# 21-849: Algebraic Geometry

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I don’t know what a sheave or a category is. ❤️

# Table of Contents

# 1. Introduction

### 1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

## 1.1. Features of algebraic geometry

Consider the two functions $e^z$ and $z^2 - 3z + 2$.

- Both are continuous in $\mathbb{R}$ or $\mathbb{C}$.
- Both are holomorphic in $\mathbb{C}$.
- Both are analytic (power series expansion at every point).
- Both are $C^\infty$.

There are differences as well.

- $f(z) = a$ has no solution or infinitely many solutions for $e^z$, but for almost all $a$, 2 solutions for $z^2 - 3z + 2$.
- $e^z$ is not definable from $\mathbb{Z} \to \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for $e^z$ but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \to \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{C} \to \hat{C}$. But $e^z$ can be periodic as the imaginary part tends to infinity.

This motivates the following result:

**Theorem 1.1** (GAGA Theorems): Compact (projective) $\mathbb{C}$-manifolds are algebraic.

Here are more cool things about algebraic geometry:

1) **Enumeration**:
   - How many solutions to $p(z)$?
   - How many points in $\{f(x, y) = g(x, y) = 0\}$?
   - How many lines meet a given set of 4 general lines in $\mathbb{C}^3$? The answer is 2.
   - How many conics ($\{f(x, y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it’s 3264…
   - Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.
2) **Birationality**:
   - Open sets are *huge*. That is, if we have $X, Y$ and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then $X$ and $Y$ are closely related.
3) **Arithmetic Geometry**:
   - Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
   - Weil conjectures: $X$ carved by polynomials with $\mathbb{Z}$-coefficients. $H^2(X_\mathbb{C}, \mathbb{Q})$ related to integer solutions.

# 2. Affine algebraic sets

## 2.1. Nullstellensatz

Notation: $\Bbbk$ is an algebraically closed field ($\Bbbk = \mathbb{C}$).

**Definition 2.1** (Affine space): An $n$-affine space $\mathbb{A}_{\Bbbk}^n$ is the set

$$\{(a_1, ..., a_n) \mid a_i \in \Bbbk, \forall i = 1, ..., n\} = \Bbbk^n.$$

An affine algebraic subset of $\mathbb{A}^n$ is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, ..., a_n) \in \mathbb{A}^n \mid f(a_1, ..., a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \Bbbk[x_1, ..., x_n]$. We write $Z = V(T)$.

**Example 2.2** *(An affine space)*:
- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \Bbbk\}$. Then note that if we project to any two dimensional plane $(xy, yz, xz)$, then we get another affine subset but on $\mathbb{A}^2$.

This leads us to the following question:

**Question**: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

Solution: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin. ■

**Definition 2.3** (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \Bbbk[x] \mid f(p) = 0, \forall p \in Z\}.$$

**Example 2.4**:

0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

**Proposition 2.5**:
1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
2) $T \subseteq \Bbbk[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, ..., f_m)$ for some $f_i$.
3) For $\mathfrak{a} \subseteq \Bbbk[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \Bbbk[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
4) Algebraic subsets of $\mathcal{A}^n$ are closed under finite unions and arbitrary intersections.

Proof: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\Bbbk[x]$ is Noetherian.
■

**Theorem 2.6** (Nullstellensatz): Let $Z$ be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \Bbbk[x]\}.$$

Proof:
1) Finite type field extensions $L \supseteq F$ are finite. Rember that finite type means that $F[x_1, ..., x_m] \twoheadrightarrow L$.
2) This implies that maximal ideals of $\Bbbk[x]$ are of the form $\langle x_1 - a_1, ..., x_n - a_n \rangle$ for $a_i \in \Bbbk$, using the fact that $\Bbbk$ is algebraically closed. So, $k[x]/\mathfrak{m} \simeq \Bbbk$.
3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \Longleftrightarrow \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq k[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq k[x, y]$. So for any $(a_1, ..., a_n, b)$ that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in k[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some $\ell$.

But also

$$k[x, y]/\langle 1 - yf \rangle \simeq k[x]\left[\frac{1}{f}\right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \cdots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal $\mathfrak{a}$ inside $R$. ■

Remark: We say $R$ is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

**Theorem 2.7**: $R$ Jacobson $\Rightarrow R[x]$ Jacobson.

**Definition 2.8** (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\Bbbk[x]/I(X)$.

1) $X \xrightarrow{f} \Bbbk$
2) $\operatorname{maxSpec} A(X) = \{\text{maximal ideals in } A(X)\} = X$.

## 3. Projective Spaces

**Definition 3.1**: $\mathbb{P}^n = (\Bbbk^{n+1} \setminus \{0\}) / \sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \Bbbk$. That is, $\mathbb{P}^n = \{1\text{-subspaces of } \Bbbk^{\{n+1\}}\}$. For $(a_0, ..., a_n) \in k^{n+1} \setminus \{0\}$, we write $[a_0 : ... : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \Bbbk^{n+1}$. $\mathbb{P}V = V \setminus \{0\} / \sim$

**Definition 3.2**: $f \in \Bbbk[\underline{x}]$ is homogeneous if $f(\lambda x_1, ..., \lambda x_n) = \lambda^\ell f(x_1, ..., x_n)$.

**Definition 3.3**: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \cdots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for $T$ a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

**Example 3.4**: Let $W \subseteq \Bbbk^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an $m$-plane in $\mathbb{P}^n$.

**Example 3.5** *(Twisted cubic curve)*: We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2 t : s t^2 : t^3] \mid [s : t\} \in \mathbb{P}^1]\}$. Then we have that $C = V(x_0 x_3 - x_1 x_2, x_1^2 - x_0 x_2, x_2^2 - x_1 x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix $A$, $\operatorname{rank}(A) \leq r \iff$ all $(r+1) \times (r+1)$ minors $= 0$.

**Question**: Can there exist $F, G$ such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \Bbbk[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of $X$.

**Exercise 3.6**:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow$$
$$\{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \Bbbk[\underline{x}] \text{ such that } \mathfrak{a} \neq \Bbbk[\underline{x}] \text{ or } \langle x_0, ..., x_n \rangle\}.$$

This last part is called the "irrelevant ideal".

**Definition 3.7** (General Position): In $\mathbb{P}^n$, any subset of size $\leq n+1$ points are linearly independent.

**Theorem 3.8**: Every set $\Gamma$ of $2n$ points in $\mathbb{P}^n$ in general position is carved out by quadrics.

Proof: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose $q$ is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\mathrm{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \mathrm{span}(\Gamma_1)$ or $q \in \mathrm{span}(\Gamma_2)$.

Let $p_1, ..., p_k$ be a minimal subset of $\Gamma$ whose span $\ni q$ ($k \leq n$). Now pick any $\Lambda$ such that $|\Lambda| = n - k + 1$ whcih does not contain any of the $p_i$. We claim that $q \notin \mathrm{span}(p_2, ..., p_k, \Lambda)$.

We then conclude that for any $|S| = n-1$, $S \subseteq \Gamma \setminus p_1, ..., p_k$, we have that $\mathrm{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \mathrm{span}(p, S)$$

is the intersection at least $n$ many hyperplanes, each of them containing $p_1, q$. But the intersection of $n$ many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ∎

**Definition 3.9**: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X$, $\exists g \in PGL_{n+1}$.

**Proposition 3.10**: Let $(M_0, ..., M_3)$ be any $\Bbbk$-basis of

$$\Bbbk[s,t]_3 = \{f \in \Bbbk[s,t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s,t) : ... : M_3(s,t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2 t : st^2 : t^3]\}$.

**Example 3.11** *(Rational normal curve)*: Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \cdots : t^n]$. Or we could map it to any basis of $\Bbbk[s : t]_n$.

**Exercise 3.12**: $I(\varphi(\mathbb{P}^1)) = ?$.

**Example 3.13**: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

**Fact:** If we pick any 3 linearly independent $M_0, M_1, M_2 \in \Bbbk[s,t]_3$. Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ by $M_0, M_1, M_2$ has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \to \mathbb{P}^3$ using 4 elements from $\Bbbk[s,t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

**Question**: Are all twisted quartic curves projectively equivalent?

Solution: No. In fact, there are infinitely many distinct families. ■

**Question** (Hartshorne's Question): Is every irreducible curve in $\mathbb{P}^3$ carved out by 2 equations?

# 4. The Zariski Topology

**Definition 4.1** (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \Bbbk[\underline{x}]\}$ form the closed sets of a tpology on $\mathbb{A}^n$ called teh Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

**Example 4.2**: Take $\mathbb{A}^1$. Two closed subsets are $\mathbb{A}^1$ and $\emptyset$. The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

**Definition 4.3**: A topological space $X$ is irreducible $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

By definition, we will also say that irreducible implies nonempty.

