item.

Let $\epsilon > 0$ and pick $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$ for all $x \in X$. Since E is totally bounded, there exist $x_1, \ldots, x_n$ in X such that $E \subseteq \bigcup_{i=1}^n B(x_i,\delta)$. Thus, $f(E) \subseteq \bigcup_{i=1}^n f(B(x_i,\delta)) \subseteq \bigcup_{i=1}^n B(f(x_i),\epsilon)$. 😊

Proposition 4.3.2

Let X be a totally bounded metric space. Then X is separable.

Proof. Since X is totally bounded, for every $n \in \mathbb{N}$ there is a finite set $\emptyset \neq E_n \subseteq X$ such that $X = \bigcup_{x \in E_n} B(x, 2^{-n})$. No we can define the countable set $E = \bigcup_{n\in\mathbb{N}} E_n \subseteq X$.

Now given any $x \in X$, we know there exists $x_n \in E_n$ such that $d(x, x_n) < 2^{-n}$. 😊

Example 4.3.2

$\mathcal{B}(X;Y)$ is not separable when X is an infinite set and $|Y| \geqslant 2$.

Theorem 4.3.2

Let X be a metric space and $E \subseteq X$. The following are equivalent:

1. E is totally bounded.

2. If $\{x_n\}_{n=\ell}^{\infty} \subseteq E$. Then there exists a Cauchy subsequence $\{x_{n_k}\}_{k=\ell}^{\infty}$.

Proof. Suppose $E \subseteq X$ is totally bounded. We will use this notation for JUST this proof:
Given two sequences $x = \{x_n\}_{n=\ell}^{\infty}, y = \{y_n\}_{n=\ell}^{\infty} \subseteq E$. We will write $x\sigma y$ to mean x is a subsequence of y. Note, if $x\sigma y$ and $y\sigma z$, then $x\sigma z$.

Set $x^\ell = x$, some given sequence $\{x_n\}_{n=\ell}^{\infty} \subseteq E$. E is totally bounded, so $E \subseteq \bigcup_{y\in F_\ell} B(y, 2^{-\ell-1})$ for F_ℓ finite. By the pigeonhole principle, $\exists y \in F_\ell$ such that $x_n^\ell \in B(y, 2^{-\ell-1})$ for infinitely many n. We may thus select $x^{\ell+1}\sigma x^\ell$ such that $x_n^{\ell+1} \in B(y, 2^{-\ell-1})$ for all n.

In particular, $d(x_n^{\ell+1}, x_m^{\ell+1}) < 2^{-\ell}$ for all $n, m \geqslant \ell$. Iterate this argument. This produces a sequence of subsequences. That is,

$$\cdots \sigma x^{m+1} \sigma x^m \sigma \cdots \sigma = x$$

such that $d(x_k^m, x_n^m) < 2^{-m+1}$ for all $m \geqslant \ell$, $n, k \geqslant \ell$.

Now let $\{y_n\}_{n=\ell}^{\infty} \subseteq E$ be given by $y_n = x_n^n$. Thus, for $m, n \geqslant N$, we know by construction that $d(y_n, y_m) = d(x_n^n, x_m^m) < 2^{-N+1}$. This easily shows that $\{y_n\}_{n=\ell}^{\infty} \subseteq E$ is Cauchy.

For the backward direction, we'll show that $\neg 1 \implies \neg 2$. Suppose E is not totally bounded. Then there exists an $\epsilon > 0$ such that E cannot be covered by finitely many ϵ-balls. Let $x_0 \in E$ be arbitrary. $E \nsubseteq B(x_0, \epsilon)$. So there exists $x \in E$ such that $d(x_0, x) \geqslant \epsilon$.

Now suppose we have $x_0, \ldots, x_n \in E$ such that $d(x_i, x_j) \geqslant \epsilon$ for $i \neq j \leqslant n$. Note that $E \nsubseteq \bigcup_{i=0}^n B(x, \epsilon)$, so we can pick an $x_{n+1} \in E$ that has a minimal distance of ϵ to all the previous points.

Proceeding via induction, we find that there exists $\{x_n\}_{n=0}^{\infty}$ such that $d(x_n, x_m) \geqslant \epsilon$ for all $m \neq n$ which cannot have a Cauchy subsequence. 😊

Corollary 4.3.1

If X is totally bounded and complete, then it is sequentially compact, meaning all sequences in X have convergent subsequences.

Definition 4.3.2

Let X be a metric space.

1. We say $\{U_\alpha\}_{\alpha\in A}$ (A is any index set) is an open cover of E if $E \subseteq \bigcup_{\alpha\in A} U_\alpha$ and each U_α is open.
2. An open subcover of the open cover $\{U_\alpha\}_{\alpha\in A}$ is a collection $\{U_\beta\}_{\beta\in B}$ for any $B \subseteq A$ such that it remains an open cover of E. We say an open subcover $\{U_\beta\}_{\beta\in B}$ is finite if B is finite.
3. We say E is compact if each open cover of E admits a finite open subcover.

Example 4.3.3

1. Let $E \subseteq X$ be finite. Then E is compact.
2. $(0, 1) \subset \mathbb{R}$ is not compact. Take $\left\{\left(0, \frac{n}{n+1}\right)\right\}_{n=1}^{\infty}$. Suppose $B \subseteq \mathbb{N} \setminus \{0\}$ is a finite set such that $\left\{\left(0, \frac{n}{n+1}\right)\right\}_{n=\in B}$ is an open subcover. Then let $N = \max B$. This implies that

$$(0, 1) \subseteq \left(0, \frac{N}{N+1}\right)$$

which is a contradiction.

3. Let $E = \{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. Pick $\alpha_0 \in A$ such that $0 \in U_{\alpha_0}$. Since $2^{-n} \longrightarrow 0$ as $n \longrightarrow \infty$, in fact U_{α_0} contains all but finitely many of the 2^{-n} terms. That is, there exists N such that $2^{-n\in U_{\alpha_0}}$ for all $n \geqslant N$. Now pick a finite subcover of $\{2^{-n} \mid 0 \leqslant n \leqslant N\}$ and we're done. E ic ompact, though E is infinite.

Proposition 4.3.3

Let X be a metric space and $K \subseteq X$ be compact. Then K is closed and totally bounded, which in particular means that K is bounded and that K is a separable metric space, provided $K \neq \emptyset$.

Proof. Let $\epsilon > 0$ and note that $\{B(x, \epsilon)\}_{x\in X}$ is a cover of K. So by compactness, there exists $x_1, \dots, x_n \in X$ such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Therefore, K is totally bounded which means it is bounded and separable.

