

# **21-849: Algebraic Geometry**

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I don't know what a sheave or a category is. ❤️

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# 1. Introduction

## 1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

## 1.1. Features of algebraic geometry

Consider the two functions  $e^z$  and  $z^2 - 3z + 2$ .

- Both are continuous in  $\mathbb{R}$  or  $\mathbb{C}$ .
- Both are holomorphic in  $\mathbb{C}$ .
- Both are analytic (power series expansion at every point).
- Both are  $C^\infty$ .

There are differences as well.

- $f(z) = a$  has no solution or infinitely many solutions for  $e^z$ , but for almost all  $a$ , 2 solutions for  $z^2 - 3z + 2$ .
- $e^z$  is not definable from  $\mathbb{Z} \rightarrow \mathbb{Z}$  but  $z^2 - 3z + 2$  is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$  for all  $\ell > 0$  for  $e^z$  but not for  $z^2 - 3z + 2$ .
- For nontrivial polynomials, as  $z \rightarrow \infty$ ,  $p(z)$  goes to infinity. So, it can be defined as a function from  $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ . But  $e^z$  can be periodic as the imaginary part tends to infinity.

This motivates the following result:

**Theorem 1.1** (GAGA Theorems): Compact (projective)  $\mathbb{C}$ -manifolds are algebraic.

Here are more cool things about algebraic geometry:

### 1) Enumeration:

- How many solutions to  $p(z)$ ?
- How many points in  $\{f(x, y) = g(x, y) = 0\}$ ?
- How many lines meet a given set of 4 general lines in  $\mathbb{C}^3$ ? The answer is 2.
- How many conics ( $\{f(x, y) = 0\}$ ,  $\deg f = 2$ ) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
- Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.

### 2) Birationality:

- Open sets are *huge*. That is, if we have  $X, Y$  and  $U \subseteq X, V \subseteq Y$  such that  $U \cong V$ , then  $X$  and  $Y$  are closely related.

### 3) Arithmetic Geometry:

- Over  $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$ , etc.
- Weil conjectures:  $X$  carved by polynomials with  $\mathbb{Z}$ -coefficients.  $H^2(X_{\mathbb{C}}, \mathbb{Q})$  related to integer solutions.

## 2. Affine algebraic sets

### 2.1. Nullstellensatz

Notation:  $\mathbb{k}$  is an algebraically closed field ( $\mathbb{k} = \mathbb{C}$ ).

**Definition 2.1** (Affine space): An  $n$ -affine space  $\mathbb{A}_{\mathbb{k}}^n$  is the set

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{k}, \forall i = 1, \dots, n\} = \mathbb{k}^n.$$

An affine algebraic subset of  $\mathbb{A}^n$  is a subset  $Z \subseteq \mathbb{A}^n$  such that

$$Z = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0, \forall f \in T\}$$

for some subset  $T \subseteq \mathbb{k}[x_1, \dots, x_n]$ . We write  $Z = V(T)$ .

**Example 2.2** (An affine space):

- $V(x^2 - y) \subset \mathbb{A}^2$ . This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$ . Note that  $x^2 + y^2 = (x + iy)(x - iy)$ , so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$ . We actually have  $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \mathbb{k}\}$ . Then note that if we project to any two dimensional plane  $(xy, yz, xz)$ , then we get another affine subset but on  $\mathbb{A}^2$ .

This leads us to the following question:

**Question:**  $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$ ?

SOLUTION: Consider  $V(1 - xy) \subseteq \mathbb{A}^2$ . If we project this to either axis, then we will miss the origin.

■

**Definition 2.3** (Ideal): Let  $Z \subseteq \mathbb{A}^n$  be an algebraic subset. Then

$$I(Z) = \{f \in \mathbb{k}[x] \mid f(p) = 0, \forall p \in Z\}.$$

**Example 2.4:**

- 0)  $Z = V(x^2) \subseteq \mathbb{A}^2$ , then  $I(Z) = \langle x \rangle$ .
- 1) If  $Z = V(x^2 - y)$ , then  $I(Z) = \langle x^2 - y \rangle$
- 2) If  $Z = V(x^2 - y, xy - z)$ , then  $I(Z) = \langle x^2 - y, xy - z \rangle$ .

**Proposition 2.5:**

- 1)  $I(Z)$  an ideal.  $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$ .
- 2)  $T \subseteq \mathbb{k}[x]$ .  $V(T) = V(\langle T \rangle)$  AND  $V(T) = V(f_1, \dots, f_m)$  for some  $f_i$ .
- 3) For  $\mathfrak{a} \subseteq \mathbb{k}[x]$  ideal,  $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ , where  $\sqrt{\mathfrak{a}} = \{f \in \mathbb{k}[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$ .
- 4) Algebraic subsets of  $\mathcal{A}^n$  are closed under finite unions and arbitrary intersections.

PROOF: We prove number 2 by using the Hilbert Basis Theorem. In particular,  $\mathbb{k}[x]$  is Noetherian.

■

**Theorem 2.6** (Nullstellensatz): Let  $Z$  be an algebraic subset. Then  $V(I(Z)) = Z$  and  $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{k}[x]\}.$$

PROOF:

- 1) Finite type field extensions  $L \supseteq F$  are finite. Remember that finite type means that  $F[x_1, \dots, x_m] \twoheadrightarrow L$ .
- 2) This implies that maximal ideals of  $\mathbb{k}[x]$  are of the form  $\langle x_1 - a_1, \dots, x_n - a_n \rangle$  for  $a_i \in \mathbb{k}$ , using the fact that  $\mathbb{k}$  is algebraically closed. So,  $\mathbb{k}[x]/\mathfrak{m} \simeq \mathbb{k}$ .
- 3) (Weak Nullstellensatz)  $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$ . That is,  $\mathfrak{a} \subsetneq \mathbb{k}[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$ .
- 4) So if  $f \in I(V(\mathfrak{a}))$ , then consider  $\mathfrak{a} + \langle 1 - yf \rangle \subseteq \mathbb{k}[x, y]$ . So for any  $(a_1, \dots, a_n, b)$  that vanishes on  $\mathfrak{a} + \langle 1 - yf \rangle$ , we realize that since  $1 - yf = 1$ , we have a unit ideal. That is, we can say  $1 = g_1 h_1 + g_2(1 - yf)$  for  $h_1 \in \mathfrak{a}$  and  $g_1, g_2 \in \mathbb{k}[x, y]$ . From here, we can conclude that  $f^\ell \in \mathfrak{a}$  for some  $\ell$ .

But also

$$\mathbb{k}[x, y]/\langle 1 - yf \rangle \simeq \mathbb{k}[x] \left[ \frac{1}{f} \right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \dots + \frac{g_\ell}{f^{\ell-1}}$$

for  $g_i \in$  ideal  $\mathfrak{a}$  inside  $R$ .

■

Remark: We say  $R$  is Jacobson if every radical ideal  $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$ .

**Theorem 2.7:**  $R$  Jacobson  $\Rightarrow R[x]$  Jacobson.

**Definition 2.8** (Coordinate ring): The coordinate ring  $A(X)$  of  $X \subseteq \mathbb{A}^n$  is  $\mathbb{k}[x]/I(X)$ .

1)  $X \xrightarrow{f} \mathbb{k}$

2)  $\text{maxSpec } A(X) = \{\text{maximal ideals in } A(X)\} = X$ .

