Category Theory

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August 25, 2025

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1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

1.1. Motivating example

Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Then let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$.

> **Theorem 1.1** (Brouwer's fixed point theorem): If $f : D^2 \to D^2$ is continuous, then f has a fixed point. That is, there is some $x \in D^2$ such that $f(x) = x$.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of $\mathbb{R}^2$) and spits out a vector space (over $\mathbb{R}$).

1) For every topological space X, there is a vector space $H(X)$ (omitting actual definition).
2) For every continuous function $f : X \to Y$, there is an "induced" linear map given by $H(f) : H(X) \to H(Y)$.
3) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps, $H(f) : H(X) \to H(Y)$, $H(g) : H(Y) \to H(Z)$ and $H(g \circ f) : H(X) \to H(Z)$, then $H(g \circ f) = H(g) \circ H(f)$.
4) For any X, $H(\mathrm{id}_X) = \mathrm{id}_{H(X)} : H(X) \to H(X)$.

Computations:

5) $H(D^2) \cong 0$.
6) $H(S^1) \cong \mathbb{R}$.

Proof: Assume $f : D^2 \to D^2$ is continuous and $f(x) = x$ for all $x \in D^2$. Define a new function $r : D^2 \to S^1$ such that $r(x) =$ intersection of the ray from $f(x)$ to x with $S^1 \subseteq D^2$.

Key fact: If $x \in S^1$, then $r(x) = x$. Check that r is also continuous.

$$\begin{tikzcd}
& D^2 \arrow[dr, "r"] & \\
S^1 \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}_{S^1}"] & & S^1
\end{tikzcd}$$

The diagram above commutes. Now we can apply homology to it.

$$\begin{tikzcd}
 & H(D^2) \arrow[dr, "H(r)"] & \\
H(S^1) \arrow[ur, "H(\iota)"] \arrow[rr, "H(\mathrm{id}_{S^1})"] & & H(S^1)
\end{tikzcd}$$

We can check that

$$\begin{aligned} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} . \end{aligned}$$

Therefore, the new diagram also commutes. So, if $w \in H(S^1)$, then

$$w = \mathrm{id}_{H(S^1)}(w) = H(r)(H(\iota)(w)) = 0.$$

This is a contradiction as $H(S^1) \neq 0$. ■

1.2. Categories

Definition 1.2 (Category): A category $\mathcal{C}$ consists of:
- a collection of objects, $\mathrm{Ob}(\mathcal{C})$. For any $A \in \mathrm{Ob}(\mathcal{C})$, we usually write $A \in \mathcal{C}$.
- for any pair of objects $A, B \in \mathcal{C}$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$, or $\mathrm{Hom}(A, B)$, or $\mathcal{C}(A, B)$. Instead of $f \in \mathcal{C}(A, B)$, we write $f : A \to B$ or $A \overset{f}{\to} B$.
- for any objects $A, B, C \in \mathcal{C}$ and morphisms $f : A \to B$ and $g : B \to C$, there is a specified composition $g \circ f : A \to C$.
- for any object $A \in \mathcal{C}$, there is a given $\mathrm{id}_A : A \to A$
- compositions are associative: $(g \circ f) \circ h = g \circ (f \circ h)$
- for any $A \overset{f}{\to} B$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$

Example 1.3:
- Set, the category of sets (& functions).

Definition 1.4 (Monoid): A monoid $(M, *)$ consists of:
- a set M
- a binary operation $* : M \times M \to M$
- an identity element $e \in M$ such that $\forall x \in M, e * x = x * e = x$.

Definition 1.5 (Monoid Homomorphism): A monoid homomorphism $f : M \to N$ is a function satisfying
- $f(xy) = f(x)f(y)$.
- $f(e) = e$.

Definition 1.6 (Functor): A functor $F : \mathcal{C} \to \mathcal{D}$ is a function satisfying
- $F(A) \in \mathcal{D}$ for all $A \in \mathcal{C}$.
- $F(f) : F(A) \to F(B)$ for all $f : A \to B$ in $\mathcal{C}$.
- $F(g \circ f) = F(g) \circ F(f)$ for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$.
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{C}$.

