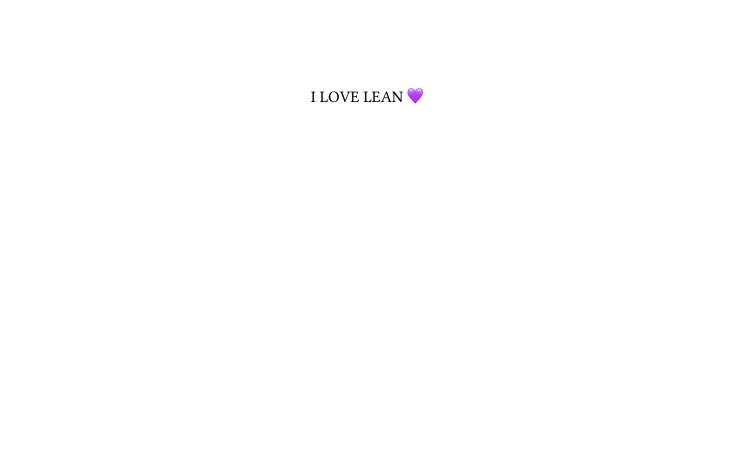
# **Category Theory**

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## 1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

#### 1.1. Motivating example

Let 
$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$$
. Then let  $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$ .

**Theorem 1.1** (Brouwer's fixed point theorem): If  $f: D^2 \to D^2$  is continuous, then f has a fixed point. That is, there is some  $x \in D^2$  such that f(x) = x.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of  $\mathbb{R}^2$ ) and spits out a vector space (over  $\mathbb{R}$ ).

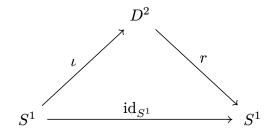
- 1) For every topological space X, there is a vector space H(X) (omitting actual definition).
- 2) For every continuous function  $f: X \to Y$ , there is an "induced" linear map given by  $H(f): H(X) \to H(Y)$ .
- 3) If  $X \to Y \to Z$  are continuous maps,  $H(f): H(X) \to H(Y), H(g): H(Y) \to H(Z)$  and  $H(g \circ f): H(X) \to H(Z)$ , then  $H(g \circ f) = H(g) \circ H(f)$ .
- 4) For any X,  $H(\mathrm{id}_X)=\mathrm{id}_{H(X)}:H(X)\to H(X).$

Computations:

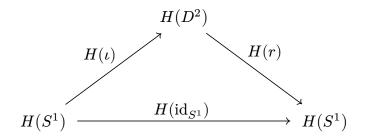
- 5)  $H(D^2) \cong 0$ .
- 6)  $H(S^1) \cong \mathbb{R}$ .

PROOF: Assume  $f:D^2\to D^2$  is continuous and f(x)=x for all  $x\in D^2$ . Define a new function  $r:D^2\to S^1$  such that r(x)= intersection of the ray from f(x) to x with  $S^1\subseteq D^2$ .

Key fact: If  $x \in S^1$ , then r(x) = x. Check that r is also continuous.



The diagram above commutes. Now we can apply homology to it.



We can check that

$$\begin{split} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} \,. \end{split}$$

Therefore, the new diagram also commutes. So, if  $w \in H(S^1)$ , then

$$w=\mathrm{id}_{H(S^1)}(w)=H(r)(H(\iota)(w))=0.$$

This is a contradiction as  $H(S^1) \neq 0$ .

### 1.2. Categories

**Definition 1.2** (Category): A category  $\mathcal{C}$  consists of:

- a collection of objects,  $\mathrm{Ob}(\mathcal{C})$ . For any  $A \in \mathrm{Ob}(\mathcal{C})$ , we usually write  $A \in \mathcal{C}$ .
- for any pair of objects  $A,B\in\mathcal{C}$ , there is a collection of morphisms  $\mathrm{Hom}_{\mathcal{C}}(A,B)$ , or  $\mathrm{Hom}(A,B)$ , or  $\mathcal{C}(A,B)$ . Instead of  $f\in\mathcal{C}(A,B)$ , we write  $f:A\to B$  or  $A\to B$ .
- for any objects  $A, B, C \in \mathcal{C}$  and morphisms  $f: A \to B$  and  $g: B \to C$ , there is a specified composition  $g \circ f: A \to C$ .
- for any object  $A \in \mathcal{C}$ , there is a given  $\mathrm{id}_A : A \to A$
- compositions are associative:  $(g\circ f)\circ h=g\circ (f\circ h)$
- for any  $A \stackrel{f}{\rightarrow} B$ ,  $f \circ id_A = f = id_B \circ f$

#### Example 1.3:

• Set, the category of sets (& functions).