### 3. Projective Spaces

**Definition 3.1:**  $\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus \{0\}) / \sim$ . That is,  $v \sim v'$  if  $v = \lambda v'$  for some  $\lambda \in \mathbb{k}$ . That is,  $\mathbb{P}^n = \{1\text{-subspaces of } \mathbb{k}^{\{n+1\}}\}$ . For  $(a_0, \dots, a_n) \in \mathbb{k}^{n+1} \setminus \{0\}$ , we write  $[a_0 : \dots : a_n] \in \mathbb{P}^n$ .

Remark:  $V \simeq \mathbb{k}^{n+1}$ .  $\mathbb{P}V = V \setminus \{0\} / \sim$

**Definition 3.2:**  $f \in \mathbb{k}[\underline{x}]$  is homogeneous if  $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\ell f(x_1, \dots, x_n)$ .

**Definition 3.3:** A projective algebraic set,  $X \subseteq \mathbb{P}^n$  is

$$V(T) = \{[x_0 : \dots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for  $T$  a set of homogeneous polynomials.

We have that  $\mathbb{P}^n \supset U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0, x_i = 1\}$ . So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

**Example 3.4:** Let  $W \subseteq \mathbb{k}^{n+1}$  of  $\dim_k W = m + 1$ . Then  $\mathbb{P}W \subseteq \mathbb{P}^n$  is a projective algebraic subset which is an  $m$ -plane in  $\mathbb{P}^n$ .

**Example 3.5 (Twisted cubic curve):** We have  $\mathbb{P}^3 \supset C = \{[s^3 : s^2t : st^2 : t^3] \mid [s : t] \in \mathbb{P}^1\}$ . Then we have that  $C = V(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$ . Then  $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$ . Additionally, we have  $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$ . Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix  $A$ ,  $\text{rank}(A) \leq r \iff \text{all } (r+1) \times (r+1) \text{ minors} = 0$ .

**Question:** Can there exist  $F, G$  such that  $V(F, G) = C$ ? (Answer is yes)

For  $X \subseteq \mathbb{P}^n$ , algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \mathbb{k}[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of  $X$ .



**Exercise 3.6:**

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow \{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \mathbb{k}[\underline{x}] \text{ such that } \mathfrak{a} \neq \mathbb{k}[\underline{x}] \text{ or } \langle x_0, \dots, x_n \rangle\}.$$

This last part is called the “irrelevant ideal”.

**Definition 3.7** (General Position): In  $\mathbb{P}^n$ , any subset of size  $\leq n + 1$  points are linearly independent.

**Theorem 3.8:** Every set  $\Gamma$  of  $2n$  points in  $\mathbb{P}^n$  in general position is carved out by quadrics.

PROOF: We want to show that if  $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$ , then  $q \in \Gamma$ . Suppose  $q$  is given. For any partition of  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ ,  $|\Gamma_i| = n$ ,  $\text{span}(\Gamma_1)$  is a hyperplane. Then for every such equi-partition,  $q \in \text{span}(\Gamma_1)$  or  $q \in \text{span}(\Gamma_2)$ .

Let  $p_1, \dots, p_k$  be a minimal subset of  $\Gamma$  whose span  $\ni q$  ( $k \leq n$ ). Now pick any  $\Lambda$  such that  $|\Lambda| = n - k + 1$  which does not contain any of the  $p_i$ . We claim that  $q \notin \text{span}(p_2, \dots, p_k, \Lambda)$ .

We then conclude that for any  $|S| = n - 1$ ,  $S \subseteq \Gamma \setminus p_1, \dots, p_k$ , we have that  $\text{span}(p_1, S) \ni q$ . Because then

$$\bigcap_S \text{span}(p, S)$$

is the intersection at least  $n$  many hyperplanes, each of them containing  $p_1, q$ . But the intersection of  $n$  many hyperplanes is a point, so  $q = p_1$ . This also concludes that in fact  $k = 1$ . ■

**Definition 3.9:** Two sets  $X, X' \subset \mathbb{P}^n$  are projectively equivalent if  $X' = g \cdot X$ ,  $\exists g \in PGL_{n+1}$ .

**Proposition 3.10:** Let  $(M_0, \dots, M_3)$  be any  $\mathbb{k}$ -basis of

$$\mathbb{k}[s, t]_3 = \{f \in \mathbb{k}[s, t] \text{ homog degree } 3\} \cup \{0\}.$$

Then  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3$  by  $\varphi : [s : t] \mapsto [M_0(s, t) : \dots : M_3(s, t)]$ . Also,  $\varphi(\mathbb{P}^1)$  is projectively equivalent to  $C = \{[s^3 : s^2t : st^2 : t^3]\}$ .

**Example 3.11** (Rational normal curve): Let  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  via  $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$ . Or we could map it to any basis of  $\mathbb{k}[s : t]_n$ .

**Exercise 3.12:**  $I(\varphi(\mathbb{P}^1)) = ?$ .

**Example 3.13:**  $[s^3 : s^2t : t^3]$  is the same as  $V(y^3 - x^2z)$ . Also take  $[st^2 - s^3 : t^3 - s^2t : s^3]$ . This is carved out by  $V(y^2z - x^3 - x^2z)$ .

**Fact:** If we pick any 3 linearly independent  $M_0, M_1, M_2 \in \mathbb{k}[s, t]_3$ . Then  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  by  $M_0, M_1, M_2$  has image projectively equivalent to one of the two curves above.

Now consider  $\mathbb{P}^1 \rightarrow \mathbb{P}^3$  using 4 elements from  $\mathbb{k}[s, t]_4$ . We consider  $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$ . This is called the twisted quartic curve.

**Question:** Are all twisted quartic curves projectively equivalent?

SOLUTION: No. In fact, there are infinitely many distinct families. ■

**Question** (Hartshorne's Question): Is every irreducible curve in  $\mathbb{P}^3$  carved out by 2 equations?

## 4. The Zariski Topology

**Definition 4.1** (Zariski topology): The sets  $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \mathbb{k}[x]\}$  form the closed sets of a topology on  $\mathbb{A}^n$  called the Zariski topology.

Given  $X \subseteq \mathbb{A}^n$ , give it the subspace topology.

**Example 4.2:** Take  $\mathbb{A}^1$ . Two closed subsets are  $\mathbb{A}^1$  and  $\emptyset$ . The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

**Definition 4.3:** A topological space  $X$  is irreducible  $X = Y_1 \cup Y_2$  (each closed) implies that  $X = Y_1$  or  $X = Y_2$ .

By definition, we will also say that irreducible implies nonempty.

**Remark:**

- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

**Proposition 4.4:** Let  $X \subseteq \mathbb{A}^n$  be a nonempty algebraic subset.  $X$  is irreducible if and only if  $I(X)$  is prime if and only if  $A(X)$  is a domain.

PROOF:

- $\Rightarrow$ : Suppose  $fg \in I(X)$ . This means  $V(f) \cup V(g) \supseteq X$ . If  $X$  is irreducible, then at least one of them completely contains  $X$ . That is,  $V(f) \supseteq X$  or  $V(g) \supseteq X$ . But this exactly means  $f$  or  $g \in I(X)$ .
- $\Leftarrow$ : Suppose for sake of contradiction that  $X$  is not irreducible. We have  $X = Y_1 \cup Y_2$  (both proper), then  $I(Y_2) \supsetneq I(X)$ . Take  $f_i \in I(Y_i) \setminus I(X)$ . Now analyze  $f_1 f_2$ .  $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$ . Therefore,  $f_1 f_2 \in I(X)$ . But this is a contradiction, so we are done.

■

**Remark:** When people say affine variety, some people mean that it is also irreducible. But for us, affine variety is the same thing as affine algebraic set.

Then a quasi-affine variety is an open subset of an affine variety.

