Question: 6

Prove that for every positive integer n, there exists an n-digit number divisible by 5^n all of whose digits are odd.

Solution: We prove this by induction. For the base case, $n = 1$, we have 5 as our number. For our inductive step, we will show that if there exists an $(n-1)$-digit number in the form $a \cdot 5^{n-1}$, then there exists an odd digit, b, such that $b \cdot 10^{n-1} + a \cdot 5^{n-1}$ is divisible by 5^n. We can rewrite this as $5^{n-1}(a + b \cdot 2^{n-1})$. This just means that we have to prove that there exists an odd b that makes $a + b \cdot 2^{n-1}$ divisible by 5. By modular arithmetic, we just need $b \equiv -3^{n-1}a \pmod 5$. Since there are 5 odd digits, we can fill the residue class which means at least one digit will work. 😊

Question: 10

A license plate has six digits from 0 to 9 and may have leading zeroes. If two plates must differ in at least two places, what is the largest number of plates possible.

Solution: We first start by noting that we can set an initial maximum on the number of license plates at 10^5. If there are $\geqslant 10^5 + 1$, then there are $\geqslant 2$ license plates with the same first 5 digits WLOG, therefore they only differ in one place.

I conjecture that the answer is indeed 10^5 and will prove so by construction. WLOG, construct all 10^5 combinations of the 5 digits. With the 6th digit, set it equal to the sum of the previous 5 but $\bmod 10$. Therefore, we have 10^5 license plates that are unique because if the first 5 digits contain different permutations of the same numbers, then they will trivially differ in at least 2 of the first 5 spots. Additionally, if you have a number that agrees with another number on the first 5 digits, then the 6th digit will be the same and they are therefore the same number. Finally, if a number disagrees with another number at one of the first 5 spots, then the 6th digit will be different and they will therefore differ in at least 2 spots. Therefore, we have constructed 10^5 license plates that differ in at least 2 spots. 😊