21-849: Algebraic Geometry

Rohan Jain

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I don't know what a sheave or a category is. ❤️

Table of Contents

1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and "multivariable calculus".
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.
- Both are continuous in $\mathbb{R}$ or $\mathbb{C}$.
- Both are holomorphic in $\mathbb{C}$.
- Both are analytic (power series expansion at every point).
- Both are C^∞.

There are differences as well.
- $f(z) = a$ has no solution or infinitely many solutions for e^z, but for almost all a, 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \to \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \to \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{C} \to \hat{C}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

> **Theorem 1.1** (GAGA Theorems): Compact (projective) $\mathbb{C}$-manifolds are algebraic.

Here are more cool things about algebraic geometry:
1) **Enumeration**:
    - How many solutions to $p(z)$?
    - How many points in $\{f(x,y) = g(x,y) = 0\}$?
    - How many lines meet a given set of 4 general lines in $\mathbb{C}^3$? The answer is 2.
    - How many conics ($\{f(x,y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
    - Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.
2) **Birationality**:
    - Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.
3) **Arithmetic Geometry**:
    - Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
    - Weil conjectures: X carved by polynomials with $\mathbb{Z}$-coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: $\Bbbk$ is an algebraically closed field ($\Bbbk = \mathbb{C}$).

Definition 2.1 (Affine space): An n-affine space $\mathbb{A}^n_\Bbbk$ is the set

$$\{(a_1, ..., a_n) \mid a_i \in \Bbbk, \forall i = 1, ..., n\} = \Bbbk^n.$$

An affine algebraic subset of $\mathbb{A}^n$ is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, ..., a_n) \in \mathbb{A}^n \mid f(a_1, ..., a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \Bbbk[x_1, ..., x_n]$. We write $Z = V(T)$.

Example 2.2 *(An affine space)*:
- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \Bbbk\}$. Then note that if we project to any two dimensional plane (xy, yz, xz), then we get another affine subset but on $\mathbb{A}^2$.

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

Solution: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin. ■

Definition 2.3 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \Bbbk[x] \mid f(p) = 0, \forall p \in Z\}.$$

Example 2.4:

0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.5:
1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
2) $T \subseteq \Bbbk[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, ..., f_m)$ for some f_i.
3) For $\mathfrak{a} \subseteq \Bbbk[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \Bbbk[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
4) Algebraic subsets of $\mathcal{A}^n$ are closed under finite unions and arbitrary intersections.

Proof: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\Bbbk[x]$ is Noetherian.
■

Theorem 2.6 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \Bbbk[x]\}.$$

Proof:
1) Finite type field extensions $L \supseteq F$ are finite. Rember that finite type means that $F[x_1, ..., x_m] \twoheadrightarrow L$.
2) This implies that maximal ideals of $\Bbbk[x]$ are of the form $\langle x_1 - a_1, ..., x_n - a_n \rangle$ for $a_i \in \Bbbk$, using the fact that $\Bbbk$ is algebraically closed. So, $k[x]/\mathfrak{m} \simeq \Bbbk$.
3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq k[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq k[x, y]$. So for any $(a_1, ..., a_n, b)$ that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in k[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some ℓ.

But also

$$k[x, y]/\langle 1 - yf \rangle \simeq k[x]\left[\frac{1}{f}\right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \cdots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal $\mathfrak{a}$ inside R. ■

Remark: We say R is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

Theorem 2.7: R Jacobson $\Rightarrow R[x]$ Jacobson.

Definition 2.8 (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\Bbbk[x]/I(X)$.

1) $X \xrightarrow{f} \Bbbk$
2) $\operatorname{maxSpec} A(X) = \{\text{maximal ideals in } A(X)\} = X$.

