21603 Model Theory I

Rohan Jain

Contents

Chapter 1

1.1　random info

rami@cmu.edu

1. Set Theory

2. Model Theory

3. Recursion Theory

4. Proof Theory

1973 book by Chang and Keisler - Model Theory - Highly recommended for elementary model theory.
What is model thoery? Model Theory = logic + universal algebra
1984 - W. Hodges - Shorter Model Theory
model theory = algebraic geometry - field theory
Algebraic structures:

1. groups

2. rings

3. vector spaces

4. fields

5. graphs - (V, E)

6. ordered structures

Around 1870, mathematicians started to layout the foundations for mathematics. One of the ideas was axiomatization. One example was Euclidean axioms for plane geometry.

1.2　Structures and Languages

> **Definition 1.2.1: Language**
>
> L is a language if $L = F \cup R \cup C$ are parameter disjoint.

> **Definition 1.2.2: L-structure**
>
> Let L be a language (similarity type/signature). Then $\mathcal{M}$ is an L-structure provided:
>
> $$\mathcal{M} = (U, \{f^{\mathcal{M}} \mid f \in F\}, \{r^{\mathcal{M}} \mid r \in R\}, \{c^{\mathcal{M}} \mid c \in C\})$$
>
> where U is a nonempty set. U is also called the universe of $\mathcal{M}$.
> For any $f \in F$ there is $U(f)$ natural number such that $f^{\mathcal{M}} : U^{n(F)} \to U$, $R^{\mathcal{M}} \subseteq U^{n(R)}$, $C^{\mathcal{M}} \subseteq U$, $\forall c \in C$.

Notation: $|\mathcal{M}| = U$. The cardinal of $\mathcal{M}$ is $|U|$. $\|\mathcal{M}\|$ denotes the cardinality of $\mathcal{M}$.

Definition 1.2.3: Theory

Let L be a language. A theory T is a set of sentences in L. A sentence is a finite set of symbols from L.

Example 1.2.1 (Sentences)

$L_{\mathrm{gr}} = \{e, \cdot\}$. $e \in C$, $\cdot \in F$. $T_{\mathrm{gr}} = \{\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \forall x (x \cdot e = x, e \cdot x = x), \forall x \exists y (x \cdot y = e, y \cdot x = e)\}$. These are the group axioms (associativity, identity, existence of inverse).

Definition 1.2.4: Term

Let L be a language. A term is:

1. c is a term for any $c \in C$.

2. x when x is a variable.

3. $\tau_1, \ldots, \tau_k$ terms, $f \in F$, $n(f) = k$, then $f(\tau_1, \ldots, \tau_k)$ is a term.

Definition 1.2.5: Term

$\mathrm{Term}(L)$ is a minimal set of finite strings of symbols from $L \cup \{(,)\} \cup X$ that contains $C \cup x$ and closed under the following rule:

$$\tau_1, \ldots, \tau_k \in \mathrm{Term}(L), f\,k-\text{place function symbol}, \text{then} f(\tau_1, \ldots, \tau_k) \in \mathrm{Term}(L)$$

Example 1.2.2 (L_r)

$L_r = \{0, 1, +, -\}$. $\mathrm{Term}(L_r) \supseteq \{\sum a_j x_1^{n_j} \mid a_j \in \mathbb{Z}, n_j \in \mathbb{N}\}$.

Example 1.2.3 (L_{gr})

$\mathrm{Term}(L_{\mathrm{gr}}) \supseteq \{x_1 \cdot x_n \cdots x_n \mid x_i \in X, n \in \omega\}$.

Definition 1.2.6: AFml

Let L be a language. The set of atomic fomrulas denotes by $\mathrm{AFml}(L)$ is the smallest set of formulas in L that contains $L \cup \{(,), =\} \cup X$ such that:

1. If $\tau_1, \tau_2 \in \mathrm{Term}(L)$, then $\tau_1 = \tau_2 \in \mathrm{AFml}(L)$.

2. Given $R(x_1, \ldots, x_n)$ relation symbol and $\tau_1, \ldots, \tau_n \in \mathrm{Term}(L)$, then $R(\tau_1, \ldots, \tau_n) \in \mathrm{AFml}(L)$.

Definition 1.2.7: Fml

$\mathrm{Fml}(L)$ is the set of (first order) formulas in L. Which is the mininmal set of finite strings of symbols from $L \cup \{(,), =, \neg, \vee, \wedge, \implies, \impliedby, \iff, \forall, \exists\} \cup X$ such that:

1. $\mathrm{Fml}(L) \supseteq \mathrm{AFml}(L)$.

2. If φ is a formula, then $\neg\varphi$ is a formula.

3. If $x \in \{\wedge, \vee, \implies, \iff\}$ and $\varphi, \psi \in \mathrm{Fml}(L)$, then $(\varphi x \psi) \in \mathrm{Fml}(L)$.