**Remark**:
- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

**Proposition 4.4**: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. $X$ is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

Proof:
- $\Longrightarrow$: Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If $X$ is irreducible, then at least one of them completely contains $X$. That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means $f$ or $g \in I(X)$.
- $\Longleftarrow$: Suppose for sake of contradiction that $X$ is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■

**Remark**: When people say affine variety, some people mean that it is also irreducible. But for us, affine variety is the same thing as affine algebraic set.

Then a quasi-affine variety is an open subset of an affine variety.

**Example 4.5**:
1) $\mathbb{A}^n$ is irreducible. ($\Bbbk[\underline{x}]$ domain)
2) $V(x^2+y^2) \subset \mathbb{A}^2$ is reducible (char $\Bbbk \neq 2$)
3) Let $f \in \Bbbk[\underline{x}]$ be square-free ($f = f_1 \ldots f_\ell$ irreducible). Then $V(f)$ is irreducible if and only if $f$ is irreducible.
4) $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then $A(X) = \frac{\Bbbk[x,y,z]}{\langle x^2 - yz \rangle}$. This is irreducible due to Eisenstein's on $f$. Now if we take $f \in A(X)$ and look at $V_X(f) \subset X$ is irreducible $\not\Leftarrow f$ irreducible element in $A(X)$.

**Definition 4.6**: A topological space $X$ is Noetherian if $\nexists X \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots$ such that each $Y_i$ is closed.

**Proposition 4.7**: An affine variety is Noetherian. (Because $A(X)$ is Noetherian).

**Theorem 4.8**: A Noetherian topological space $X$ is uniquely a finite union of maximal irreducible closed subsets.

Proof: Consider

$$\{\text{nonempty closed subsets of } X \text{ that does not admit a decomposition into irreducible closed subsets.}\}.$$

Suppose it is nonempty. Then it has a minimal element $Y$. $Y$ is not irreducible, so $Y = Y_1 \cup Y_2$ (both proper and closed). Since $Y$ is minimal, $Y_1$ and $Y_2$ both have decompositions into irreducible closed subsets. So if we just union those decompositions, then we contradict $Y$'s membership in the set. As such, the original set must have actually been empty.

Uniqueness and maximality are left as an exercise. ■

**Proposition 4.9**:
1) $X$ irreducible and $U \subseteq X$ open. Then $\overline{U} = X$.
2) $V \subseteq X$, $V$ irreducible $\Longrightarrow \overline{V}$ irreducible.
3) $f : X \to Y$ continuous. Image of irreducible set under $f$ is irreducible. (Irreducibility is a topological property).

**Example 4.10**: Let's have $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ by $\varphi(\underline{x}) = (f_1(\underline{x}), \ldots, f_m(\underline{x}))$ for some $f_1, \ldots, f_m \in \Bbbk[\underline{x}]$. Then $\operatorname{im}(\varphi)$ is irreducible. It is left to show that $\varphi$ is a continuous map.

**Definition 4.11**: Let $X$ be a nonempty topological space.

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n, \text{each } Y_i \text{ irreducible and closed}\}.$$

Then let $Y \subseteq X$ closed irreducible subset.

$$\operatorname{codim}_X Y := \sup\{n \mid \exists Y \subseteq Y_0 \subsetneq \cdots \subsetneq Y_n, \text{each } Y_i \text{ irreducible and closed}\}.$$

**Example 4.12**:
1) $\dim \mathbb{A}^1 = 1$.
2) $X = V(xz, yz) \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Let $p$ be a point on the axis not touching the $x$-$y$ plane. Then let $q$ be the origin. We have that $\operatorname{codim}_X p = 1$ and $\operatorname{codim}_X q = 2$. Also $\dim p = \dim q = 0$.

**Definition 4.13**: Height of a prime $\mathfrak{p} \subset R$ is

$$\text{ht } \mathfrak{p} := \sup\{n \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n, \text{each } \mathfrak{p}_i \text{ prime}\}.$$

Then Krull dimension of $R$ is

$$\dim R := \sup\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ prime}\}.$$

**Definition 4.14**: For an ideal $I$, we have that

$$\text{ht } I := \inf\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}.$$

"inf of sup".

From these, we can basically show from definition that

$$\text{ht } I + \dim R/I \leq \dim R.$$

The $<$ case is possible if $R$ is not a domain. For example, if we have that $R = \Bbbk[x, y, z]/\langle xz, yz\rangle$ and then $I = \langle x, y, z - 1\rangle$.

But the $<$ case is also possible even if $R$ is a domain and $I$ prime.

Before we cover the next theorem, we note that

$$\{\text{minimal primes over } I\} = \{\mathfrak{p} \text{ prime } \mathfrak{p} \supseteq I, \text{and } \nexists \mathfrak{p} \supsetneq \mathfrak{q} \supseteq I, \text{prime } \mathfrak{q}\}$$

**Theorem 4.15** (Krull Principal Ideal Theorem / Height Theorem): Let $R$ be a Noetherian ring and $f_1, ..., f_c \in R$.

1) Minimal primes over $\langle f_1 \rangle$ have height $\leq 1$. And the height is equal to 1 if $f_1$ is nonzerodivisor and nonunit.
2) Minimal primes over $\langle f_1, ..., f_c \rangle$ have height $\leq c$.

"We could do this proof, but it's like proving that there exists a complete ordered field satisfying the least upper bound property."

**Theorem 4.16**: Let $X \subseteq \mathbb{A}^n$ and $\mathbb{Y} \subseteq \mathbb{A}^m$ irreducible affine varieties.

1) $\dim(X \times Y) = \dim X + \dim Y$.
2) If $Y \subseteq X$, then $\dim Y + \operatorname{codim}_X Y = \dim X$.

**Remark** (Noether normalization): For $X \subseteq \mathbb{A}^n$ irreducible affine variety. There exists $y_1, ..., y_d \in A(X)$ such that $\Bbbk[Y_1, ..., Y_d] \to A(X)$ with $Y_i \mapsto y_i$ which is a finite extension (injective and $A(X)$ is finitely generated $\Bbbk[Y]$-module) and $d = \dim X$.

**Corollary 4.17**:

1) $\dim \mathbb{A}^n = n$.
2) $X \subseteq \mathbb{A}^n$ irreducible affine variety. $0 \neq f \in A(X)$ non unit. Then $V_X(f) = V(f) \cap X$ has dimension $\dim X - 1$.

**Exercise 4.18**: Let $U \subseteq X$ be open for $X$ affine variety irreducible. Then $\dim U = \dim X$.

**Proposition 4.19**: Let $R$ be Noetherian domain. Then $R$ UFD $\Longleftrightarrow$ every $\mathrm{ht} = 1$ prime is principal.

Proof: $R$ being a UFD implies that $\mathfrak{p}$ has height 1. So let $f = f_1, ..., f_\ell \in \mathfrak{p}$. Suppose $f_1 \in \mathfrak{p}$. So then $0 \neq \langle f_1 \rangle \subseteq \mathfrak{p}$. But as $\mathrm{ht}\ \mathfrak{p} = 1$, we have that $\langle f_1 \rangle = \mathfrak{p}$.

Conversely, we need to show that irreducible implies prime. That is, recall that (ACCP + irreducible = prime) implies that we have a UFD.

So let $f \in$ irred. Krull's PIT says $\langle f \rangle \subseteq \mathfrak{p}$ where $\mathfrak{p}$ has height 1. So by definition, $\mathfrak{p} = \langle g \rangle$, but $\langle f \rangle \subseteq \langle g \rangle$ implies that $f = g$ because $f$ is irreducible. ■

**Example 4.20**: Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then let $Y = V(x, y) \subseteq X \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Then $\dim Y = 1$. So can we find $f$ such that $\langle f, x^2 - yz \rangle = I(Y)$? The answer to this is no.