We now prove that K is closed. If $K = \emptyset$ or $K^c = \emptyset$, then we are done. So, suppose otherwise. Let $x \in K^c$. For each $y \in K$, note that $B(y, d(x, y)/2) \cap B(x, d(x, y)2) = \emptyset$. Also, $\{B(y, d(x, y)/2)\}_{y\in K}$ is an open cover of K.

K being compact implies that $K \subseteq \bigcup_{i=1}^{n} B(y_i, d(x, y_i)/2)$ for some $y_1, \dots, y_n \in K$.

Let $\delta = \min\{d(x, y_i)/2 \mid 1 \leqslant i \leqslant n\}$. Then, $K \cap B(x, \delta) \subseteq \bigcup_{i=1}^{n} B(y_i, d(x, y_i)/2) \cap B(x, \delta) = \emptyset$. This means that $B(x, \delta) \subseteq K^c$. As such, K^c is open and K is closed. 😊

Proposition 4.3.4

Let X be a metric space and $\emptyset \neq Y \subseteq X$ and $K \subseteq Y \subseteq X$. Then K is compact in Y if and only if K is compact in X.

Proof. Suppose K is compact in Y. Then let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of K in X. Then, $\{Y \cap U_\alpha\}_{\alpha\in A}$ is an open cover of K in Y. By compactness, $K \subseteq \bigcup_{i=1}^{n}(Y \cap U_{\alpha_i}) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. So, K is compact in X.

Now suppose K is compact in X. Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of K in Y. Then, there exists an open (in X) U_α such that $V_\alpha = Y \cap U_\alpha$ for all $\alpha \in A$. THus, $\{U_\alpha\}_{\alpha\in A}$ is an open cover of K in X. So,

$$K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \implies K = K \cap Y \subseteq \bigcup_{i=1}^{n} Y_n \cap U_{\alpha_i} = \bigcup_{i=1}^{n} V_{\alpha_i}.$$

So, K is compact in Y. 😃

Note:
In light of this, we will mostly study compact metric spaces rather than compact subsets.

Theorem 4.3.3
Let X and Y be metric spaces with X compact and $f : X \to Y$ be continuous. Then $f(X) \subseteq Y$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $f(X)$. Note that f being continuous implies that $f^{-1}(U_\alpha) \subseteq X$ is open for all $\alpha \in A$. Thus, $\{f^{-1}(U_\alpha)\}_{\alpha in A}$ is an open cover of X. So since X is compact, we have that $X = \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Therefore, $f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. So, $f(X)$ is compact. 😃

Corollary 4.3.2
Compactness is a topological property.

Proposition 4.3.5
Let X be a compact metric space. Then $C \subseteq X$ si compact if and only if C is closed.

Proof. Forward direction has already been proved. So now assume C is closed and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of C. Then, $C^c \cup \{U_\alpha\}_{\alpha in A}$ is an open cover of X. So, $X = C^c \cup \bigcup_{i=1}^{n} U_{\alpha_i} \implies C \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, so C is compact. 😃

Theorem 4.3.4
Let X, Y be metric spaces with X compact and suppose $f : X \to Y$ is a continuous bijection. Then $f^{-1} : Y \to X$ is continuous and f is a homeomorphism.

Proof. First note that if $C \subseteq X$ is closed, then C is compact by the above proposition. So, $f(C)$ is compact and hence closed in Y. Define $g = f^{-1} : Y \to X$ and note that for $C \subseteq X$ closed, $g^{-1}(C) = f(C)$ which is closed for all such C. As such, g is continuous by the closet set characterization of continuity. 😃

Example 4.3.4
The theorem fails if X is not compact:

1. $[0, 2\pi) \ni t \mapsto (\cos(t), \sin(t)) \in \mathbb{S}^1$. This is continuous, but its inverse is not.
2. $(0, 2\pi) \ni t \mapsto (\sin(2t), \sin(t)) \in \mathbb{R}^2$. This sort of creates a figure 8 as the image.

Theorem 4.3.5
Let X be a metric space, $f : X \to Y$ continuous. The following hold:

1. Suppose $\emptyset \neq K \subseteq X$ is compact. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in K$ and $y \in X$ and $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$.
2. If X is compact, then f is uniformly continuous.

Proof. Clearly $1 \implies 2$, so weonly need to prove 1. Let $\emptyset \neq K \subseteq X$ be compact and $\epsilon > 0$. Since f is continuous, at each $x \in K$, $\exists \delta_x > 0$ such that

$$y \in X \wedge d_X(x, y) < \delta_x \implies d_Y(f(x), f(y)) < \frac{\epsilon}{2}.$$

So then, $\{B(x, \delta_x/2)\}_{x \in K}$ is an open cover in K. But since K is compact, we have that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_i/2).$$

Now let $\delta = \min\{\delta_1/2, \ldots, \delta_n/2\} > 0$. Now let $x \in K$ and $y \in X$ be such that $d_X(x, y) < \delta$. Since $x \in K$, there eixsts $1 \leqslant m \leqslant n$ such that $d_X(x, x_n) < \delta_m/2$. Thus,

$$\begin{aligned} d_X(x_m, y) &\leqslant d_X(x_m, x) + d_X(x, y) \\ &\leqslant \frac{\delta_m}{2} + \delta \\ &\leqslant \frac{2\delta_m}{2} = \delta_m. \end{aligned}$$

As such,

$$\begin{aligned} d_Y(f(x), f(y)) &\leqslant d_Y(f(x), f(x_m)) + d_Y(f(x_m), f(y)) \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

as desired. 😊

Corollary 4.3.3
Suppose X is a compact metric space. Then

$$[X]_{hom} = [X]_{uni}.$$

Theorem 4.3.6 Borel-Lebesgue

Let X be a metric space. The following are equivalent:

1. X satisfies the Bolzano-Weierstrass limit point property: If $E \subseteq X$ is infinite, then it has a limit point.
2. X is sequentially compact: all sequences in X have convergent subsequences.
3. X is complete and totally bounded.
4. X is compact.

Proof. (BW $\implies$ sequentially compact): Let $\{x_n\}_{n=\ell}^{\infty} \subseteq X$ and let $F = \{x_n \mid n \geqslant \ell\} \subseteq X$. If F is finite then we can extract a constant subsequence, which trivially converges. Assume then that F is an infinite set, which means that $\exists x \in F'$. Pick $n_\ell \geqslant \ell$ such that $x_{n_\ell} \in B(x, 2^{-\ell}) \cap F \setminus \{x\}$. Now suppose we have $n_\ell < n_{\ell+1} < \cdots < n_k$ and $x_{n_\ell}, \ldots, x_{n_k} \in X$ such that

$$x_{n_j} \in B(x, 2^{-j}) \cap F \setminus \{x\}.$$

Note that $B(x, 2^{-(k+1)}) \cap F \setminus \{x\} \neq \emptyset$. Note that this also must be infinite as otherwise we could just shrink the interval as much as we want to not include any of the points. Thus, we can choose $n_{k+1} > n_k$ and $x_{n_{k+1}} \in B(x, 2^{-k-1}) \cap F \setminus \{x\}$. By induction we know have $\{x_{n_k}\}_{k=\ell}^{\infty}$ such that $d(x, x_{n_k}) < 2^{-k}$ for all $k \geqslant \ell$. Therefore $x_{n_k} \to x$ as $k \to \infty$. Therefore, X is sequentially compact.