4. If $\varphi \in \mathrm{Fml}(L)$, $Q \in \{\forall, \exists\}$, and $x \in X$, then $Qx\varphi \in \mathrm{Fml}(L)$.

5. If $\varphi \in \mathrm{Fml}(L)$, $\mathrm{FV}(\varphi)$ is the set of free variables in φ defined by induction on the structure of φ.
   Case 1: $\varphi \in \mathrm{AFml}(L)$.

   (a) φ is $\tau_1 = \tau_2$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \mathrm{FV}(\tau_2)$.
   (b) φ is $R(\tau_1, \ldots, \tau_n)$. $\mathrm{FV}(\varphi) = \mathrm{FV}(\tau_1) \cup \ldots \cup \mathrm{FV}(\tau_n)$.

   Case 2:

   (a) if φ is $\neg\psi$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$.
   (b) if $\varphi = \psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi_1) \cup \mathrm{FV}(\psi_2)$.

   Case 3: φ is $Qx\psi$, $Q \in \{\forall, \exists\}$. Then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \setminus \{x\}$.

6. $\mathrm{Sent}(L)$ are the sentences in L. $\mathrm{Sent}(L) = \{\varphi \in \mathrm{Fml}(L) \mid \mathrm{FV}(\varphi) = \emptyset\}$.

Example 1.2.4

If $L_f = \{+, \cdot, 0, 1\}$, then $T_f = \{$

- $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$,

- $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$,

- $\forall x \forall y (x + y = y + x)$,

- $\forall x \forall y (x \cdot y = y \cdot x)$,

- $\forall x (x \cdot 1 = x, 1 \cdot x = x)$,

- $\forall x (x + 0 = x, 0 + x = x)$,

- $\forall x \exists y (x \cdot y = 1, y \cdot x = 1)$,

- $\forall x \exists y (x + y = 0, y + x = 0)$,

- $\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

$\}$.

Definition 1.2.8: L-theory

T is an L-theory if $T \subseteq \mathrm{Sent}(L)$.

The example above is "field theory".

Definition 1.2.9

Let M be an L-structure. $\tau(\bar{x})$ is a term, $\bar{a} \in |M|^{\ell(n)}$. T
Case 1: $\tau(\bar{x}) = c$ for some constant symbol. Then $\tau^M(\bar{a}) = c^M$.
Case 2: $\tau(\bar{x}) = x_i$. Then $\tau^M(\bar{a}) = a_i$.
Case 3: $\tau(\bar{x}) = f(\tau_1, \ldots, \tau_k)$. Then $\tau^M(\bar{a}) = f^M(\tau_1^M(\bar{a}), \ldots, \tau_k^M(\bar{a}))$.

Definition 1.2.10: ⊨

Let L be a language, $\varphi \in \text{Fml}(L)$, M and L-structure, $n = \ell(\bar{x})$, $\bar{a} \in |M|^n$. Define $M \models \varphi(\bar{a})$ at $\bar{a}$ by induction on the structure of φ:

- If φ is atomic,

    - when $\varphi(x)$ is $\tau_1 = \tau_2$, then $M \models \varphi(\bar{a})$ iff $\tau_1(\bar{a}) = \tau_2(\bar{a})$.
    - when $\varphi(x)$ is $R(\tau_1, \ldots, \tau_k)$, then $M \models \varphi(\bar{a})$ iff $(\tau_1(\bar{a}), \ldots, \tau_k(\bar{a})) \in R^M$.

- If φ is not atomic, then:

    - if φ is $\neg\psi$, then $M \models \varphi(\bar{a})$ iff $M \models \psi(\bar{a})$ is false.
    - if φ is $\psi_1 * \psi_2$ for $* \in \{\wedge, \vee, \implies, \iff\}$, then $M \models \varphi(\bar{a})$ iff $M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a})$.
    - if φ is $\exists y\psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff there is $b \in |M|$ such that $M \models \psi(b, \bar{a})$.
    - if φ is $\forall y\psi(y, \bar{x})$, then $M \models \varphi(\bar{a})$ iff for all $b \in |M|$, $M \models \psi(b, \bar{a})$.

Definition 1.2.11

Let M be an L-structure and T an L-theory. $M \models T$ iff for every $\varphi \in T$, $M \models \varphi$. We say T "satisfies" M.

Example 1.2.5 (Models)
$M \models T_f \iff (|M|, +^M, \cdot^M, 0^M, 1^M)$ is a field.

Definition 1.2.12: Mod

$\text{Mod}(T) = \{M \ L\text{-structure} \mid M \models T\}$.

Example 1.2.6
$\text{Mod}(T_f)$ is the class of all fields and $\text{Mod}(T_{\text{gr}})$ is the class of all groups.

Definition 1.2.13: Structure Isomorphism

Let M, N both be L-structures. f is an isomorphism from M onto N if $f : |M| \to |N|$ is a bijection such that:

- $f(c^M) = c^N$ for all $c \in C$.

- $G(x_1, \ldots, x_k)$ function symbol. $a_1, \ldots, a_k \in |M|$, then $f(G^M(a_1, \ldots, a_k)) = G^N(f(a_1), \ldots, f(a_k))$.

