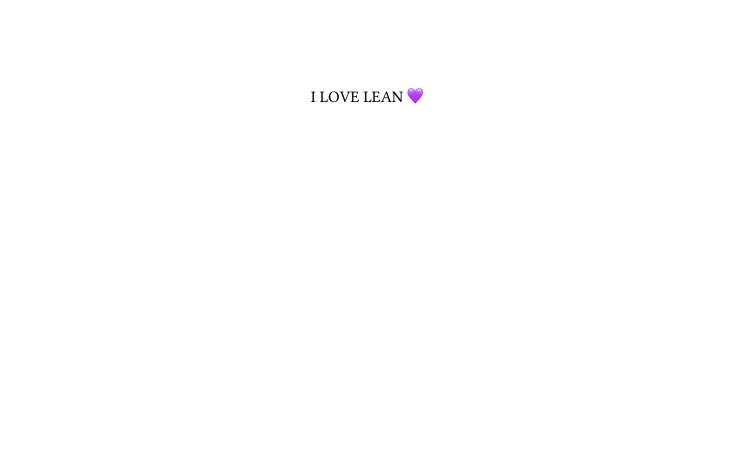
# **Category Theory**

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## 1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

## 1.1. Motivating example

Let 
$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$$
. Then let  $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$ .

**Theorem 1.1** (Brouwer's fixed point theorem): If  $f: D^2 \to D^2$  is continuous, then f has a fixed point. That is, there is some  $x \in D^2$  such that f(x) = x.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of  $\mathbb{R}^2$ ) and spits out a vector space (over  $\mathbb{R}$ ).

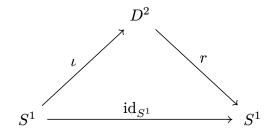
- 1) For every topological space X, there is a vector space H(X) (omitting actual definition).
- 2) For every continuous function  $f: X \to Y$ , there is an "induced" linear map given by  $H(f): H(X) \to H(Y)$ .
- 3) If  $X \to Y \to Z$  are continuous maps,  $H(f): H(X) \to H(Y), H(g): H(Y) \to H(Z)$  and  $H(g \circ f): H(X) \to H(Z)$ , then  $H(g \circ f) = H(g) \circ H(f)$ .
- 4) For any X,  $H(\mathrm{id}_X)=\mathrm{id}_{H(X)}:H(X)\to H(X).$

Computations:

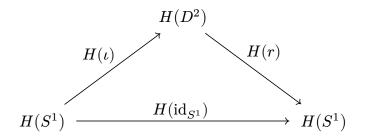
- 5)  $H(D^2) \cong 0$ .
- 6)  $H(S^1) \cong \mathbb{R}$ .

PROOF: Assume  $f:D^2\to D^2$  is continuous and f(x)=x for all  $x\in D^2$ . Define a new function  $r:D^2\to S^1$  such that r(x)= intersection of the ray from f(x) to x with  $S^1\subseteq D^2$ .

Key fact: If  $x \in S^1$ , then r(x) = x. Check that r is also continuous.



The diagram above commutes. Now we can apply homology to it.



We can check that

$$\begin{split} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} \,. \end{split}$$

Therefore, the new diagram also commutes. So, if  $w \in H(S^1)$ , then

$$w=\mathrm{id}_{H(S^1)}(w)=H(r)(H(\iota)(w))=0.$$

This is a contradiction as  $H(S^1) \neq 0$ .