3. Projective Spaces

Definition 3.1: $\mathbb{P}^n = (\Bbbk^{n+1} \setminus \{0\}) / \sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \Bbbk$. That is, $\mathbb{P}^n = \{1\text{-subspaces of } \Bbbk^{\{n+1\}}\}$. For $(a_0, ..., a_n) \in k^{n+1} \setminus \{0\}$, we write $[a_0 : ... : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \Bbbk^{n+1}$. $\mathbb{P}V = V \setminus \{0\} / \sim$

Definition 3.2: $f \in \Bbbk[\underline{x}]$ is homogeneous if $f(\lambda x_1, ..., \lambda x_n) = \lambda^\ell f(x_1, ..., x_n)$.

Definition 3.3: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \cdots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for T a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

Example 3.4: Let $W \subseteq \Bbbk^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an m-plane in $\mathbb{P}^n$.

Example 3.5 *(Twisted cubic curve)*: We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2 t : s t^2 : t^3] \mid [s : t\} \in \mathbb{P}^1]\}$. Then we have that $C = V(x_0 x_3 - x_1 x_2, x_1^2 - x_0 x_2, x_2^2 - x_1 x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix A, $\operatorname{rank}(A) \leq r \iff$ all $(r + 1) \times (r + 1)$ minors $= 0$.

Question: Can there exist F, G such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \Bbbk[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of X.

Exercise 3.6:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow$$
$$\{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \Bbbk[\underline{x}] \text{ such that } \mathfrak{a} \neq \Bbbk[\underline{x}] \text{ or } \langle x_0, ..., x_n \rangle\}.$$

This last part is called the "irrelevant ideal".

Definition 3.7 (General Position): In $\mathbb{P}^n$, any subset of size $\leq n+1$ points are linearly independent.

Theorem 3.8: Every set Γ of $2n$ points in $\mathbb{P}^n$ in general position is carved out by quadrics.

Proof: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose q is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\operatorname{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \operatorname{span}(\Gamma_1)$ or $q \in \operatorname{span}(\Gamma_2)$.

Let $p_1, ..., p_k$ be a minimal subset of Γ whose span $\ni q$ ($k \leq n$). Now pick any Λ such that $|\Lambda| = n - k + 1$ whcih does not contain any of the p_i. We claim that $q \notin \operatorname{span}(p_2, ..., p_k, \Lambda)$.

We then conclude that for any $|S| = n - 1$, $S \subseteq \Gamma \setminus p_1, ..., p_k$, we have that $\operatorname{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \operatorname{span}(p, S)$$

is the intersection at least n many hyperplanes, each of them containing p_1, q. But the intersection of n many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ∎

Definition 3.9: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X, \exists g \in PGL_{n+1}$.

Proposition 3.10: Let $(M_0, ..., M_3)$ be any $\Bbbk$-basis of

$$\Bbbk[s,t]_3 = \{f \in \Bbbk[s,t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s,t) : ... : M_3(s,t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2 t : s t^2 : t^3]\}$.

Example 3.11 *(Rational normal curve)*: Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1} t : \cdots : t^n]$. Or we could map it to any basis of $\Bbbk[s : t]_n$.

Exercise 3.12: $I(\varphi(\mathbb{P}^1)) = ?$.

Example 3.13: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

Fact: If we pick any 3 linearly independent $M_0, M_1, M_2 \in \Bbbk[s,t]_3$. Then $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ by M_0, M_1, M_2 has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \to \mathbb{P}^3$ using 4 elements from $\Bbbk[s,t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

Question: Are all twisted quartic curves projectively equivalent?

Solution: No. In fact, there are infinitely many distinct families. ■

Question (Hartshorne's Question): Is every irreducible curve in $\mathbb{P}^3$ carved out by 2 equations?

4. The Zariski Topology

Definition 4.1 (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \Bbbk[\underline{x}]\}$ form the closed sets of a tpology on $\mathbb{A}^n$ called teh Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

Example 4.2: Take $\mathbb{A}^1$. Two closed subsets are $\mathbb{A}^1$ and $\emptyset$. The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

Definition 4.3: A topological space X is irreducible $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

By definition, we will also say that irreducible implies nonempty.

