Question: 1

Find a polynomial with integer coefficients that has the root $\sqrt{2} + \sqrt[3]{3}$.

Solution: We can easily construct a polynomial that has the desired zero. An example would be $(x-(\sqrt{2}+\sqrt[3]{3}))$, but obviously this doesn't have integer coefficients. We can try something different like $(x-\sqrt{2})^3 - 3$, which removes the $\sqrt{2}$ inside the parentheses and cubes the cuberoot then subtracts it. This seems more promising, so we can try to multiply by its conjugate to maybe get integer coefficients. Turns out:

$$((x-\sqrt{2})^3 - 3)((x+\sqrt{2})^3 - 3) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1.$$

So, the problem is done.

Question: 3

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that $P(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a^n$ has a unique positive zero.

Solution: By Descartes's Rule of Signs, there is exactly 1 positive zero.