- $R(x_1, \ldots, x_k)$ predicate symbol. $a_1, \ldots, a_k \in |M|$, then $(a_1, \ldots, a_k) \in R^M$ iff $(f(a_1), \ldots, f(a_k)) \in R^N$.

We write $f : M \cong N$. Also $M \cong N \iff \exists f : M \cong N$.

> **Definition 1.2.14**
>
> Let $\lambda \geqslant \aleph_0$, T an L-theory. T is λ-categorical if for all $M, N \models T$ of cardinality λ, $M \cong N$.

> **Theorem 1.2.1** Los Conjecture (1954)
>
> Let L be a language, T a first order L-theory, in an at most countable language. If $\exists \lambda > \aleph_0$ such that T is λ-categorical, then for all $\mu > \aleph_0$, T is μ-categorical.

Somewhere around 1961-1965, Morley proved this conjecture.

Chapter 2

Basic Concepts

Lenma 2.0.1

1. $M \cong N \implies N \cong M$.

2. $M \cong M$, $f = \mathrm{id}_{|M|}$.

3. Let M_1, M_2, M_3 be all L-structures. Then $f_1 : M_1 \cong M_2$ and $f_2 : M_2 \cong M_3 \implies f_2 \circ f_1 : M_1 \cong M_3$.

In other words, $\cong$ is an equivalence relation on $\mathrm{Struct}(L)$.

$M/\!\!\cong\; = \{N \text{ is an } L(M)\text{-structure} \mid N \cong M\}$.

Definition 2.0.1: Spectrum function of T

Let T be a first order theory ($T \subseteq \mathrm{Sent}(L)$) of cardinality λ. Then $I(\lambda, T)$ is the number of pairwise nonisomorphic models of T of cardinality λ. We have

$$I(\lambda, T) = |M/\!\!\cong|$$

where $M \models T$ and $\|M\| = \lambda$.

Consider $\lambda \mapsto I(\lambda, T)$, $\lambda \in \mathrm{Card}$ (the class of cardinal numbers). But what is the shape of $\lambda \mapsto I(\lambda, T)$. Is it weakly monotone? That is, $\mu > \lambda \implies I(\mu, T) \geqslant I(\lambda, T)$?

Theorem 2.0.1 Morley's Conjecture (~1965)

Suppose T is first order and $|L(T)| \leqslant \aleph_0$. Then $\mu > \lambda > \aleph_0 \implies I(\mu, T) \geqslant I(\lambda, T)$.

The basic problem is that given M and N both of cardinality λ, $M \not\cong N$, find M', N' both of cardinality μ such that $M' \cong N'$. In 1990, Shelah solved Morley's Conjecture. However, this is an open question for uncountable T.

Theorem 2.0.2 Morley's Category Theorem

Let T be a first order theory for $|L(T)| \leqslant \aleph_0$. Then $\exists \lambda > \aleph_0, I(\lambda, T) = 1$ then $\forall \mu > \aleph_0, I(\mu, T) = 1$.

Shelah listed all possible functions $\lambda \mapsto I(\lambda, T)$ and, by hand, verified that they were weakly monotone.

Example 2.0.1

1. $I(\lambda, T) = 1$ for all $\lambda > \aleph_0$.

2. $I(\lambda, T) = 2^\lambda$ for all $\lambda > \aleph_0$.

Hart, Hrushovski, and Laskowski found all the 13 functions.

Definition 2.0.2: Submodel

Let M, N be L-structures. M is a submodel of N if:

1. $|M| \leqslant |N|$

2. $\forall a_1, \ldots, a_n \in |M|$ and $F(x_1, \ldots, x_n)$, $F^M(a_1, \ldots, a_n) = F^N(a_1, \ldots, a_n)$.

3. $c^M = c^N$ for all constant symbols c.

4. $R^M = R^N \cap (|M| \times \cdots \times |M|)$.

Definition 2.0.3: Elementarily Equivalent

Let M, N be L-structure. M is elementarily equivalent to N denoted by $M \equiv N$ provided $M \models \varphi \iff N \models \varphi$ for any $\varphi \in \text{Sent}(L)$.

Definition 2.0.4

Let M be an L-structure. The theory of M is denoted $(M) = \{\text{Th}(M)\varphi \in \text{Sent}(L) \mid M \models \varphi\}$.

Let $N := (\omega, +, \cdot, 0, 1)$. Then TA = Th$(N)$ "True Arithmetic". For example the twin primes conjecture is $\{p \mid p$ and $p + 2$ are both primes$\}$ is infinite. If it is true, then it is a member of TA.

Theorem 2.0.3

Let M, N be L-sustructres. If $M \cong N$, then $M \equiv N$.

Theorem 2.0.4

Let M, N be L-structures. Suppose $f : M \cong N$. Then for any $\bar{a} \in |M|$ and any $\varphi(\bar{x}) \in \text{Fml}(L)$ with $\ell(\bar{x}) = \ell(\bar{a})$, $M \models \varphi[\bar{a}] \iff N \models \varphi[f(\bar{a})]$.

Proof. Suppose $\varphi(\bar{x})$ is atomic.

