21-610
Algebra I

Rohan Jain

Contents

Chapter 1

1.1 1/17 - Group Actions

Definition 1.1.1: Action

With a group G and set X, an *action* of G on X is a HM from G to Σ_X (the group of permutations of X).

Definition 1.1.2: $g \cdot x$

If $\phi : G \to \Sigma_X$ is an action, then for $g \in G$ and $x \in X$, we write $g \cdot x$ for $\phi(g)(x)$.

Note:
People will eventually lose the $\cdot$. So, $g \cdot x$ will be written as gx.

Example 1.1.1 (actions)

- $1 \cdot x = x$ $(*)$
- $g \cdot (h \cdot x) = \phi(g)(\phi(h)(x)) = (\phi(g) \circ \phi(h))(x) = \phi(gh)(x) = (gh) \cdot x$ $(**)$

If $\cdot : G \times X \to X$ satisfies $(*) \cdot \&(**)$, then there's unique action $\phi : G \to \Sigma_X$ such that $g \cdot x = \phi(g)(x)$.

Proof. Define $\phi : G \to^X X$ by $\phi(g)(x) = g \cdot x$.
$\phi(g^{-1})$ is 2-sided inverse of $\phi(g)$, $\phi(g) \in \Sigma_X$. So ϕ is an HM by $(**)$. 😄

Definition 1.1.3: Orbit Equivalence Relation

Let G act on X. The *orbit equivalence relation* on X is induced by action: $x \sim y$ if $\exists g \in G$ such that $g \cdot x = y$.

Definition 1.1.4: Orbits

The equivalence classes of this relation are called *orbits*. They are defined as

$$O_x = \{y : x \sim y\} = \{y : \exists g, g \cdot x = y\}$$

Definition 1.1.5: Stabilizer

Let G act on X. The *stabilizer* of $x \in X$ is the subgroup of G defined as

$$G_x = \{g \in G : g \cdot x = x\}$$

Note that $G_x \leqslant G$.

Proof. We need to show that G_x is a subgroup of G.

- $1 \cdot x = x$, so $1 \in G_x$.
- $g \in G_x \implies g \cdot x = x \implies g^{-1} \cdot x = x \implies g^{-1} \in G_x$.
- $g, h \in G_x$. $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, so $gh \in G_x$.

😄

A calculation:

$$\begin{aligned} g_1 \cdot x = g_2 \cdot x &\iff g_2^{-1} \cdot (g_1 \cdot x) = x \\ &\iff (g_2^{-1} g_1) \cdot x = x \\ &\iff g_2^{-1} g_1 \in G_x \\ &\iff g_1 \in g_2 G_x \\ &\iff g_1 G_x = g_2 G_x \end{aligned}$$

This gives a bijection between O_x and set of left cosets of G_x. So, we have the orbit-stabilizer theorem:

Theorem 1.1.1 Orbit-Stabilizer Theorem

Let G act on X. Then for all $x \in X$, $|O_x| = [G : G_x]$.

Definition 1.1.6: Fixed Point

Let G act on X. A *fixed point* of the action is an $x \in X$ such that $g \cdot x = x$ for all $g \in G$. That is, $G_x = G$.

Definition 1.1.7: Fixed-Point Set

Let G act on X. Choose a $g \in G$. The *fixed-point set* of g is the set of all $x \in X$ such that $g \cdot x = x$ and is denoted X_g.

1.2 1/19 - Group Actions

Example 1.2.1 (Automorphism Groups)

$$\operatorname{Aut}(G) = \{f : G \to G : f \text{ is an isomorphism}\}$$

$\phi \in \Sigma_G$, $\phi(ab) = \phi(a)\phi(b)$. Recall conjugate of h by g is $h^g = ghg^{-1}$.
Fact 1: For any $g \in G$, $h \mapsto h^g$ is an automorphism of G.
Fact 2: If $\phi : G \to \operatorname{Aut}(G)$, $\phi : g \mapsto (h \mapsto h^g)$, then ϕ is an HM for G to $\operatorname{Aut}(G)$.
G acts on G by automorphisms. $g \cdot h = h^g = ghg^{-1}$.
In this setting:

1. Orbit equivalence relation is conjugacy.
2. Orbits are conjugacy classes.
3. For $h \in G$, the stabilizer of h for conjugation action $= \{g : h^g = g\}$.

$$\begin{aligned} h^g = h &\iff ghg^{-1} = h \\ &\iff gh = hg \\ &\iff g \in C_G(h) \end{aligned}$$

Definition 1.2.1: Centralizer

Let G act on X. The *centralizer* of $x \in X$ is the subgroup of G defined as

$$C_G(x) = \{g \in G : g \cdot x = x \cdot g\}$$

Theorem 1.2.1 Orbit-Stablizer Equation for conjugation action

$$|\text{conj class of } h| = [G : C_G(h)]$$

$$|G| = \sum_{C \text{ conj. class}} |C|$$

So, if C = class of h, $|C| = [G : C_G(h)]$.

Recall the definition of a fixed-point. So, for G acting on G by conjugation, $X_g = C_G(g)$. That is,

$$h \text{ fixed point} \iff h^g = h \iff hg = gh \qquad \text{(for all } g\text{)}$$

Definition 1.2.2: Center

The *center* of G is $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$. In fact, $Z(G)$ is normal in G. That is, $Z(G) \trianglelefteq G$.