**Example 4.5:**

- 1)  $\mathbb{A}^n$  is irreducible. ( $\mathbb{k}[\underline{x}]$  domain)
- 2)  $V(x^2 + y^2) \subset \mathbb{A}^2$  is reducible ( $\text{char } \mathbb{k} \neq 2$ )
- 3) Let  $f \in \mathbb{k}[\underline{x}]$  be square-free ( $f = f_1 \dots f_\ell$  irreducible). Then  $V(f)$  is irreducible if and only if  $f$  is irreducible.
- 4)  $X = V(x^2 - yz) \subseteq \mathbb{A}^3$ . Then  $A(X) = \frac{\mathbb{k}[x, y, z]}{\langle x^2 - yz \rangle}$ . This is irreducible due to Eisenstein's on  $f$ . Now if we take  $f \in A(X)$  and look at  $V_X(f) \subset X$  is irreducible  $\Leftrightarrow f$  irreducible element in  $A(X)$ .

**Definition 4.6:** A topological space  $X$  is Noetherian if  $\nexists X \supseteq Y_0 \supsetneq Y_1 \supsetneq \dots$  such that each  $Y_i$  is closed.

**Proposition 4.7:** An affine variety is Noetherian. (Because  $A(X)$  is Noetherian).

**Theorem 4.8:** A Noetherian topological space  $X$  is uniquely a finite union of maximal irreducible closed subsets.

PROOF: Consider

{nonempty closed subsets of  $X$  that does not admit a decomposition into irreducible closed subsets.}.

Suppose it is nonempty. Then it has a minimal element  $Y$ .  $Y$  is not irreducible, so  $Y = Y_1 \cup Y_2$  (both proper and closed). Since  $Y$  is minimal,  $Y_1$  and  $Y_2$  both have decompositions into irreducible closed subsets. So if we just union those decompositions, then we contradict  $Y$ 's membership in the set. As such, the original set must have actually been empty.

Uniqueness and maximality are left as an exercise. ■

**Proposition 4.9:**

- 1)  $X$  irreducible and  $U \subseteq X$  open. Then  $\overline{U} = X$ .
- 2)  $V \subseteq X$ ,  $V$  irreducible  $\implies \overline{V}$  irreducible.
- 3)  $f : X \rightarrow Y$  continuous. Image of irreducible set under  $f$  is irreducible. (Irreducibility is a topological property).

**Example 4.10:** Let's have  $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^m$  by  $\varphi(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x}))$  for some  $f_1, \dots, f_m \in \mathbb{k}[\underline{x}]$ . Then  $\text{im}(\varphi)$  is irreducible. It is left to show that  $\varphi$  is a continuous map.

**Definition 4.11:** Let  $X$  be a nonempty topological space.

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

Then let  $Y \subseteq X$  closed irreducible subset.

$$\operatorname{codim}_X Y := \sup\{n \mid \exists Y \subseteq Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

**Example 4.12:**

- 1)  $\dim \mathbb{A}^1 = 1$ .
- 2)  $X = V(xz, yz) \subseteq \mathbb{A}^3$ . Then  $\dim X = 2$ . Let  $p$  be a point on the axis not touching the  $x$ - $y$  plane. Then let  $q$  be the origin. We have that  $\operatorname{codim}_X p = 1$  and  $\operatorname{codim}_X q = 2$ . Also  $\dim p = \dim q = 0$ .

**Definition 4.13:** Height of a prime  $\mathfrak{p} \subset R$  is

$$\operatorname{ht} \mathfrak{p} := \sup\{n \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n, \text{ each } \mathfrak{p}_i \text{ prime}\}.$$

Then Krull dimension of  $R$  is

$$\dim R := \sup\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ prime}\}.$$

**Definition 4.14:** For an ideal  $I$ , we have that

$$\operatorname{ht} I := \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}.$$

“inf of sup”.

From these, we can basically show from definition that

$$\operatorname{ht} I + \dim R/I \leq \dim R.$$

The  $<$  case is possible if  $R$  is not a domain. For example, if we have that  $R = \mathbb{k}[x, y, z]/\langle xz, yz \rangle$  and then  $I = \langle x, y, z - 1 \rangle$ .

But the  $<$  case is also possible even if  $R$  is a domain and  $I$  prime.

Before we cover the next theorem, we note that

$$\{\text{minimal primes over } I\} = \{\mathfrak{p} \text{ prime } \mathfrak{p} \supseteq I, \text{ and } \nexists \mathfrak{p} \supsetneq \mathfrak{q} \supseteq I, \text{ prime } \mathfrak{q}\}$$

**Theorem 4.15** (Krull Principal Ideal Theorem / Height Theorem): Let  $R$  be a Noetherian ring and  $f_1, \dots, f_c \in R$ .

- 1) Minimal primes over  $\langle f_1 \rangle$  have height  $\leq 1$ . And the height is equal to 1 if  $f_1$  is nonzerodivisor and nonunit.
- 2) Minimal primes over  $\langle f_1, \dots, f_c \rangle$  have height  $\leq c$ .

“We could do this proof, but it’s like proving that there exists a complete ordered field satisfying the least upper bound property.”

**Theorem 4.16:** Let  $X \subseteq \mathbb{A}^n$  and  $Y \subseteq \mathbb{A}^m$  irreducible affine varieties.

- 1)  $\dim(X \times Y) = \dim X + \dim Y$ .
- 2) If  $Y \subseteq X$ , then  $\dim Y + \operatorname{codim}_X Y = \dim X$ .

**Remark** (Noether normalization): For  $X \subseteq \mathbb{A}^n$  irreducible affine variety. There exists  $y_1, \dots, y_d \in A(X)$  such that  $\mathbb{k}[Y_1, \dots, Y_d] \rightarrow A(X)$  with  $Y_i \mapsto y_i$  which is a finite extension (injective and  $A(X)$  is finitely generated  $\mathbb{k}[Y]$ -module) and  $d = \dim X$ .

**Corollary 4.17:**

- 1)  $\dim \mathbb{A}^n = n$ .
- 2)  $X \subseteq \mathbb{A}^n$  irreducible affine variety.  $0 \neq f \in A(X)$  non unit. Then  $V_X(f) = V(f) \cap X$  has dimension  $\dim X - 1$ .

**Exercise 4.18:** Let  $U \subseteq X$  be open for  $X$  affine variety irreducible. Then  $\dim U = \dim X$ .

**Proposition 4.19:** Let  $R$  be Noetherian domain. Then  $R$  UFD  $\iff$  every ht = 1 prime is principal.

PROOF:  $R$  being a UFD implies that  $\mathfrak{p}$  has height 1. So let  $f = f_1, \dots, f_\ell \in \mathfrak{p}$ . Suppose  $f_1 \in \mathfrak{p}$ . So then  $0 \neq \langle f_1 \rangle \subseteq \mathfrak{p}$ . But as  $\operatorname{ht} \mathfrak{p} = 1$ , we have that  $\langle f_1 \rangle = \mathfrak{p}$ .

Conversely, we need to show that irreducible implies prime. That is, recall that (ACCP + irreducible = prime) implies that we have a UFD.

So let  $f \in \text{irred}$ . Krull’s PIT says  $\langle f \rangle \subseteq \mathfrak{p}$  where  $\mathfrak{p}$  has height 1. So by definition,  $\mathfrak{p} = \langle g \rangle$ , but  $\langle f \rangle \subseteq \langle g \rangle$  implies that  $f = g$  because  $f$  is irreducible. ■

**Example 4.20:** Let  $X = V(x^2 - yz) \subseteq \mathbb{A}^3$ . Then let  $Y = V(x, y) \subseteq X \subseteq \mathbb{A}^3$ . Then  $\dim X = 2$ . Then  $\dim Y = 1$ . So can we find  $f$  such that  $\langle f, x^2 - yz \rangle = I(Y)$ ? The answer to this is no.