(sequentially compact $\implies$ complete and totally bounded): Let $\{x_n\}_{n=\ell}^{\infty} \subseteq X$ be Cauchy. Sequential compactness give us a convergent subsequence $\{x_{n_k}\}_{k=\ell}^{\infty}$. We know that Cauchy and convergent subsequence implies convergence, so $\{x_n\}_{n=\ell}^{\infty}$ is convergent. Also, all convergence subsequences are Cauchy, so all sequences in X have Cauchy subsequences, so X is totally bounded.

(complete + totally bounded $\implies$ compact): Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Claim: $\exists \epsilon > 0$ such that $\forall x \in X$, $\exists \alpha \in A$ such that $B(x, \epsilon) \subseteq U_\alpha$. We now prove this claim.

Suppose the contrapositive. Then for each $n \in \mathbb{N}$, there is an $x_n \in X$ such that $B(x_n, 2^{-n}) \cap U_\alpha^c \neq \emptyset$ for all $\alpha \in A$. X is totally bounded, so there exists a Cauchy subsequence $\{x_{n_k}\}_{k=0}^{\infty}$, so this sequence has to be convergent by completeness of X. Pick an $\alpha \in A$ such that $x \in U_\alpha$. Since U_α is open, we can pick an $r > 0$ such that $B(x, r) \subseteq U_\alpha$. Now pick $M \geqslant 0$ such that if $k \geqslant M$, then $d(x_{n_k}, x) < r/2$ and $2^{-n_k} < r/2$. Now let $y \in B(x_{n_M}, r/2)$. Then $d(x, y) \leqslant d(x, x_{n_M}) + d(x_{n_M}, y) < r/2 + r/2 = r$. Therefore, $y \in B(x, r)$ and $B(x_{n_M}, r/2) \subseteq B(x, r) \subseteq U_\alpha$. In turn, $B(x_{n_M}, 2^{-n_k}) \subseteq B(x_{n_M}, r/2) \subseteq U_\alpha$ which is a contradiction.

So back to the main proof: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover and let $\epsilon > 0$ be as given by the claim. Then, $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$ thanks to totally boundedness. But the claim guarantees that $\exists \alpha_i \in A$ such that $B(x_i, \epsilon) \subseteq U_{\alpha_i}$, so $X = \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore X is compact.

(compact $\implies$ BW): Let $E \subseteq X$ and suppose that $E' = \emptyset$. Then $\forall x \in X$, $\exists \epsilon_x > 0$ such that $E \cap B(x, \epsilon_x) \setminus \{x\} = \emptyset$. Then, $\{B(x, \epsilon_x)\}_{x \in X}$ is an open cover of X. So, by compactness, $X = \bigcup_{i=1}^{n} B(x_i, \epsilon_{x_i})$. Thus, $E = E \cap X = \bigcup_{i=1}^{n} B(x_i, \epsilon_{x_i}) \subseteq \bigcup_{i+1}^{n} \{x_i\}$. Therefore E is finite. But now we're done as if E is infinite, E' cannot be empty. ☺

Corollary 4.3.4 Heine-Borel

Let V be a finite dimensional normed vector space and $E \subseteq V$. Then E is compact if and only if E is closed and bounded.

Proof. We saw that sequential compactness is true if and only if closed and bounded when $V = \mathbb{R}^n$. Pick your favorite linear homeomorphism $T : V \to \mathbb{R}^n$. We can do this because we can always consider this as V as a vector space over $\mathbb{R}$. Now $E \subseteq V$ is compact if and only if $E \subseteq V$ is sequentially compact if and only if $T(E) \subseteq \mathbb{R}^n$ is sequentially compact if and only if $T(E) \subseteq \mathbb{R}^n$ is closed and bounded.

Heavily note that since T is linear, it is bi-Lipschitz, which means $T(E)$ is closed and bounded if and only if E is closed and bounded. ☺

Example 4.3.5

$B[x,r] \subseteq V$ are totally bounded.

Lemma 4.3.1

Suppose $\emptyset \neq K \subseteq \mathbb{R}$ be compact. Then there exists $x_0 = \min k$ and $x_1 = \max K$.

Proof. K compact implies K is closed and bounded. Let $M = \sup K$ and $m = \inf K$. Now pick $\{m_n\}_n$ and $\{M_n\}_n$ such that $m_n \to m$ and $M_n \to M$ as $n \to \infty$. By closedness, $m, M \in K$ so we have a minimum and maximum as desired. 😊

Theorem 4.3.7 Extreme Value Theorem

Let X be a compact metric space and $f : X \to \mathbb{R}$ be continuous. Then there exists $x_0, x_1 \in X$ such that $f(x_0) \leqslant f(x) \leqslant f(x_1)$ for all $x \in X$.

Proof. $f(X) \subseteq \mathbb{R}$ is compact because f is continuous and X is compact. Therefore, the lemma above shows that there is $y_0, y_1 \in f(X)$ such that $y_0 < f(x) < y_1$ for all $x \in X$. So we can write this as

$$f(x_0) \leqslant f(x) \leqslant f(x_1)$$

where $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. 😊

Definition 4.3.3

Let X be a metric space.

1. We say X is connected if the only sets in X that are clopen are $\emptyset$ and X.
2. $E \subseteq X$ is a connected subset if

$$\begin{cases} E = \emptyset \\ E \text{ is a connected metric space when endowed with the metric from } X \end{cases}$$

Note:

Connectness is automatically intrinsic. That is, if $E \subseteq X \subseteq Y$ for Y a metric space, $X \neq \emptyset$, then E is a connected subset of X.

Example 4.3.6

1. $(0,1) \cup (1,2)$ with the metric from $\mathbb{R}$ is disconnected.
2. Let X be a metric space and $E = \{x\} \subseteq X$. Then E is trivially connected.
3. Let $\mathbb{F} \subset \mathbb{R}$ be an archimedean field. Pick $z \in \mathbb{R} \setminus \mathbb{F}$ and let $L_z = \mathbb{F} \cap (-\infty, z)$ and $R_z = \mathbb{F} \cap (z, \infty)$. Then $R_z = L_z^c$ so they are both clopen.
4. Let X be a discrete metric space with cardinality greater than 2. X is disconnected. In fact, the only connected subsets are singletons.

