Real Analysis II

Rohan Jain

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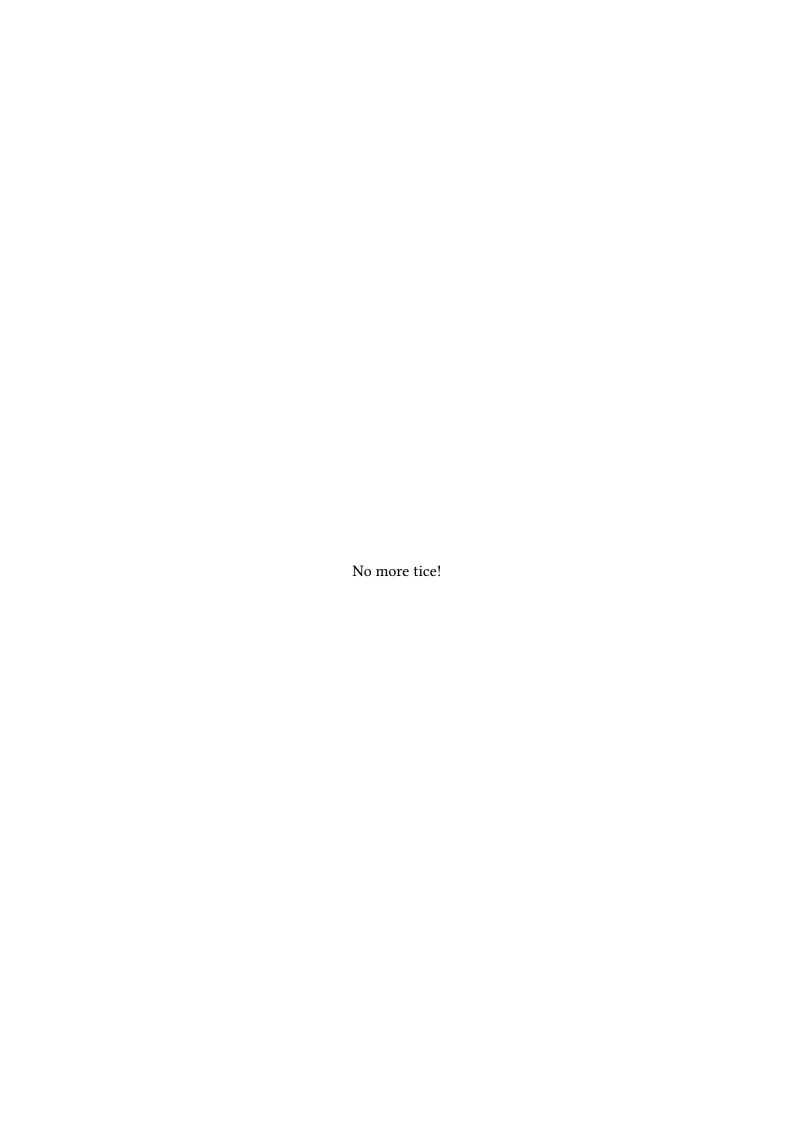


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1. Sequences

Theorem 1.1 (Bolzano Weierstrass): If a sequence (x_n) in \mathbb{R}^d is bounded, then it has a convergent subsequence.

PROOF: We take $(x_{n,1})$. By the d=1 case, it has a convergent subsequence, let's say $(x_{n_k,1})$. Then we take $(x_{n_k,2})$, it has a convergent subsequence, let's say $(x_{n_{k_\ell},2})$. Etc. Once we have reached the dth coordinate, we can guarantee that we have convergent on every coordinate.

This shows that we have a convergent subsequence because $x_n \to x^*$ iff $\forall j \leq d$, we have that $x_{n,j} \to x_j^*$.

Definition 1.2: We say a sequence (x_n) is Cauchy if $\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n, m \geq N$, we have that $|x_n - x_m| < \varepsilon$.

Definition 1.3: We say that (X, ρ) is complete if every Cauchy sequence (x_n) in X is convergent.

Theorem 1.4: $(\mathbb{R}^d, |\cdot|)$ is complete.

PROOF: d=1 is known. Coordinate wise application of the d=1 case shows that $(\mathbb{R}^d,|\cdot|)$ is complete.

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2. Continuity

Definition 2.1: Let $D \subset \mathbb{R}^d$, $F: D - < \mathbb{R}^\ell$ is continuous at a point $a \in D$ if $\forall \varepsilon > 0$, there is $\delta > 0$ such that for all $x \in D$,

$$|x-a| < \delta \Rightarrow |F(x) - F(a)| < \varepsilon.$$

Lemma 2.2: F being continuous at a is equivalent to $\forall (x_n)$ in $D, x_n \to A \Rightarrow F(x_n) \to F(a)$.