**Question: 11**

Let $\mathbb{Z}[\sqrt{2}\,] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

a. Prove that $\mathbb{Z}[\sqrt{2}\,]$ is an integral domain.

b. Find all the units of $\mathbb{Z}[\sqrt{2}\,]$.

c. Determine the field of fractions of $\mathbb{Z}[\sqrt{2}\,]$.

d. Prove that $\mathbb{Z}[\sqrt{2}i\,]$ is a Euclidean domain under the Euclidean valuation $\nu(a + b\sqrt{2}\,i) = a^2 + 2b^2$.

***Solution:***

a. Consider $(a+b\sqrt{2})(c+d\sqrt{2}) = 0$. This means that $(a+b\sqrt{2})(a-b\sqrt{2})(c+d\sqrt{2})(c-d\sqrt{2}) = (a^2-2b^2)(c^2-2d^2) = 0$. The irrationality of $\sqrt{2}$ and that fact that $a, b, c, d$ are all integers tells us that either $a = b = 0$ or $c = d = 0$. 😄

b.

**Question: 17**

Prove or disprove: Every subdomain of a UFD is also a UFD.

***Solution:*** $\mathbb{Z}[3i] \subseteq \mathbb{C}$ is a subdomain of a UFD, but is not a UFD. 😄

**Question: 18**

An ideal of a commutative ring $R$ is said to be ***finitely generated*** if there exist elements $a_1, \ldots, a_n$ in $R$ such that every element $r$ in the ideal can be written as $a_1r_1 + \cdots + a_nr_n$ for some $r_1, \ldots, r_n$ in $R$. Prove that $R$ satisfies the ascending chain condition if and only if every ideal of $R$ is finitely generated.

***Solution:***

**Question: 19**

Let $D$ be an integral domain with a descending chain of ideals $I_1 \supset I_2 \supset I_3 \supset \cdots$. Suppose that there exists $N$ such that $I_k = I_N$ for all $k \geqslant N$. A ring satisfying this condition is said to satisfy the ***descending chain condition***, or DCC. Rings satisfying the DCC are called ***Artinian rings***, after Emil Artin. Show that if $D$ satisfies the descending chain condition, it must satisfy the ascending chain condition.

***Solution:***