But can we find $f$ such that $\sqrt{\langle f, x^2 - yz \rangle} = \langle x, y \rangle$? Take $f = y$ and analyze $\langle y, x^2 - yz \rangle$. This is the same as $\langle y, x^2 \rangle$, whose radical is $\langle x, y \rangle$ as we desire.

**Example 4.21**: Now consider $X = V(xw - yz) \subseteq \mathbb{A}^4$. $\dim X = 3$ and let $Y = V(x, y)$. Now does there exist $f$ such that $\sqrt{\langle f, xw - yz \rangle} = \langle x, y \rangle$?

This is false, but we don't have the tools to prove it.

**Definition 4.22**: Zariski topology on $\mathbb{P}^n$ has projective algebraic sets as its closed subsets.

Two ways: projective varieties $\to$ affine varieties.
1) $U_i = \{x_i \neq 0\} = \{[x_0 : \ldots : x_i = 1 : \ldots : x_n]\} \simeq \mathbb{A}^n$.

**Proposition 4.23**: $\forall i = 0, \ldots, n$, say $i = 0$, $\mathbb{A}^n \longrightarrow U_0$, $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$ is a homeomorphism.

Proof:
- *Homogenization*: let $f \in \mathbb{k}[x_1, \ldots x_n]$. Then we have

$$f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{k}[x_1, \ldots, x_n].$$

If $Z = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$, $\varphi(Z) = U_0 \cap V(f_1^h, \ldots f_m^h)$ is closed.

If $Z' = V(F_1, \ldots, F_\ell) \cap U_0$, then $\varphi(Z') = V(F_1(1, x_2, \ldots, x_n), \ldots, F_\ell(1, x_2, \ldots, x_n))$.

Now $U_0 \cup \ldots \cup U_n = \mathbb{P}^n$.

■

**Exercise 4.24**: Let $Y \subseteq \mathbb{A}^n \simeq U_0$ be an affine variety. $\overline{Y} = V(?)$. Suppose $V(f_1, \ldots, f_m) = Y$. It is tempting to say $\overline{Y} = V(f_1^h, \ldots, f_m^h)$.

**Corollary 4.25**:
1) $\dim \mathbb{P}^n = n$.
2) If $H_i = V(x_i) \subseteq \mathbb{P}^n$ does not contain any irreducible components of $Y \subseteq \mathbb{P}^n$, then $\dim Y = \dim Y \cap U_i$.

**Definition 4.26**: Let $Y \subseteq \mathbb{P}^n$ be a projective variety. The affine cone $\hat{Y} = C(Y)$ is

$$\theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}$$

where

$$\theta : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n.$$

**Proposition 4.27**:
1) $\hat{Y} = V(I(Y))$. In fact, $I\left(\hat{Y}\right) = I(Y)$.
2) $\dim \hat{Y} = \dim Y + 1$.
3) $\hat{Y}$ is irreducible if and only if $Y$ is irreducible.

**Theorem 4.28**: If $X, Y \subseteq \mathbb{P}^n$ are projective varieties and $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

**Lemma 4.29**: If $X, Y \subseteq \mathbb{A}^n$ affine varieties, then $X \cap Y = \emptyset$ or every irreducible component of $X \cap Y$ has $\dim \geq \dim X + \dim Y - n$.

Proof: Let $\Delta = V(x_1 - y_1, ..., x_n - y_n) \subseteq \mathbb{A}^{n+n}$. Note that

$$X \times Y \cap \Delta \simeq X \cap Y.$$

So, $\dim(X \times Y \cap \Delta) \geq \dim X + \dim Y - n$ by Krull's height theorem.

If $\underline{a} = (a_1, ...a_n)$ are varieties, then $I_{a(X)} = \{f(\underline{a} \mid f \in I(X)\}$. Then,

$$A(X \cap Y) = \frac{\Bbbk[\underline{z}]}{\sqrt{\langle I_{z(X)} + I_{z(Y)} \rangle}}$$

and

$$A(X \times Y \cap \Delta) = \frac{\Bbbk\left[\underline{x}, \underline{y}\right]}{\sqrt{\langle I_{x(X)} + I_{y(Y)} + I(A) \rangle}}.$$

So this implies that $x_i = y_i$ for all $i$, meaning they are isomorphic rings. ∎

Proof *of Theorem 4.28*: $X, Y$ irreducible implies that $\hat{X}$ and $\hat{Y}$ are irreducible. So, then

$$\dim\left(\hat{X} \cap \hat{Y}\right) \geq \dim X + 1 + \dim Y + 1 - (n + 1) \geq \dim X + \dim Y - n + 1.$$

$\hat{X} \cap \hat{Y}$ contains origin by construction, but it has at least one other point because dimension. ∎

# 5. Morphisms

**Definition 5.1**: For $U \subseteq \mathbb{R}^n$, $U' \subseteq \mathbb{R}^m$ open, $\varphi : U \to U'$ is continuous/continuously differentiable/smooth if $f \circ \varphi$ is smooth for any smooth $f : U' \to \mathbb{R}$.

$f' : U' \to \mathbb{R}$ is smooth if $f$ is smooth at every point $p \in U'$.

**Definition 5.2**: For affine variety $X \subseteq \mathbb{A}^n$ and $U \subseteq X$ open, a function $\varphi : U \to \Bbbk = \mathbb{A}^1$ is regular if $\forall p \in U$, $\exists U_p \ni p$ open and $f_p, g_p \in A(X)$ such that $\varphi(x) = \frac{f_p(x)}{g_p(x)}$ for all $x \in U_p$. In particular, $g_p(x) \neq 0$ for all $x \in U_p$.

$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$. This is also a $\Bbbk$-algebra.

**Example 5.3**: Let $U \subseteq X$, $\varphi : U \to \mathbb{A}^1$ regular $\not\Rightarrow \varphi = \frac{f}{g}$ globally for some $f, g \in A(X)$.

$X = V(xw - yz) \subset \mathbb{A}^4$, $U = X \setminus V(x, y)$.

$$\varphi(x, y, z, w) = \begin{cases} \frac{z}{x} \text{ if } x \neq 0 \\ \frac{w}{y} \text{ if } y \neq 0 \end{cases}.$$

**Lemma 5.4**: $\varphi : U \to \mathbb{A}^1$ regular, then $V(\varphi) = \{x \in U \mid \varphi(x) = 0\}$ is closed in $U$. In particular $\varphi$ is continuous.

Proof: Closedness is a local condition, and around any $p \in U$, $\{\varphi \restriction U_p : 0\} = V(f_p) \cap U_p$. ■

**Remark**: If $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ for $U$ irreducible, and $\varphi_1 \restriction U' = \varphi_2 \restriction U'$ for some $\emptyset \neq U' \subseteq U$, then $\varphi_1 = \varphi_2$.

**Definition 5.5**: $X \subseteq \mathbb{A}^n$ affine variety. A distinguished open subset $U$ of $X$ is an open subset of the form $X \setminus V(f)$ for some $f \in A(X)$, denoted $D(f)$, $D_f$, $U_f$, $X_f$. $X_f$ is probably the most descriptive as it actually mentions $X$.

**Remark**: $\{D(f)\}_{f \in A(X)}$ form a basis for Zariski topology. What that means is that any $U \subseteq X$ is a union for $D(f)$'s.

**Exercise 5.6**: $D(f)$ is homeomorphic to $V(I(X) + \langle 1 - yf \rangle) \subseteq \mathbb{A}^{n+1}$.

**Theorem 5.7**: $\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} \mid g \in A(X), m \in \mathbb{Z}_{\geq 0} \right\}$. In fact, $\mathcal{O}_X(D(f)) = A(X)_f$.

**Example 5.8**: $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = A(\mathbb{A}^2) = \mathbb{k}[x, y]$. Then,

$$\mathbb{A}^2 \ni \varphi = \begin{cases} \frac{f}{x^m} \text{ for some } f \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(x) \\ \frac{g}{y^\ell} \text{ for some } g \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(y) \end{cases}.$$

Then we say $y^\ell f = x^m g$ on $\mathbb{A}^2$. Because we are in a UFD, this means that $x^m \mid f$ and $y^\ell \mid g$. But this implies $m = \ell = 0$, so $f = g = \varphi$.