## 1.2. Categories

**Definition 1.2** (Category): A category  $\mathcal{C}$  consists of:

- a collection of objects,  $\mathrm{Ob}(\mathcal{C})$ . For any  $A \in \mathrm{Ob}(\mathcal{C})$ , we usually write  $A \in \mathcal{C}$ .
- for any pair of objects  $A,B\in\mathcal{C}$ , there is a collection of morphisms  $\mathrm{Hom}_{\mathcal{C}}(A,B)$ , or  $\mathrm{Hom}(A,B)$ , or  $\mathcal{C}(A,B)$ . Instead of  $f\in\mathcal{C}(A,B)$ , we write  $f:A\to B$  or  $A\to B$ .
- for any objects  $A, B, C \in \mathcal{C}$  and morphisms  $f: A \to B$  and  $g: B \to C$ , there is a specified composition  $g \circ f: A \to C$ .
- for any object  $A \in \mathcal{C}$ , there is a given  $\mathrm{id}_A : A \to A$
- compositions are associative:  $(g\circ f)\circ h=g\circ (f\circ h)$
- for any  $A \stackrel{f}{\rightarrow} B$ ,  $f \circ id_A = f = id_B \circ f$

#### Example 1.3:

• Set, the category of sets (& functions).

**Definition 1.4** (Monoid): A monoid (M, \*) consists of:

- a set M
- a binary operation  $*: M \times M \to M$
- an identity element  $e \in M$  such that  $\forall x \in M, e * x = x * e = x$ .

**Definition 1.5** (Monoid Homomorphism): A monoid homomorphism  $f: M \to N$  is a function satisfying

- f(xy) = f(x)f(y).
- f(e) = e.

**Definition 1.6** (Functor): A functor  $F: \mathcal{C} \to \mathcal{D}$  is a function satisfying

- $F(A) \in \mathcal{D}$  for all  $A \in \mathcal{C}$ .
- $F(f): F(A) \to F(B)$  for all  $f: A \to B$  in  $\mathcal{C}$ .
- $F(g \circ f) = F(g) \circ F(f)$  for all  $f: A \to B$  and  $g: B \to C$  in  $\mathcal{C}$ .
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$  for all  $A \in \mathcal{C}$ .

#### 1.3. 09/02/2025

Two "sorts" of categories:

- "concrete" categories: sets with some sort of familiar structure (groups, rings, modules, etc.)
- "abstract" categories: 1, 2, 3, etc. More formal symbols than not.

**Definition 1.7** (Endomorphism): An endomorphism  $f: A \to A$  is a morphism from an object to itself.

New categories from old:

- 1) Product category.
  - input: two categories  $\mathcal C$  and  $\mathcal D$
  - output:  $\mathcal{C} \times \mathcal{D}$
  - objects: (A, B) where  $A \in Ob(\mathcal{C})$  and  $B \in Ob(\mathcal{D})$
  - morphisms: (f,g) where  $f:A\to A'$  in  $\mathcal C$  and  $g:B\to B'$  in  $\mathcal D$
  - composition:  $(f,g) \circ (f',g') = (f \circ f', g \circ g')$
  - identity:  $(id_A, id_B)$

Projection functors on  $\mathcal{C} \times \mathcal{D}$ :

- $\pi_1: \mathcal{C} \times \mathcal{D} \to C, \pi_2: \mathcal{C} \times \mathcal{D} \to \mathcal{D}.$
- on objects:  $\pi_1((A, B)) = A$
- on morphisms:  $\pi_1((f,g)) = f: A \to A'$ .
- 2) Slice categories, coslice categories
  - input: a category  $\mathcal{C}$  and an object  $X \in \mathrm{Ob}(\mathcal{C})$
  - output:  $\mathcal{C}/X$  or  $X/\mathcal{C}$

description of coslice:

- objects: pair (A, f), where  $A \in \mathrm{Ob}(\mathcal{C})$  and  $f : A \to X$  in  $\mathcal{C}$
- morphisms: from  $(A, f) \to (B, g)$ : morphism  $k : A \to B$  of  $\mathcal C$  such that  $k \circ f = g$ .

• composition:  $(A, f) \xrightarrow{k} (B, g) \xrightarrow{l} (C, h)$  is  $(A, f) \xrightarrow{l \circ k} (C, h)$ . We can check that  $(l \circ k) \circ f = h$ . The TLDR for this is that you can copy and paste commutative diagrams and get another commutative diagram.