Lenma 2.0.2

Suppose $f : M \cong N$ and $\tau(\bar{x})$ sequence of terms. $\bar{a} \in |M|$, $\ell(\bar{x}) = \ell(\bar{a})$. Then $f(\tau(\bar{a})) = \tau(f(\bar{a}))$.

Proof. By induction on the length of τ.
  Case 1: $\tau(\bar{x})$ is x. Then $f(\tau(\bar{a})) = f(a) = \tau(f(\bar{a}))$.
  Case 2: $\tau(\bar{x}) = c$. then $f(c^M) = c^N$ by definition of isomorphism.
  Case 3: $\tau(\bar{x}) = G(y_1, \ldots, y_n)$ function symbol. Then $\tau_1(\bar{x}), \ldots, \tau_n(\bar{x})$ are terms. By induction, $f(\tau(\bar{a})) = f(G^M(\tau_1(\bar{a}), \ldots, \tau_n(\bar{a}))) = G^N(f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) = \tau^N(f(\bar{a}))$. ☺

  Now returning to the proof:
  Case 1: $\varphi(\bar{x})$ is $\tau_1(\bar{x}) = \tau_2(\bar{x})$. Then, we have $M \models \varphi(\bar{a}) \iff \tau_1(\bar{a}) = \tau_2(\bar{a}) \iff f(\tau_1(\bar{a})) = f(\tau_2(\bar{a})) \iff \tau_1^N(f(\bar{a})) = \tau_2^N(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.
  Case 2: $\varphi(\bar{x})$ is $R(\tau_1(\bar{x}), \ldots, \tau_n(\bar{x}))$. When $R(y_1, \ldots, y_n)$ is a relation symbol and $\tau_i(\bar{x})$ are terms. Then $M \models \varphi(\bar{a}) \iff (\tau_1(\bar{a}), \ldots, \tau_n(\bar{a})) \in R^M \iff (f(\tau_1(\bar{a})), \ldots, f(\tau_n(\bar{a}))) \in R^N \iff (\tau_1(f(\bar{a})), \ldots, \tau_n(f(\bar{a}))) \iff N \models \varphi(f(\bar{a}))$.
  Suppose φ is $\neg\psi$. Then $M \models \varphi(\bar{a}) \iff M \not\models \psi(\bar{a}) \iff N \not\models \psi(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.
  Suppose φ is $\psi_1 \wedge \psi_2$. Then $M \models \varphi(\bar{a}) \iff M \models \psi_1(\bar{a})$ and $M \models \psi_2(\bar{a}) \iff N \models \psi_1(f(\bar{a}))$ and $N \models \psi_2(f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$.
  Suppose $\varphi(\bar{x})$ is $\exists y \psi(y, \bar{x})$. Then $M \models \varphi(\bar{a}) \iff$ there is $b \in |M|$ such that $M \models \psi(b, \bar{a}) \iff$ there is $c \in |N|$ such that $N \models \psi(c, f(\bar{a})) \iff N \models \exists \psi(y, f(\bar{a})) \iff N \models \varphi(f(\bar{a}))$. ☺

General Remark:

$$M \models \exists y \varphi(y, \bar{a}) \iff M \models \neg \forall y \neg \varphi(y, \bar{a})$$
$$M \models \neg \exists y \varphi(y, \bar{x}) \iff \forall y \neg \varphi(y, \bar{a}).$$

> **Example 2.0.2**
>
> $L_{gr} = \{\cdot, 1\}$. $(\mathbb{Q}, +, 0), (\mathbb{R}, +, 0)$ are not isomorphic because diff cardinality. $(\mathbb{Q}, +, 0), (\mathbb{Z}, +, 0)$ are not isomorphic because:
>
> $$(\mathbb{Q}, +, 0) \models \forall x \exists y (x = y + y).$$
>
> This sentence is not true for $\mathbb{Z}$ under addition.

$N = (\omega, +, \cdot, 0, 1)$ is called the standard model of arithmetic. TA = Th(N), true arithmetic.

> **Question**
>
> Given $M_1, M_2 \models$ TA both countable. Are they isomorphic?

> **Question**
>
> What is $I(\aleph_0, \text{TA})$? Voted on $2^{\aleph_0}$.... and it is.

Let T be a theory and $\varphi \in \text{Sent}(L)$. We say T proves φ (denoted $T \vdash \varphi$) if there exists a finite set of sequences from L, $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for all i, $\varphi_i \in T$ or φ_i is a tautology or there are $j, k < i$ where $\varphi_j = (\varphi_k \implies \varphi_i)$.

1. $Q \to P$: the rule of inference. "modus ponens".

2.

Other rules (possible members of φ):

- $x = y$, $y = z$ then $x = z$.

- If $\varphi_i = \forall x \varphi(x)$, then also $\forall y \varphi(y)$ in the sequence.

- If $\forall x \varphi(x)$ also $\varphi(\tau(\bar{c}))$.

> **Definition 2.0.5**
>
> A set of sentences is a consistent theory if there is no φ such that φ and $\neg \varphi$ are both in the theory. T is inconsistent if it is not consistent.