Proof *of Theorem 5.7*: $\supseteq$ is clear. So we only prove the $\subseteq$ case.

Suppose we have $\varphi \in \mathcal{O}_X(D(f))$. Then for all $p \in D(f)$, $\exists U_p \ni p$ and $\varphi \upharpoonright U_p = \frac{g_{p'}}{f_{p'}}$ for $g_{p'}, f_{p'} \in A(X)$.

Take a nonempty $D(h_p) \subseteq U_p$ and write $g_p = g_{p'} h_p$ and $f_p = f_{p'} h_p$.

Then $\varphi \upharpoonright D(f_p) = \frac{g_p}{f_p} = \frac{g_p f_p}{f_p^2}$. So assume $g_p = 0$ on $V(f_p)$.

Now we claim that $\forall p, q \in D(f)$, we have $g_p f_q = g_q f_p$ in $A(X)$.

Then $D(f) = \bigcup_p D(f_p)$. Then $V(f) = \bigcap_p V(f_p)$. Nullstellensatz says that $\sqrt{\langle f \rangle} = \sqrt{\langle f_p : p \in D(f) \rangle}$ as ideals in $A(X)$. But then, $f^m = \sum k_p f_p$. By Noetherian-ness, this is a finite sum. We claim that $g = \sum k_p f_p$.

Then $\frac{g}{f^m} = \frac{g_q}{f_q}$ on $D(f_q)$ for all $q \in D(f)$. So,

$$g f_q = \sum_p k_p g_p f_q = \sum_p k_p f_p g_q = g_q f^m.$$

■

# 6. Sheaves

Let $\mathcal{A}$ be a category: AbGrp, Rings, $\Bbbk$-algebras. Given a topological space $X$, $\mathrm{Top}(X)$ is a category where the objects are open subsets $U \subseteq X$ and morphisms are inclusions between $U \subseteq V$ open subsets.

**Definition 6.1**: A presheaf (with values in $\mathcal{A}$) on $X$ is a contravariant functor $\mathcal{F} : \mathrm{Top}(X) \to \mathcal{A}$.

$\mathcal{F}$ is further a sheaf if for every open cover $\{U_i\}_i$ of any open subset $U \subseteq X$ if

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

**Translation**:
1) Assignment $U \mapsto \mathcal{F}(U) \in \mathrm{obj}(\mathcal{A})$ such that $\forall U \subseteq V \subseteq X$ open,

$$\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that $\mathrm{res}_{U,U} = \mathrm{id}$ and $\mathrm{res}(V,U) \circ \mathrm{res}(W,V) = \mathrm{res}(W,U)$.
2) If $(f_i)_i \in \prod_i \mathcal{F}(U_i)$ such that $\mathrm{res}_{U_i,U_i \cap U_j}(f_i) = \mathrm{res}_{U_j,U_i \cap U_j}(f_j)$, then $\exists! f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i}(f) = f_i$. Also $\mathcal{F}(\emptyset) = 0$ as a consequence.

$f \upharpoonright V := \mathrm{res}_{U,V}(f)$, $f \in \mathcal{F}(U)$. $\mathcal{F}(U)$ elements are called sections of $\mathcal{F}$ over $U$.

**Example 6.2**: Note that throughout these examples, $X$ is a topological space and $U \subseteq X$.
1) $\mathcal{F}_{\mathrm{ct}}(U) := \{\varphi : U \to \mathbb{R}\}$. Then if $U' \subseteq U$, $\mathrm{res}_{U,U'}(f) := f \upharpoonright U'$.
2) $C(U) := \{\varphi : U \to \mathbb{R} \text{ cts}\}$.
3) $C^\infty(U) := \{\varphi : U \to \mathbb{R} \text{ smooth}\}$ ($X \subseteq \mathbb{R}^n$ open).
4) $\underline{\mathbb{R}}(U) := \{\varphi : U \to \mathbb{R}, \text{constant}\}$. This is not a sheaf. If we consider a constant function that takes the value $a$ on $U$ and $b$ on $U'$, then there is no value $c$ such that they can be glued together to be equal on both sets.
5) $\mathcal{O}_{X(U)} := \{\varphi : U \to \Bbbk \text{ regular}\}$

**Remark**: A constant sheaf $A_{X(U \text{ conn})} = A$. A locally constant sheaf (locally $\mathcal{F} \upharpoonright U$ is constant). Locally constant does not imply constant.

**Definition 6.3**: $U \subseteq X$, $\mathcal{F}$ sheaf on $X$, $\mathcal{F} \upharpoonright U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

**Definition 6.4**: $\mathcal{F}$ sheaf on $X$. $p \in X$. The stalk of $\mathcal{F}$ at $p$, $\mathcal{F}_p := \lim_{U \ni p} \mathcal{F}(U)$. This is actually just equal to $\{(U, f) \mid U \ni p, f \in \mathcal{F}(U)/\sim\}$ where $(U_1, f_1) \sim (U_2, f_2)$ if $\exists V \ni p$ such that $f_1 \upharpoonright V = f_2 \upharpoonright V$.

Remember that $\mathcal{F}\text{ct}_S(U) = \{f : U \to S\}$ and $C(U) = \{f : U \to R \text{ cts}\}$.

**Remark**: $\mathcal{F}(V) \to \mathcal{F}(U)$ by $'\text{res}_{V,U}$. This map need not be surjective.

**Theorem 6.5**: Let $X$ be an affine variety and $x \in X$. $\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_X}$.

Proof: Consider the ring map $A(X)_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$ where we map $\frac{f}{g} \mapsto \frac{f}{g}$.

Now if $\frac{f}{g} = \frac{f'}{g'}$ in $A(X)_{\mathfrak{m}_x}$, then we need to check that the same is true around $x$ in $\mathcal{O}_{X,x}$.

Now if $\frac{f}{g}$ is 0 around $x$ (in $D(h)$), we can deduce that $\frac{f}{g} = 0$ in $A(X)_{\mathfrak{m}_x}$. ■

**Definition 6.6**: A ringed space $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf on $X$ with values in Ring. We call $\mathcal{O}_X$ the structure sheaf of this ringed space.

**Definition 6.7**: $f : X \to Y$ continuous and $\mathcal{F}$ a sheaf on $X$.

$$\text{pushforward } f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

where $V \supset Y$ is open.

**Definition 6.8**: Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$. $\Phi : \mathcal{F} \to \mathcal{G}$ means that for each $U \subseteq X$, we specify $\Phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ where for $U \subseteq V$, we have the following diagram commuting:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\Phi(U)"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u, "\text{res}"] \arrow[r, "\Phi(V)"] & \mathcal{G}(V) \arrow[u, "\text{res}"'] \end{tikzcd}$$

where $V \supset Y$ is open.

**Definition 6.9**: A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ where $f : X \to Y$ is continuous and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$.

**Example 6.10**: $(U \subseteq \mathbb{R}^n, C^1) \to (V \subseteq \mathbb{R}^m, C^1)$.

**Remark**: When we say $(X, \mathcal{O}_X)$, we mean that $\mathcal{O}_X$ is a subsheaf of $\mathcal{F}\mathrm{ct}_{\Bbbk}$ for some fixed $\Bbbk$. Then $\mathcal{O}_X = \{\varphi : U \to \Bbbk \mid \varphi \text{ satisfies some condition}\}$. And, $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is always given by precomposition.

## 6.1. Category of quasi Affine varieties "qAffVar".

To define this cateogry, we'll have the objects be open subsets of an affine variety consdiered as a ringed space. The morphisms will be maps of ringed spaces with $f^{\#}$ being the precomposition as our convention above dictates.

**Theorem 6.11**: $X, Y$ both affine varieties. $U \subseteq X$ open. Then there exists a natural bijection.

$$\mathrm{Mor}(U, Y) \simeq \mathrm{Hom}_{\Bbbk\text{-alg}}(A(Y), \mathcal{O}(U)).$$

# 7. Projective morphisms

### 7.0.1. Ok unfortunately i was forced to miss two classes so there is gap here

**Proposition 7.1**: Suppose $X, Y$ are prevarieties with affine covers $\{U_i\}$ and $\{V_j\}$ respectively. Then $X \times Y$ is a product in the category of prevarieties constructed by gluing together $U_i \times V_j$ and $U_{i'} \times V_{j'}$,

$$(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

for all such pairs.