**Example 1.8** (Coslice): Let  $\mathcal{C} = \operatorname{Set}$ ,  $X = 1 = \{*\}$ . So coslice  $X/\mathcal{C} = 1/\operatorname{Set} = ?$ .

- objects: pairs (A, f) of a set A and a function  $f: 1 \to A$ .
- morphisms: functions k such that  $k \circ f = g$ .

Elements of sets categorically. A is a set. How do we express  $a \in A$  in terms of the category Set?

elements of 
$$A \longleftrightarrow$$
 functions  $f: 1 \to A$  
$$a \in A \longleftrightarrow f: 1 \to A, f(*) = a$$
 
$$f(x) \in A \longleftrightarrow f: 1 \to A.$$

- 3) Opposite category.
  - input: a category  $\mathcal C$
  - output:  $\mathcal{C}^{\mathrm{op}}$
  - objects of  $\mathcal{C}^{\text{op}}$ :  $A^*$  for  $A \in \mathcal{C}$ .
  - morphisms of  $\mathcal{C}^{op}(A^*, B^*)$ :  $f^*$  for  $f: A \to B$  in  $\mathcal{C}$ .
  - composition:  $(f^* \circ g^*) = (g \circ f)^*$

## 1.4. 09/04/2025

Examples of functors between concrete categories:

- 1) Forgetful functors. E.g.  $U: \operatorname{Mon} \to \operatorname{Set}.\ U(M) = M.$  And if  $f: M \to N$  is a monoid homomorphism. Then  $U(f): UM \to UN$ , so we just take U(f) = f. Then we just have to check that  $U(g \circ f) = U(g) \circ U(f)$  but this is obvious. There are other similar examples like  $\operatorname{Vect}_k \to \operatorname{Set}$  or  $\operatorname{Top} \to \operatorname{Set}$ . Basically it's just "forgetting" some sort of structure from the original category.
- 2) Free functors. E.g.  $F : Set \to Mon$  which is the free monoid functor.

Let A be a set,  $\mathrm{List}(A)=\{\mathrm{strings}\ a_1,...,a_n\mid n\geq 0, a_i\in A\}.$  So if  $A=\{\mathrm{a,b,c}\},$  then we have that

$$List(A) = {<>, a, b, c, aa, ab, ac...}.$$

Define concatenation as · where

$$(a_1a_2...a_n)\cdot (b_1b_2...b_m)=(a_1a_2...a_nb_1b_2...b_m).$$

We claim that List(A) is a monoid with unit <>. Call that monoid  $FA \in Mon$ .

On morphisms: given  $f:A\to B$ , get monoid homomorphism  $F(f)=FA\to FB$ , we define

$$F(f)(a_1a_2...a_n) = f(a_1)f(a_2)...f(a_n).$$

We can also check that  $F(f \circ g) = F(f) \circ F(g)$  and  $F(\mathrm{id}_A) = \mathrm{id}_{FA}$ .

**Definition 1.9** (Contravariant Functor): A contravariant functor from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor  $F:\mathcal{C}^{\mathrm{op}}\to\mathcal{D}$ .

## **Universal Mapping Property**

Idea: universal property of X is a description of morphisms into/out of X.

#### 1.5. 09/11/2025

#### **Natural Transformations**

Let  $\mathcal C$  and  $\mathcal D$  be categories,  $F,G:\mathcal C\to\mathcal D$ . A natural transformation  $\alpha:F\to G$  consists of components  $\alpha_A:F(A)\to G(A)$  for each  $A\in\mathcal C$ , such that for any  $f:A\to B$  in  $\mathcal C$ , we have that  $G(f)\circ\alpha_A=\alpha_B\circ F(f)$ . This latter condition is called the naturality condition.

**Definition 1.10**: The category of graphs is  $[J^{op}, Set]$ . The objects of graphs are all the functors  $F: J^{op} \to Set$ , which consists of:

- a set F(0) "vertices"
- a set F(1) "edges"
- a function  $F(\sigma): F(1) \to F(0)$  "source"
- a function  $F(\tau): F(1) \to F(0)$  "target"

**Definition 1.11**: A category  $\mathcal{C}$  is small if  $\mathrm{Ob}(\mathcal{C})$  and every  $\mathcal{C}(A,B)$  is a set.

