

# **21-849: Algebraic Geometry**

## **I**

Rohan Jain

January 13, 2025

I don't know what a sheave or a category is. ❤️

# Table of Contents

<b>1. Introduction</b>	<b>1</b>
1.0.1. Administrivia	1
1.1. Features of algebraic geometry	1
<b>2. Affine algebraic sets</b>	<b>2</b>
2.1. Nullstellensatz	2
<b>3. Projective Spaces</b>	<b>5</b>

# 1. Introduction

## 1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

## 1.1. Features of algebraic geometry

Consider the two functions  $e^z$  and  $z^2 - 3z + 2$ .

- Both are continuous in  $\mathbb{R}$  or  $\mathbb{C}$ .
- Both are holomorphic in  $\mathbb{C}$ .
- Both are analytic (power series expansion at every point).
- Both are  $C^\infty$ .

There are differences as well.

- $f(z) = a$  has no solution or infinitely many solutions for  $e^z$ , but for almost all  $a$ , 2 solutions for  $z^2 - 3z + 2$ .
- $e^z$  is not definable from  $\mathbb{Z} \rightarrow \mathbb{Z}$  but  $z^2 - 3z + 2$  is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$  for all  $\ell > 0$  for  $e^z$  but not for  $z^2 - 3z + 2$ .
- For nontrivial polynomials, as  $z \rightarrow \infty$ ,  $p(z)$  goes to infinity. So, it can be defined as a function from  $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ . But  $e^z$  can be periodic as the imaginary part tends to infinity.

This motivates the following result:

**Theorem 1.1** (GAGA Theorems): Compact (projective)  $\mathbb{C}$ -manifolds are algebraic.

Here are more cool things about algebraic geometry:

### 1) Enumeration:

- How many solutions to  $p(z)$ ?
- How many points in  $\{f(x, y) = g(x, y) = 0\}$ ?
- How many lines meet a given set of 4 general lines in  $\mathbb{C}^3$ ? The answer is 2.
- How many conics ( $\{f(x, y) = 0\}$ ,  $\deg f = 2$ ) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
- Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.

### 2) Birationality:

- Open sets are *huge*. That is, if we have  $X, Y$  and  $U \subseteq X, V \subseteq Y$  such that  $U \cong V$ , then  $X$  and  $Y$  are closely related.

### 3) Arithmetic Geometry:

- Over  $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$ , etc.
- Weil conjectures:  $X$  carved by polynomials with  $\mathbb{Z}$ -coefficients.  $H^2(X_{\mathbb{C}}, \mathbb{Q})$  related to integer solutions.

## 2. Affine algebraic sets

### 2.1. Nullstellensatz

Notation:  $\mathbb{k}$  is an algebraically closed field ( $\mathbb{k} = \mathbb{C}$ ).

**Definition 2.1** (Affine space): An  $n$ -affine space  $\mathbb{A}_{\mathbb{k}}^n$  is the set

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{k}, \forall i = 1, \dots, n\} = \mathbb{k}^n. \quad (1)$$

An affine algebraic subset of  $\mathbb{A}^n$  is a subset  $Z \subseteq \mathbb{A}^n$  such that

$$Z = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0, \forall f \in T\} \quad (2)$$

for some subset  $T \subseteq \mathbb{k}[x_1, \dots, x_n]$ . We write  $Z = V(T)$ .

**Example 2.1** (An affine space):

- $V(x^2 - y) \subset \mathbb{A}^2$ . This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$ . Note that  $x^2 + y^2 = (x + iy)(x - iy)$ , so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$ . We actually have  $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \mathbb{k}\}$ . Then note that if we project to any two dimensional plane  $(xy, yz, xz)$ , then we get another affine subset but on  $\mathbb{A}^2$ .

This leads us to the following question:

**Question:**  $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$ ?

SOLUTION: Consider  $V(1 - xy) \subseteq \mathbb{A}^2$ . If we project this to either axis, then we will miss the origin.

■

**Definition 2.2** (Ideal): Let  $Z \subseteq \mathbb{A}^n$  be an algebraic subset. Then

$$I(Z) = \{f \in \mathbb{k}[x] \mid f(p) = 0, \forall p \in Z\}. \quad (3)$$

**Example 2.2:**

- 0)  $Z = V(x^2) \subseteq \mathbb{A}^2$ , then  $I(Z) = \langle x \rangle$ .
- 1) If  $Z = V(x^2 - y)$ , then  $I(Z) = \langle x^2 - y \rangle$
- 2) If  $Z = V(x^2 - y, xy - z)$ , then  $I(Z) = \langle x^2 - y, xy - z \rangle$ .