Remark:
- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

Proposition 4.4: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. X is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

Proof:
- $\Longrightarrow$: Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If X is irreducible, then at least one of them completely contains X. That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means f or $g \in I(X)$.
- $\Longleftarrow$: Suppose for sake of contradiction that X is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■

Remark: When people say affine variety, some people mean that it is also irreducible. But for us, affine variety is the same thing as affine algebraic set.

Then a quasi-affine variety is an open subset of an affine variety.

Example 4.5:
1) $\mathbb{A}^n$ is irreducible. ($\Bbbk[\underline{x}]$ domain)
2) $V(x^2 + y^2) \subset \mathbb{A}^2$ is reducible (char $\Bbbk \neq 2$)
3) Let $f \in \Bbbk[\underline{x}]$ be square-free ($f = f_1 \dots f_\ell$ irreducible). Then $V(f)$ is irreducible if and only if f is irreducible.
4) $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then $A(X) = \frac{\Bbbk[x,y,z]}{\langle x^2 - yz \rangle}$. This is irreducible due to Eisenstein's on f. Now if we take $f \in A(X)$ and look at $V_X(f) \subset X$ is irreducible $\not\Leftarrow$ f irreducible element in $A(X)$.

Definition 4.6: A topological space X is Noetherian if $\nexists X \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots$ such that each Y_i is closed.

Proposition 4.7: An affine variety is Noetherian. (Because $A(X)$ is Noetherian).

Theorem 4.8: A Noetherian topological space X is uniquely a finite union of maximal irreducible closed subsets.

Proof: Consider

$$\{\text{nonempty closed subsets of } X \text{ that does not admit a decomposition into irreducible closed subsets.}\}.$$

Suppose it is nonempty. Then it has a minimal element Y. Y is not irreducible, so $Y = Y_1 \cup Y_2$ (both proper and closed). Since Y is minimal, Y_1 and Y_2 both have decompositions into irreducible closed subsets. So if we just union those decompositions, then we contradict Y's membership in the set. As such, the original set must have actually been empty.

Uniqueness and maximality are left as an exercise. ■

Proposition 4.9:
1) X irreducible and $U \subseteq X$ open. Then $\overline{U} = X$.
2) $V \subseteq X$, V irreducible $\Longrightarrow \overline{V}$ irreducible.
3) $f : X \to Y$ continuous. Image of irreducible set under f is irreducible. (Irreducibility is a topological property).

Example 4.10: Let's have $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ by $\varphi(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x}))$ for some $f_1, \dots, f_m \in \Bbbk[\underline{x}]$. Then $\operatorname{im}(\varphi)$ is irreducible. It is left to show that φ is a continuous map.

Definition 4.11: Let X be a nonempty topological space.

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n, \text{each } Y_i \text{ irreducible and closed}\}.$$

Then let $Y \subseteq X$ closed irreducible subset.

$$\operatorname{codim}_X Y := \sup\{n \mid \exists Y \subseteq Y_0 \subsetneq \cdots \subsetneq Y_n, \text{each } Y_i \text{ irreducible and closed}\}.$$

Example 4.12:
1) $\dim \mathbb{A}^1 = 1$.
2) $X = V(xz, yz) \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Let p be a point on the axis not touching the x-y plane. Then let q be the origin. We have that $\operatorname{codim}_X p = 1$ and $\operatorname{codim}_X q = 2$. Also $\dim p = \dim q = 0$.

Definition 4.13: Height of a prime $\mathfrak{p} \subset R$ is

$$\text{ht } \mathfrak{p} := \sup\{n \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n, \text{each } \mathfrak{p}_i \text{ prime}\}.$$

Then Krull dimension of R is

$$\dim R := \sup\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ prime}\}.$$

Definition 4.14: For an ideal I, we have that

$$\text{ht } I := \inf\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}.$$

"inf of sup".

From these, we can basically show from definition that

$$\text{ht } I + \dim R/I \leq \dim R.$$

The $<$ case is possible if R is not a domain. For example, if we have that $R = \Bbbk[x, y, z]/\langle xz, yz\rangle$ and then $I = \langle x, y, z - 1\rangle$.