> **Theorem 2.0.5** Godel's Completeness Theorem
>
> Let T be some set of sentences in L. Then T is consistent iff T has a model.

Godel only proved it for when $|L| \leqslant \aleph_0$.

> **Theorem 2.0.6** Compactness Theorem
>
> Let $T \subseteq \text{Sent}(L)$. If for any finite $T_0 \subseteq T$, T_0 has a model, then T has a model.

Proof. Enough to show by completeness that is consistent. If T inconsistent, $T \vdash \varphi$ and $T \vdash \neg \varphi$, there are $T_1, T_2 \subset T$ finite such that $T_1 \vdash \varphi$ and $T_2 \vdash \neg \varphi$. Then $T_1 \cup T_2 \vdash \varphi \wedge \neg \varphi$. By assumption on T, $\exists M_0 \models T_1 \cup T_2$. Then $M_0 \models \varphi \wedge \neg \varphi$ which is a contradiction. ☺

ZF cannot prove the compactness theorem.

Let G be a group, $A \subseteq G$. Then subgroup generated by A is denoted:

$$\langle A \rangle := \bigcap \{H \mid H \leqslant G, H \supseteq A\}.$$

Proposition 2.0.1

$\langle A \rangle = \{a_1^{\epsilon_1} \cdot a_2^{\epsilon_2}, \ldots, a_n^{\epsilon_n} \mid n \in \mathbb{N}, a_i \in A, \epsilon_i \in \{1, -1\}\}$.

Theorem 2.0.7 Submodel Theorem

Let M be an L-structure. Denote by $\lambda := |L| + \aleph_0$. For any $A \subseteq |M|$, there is $N \subseteq M$ such that $|N| \supseteq A$ and $\|N\| \leqslant |A| + \lambda$.

Remark: When $|L| \leqslant \aleph_0$, then $\lambda = \aleph_0$. For infinite A we have $|A| \geqslant \|N\| \geqslant |A|$, so by CB, $\|N\| = |A|$.

Proof. By induction on $n < \omega$, define $\{A_n \subseteq |M| \mid n < \omega\}$ such that $A_0 = A \cup \{c^M \mid c \text{ constant symbols}\}$. For $n+1$, let $A_{n+1} = A_n \cup \{f^M(a_1, \ldots, a_k) \mid f \text{ function symbol}\}$. Take $B = \bigcup_{n < \omega} A_n$. Now let $N = (B, F^M, R^M, c^M)_{F,R,c \in L}$.

We claim that N is as required.

$|N| \supseteq A$: $B = \bigcup_{n < \omega} A_n \supseteq A_0 \supseteq A$.

$N \subseteq M$: Enough to show $F(x_1, \ldots, x_k)$ is a function symbol for $a_1, \ldots, a_k \in B$.

$F(a_1, \ldots, a_k) \in B$: Given $a_1, \ldots, a_k \in B$, forall $1 \leqslant n \leqslant k$, $\exists n_i < \omega$, $a_i \in A_{n_i}$. Let $\mu = \max\{n_1, \ldots, n_k\}$. By $A_{n+1} \supseteq A_n$ for all n, we have $A_\mu \supseteq A_{n_i}$ for all $i \leqslant k$. So $a_1, \ldots, a_k \in A_\mu$. By definition of $A_{\mu+1}$, $F(a_1, \ldots, a_k) \in A_{\mu+1} \subseteq B = |N|$.

$\|N\| \leqslant |A| + \mu$: We proceed with induction on $n < \omega$.

$|A_n| \leqslant \lambda + |A|$: $n = 0$. By definition of A_0, $|A_0| \leqslant |A| + |L| \leqslant |A| + \lambda \implies |L| \leqslant \lambda$.

So $|A_{n+1}| \leqslant |A_n| + |L| + \sum_{k \leqslant \omega} |A_n|^k \leqslant \mu + \sum_{k < \omega} \mu^k = \sum_{k < \omega} \mu = \mu + \aleph_0 \mu = \mu = |A| + \lambda$. ☺

Definition 2.0.6

Let M be an L structure, $L_0 \subseteq L$, $M \restriction L_0 := \langle |M|, F^M, R^M, c^M \rangle_{F,R,c \in L_0}$. We can also say M is an expansion of $M \restriction L_0$.

Example 2.0.3

Suppose you have a field $(F, +, \cdot, 0, 1)$, so $L = (+, \cdot, 0, 1)$. Then let $L_0 = \{+, 0\}$. Then $F \restriction L_0$ is the additive group of F.

Theorem 2.0.8

Let T be a first order theory with $\lambda \geqslant |L(T)| + \aleph_0$. If T has an infinite model, then $\exists N \models T$ such that $\|N\| \geqslant \lambda$.

Remark: This is a very simple version of the Upward Lowenheim-Skolem Theorem.

Proof. Let $\{c_i \mid i < \lambda\}$ be a set of constant symbols not in $L(T)$. Let $T_1 = T \cup \{c_i \neq c_j \mid i \neq j, i, j < \lambda\}$. We claim that if $N_1 \models T_1$, then $N := N_1 \restriction L(T)$ is as required.