But can we find  $f$  such that  $\sqrt{\langle f, x^2 - yz \rangle} = \langle x, y \rangle$ ? Take  $f = y$  and analyze  $\langle y, x^2 - yz \rangle$ . This is the same as  $\langle y, x^2 \rangle$ , whose radical is  $\langle x, y \rangle$  as we desire.

**Example 4.21:** Now consider  $X = V(xw - yz) \subseteq \mathbb{A}^4$ .  $\dim X = 3$  and let  $Y = V(x, y)$ . Now does there exist  $f$  such that  $\sqrt{\langle f, xw - yz \rangle} = \langle x, y \rangle$ ?

This is false, but we don't have the tools to prove it.

**Definition 4.22:** Zariski topology on  $\mathbb{P}^n$  has projective algebraic sets as its closed subsets.

Two ways: projective varieties  $\rightarrow$  affine varieties.

1)  $U_i = \{x_i \neq 0\} = \{[x_0 : \dots : x_i = 1 : \dots : x_n]\} \simeq \mathbb{A}^n$ .

**Proposition 4.23:**  $\forall i = 0, \dots, n$ , say  $i = 0$ ,  $\mathbb{A}^n \rightarrow U_0, (x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$  is a homeomorphism.

PROOF:

• *Homogenization:* let  $f \in \mathbb{k}[x_1, \dots, x_n]$ . Then we have

$$f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{k}[x_1, \dots, x_n].$$

If  $Z = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n$ ,  $\varphi(Z) = U_0 \cap V(f_1^h, \dots, f_m^h)$  is closed.

If  $Z' = V(F_1, \dots, F_\ell) \cap U_0$ , then  $\varphi(Z') = V(F_1(1, x_2, \dots, x_n), \dots, F_\ell(1, x_2, \dots, x_n))$ .

Now  $U_0 \cup \dots \cup U_n = \mathbb{P}^n$ .

■

**Exercise 4.24:** Let  $Y \subseteq \mathbb{A}^n \simeq U_0$  be an affine variety.  $\overline{Y} = V(?)$ . Suppose  $V(f_1, \dots, f_m) = Y$ . It is tempting to say  $\overline{Y} = V(f_1^h, \dots, f_m^h)$ .

**Corollary 4.25:**

- 1)  $\dim \mathbb{P}^n = n$ .
- 2) If  $H_i = V(x_i) \subseteq \mathbb{P}^n$  does not contain any irreducible components of  $Y \subseteq \mathbb{P}^n$ , then  $\dim Y = \dim Y \cap U_i$ .

**Definition 4.26:** Let  $Y \subseteq \mathbb{P}^n$  be a projective variety. The affine cone  $\hat{Y} = C(Y)$  is

$$\theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}$$

where

$$\theta : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n.$$

**Proposition 4.27:**

- 1)  $\hat{Y} = V(I(Y))$ . In fact,  $I(\hat{Y}) = I(Y)$ .
- 2)  $\dim \hat{Y} = \dim Y + 1$ .
- 3)  $\hat{Y}$  is irreducible if and only if  $Y$  is irreducible.

**Theorem 4.28:** If  $X, Y \subseteq \mathbb{P}^n$  are projective varieties and  $\dim X + \dim Y \geq n$ , then  $X \cap Y \neq \emptyset$ .

**Lemma 4.29:** If  $X, Y \subseteq \mathbb{A}^n$  affine varieties, then  $X \cap Y = \emptyset$  or every irreducible component of  $X \cap Y$  has  $\dim \geq \dim X + \dim Y - n$ .

PROOF: Let  $\Delta = V(x_1 - y_1, \dots, x_n - y_n) \subseteq \mathbb{A}^{n+n}$ . Note that

$$X \times Y \cap \Delta \simeq X \cap Y.$$

So,  $\dim(X \times Y \cap \Delta) \geq \dim X + \dim Y - n$  by Krull's height theorem.

If  $\underline{a} = (a_1, \dots, a_n)$  are varieties, then  $I_{\underline{a}(X)} = \{f(\underline{a} \mid f \in I(X)\}$ . Then,

$$A(X \cap Y) = \frac{\mathbb{k}[\underline{z}]}{\sqrt{\langle I_{z(X)} + I_{z(Y)} \rangle}}$$

and

$$A(X \times Y \cap \Delta) = \frac{\mathbb{k}[\underline{x}, \underline{y}]}{\sqrt{\langle I_{x(X)} + I_{y(Y)} + I(A) \rangle}}.$$

So this implies that  $x_i = y_i$  for all  $i$ , meaning they are isomorphic rings. ■

PROOF of Theorem 4.28:  $X, Y$  irreducible implies that  $\hat{X}$  and  $\hat{Y}$  are irreducible. So, then

$$\dim(\hat{X} \cap \hat{Y}) \geq \dim X + 1 + \dim Y + 1 - (n + 1) \geq \dim X + \dim Y - n + 1.$$

$\hat{X} \cap \hat{Y}$  contains origin by construction, but it has at least one other point because dimension. ■



## 5. Morphisms

**Definition 5.1:** For  $U \subseteq \mathbb{R}^n$ ,  $U' \subseteq \mathbb{R}^m$  open,  $\varphi : U \rightarrow U'$  is continuous/continuously differentiable/smooth if  $f \circ \varphi$  is smooth for any smooth  $f : U' \rightarrow \mathbb{R}$ .

$f' : U' \rightarrow \mathbb{R}$  is smooth if  $f$  is smooth at every point  $p \in U'$ .

**Definition 5.2:** For affine variety  $X \subseteq \mathbb{A}^n$  and  $U \subseteq X$  open, a function  $\varphi : U \rightarrow \mathbb{k} = \mathbb{A}^1$  is regular if  $\forall p \in U$ ,  $\exists U_p \ni p$  open and  $f_p, g_p \in A(X)$  such that  $\varphi(x) = \frac{f_p(x)}{g_p(x)}$  for all  $x \in U_p$ . In particular,  $g_p(x) \neq 0$  for all  $x \in U_p$ .

$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$ . This is also a  $\mathbb{k}$ -algebra.

**Example 5.3:** Let  $U \subseteq X$ ,  $\varphi : U \rightarrow \mathbb{A}^1$  regular  $\nRightarrow \varphi = \frac{f}{g}$  globally for some  $f, g \in A(X)$ .

$X = V(xw - yz) \subset \mathbb{A}^4$ ,  $U = X \setminus V(x, y)$ .

$$\varphi(x, y, z, w) = \begin{cases} \frac{z}{x} & \text{if } x \neq 0 \\ \frac{w}{y} & \text{if } y \neq 0 \end{cases}$$

**Lemma 5.4:**  $\varphi : U \rightarrow \mathbb{A}^1$  regular, then  $V(\varphi) = \{x \in U \mid \varphi(x) = 0\}$  is closed in  $U$ . In particular  $\varphi$  is continuous.

PROOF: Closedness is a local condition, and around any  $p \in U$ ,  $\{\varphi \upharpoonright U_p : 0\} = V(f_p) \cap U_p$ . ■

**Remark:** If  $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$  for  $U$  irreducible, and  $\varphi_1 \upharpoonright U' = \varphi_2 \upharpoonright U'$  for some  $\emptyset \neq U' \subseteq U$ , then  $\varphi_1 = \varphi_2$ .