We did not prove that gluing gives you a prevariety, but we will believe it. Also note that $X \times Y$ is a prevariety by affine cover $\{U_i \times V_j\}$.

**Proposition 7.2**: For $Y \subseteq X$ closed where $\iota : Y \to X$ and $X$ is a prevariety. Then for $U' \subseteq Y$ open,

$$\iota^* \mathcal{O}_X(U') = \{f : U' \to \Bbbk \mid \forall y \in U', \exists U_y \subset X \text{ with } \varphi \in \mathcal{O}_X(U_y) \text{ such that } f \upharpoonright U_y \cap Y = \varphi\}.$$

Then $\iota^* \mathcal{O}_X$ is a sheaf and $(Y, \iota^* \mathcal{O}_X)$ is a prevariety, and $\iota : Y \to X$ is a morphism.

Proof: We will believe that $\iota^* \mathcal{O}_X$ is a sheaf. Also $\iota$ is a morphism of prevarieties.

Let $U \subseteq X$ be affine open. We claim that $(U \cap Y, \iota^* \mathcal{O}_X \upharpoonright U \cap Y)$ is affine. We claim that $Y = V(I)$ for some $I \subseteq A(X)$.

Then, $\iota^* \mathcal{O}_X$ are the functions that are locally restrictions of regular functions on $X$. Then $\mathcal{O}_{V(I)}$ are functions that are locally quotient of polynomials on $\mathbb{A}^n$. These are equal. ■

We say that $\iota$ is a closed embedding.

**Example 7.3**:
1) $\mathbb{A}^2 \to \mathbb{A}^2$ via $(x, y) \mapsto (x, xy)$. This maps $\mathbb{A}^2$ to itself without the $y$-axis, but still including the origin. Note that the image of this map is neither open nor closed in $\mathbb{A}^2$.

   **Remark**: $Y \subseteq X$ is locally closed if it is $U \cap V$ where $U$ is open in $X$ and $V$ is closed in $X$.

   This example is not even locally closed.
2) Glue $\mathbb{A}^1$ and $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ via the identity. This is basically a line with two origins. Let this line be called $X$. Consider $g : X \to X$ via switching the origins and keeping the other points the same. This is a morphism where our open subsets are lines including only one

of the origins, and it is not hard to check that this is actually a morphism. Then, $\{g(x) = x)\} \simeq \mathbb{A}^1 \setminus \{0\}$ which is not closed in $X$.

**Definition 7.4**: A prevariety of $X$ is a variety of the diagonal map $\Delta : X \to X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

**Lemma 7.5**: For checking if something is a variety, $\Delta$ being a topologically closed embedding is a sufficient condition.

**Corollary 7.6**: Open and closed subprevarieties of varieties are varieties.

**Lemma 7.7**: Let $X$ and $Y$ be affine varieties and $f : X \to Y$. If $f^\# : A(Y) \twoheadrightarrow A(X)$, then $f$ is a closed embedding.

Proof: Let $J$ be the kernel. And consider the surjective map

$$\frac{k[\underline{y}]}{I(Y)} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}.$$

This is also surjective onto $\frac{k[\underline{y}]}{I(Y)+J}$. This is $V_Y(J)$, so we get that $A(V_Y(J)) \simeq A(X)$ as desired. ■

**Lemma 7.8**: Let $X$ be an affine variety, then $\Delta : X \to X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

Proof: $A(X) = \frac{k[\underline{x}]}{I(X)}$, $A(X \times X) \simeq A(X) \otimes A(X)$. THen we can map

$$\frac{k[\underline{x}, \underline{y}]}{I_{x(X)} + I_{y(Y)}} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}$$

via $x_i \mapsto x_i$ and $y_i \mapsto x_j$. ■

**Proposition 7.9**: $X$ prevariety is a variety if $\Delta(X) \subseteq X \times X$ is closed.

Proof: We claim that $\forall x \in X$, take any $x \in U \subset X$ affine open. Then $\Delta \restriction U : U \to U \times U$ is a closed embedding.

Since $\Delta(X) \subseteq X \times X$ is closed (topologically closed embedding, locally closed embedding implies closed embedding. So $\mathcal{O}_X \simeq \mathcal{O}_{\Delta(X)}$.

It is important theat $\Delta \restriction U$ is closed because of the following:

$Y \subseteq X$ such that $\forall y \in Y, \exists U_y \subset X$ containing $y$ such that $U_y \cap Y$ is closed in $U_y$ does not imply that $Y \subseteq X$ is closed. However, under our assumption, it would be that way. ■

**Corollary 7.10**: qAffVar are varieties.

**Corollary 7.11**: $X \xrightarrow{f} Y$ morphism of varieties. The graph $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

Proof: $X \times Y \xrightarrow{f \times \mathrm{id}} Y \times Y$. Then $(f \circ \mathrm{id})^{-1}(\Delta(Y)) = \Gamma_f$. ■

**Exercise 7.12**: Let $X$ be a variety and $U, U' \subseteq X$ open affine subsets. Then $U \cap U'$ is also affine open.

**Definition 7.13**: Let $X \subseteq \mathbb{P}^n$ be a projective algebraic subset. $\mathcal{O}_X(U) = \left\{\varphi : U \to \Bbbk \mid \text{locally encrypted } (\varphi \restriction U') = \frac{F}{G} \text{ for some homogeneous poly } F, G \text{ of same degree.}\right\}$.

**Proposition 7.14**: $X \subseteq \mathbb{P}^n$ projected algebraic set $\rightsquigarrow (X, \mathcal{O}_X)$ if for all $i = 0, 1, ..., n$, let $U_i = \mathbb{P}^n \setminus V(x_i) \overset{\text{homeo}}{\simeq} \mathbb{A}^n$. Then $(X \cap U_i, \mathcal{O}_X \restriction U_i)$ is isomorphic as a ringed space to $X \cap U_i$ considered as a closed subset in $\mathbb{A}^n$ under $\mathbb{A}^n \simeq U_i$ and hence an affine variety. In particular, $X$ is a prevariety.

Proof: $X = V(F_1, ..., F_m)$. Then $X \cap U_i \subset \mathbb{A}^n$ is $V(F_1(x_0, ..., x_i = i, ..., x_n), ..., F_m)$.

If we have $\frac{F}{G} \in (X \cap U_i, \mathcal{O}_X \restriction U_i)$, then we can just dehomogenize to get $\frac{f}{g} \in \left(X \cap U_i, \mathcal{O}_{X \cap U_i \subseteq \mathbb{A}^n}\right)$. ■

**Lemma 7.15**: $X \subset \mathbb{P}^n$ a projective variety. Let $F_0, ..., F_m$ be homogeneous polynomials on $\mathbb{P}^n$ of same degree. Then $F : X \setminus V(F_0, ..., F_m) \to \mathbb{P}^m$ by $x \mapsto (F_0(x), ..., F_m(x))$ is a morphism.

Proof: As a set map, this is well-defined. We will verify that $\forall j = 0, ..., m$, the distinguished open subset $U_j = \left\{[\underline{Y}] \in \mathbb{P}^m \mid Y_j \neq 0\right\}$. We have

$$F^{-1}(U_j) = \mathbb{P}^n \setminus V(F_j).$$

Then $F^{-1}(U_j) \to U_j$ where $U_j \simeq \mathbb{A}^m$. We'll call the coordinates of $\mathbb{A}^m$ by $\left(\frac{Y_0}{Y_j}, ..., \frac{Y_m}{Y_j}\right)$. So if we have a point $x \in F^{-1}(U_j)$, then the associated point in $\mathbb{A}^m$ would attained by sending

$$x \mapsto \left(\frac{F_0(x)}{F_j(x)}, ..., \frac{F_m(x)}{F_j(x)}\right).$$

We showed that if we had a map from a quasi affine variety $W \to X$ where $X$ is an affine variety, we just had to map $A(X) \to \mathcal{O}(W)$. In an exercise, we showed that you can replace "quasi affine variety" with "prevariety" and get the same result. ■

**Example 7.16**:

1) $\mathbb{P}^1 \to \mathbb{P}^n$ where we map $[s : t] \mapsto [s^n : s^{n-1}t : \cdots : t^n]$. We know this will be a morphism by our lemma as long as we verify that it is full of zeroes only when $s = t = 0$, but this is clear.