Theorem 1.2.2

Let p be prime. Let G be a group of order p^n. Then $Z(G) \neq 1$.

Proof. Let G act on G by conjugation. G is partitioned into orbits (i.e. conjugacy classes).

For any h, we know that the size of the class of h is $[G : C_G(h)] = \dfrac{p^n}{|C_G(h)|}$. Each orbit has size 1 or a power of p. So, $|C_G(h)|$ is a power of p.

Note in any action of G onto X, x being a fixed point implies $O_x = \{x\}$. So, $|O_x| = 1$.

So, $|G| = A + B$ where A is the number of orbits of size 1 and $B = \sum |C|$ where C is a conjugacy class of size p^n for $n > 0$.

So, $A = p^n - B$. So $p|A$. As $Z(G) \neq \emptyset$, $|Z(G)| > 0, p||Z(G)|$. So, $|Z(G)| \geqslant p$, which is at least 2, so $Z(G) \neq 1$. 😃

Theorem 1.2.3 Cauchy's Theorem

Let G be a finite group. If p is a prime dividing $|G|$, then G has an element or subgroup of order p.

Facts to remember from undergraduate group theory:

- Let $N \trianglelefteq G$. Then subgroups of G/N are in bijection with $\{H : N \leqslant H \leqslant G\}$. In fact $H \mapsto H/N$ is a bijection.
- Normal subgroups of G/N are uniquely of the form H/N where $H \trianglelefteq G$ and $N \leqslant H$.
- $H/N \trianglelefteq G/N$, $\frac{G/N}{H/N} \cong G/H$.

1.3 1/22 - Using Group Actions to Prove Theorems

Now we prove Cauchy's Theorem:

Proof. Let $X = \{(g_1, \dots, g_p) \in G^p : g_1 \cdots g_p = 1\}$. Some remarks:

- $(g_1, \dots, g_p) \in X \iff (g_1 \dots g_{p-1})g_p = 1$. So, $g_p = (g_1 \dots g_{p-1})^{-1}$ and $(g_p, g_1, \dots, g_{p-1}) \in X$. So, $|X| = |G|^{p-1}$.
- $X \neq \emptyset$ as $(1, \dots, 1) \in X$.

So now it's easy to define an action of C_p(cyclic group of order p) on X. Explicitly, if $C_p = \langle a \rangle >$, then $a \cdot (g_1, \dots, g_p) = (g_2, \dots, g_p, g_1)$.

Now we analyze the fixed-points. $(g_1, \dots, g_p)$ if and only if all the g_i are equal. So, fixed points in the action of C_p on X are $(g, \dots, g) \in X$ where $g^p = 1$.

As $p \mid |G|$, $p \mid |X| = |G|^{p-1}$. As p is prime, $|C_p| = p$. So all orbits have size 1 or p. X is partitioned into orbits, say

$$|X| = C + D_p$$

So $p|C$ where C is the number of fixed-points for this action. As $(1, \dots, 1)$ is a fixed-point, $C > 0$, so $C \geqslant p > 1$. So, there is a fixed-point $(g, \dots, g) \in X$ where $g^p = 1$. So, g has order p. 😁

Definition 1.3.1: $\mathrm{Syl}_p(G)$

$\mathrm{Syl}_p(G) = \{H : H \leqslant G, |H| = p^k \text{ for some } k \geqslant 1 \text{ for largest } k\}$

Note:

If $p \nmid |G|$, then $\mathrm{Syl}_p(G) = \{1\}$.

Theorem 1.3.1 Sylow's Theorem

Let G be a finite group. Let p be a prime dividing $|G|$.

1. If $H \leqslant G$ and $|H|$ is a power of p, there is $K \in \mathrm{Syl}_p(G)$ such that $H \leqslant K$.
2. If $K_1, K_2 \in \mathrm{Syl}_p(G)$, then K_1 and K_2 are conjugate.
3. $|\mathrm{Syl}_p(G)| \equiv 1 \pmod p$ and divides $|G|$.

Notes before proof:

- Let G be a group. $\alpha \in \mathrm{Aut}(G), H \leqslant G$. Then $\alpha[H] = \{\alpha(h) : h \in H\}$ is a subgroup of G and $\alpha[H] \cong H$. α is a bijection from H to $\alpha[H]$.
- In particular, for $g \in G$, if α is "conjugation by g", then $\alpha[H] = gHg^{-1}$ or H goes to H^g. We can check: G acts on $\{H : H \leqslant G\}$. $g \cdot H = H^g = gHg^{-1}$.
- H is a fixed point of this action if and only if $H^g = H$ for all $g \in G$. That is, $H \trianglelefteq G$.
- For any H, stabilizer of H for this action is $N_G(H) = \{g \in G : gHg^{-1} = H\}$.

Definition 1.3.2: Normalizer

Let $H \leqslant G$. The *normalizer* of H in G is $N_G(H) = \{g \in G : gHg^{-1} = H\}$.

- Let G act on X. Then we know that if $Y \subseteq X$, and $g \cdot y \in Y$ for all $g \in G$ and $y \in Y$, then Y is a union of orbits and we then get an action for G onto Y.
- Let G act on X. Let $H \leqslant G$, now easily H acts on X. Each G-orbit breaks up as a union of H-orbits.

1.4 1/29 - Using Group Actions to Prove Theorems

Now we finally prove Sylow's.

Proof. 😄