21-849: Algebraic Geometry I

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I don’t know what a sheave or a category is. ❤️

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1.

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and "multivariable calculus".
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.
- Both are continuous in $\mathbb{R}$ or $\mathbb{C}$.
- Both are holomorphic in $\mathbb{C}$.
- Both are analytic (power series expansion at every point).
- Both are C^∞.

There are differences as well.
- $f(z) = a$ has no solution or infinitely many solutions for e^z, but for almost all a, 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \to \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \to \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{C} \to \hat{C}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

Theorem 1.1 (GAGA Theorems): Compact (projective) $\mathbb{C}$-manifolds are algebraic.

Here are more cool things about algebraic geometry:
1) **Enumeration**:
   - How many solutions to $p(z)$?
   - How many points in $\{f(x,y) = g(x,y) = 0\}$?
   - How many lines meet a given set of 4 general lines in $\mathbb{C}^3$? The answer is 2.
   - How many conics ($\{f(x,y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264…
   - Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.
2) **Birationality**:
   - Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.
3) **Arithmetic Geometry**:
   - Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
   - Weil conjectures: X carved by polynomials with $\mathbb{Z}$-coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Chapter 2

2.1. Section 2