1.3. 09/02/2025

Two "sorts" of categories:
- "concrete" categories: sets with some sort of familiar structure (groups, rings, modules, etc.)
- "abstract" categories: $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$, etc. More formal symbols than not.

Definition 1.7 (Endomorphism): An endomorphism $f : A \to A$ is a morphism from an object to itself.

New categories from old:

1) Product category.
   - input: two categories $\mathcal{C}$ and $\mathcal{D}$
   - output: $\mathcal{C} \times \mathcal{D}$
   - objects: (A, B) where $A \in \mathrm{Ob}(\mathcal{C})$ and $B \in \mathrm{Ob}(\mathcal{D})$
   - morphisms: (f, g) where $f : A \to A'$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$
   - composition: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$
   - identity: $(\mathrm{id}_A, \mathrm{id}_B)$

   Projection functors on $\mathcal{C} \times \mathcal{D}$:
   - $\pi_1 : \mathcal{C} \times \mathcal{D} \to C$, $\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.
   - on objects: $\pi_1((A, B)) = A$
   - on morphisms: $\pi_1((f, g)) = f : A \to A'$.
2) Slice categories, coslice categories
   - input: a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$
   - output: $\mathcal{C}/X$ or $X/\mathcal{C}$

   description of coslice:
   - objects: pair (A, f), where $A \in \mathrm{Ob}(\mathcal{C})$ and $f : A \to X$ in $\mathcal{C}$
   - morphisms: from $(A, f) \to (B, g)$: morphism $k : A \to B$ of $\mathcal{C}$ such that $k \circ f = g$.

- composition: $(A,f) \xrightarrow{k} (B,g) \xrightarrow{l} (C,h)$ is $(A,f) \xrightarrow{l\circ k} (C,h)$. We can check that $(l \circ k) \circ f = h$. The TLDR for this is that you can copy and paste commutative diagrams and get another commutative diagram.

Example 1.8 *(Coslice)*: Let $\mathcal{C} = \text{Set}$, $X = 1 = \{*\}$. So coslice $X/\mathcal{C} = 1/\text{Set} = ?$.
- objects: pairs (A, f) of a set A and a function $f : 1 \to A$.
- morphisms: functions k such that $k \circ f = g$.

Elements of sets categorically. A is a set. How do we express $a \in A$ in terms of the category Set?

$$\text{elements of } A \longleftrightarrow \text{functions } f : 1 \to A$$
$$a \in A \longleftrightarrow f : 1 \to A, f(*) = a$$
$$f(x) \in A \longleftrightarrow f : 1 \to A.$$

3) Opposite category.
   - input: a category $\mathcal{C}$
   - output: $\mathcal{C}^{\text{op}}$
   - objects of $\mathcal{C}^{\text{op}}$: A^* for $A \in \mathcal{C}$.
   - morphisms of $\mathcal{C}^{\text{op}}(A^*, B^*)$: f^* for $f : A \to B$ in $\mathcal{C}$.
   - composition: $(f^* \circ g^*) = (g \circ f)^*$

1.4. 09/04/2025

Examples of functors between concrete categories:

1) Forgetful functors. E.g. $U : \text{Mon} \to \text{Set}$. $U(M) = M$. And if $f : M \to N$ is a monoid homomorphism. Then $U(f) : UM \to UN$, so we just take $U(f) = f$. Then we just have to check that $U(g \circ f) = U(g) \circ U(f)$ but this is obvious. There are other similar examples like $\text{Vect}_k \to \text{Set}$ or $\text{Top} \to \text{Set}$. Basically it's just "forgetting" some sort of structure from the original category.
2) Free functors. E.g. $F : \text{Set} \to \text{Mon}$ which is the free monoid functor.

   Let A be a set, $\text{List}(A) = \{\text{strings } a_1, ..., a_n \mid n \geq 0, a_i \in A\}$. So if $A = \{\text{a}, \text{b}, \text{c}\}$, then we have that

   $$\text{List}(A) = \{<>, \text{a}, \text{b}, \text{c}, \text{aa}, \text{ab}, \text{ac}...\}.$$

   Define concatenation as $\cdot$ where

   $$(a_1 a_2 ... a_n) \cdot (b_1 b_2 ... b_m) = (a_1 a_2 ... a_n b_1 b_2 ... b_m).$$

   We claim that $\text{List}(A)$ is a monoid with unit $<>$. Call that monoid $FA \in \text{Mon}$.