As $N_1 \models T_1$ and $T \subseteq T_1$, $N_1 \models T$, so $N \models T$.

Let $a_i := c_i$ for all $i < \lambda$. Let $i < j < \lambda$. Since $N_1 \models c_i \neq c_j$, by definition of $\models$, $a_i \neq a_j$. But $\{a_i \mid i < \lambda\} \subseteq |N_1| = |N|$.

So by claim, it is enought o show that there exists $N_1 \models T_1$. We apply the compactness theorem to T_1. Let $T_0 \subseteq T_1$ be finite.

Let $i_1, \ldots i_n < \lambda$ such that $T_0 \subseteq T \cup \{c_{i_\ell} \neq c_{i_k} \mid \ell \neq k, \ell, k \leqslant n\}$. As T has an infinite model M, pick $\{a_1, \ldots, a_n\} \subseteq |M|$. Let $M_0 = \langle |M|, R^M, F^M, c^M, a_1, \ldots, a_n \rangle_{R,F,c \in L(T)}$. $c_i^{M_0} := a_i$. Clearly $M_0 \models T_0$. ☺

CT - the statement of the compactness theorem. Facts:

- Tychonov's theorem. Product of compact topological spaces is compact. This is known to be equivalent to the axiom of choice.

- Ty H $\iff$ Tychonov's theorem for Hausdorff spaces.

- BPI: Boolean Prime Ideal Theorem.

> **Theorem 2.0.9**
>
> BPI $\iff$ CT $\iff$ Ty H.

This was proved by ZF. More facts:

- ZFC $\vdash$ CT.

- $\exists M \models (ZF + \neg BPI)$. So, ZF $\nvdash$ CT.

Remark: $M \subseteq N \nRightarrow M \equiv N$.

> **Example 2.0.4**
>
> $N = \langle \omega, +, 0 \rangle$. Then $M = \langle \mathbb{Z}, +, 0 \rangle$. $N \subseteq M$, but M is a group and N isn't.

> **Example 2.0.5**
>
> Suppose $\{M_k \mid k < \omega\}$. We have that $M_k \subseteq M_{k+1}$ for all k. Then $N := \bigcup_{k<\omega} M_k$. Then,
>
> - $F^N := \bigcup_{k<\omega} F_k^M$.
>
> - $R^N := \bigcup_{k<\omega} R_k^M$.

More generally, let $(I, <)$ be a linearly ordered set. Suppose $\{M_i \mid i \in I\}$ is such that $i \leqslant j \implies M_i \subseteq M_j$. Then $N := \bigcup_{i\in I} M_i$.
Question: Does $N \equiv M_k$ for any k?
Answer: No.

Suppose we know in addition that for all k, $M_k \equiv M_{k+1}$. The answer remains no.

Now let $L = \{<\}$. For $k \in \omega$, $M_k = [-k-1, k+1]$. Clearly $M_k \cong M_{k+1} \models \exists x \forall y [y > x \lor y = x]$. But $N \equiv \mathbb{R} \models \forall x \neg \exists y [y > x \lor y = x]$.

> **Definition 2.0.7**
>
> Let M and N be L-structures. M is an elementary submodel of N (denoted by $M \prec N$) if:
>
> 1. $M \subseteq N$.
>
> 2. for every $\varphi(x) \in \text{Fml}(L)$ and every $\bar{a} \in |M|$ with $\ell(x) = \ell(\bar{a})_{\text{i}}$ we know $M \models \varphi[\bar{a}] \iff N \models \varphi[\bar{a}]$.

> **Theorem 2.0.10** Tarski Vaught Chain Theorem (1956)
>
> If $(I, <)$ linearly ordered set and $\{M_i \mid i \in I\}$ elementary chain and $M := \bigcup_{i\in I} M_i$. Then $M_i \prec N$ for all $i \in I$.

Proof. Clearly $M_i \subseteq N$ for all $i \in I$. So now we proceed by induction on $\varphi(x) \in \text{Fml}(L)$.

For $\varphi \in \text{Fml}(L)$, $\bar{a} \in |M|$, then $M_i \models \varphi[\bar{a}] \iff N \models \varphi[\bar{a}]$ for every $i \in I$ and $\bar{a} \in |M_i|$.

> **Lenma 2.0.3**
>
> $M \subseteq N \implies M \models (\varphi[\bar{a}] \iff N \models \varphi[\bar{a}] \forall \bar{a} \in |M|$ and every atomic formula $\varphi \in \text{Fml}(L))$.

We check that when $\varphi(\bar{x}) \in \text{AFml}(L)$:

(a) $\varphi(\bar{x})$ is $\tau_1(\bar{x}) = \tau_2(\bar{x})$.

$M \models \tau_1(\bar{a}) = \tau_2(\bar{a})$, so $\tau_1^M[\bar{a}] = \tau_2^M[\bar{a}] \iff \tau_1^N[\bar{a}] = \tau_2^N[\bar{a}]$. As such, $M \models \varphi(\bar{a})$.