Examples of small categories: 2, J.

Large or non-small categories: Set, Mon, Top.

**Definition 1.12**: Cat is the category of small categories. The objects of Cat are small categories, and the morphisms are functors.

## 2. Limits

#### 2.1. 09/16/2025

We start by talking about the construction of objects. For sets A, B, we can form:

- Disjoint union A + B, which is a coproduct (colimit).
- Cartesian product  $A \times B$ , which is a product (limit).
- Set of functions  $B^A$ , which is exponential (adjunctions).

#### Products of sets.

**Definition 2.1**: Let A, B be sets. Their Cartesian product  $A \times B$  is the set of pairs (a, b) where  $a \in A$  and  $b \in B$ .

We write  $\pi_1:A\times B\to A$  and  $\pi_2:A\times B\to B$  for the projection maps.

What is UMP of  $A \times B$ ? For a set S, giving a function  $f: S \to A \times B$  is the same thing as giving for each  $s \in S$ , an element  $f(s) \in A \times B$ , which is the same thing as giving each  $s \in S$  an element  $a(s) \in A$  and an element  $b(s) \in B$ . Explicitly,  $a = \pi_1 \circ f$  and  $b = \pi_2 \circ f$ .

#### UMP of $A \times B$

For any set S and  $f_1: S \to A$  and  $f_2: S \to B$ , there is a unique  $u: S \to A \times B$  such that  $f_1 = \pi_1 \circ u$  and  $f_2 = \pi_2 \circ u$ .

**Definition 2.2**:  $\mathcal C$  a category,  $A,B\in\mathcal C$ . A diagram  $A \underset{p_1}{\leftarrow} P \underset{p_2}{\rightarrow} B$  is a product diagram if: for any object X and  $f_1:X\to A$  and  $f_2:X\to B$ , there is a unique  $u:X\to P$  such that  $f_1=p_1\circ u$  and  $f_2=p_2\circ u$ .

#### Terminology:

- $p_1, p_2$  are "projections" and P is the "product" of A and B.
- $u: X \to P$  is the map induced by  $f_1, f_2$ . Write  $u = \langle f_1, f_2 \rangle$  or  $u = (f_1, f_2)$ .
- *P* is "the product" of *A* and *B*, but:
  - ▶ being "a product" is a property of the whole diagram and not just a property of *P*,
  - "the" product may not be unique,
  - it also may not exist.

**Definition 2.3**: Given  $\mathcal{C}$ , and  $A, B \in \mathcal{C}$ , define the double slice category  $\mathcal{C}/(A, B)$  by:

- objects:  $(X\in\mathcal{C},\,f_1;X\to A,\,f_2:X\to B)$ . That is,  $A\xleftarrow{f_1}X\xrightarrow{f_2}B$ .
- morphisms: from  $(X, f_1, f_2)$  to  $(X', f_1', f_2')$  is a morphism  $f: X \to X'$  such that  $f \circ f_1 = f_1'$  and  $f \circ f_2 = f_2'$ .

Fact: a diagram in  $\mathcal{C}$   $A \leftarrow P \xrightarrow{p_1} B$  is a product diagram iff in  $\mathcal{C}/(A,B)$ , it is a terminal object: for every object of  $\mathcal{C}/(A,B) \ni \left(A \leftarrow P \xrightarrow{p_2} B\right)$ , there is a unique morphism of  $\mathcal{C}/(A,B)$  from it to  $\left(A \leftarrow P \xrightarrow{p_1} B\right)$ .

**Proposition 2.4**:  $\mathcal{D}$  category. If  $X,Y\in\mathcal{D}$  are both terminal objects, then there is a unique isomorphism  $X\to Y$ .