   On morphisms: given $f : A \to B$, get monoid homomorphism $F(f) = FA \to FB$, we define

   $$F(f)(a_1 a_2 ... a_n) = f(a_1) f(a_2) ... f(a_n).$$

   We can also check that $F(f \circ g) = F(f) \circ F(g)$ and $F(\text{id}_A) = \text{id}_{FA}$.

Definition 1.9 (Contravariant Functor): A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$.

Universal Mapping Property

Idea: universal property of X is a description of morphisms into/out of X.

1.5. 09/11/2025

Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\alpha : F \to G$ consists of components $\alpha_A : F(A) \to G(A)$ for each $A \in \mathcal{C}$, such that for any $f : A \to B$ in $\mathcal{C}$, we have that $G(f) \circ \alpha_A = \alpha_B \circ F(f)$. This latter condition is called the naturality condition.

Definition 1.10: The category of graphs is $[J^{\text{op}}, \text{Set}]$. The objects of graphs are all the functors $F : J^{\text{op}} \to \text{Set}$, which consists of:
- a set $F(0)$ "vertices"
- a set $F(1)$ "edges"
- a function $F(\sigma) : F(1) \to F(0)$ "source"
- a function $F(\tau) : F(1) \to F(0)$ "target"

Definition 1.11: A category $\mathcal{C}$ is small if $\text{Ob}(\mathcal{C})$ and every $\mathcal{C}(A, B)$ is a set.

Examples of small categories: $\mathbb{2}$, J.

Large or non-small categories: Set, Mon, Top.

Definition 1.12: Cat is the category of small categories. The objects of Cat are small categories, and the morphisms are functors.

2. Limits

2.1. 09/16/2025

We start by talking about the construction of objects. For sets A, B, we can form:
- Disjoint union $A + B$, which is a coproduct (colimit).
- Cartesian product $A \times B$, which is a product (limit).
- Set of functions B^A, which is exponential (adjunctions).

Products of sets.

> **Definition 2.1**: Let A, B be sets. Their Cartesian product $A \times B$ is the set of pairs (a, b) where $a \in A$ and $b \in B$.

We write $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ for the projection maps.

What is UMP of $A \times B$? For a set S, giving a function $f : S \to A \times B$ is the same thing as giving for each $s \in S$, an element $f(s) \in A \times B$, which is the same thing as giving each $s \in S$ an element $a(s) \in A$ and an element $b(s) \in B$. Explicitly, $a = \pi_1 \circ f$ and $b = \pi_2 \circ f$.

UMP of $A \times B$

For any set S and $f_1 : S \to A$ and $f_2 : S \to B$, there is a unique $u : S \to A \times B$ such that $f_1 = \pi_1 \circ u$ and $f_2 = \pi_2 \circ u$.

> **Definition 2.2**: $\mathcal{C}$ a category, $A, B \in \mathcal{C}$. A diagram $A \underset{p_1}{\leftarrow} P \underset{p_2}{\rightarrow} B$ is a product diagram if: for any object X and $f_1 : X \to A$ and $f_2 : X \to B$, there is a unique $u : X \to P$ such that $f_1 = p_1 \circ u$ and $f_2 = p_2 \circ u$.

Terminology:
- p_1, p_2 are "projections" and P is the "product" of A and B.
- $u : X \to P$ is the map induced by f_1, f_2. Write $u = \langle f_1, f_2 \rangle$ or $u = (f_1, f_2)$.
- P is "the product" of A and B, but:
  - being "a product" is a property of the whole diagram and not just a property of P,
  - "the" product may not be unique,
  - it also may not exist.

> **Definition 2.3**: Given $\mathcal{C}$, and $A, B \in \mathcal{C}$, define the double slice category $\mathcal{C}/(A, B)$ by:
> - objects: $(X \in \mathcal{C},\ f_1 ; X \to A,\ f_2 : X \to B)$. That is, $A \underset{f_1}{\leftarrow} X \underset{f_2}{\rightarrow} B$.
> - morphisms: from (X, f_1, f_2) to $(X', f_1{}', f_2{}')$ is a morphism $f : X \to X'$ such that $f \circ f_1 = f_1{}'$ and $f \circ f_2 = f_2{}'$.

