

Real Analysis II

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Table of Contents

1. Sequences	1
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1. Sequences

Theorem 1.1 (Bolzano Weierstrass): If a sequence (x_n) in \mathbb{R}^d is bounded, then it has a convergent subsequence.

PROOF: We take $(x_{n,1})$. By the $d = 1$ case, it has a convergent subsequence, let's say $(x_{n_k,1})$. Then we take $(x_{n_k,2})$, it has a convergent subsequence, let's say $(x_{n_{k_\ell},2})$. Etc. Once we have reached the d th coordinate, we can guarantee that we have convergent on every coordinate.

This shows that we have a convergent subsequence because $x_n \rightarrow x^*$ iff $\forall j \leq d$, we have that $x_{n,j} \rightarrow x_j^*$. ■

Definition 1.2: We say a sequence (x_n) is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t. $\forall n, m \geq N$, we have that $|x_n - x_m| < \varepsilon$.

Definition 1.3: We say that (X, ρ) is complete if every Cauchy sequence (x_n) in X is convergent.

Theorem 1.4: $(\mathbb{R}^d, |\cdot|)$ is complete.

PROOF: $d = 1$ is known. Coordinate wise application of the $d = 1$ case shows that $(\mathbb{R}^d, |\cdot|)$ is complete. ■