Theorem 4.3.8

Let V be a normed vector space and $C \subseteq V$ be convex. That is $x, y \in C$ and $t \in [0,1] \implies (1-t)x+ty \in C$. Then C is connected. In particular, V itself is connected.

Proof. Suppose BWOC C is disconnected. Then $\exists \emptyset \neq E \subset C$ that's open and closed. That is, $\emptyset \neq E^c \subset C$ is also open and closed. Let $x \in E$ and $y \in E^c$. Define

$$S = \{s \in [0,1] \mid (1-t)x + ty \in E \text{ for all } 0 \leqslant t \leqslant s\}.$$

Note $x \in E$ implies that $0 \in S$. So we can define $s = \sup S \in [0,1]$. Now define $z := (1-s) + sy \in C$. It must be the case that $z \in E$ or $z \in E^c$.

- Let $z \in E$. Then E is open, so $B(z, \epsilon) \subseteq E$. Then for $0 < \delta < \epsilon$, the point $w_\delta = x + \left(s + \frac{d}{\|x-y\|}\right)(y-x)$ is such that

  $$\|w_\delta - z\| = \delta < \epsilon \implies w_\delta \subseteq B(z, eps) \subseteq E.$$

  Then, $s + \frac{\epsilon}{2\|x-y\|} \in S$, so we have a contradiction to s being an upper bound.

- Let $z \in E^c$, but E^c is also open so $B(z,\epsilon) \subseteq E^c$. Arguing as above shoes that the point $x + \left(s - \frac{\delta}{\|x-y\|}\right)(y-x) \in B(z,\epsilon) \subseteq E^c$ for all $0 < \delta < \epsilon$. Thus, $s - \frac{\epsilon}{2\|x-y\|}$ is an upper bound of s, a contradiction.

Therefore, C is connected. 😊

Theorem 4.3.9

Let X and Y be metric spaces with X connected. Let $f : X \to Y$ be continuous. Then $f(X) \subseteq Y$ is connected.

Proof. Suppose not, i.e. $f(X)$ is a disconnected metric space. Pick $\emptyset \neq V \subset f(X)$ such that V is clopen. Note $f : X \to f(X)$ is still continuous. Thus, $f^{-1}(V) \subseteq X$ is clopen. Since $\emptyset \neq V \subset f(X)$, we must have that $\emptyset \neq f^{-1}(V) \subset X$. Therefore, X is disconnected, which is a contradiction. 😊

Corollary 4.3.5

Connectedness is a topological property.

Example 4.3.7

Let $f : [0, 2\pi] \to \mathbb{R}^2$ via $f(t) = (\cos t, \sin t)$. Note that the image is connected, but not convex. Specifically, $\mathbb{S}^1$ is not convex.

Theorem 4.3.10 Characterizations of disconnectedness

Let X be a metric space. The following are equivalent:

1. X is disconnected.
2. $\exists A, B \subseteq X$ nonempty such that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$ and $A \cup B = X$.
3. $\exists$ nonempty closed sets $C, D \subseteq X$ such that $C \cup D = X$ and $C \cap D = \emptyset$.
4. $\exists$ nonempety open sets $U, V \subseteq X$ such that $U \cap V = X$ and $U \cap V = \emptyset$.
5. $\exists$ continuous surjection $f : \mathrm{X} \to \{0, 1\}$.
6. $\exists$ a discrete metric space Y with cardinality greater than 1 and a continuous surjection $f : X \to Y$.

Proof. We go in order.

(1) $\implies$ (2): X is disconnected, so there exists a clopen set $\emptyset \neq A \subset X$. Let $B := A^c$, which is also clopen and such that $\emptyset \neq B \subset X$. Obviously $A \cup B = X$. Now we check that $A \cap \overline{B} = A \cap B = \emptyset = \overline{A} \cap B$ as desired.

(2) $\implies$ (3): Let A and B as in (2). Note that $\overline{A} = \overline{A} \cap (A \cup B) = [\overline{A} \cap A] \cup [\overline{A} \cap B] = \overline{A} \cap A = A$. As such, A is closed. Similarly, B is also closed. Let $C = A, D = B$.

(3) $\implies$ (4): Let C, D as in (3). Let $U = C^c, V = D^c$.

(4) $\implies$ (5): Let U, V be the open sets from (4). Define $f : \mathrm{X} \to \{0, 1\}$ via

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \in V \end{cases}$$

f is obviously surjective. Note,

- $f^{-1}(\emptyset) = \emptyset$.
- $f^{-1}(\{0\}) = U$.
- $f^{-1}(\{1\}) = V$.
- $f^{-1}(\{0, 1\}) = U \cup V = X$.

These are all open sets, so f is continuous.

(5) $\implies$ (6): Let $f : \mathrm{X} \to Y$ as in (5). Let $Y = \{0, 1\}$.

(6) $\implies$ (1): Let $f : \mathrm{X} \to Y$ as in (6) with Y a discrete space (all sets are clopen) and cardinality of Y at least 2. Pick $y, z \in Y$ such that $y \neq z$. Let $E = f^{-1}(\{y\})$, which is clopen because f is continuous. As f is surjective, E is nonempty. Also, $z \neq y \implies E \neq X$. Therefore, E is a nonempty clopen set. As such, X is disconnected. 😊

Corollary 4.3.6

$C \subseteq \mathbb{R}$ is connected if and only if C is convex.

Proof. The backwards direction was done already, so now we prove the forward. Let $x, y \in C$ and suppose $x < z < y$ but $z \notin C$. Let

$$\begin{cases} L_z = C \cap (-\infty, z) \\ R_z = C \cap (z, \infty) \end{cases}.$$

Both L_z and R_z are open and nonempty, and $L_z \cap R_z = C$, therefore C is disconnected. This is a contradiction, so $x, y \in C \implies (1 - t)x + ty \in C$ for all $t \in [0, 1]$, so C is convex. 😊

Theorem 4.3.11 Intermediate Value Theorem

Let X be a metric space. Then the following are equivalent:

1. X is connected.
2. X satisfies the intermediate value theorem: if $f : X \to \mathbb{R}$ is continuous and $\alpha, \beta \in f(X)$ with $\alpha < \beta$ and $\alpha < \gamma < \beta$, then there exists $x \in X$ such that $f(x) = \gamma$.

Proof. (1) $\implies$ (2): Let X be connected. Since f is continuous, $f(X) \subseteq \mathbb{R}$ is connected and therefore convex.