**Definition 5.5:**  $X \subseteq \mathbb{A}^n$  affine variety. A distinguished open subset  $U$  of  $X$  is an open subset of the form  $X \setminus V(f)$  for some  $f \in A(X)$ , denoted  $D(f)$ ,  $D_f$ ,  $U_f$ ,  $X_f$ .  $X_f$  is probably the most descriptive as it actually mentions  $X$ .

**Remark:**  $\{D(f)\}_{f \in A(X)}$  form a basis for Zariski topology. What that means is that any  $U \subseteq X$  is a union for  $D(f)$ 's.

**Exercise 5.6:**  $D(f)$  is homeomorphic to  $V(I(X) + \langle 1 - yf \rangle) \subseteq \mathbb{A}^{n+1}$ .

**Theorem 5.7:**  $\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} \mid g \in A(X), m \in \mathbb{Z}_{\geq 0} \right\}$ . In fact,  $\mathcal{O}_X(D(f)) = A(X)_f$ .

**Example 5.8:**  $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = A(\mathbb{A}^2) = \mathbb{k}[x, y]$ . Then,

$$\mathbb{A}^2 \ni \varphi = \begin{cases} \frac{f}{x^m} \text{ for some } f \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(x) \\ \frac{g}{y^\ell} \text{ for some } g \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(y) \end{cases}.$$

Then we say  $y^\ell f = x^m g$  on  $\mathbb{A}^2$ . Because we are in a UFD, this means that  $x^m \mid f$  and  $y^\ell \mid g$ . But this implies  $m = \ell = 0$ , so  $f = g = \varphi$ .

PROOF of Theorem 5.7:  $\supseteq$  is clear. So we only prove the  $\subseteq$  case.

Suppose we have  $\varphi \in \mathcal{O}_X(D(f))$ . Then for all  $p \in D(f)$ ,  $\exists U_p \ni p$  and  $\varphi \upharpoonright U_p = \frac{g_{p'}}{f_{p'}}$  for  $g_{p'}, f_{p'} \in A(X)$ .

Take a nonempty  $D(h_p) \subseteq U_p$  and write  $g_p = g_{p'} h_p$  and  $f_p = f_{p'} h_p$ .

Then  $\varphi \upharpoonright D(f_p) = \frac{g_p}{f_p} = \frac{g_{p'} f_{p'}}{f_{p'}^2}$ . So assume  $g_p = 0$  on  $V(f_p)$ .

Now we claim that  $\forall p, q \in D(f)$ , we have  $g_p f_q = g_q f_p$  in  $A(X)$ .

Then  $D(f) = \bigcup_p D(f_p)$ . Then  $V(f) = \bigcap_p V(f_p)$ . Nullstellensatz says that  $\sqrt{\langle f \rangle} = \sqrt{\langle f_p : p \in D(f) \rangle}$  as ideals in  $A(X)$ . But then,  $f^m = \sum k_p f_p$ . By Noetherian-ness, this is a finite sum. We claim that  $g = \sum k_p f_p$ .

Then  $\frac{g}{f^m} = \frac{g_q}{f_q^m}$  on  $D(f_q)$  for all  $q \in D(f)$ . So,

$$g f_q = \sum_p k_p g_p f_q = \sum_p k_p f_p g_q = g_q f^m.$$

■

## 6. Sheaves

Let  $\mathcal{A}$  be a category: AbGrp, Rings,  $\mathbb{k}$ -algebras. Given a topological space  $X$ ,  $\text{Top}(X)$  is a category where the objects are open subsets  $U \subseteq X$  and morphisms are inclusions between  $U \subseteq V$  open subsets.

**Definition 6.1:** A presheaf (with values in  $\mathcal{A}$ ) on  $X$  is a contravariant functor  $\mathcal{F} : \text{Top}(X) \rightarrow \mathcal{A}$ .

$\mathcal{F}$  is further a sheaf if for every open cover  $\{U_i\}_i$  of any open subset  $U \subseteq X$  if

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

**Translation:**

- 1) Assignment  $U \mapsto \mathcal{F}(U) \in \text{obj}(\mathcal{A})$  such that  $\forall U \subseteq V \subseteq X$  open,

$$\text{res}_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$$

such that  $\text{res}_{U,U} = \text{id}$  and  $\text{res}(V, U) \circ \text{res}(W, V) = \text{res}(W, U)$ .

- 2) If  $(f_i)_i \in \prod_i \mathcal{F}(U_i)$  such that  $\text{res}_{U_i, U_i \cap U_j}(f_i) = \text{res}_{U_j, U_i \cap U_j}(f_j)$ , then  $\exists! f \in \mathcal{F}(U)$  such that  $\text{res}_{U, U_i}(f) = f_i$ . Also  $\mathcal{F}(\emptyset) = 0$  as a consequence.

$f \upharpoonright V := \text{res}_{U,V}(f)$ ,  $f \in \mathcal{F}(U)$ .  $\mathcal{F}(U)$  elements are called sections of  $\mathcal{F}$  over  $U$ .

**Example 6.2:** Note that throughout these examples,  $X$  is a topological space and  $U \subseteq X$ .

- 1)  $\mathcal{F}_{\text{ct}}(U) := \{\varphi : U \rightarrow \mathbb{R}\}$ . Then if  $U' \subseteq U$ ,  $\text{res}_{U,U'}(f) := f \upharpoonright U'$ .
- 2)  $C(U) := \{\varphi : U \rightarrow \mathbb{R} \text{ cts}\}$ .
- 3)  $C^\infty(U) := \{\varphi : U \rightarrow \mathbb{R} \text{ smooth}\}$  ( $X \subseteq \mathbb{R}^n$  open).
- 4)  $\underline{\mathbb{R}}(U) := \{\varphi : U \rightarrow \mathbb{R}, \text{ constant}\}$ . This is not a sheaf. If we consider a constant function that takes the value  $a$  on  $U$  and  $b$  on  $U'$ , then there is no value  $c$  such that they can be glued together to be equal on both sets.
- 5)  $\mathcal{O}_{X(U)} := \{\varphi : U \rightarrow \mathbb{k} \text{ regular}\}$

**Remark:** A constant sheaf  $A_{X(U \text{ conn})} = A$ . A locally constant sheaf (locally  $\mathcal{F} \upharpoonright U$  is constant). Locally constant does not imply constant.

**Definition 6.3:**  $U \subseteq X$ ,  $\mathcal{F}$  sheaf on  $X$ ,  $\mathcal{F} \upharpoonright U(V) = \mathcal{F}(V)$  for  $V \subseteq U$  open.

**Definition 6.4:**  $\mathcal{F}$  sheaf on  $X$ .  $p \in X$ . The stalk of  $\mathcal{F}$  at  $p$ ,  $\mathcal{F}_p := \lim_{U \ni p} \mathcal{F}(U)$ . This is actually just equal to  $\{(U, f) \mid U \ni p, f \in \mathcal{F}(U) / \sim\}$  where  $(U_1, f_1) \sim (U_2, f_2)$  if  $\exists V \ni p$  such that  $f_1 \upharpoonright V = f_2 \upharpoonright V$ .

Remember that  $\mathcal{F}ct_S(U) = \{f : U \rightarrow S\}$  and  $C(U) = \{f : U \rightarrow R \text{ cts}\}$ .

**Remark:**  $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$  by  $'\text{res}_{V,U}$ . This map need not be surjective.

**Theorem 6.5:** Let  $X$  be an affine variety and  $x \in X$ .  $\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_X}$ .

PROOF: Consider the ring map  $A(X)_{\mathfrak{m}_x} \rightarrow \mathcal{O}_{X,x}$  where we map  $\frac{f}{g} \mapsto \frac{f}{g}$ .