But the $<$ case is also possible even if R is a domain and I prime.

Before we cover the next theorem, we note that

$$\{\text{minimal primes over } I\} = \{\mathfrak{p} \text{ prime } \mathfrak{p} \supseteq I, \text{and } \nexists \mathfrak{p} \supsetneq \mathfrak{q} \supseteq I, \text{prime } \mathfrak{q}\}$$

Theorem 4.15 (Krull Principal Ideal Theorem / Height Theorem): Let R be a Noetherian ring and $f_1, ..., f_c \in R$.

1) Minimal primes over $\langle f_1 \rangle$ have height ≤ 1. And the height is equal to 1 if f_1 is nonzerodivisor and nonunit.
2) Minimal primes over $\langle f_1, ..., f_c \rangle$ have height $\leq c$.

"We could do this proof, but it's like proving that there exists a complete ordered field satisfying the least upper bound property."

Theorem 4.16: Let $X \subseteq \mathbb{A}^n$ and $\mathbb{Y} \subseteq \mathbb{A}^m$ irreducible affine varieties.

1) $\dim(X \times Y) = \dim X + \dim Y$.
2) If $Y \subseteq X$, then $\dim Y + \operatorname{codim}_X Y = \dim X$.

Remark (Noether normalization): For $X \subseteq \mathbb{A}^n$ irreducible affine variety. There exists $y_1, ..., y_d \in A(X)$ such that $\Bbbk[Y_1, ..., Y_d] \to A(X)$ with $Y_i \mapsto y_i$ which is a finite extension (injective and $A(X)$ is finitely generated $\Bbbk[Y]$-module) and $d = \dim X$.

Corollary 4.17:

1) $\dim \mathbb{A}^n = n$.
2) $X \subseteq \mathbb{A}^n$ irreducible affine variety. $0 \neq f \in A(X)$ non unit. Then $V_X(f) = V(f) \cap X$ has dimension $\dim X - 1$.

Exercise 4.18: Let $U \subseteq X$ be open for X affine variety irreducible. Then $\dim U = \dim X$.

Proposition 4.19: Let R be Noetherian domain. Then R UFD $\Longleftrightarrow$ every $\mathrm{ht} = 1$ prime is principal.

Proof: R being a UFD implies that $\mathfrak{p}$ has height 1. So let $f = f_1, ..., f_\ell \in \mathfrak{p}$. Suppose $f_1 \in \mathfrak{p}$. So then $0 \neq \langle f_1 \rangle \subseteq \mathfrak{p}$. But as $\mathrm{ht}\ \mathfrak{p} = 1$, we have that $\langle f_1 \rangle = \mathfrak{p}$.

Conversely, we need to show that irreducible implies prime. That is, recall that (ACCP + irreducible = prime) implies that we have a UFD.

So let $f \in$ irred. Krull's PIT says $\langle f \rangle \subseteq \mathfrak{p}$ where $\mathfrak{p}$ has height 1. So by definition, $\mathfrak{p} = \langle g \rangle$, but $\langle f \rangle \subseteq \langle g \rangle$ implies that $f = g$ because f is irreducible. ■

Example 4.20: Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then let $Y = V(x, y) \subseteq X \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Then $\dim Y = 1$. So can we find f such that $\langle f, x^2 - yz \rangle = I(Y)$? The answer to this is no.

But can we find f such that $\sqrt{\langle f, x^2 - yz \rangle} = \langle x, y \rangle$? Take $f = y$ and analyze $\langle y, x^2 - yz \rangle$. This is the same as $\langle y, x^2 \rangle$, whose radical is $\langle x, y \rangle$ as we desire.

Example 4.21: Now consider $X = V(xw - yz) \subseteq \mathbb{A}^4$. $\dim X = 3$ and let $Y = V(x, y)$. Now does there exist f such that $\sqrt{\langle f, xw - yz \rangle} = \langle x, y \rangle$?