(2) $\implies$ (1): We'll show $\neg(1) \implies \neg(2)$. Suppose X is disconnected. Then, there exists a continuous surjection $f : X \to \{0, 1\}$. Therefore (2) is false as f does not take on values in $(0, 1)$. 😄

Proposition 4.3.6

Let X be a connected metric space with cardinality of X greater than or equal to 2. Then X is uncountable.

Proof. Let $y, z \in X$ be distinct and define $f : X \to \mathbb{R}$ via $f(x) = d(x, z)$. Note, f is continuous, $f(z) = 0$, and $f(y) = d(y, z) > 0$. Thus, by the intermediate value theorem, we have the inclusion $[0, d(y, z)] \subseteq f(X)$. Therefore, $f(X)$ is uncountable. Now if X were countable, then there would exist $g : \mathbb{N} \to X$ that's surjective, meaning $f \circ g : \mathbb{N} \to f(X)$ is a surjection, implying $f(X)$ is countable which is a contradiction. As such, X is uncountable. 😄

Chapter 5

Spaces of Functions

Definition 5.0.1

Let $X \neq \emptyset$ be a set, Y a metric space, and $f, f_n : X \to Y$ for $(n \geqslant \ell)$.

1. We say $f_n \to f$ uniformly if $\forall \epsilon > 0$ there exists $N \geqslant \ell$ such that
$$n \geqslant N \implies d(f(x), f_n(x)) < \epsilon \text{ for all } x \in X.$$
2. We say $f_n \to f$ pointwise if for all x, $\forall \epsilon > 0$, there exists $N \geqslant \ell$ such that
$$n \geqslant N \implies d(f(x), f_n(x)) < \epsilon \text{ for all } x \in X.$$
3. We say $\{f_n\}_{n=\ell}^{\infty}$ is uniformly Cauchy if for every $\epsilon > 0$, there is $N \geqslant \ell$ such that
$$n, m \geqslant N \implies d(f_n(x), f_m(x)) < \epsilon \text{ for all } x \in X.$$

Clearly, uniformly convergent implies uniformly Cauchy and pointwise convergent.

Note:

Uniform convergence is the same as metric convergence in $\mathcal{B}(X;Y)$ if we know the sequence is in $\mathcal{B}(X:Y)$. Otherwise it's not really metric convergence. Can we hack this?

Idea: An extended metric space is a set X equipped with a map $d : X \times X \to [0, \infty]$ such that

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) \leqslant d(x, z) + d(z, y)$
3. $d(x, y) = d(y, x)$

Example 5.0.1

Let $X \neq \emptyset$ be a set and Y a metric space.
$$\mathcal{F}(X;Y) = \{f : X \to Y\}$$
equipped with $d(f, g) = \sup_{x \in X} d(f(x), g(x))$ is an extended metric.

Idea: In an extended metric space, let $x \sim y \iff d(x, y) < \infty$. This is an equivalence relation. So d restricted to each equivalence class $[x]$ is a metric space. Therefore
$$X = \bigsqcup [x].$$

The moral is that uniform convergence is convergence in $\mathcal{F}(X;Y)$ with the extended metric. Ditto for Cauchy. Things preserved by uniform limits. Suppose $f_n : X \to Y$, $f : X \to Y$ such that $f_n \to f$ uniformly as $n \to \infty$.

1. If each f_n is bounded, then f is bounded.

2. If each f_n is continuous at $x \in X$, then f is continuous at x.

3. If each f_n is uniformly continuous, then f is uniformly continuous.

This breaks with just pointwise:

1. Suppose $f : X \to Y$ is an unbounded function. Fix a point $z \in X$. Let $f_n : X \to Y$ via

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \in B(f(z), 2^n) \\ f(z) & \text{otherwise} \end{cases}.$$

Each f_n is bounded and converges to f. So f is not necessarily bounded.

2. Let $f_n : [0,1] \to \mathbb{R}$ where $f_n(x) > x^n$ for $n \in \mathbb{N}$. Then $f_n \to f$ pointwise as $n \to \infty$ where

$$f(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & x = 1 \end{cases}.$$

However, each f_n is uniformly continuous but f is not.

3. Suppose X, Y are metric spaces, $M \geqslant 0$. Let $L_M = \{f : X \to Y \mid K(f) \leqslant M\}$. Then L_M is closed under pointwise limits. Indeed, suppose $\{f_n\}_{n=\ell}^\infty \subseteq L_M$ and $f_n \to f$ pointwise as $n \to \infty$. By assumption, $d_Y(f_n(x), f_n(y)) \leqslant M d_X(x,y)$ for all $x, y \in X$. Fix a pair $x, y \in X$. Then,

$$\begin{cases} f_n(x) \to f(x) \\ f_n(y) \to f(y) \end{cases}$$

as $n \to \infty$. Thus, $d_Y(f(x), f(y)) = \lim_{n\to\infty} d_Y(f_n(x), f_n(y)) \leqslant M d_X(x,y)$. As such, f is Lipschitz with $K(f) \leqslant M$.

Definition 5.0.2

Let $X \neq \emptyset$ and $f_n : X \to \overline{\mathbb{R}}$ for $n \geqslant \ell$. We say $\{f_n\}_{n=\ell}^\infty$ is

- non-decreasing if $m < n \implies f_m(x) \leqslant f_n(x)$ for all $x \in X$.
- non-increasing if $m < n \implies f_n(x) \leqslant f_m(x)$ for all $x \in X$.
- monotone if either are true.

Theorem 5.0.1 Dini

Let X be a compact metric space and suppose $\{f_n\}_{n=\ell}^\infty \subseteq C^0(X; \mathbb{R})$ is monotone. Further suppose that there exists a continuous $f : X \to \mathbb{R}$ such that $f_n \to f$ pointwise as $n \to \infty$. Then $f_n \to f$ as $n \to \infty$.

