

# Category Theory

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# 1. What is Category Theory?

Category theory is a language for talking about structuralist mathematics.

- materialism: an object is understood in terms of what it consists of
- structuralism: an object is understood in terms of its relationships to other objects

## 1.1. Motivating example

Let  $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ . Then let  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq D^2$ .

**Theorem 1.1** (Brouwer's fixed point theorem): If  $f : D^2 \rightarrow D^2$  is continuous, then  $f$  has a fixed point. That is, there is some  $x \in D^2$  such that  $f(x) = x$ .

The proof uses a trick and facts about homology. Effectively, there is a machine that takes a topological space (subsets of  $\mathbb{R}^2$ ) and spits out a vector space (over  $\mathbb{R}$ ).

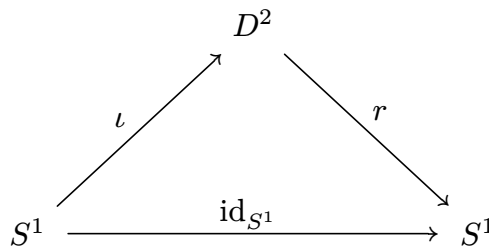
- 1) For every topological space  $X$ , there is a vector space  $H(X)$  (omitting actual definition).
- 2) For every continuous function  $f : X \rightarrow Y$ , there is an "induced" linear map given by  $H(f) : H(X) \rightarrow H(Y)$ .
- 3) If  $X \xrightarrow{f} Y \xrightarrow{g} Z$  are continuous maps,  $H(f) : H(X) \rightarrow H(Y)$ ,  $H(g) : H(Y) \rightarrow H(Z)$  and  $H(g \circ f) : H(X) \rightarrow H(Z)$ , then  $H(g \circ f) = H(g) \circ H(f)$ .
- 4) For any  $X$ ,  $H(\text{id}_X) = \text{id}_{H(X)} : H(X) \rightarrow H(X)$ .

Computations:

- 5)  $H(D^2) \cong 0$ .
- 6)  $H(S^1) \cong \mathbb{R}$ .

PROOF: Assume  $f : D^2 \rightarrow D^2$  is continuous and  $f(x) = x$  for all  $x \in D^2$ . Define a new function  $r : D^2 \rightarrow S^1$  such that  $r(x)$  = intersection of the ray from  $f(x)$  to  $x$  with  $S^1 \subseteq D^2$ .

Key fact: If  $x \in S^1$ , then  $r(x) = x$ . Check that  $r$  is also continuous.



The diagram above commutes. Now we can apply homology to it.

$$\begin{array}{ccc}
 & H(D^2) & \\
 H(\iota) \nearrow & & \searrow H(r) \\
 H(S^1) & \xrightarrow{H(\text{id}_{S^1})} & H(S^1)
 \end{array}$$

We can check that

$$\begin{aligned}
 H(r) \circ H(\iota) &= H(r \circ \iota) \\
 &= H(\text{id}_{S^1}) \\
 &= \text{id}_{H(S^1)}.
 \end{aligned} \tag{1}$$

Therefore, the new diagram also commutes. So, if  $w \in H(S^1)$ , then

$$w = \text{id}_{H(S^1)}(w) = H(r)(H(\iota)(w)) = 0. \tag{2}$$

This is a contradiction as  $H(S^1) \neq 0$ . ■

## 1.2. Categories

**Definition 1.2** (Category): A category  $\mathcal{C}$  consists of:

- a collection of objects,  $\text{Ob}(\mathcal{C})$ . For any  $A \in \text{Ob}(\mathcal{C})$ , we usually write  $A \in \mathcal{C}$ .
- for any pair of objects  $A, B \in \mathcal{C}$ , there is a collection of morphisms  $\text{Hom}_{\mathcal{C}}(A, B)$ , or  $\text{Hom}(A, B)$ , or  $\mathcal{C}(A, B)$ . Instead of  $f \in \mathcal{C}(A, B)$ , we write  $f : A \rightarrow B$  or  $A \xrightarrow{f} B$ .
- for any objects  $A, B, C \in \mathcal{C}$  and morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , there is a specified composition  $g \circ f : A \rightarrow C$ .
- for any object  $A \in \mathcal{C}$ , there is a given  $\text{id}_A : A \rightarrow A$
- compositions are associative:  $(g \circ f) \circ h = g \circ (f \circ h)$
- for any  $A \xrightarrow{f} B$ ,  $f \circ \text{id}_A = f = \text{id}_B \circ f$

**Example 1.3:**

- Set, the category of sets (& functions).