**Proposition 2.1:**

- 1)  $I(Z)$  an ideal.  $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$ .
- 2)  $T \subseteq \mathbb{k}[x]$ .  $V(T) = V(\langle T \rangle)$  AND  $V(T) = V(f_1, \dots, f_m)$  for some  $f_i$ .
- 3) For  $\mathfrak{a} \subseteq \mathbb{k}[x]$  ideal,  $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ , where  $\sqrt{\mathfrak{a}} = \{f \in \mathbb{k}[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$ .
- 4) Algebraic subsets of  $\mathcal{A}^n$  are closed under finite unions and arbitrary intersections.

PROOF: We prove number 2 by using the Hilbert Basis Theorem. In particular,  $\mathbb{k}[x]$  is Noetherian.

■

**Theorem 2.2** (Nullstellensatz): Let  $Z$  be an algebraic subset. Then  $V(I(Z)) = Z$  and  $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{k}[x]\}. \quad (4)$$

PROOF:

- 1) Finite type field extensions  $L \supseteq F$  are finite. Remember that finite type means that  $F[x_1, \dots, x_m] \twoheadrightarrow L$ .
- 2) This implies that maximal ideals of  $\mathbb{k}[x]$  are of the form  $\langle x_1 - a_1, \dots, x_n - a_n \rangle$  for  $a_i \in \mathbb{k}$ , using the fact that  $\mathbb{k}$  is algebraically closed. So,  $\mathbb{k}[x]/\mathfrak{m} \simeq \mathbb{k}$ .
- 3) (Weak Nullstellensatz)  $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$ . That is,  $\mathfrak{a} \subsetneq \mathbb{k}[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$ .
- 4) So if  $f \in I(V(\mathfrak{a}))$ , then consider  $\mathfrak{a} + \langle 1 - yf \rangle \subseteq \mathbb{k}[x, y]$ . So for any  $(a_1, \dots, a_n, b)$  that vanishes on  $\mathfrak{a} + \langle 1 - yf \rangle$ , we realize that since  $1 - yf = 1$ , we have a unit ideal. That is, we can say  $1 = g_1 h_1 + g_2(1 - yf)$  for  $h_1 \in \mathfrak{a}$  and  $g_1, g_2 \in \mathbb{k}[x, y]$ . From here, we can conclude that  $f^\ell \in \mathfrak{a}$  for some  $\ell$ .

But also

$$\mathbb{k}[x, y]/\langle 1 - yf \rangle \simeq \mathbb{k}[x] \left[ \frac{1}{f} \right] = R. \quad (5)$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \dots + \frac{g_\ell}{f^{\ell-1}} \quad (6)$$

for  $g_i \in$  ideal  $\mathfrak{a}$  inside  $R$ .

■

Remark: We say  $R$  is Jacobson if every radical ideal  $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$ .

**Theorem 2.3:**  $R$  Jacobson  $\Rightarrow R[x]$  Jacobson.

**Definition 2.3** (Coordinate ring): The coordinate ring  $A(X)$  of  $X \subseteq \mathbb{A}^n$  is  $\mathbb{k}[x]/I(X)$ .

1)  $X \xrightarrow{f} \mathbb{k}$

2)  $\text{maxSpec } A(X) = \{\text{maximal ideals in } A(X)\} = X$ .

### 3. Projective Spaces

**Definition 3.1:**  $\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus \{0\}) / \sim$ . That is,  $v \sim v'$  if  $v = \lambda v'$  for some  $\lambda \in \mathbb{k}$ . That is,  $\mathbb{P}^n = \{1\text{-subspaces of } \mathbb{k}^{\{n+1\}}\}$ . For  $(a_0, \dots, a_n) \in \mathbb{k}^{n+1} \setminus \{0\}$ , we write  $[a_0 : \dots : a_n] \in \mathbb{P}^n$ .

Remark:  $V \simeq \mathbb{k}^{n+1}$ .  $\mathbb{P}V = V \setminus \{0\} / \sim$

**Definition 3.2:**  $f \in \mathbb{k}[x]$  is homogeneous if  $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\ell f(x_1, \dots, x_n)$ .

**Definition 3.3:** A projective algebraic set,  $X \subseteq \mathbb{P}^n$  is

$$V(T) = \{[x_0 : \dots : x_n] \mid f(x) = 0, \forall f \in T\} \quad (7)$$

for  $T$  a set of homogeneous polynomials.