

21-849: Algebraic Geometry

Rohan Jain

January 13, 2025

I don't know what a sheave or a category is. ❤️

Table of Contents

1. Introduction	1
1.0.1. Administrivia	1
1.1. Features of algebraic geometry	1
2. Affine algebraic sets	2
2.1. Nullstellensatz	2
3. Projective Spaces	5
4. The Zariski Topology	8

1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.

- Both are continuous in \mathbb{R} or \mathbb{C} .
- Both are holomorphic in \mathbb{C} .
- Both are analytic (power series expansion at every point).
- Both are C^∞ .

There are differences as well.

- $f(z) = a$ has no solution or infinitely many solutions for e^z , but for almost all a , 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \rightarrow \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \rightarrow \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

Theorem 1.1 (GAGA Theorems): Compact (projective) \mathbb{C} -manifolds are algebraic.

Here are more cool things about algebraic geometry:

1) Enumeration:

- How many solutions to $p(z)$?
- How many points in $\{f(x, y) = g(x, y) = 0\}$?
- How many lines meet a given set of 4 general lines in \mathbb{C}^3 ? The answer is 2.
- How many conics ($\{f(x, y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
- Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.

2) Birationality:

- Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.

3) Arithmetic Geometry:

- Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
- Weil conjectures: X carved by polynomials with \mathbb{Z} -coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: \mathbb{k} is an algebraically closed field ($\mathbb{k} = \mathbb{C}$).

Definition 2.1 (Affine space): An n -affine space $\mathbb{A}_{\mathbb{k}}^n$ is the set

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{k}, \forall i = 1, \dots, n\} = \mathbb{k}^n.$$

An affine algebraic subset of \mathbb{A}^n is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \mathbb{k}[x_1, \dots, x_n]$. We write $Z = V(T)$.

Example 2.2 (An affine space):

- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \mathbb{k}\}$. Then note that if we project to any two dimensional plane (xy, yz, xz) , then we get another affine subset but on \mathbb{A}^2 .

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

SOLUTION: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin.

■

Definition 2.3 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \mathbb{k}[x] \mid f(p) = 0, \forall p \in Z\}.$$

Example 2.4:

- 0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
- 1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
- 2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.5:

- 1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
- 2) $T \subseteq \mathbb{k}[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, \dots, f_m)$ for some f_i .
- 3) For $\mathfrak{a} \subseteq \mathbb{k}[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \mathbb{k}[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
- 4) Algebraic subsets of \mathcal{A}^n are closed under finite unions and arbitrary intersections.

PROOF: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\mathbb{k}[x]$ is Noetherian.

■

Theorem 2.6 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{k}[x]\}.$$

PROOF:

- 1) Finite type field extensions $L \supseteq F$ are finite. Remember that finite type means that $F[x_1, \dots, x_m] \twoheadrightarrow L$.
- 2) This implies that maximal ideals of $\mathbb{k}[x]$ are of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ for $a_i \in \mathbb{k}$, using the fact that \mathbb{k} is algebraically closed. So, $\mathbb{k}[x]/\mathfrak{m} \simeq \mathbb{k}$.
- 3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq \mathbb{k}[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
- 4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq \mathbb{k}[x, y]$. So for any (a_1, \dots, a_n, b) that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in \mathbb{k}[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some ℓ .

But also

$$\mathbb{k}[x, y]/\langle 1 - yf \rangle \simeq \mathbb{k}[x] \left[\frac{1}{f} \right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \dots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal \mathfrak{a} inside R .

■

Remark: We say R is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

Theorem 2.7: R Jacobson $\Rightarrow R[x]$ Jacobson.

Definition 2.8 (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\mathbb{k}[x]/I(X)$.

1) $X \xrightarrow{f} \mathbb{k}$

2) $\text{maxSpec } A(X) = \{\text{maximal ideals in } A(X)\} = X$.