Proof. Suppose WLOG that $\{f_n\}_{n=\ell}^\infty$ is nondecreasing. Let $\epsilon > 0$. First note that for all $x \in X$, the pointwise converge guarantees $\exists N_x \geqslant \ell$ such that

$$n \geqslant N_x \implies \|f(x) - f_n(x)\| < \frac{\epsilon}{3}.$$

In turn, since both f and f_{N_x} are continuous at x, $\exists \delta_x > 0$ such that

$$y \in B(x, \delta_x) \implies \begin{cases} \|f(x) - f(y)\| < \frac{\epsilon}{3} \\ \|f_{N_x}(x) - f_{N_x}(y)\| < \frac{\epsilon}{3} \end{cases}.$$

X is compact and $\{B(x,\delta_x)\}_{x\in X}$ is an open cover of X. So, $\exists x_1,\dots,x_n$ such that

$$X = \bigcup_{i=1}^{n} B(x_i,\delta_i).$$

Note that if $y \in B(x,\delta_x)$ for some $x \in X$, then

$$\|f(y) - f_{N_x}(y)\| \leqslant \|f(y) - f(x)\| + \|f(x) - f_{N_x}(x)\| + \|f_{N_x}(x) - f_{N_x}(y)\| < \epsilon.$$

Given any $y \in X$, we know $y \in B(x_i,\delta_i)$ for some $1 \leqslant i \leqslant n$. So,

$$\|f(y) - f_{N_{x_i}}(y)| < \epsilon.$$

Thus for $n \geqslant N$, we have that

$$\|f(y) - f_n(y)\| = f(y) - f_n(y) \leqslant f(y) - f_{N_{x_i}}(y) \leqslant \|f(y) - f_{N_{x_i}}(y) < \epsilon.$$

As such, we have uniform convergence of $f_n \to f$. 😄

Definition 5.0.3

Let X, Y be metric spaces. Recall that

$$UC^0(X;Y) \subseteq C^0(X;Y) \subseteq \mathcal{F}(X;Y)$$

and that

$$UC_b^0(X;Y) \subseteq C_b^0(X;Y) \subseteq \mathcal{B}(X;Y) \subseteq \mathcal{F}(X;Y).$$

We also have

$$C^{0,1}(X;Y) \subseteq C^0(X;Y) C_b^{0,1}(X;Y) \qquad\qquad \subseteq C_b^0(X;Y)$$

where

$$C^{0,1}(X;Y) = \{f : X \to Y \mid\ f \text{ is Lipschitz}\}$$

and

$$C_b^{0,1}(X;Y) = \{f : X \to Y \mid\ f \text{ is Lipschitz and bounded}\}.$$

Fact:

1. Y is complete $\iff$ any of $C_b^0, UC_b^0, \mathcal{B}$ are complete.

But what about $C_b^{0,1}(X;Y)$? Unfortunately this is not true :(

Theorem 5.0.2 Weierstrass' Monster

There exists $\{f_n\}_{n=0}^{\infty} \subseteq C_b^{0,1}(\mathbb{R};[0,1])$ such that

$$f_n \to f \in UC_b^0(\mathbb{R};[0,1])$$

but $f \notin C_b^{0,1}(\mathbb{R};[0,1])$. More precisely, for every $x \in \mathbb{R}$ and $0 < m \in \mathbb{N}$, there will exist $\delta_m^{\pm} \in \left\{\frac{8^{-m}}{2}, \frac{3\cdot 8^{-m}}{2}\right\}$ such that

$$\frac{|f(x \pm \delta_m^{\pm}) - f(x)|}{|\delta_m^{\pm}|} \geqslant \frac{7^m + 1}{48} \tag{5.1}$$

and

$$\left(\frac{f(x+\delta_m^+) - f(x)}{\delta_m^+}\right)\left(\frac{f(x-\delta_m^-) - f(x)}{\delta_m^-}\right) < 0. \tag{5.2}$$

Corollary 5.0.1

$\exists f \in UC_b^0(\mathbb{R};[0,1])$ such that f is nowhere differentiable. Moreover, for $x \in \mathbb{R}$, if $\lim_{y\to 0^-} \frac{f(x+y)-f(x)}{y} = L \in \overline{\mathbb{R}}$ and $\lim_{y\to 0^+} \frac{f(x+y)-f(x)}{y} = R \in \overline{\mathbb{R}}$, then $L, R \in \{+\infty, -\infty\}$ and $LR < 0$.

Proof. The fact that f is not differentiable at any given $x \in \mathbb{R}$ follows from (5.1). For the second point, (5.1) also implies that $|L| = |R| = \infty$. And (5.2) implies that $LR < 0$. 😊

Proof of Theorem 5.0.2. We begin by defining $\varphi : \mathbb{R} \to [0,1]$ via

$$\varphi(x) = \operatorname{dist}(x, 2\mathbb{Z}).$$

That is, the distance between x and the closest even integer. Also,

$$|\varphi(x) - \varphi(y)| < 1 \quad \forall x, y \in \mathbb{R} \tag{5.3}$$

and

$$\varphi(x+2) = \varphi(x) \quad \forall x \in \mathbb{R}. \tag{5.4}$$

Now for $N \in \mathbb{N}$, we let $f_N \in C_b^{0,1}(\mathbb{R};\mathbb{R})$ be given by

$$f_N(x) = \frac{1}{8}\sum_{n=0}^{N}\left(\frac{7}{8}\right)^n \varphi(8^n x).$$

This is Lipschitz because it is the finite sum of Lipschitz functions. Notice that

$$0 \leqslant f_N(x) \leqslant \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{7}{8}\right)^n = \frac{1}{8}\cdot\frac{1}{1-\frac{7}{8}} = 1.$$

Also if $M \geqslant N \geqslant 0$, then

$$|f_M(x) - f_N(x)| \leqslant \frac{1}{8}\sum_{n=N+1}^{M}\left(\frac{7}{8}\right)^n$$

$$\sup_{x\in\mathbb{R}} |f_M(x) - f_N(x) \leqslant \frac{1}{8}\sum_{n=N+1}^{M}\left(\frac{7}{8}\right)^n.$$

Therefore, $\{f_N\}_{N=0}^{\infty}$ is Cauchy in $UC_b^0(\mathbb{R};[0,1])$. This is complete set, so

$$\exists f \in UC_b^0(\mathbb{R};[0,1]) \quad \text{such that} \quad f_N \to f.$$

So,

$$f(x) = \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n \varphi(8^n x).$$

Now fix $x \in \mathbb{R}$ and $0 < m \in \mathbb{N}$. Pick $!k \in \mathbb{Z}$ such that $8^m x \in [2k, 2k+2)$, and set

$$\delta_m^+ = 8^{-m} \begin{cases} \frac{1}{2} & 8^m x \in [2k, 2k+\frac{1}{2}) \\ \frac{3}{2} & 8^m x \in [2k+\frac{1}{2}, 2k+1) \\ \frac{1}{2} & 8^m x \in [2k+1, 2k+\frac{3}{2}) \\ \frac{3}{2} & 8^m x \in [2k+\frac{3}{2}, 2k+2) \end{cases}$$

$$\delta_m^- = 8^{-m} \begin{cases} \frac{3}{2} & 8^m x \in [2k, 2k+\frac{1}{2}) \\ \frac{1}{2} & 8^m x \in [2k+\frac{1}{2}, 2k+1) \\ \frac{3}{2} & 8^m x \in [2k+1, 2k+\frac{3}{2}) \\ \frac{1}{2} & 8^m x \in [2k+\frac{3}{2}, 2k+2) \end{cases}.$$