   We can also map $\mathbb{P}^1 \to \mathbb{P}^2$ by $[s : t] \mapsto [s^3 : s^2t : t^3]$. This is because it maps nicely as above to $[s^3 : s^2t : st^2 : t^3]$, then we can project to drop the third coordinate to get the map we are describing. We are left to show that projections are morphisms
2) Projections: $\mathbb{P}^n \setminus \{[1 : 0 : \cdots : 0]\} \to \mathbb{P}^{n-1}$ by mapping

   $$[x_0 : \cdots x_n] \mapsto [x_1 : \cdots : x_n].$$

   More formally, we can consider $\mathbb{P}V \setminus \{[v]\} \to \mathbb{P}\left(\frac{V}{\text{span}(v)}\right)$, or $\mathbb{P}V \setminus \mathbb{P}W \to \mathbb{P}(V/W)$ where $W \subset V$.

   So, the second example above becomes $\mathbb{P}^3 \setminus [0 : 0 : 1 : 0] \to \mathbb{P}^2$.
3) Veronese embedding.

   $$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

   by $[x] \mapsto$ [every monomial of $x$ of degree $d$].

   **Exercise 7.17**: $\nu_d$ is a closed embedding.

4) Segre embedding:

   $$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)}$$

   by $([\underline{X}], [\underline{Y}]) \mapsto \begin{pmatrix} x_0y_0 & \cdots & x_0y_m \\ \vdots & & \vdots \\ x_ny_0 & \cdots & x_ny_m \end{pmatrix}$ where this really should just be one long vector, but it is easier to represent as such. We will prove that this is a closed embedding.

   Proof: Fix some $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we have

   $$\begin{aligned} U_{ij} &\simeq \mathbb{A}^{mn+m+n} \\ &= \left\{[z_{ab}] \in \mathbb{P}^{(n+1)(m+1)-1} \mid z_{ij} \neq 0\right\}. \end{aligned}$$

   Then $S^{-1}(U_{ij}) = U_i \times U_j$ where $U_i \subset \mathbb{P}^n$ and $U_j \subset \mathbb{P}^m$.The coordinates are $\frac{x_a}{x_i}$'s and $\frac{y_b}{y_j}$'s. This maps $\mathbb{A}^{n+m} \to \mathbb{A}^{n+m+nm}$ where the coordinates are $\frac{z_{ab}}{z_{ij}}$'s. We could map

$$\frac{z_{ab}}{z_{ij}} \mapsto \frac{x_a y_b}{x_i y_j}.$$

We claim that this is surjective. This is clear, as for example $\frac{z_{aj}}{z_{ij}} \mapsto \frac{x_a}{x_i}$. ■

5) $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ maps $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$. The matrix is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then the image is the same as $V(xw - yz)$ where $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.
6) $X \subset \mathbb{P}^n$ and $V(F_0, ..., F_m) \cap X = \emptyset$. Then $F : X \to \mathbb{P}^m$ is a well-defined morphism. The question is: do all maps from $X \to \mathbb{P}^m$ arise in this way?

   Well the answer is no because $P^1 \times P^1 \to \mathbb{P}^3$ as defined in the last example works. We can project $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. And the counterexample arises because there is no $F_0, F_1$ of the same degree such that there is no map $\mathbb{P}^3 \setminus V(F_0, F_1) \to \mathbb{P}^1$ that makes the diagram commute.

   Let $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ and let $Q = \mathrm{im}(S)$. We want $Q \cap V(F_0, F_1) = \emptyset$. However, $V(Q, F_0, F_1)$ has codimension at most 3, so dimension at least 0, in particular non-empty. This comes from Krull's height theorem.
7) If we are given four random lines in $\mathbb{P}^3$, how many meets all 4? The answer is 2.

   As an exercise, consider 3 random lines in $\mathbb{P}^3$, we can consider the union of all lines that touch all 3 and show that it is a projective variety.

#### 7.0.2. I skipped class again oops

# 8. Rational maps

For today and the rest of the week, we assume that every variety is irreducible.

**Warm-up**: Let $f, g : X \to Y$ be maps of varieties such that $f \restriction U = g \restriction U$, $\exists\ \emptyset \neq U \overset{\text{open}}{\subseteq} X$, then $f = g$.

Proof: Let $X \to X \times X$ by the diagonal map $\Delta$. Then let $X \times X \to Y \times Y$ by $f \times g$. The inverse image of $\Delta(Y)$ is $\{x \mid f(x) = g(x)\}$. Since they agree on an open subset and it is dense, they are actually equal. ∎

**Definition 8.1**: A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs $(U, \varphi_U)$ where we have that $\emptyset \neq U \subset X$ is open and $\varphi_U : U \to Y$ is a morphism. Then we have $(U, \varphi_U) \sim (V, \psi_V)$ if we have that $\varphi_U \restriction U \cap V = \psi_V \restriction U \cap V$.

**Definition 8.2**: $\varphi : X \dashrightarrow Y$ is dominant if $\varphi(U)$ is dense in $Y$ or some/every rep $(U, \varphi_U)$.

$\varphi$ is birational if $\exists \psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.

Two varieties are birational if there exists a birational map between them.

**Remark**: In general, you cannot compose rational maps.

**Example 8.3**:
1) $\mathbb{P}^{n+m}$ and $\mathbb{P}^n \times \mathbb{P}^m$ are birational. This is because there is a copy of $\mathbb{A}^{n+m}$ in both of them.
2) $\mathbb{A}^1$ and $V(x^3 - y^2)$ are birational. Consider $t \mapsto (t^2, t^3)$, or rather $(x, y) \mapsto \frac{y}{x}$ in the opposite direction.
3) $\mathbb{P}^1$ and $V(y^2 z - x^3 - x^2 z) \subset \mathbb{P}^2$. Take $[x : y] \mapsto \left[x : y : \frac{x^3}{y^2 - x^2}\right]$.

**Remark**: A variety $X$ is rational if it is birational to $\mathbb{A}^n$ for some $n$.

**Question**: Is there a non-rational variety? (Yes.)

Is $\varphi : X \dashrightarrow Y$ dominant and injective in a nonempty open subset, is it birational? This is true for characteristic zero, but false for characteristic $> 0$.

**Remark**: $X$ is unirational if $\exists$ dominant $\mathbb{A}^n \dashrightarrow X$. Rational and unirational are not equivalent. There are also non unirational varieties.

**Definition 8.4**: A rational function on $X$ is a rational map from $X \dashrightarrow \mathbb{A}^1$. We denote

$$K(X) := \{\text{rational functions on } X\}$$

and call it the (rational) function field.

**Theorem 8.5**:

$$\{\text{dominant rational maps } X \dashrightarrow Y\} \longleftrightarrow \{k\text{-alg extensions } K(Y) \subseteq K(X)\}$$

by the map $f \mapsto (\varphi \mapsto \varphi \circ f)$.

Proof: Let $\Theta : K(Y) \hookrightarrow K(X)$. We may assume $Y$ is affine, $Y \subseteq \mathbb{A}^n$. Now look at the functions $\Theta(y_1), ..., \Theta(y_m)$, where $A(Y) = \frac{\Bbbk[y]}{I}$ for some ideal $I$. All of the functions listed are regular on some open $U \subset X$.

So we have made a map from $A(Y) \to \mathcal{O}_X(U)$ by $y_i \mapsto \Theta(y_i)$, which defines a morphism (there is some theorem that says having a map from a coordinate ring to the structure sheaf defines a morphism). ■

**Corollary 8.6**: This bijection is an equivalence of categories:

$$\{\text{vars and rational dominant maps}\} \leftrightarrow \{\text{finitely generated field extensions over } \Bbbk\}.$$

**Corollary 8.7**: $X$ and $Y$ varieties. The following are equivalent:
1) $X$ and $Y$ are birational.
2) $\exists \emptyset \neq U \overset{\text{open}}{\subseteq} X, V \overset{\text{open}}{\subseteq} Y$ such that $U \simeq V$ isomorphic.
3) $K(X) \simeq K(Y)$.

1 to 2 can be verified. 2 to 3 uses the theorem above.

