Category Theory

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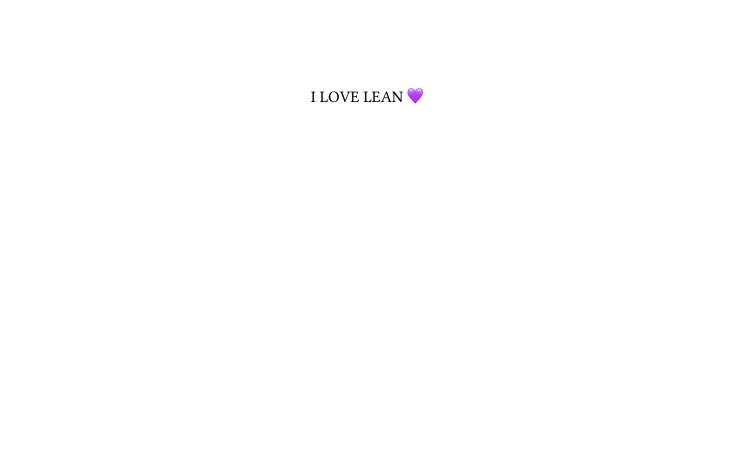


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1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

1.1. Motivating example

Let
$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$$
. Then let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$.

Theorem 1.1 (Brouwer's fixed point theorem): If $f: D^2 \to D^2$ is continuous, then f has a fixed point. That is, there is some $x \in D^2$ such that f(x) = x.

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of \mathbb{R}^2) and spits out a vector space (over \mathbb{R}).

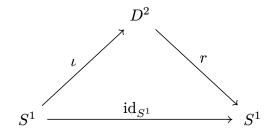
- 1) For every topological space X, there is a vector space H(X) (omitting actual definition).
- 2) For every continuous function $f: X \to Y$, there is an "induced" linear map given by $H(f): H(X) \to H(Y)$.
- 3) If $X \to Y \to Z$ are continuous maps, $H(f): H(X) \to H(Y), H(g): H(Y) \to H(Z)$ and $H(g \circ f): H(X) \to H(Z)$, then $H(g \circ f) = H(g) \circ H(f)$.
- 4) For any X, $H(\mathrm{id}_X)=\mathrm{id}_{H(X)}:H(X)\to H(X).$

Computations:

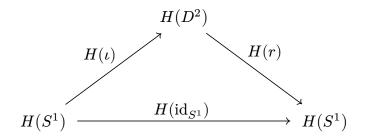
- 5) $H(D^2) \cong 0$.
- 6) $H(S^1) \cong \mathbb{R}$.

PROOF: Assume $f:D^2\to D^2$ is continuous and f(x)=x for all $x\in D^2$. Define a new function $r:D^2\to S^1$ such that r(x)= intersection of the ray from f(x) to x with $S^1\subseteq D^2$.

Key fact: If $x \in S^1$, then r(x) = x. Check that r is also continuous.



The diagram above commutes. Now we can apply homology to it.



We can check that

$$\begin{split} H(r) \circ H(\iota) &= H(r \circ \iota) \\ &= H(\mathrm{id}_{S^1}) \\ &= \mathrm{id}_{H(S^1)} \,. \end{split}$$

Therefore, the new diagram also commutes. So, if $w \in H(S^1)$, then

$$w=\mathrm{id}_{H(S^1)}(w)=H(r)(H(\iota)(w))=0.$$

This is a contradiction as $H(S^1) \neq 0$.

1.2. Categories

Definition 1.2 (Category): A category \mathcal{C} consists of:

- a collection of objects, $\mathrm{Ob}(\mathcal{C})$. For any $A \in \mathrm{Ob}(\mathcal{C})$, we usually write $A \in \mathcal{C}$.
- for any pair of objects $A,B\in\mathcal{C}$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$, or $\mathrm{Hom}(A,B)$, or $\mathcal{C}(A,B)$. Instead of $f\in\mathcal{C}(A,B)$, we write $f:A\to B$ or $A\to B$.
- for any objects $A, B, C \in \mathcal{C}$ and morphisms $f: A \to B$ and $g: B \to C$, there is a specified composition $g \circ f: A \to C$.
- for any object $A \in \mathcal{C}$, there is a given $\mathrm{id}_A : A \to A$
- compositions are associative: $(g\circ f)\circ h=g\circ (f\circ h)$
- for any $A \stackrel{f}{\rightarrow} B$, $f \circ id_A = f = id_B \circ f$

Example 1.3:

• Set, the category of sets (& functions).

Definition 1.4 (Monoid): A monoid (M, *) consists of:

- a set M
- a binary operation $*: M \times M \to M$
- an identity element $e \in M$ such that $\forall x \in M, e * x = x * e = x$.

Definition 1.5 (Monoid Homomorphism): A monoid homomorphism $f: M \to N$ is a function satisfying

- f(xy) = f(x)f(y).
- f(e) = e.

Definition 1.6 (Functor): A functor $F: \mathcal{C} \to \mathcal{D}$ is a function satisfying

- $F(A) \in \mathcal{D}$ for all $A \in \mathcal{C}$.
- $F(f): F(A) \to F(B)$ for all $f: A \to B$ in \mathcal{C} .
- $F(g \circ f) = F(g) \circ F(f)$ for all $f: A \to B$ and $g: B \to C$ in \mathcal{C} .
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{C}$.