Now if  $\frac{f}{g} = \frac{f'}{g'}$  in  $A(X)_{\mathfrak{m}_x}$ , then we need to check that the same is true around  $x$  in  $\mathcal{O}_{X,x}$ .

Now if  $\frac{f}{g}$  is 0 around  $x$  (in  $D(h)$ ), we can deduce that  $\frac{f}{g} = 0$  in  $A(X)_{\mathfrak{m}_x}$ . ■

**Definition 6.6:** A ringed space  $(X, \mathcal{O}_X)$  where  $X$  is a topological space and  $\mathcal{O}_X$  is a sheaf on  $X$  with values in Ring. We call  $\mathcal{O}_X$  the structure sheaf of this ringed space.

**Definition 6.7:**  $f : X \rightarrow Y$  continuous and  $\mathcal{F}$  a sheaf on  $X$ .

$$\text{pushforward } f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

where  $V \supset Y$  is open.

**Definition 6.8:** Let  $\mathcal{F}$  and  $\mathcal{G}$  be sheaves on  $X$ .  $\Phi : \mathcal{F} \rightarrow \mathcal{G}$  means that for each  $U \subseteq X$ , we specify  $\Phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  where for  $U \subseteq V$ , we have the following diagram commuting:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi(U)} & \mathcal{G}(U) \\ \uparrow \text{res} & & \uparrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{\Phi(V)} & \mathcal{G}(V) \end{array}$$

where  $V \supset U$  is open.

**Definition 6.9:** A morphism of ringed spaces  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a pair  $(f, f^\#)$  where  $f : X \rightarrow Y$  is continuous and  $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ .

**Example 6.10:**  $(U \subseteq \mathbb{R}^n, C^1) \rightarrow (V \subseteq \mathbb{R}^m, C^1)$ .

**Remark:** When we say  $(X, \mathcal{O}_X)$ , we mean that  $\mathcal{O}_X$  is a subsheaf of  $\mathcal{F}ct_{\mathbb{k}}$  for some fixed  $\mathbb{k}$ . Then  $\mathcal{O}_X = \{\varphi : U \rightarrow \mathbb{k} \mid \varphi \text{ satisfies some condition}\}$ . And,  $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$  is always given by precomposition.

## 6.1. Category of quasi Affine varieties “qAffVar”.

To define this category, we'll have the objects be open subsets of an affine variety considered as a ringed space. The morphisms will be maps of ringed spaces with  $f^\#$  being the precomposition as our convention above dictates.

**Theorem 6.11:**  $X, Y$  both affine varieties.  $U \subseteq X$  open. Then there exists a natural bijection.

$$\text{Mor}(U, Y) \simeq \text{Hom}_{\mathbb{k}\text{-alg}}(A(Y), \mathcal{O}(U)).$$

## 7. Projective morphisms

7.0.1. Ok unfortunately i was forced to miss two classes so there is gap here

**Proposition 7.1:** Suppose  $X, Y$  are prevarieties with affine covers  $\{U_i\}$  and  $\{V_j\}$  respectively. Then  $X \times Y$  is a product in the category of prevarieties constructed by gluing together  $U_i \times V_j$  and  $U_{i'} \times V_{j'}$ ,

$$(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

for all such pairs.

We did not prove that gluing gives you a prevariety, but we will believe it. Also note that  $X \times Y$  is a prevariety by affine cover  $\{U_i \times V_j\}$ .

**Proposition 7.2:** For  $Y \subseteq X$  closed where  $\iota : Y \rightarrow X$  and  $X$  is a prevariety. Then for  $U' \subseteq Y$  open,

$$\iota^* \mathcal{O}_X(U') = \{f : U' \rightarrow \mathbb{k} \mid \forall y \in U', \exists U_y \subset X \text{ with } \varphi \in \mathcal{O}_X(U_y) \text{ such that } f|_{U_y \cap Y} = \varphi\}.$$

Then  $\iota^* \mathcal{O}_X$  is a sheaf and  $(Y, \iota^* \mathcal{O}_X)$  is a prevariety, and  $\iota : Y \rightarrow X$  is a morphism.

PROOF: We will believe that  $\iota^* \mathcal{O}_X$  is a sheaf. Also  $\iota$  is a morphism of prevarieties.

Let  $U \subseteq X$  be affine open. We claim that  $(U \cap Y, \iota^* \mathcal{O}_X|_{U \cap Y})$  is affine. We claim that  $Y = V(I)$  for some  $I \subseteq A(X)$ .

Then,  $\iota^* \mathcal{O}_X$  are the functions that are locally restrictions of regular functions on  $X$ . Then  $\mathcal{O}_{V(I)}$  are functions that are locally quotient of polynomials on  $\mathbb{A}^n$ . These are equal. ■

We say that  $\iota$  is a closed embedding.

**Example 7.3:**

- 1)  $\mathbb{A}^2 \rightarrow \mathbb{A}^2$  via  $(x, y) \mapsto (x, xy)$ . This maps  $\mathbb{A}^2$  to itself without the  $y$ -axis, but still including the origin. Note that the image of this map is neither open nor closed in  $\mathbb{A}^2$ .

**Remark:**  $Y \subseteq X$  is locally closed if it is  $U \cap V$  where  $U$  is open in  $X$  and  $V$  is closed in  $X$ .

This example is not even locally closed.

- 2) Glue  $\mathbb{A}^1$  and  $\mathbb{A}^1$  along  $\mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus \{0\}$  via the identity. This is basically a line with two origins. Let this line be called  $X$ . Consider  $g : X \rightarrow X$  via switching the origins and keeping the other points the same. This is a morphism where our open subsets are lines including only one

of the origins, and it is not hard to check that this is actually a morphism. Then,  $\{g(x) = x\} \simeq \mathbb{A}^1 \setminus \{0\}$  which is not closed in  $X$ .

**Definition 7.4:** A prevariety of  $X$  is a variety of the diagonal map  $\Delta : X \rightarrow X \times X$  defined by  $x \mapsto (x, x)$  is a closed embedding.

**Lemma 7.5:** For checking if something is a variety,  $\Delta$  being a topologically closed embedding is a sufficient condition.

**Corollary 7.6:** Open and closed subprevarieties of varieties are varieties.

**Lemma 7.7:** Let  $X$  and  $Y$  be affine varieties and  $f : X \rightarrow Y$ . If  $f^\# : A(Y) \twoheadrightarrow A(X)$ , then  $f$  is a closed embedding.

PROOF: Let  $J$  be the kernel. And consider the surjective map

$$\frac{k[\underline{y}]}{I(Y)} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}.$$

This is also surjective onto  $\frac{k[\underline{y}]}{I(Y)+J}$ . This is  $V_Y(J)$ , so we get that  $A(V_Y(J)) \simeq A(X)$  as desired. ■

**Lemma 7.8:** Let  $X$  be an affine variety, then  $\Delta : X \rightarrow X \times X$  defined by  $x \mapsto (x, x)$  is a closed embedding.

PROOF:  $A(X) = \frac{k[\underline{x}]}{I(X)}$ ,  $A(X \times X) \simeq A(X) \otimes A(X)$ . Then we can map

$$\frac{k[\underline{x}, \underline{y}]}{I_{x(X)} + I_{y(Y)}} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}$$

via  $x_i \mapsto x_i$  and  $y_i \mapsto x_j$ . ■

**Proposition 7.9:**  $X$  prevariety is a variety if  $\Delta(X) \subseteq X \times X$  is closed.

PROOF: We claim that  $\forall x \in X$ , take any  $x \in U \subset X$  affine open. Then  $\Delta \upharpoonright U : U \rightarrow U \times U$  is a closed embedding.