**Theorem 8.8**: Let $f : X \dashrightarrow Y$ be a dominant map. Then $f$ is generically finite (i.e. for any representative $f : U \to Y$, general fiber is finite) if and only if $K(Y) \subseteq K(X)$ is a finite extension. Further, if gen. fin. and char $k = 0$, then general fiber has exactly $[K(X) : K(Y)]$.

**Corollary 8.9**: In characteristic zero, a rational dominant map that is generically one to one is birational. This is very false in positive characteristic.

Proof *of Theorem 8.8*: Reduce to $X, Y$ affine, $X \xrightarrow{f} Y$ where $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ where $m \leq n$. Reduce to $m = n - 1$, where this map is now $(z_1, ..., z_n) \mapsto (z_1, ..., z_{n-1})$.

Now we split into cases:

1) $z_n \in K(X) = \Bbbk[z_1, ..., z_n]/I$ is algebraic over $K(Y)$. By definition, there is a minimal polynomial $G = a_d(z_1, ..., z_{n-1})z_n^d + \cdots + a_1(z_1, ..., z_{n-1})z_n + a_o(z_1, ..., z_{n-1}) \in K(Y)[z_n]$. We may assume that $G \in A(Y)[z_n]$.

   $D_Y(a_d) \neq \emptyset$ open in $Y$, $f$ is finite over $D(a_d)$. The discrimnant $\Delta$ of $G$ will be nonzero on $Y$. In other words, on $D_{y(a_d \cdot \Delta)}$, $|\text{fiber}| = d$.

■

## 8.1. Wasn't here for first part of blowups

**Definition 8.10**: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $I = \langle f_0, ..., f_m \rangle \subset A(X)$. The blowup, which we define as $\tilde{X} = B\ell_I X$, is the subvariety of $X \times \mathbb{P}^m (\subset \mathbb{A}^n \times \mathbb{P}^m)$ given by $u$-homogeneous elements of $\ker(\Bbbk[\underline{x}][\underline{u}] \twoheadrightarrow A(X)[tI] \subseteq A(X)[t])$. $\pi : \tilde{X} \to X$ the "blow-down" map.

**Proposition 8.11**: $B\ell_I X$ is independent of the choice of generators $f_0, ..., f_m$.

Proof: $B\ell_{\langle f_0, ..., f_m \rangle} X \simeq B\ell_{\langle f_0, ..., f_m, g \rangle} X$. ■

**Proposition 8.12**: $I = \langle f_0, ..., f_m \rangle \subset A(X)$. Then $\tilde{X} = B\ell_I X \simeq$ closure in $X \times \mathbb{P}^m$ of the image of

$$(X \setminus V(I)) \to X \times \mathbb{P}^m$$

given by

$$x \mapsto \left(x, \left[f_0(x), ..., f_{m(x)}\right]\right).$$

Proof:

1) $\overline{X} \subseteq \tilde{X}$:

2) $\tilde{X} \subseteq \overline{X}$:

$\blacksquare$

## 8.2. im retarded

# 9. Smoothness/Nonsingularness

What is a tangent vector? Rather, for $0 \in X \subseteq \mathbb{A}^n$, how can we find a tangent vector to $X$?

1) Something in the tangent space?

**Example 9.1**: Suppose char $k \neq 2$, $X = V\big((x-1)^2 + (y-1)^2 - 2\big)$. Then we can say that

$$T_0 X = V(x+y).$$

But how do we get this? We can see that

$$(x-1)^2 + (y-1)^2 - 2 = x^2 - 2x + y^2 - 2y.$$

And we see $-2(x+y)$ is the gradient of this function or something idk.

**Definition 9.2**: For $0 \in X \subseteq \mathbb{A}^n$,

$$T_0 X := V\big(f^{\text{linear}} \mid f \in I(X)\big).$$

For $p \in X \subseteq \mathbb{A}^n$,

$$T_p K := \ker \operatorname{Jac} \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{i,j}$$

for any generating set $f_1, ..., f_m$ of $I(X)$.

**Definition 9.3**: Let $A$ be a $k$-alg and $M$ an $A$-module. Then

$$\operatorname{Der}_k(A, M) := \{\delta \in \operatorname{Hom}_k(A, M) \mid \forall f, g \in A, \delta(fg) = f\delta(g) + g\delta(f)\}.$$

**Example 9.4**: Let $A = k[x, y]$ and $M = k[x, y]/\langle x - a, y - b\rangle$ for $a, b \in k$. Then this is isomorphism by $k$ with the group action of acting by $a$ on $x$.

Then

$$\operatorname{Der}_k(A, M) = k\left[\frac{\partial}{\partial x}|_{\substack{x=a \\ y=b}}, \frac{\partial}{\partial y}|_{\substack{x=a \\ y=b}}\right].$$

So for $\delta \in A$, we have $\delta(x^n) = nx^{n-1} \cdot \delta(x) = na^{n-1}\delta(x)$.

**Definition 9.5**: Zariski cotangent space of $p \in X$ is $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes A(X)/\mathfrak{m}$.

**Proposition 9.6**: Let $0 \in X \subseteq \mathbb{A}^n$. Then

$$T_0 X \simeq \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \simeq \mathrm{Der}_k(A(X), A(X)/\mathfrak{m} \simeq k)$$

where $\mathfrak{m} = \langle \overline{x}_1, ..., \overline{x}_n \rangle$. We can accomplish this with the maps

$$v \mapsto \left(\overline{f} \mapsto f^{\text{linear}}(v)\right)$$

from first to second and

$$\delta \mapsto \left(\overline{f} \mapsto \delta(f)\right)$$

for third to second.

Proof: Let $\varphi(\delta) = \overline{f} \mapsto \delta(f)$. Then $\delta(\mathfrak{m}^2) = 0$ implies that $\delta = 0$. Also $\delta(1) = 0$, so it is injective.

I'm too lazy to write this whole thing out ■

**Remark**: Note that $\dim T_p X \geq \operatorname{codim}_X p$.

**Definition 9.7**: $X$ variety is nonsingular at a point $p \in X$ if

$$\dim T_p X = \operatorname{codim}_X p.$$

**Proposition 9.8**: $p \in X$ is nonsingular if and only if $\dim T_p X \leq \operatorname{codim}_X p$. Also if and only if $T_p X = TC_p X$. Also if and only if $\dim\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \dim \mathcal{O}_{X,p}$. Also if and only if $\operatorname{rank} \mathrm{Jac}\left[\left(\frac{\partial f_i}{\partial x_j}(p)\right)\right]_{i,j} = n - \operatorname{codim}_X p$ when $X \subseteq \mathbb{A}^n$.

**Example 9.9**: $X = V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$. $TC_0 X = V(x^2 - y^2)$. $T_0 X = \mathbb{A}^2$.

**Remark**: A Noetherian ring $(R, \mathfrak{m})$ is a regular local ring if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

**Theorem 9.10**: Regular local rings are UFDs.

Proof: Something something Nakayama. ■

**Proposition 9.11**: Let $p \in X \subseteq \mathbb{A}^n$ is nonsingular if $\operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_m) \geq n - \operatorname{codim}_X p$ for any $f_1, ..., f_m$ such that $V(f_1, ..., f_m) = X$.

**Exercise 9.12**: Suppose $Y \subset \mathbb{P}^n$, $\langle F_1, ..., F_m \rangle = I(Y)$. To test $X \subset \mathbb{A}^n$ is nonsingular,

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} = (\deg F) F.$$

**Theorem 9.13**: Let $X$ be a variety. Then the nonsingular loci is open and nonempty in $X$.

Proof: Reduce to $X$ being affine. $X$ is irreducible, $X \subseteq \mathbb{A}^n$, $V(f_1, ..., f_m) = X$. Then

$$\operatorname{rank}\left[\operatorname{Jac}_p(f_1, ..., f_m)\right] \geq n - \dim X$$

so $X_{\text{sing}} = V(\operatorname{codim} X + 1 \text{ minors of } \operatorname{Jac}) \cap X$. ∎

**Lemma 9.14**: Any irreducible variety $X$ is birational to an irreducible hypersurface $V(f) \subset \mathbb{A}^n$.

Proof: Take $K(X)/k$ which is separable and finitely generated and separably generated. So there exists $x_1, ..., x_d \in K(X)$ such that $K(X) \supset k(x_1, ..., x_d)$ where $d = \dim X$. Then

$$K(X) \simeq \frac{k(x_1, ..., x_d)[y]}{f(y)}$$

where the coefficients of $f$ are in $x_1, ..., x_d$. ∎

**Definition 9.15**: $\mathbb{P}$ irreducible smooth variety, $X, Y$ irreducible subvarieties of $\mathbb{P}$, and $X \cap Y \supset Z$ an irreducible comp. We say $X$ and $Y$ intersect transversally at $p \in Z$ if

1) $p$ is smooth on $X$ and $Y$.
2) $\operatorname{codim}_{T_p\mathbb{P}} T_p X + \operatorname{codim}_{T_p\mathbb{P}} T_p Y = \operatorname{codim}_{T_p\mathbb{P}}(T_p X \cap T_p Y)$.