(b) $\varphi(\bar{x})$ is $R(\tau_1, \ldots, \tau_k)$.

Then $M \models \varphi[\bar{a}] \implies \langle \tau_1^M[\bar{a}], \ldots, \tau_k^M[\bar{a}] \rangle \in R^M \implies \langle \tau_1^N[\bar{a}], \ldots, \tau_k^N[\bar{a}] \rangle \in R^N \implies N \models \varphi[\bar{a}]$.

So if φ is atomic, we have $(*)$

If $\varphi(\bar{x}) = \neg\psi(\bar{x})$, then $M_i \models \varphi[\bar{a}] \iff$ not true $M_i \models \psi(\bar{a}) \iff$ not true $N \models \psi[\bar{a}] \iff N \models \neg\psi[\bar{a}]$.

Then if $\varphi(\bar{x})$ is $\psi_1(\bar{x}) \wedge \psi_2(\bar{x})$,

$$M_i \models \varphi[\bar{a}] \iff M_i \models \psi_1[\bar{a}] \wedge M_i \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \wedge N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}].$$

If $\varphi(\bar{x})$ is $\exists y \psi(y, \bar{x})$,

$$M_i \models \varphi[\bar{a}] \iff \text{there is } b \in |M_i| \text{such that} M_i \models \psi[b, \bar{a}].$$

So by the inductive hypothesis,

$$N \models \psi[b, \bar{a}] \implies b \in |N|.$$

As such, $N \models \exists y \psi(y, \bar{x}) \implies N \models \varphi[\bar{x}]$.

Suppose $N \models \varphi[\bar{x}] \implies \exists b \in |N| \psi[b, \bar{a}]$. As $N = \bigcup M_i$, $\exists j \in I$ such that $b \in M_j$. So let $k = \max(i, j)$. Since $i_1 < i_2 \implies M_{i_1} \subseteq M_{i_2}$, we have $M_i \subseteq M_k$ and $M_j \subseteq M_k$. So $\bar{a} \in |M_i|, b \in |M_j| \implies \bar{a}, b \in |M_k|$.

Apply $(*)$ to $i \leftarrow k$, so $M_k \models \exists y \psi(y, \bar{a})$. Since $M_i \subseteq M_k$ and $\bar{a} \in M_i$, we have $M_i \models \exists y \psi(y, \bar{a}) \implies M_k \models \exists y \psi(y, \bar{a})$.

☺

> **Theorem 2.0.11** Tarski Vaught Test
>
> Suppose $M \subseteq N$. TFAE:
>
> 1. $M \prec N$
>
> 2. For every $\varphi(y, \bar{x}) \in \mathrm{Fml}(L)$, $\forall \bar{a} \in |M|$, if $N \models \exists y \varphi(y, \bar{a})$, then there is $b \in |M|$ such that $N \models \varphi[b, \bar{a}]$.

Proof. Obviously $1 \implies 2$. For the other direction, we know that for atomic $\varphi(\bar{x})$, $M \subseteq N \implies (M \models \varphi[\bar{a}] \iff N \models \varphi(\bar{a}))$ for every $\bar{a} \in |M|$.

We proceed by induction on φ.

Suppose φ is $\neg\psi$. $M \models \varphi[\bar{a}] \iff M \not\models \psi[\bar{a}] \iff N \not\models \psi[\bar{a}] \iff N \models \varphi[\bar{a}]$.

Suppose φ is $\psi_1 \wedge \psi_2$.

$$M \models \varphi[\bar{a}] \iff M \models \psi_1[\bar{a}] \wedge M \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \wedge N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}]$$

.

Suppose φ is $\psi_1 \vee \psi_2$.

$$M \models \varphi[\bar{a}] \iff M \models \psi_1[\bar{a}] \vee M \models \psi_2[\bar{a}] \iff N \models \psi_1[\bar{a}] \vee N \models \psi_2[\bar{a}] \iff N \models \varphi[\bar{a}].$$

Suppose φ is $\exists y \psi(y, \bar{x})$. Then,

$$M \models \varphi(\bar{a}) \iff \exists b \in |M|, M \models \psi(b, \bar{a}) \implies \exists b \in |M|, N \models \psi(b, \bar{a}) \implies b \in |N|, N \models \psi[b, \bar{a}] \implies N \models \varphi[\bar{a}].$$

Suppose $N \models \varphi[\bar{a}]$,

$$N \models \exists y \psi(y, \bar{a}).$$

By the assumption, there is $b \in |M|$ such that $N \models \psi[b, \bar{a}]$. As $b, \bar{a} \in |M|$, there is $b \in |M|$ such that $M \models \psi[b, \bar{a}] \implies M \models \exists y \psi(y, \bar{a}) \implies M \models \varphi[\bar{a}]$.

☺

> **Theorem 2.0.12** Downward Lowenheim Skolem-Tarski
>
> Let M be L-structure, $A \subseteq |M|$. Then there is $N \prec M$, $|N| \supseteq A$, and $\|M\| = |A| + \lambda$.