Since  $\Delta(X) \subseteq X \times X$  is closed (topologically closed embedding, locally closed embedding implies closed embedding). So  $\mathcal{O}_X \simeq \mathcal{O}_{\Delta(X)}$ .

It is important that  $\Delta \upharpoonright U$  is closed because of the following:

$Y \subseteq X$  such that  $\forall y \in Y, \exists U_y \subset X$  containing  $y$  such that  $U_y \cap Y$  is closed in  $U_y$  does not imply that  $Y \subseteq X$  is closed. However, under our assumption, it would be that way. ■

**Corollary 7.10:**  $\text{qAffVar}$  are varieties.

**Corollary 7.11:**  $X \xrightarrow{f} Y$  morphism of varieties. The graph  $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$  is closed in  $X \times Y$ .

PROOF:  $X \times Y \xrightarrow{f \times \text{id}} Y \times Y$ . Then  $(f \circ \text{id})^{-1}(\Delta(Y)) = \Gamma_f$ . ■

**Exercise 7.12:** Let  $X$  be a variety and  $U, U' \subseteq X$  open affine subsets. Then  $U \cap U'$  is also affine open.

**Definition 7.13:** Let  $X \subseteq \mathbb{P}^n$  be a projective algebraic subset.  $\mathcal{O}_X(U) = \{\varphi : U \rightarrow \mathbb{k} \mid \text{locally encrypted } (\varphi \upharpoonright U') = \frac{F}{G} \text{ for some homogeneous poly } F, G \text{ of same degree.}\}$ .

**Proposition 7.14:**  $X \subseteq \mathbb{P}^n$  projected algebraic set  $\leadsto (X, \mathcal{O}_X)$  if for all  $i = 0, 1, \dots, n$ , let  $U_i = \mathbb{P}^n \setminus V(x_i) \xrightarrow{\text{homeo}} \mathbb{A}^n$ . Then  $(X \cap U_i, \mathcal{O}_X \upharpoonright U_i)$  is isomorphic as a ringed space to  $X \cap U_i$  considered as a closed subset in  $\mathbb{A}^n$  under  $\mathbb{A}^n \simeq U_i$  and hence an affine variety. In particular,  $X$  is a prevariety.

PROOF:  $X = V(F_1, \dots, F_m)$ . Then  $X \cap U_i \subset \mathbb{A}^n$  is  $V(F_1(x_0, \dots, x_i = 1, \dots, x_n), \dots, F_m)$ .

If we have  $\frac{F}{G} \in (X \cap U_i, \mathcal{O}_X \upharpoonright U_i)$ , then we can just dehomogenize to get  $\frac{f}{g} \in (X \cap U_i, \mathcal{O}_{X \cap U_i \subseteq \mathbb{A}^n})$ . ■

**Lemma 7.15:**  $X \subset \mathbb{P}^n$  a projective variety. Let  $F_0, \dots, F_m$  be homogeneous polynomials on  $\mathbb{P}^n$  of same degree. Then  $F : X \setminus V(F_0, \dots, F_m) \rightarrow \mathbb{P}^m$  by  $x \mapsto (F_0(x), \dots, F_m(x))$  is a morphism.

PROOF: As a set map, this is well-defined. We will verify that  $\forall j = 0, \dots, m$ , the distinguished open subset  $U_j = \{[\underline{Y}] \in \mathbb{P}^m \mid Y_j \neq 0\}$ . We have

$$F^{-1}(U_j) = \mathbb{P}^n \setminus V(F_j).$$



Then  $F^{-1}(U_j) \rightarrow U_j$  where  $U_j \simeq \mathbb{A}^m$ . We'll call the coordinates of  $\mathbb{A}^m$  by  $\left(\frac{Y_0}{Y_j}, \dots, \frac{Y_m}{Y_j}\right)$ . So if we have a point  $x \in F^{-1}(U_j)$ , then the associated point in  $\mathbb{A}^m$  would be attained by sending

$$x \mapsto \left(\frac{F_0(x)}{F_j(x)}, \dots, \frac{F_m(x)}{F_j(x)}\right).$$

We showed that if we had a map from a quasi affine variety  $W \rightarrow X$  where  $X$  is an affine variety, we just had to map  $A(X) \rightarrow \mathcal{O}(W)$ . In an exercise, we showed that you can replace “quasi affine variety” with “prevariety” and get the same result. ■

**Example 7.16:**

- 1)  $\mathbb{P}^1 \rightarrow \mathbb{P}^n$  where we map  $[s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$ . We know this will be a morphism by our lemma as long as we verify that it is full of zeroes only when  $s = t = 0$ , but this is clear.

We can also map  $\mathbb{P}^1 \rightarrow \mathbb{P}^2$  by  $[s : t] \mapsto [s^3 : s^2t : t^3]$ . This is because it maps nicely as above to  $[s^3 : s^2t : st^2 : t^3]$ , then we can project to drop the third coordinate to get the map we are describing. We are left to show that projections are morphisms

- 2) Projections:  $\mathbb{P}^n \setminus \{[1 : 0 : \dots : 0]\} \rightarrow \mathbb{P}^{n-1}$  by mapping

$$[x_0 : \dots : x_n] \mapsto [x_1 : \dots : x_n].$$

More formally, we can consider  $\mathbb{P}V \setminus \{[v]\} \rightarrow \mathbb{P}\left(\frac{V}{\text{span}(v)}\right)$ , or  $\mathbb{P}V \setminus \mathbb{P}W \rightarrow \mathbb{P}(V/W)$  where  $W \subset V$ .

So, the second example above becomes  $\mathbb{P}^3 \setminus [0 : 0 : 1 : 0] \rightarrow \mathbb{P}^2$ .

- 3) Veronese embedding.

$$\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$

by  $[x] \mapsto [\text{every monomial of } x \text{ of degree } d]$ .

**Exercise 7.17:**  $\nu_d$  is a closed embedding.

- 4) Segre embedding:

$$\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)}$$

by  $([X], [Y]) \mapsto \begin{pmatrix} x_0y_0 & \dots & x_0y_m \\ \vdots & & \vdots \\ x_ny_0 & \dots & x_ny_m \end{pmatrix}$  where this really should just be one long vector, but it is easier to represent as such. We will prove that this is a closed embedding.

PROOF: Fix some  $0 \leq i \leq n$  and  $0 \leq j \leq m$ . Then we have

$$\begin{aligned} U_{ij} &\simeq \mathbb{A}^{mn+m+n} \\ &= \{[z_{ab}] \in \mathbb{P}^{(n+1)(m+1)-1} \mid z_{ij} \neq 0\}. \end{aligned}$$

Then  $S^{-1}(U_{ij}) = U_i \times U_j$  where  $U_i \subset \mathbb{P}^n$  and  $U_j \subset \mathbb{P}^m$ . The coordinates are  $\frac{x_a}{x_i}$ 's and  $\frac{y_b}{y_j}$ 's. This maps  $\mathbb{A}^{n+m} \rightarrow \mathbb{A}^{n+m+nm}$  where the coordinates are  $\frac{z_{ab}}{z_{ij}}$ 's. We could map

$$\frac{z_{ab}}{z_{ij}} \mapsto \frac{x_a y_b}{x_i y_j}.$$

We claim that this is surjective. This is clear, as for example  $\frac{z_{aj}}{z_{ij}} \mapsto \frac{x_a}{x_i}$ . ■

- 5)  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$  maps  $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ . The matrix is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then the image is the same as  $V(xw - yz)$  where  $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ .
- 6)  $X \subset \mathbb{P}^n$  and  $V(F_0, \dots, F_m) \cap X = \emptyset$ . Then  $F : X \rightarrow \mathbb{P}^m$  is a well-defined morphism. The question is: do all maps from  $X \rightarrow \mathbb{P}^m$  arise in this way?