This is false, but we don't have the tools to prove it.

Definition 4.22: Zariski topology on $\mathbb{P}^n$ has projective algebraic sets as its closed subsets.

Two ways: projective varieties $\to$ affine varieties.

1) $U_i = \{x_i \neq 0\} = \{[x_0 : \ldots : x_i = 1 : \ldots : x_n]\} \simeq \mathbb{A}^n$.

Proposition 4.23: $\forall i = 0, \ldots, n$, say $i = 0$, $\mathbb{A}^n \longrightarrow U_0$, $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$ is a homeomorphism.

Proof:

- *Homogenization*: let $f \in \Bbbk[x_1, \ldots x_n]$. Then we have

$$f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \Bbbk[x_1, \ldots, x_n].$$

If $Z = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$, $\varphi(Z) = U_0 \cap V\left(f_1^h, \ldots f_m^h\right)$ is closed.

If $Z' = V(F_1, \ldots, F_\ell) \cap U_0$, then $\varphi(Z') = V(F_1(1, x_2, \ldots, x_n), \ldots, F_\ell(1, x_2, \ldots, x_n))$.

Now $U_0 \cup \ldots \cup U_n = \mathbb{P}^n$.

■

Exercise 4.24: Let $Y \subseteq \mathbb{A}^n \simeq U_0$ be an affine variety. $\overline{Y} = V(?)$. Suppose $V(f_1, \ldots, f_m) = Y$. It is tempting to say $\overline{Y} = V\left(f_1^h, \ldots, f_m^h\right)$.

Corollary 4.25:

1) $\dim \mathbb{P}^n = n$.
2) If $H_i = V(x_i) \subseteq \mathbb{P}^n$ does not contain any irreducible components of $Y \subseteq \mathbb{P}^n$, then $\dim Y = \dim Y \cap U_i$.

Definition 4.26: Let $Y \subseteq \mathbb{P}^n$ be a projective variety. The affine cone $\hat{Y} = C(Y)$ is

$$\theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}$$

where

$$\theta : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n.$$

Proposition 4.27:
1) $\hat{Y} = V(I(Y))$. In fact, $I\left(\hat{Y}\right) = I(Y)$.
2) $\dim \hat{Y} = \dim Y + 1$.
3) $\hat{Y}$ is irreducible if and only if Y is irreducible.

Theorem 4.28: If $X, Y \subseteq \mathbb{P}^n$ are projective varieties and $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

Lemma 4.29: If $X, Y \subseteq \mathbb{A}^n$ affine varieties, then $X \cap Y = \emptyset$ or every irreducible component of $X \cap Y$ has $\dim \geq \dim X + \dim Y - n$.

Proof: Let $\Delta = V(x_1 - y_1, ..., x_n - y_n) \subseteq \mathbb{A}^{n+n}$. Note that

$$X \times Y \cap \Delta \simeq X \cap Y.$$

So, $\dim(X \times Y \cap \Delta) \geq \dim X + \dim Y - n$ by Krull's height theorem.

If $\underline{a} = (a_1, ...a_n)$ are varieties, then $I_{a(X)} = \{f(\underline{a} \mid f \in I(X)\}$. Then,

$$A(X \cap Y) = \frac{\mathbb{k}[\underline{z}]}{\sqrt{\langle I_{z(X)} + I_{z(Y)} \rangle}}$$

and

$$A(X \times Y \cap \Delta) = \frac{\mathbb{k}\left[\underline{x}, \underline{y}\right]}{\sqrt{\langle I_{x(X)} + I_{y(Y)} + I(A) \rangle}}.$$

So this implies that $x_i = y_i$ for all i, meaning they are isomorphic rings. ■

Proof *of Theorem 4.28*: X, Y irreducible implies that $\hat{X}$ and $\hat{Y}$ are irreducible. So, then

$$\dim\left(\hat{X} \cap \hat{Y}\right) \geq \dim X + 1 + \dim Y + 1 - (n + 1) \geq \dim X + \dim Y - n + 1.$$

$\hat{X} \cap \hat{Y}$ contains origin by construction, but it has at least one other point because dimension. ■

5. Morphisms

Definition 5.1: For $U \subseteq \mathbb{R}^n$, $U' \subseteq \mathbb{R}^m$ open, $\varphi : U \to U'$ is continuous/continuously differentiable/smooth if $f \circ \varphi$ is smooth for any smooth $f : U' \to \mathbb{R}$.