Fact: a diagram in $\mathcal{C}$ $A \underset{p_1}{\leftarrow} P \underset{p_2}{\rightarrow} B$ is a product diagram iff in $\mathcal{C}/(A, B)$, it is a terminal object: for every object of $\mathcal{C}/(A, B) \ni \left(A \underset{p_1}{\leftarrow} P \underset{p_2}{\rightarrow} B\right)$, there is a unique morphism of $\mathcal{C}/(A, B)$ from it to $\left(A \underset{p_1}{\leftarrow} P \underset{p_2}{\rightarrow} B\right)$.

Proposition 2.4: $\mathcal{D}$ category. If $X, Y \in \mathcal{D}$ are both terminal objects, then there is a unique isomorphism $X \to Y$.

Proof: Get (unique) morphism $f : X \to Y$ since Y is terminal. Get (unique) morphism $g : Y \to X$ since X is terminal. We have that $g \circ f : X \to X$ and want to show that it is the identity. But since X is terminal, there is only one map from $X \to X$, so therefore $g \circ f = \mathrm{id}_X$. Likewise with $f \circ g$ and id_Y. Therefore, f is an isomorphism with g as its inverse. ■

"Product diagrams are unique up to unique isomorphism."

2.2. 09/18/2025

Theorem 2.5: $\mathcal{C}$ has finite products if and only if $\mathcal{C}$ has binary products and a terminal object.

Proof *sketch*:

$\Longrightarrow$: binary products, terminal object are finite products

$\Longleftarrow$: Given a finite family $(A_i)_{i \in I}$, need to build a product.

if $I = \emptyset$: terminal object.

if $|I| = 1$: then A is the product of (A).

if $|I| = 2$: binary product. ■

Equalizers

Definition 2.6: Given $A \overset{f}{\underset{g}{\rightrightarrows}} B$, form $E = \{a \in A \mid f(a) = g(a)\} \subseteq A$ and the inclusion map $e : E \to A$ defined as $e(a) = a$.

In $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$, we have $f \circ e = g \circ e$. If FINISH LATER

Definition 2.7: $\mathcal{C}$ a category, $\left(A \overset{f}{\underset{g}{\rightrightarrows}} B\right) = P$ “parallel pair” in $\mathcal{C}$. A fork on P is (E, e) where $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$ such that $f \circ e = g \circ e$.

A fork $E \xrightarrow{e} P$ is an equalizer (diagram) if for any $X \xrightarrow{f} A$ such that $f \circ h = g \circ h$, there is a unique u such that $e \circ u = h$.

2.3. 09/23/2025

Definition 2.8: A commutative square is called a pullback if for every T, $q_1 : T \to X$, $q_2 : T \to Y$ such that $f \circ q_1 = g \circ q_2$, there is a unique $u : T \to P$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Fact: In Set, a square is a pullback iff for every $x \in X$ and $y \in Y$ with $f(x) = g(y)$, there is a unique $a \in P$ such that $p_1(a) = x$ and $p_2(a) = y$.

Proof: Elements correspond to map from $1 =: T$. Also, given $q_1 : T \to X$ and $q_2 : T \to Y$, define $u : T \to P$ by $u(t) =$ the unique $a \in P$ such that $p_1(a) = q_1(t)$ and $p_2(a) = q_2(t)$. ■

Definition 2.9: Given $f : X \to Z$ and $z \in Z$, the fiber of f (or X) over z is

$$X_z \coloneqq \mathrm{fib}_f(z) \coloneqq \{x \in X \mid f(x) = z\} \subseteq X.$$

Lemma 2.10 (Two pullbacks lemma): In any category $\mathcal{C}$, given a diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p"] & Y' \arrow[r, "g'"] \arrow[d, "q"] & Z' \arrow[d, "r"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[r, "g'f'"] \arrow[d, "p"] & Z' \arrow[d, "r"] \\ X \arrow[r, "gf"] & Z \end{tikzcd}$$

then

- if the first and second squares are pullbacks, then so is the third square.
- if the second and third squares are pullbacks, then so is the first square.