Now, to write the class equation for the direct product of A_4 and Z_2, we need to find the conjugacy classes of the elements in $A_4 \times Z_2$.

First, note that Z_2 is the cyclic group of order 2 and has only two elements: the identity element and a non-identity element.

Next, recall that A_4 is the alternating group of degree 4 and has 12 elements. We can list its elements as:

$$\begin{aligned} A_4 = \{&(1), (12)(34), (13)(24), (14)(23), \\ &(123), (132), (124), (142), (134), (143), (234), (243)\} \end{aligned}$$

Now, we can form the elements of $A_4 \times Z_2$ by taking the direct product of each element in A_4 with both the identity element and the non-identity element of Z_2. This gives us 24 elements, which we can list as:

$$\begin{aligned} &(1, 0), (1, 1), \\ &((12)(34), 0), ((12)(34), 1), \\ &((13)(24), 0), ((13)(24), 1), \\ &((14)(23), 0), ((14)(23), 1), \\ &((123), 0), ((123), 1), \\ &((243), 0), ((243), 1) \\ &((142), 0), ((142), 1), \\ &((134), 0), ((134), 1), \\ &((132), 0), ((132), 1), \\ &((124), 0), ((124), 1), \\ &((143), 0), ((143), 1), \\ &((234), 0), ((234), 1), \end{aligned}$$

To find the conjugacy classes of $A_4 \times Z_2$, we need to determine which elements are conjugate to each other under the group operation. Two elements (g, h) and (g', h') are conjugate if there exists an element $(x, y) \in A_4 \times Z_2$ such that $(g, h) = (x, y)(g', h')(x^{-1}, y^{-1})$. Since the group operation in $A_4 \times Z_2$ is defined componentwise, we can write this condition as:

$$(gxg^{-1}, hyh^{-1}) = (g', h')(x, y)(g'^{-1}, h'^{-1})$$

This gives us two conditions: $gxg^{-1} = g'$ and $hyh^{-1} = h'$. Using these conditions, we can form the following conjugacy classes:

$$\{0, \text{conj classes of } A_4\}, \{1, \text{conj classes of } A4\}$$

Therefore, the class equation for $A_4 \times Z_2$ is:

$$|A_4 \times Z_2| = 1 + 3 + 4 + 4 + 1 + 3 + 4 + 4 = 24$$

Here, we write the normal subgroups of A_4:

- $\{e\}$

- K_4
- A_4

Then, the normal subgroups of $\mathbb{Z}_2$:

- $\{e\}$
- $\mathbb{Z}_2$

We can combine any of the subgroups (one from each subgroup list) to get a normal subgroup of $A_4 \times \mathbb{Z}_2$. This gives us the following list:

- $\{e\} \times \{e\} = \{e\}$
- $\{e\} \times \mathbb{Z}_2 = \mathbb{Z}_2$
- $K_4 \times \{e\} = K_4$
- $K_4 \times \mathbb{Z}_2$
- $A_4 \times \{e\} = A_4$
- $A_4 \times \mathbb{Z}_2$

Using these, we can find the quotient groups:

- $(A_4 \times \mathbb{Z}_2)/\{e\} = A_4 \times \mathbb{Z}_2$
- $(A_4 \times \mathbb{Z}_2)/\mathbb{Z}_2 = A_4$
- $(A_4 \times \mathbb{Z}_2)/K_4 = \mathbb{Z}_6$
- $(A_4 \times \mathbb{Z}_2)/(K_4 \times \mathbb{Z}_2) = \mathbb{Z}_3$
- $(A_4 \times \mathbb{Z}_2)/A_4 = \mathbb{Z}_2$
- $(A_4 \times \mathbb{Z}_2)/(A_4 \times \mathbb{Z}_2) = \{e\}$

Additionally, here are the elements and conjugacy classes of $A_4 \times \mathbb{Z}_2$ but written as generated by r and s.

```
In [41]: ConjugacyClasses={}
         for i in range(len(list(D.keys()))):
             b=[]
             for j in range(len(list(D.keys()))):
                 p=list(D.values())[j].inverse()*list(D.val
                 I=list(D.values()).index(p)
                 if(list(D.keys())[I] not in b):
                     b.append(list(D.keys())[I])
             ConjugacyClasses[list(D.keys())[i]]=b
```

```
In [42]: ConjugacyClasses

Out[42]: {'r': ['r', 'srs', 'rrsrsr', 'rrsrrs'],
          'rr': ['rr', 'srrs', 'rsrs', 'srsr'],
          'rs': ['sr', 'rrsrr', 'rs', 'rrsrsrs'],
          'rrs': ['rsr', 'srr', 'srsrs', 'rrs'],
          's': ['rrsr', 'rsrr', 's'],
          'sr': ['rrsrr', 'rs', 'rrsrsrs', 'sr'],
          'rsr': ['srr', 'rrs', 'srsrs', 'rsr'],
          'rrsr': ['rsrr', 's', 'rrsr'],
          'srr': ['rrs', 'rsr', 'srr', 'srsrs'],
          'rsrr': ['s', 'rrsr', 'rsrr'],
          'rrsrr': ['rs', 'sr', 'rrsrsrs', 'rrsrr'],
          'srs': ['rrsrsr', 'rrsrrs', 'r', 'srs'],
          'rsrs': ['srsr', 'srrs', 'rsrs', 'rr'],
          'rrsrs': ['rsrsr', 'rsrrs', 'rrsrs'],
          'srrs': ['rsrs', 'srsr', 'rr', 'srrs'],
          'rsrrs': ['rrsrs', 'rsrsr', 'rsrrs'],
          'rrsrrs': ['srs', 'rrsrsr', 'r', 'rrsrrs'],
          'srsr': ['srrs', 'rsrs', 'rr', 'srsr'],
          'rsrsr': ['rsrrs', 'rrsrs', 'rsrsr'],
          'rrsrsr': ['rrsrrs', 'srs', 'rrsrsr', 'r'],
          'rsrsrs': ['rsrsrs'],
          'rrsrsrs': ['rrsrsrs', 'rrsrr', 'rs', 'sr'],
          'srsrs': ['srsrs', 'rsr', 'srr', 'rrs']}
```

As practice, we write the closed form formulas for various Burnside lemma problems. We start with the counting how many ways you can paint the seats of the ferris wheel with n colors.

$$\frac{n^6 + 3n^5 + 3n^4 + n^3 + 2n^2 + 14n}{24}$$

Next, we count how many ways you can color the carts of the ferris wheel with n colors.

$$\frac{n^3 + 2n}{3}$$