# Real Analysis II

Rohan Jain

August 25, 2025

No more tice!

# Table of Contents

# 1. Sequences

**Theorem 1.1** (Bolzano Weierstrass): If a sequence $(x_n)$ in $\mathbb{R}^d$ is bounded, then it has a convergent subsequence.

Proof: We take $(x_{n,1})$. By the $d = 1$ case, it has a convergent subsequence, let's say $(x_{n_k,1})$. Then we take $(x_{n_k,2})$, it has a convergent subsequence, let's say $\left(x_{n_{k_\ell},2}\right)$. Etc. Once we have reached the $d$th coordinate, we can guarantee that we have convergent on every coordinate.

This shows that we have a convergent subsequence because $x_n \to x^*$ iff $\forall j \leq d$, we have that $x_{n,j} \to x^*_j$. ■

**Definition 1.2**: We say a sequence $(x_n)$ is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t. $\forall n, m \geq N$, we have that $|x_n - x_m| < \varepsilon$.

**Definition 1.3**: We say that $(X, \rho)$ is complete if every Cauchy sequence $(x_n)$ in $X$ is convergent.

**Theorem 1.4**: $(\mathbb{R}^d, |\cdot|)$ is complete.

Proof: $d = 1$ is known. Coordinate wise application of the $d = 1$ case shows that $(\mathbb{R}^d, |\cdot|)$ is complete. ■

# 2. Continuity

**Definition 2.1**: Let $D \subset \mathbb{R}^d$, $F : D -< \mathbb{R}^\ell$ is continuous at a point $a \in D$ if $\forall \varepsilon > 0$, there is $\delta > 0$ such that for all $x \in D$,

$$|x - a| < \delta \Rightarrow |F(x) - F(a)| < \varepsilon.$$

**Lemma 2.2**: $F$ being continuous at $a$ is equivalent to $\forall (x_n)$ in $D, x_n \to A \Rightarrow F(x_n) \to F(a)$.