The point is that

$$\begin{cases} |\varphi(y+1/2) - \varphi(y)| = \frac{1}{2} \\ |\varphi(y+3/2) - \varphi(y)| = \frac{1}{2} \end{cases}.$$

Next, set

$$\gamma_n^\pm = \frac{\varphi(8^n(x \pm \delta_m^\pm)) - \varphi(8^n x)}{\delta_m^\pm}$$

for $n \in \mathbb{N}$. Note that $0 \leqslant n < m \implies |\gamma_n^\pm| \leqslant \frac{|8^n \delta_m^\pm|}{|\delta_m^\pm|} = 8^n$. Also, $n = m \implies |\gamma_m^\pm| = \frac{1}{2}/|\delta_m^\pm \in \{8^m, 8^m/3\}$.
Now we consider $m < n$. In this situation, $8^n \delta_m^\pm \in 2\mathbb{Z}$, which means that $\gamma_m^\pm = 0$ due to the 2-periodicity of φ. Thus, for $N > m$:

$$\frac{8|f_N(x \pm \delta_m^\pm) - f_N(x)|}{|\delta_m^\pm|} = \left|\sum_{n=0}^{N} \left(\frac{7}{8}\right)^n \gamma_n^\pm\right| = \left|\sum_{n=0}^{m} \left(\frac{7}{8}\right)^n \gamma_n^\pm\right| \geqslant \left(\frac{7}{8}\right)^m |\gamma_m^\pm| - \sum_{n=0}^{m-1} \left(\frac{7}{8}\right)^n |\gamma_n^\pm|.$$

But this is also greater than

$$\left(\frac{7}{8}\right)^m \cdot \frac{8^m}{3} - \sum_{n=0}^{m-1} 7^n = \frac{7^m}{3} - \frac{7^m - 1}{6} = \frac{7^m + 1}{6}.$$

Sending N to infinity, we get

$$\frac{|f(x \pm \delta_m^\pm) - f(x)|}{|\delta_m^\pm|} \geqslant \frac{1}{8} \cdot \frac{7^m + 1}{6}.$$

In particular, $K(f) \geqslant \liminf''_{m \to \infty} \geqslant \liminf_{m \to \infty} \frac{7m+1}{48} = \infty$. So, $f \notin C_b^{0,1}$, but $f \in UC_b^0$. A similar argument shows the other inequality. 😄

Note:

Remarks:

1. Weierstrass' original proof used trigonometric functions in place of φ. The upshot is that there exists sequences of smooth functions that converge uniformly to a uniformly continuous and bounded function that is nowhere differentiable.

2. There's room to play.

Definition 5.0.4

Let V be a normed vector space, X a metric space. We define a seminorm on $C^{0,1}(X;V)$ by

$$K(f) = [f]_1 = \begin{cases} \sup_{x \neq y} \frac{\|f(x)-f(y)\|_V}{d(x,y)} & |X| \geqslant 2 \\ 0 & \text{otherwise} \end{cases}.$$

This is a seminorm: [constant] = 0. $C_b^{0,1}(X;V)$ is normed:

$$\|f\|_{C_b^{0,1}} = \|f\|_{C_b^0} + [f]_1.$$

Theorem 5.0.3

Let X be a metric space and V a normed vector space. Then $C_b^{0,1}(X;V)$ is complete if and only if V is complete.

Proof. We only do the backwards direction. Suppose V is Banach and $\{f_n\}_{n=\ell}^{\infty} C_b^{0,1}$ is Cauchy. Then it's Cauchy in C_b^0 as well, and hence convergent in C_b^0 to some $f \in C_b^0(X;V)$.

Let $\epsilon > 0$ and pick $N \geqslant \ell$ such that $m, n \geqslant N \implies \|f_n - f_m\|_{C_b^{0,1}} < \frac{\epsilon}{2} \implies [f_n - f_m] < \frac{\epsilon}{2}$. As such,

$$\|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))\|_V \leqslant \frac{\epsilon}{2} d(x,y)$$

for all $x, y \in X$. Send $m \to \infty$ for $n \geqslant N$, we have $[f_n - f] \leqslant \frac{\epsilon}{2}$. 😄

5.1 Stone-Weierstrass

Definition 5.1.1

A vector space V is an algebra if it is equipped with an associative multiplication. That is,

$$\begin{aligned} v \cdot w \in V &\text{ for } v, w \in V \\ v \cdot (w \cdot z) &= (v \cdot w) \cdot z \\ \alpha(v \cdot w) &= (\alpha v) \cdot w = v \cdot (\alpha w) \\ v(w+z) &= vw + vz \\ (w+z)v &= wv + zv. \end{aligned}$$

We say V is unital if there exists an $e \in V$ such that $ev = ve = v$ for all $v \in V$.
A normed algebra is an algebra V that's a normed vector space such that

$$\|vw\| \leqslant \|v\|\|w\|.$$

A Banach algebra is a complete normed algebra.

Example 5.1.1 (Normed Algebras)

1. $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ a field.
2. $\mathbb{F}^{n \times m}$ with the operator norm.
3. Let V be a normed vector space. Then $\mathcal{L}(V)$ is a normed algebra. $T, S \in \mathcal{L}(V)$, $TS = T \circ S$.
4. Let A be a normed algebra. $\mathcal{B}(X; A)$ is a normed algebra. $f, g \in (X; A)$, $fg \in (B; X)$, $(fg)(x) = f(x)g(x)$. We can check that A unital $\iff$ $\mathcal{B}(X; A)$ unital.
5. $\ell^p(\mathbb{N}; A)$ is a normed algebra. $x, y \in \ell^p \implies (xy)_n = x_n y_n \in A$. Also,

$$\|xy\|_p = \left(\sum_{n=0}^{\infty} \|x_n y_n\|_A^p\right)^{\frac{1}{p}} \leqslant \left(\sum \|x_n\|^p \|y_n\|^p\right)^{\frac{1}{p}} \leqslant \|x\|_\infty \|y\|_\infty \leqslant \|x\|^p \|y\|^p.$$

6. Let X be a metric space. Then $C^0(b(X; A))$, $UC_b^0(X; A)$ and $C_b^{0,1}(X; A)$ are normed algebras.
7. Now let $Y \subset X$ as above. $S = \{f \in C_b^0(X; A) \mid f(y) = 0 \,\forall y \in Y\}$ is a normed algebra.

Remarks:

1. Suppose A is a normed algebra. If we complete A to get A^*, then A^* is a Banach algebra.
2. Suppose A is a normed algebra and $B \subseteq A$ is a subalgebra. Then $\overline{B}$ is also a normed algebra.