1.3. 09/02/2025

Two "sorts" of categories:

- "concrete" categories: sets with some sort of familiar structure (groups, rings, modules, etc.)
- "abstract" categories: 1, 2, 3, etc. More formal symbols than not.

Definition 1.7 (Endomorphism): An endomorphism $f: A \to A$ is a morphism from an object to itself.

New categories from old:

- 1) Product category.
 - input: two categories $\mathcal C$ and $\mathcal D$
 - output: $\mathcal{C} \times \mathcal{D}$
 - objects: (A, B) where $A \in Ob(\mathcal{C})$ and $B \in Ob(\mathcal{D})$
 - morphisms: (f,g) where $f:A\to A'$ in $\mathcal C$ and $g:B\to B'$ in $\mathcal D$
 - composition: $(f,g) \circ (f',g') = (f \circ f', g \circ g')$
 - identity: (id_A, id_B)

Projection functors on $\mathcal{C} \times \mathcal{D}$:

- $\pi_1: \mathcal{C} \times \mathcal{D} \to C, \pi_2: \mathcal{C} \times \mathcal{D} \to \mathcal{D}.$
- on objects: $\pi_1((A, B)) = A$
- on morphisms: $\pi_1((f,g)) = f: A \to A'$.
- 2) Slice categories, coslice categories
 - input: a category \mathcal{C} and an object $X \in \mathrm{Ob}(\mathcal{C})$
 - output: \mathcal{C}/X or X/\mathcal{C}

description of coslice:

- objects: pair (A, f), where $A \in \mathrm{Ob}(\mathcal{C})$ and $f : A \to X$ in \mathcal{C}
- morphisms: from $(A, f) \to (B, g)$: morphism $k : A \to B$ of $\mathcal C$ such that $k \circ f = g$.

• composition: $(A, f) \xrightarrow{k} (B, g) \xrightarrow{l} (C, h)$ is $(A, f) \xrightarrow{l \circ k} (C, h)$. We can check that $(l \circ k) \circ f = h$. The TLDR for this is that you can copy and paste commutative diagrams and get another commutative diagram.

Example 1.8 (Coslice): Let $\mathcal{C} = \operatorname{Set}$, $X = 1 = \{*\}$. So coslice $X/\mathcal{C} = 1/\operatorname{Set} = ?$.

- objects: pairs (A, f) of a set A and a function $f: 1 \to A$.
- morphisms: functions k such that $k \circ f = g$.

Elements of sets categorically. A is a set. How do we express $a \in A$ in terms of the category Set?

elements of
$$A \longleftrightarrow$$
 functions $f: 1 \to A$
 $a \in A \longleftrightarrow f: 1 \to A, f(*) = a$
 $f(x) \in A \longleftrightarrow f: 1 \to A.$

- 3) Opposite category.
 - input: a category $\mathcal C$
 - output: $\mathcal{C}^{\mathrm{op}}$
 - objects of \mathcal{C}^{op} : A^* for $A \in \mathcal{C}$.
 - morphisms of $\mathcal{C}^{op}(A^*, B^*)$: f^* for $f: A \to B$ in \mathcal{C} .
 - composition: $(f^* \circ g^*) = (g \circ f)^*$

1.4. 09/04/2025

Examples of functors between concrete categories:

- 1) Forgetful functors. E.g. $U: \operatorname{Mon} \to \operatorname{Set}.\ U(M) = M.$ And if $f: M \to N$ is a monoid homomorphism. Then $U(f): UM \to UN$, so we just take U(f) = f. Then we just have to check that $U(g \circ f) = U(g) \circ U(f)$ but this is obvious. There are other similar examples like $\operatorname{Vect}_k \to \operatorname{Set}$ or $\operatorname{Top} \to \operatorname{Set}$. Basically it's just "forgetting" some sort of structure from the original category.
- 2) Free functors. E.g. $F : Set \to Mon$ which is the free monoid functor.

Let A be a set, $\mathrm{List}(A)=\{\mathrm{strings}\ a_1,...,a_n\mid n\geq 0, a_i\in A\}.$ So if $A=\{\mathrm{a,b,c}\},$ then we have that

$$\operatorname{List}(A) = \{<>, \operatorname{a}, \operatorname{b}, \operatorname{c}, \operatorname{aa}, \operatorname{ab}, \operatorname{ac}...\}.$$

Define concatenation as · where

$$(a_1a_2...a_n)\cdot (b_1b_2...b_m)=(a_1a_2...a_nb_1b_2...b_m).$$

We claim that List(A) is a monoid with unit <>. Call that monoid $FA \in Mon$.

On morphisms: given $f:A\to B$, get monoid homomorphism $F(f)=FA\to FB$, we define

$$F(f)(a_1a_2...a_n)=f(a_1)f(a_2)...f(a_n). \\$$

We can also check that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{id}_A) = \mathrm{id}_{FA}$.

Definition 1.9 (Contravariant Functor): A contravariant functor from $\mathcal C$ to $\mathcal D$ is a functor $F:\mathcal C^{\mathrm{op}}\to\mathcal D$.

Universal Mapping Property

Idea: universal property of X is a description of morphisms into/out of X.