PROOF: Get (unique) morphism  $f: X \to Y$  since Y is terminal. Get (unique) morphism  $g: Y \to X$  since X is terminal. We have that  $g \circ f: X \to X$  and want to show that it is the identity. But since X is terminal, there is only one map from  $X \to X$ , so therefore  $g \circ f = \mathrm{id}_X$ . Likewise with  $f \circ g$  and  $\mathrm{id}_Y$ . Therefore, f is an isomorphism with g as its inverse.

"Product diagrams are unique up to unique isomorphism."

## 2.2. 09/18/2025

**Theorem 2.5**:  $\mathcal{C}$  has finite products if and only if  $\mathcal{C}$  has binary products and a terminal object.

Proof sketch:

⇒: binary products, terminal object are finite products

 $\Longleftarrow$ : Given a finite family  $\left(A_{i}\right)_{i\in I}$ , need to build a product.

if  $I = \emptyset$ : terminal object.

if |I| = 1: then A is the product of (A).

if |I| = 2: binary product.

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#### **Equalizers**

**Definition 2.6**: Given  $A \stackrel{f}{\underset{g}{\Longrightarrow}} B$ , form  $E = \{a \in A \mid f(a) = g(a)\} \subseteq A$  and the inclusion map  $e : E \to A$  defined as e(a) = a.

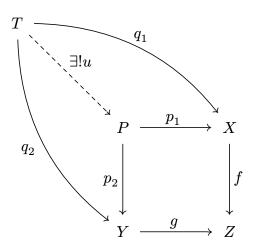
In  $E \stackrel{e}{\rightarrow} A \stackrel{f}{\underset{g}{\Longrightarrow}} B$ , we have  $f \circ e = g \circ e$ . If FINISH LATER

**Definition 2.7**:  $\mathcal{C}$  a category,  $\left(A \overset{f}{\underset{g}{\Longrightarrow}} B\right) = P$  "parallel pair" in  $\mathcal{C}$ . A fork on P is (E,e) where  $E \overset{e}{\underset{g}{\Longrightarrow}} A \overset{f}{\underset{g}{\Longrightarrow}} B$  such that  $f \circ e = g \circ e$ .

A fork  $E \stackrel{e}{\to} P$  is an equalizer (diagram) if for any  $X \stackrel{f}{\to} A$  such that  $f \circ h = g \circ h$ , there is a unique u such that  $e \circ u = h$ .

#### 2.3. 09/23/2025

**Definition 2.8**: A commutative square is called a pullback if for every  $T,q_1:T\to X,$   $q_2:T\to Y$  such that  $f\circ q_1=g\circ q_2$ , there is a unique  $u:T\to P$  such that  $p_1\circ u=q_1$  and  $p_2\circ u=q_2$ .



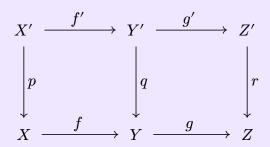
**Fact**: In Set, a square is a pullback iff for every  $x \in X$  and  $y \in Y$  with f(x) = g(y), there is a unique  $a \in P$  such that  $p_1(a) = x$  and  $p_2(a) = y$ .

PROOF: Elements correspond to map from 1=:T. Also, given  $q_1:T\to X$  and  $q_2:T\to Y,$  define  $u:T\to P$  by u(t)= the unique  $a\in P$  such that  $p_1(a)=q_1(t)$  and  $p_2(a)=q_2(t).$ 

**Definition 2.9**: Given  $f: X \to Z$  and  $z \in Z$ , the fiber of f (or X) over z is

$$X_z \coloneqq \mathrm{fib}_f(z) \coloneqq \{x \in X \mid f(x) = z\} \subseteq X.$$

**Lemma 2.10** (Two pullbacks lemma): In any category  $\mathcal{C}$ , given a diagram



$$X' \xrightarrow{g'f'} Z'$$

$$\downarrow^p \qquad \qquad \downarrow^r$$

$$X \xrightarrow{gf} Z$$

#### then

- if the first and second squares are pullbacks, then so is the third square.
- if the second and third squares are pullbacks, then so is the first square.