Theorem 5.1.1

Let X be a unital Banach algebra. Then for every $x \in X$, the sequence $\{\sum_{n=0}^{N} \frac{x^n}{n!}\}_{N=0}^{\infty} \subseteq X$ converges. Thus, we may define the map $\exp : X \longrightarrow X$ via $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Proof. Recall that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \in \mathbb{R}$$

is well defined (convergent) because it is equal to

$$\lim_{N\to\infty}\sum_{n=0}^{N}\frac{r^n}{n!}$$

for all $r \in \mathbb{R}$. Now note for $N > M \geqslant 0$ and some $x \in X$,

$$\left\|\sum_{n=M+1}^{N}\frac{x^n}{n!}\right\| \leqslant \sum_{n=M+1}^{N}\frac{\|x^n\|}{n!} \leqslant \sum_{n=M+1}^{N}\frac{\|x\|^n}{n!}$$

Therefore, $\left\{\sum_{n=0}^{N}\frac{x^n}{n!}\right\}_{N=0}^{\infty}$ is Cauchy and therefore converges. 😄

Question: Suppose $A \subseteq C_b^0(X;\mathbb{F})$ is a subalgebra. Are there algebraic conditions on A that guarantee that A is dense in $C_b^0(X;\mathbb{F})$?
Answer: Yes.

Lenma 5.1.1

There eixst $\{f_n\}_{n=0}^{\infty} \subseteq UC_b^0([0,1];\mathbb{R})$ such that the following hold:

1. Each f_n is a polynomial satisfying $0 \leqslant f_n(x) \leqslant \sqrt{x}$ for all $x \in [0,1]$. Also, $f_n \leqslant f_{n+1}$.
2. $f_n \to f$ uniformly as $n \to \infty$ where $f(x) = \sqrt{x}$.

Proof. We define $\{f_n\}_{n=0}^{\infty}$ inductively as follows:

- $f_0(x) = 0$.
- Given f_n, define $f_{n+1}(x) = f_n(x) + \frac{x-(f_n(x))^2}{2}$.

We claim $0 \leqslant f_n(x) \leqslant \sqrt{x}$ for all $n \in \mathbb{N}$ and $x \in [0,1]$. We proceed by induction on n:

Suppose true for n, we show truth for $n+1$. Then $0 \leqslant f_n(x) \leqslant \sqrt{x} \implies 0 \leqslant \frac{f_n(x)+\sqrt{x}}{2} \leqslant \sqrt{x} \leqslant 1$. So,

$$\sqrt{x} - f_{n+1}(x) = \sqrt{x} - f_n(x) - \frac{x - f_n^2(x)}{2} = (\sqrt{x} - f_n(x))\left(1 - \frac{\sqrt{x} + f_n(x)}{2}\right)^2 \geqslant 0,$$

so $\sqrt{x} \geqslant f_{n+1}(x) \geqslant f_n(x) \geqslant 0$. Next, note that

$$f_{n+1}(x) = f_n(x) + \frac{x - f_n^2(x)}{2} \geqslant f_n(x)$$

so the sequence of f_n's is non-decreasing. Thus, for all $x \in [0,1]$,

$$\begin{aligned} 0 \leqslant f_n(x) &\leqslant \sqrt{x} \\ f_n(x) &\leqslant f_{n+1}(x). \end{aligned}$$

Now since the sequence is bounded and monotone, it converges pointwise to its supremum. Now,

$$f_{n+1}(x) = f_n(x) + \frac{x - f_n^2(x)}{2} \implies f(x) = f(x) + \frac{x + f^2(x)}{2} \implies f(x) = \sqrt{x}$$

as desired. Therefore, Dini tells us the convergence of $f_n \to f$ is uniform, since $[0,1]$ is compact. 😄

Theorem 5.1.2
Let X be a set and $A \subseteq \mathcal{B}(X;\mathbb{R})$ a subalgebra. The following hold:

1. If $f \in A$, then $|f| \in \overline{A}$.
2. If $f \in \overline{A}$, then $|f| \in \overline{A}$.
3. If $f, g \in \overline{A}$, then $f \vee g, f \wedge g \in \overline{A}$ where

$$f \vee g(x) = \max\{f(x), g(x)\}$$
$$f \wedge g(x) = \min\{f(x), g(x)\}.$$

Proof. We go in order:

1. Let $\{p_n\}_{n=0}^{\infty} \subseteq UC_b^0([0,1],[0,1])$ be as in the previous lemma. That is, p_n are polynomials such that $p_n \to \sqrt{\cdot}$ uniformly. If $f = 0$, then $|f| = 0 = |f| \in A \subseteq \overline{A}$, so suppose $f \neq 0$. Then let
$$g = \left(\frac{f}{\|f\|_{\mathcal{B}}}\right)^2 \in A.$$
Note that $0 \leqslant |g(x)| \leqslant 1$ for all $x \in X$. Also, $p_n(0) = 0$ for all $n \in \mathbb{N}$. So, $p_n \circ g \in A$ as well. Now note that $d_{\mathcal{B}}(p_n \circ g, \sqrt{g}) = \sup_{x \in X} |p_n(g(x)) - \sqrt{g(x)}| \leqslant \sup_{y \in [0,1]} |p_n(y) - \sqrt{y}|$, which tends to 0 as $n \to \infty$ by construction.
Therefore, $p_n \circ g \to \sqrt{g} \in \mathcal{B}(X;\mathbb{R})$, so $\sqrt{g} \in \overline{A}$. But, $\sqrt{g} = \frac{|f|}{\|f\|_{\mathcal{B}}}$. So,
$$|f| = \|f\|_{\mathcal{B}} \sqrt{g}.$$
So since $\sqrt{g} \in \overline{A}$, so is $|f|$.
2. Let $f \in \overline{A}$. Pick $\{f_n\}_{n=0}^{\infty} \subseteq A$ such that $f_n \to f$ in $\mathcal{B}$. By 1, $|f_n| \in \overline{A}$ for all n. Now notice
$$||f_n| - |f|| \leqslant |f_n - f|,$$
so $|f_n| \to |f|$ as $n \to \infty$, but $\overline{A}$ is closed, so $|f| in \overline{A}$.
3. Note that $f \vee g = \frac{f+g+|f-g|}{2}$, $f \wedge g = \frac{f+g-|f-g|}{2}$. Because of algebra, $f, g \in A \implies f \vee g, f \wedge g \in \overline{A}$.

😊

Remark: There's nor eason to stick with a single min/max. 3. + induction shows that if $f_1, \ldots f_n \in A$, then $\min_i f_i, \max_i f_i \in \overline{A}$.