21-849: Algebraic Geometry

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I don’t know what a sheave or a category is. ❤️

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1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and "multivariable calculus".
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.
- Both are continuous in $\mathbb{R}$ or $\mathbb{C}$.
- Both are holomorphic in $\mathbb{C}$.
- Both are analytic (power series expansion at every point).
- Both are C^∞.

There are differences as well.
- $f(z) = a$ has no solution or infinitely many solutions for e^z, but for almost all a, 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \to \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \to \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{C} \to \hat{C}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

> **Theorem 1.1** (GAGA Theorems): Compact (projective) $\mathbb{C}$-manifolds are algebraic.

Here are more cool things about algebraic geometry:
1) **Enumeration**:
   - How many solutions to $p(z)$?
   - How many points in $\{f(x,y) = g(x,y) = 0\}$?
   - How many lines meet a given set of 4 general lines in $\mathbb{C}^3$? The answer is 2.
   - How many conics ($\{f(x,y) = 0\}$, deg $f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264…
   - Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.
2) **Birationality**:
   - Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.
3) **Arithmetic Geometry**:
   - Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
   - Weil conjectures: X carved by polynomials with $\mathbb{Z}$-coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: $\Bbbk$ is an algebraically closed field ($\Bbbk = \mathbb{C}$).

Definition 2.1 (Affine space): An n-affine space $\mathbb{A}^n_{\Bbbk}$ is the set

$$\{(a_1, ..., a_n) \mid a_i \in \Bbbk, \forall i = 1, ..., n\} = \Bbbk^n.$$

An affine algebraic subset of $\mathbb{A}^n$ is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, ..., a_n) \in \mathbb{A}^n \mid f(a_1, ..., a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \Bbbk[x_1, ..., x_n]$. We write $Z = V(T)$.

Example 2.2 *(An affine space)*:
- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \Bbbk\}$. Then note that if we project to any two dimensional plane (xy, yz, xz), then we get another affine subset but on $\mathbb{A}^2$.

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

Solution: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin. ■

Definition 2.3 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \Bbbk[x] \mid f(p) = 0, \forall p \in Z\}.$$

Example 2.4:
0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.5:
1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
2) $T \subseteq \Bbbk[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, ..., f_m)$ for some f_i.
3) For $\mathfrak{a} \subseteq \Bbbk[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \Bbbk[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
4) Algebraic subsets of $\mathcal{A}^n$ are closed under finite unions and arbitrary intersections.

Proof: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\Bbbk[x]$ is Noetherian. ■

Theorem 2.6 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \Bbbk[x]\}.$$

Proof:
1) Finite type field extensions $L \supseteq F$ are finite. Rember that finite type means that $F[x_1, ..., x_m] \twoheadrightarrow L$.
2) This implies that maximal ideals of $\Bbbk[x]$ are of the form $\langle x_1 - a_1, ..., x_n - a_n \rangle$ for $a_i \in \Bbbk$, using the fact that $\Bbbk$ is algebraically closed. So, $k[x]/\mathfrak{m} \simeq \Bbbk$.
3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \Longleftrightarrow \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq k[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq k[x, y]$. So for any $(a_1, ..., a_n, b)$ that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in k[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some ℓ.

But also

$$k[x, y]/\langle 1 - yf \rangle \simeq k[x]\left[\frac{1}{f}\right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \cdots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal $\mathfrak{a}$ inside R. ■

Remark: We say R is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

Theorem 2.7: R Jacobson $\Rightarrow R[x]$ Jacobson.

Definition 2.8 (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\Bbbk[x]/I(X)$.

1) $X \xrightarrow{f} \Bbbk$
2) $\operatorname{maxSpec} A(X) = \{\text{maximal ideals in } A(X)\} = X$.

