Category Theory

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1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

1.1. Motivating example

Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Then let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$.

> **Theorem 1.1** (Brouwer's fixed point theorem): If $f : D^2 \to D^2$ is continuous, then f has a fixed point. That is, there is some $x \in D^2$ such that $f(x) = x$.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of $\mathbb{R}^2$) and spits out a vector space (over $\mathbb{R}$).

1) For every topological space X, there is a vector space $H(X)$ (omitting actual definition).
2) For every continuous function $f : X \to Y$, there is an "induced" linear map given by $H(f) : H(X) \to H(Y)$.
3) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps, $H(f) : H(X) \to H(Y)$, $H(g) : H(Y) \to H(Z)$ and $H(g \circ f) : H(X) \to H(Z)$, then $H(g \circ f) = H(g) \circ H(f)$.
4) For any X, $H(\mathrm{id}_X) = \mathrm{id}_{H(X)} : H(X) \to H(X)$.

Computations:

5) $H(D^2) \cong 0$.
6) $H(S^1) \cong \mathbb{R}$.

Proof: Assume $f : D^2 \to D^2$ is continuous and $f(x) = x$ for all $x \in D^2$. Define a new function $r : D^2 \to S^1$ such that $r(x) =$ intersection of the ray from $f(x)$ to x with $S^1 \subseteq D^2$.

Key fact: If $x \in S^1$, then $r(x) = x$. Check that r is also continuous.

$$\begin{tikzcd}
 & D^2 \arrow[dr, "r"] & \\
S^1 \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}_{S^1}"] & & S^1
\end{tikzcd}$$

The diagram above commutes. Now we can apply homology to it.

$$\begin{tikzcd}
& H(D^2) \arrow[dr, "H(r)"] & \\
H(S^1) \arrow[ur, "H(\iota)"] \arrow[rr, "H(\mathrm{id}_{S^1})"] & & H(S^1)
\end{tikzcd}$$

We can check that

$$\begin{aligned} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} . \end{aligned}$$

Therefore, the new diagram also commutes. So, if $w \in H(S^1)$, then

$$w = \mathrm{id}_{H(S^1)}(w) = H(r)(H(\iota)(w)) = 0.$$

This is a contradiction as $H(S^1) \neq 0$. ■

1.2. Categories

Definition 1.2 (Category): A category $\mathcal{C}$ consists of:
- a collection of objects, $\mathrm{Ob}(\mathcal{C})$. For any $A \in \mathrm{Ob}(\mathcal{C})$, we usually write $A \in \mathcal{C}$.
- for any pair of objects $A, B \in \mathcal{C}$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, or $\mathrm{Hom}(A, B)$, or $\mathcal{C}(A, B)$. Instead of $f \in \mathcal{C}(A, B)$, we write $f : A \to B$ or $A \xrightarrow{f} B$.
- for any objects $A, B, C \in \mathcal{C}$ and morphisms $f : A \to B$ and $g : B \to C$, there is a specified composition $g \circ f : A \to C$.
- for any object $A \in \mathcal{C}$, there is a given $\mathrm{id}_A : A \to A$
- compositions are associative: $(g \circ f) \circ h = g \circ (f \circ h)$
- for any $A \xrightarrow{f} B$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$

Example 1.3:
- Set, the category of sets (& functions).

Definition 1.4 (Monoid): A monoid $(M, *)$ consists of:
- a set M
- a binary operation $* : M \times M \to M$
- an identity element $e \in M$ such that $\forall x \in M, e * x = x * e = x$.

Definition 1.5 (Monoid Homomorphism): A monoid homomorphism $f : M \to N$ is a function satisfying
- $f(xy) = f(x)f(y)$.
- $f(e) = e$.

Definition 1.6 (Functor): A functor $F : \mathcal{C} \to \mathcal{D}$ is a function satisfying
- $F(A) \in \mathcal{D}$ for all $A \in \mathcal{C}$.
- $F(f) : F(A) \to F(B)$ for all $f : A \to B$ in $\mathcal{C}$.
- $F(g \circ f) = F(g) \circ F(f)$ for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$.
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{C}$.

1.3. 09/02/2025

Two "sorts" of categories:
- "concrete" categories: sets with some sort of familiar structure (groups, rings, modules, etc.)
- "abstract" categories: $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$, etc. More formal symbols than not.

Definition 1.7 (Endomorphism): An endomorphism $f : A \to A$ is a morphism from an object to itself.

