21-849: Algebraic Geometry I

Rohan Jain

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I don’t know what a sheave or a category is. ❤️

Table of Contents

1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and "multivariable calculus".
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.
- Both are continuous in $\mathbb{R}$ or $\mathbb{C}$.
- Both are holomorphic in $\mathbb{C}$.
- Both are analytic (power series expansion at every point).
- Both are C^∞.

There are differences as well.
- $f(z) = a$ has no solution or infinitely many solutions for e^z, but for almost all a, 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \to \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \to \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{C} \to \hat{C}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

Theorem 1.1 (GAGA Theorems): Compact (projective) $\mathbb{C}$-manifolds are algebraic.

Here are more cool things about algebraic geometry:
1) **Enumeration**:
   - How many solutions to $p(z)$?
   - How many points in $\{f(x, y) = g(x, y) = 0\}$?
   - How many lines meet a given set of 4 general lines in $\mathbb{C}^3$? The answer is 2.
   - How many conics ($\{f(x, y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264…
   - Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.
2) **Birationality**:
   - Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.
3) **Arithmetic Geometry**:
   - Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
   - Weil conjectures: X carved by polynomials with $\mathbb{Z}$-coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: $\Bbbk$ is an algebraically closed field ($\Bbbk = \mathbb{C}$).

Definition 2.1 (Affine space): An n-affine space $\mathbb{A}^n_\Bbbk$ is the set

$$\{(a_1, ..., a_n) \mid a_i \in \Bbbk, \forall i = 1, ..., n\} = \Bbbk^n. \quad (1)$$

An affine algebraic subset of $\mathbb{A}^n$ is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, ..., a_n) \in \mathbb{A}^n \mid f(a_1, ..., a_n) = 0, \forall f \in T\} \quad (2)$$

for some subset $T \subseteq \Bbbk[x_1, ..., x_n]$. We write $Z = V(T)$.

Example 2.1 *(An affine space)*:
- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \Bbbk\}$. Then note that if we project to any two dimensional plane (xy, yz, xz), then we get another affine subset but on $\mathbb{A}^2$.

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

Solution: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin. ■

Definition 2.2 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \Bbbk[x] \mid f(p) = 0, \forall p \in Z\}. \quad (3)$$

Example 2.2:
0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.1:
1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
2) $T \subseteq \Bbbk[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, ..., f_m)$ for some f_i.
3) For $\mathfrak{a} \subseteq \Bbbk[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \Bbbk[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
4) Algebraic subsets of $\mathcal{A}^n$ are closed under finite unions and arbitrary intersections.

Proof: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\Bbbk[x]$ is Noetherian. ∎

Theorem 2.2 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \Bbbk[x]\}. \tag{4}$$

Proof: Finite type field extensions are finite, which implies that maximal ideals of $\Bbbk[x]$ are of the form $\langle x_1 - a_1, ..., x_n - a_n \rangle$ for $a_i \in \Bbbk$, using the fact that $\Bbbk$ is algebraically closed. ∎