Well the answer is no because  $P^1 \times P^1 \rightarrow \mathbb{P}^3$  as defined in the last example works. We can project  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}^1$ . And the counterexample arises because there is no  $F_0, F_1$  of the same degree such that there is no map  $\mathbb{P}^3 \setminus V(F_0, F_1) \rightarrow \mathbb{P}^1$  that makes the diagram commute.

Let  $S : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$  and let  $Q = \text{im}(S)$ . We want  $Q \cap V(F_0, F_1) = \emptyset$ . However,  $V(Q, F_0, F_1)$  has codimension at most 3, so dimension at least 0, in particular non-empty. This comes from Krull's height theorem.

- 7) If we are given four random lines in  $\mathbb{P}^3$ , how many meets all 4? The answer is 2.

As an exercise, consider 3 random lines in  $\mathbb{P}^3$ , we can consider the union of all lines that touch all 3 and show that it is a projective variety.

### 7.0.2. I skipped class again oops

## 8. Rational maps

For today and the rest of the week, we assume that every variety is irreducible.

**Warm-up:** Let  $f, g : X \rightarrow Y$  be maps of varieties such that  $f \upharpoonright U = g \upharpoonright U$ ,  $\exists \emptyset \neq U \overset{\text{open}}{\subseteq} X$ , then  $f = g$ .

**PROOF:** Let  $X \rightarrow X \times X$  by the diagonal map  $\Delta$ . Then let  $X \times X \rightarrow Y \times Y$  by  $f \times g$ . The inverse image of  $\Delta(Y)$  is  $\{x \mid f(x) = g(x)\}$ . Since they agree on an open subset and it is dense, they are actually equal. ■

**Definition 8.1:** A rational map  $\varphi : X \dashrightarrow Y$  is an equivalence class of pairs  $(U, \varphi_U)$  where we have that  $\emptyset \neq U \subset X$  is open and  $\varphi_U : U \rightarrow Y$  is a morphism. Then we have  $(U, \varphi_U) \sim (V, \psi_V)$  if we have that  $\varphi_U \upharpoonright U \cap V = \psi_V \upharpoonright U \cap V$ .

**Definition 8.2:**  $\varphi : X \dashrightarrow Y$  is dominant if  $\varphi(U)$  is dense in  $Y$  or some/every rep  $(U, \varphi_U)$ .

$\varphi$  is birational if  $\exists \psi : Y \dashrightarrow X$  such that  $\varphi \circ \psi = \text{id}_Y$  and  $\psi \circ \varphi = \text{id}_X$ .

Two varieties are birational if there exists a birational map between them.

**Remark:** In general, you cannot compose rational maps.

**Example 8.3:**

- 1)  $\mathbb{P}^{n+m}$  and  $\mathbb{P}^n \times \mathbb{P}^m$  are birational. This is because there is a copy of  $\mathbb{A}^{n+m}$  in both of them.
- 2)  $\mathbb{A}^1$  and  $V(x^3 - y^2)$  are birational. Consider  $t \mapsto (t^2, t^3)$ , or rather  $(x, y) \mapsto \frac{y}{x}$  in the opposite direction.
- 3)  $\mathbb{P}^1$  and  $V(y^2z - x^3 - x^2z) \subset \mathbb{P}^2$ . Take  $[x : y] \mapsto [x : y : \frac{x^3}{y^2 - x^2}]$ .

**Remark:** A variety  $X$  is rational if it is birational to  $\mathbb{A}^n$  for some  $n$ .

**Question:** Is there a non-rational variety? (Yes.)

Is  $\varphi : X \dashrightarrow Y$  dominant and injective in a nonempty open subset, is it birational? This is true for characteristic zero, but false for characteristic  $> 0$ .

**Remark:**  $X$  is unirational if  $\exists$  dominant  $\mathbb{A}^n \dashrightarrow X$ . Rational and unirational are not equivalent. There are also non unirational varieties.

**Definition 8.4:** A rational function on  $X$  is a rational map from  $X \dashrightarrow \mathbb{A}^1$ . We denote

$$K(X) := \{\text{rational functions on } X\}$$

and call it the (rational) function field.

**Theorem 8.5:**

$$\{\text{dominant rational maps } X \dashrightarrow Y\} \longleftrightarrow \{k\text{-alg extensions } K(Y) \subseteq K(X)\}$$

by the map  $f \mapsto (\varphi \mapsto \varphi \circ f)$ .

PROOF: Let  $\Theta : K(Y) \hookrightarrow K(X)$ . We may assume  $Y$  is affine,  $Y \subseteq \mathbb{A}^n$ . Now look at the functions  $\Theta(y_1), \dots, \Theta(y_m)$ , where  $A(Y) = \frac{\mathbb{k}[y]}{I}$  for some ideal  $I$ . All of the functions listed are regular on some open  $U \subset X$ .

So we have made a map from  $A(Y) \rightarrow \mathcal{O}_X(U)$  by  $y_i \mapsto \Theta(y_i)$ , which defines a morphism (there is some theorem that says having a map from a coordinate ring to the structure sheaf defines a morphism). ■

**Corollary 8.6:** This bijection is an equivalence of categories:

$$\{\text{vars and rational dominant maps}\} \leftrightarrow \{\text{finitely generated field extensions over } \mathbb{k}\}.$$

**Corollary 8.7:**  $X$  and  $Y$  varieties. The following are equivalent:

- 1)  $X$  and  $Y$  are birational.
- 2)  $\exists \emptyset \neq U \subseteq_{\text{open}} X, V \subseteq_{\text{open}} Y$  such that  $U \simeq V$  isomorphic.
- 3)  $K(X) \simeq K(Y)$ .

1 to 2 can be verified. 2 to 3 uses the theorem above.

**Theorem 8.8:** Let  $f : X \dashrightarrow Y$  be a dominant map. Then  $f$  is generically finite (i.e. for any representative  $f : U \rightarrow Y$ , general fiber is finite) if and only if  $K(Y) \subseteq K(X)$  is a finite extension. Further, if gen. fin. and char  $k = 0$ , then general fiber has exactly  $[K(X) : K(Y)]$ .

**Corollary 8.9:** In characteristic zero, a rational dominant map that is generically one to one is birational. This is very false in positive characteristic.

PROOF of Theorem 8.8: Reduce to  $X, Y$  affine,  $X \xrightarrow{f} Y$  where  $X \subseteq \mathbb{A}^n$  and  $Y \subseteq \mathbb{A}^m$  where  $m \leq n$ . Reduce to  $m = n - 1$ , where this map is now  $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1})$ .

Now we split into cases:

- 1)  $z_n \in K(X) = \mathbb{k}[z_1, \dots, z_n]/I$  is algebraic over  $K(Y)$ . By definition, there is a minimal polynomial  $G = a_d(z_1, \dots, z_{n-1})z_n^d + \dots + a_1(z_1, \dots, z_{n-1})z_n + a_0(z_1, \dots, z_{n-1}) \in K(Y)[z_n]$ . We may assume that  $G \in A(Y)[z_n]$ .

$D_Y(a_d) \neq \emptyset$  open in  $Y$ ,  $f$  is finite over  $D(a_d)$ . The discriminant  $\Delta$  of  $G$  will be nonzero on  $Y$ . In other words, on  $D_{Y(a_d \cdot \Delta)}$ ,  $|\text{fiber}| = d$ .

■