New categories from old:

1) Product category.
   - input: two categories $\mathcal{C}$ and $\mathcal{D}$
   - output: $\mathcal{C} \times \mathcal{D}$
   - objects: (A, B) where $A \in \mathrm{Ob}(\mathcal{C})$ and $B \in \mathrm{Ob}(\mathcal{D})$
   - morphisms: (f, g) where $f : A \to A'$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$
   - composition: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$
   - identity: $(\mathrm{id}_A, \mathrm{id}_B)$

   Projection functors on $\mathcal{C} \times \mathcal{D}$:
   - $\pi_1 : \mathcal{C} \times \mathcal{D} \to C$, $\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.
   - on objects: $\pi_1((A, B)) = A$
   - on morphisms: $\pi_1((f, g)) = f : A \to A'$.
2) Slice categories, coslice categories
   - input: a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$
   - output: $\mathcal{C}/X$ or $X/\mathcal{C}$

   description of coslice:
   - objects: pair (A, f), where $A \in \mathrm{Ob}(\mathcal{C})$ and $f : A \to X$ in $\mathcal{C}$
   - morphisms: from $(A, f) \to (B, g)$: morphism $k : A \to B$ of $\mathcal{C}$ such that $k \circ f = g$.

- composition: $(A, f) \xrightarrow{k} (B, g) \xrightarrow{l} (C, h)$ is $(A, f) \xrightarrow{l \circ k} (C, h)$. We can check that $(l \circ k) \circ f = h$. The TLDR for this is that you can copy and paste commutative diagrams and get another commutative diagram.

Example 1.8 *(Coslice)*: Let $\mathcal{C} = \mathrm{Set}$, $X = 1 = \{*\}$. So coslice $X/\mathcal{C} = 1/\mathrm{Set} = ?$.
- objects: pairs (A, f) of a set A and a function $f : 1 \to A$.
- morphisms: functions k such that $k \circ f = g$.

Elements of sets categorically. A is a set. How do we express $a \in A$ in terms of the category Set?

$$\text{elements of } A \longleftrightarrow \text{functions } f : 1 \to A$$
$$a \in A \longleftrightarrow f : 1 \to A, f(*) = a$$
$$f(x) \in A \longleftrightarrow f : 1 \to A.$$

3) Opposite category.
   - input: a category $\mathcal{C}$
   - output: $\mathcal{C}^{\mathrm{op}}$
   - objects of $\mathcal{C}^{\mathrm{op}}$: A^* for $A \in \mathcal{C}$.
   - morphisms of $\mathcal{C}^{\mathrm{op}}(A^*, B^*)$: f^* for $f : A \to B$ in $\mathcal{C}$.
   - composition: $(f^* \circ g^*) = (g \circ f)^*$

1.4. 09/04/2025

Examples of functors between concrete categories:

1) Forgetful functors. E.g. $U : \mathrm{Mon} \to \mathrm{Set}$. $U(M) = M$. And if $f : M \to N$ is a monoid homomorphism. Then $U(f) : UM \to UN$, so we just take $U(f) = f$. Then we just have to check that $U(g \circ f) = U(g) \circ U(f)$ but this is obvious. There are other similar examples like $\mathrm{Vect}_k \to \mathrm{Set}$ or $\mathrm{Top} \to \mathrm{Set}$. Basically it's just "forgetting" some sort of structure from the original category.
2) Free functors. E.g. $F : \mathrm{Set} \to \mathrm{Mon}$ which is the free monoid functor.

   Let A be a set, $\mathrm{List}(A) = \{\text{strings } a_1, ..., a_n \mid n \geq 0, a_i \in A\}$. So if $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, then we have that

   $$\mathrm{List}(A) = \{<>, \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{aa}, \mathrm{ab}, \mathrm{ac}...\}.$$

   Define concatenation as $\cdot$ where

   $$(a_1 a_2 ... a_n) \cdot (b_1 b_2 ... b_m) = (a_1 a_2 ... a_n b_1 b_2 ... b_m).$$

   We claim that $\mathrm{List}(A)$ is a monoid with unit $<>$. Call that monoid $FA \in \mathrm{Mon}$.

   On morphisms: given $f : A \to B$, get monoid homomorphism $F(f) = FA \to FB$, we define

   $$F(f)(a_1 a_2 ... a_n) = f(a_1) f(a_2) ... f(a_n).$$

   We can also check that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{id}_A) = \mathrm{id}_{FA}$.

Definition 1.9 (Contravariant Functor): A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$.

Universal Mapping Property

Idea: universal property of X is a description of morphisms into/out of X.