3. Projective Spaces

Definition 3.1: $\mathbb{P}^n = (\Bbbk^{n+1} \setminus \{0\})/\sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \Bbbk$. That is, $\mathbb{P}^n = \{\text{1-subspaces of } \Bbbk^{\{n+1\}}\}$. For $(a_0, ..., a_n) \in k^{n+1} \setminus \{0\}$, we write $[a_0 : ... : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \Bbbk^{n+1}$. $\mathbb{P}V = V \setminus \{0\}/\sim$

Definition 3.2: $f \in \Bbbk[\underline{x}]$ is homogeneous if $f(\lambda x_1, ..., \lambda x_n) = \lambda^\ell f(x_1, ..., x_n)$.

Definition 3.3: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \cdots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for T a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

Example 3.4: Let $W \subseteq \Bbbk^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an m-plane in $\mathbb{P}^n$.

Example 3.5 *(Twisted cubic curve)*: We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2 t : st^2 : t^3] \mid [s : t\} \in \mathbb{P}^1]\}$. Then we have that $C = V(x_0 x_3 - x_1 x_2, x_1^2 - x_0 x_2, x_2^2 - x_1 x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix A, $\operatorname{rank}(A) \leq r \iff$ all $(r+1) \times (r+1)$ minors $= 0$.

Question: Can there exist F, G such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \Bbbk[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of X.

Exercise 3.6:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow$$
$$\{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \Bbbk[\underline{x}] \text{ such that } \mathfrak{a} \neq \Bbbk[\underline{x}] \text{ or } \langle x_0, ..., x_n \rangle\}.$$

This last part is called the "irrelevant ideal".

Definition 3.7 (General Position): In $\mathbb{P}^n$, any subset of size $\leq n+1$ points are linearly independent.

Theorem 3.8: Every set Γ of $2n$ points in $\mathbb{P}^n$ in general position is carved out by quadrics.

Proof: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose q is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\operatorname{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \operatorname{span}(\Gamma_1)$ or $q \in \operatorname{span}(\Gamma_2)$.

Let $p_1, ..., p_k$ be a minimal subset of Γ whose span $\ni q$ ($k \leq n$). Now pick any Λ such that $|\Lambda| = n - k + 1$ whcih does not contain any of the p_i. We claim that $q \notin \operatorname{span}(p_2, ..., p_k, \Lambda)$.

We then conclude that for any $|S| = n - 1$, $S \subseteq \Gamma \setminus p_1, ..., p_k$, we have that $\operatorname{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \operatorname{span}(p, S)$$

is the intersection at least n many hyperplanes, each of them containing p_1, q. But the intersection of n many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ∎

Definition 3.9: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X, \exists g \in PGL_{n+1}$.

Proposition 3.10: Let $(M_0, ..., M_3)$ be any $\Bbbk$-basis of

$$\Bbbk[s,t]_3 = \{f \in \Bbbk[s,t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s,t) : ... : M_3(s,t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2t : st^2 : t^3]\}$.

Example 3.11 *(Rational normal curve)*: Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \cdots : t^n]$. Or we could map it to any basis of $\Bbbk[s : t]_n$.

Exercise 3.12: $I(\varphi(\mathbb{P}^1)) = ?$.

Example 3.13: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

Fact: If we pick any 3 linearly independent $M_0, M_1, M_2 \in \Bbbk[s,t]_3$. Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ by M_0, M_1, M_2 has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \to \mathbb{P}^3$ using 4 elements from $\Bbbk[s,t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

Question: Are all twisted quartic curves projectively equivalent?

Solution: No. In fact, there are infinitely many distinct families. ■

Question (Hartshorne's Question): Is every irreducible curve in $\mathbb{P}^3$ carved out by 2 equations?

4. The Zariski Topology

Definition 4.1 (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \Bbbk[\underline{x}]\}$ form the closed sets of a tpology on $\mathbb{A}^n$ called teh Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

Example 4.2: Take $\mathbb{A}^1$. Two closed subsets are $\mathbb{A}^1$ and $\emptyset$. The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

Definition 4.3: A topological space X is irreducible $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

Remark:
- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

Proposition 4.4: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. X is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

Proof:
- $\Longrightarrow$: Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If X is irreducible, then at least one of them completely contains X. That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means f or $g \in I(X)$.
- $\Longleftarrow$: Suppose for sake of contradiction that X is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■