Category Theory

Rohan Jain

August 25, 2025

I LOVE LEAN 💜

Table of Contents

1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

1.1. Motivating example

Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Then let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$.

> **Theorem 1.1** (Brouwer's fixed point theorem): If $f : D^2 \to D^2$ is continuous, then f has a fixed point. That is, there is some $x \in D^2$ such that $f(x) = x$.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of $\mathbb{R}^2$) and spits out a vector space (over $\mathbb{R}$).

1) For every topological space X, there is a vector space $H(X)$ (omitting actual definition).
2) For every continuous function $f : X \to Y$, there is an "induced" linear map given by $H(f) : H(X) \to H(Y)$.
3) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps, $H(f) : H(X) \to H(Y)$, $H(g) : H(Y) \to H(Z)$ and $H(g \circ f) : H(X) \to H(Z)$, then $H(g \circ f) = H(g) \circ H(f)$.
4) For any X, $H(\mathrm{id}_X) = \mathrm{id}_{H(X)} : H(X) \to H(X)$.

Computations:

5) $H(D^2) \cong 0$.
6) $H(S^1) \cong \mathbb{R}$.

PROOF: Assume $f : D^2 \to D^2$ is continuous and $f(x) = x$ for all $x \in D^2$. Define a new function $r : D^2 \to S^1$ such that $r(x) =$ intersection of the ray from $f(x)$ to x with $S^1 \subseteq D^2$.

Key fact: If $x \in S^1$, then $r(x) = x$. Check that r is also continuous.

$$\begin{tikzcd} & D^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}_{S^1}"] & & S^1 \end{tikzcd}$$

The diagram above commutes. Now we can apply homology to it.

$$\begin{tikzcd}
 & H(D^2) \arrow[dr, "H(r)"] & \\
H(S^1) \arrow[ur, "H(\iota)"] \arrow[rr, "H(\mathrm{id}_{S^1})"] & & H(S^1)
\end{tikzcd}$$

We can check that

$$\begin{aligned} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} . \end{aligned} \tag{1}$$

Therefore, the new diagram also commutes. So, if $w \in H(S^1)$, then

$$w = \mathrm{id}_{H(S^1)}(w) = H(r)(H(\iota)(w)) = 0. \tag{2}$$

This is a contradiction as $H(S^1) \neq 0$. ■

1.2. Categories

Definition 1.2 (Category): A category $\mathcal{C}$ consists of:
- a collection of objects, $\mathrm{Ob}(\mathcal{C})$. For any $A \in \mathrm{Ob}(\mathcal{C})$, we usually write $A \in \mathcal{C}$.
- for any pair of objects $A, B \in \mathcal{C}$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, or $\mathrm{Hom}(A, B)$, or $\mathcal{C}(A, B)$. Instead of $f \in \mathcal{C}(A, B)$, we write $f : A \to B$ or $A \xrightarrow{f} B$.
- for any objects $A, B, C \in \mathcal{C}$ and morphisms $f : A \to B$ and $g : B \to C$, there is a specified composition $g \circ f : A \to C$.
- for any object $A \in \mathcal{C}$, there is a given $\mathrm{id}_A : A \to A$
- compositions are associative: $(g \circ f) \circ h = g \circ (f \circ h)$
- for any $A \xrightarrow{f} B$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$

Example 1.3:
- Set, the category of sets (& functions).