**Lemma 9.16**: If $X \pitchfork Y$ at $p \in Z$, then $p$ is nonsingular on $Z$ and $\operatorname{codim} Z = \operatorname{codim} X + \operatorname{codim} Y$.

Proof: Reduce to $\mathbb{P} = \mathbb{A}^n$. Then $I(X) = \langle f_1, ..., f_k \rangle$ and $I(Y) = \langle g_1, ..., g_\ell \rangle$. Then

$$\operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_k, g_1, ..., g_\ell) \leq \operatorname{codim} T_p Z = \operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_k) + \operatorname{rank} \operatorname{Jac}_p(g_1, ..., g_\ell)$$
$$\operatorname{codim} X + \operatorname{codim} Y \leq \operatorname{codim} T_P Z \leq \operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

Therefore they are all equal. ■

**Theorem 9.17** (Bertini): Fix $X \subseteq \mathbb{P}^n$. Then a general hyperplane $H \subseteq \mathbb{P}^n$ intersects $X$ transversally at all nonsingular points in $X$.

Proof: Let $\Gamma = \{(x, H) \mid H \text{ not transversally intersecting } X \text{ at } x\} \subseteq X_{\text{sm}} \times (\mathbb{P}^n)^\curlyvee$. We claim this is closed.

Fact 1: Let $X \twoheadrightarrow Y$ and general fiber has $\dim X - \dim Y$. Then

$$\dim \Gamma = \dim X + \operatorname{codim} X_1 = n - 1.$$

So then

$$\dim \pi_2(\Gamma) \leq n - 1$$

which shows us that $\pi_2(\Gamma) \subsetneq (\mathbb{P}^n)^\curlyvee$. Then another fact is that image of a morphism of varieties is constructible. ■

**Remark**: Let $f : X \to \mathbb{P}^n$ and $X$ smooth. For general $H \subseteq \mathbb{P}^n$, is $f^{-1}(H)$ smooth? This is true for char $= 0$ and false otherwise.

**Remark**: If $X \subseteq \mathbb{P}^n$ is smooth and irreducible with $\dim X > 1$. Then for general $H \subseteq \mathbb{P}^n$, $X \cap H$ is smooth, but is it irreducible?

This is true but very hard to prove.

# 10. Hilbert functions

Given $X \subseteq \mathbb{P}^n$, what is $\deg(X \subseteq \mathbb{P}^n)$?

Requirements:
1) $\deg(X \text{ finite}) = \# |X|$.
2) $\deg(V(F)) = \deg(F)$.
5) $(\deg X)(\deg Y) = \deg(X \cap Y)$ when $X$ transversally intersects $Y$.
6) $\deg(I(X) + I(Y)) = \sum_{z \text{ irred comp } X \cap Y} \operatorname{mult}_z(\deg z)$.

idk i cant read the rest of the board

**Definition 10.1**: Let $S = k[x_0, ..., x_n]$ and $I$ a homogeneous ideal. The Hilbert function of the ideal $I$ is defined as $h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ where

$$h_{I(d)} := \dim_k (S/I)_d.$$

For $X \subseteq \mathbb{P}^n$, $h_{X(d)} := h_{I(X)}(d)$.

**Example 10.2**:
1) $h_{\mathbb{P}^n}(d) = \binom{d+n}{n}$.
2) $h_{\text{pt}}(d) = 1$.
3) Suppose $V(I) = \emptyset$. Then $h_I(d) = 0$ for $d \gg 0$.
4) Let $I = \langle x_0^3, x_0^2 x_1 \rangle \subseteq k[x_0, x_1]$.

**Lemma 10.3**: Let $I, J \subseteq S$. Then $h_{I \cap J} + h_{I+J} = h_I + h_J$.

Proof: $0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I+J) \to 0$. The first map is $f \mapsto (f, f)$ and the second is $(f, g) \mapsto f - g$. ∎

**Lemma 10.4**: Let $I \subseteq S$. Let $f \in S$ be a nonzerodivisor on $(S/I)_d$ for $d \gg 0$. Then $h_{I+\langle f \rangle} = h_I(d) - h_I(d - \deg f)$.

Proof: Consider the sequence $0 \to (S/I)_{d - \deg f} \to (S/I)_d \to (S/I + f)_d \to 0$. ∎

**Theorem 10.5**: Let $I \subseteq S$ be homoegenous. There exists a unique Hilbert polynomial $\chi_I(t)$ such that $h_I(d) = \chi_I(d)$ for $d \gg 0$. Moreover $\deg \chi_I(t) = \dim V(I) \subseteq \mathbb{P}^n$. Note that we will say $\deg 0 := -1$ and $\dim \emptyset := -1$.

**Definition 10.6**: A polynomial $P(t) \in \mathbb{Q}(t)$ is numerical if $P(d) \in \mathbb{Z}$ for all $d \gg 0$, $d \in \mathbb{Z}$.

**Lemma 10.7**:
1) If $P(t)$ numerical of degree $r$, then there exists unique $c_0, \dots, c_r \in \mathbb{Z}$ such that

$$P(t) = c_r \binom{t}{r} + c_{r-1} \binom{t}{r-1} + \cdots + c_0.$$

2) If $f : \mathbb{Z} \to \mathbb{Z}$ such that $\Delta f(d) := f(d+1) - f(d)$ is a numerical polynmoial for $d \gg 0$, then so is $f$ for $d \gg 0$.

Proof:
1) Induct on $\deg P$.

$$P = c_r \binom{t}{r} + \cdots + c_1 \binom{t}{1} + c_0$$
$$\Delta P = c_r \binom{t}{r-1} + \cdots + c_1.$$

In particular, $P(d) \in \mathbb{Z}$ for all $d$.
2) Let $Q(t)$ such that $Q(d) = \Delta f(d)$ for all $d \gg 0$.

$$Q(t) = c_r \binom{t}{r} + \cdots + c_1 \binom{t}{1} + c_0$$
$$P(t) = c_r \binom{t}{r+1} + \cdots + c_1 \binom{t}{2} + c_0 \binom{t}{1} + C$$
$$\Delta(P - f)(d) = 0.$$

This last statement implies $P - f$ is a constant. Now we induct on $V(I)$.
- Base case: $V(I) = \emptyset$. Then $\chi_I = 0$.
- Now pick $f$ homogeneous linear such that the lemma applies. So the lemma then tells us that $\Delta h_I(d) = h_{I+\langle f \rangle}(d+1) = \chi_{I+\langle f \rangle}(d+1)$. We would be done, but this relies on the claim that $\dim V(I + \langle f \rangle) = \dim V(I) - 1$.

So let $f$ be a nonzerodivisor, $S/\langle I_d \rangle$, $\dim V(\langle I_d \rangle + f) = \dim V(I_d) - 1$. We claim that in fact $\dim V(I_d) = \dim V(I)$. ■

**Definition 10.8**: For $I \subseteq S$, $\deg(I) :=$ coefficient of $\binom{t}{\dim V(I)}$ in $\chi_I$. This is also the leading coefficient of $t^{\dim V(I)}$. Then $\deg(X \subseteq \mathbb{P}^n) := \deg I(X)) \cdot (\dim V(I))!$.

**Example 10.9**:

1) $\chi_{\mathbb{P}^n}(d) = \binom{d+n}{n} = h_{\mathbb{P}^n}(d)$. Also $\deg(\mathbb{P}^n \subseteq \mathbb{P}^n) = 1$.
2)