**Definition 1.4** (Monoid): A monoid (M, \*) consists of:

- a set M
- a binary operation  $*: M \times M \to M$
- an identity element  $e \in M$  such that  $\forall x \in M, e * x = x * e = x$ .

**Definition 1.5** (Monoid Homomorphism): A monoid homomorphism  $f: M \to N$  is a function satisfying

- f(xy) = f(x)f(y).
- f(e) = e.

**Definition 1.6** (Functor): A functor  $F: \mathcal{C} \to \mathcal{D}$  is a function satisfying

- $F(A) \in \mathcal{D}$  for all  $A \in \mathcal{C}$ .
- $F(f): F(A) \to F(B)$  for all  $f: A \to B$  in  $\mathcal{C}$ .
- $F(g \circ f) = F(g) \circ F(f)$  for all  $f: A \to B$  and  $g: B \to C$  in  $\mathcal{C}$ .
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$  for all  $A \in \mathcal{C}$ .

#### 1.3. 09/02/2025

Two "sorts" of categories:

- "concrete" categories: sets with smoe sort of familiar structure (groups, rings, modules, etc.)
- "abstract" categories: 1, 2, 3, etc. More formal symbols than not.

**Definition 1.7** (Endomorphism): An endomorphism  $f: A \to A$  is a morphism from an object to itself.

New categories from old:

- 1) Product category.
  - input: two categories  $\mathcal C$  and  $\mathcal D$
  - output:  $\mathcal{C} \times \mathcal{D}$
  - objects: (A, B) where  $A \in Ob(\mathcal{C})$  and  $B \in Ob(\mathcal{D})$
  - morphisms: (f,g) where  $f:A\to A'$  in  $\mathcal C$  and  $g:B\to B'$  in  $\mathcal D$
  - composition:  $(f,g) \circ (f',g') = (f \circ f', g \circ g')$
  - identity:  $(id_A, id_B)$

Projection functors on  $\mathcal{C} \times \mathcal{D}$ :

- $\pi_1: \mathcal{C} \times \mathcal{D} \to C, \pi_2: \mathcal{C} \times \mathcal{D} \to \mathcal{D}.$
- on objects:  $\pi_1((A, B)) = A$
- on morphisms:  $\pi_1((f,g)) = f: A \to A'$ .
- 2) Slice categories, coslice categories
  - input: a category  $\mathcal{C}$  and an object  $X \in \mathrm{Ob}(\mathcal{C})$
  - output:  $\mathcal{C}/X$  or  $X/\mathcal{C}$

description of coslice:

- objects: pair (A, f), where  $A \in Ob(\mathcal{C})$  and  $f : A \to X$  in  $\mathcal{C}$
- morphisms: from  $(A, f) \to (B, g)$ : morphism  $k : A \to B$  of  $\mathcal C$  such that  $k \circ f = g$ .

• composition:  $(A,f) \stackrel{k}{\to} (B,g) \stackrel{l}{\to} (C,h)$  is  $(A,f) \stackrel{l \circ k}{\to} (C,h)$ . We can check that  $(l \circ k) \circ f = h$ . The TLDR for this is that you can copy and paste commutative diagrams and get another commutative diagram.

**Example 1.8** (Coslice): Let  $\mathcal{C} = \operatorname{Set}$ ,  $X = 1 = \{*\}$ . So coslice  $X/\mathcal{C} = 1/\operatorname{Set} = ?$ .

- objects: pairs (A, f) of a set A and a function  $f: 1 \to A$ .
- morphisms: functions k such that  $k \circ f = g$ .

Elements of sets categorically. A is a set. How do we express  $a \in A$  in terms of the category Set?

elements of 
$$A \longleftrightarrow$$
 functions  $f: 1 \to A$  
$$a \in A \longleftrightarrow f: 1 \to A, f(*) = a$$
 
$$f(x) \in A \longleftrightarrow f: 1 \to A.$$

- 3) Opposite category.
  - input: a category  $\mathcal C$
  - output:  $\mathcal{C}^{\text{op}}$
  - objects of  $\mathcal{C}^{\text{op}}: A^*$  for  $A \in \mathcal{C}$ .
  - morphisms of  $\mathcal{C}^{\text{op}}(A^*, B^*)$ :  $f^*$  for  $f: A \to B$  in  $\mathcal{C}$ .
  - composition:  $(f^* \circ g^*) = (g \circ f)^*$