3. Projective Spaces

Definition 3.1: $\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus \{0\}) / \sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \mathbb{k}$. That is, $\mathbb{P}^n = \{1\text{-subspaces of } \mathbb{k}^{\{n+1\}}\}$. For $(a_0, \dots, a_n) \in \mathbb{k}^{n+1} \setminus \{0\}$, we write $[a_0 : \dots : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \mathbb{k}^{n+1}$. $\mathbb{P}V = V \setminus \{0\} / \sim$

Definition 3.2: $f \in \mathbb{k}[\underline{x}]$ is homogeneous if $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\ell f(x_1, \dots, x_n)$.

Definition 3.3: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \dots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for T a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

Example 3.4: Let $W \subseteq \mathbb{k}^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an m -plane in \mathbb{P}^n .

Example 3.5 (Twisted cubic curve): We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2t : st^2 : t^3] \mid [s : t] \in \mathbb{P}^1\}$. Then we have that $C = V(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix A , $\text{rank}(A) \leq r \iff \text{all } (r+1) \times (r+1) \text{ minors} = 0$.

Question: Can there exist F, G such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \mathbb{k}[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of X .

Exercise 3.6:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow \{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \mathbb{k}[\underline{x}] \text{ such that } \mathfrak{a} \neq \mathbb{k}[\underline{x}] \text{ or } \langle x_0, \dots, x_n \rangle\}.$$

This last part is called the “irrelevant ideal”.

Definition 3.7 (General Position): In \mathbb{P}^n , any subset of size $\leq n + 1$ points are linearly independent.

Theorem 3.8: Every set Γ of $2n$ points in \mathbb{P}^n in general position is carved out by quadrics.

PROOF: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose q is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\text{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \text{span}(\Gamma_1)$ or $q \in \text{span}(\Gamma_2)$.

Let p_1, \dots, p_k be a minimal subset of Γ whose span $\ni q$ ($k \leq n$). Now pick any Λ such that $|\Lambda| = n - k + 1$ which does not contain any of the p_i . We claim that $q \notin \text{span}(p_2, \dots, p_k, \Lambda)$.

We then conclude that for any $|S| = n - 1$, $S \subseteq \Gamma \setminus p_1, \dots, p_k$, we have that $\text{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \text{span}(p, S)$$

is the intersection at least n many hyperplanes, each of them containing p_1, q . But the intersection of n many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ■

Definition 3.9: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X$, $\exists g \in PGL_{n+1}$.

Proposition 3.10: Let (M_0, \dots, M_3) be any \mathbb{k} -basis of

$$\mathbb{k}[s, t]_3 = \{f \in \mathbb{k}[s, t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s, t) : \dots : M_3(s, t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2t : st^2 : t^3]\}$.

Example 3.11 (Rational normal curve): Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$. Or we could map it to any basis of $\mathbb{k}[s : t]_n$.

Exercise 3.12: $I(\varphi(\mathbb{P}^1)) = ?$.

Example 3.13: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

Fact: If we pick any 3 linearly independent $M_0, M_1, M_2 \in \mathbb{k}[s, t]_3$. Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ by M_0, M_1, M_2 has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ using 4 elements from $\mathbb{k}[s, t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

Question: Are all twisted quartic curves projectively equivalent?

SOLUTION: No. In fact, there are infinitely many distinct families. ■

Question (Hartshorne's Question): Is every irreducible curve in \mathbb{P}^3 carved out by 2 equations?

4. The Zariski Topology

Definition 4.1 (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \mathbb{k}[\underline{x}]\}$ form the closed sets of a topology on \mathbb{A}^n called the Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

Example 4.2: Take \mathbb{A}^1 . Two closed subsets are \mathbb{A}^1 and \emptyset . The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

Definition 4.3: A topological space X is irreducible if $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

Remark:

- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

Proposition 4.4: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. X is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

PROOF:

- \Rightarrow : Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If X is irreducible, then at least one of them completely contains X . That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means f or $g \in I(X)$.
- \Leftarrow : Suppose for sake of contradiction that X is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■