Proof. For every $\psi(y, \bar{x}) \in \mathrm{Fml}(L)$, let G_ψ be a new function symbol, $L_1 = L \cup \{G_\psi \mid \psi \in \mathrm{Fml}(L)\}$. M_1 is an expansion of M to L_1.

$G_\psi : |M|^{\ell(\bar{x})} \to |M|$. Fix $<$ a well-ordering of $|M|$. For $\bar{a} \in |M|^{\ell(\bar{x})}$,

$$G_\psi^{M_1}(\bar{a}) := \begin{cases} b_0 & M \not\models \exists y \psi(y, \bar{a}) \\ \min\{b \in |M| \mid M \models \varphi[b, \bar{a}]\} \end{cases}.$$

Apply the submodel theorem to find $N_1 \subseteq M_1$ containing A of cardinality $\lambda + |A|$. Take $N := N_1 \cap L_1$.

Verify $N \prec M$. Follows for $N_1 \prec M_1$. Clearly $N_1 \subseteq M_1$. Verify the Tarski-Vaught test (condition 2). Given $\varphi(y, \bar{x}), \bar{a} \in |N|$,

$$M_1 \models \exists y \psi(y, \bar{a}) \text{ by definition of } G_\varphi$$
$$M_1 \models \varphi(G_\varphi(\bar{a}), \bar{a}) \text{ since } N_1 \subseteq M_1$$
$$G_\varphi^{N_1}(\bar{a}) = G_\varphi^{M_1}(\bar{a}).$$

☺

> **Corollary 2.0.1** Upward Lowenheim Skolem Tarski
>
> Given L, M an infinite L-structure. For every $\lambda \geqslant |L| + \|M\|$, there exists $N > M$ of cardinality λ.

> **Corollary 2.0.2** Downward Lowenheim Skolem Tarski (stronger)
>
> For all M L-structure, $\lambda \leqslant \|M\|$, $\lambda \geqslant |L| + \aleph_0$, there exists $N \prec M$ of cardinality λ.

Question Suppose T is first order with $|L(T)| \leqslant \aleph_0$. If $I(\lambda, T) \neq 0$ for some $\lambda \geqslant \aleph_0$, then $I(\mu, T) \neq 0$ for all $\mu \geqslant \aleph_0$.

Answer By the two above corralaries, yes.

Proof. Given M and λ, pick $A \subseteq |M|$, $|A| = \lambda$. Apply the last theorem to find $N \prec M$, $|N| \supseteq A$ of cardinality λ. ☺

Fact $\exists f : \omega \times \omega \to \omega$. So $M = \langle omega, f \rangle \models f$ is a bijection.

By compactness and Downward Lowenheim Skolem, T is first order such that $|L(T)| \leqslant \aleph_0$. Then if T has an inifnite model, then for all infinite A, $\exists M \models T$, $\|M\| = |A|$.

Given A infinite, apply the above to M to find $N \models \mathrm{Th}(M)$, $\|N\| = |A|$ such that f^N induces a bijection for $A \times A \to A$.

> **Definition 2.0.8**
>
> $\lambda \geqslant \aleph_0$ is regular provided for every A, $|A| = \lambda$ and every $\mu < \lambda$, B, $|B| = \mu$ and every $f : A \to B$, there exists $b \in B$ such that $|f^{-1}(b)| = \lambda$.

Above, A are the pigeons and B are the pigeon holes.

> **Definition 2.0.9**
>
> $\lambda \geqslant \aleph_0$, $\lambda^+ = \min\{\mu \text{ cardinality} \mid \mu > \lambda\}$. For examples, $\aleph_0 = |\mathbb{N}|$ and $\aleph_0^+ = \aleph_1$.

Theorem 2.0.13 Erdos-Rado Theorem (1952)

For all $\lambda \geqslant \aleph_0$, λ^+ is regular.

Proof. Assume λ^+ is not regular. Then there is $f : \lambda^+ \to \lambda$ such that for all $\alpha in \lambda$, $|f^{-1}(\alpha)| < \lambda^+$ or that $|f^{-1}(\alpha)| \leqslant \lambda$. Since the domain of f is λ^+, So,

$$\lambda^+ = \operatorname{dom} f = \cup_{\alpha < \lambda} f^{-1}(\alpha).$$

So, $\lambda^+ = |\lambda^+| \leqslant \sup_{\alpha \leqslant \lambda} |f^{-1}(\alpha)| = \sup_{\alpha < \lambda} \lambda = \lambda$. This is a contradiction. ☺

Definition 2.0.10

λ, μ, κ cardinals, $n < \omega$. $\lambda \to (\mu)^n_\kappa$ is true or false. For all $F : [\lambda]^n \to \kappa$, $\exists A \subseteq \lambda, |A| = \mu$ such that for all $\bar{a} \in [A]^n, F(\bar{a}) = \beta$. This would mean A is monochromatic in β.

Theorem 2.0.14 Infinite Ramsey

$\aleph_0 \to (\aleph_0)^2_2$.

Theorem 2.0.15 Sierpinski

ZFC $\vdash \aleph_1 \nrightarrow (\aleph_1)^2_2$.