$f' : U' \to \mathbb{R}$ is smooth if f is smooth at every point $p \in U'$.

Definition 5.2: For affine variety $X \subseteq \mathbb{A}^n$ and $U \subseteq X$ open, a function $\varphi : U \to \Bbbk = \mathbb{A}^1$ is regular if $\forall p \in U$, $\exists U_p \ni p$ open and $f_p, g_p \in A(X)$ such that $\varphi(x) = \frac{f_p(x)}{g_p(x)}$ for all $x \in U_p$. In particular, $g_p(x) \neq 0$ for all $x \in U_p$.

$\mathcal{O}_X(U) \coloneqq \{\text{regular functions on } U\}$. This is also a $\Bbbk$-algebra.

Example 5.3: Let $U \subseteq X$, $\varphi : U \to \mathbb{A}^1$ regular $\not\Rightarrow \varphi = \frac{f}{g}$ globally for some $f, g \in A(X)$.

$X = V(xw - yz) \subset \mathbb{A}^4$, $U = X \setminus V(x, y)$.

$$\varphi(x, y, z, w) = \begin{cases} \frac{z}{x} \text{ if } x \neq 0 \\ \frac{w}{y} \text{ if } y \neq 0 \end{cases}.$$

Lemma 5.4: $\varphi : U \to \mathbb{A}^1$ regular, then $V(\varphi) = \{x \in U \mid \varphi(x) = 0\}$ is closed in U. In particular φ is continuous.

Proof: Closedness is a local condition, and around any $p \in U$, $\{\varphi \restriction U_p : 0\} = V(f_p) \cap U_p$. ■

Remark: If $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ for U irreducible, and $\varphi_1 \restriction U' = \varphi_2 \restriction U'$ for some $\emptyset \neq U' \subseteq U$, then $\varphi_1 = \varphi_2$.

Definition 5.5: $X \subseteq \mathbb{A}^n$ affine variety. A distinguished open subset U of X is an open subset of the form $X \setminus V(f)$ for some $f \in A(X)$, denoted $D(f)$, D_f, U_f, X_f. X_f is probably the most descriptive as it actually mentions X.

Remark: $\{D(f)\}_{f \in A(X)}$ form a basis for Zariski topology. What that means is that any $U \subseteq X$ is a union for $D(f)$'s.

Exercise 5.6: $D(f)$ is homeomorphic to $V(I(X) + \langle 1 - yf \rangle) \subseteq \mathbb{A}^{n+1}$.

Theorem 5.7: $\mathcal{O}_X(D(f)) = \left\{\frac{g}{f^m} \mid g \in A(X), m \in \mathbb{Z}_{\geq 0}\right\}$. In fact, $\mathcal{O}_X(D(f)) = A(X)_f$.

Example 5.8: $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = A(\mathbb{A}^2) = \Bbbk[x, y]$. Then,

$$\mathbb{A}^2 \ni \varphi = \begin{cases} \frac{f}{x^m} \text{ for some } f \in \Bbbk[x, y] \text{ on } \mathbb{A}^2 \setminus V(x) \\ \frac{g}{y^\ell} \text{ for some } g \in \Bbbk[x, y] \text{ on } \mathbb{A}^2 \setminus V(y) \end{cases}.$$

Then we say $y^\ell f = x^m g$ on $\mathbb{A}^2$. Because we are in a UFD, this means that $x^m \mid f$ and $y^\ell \mid g$. But this implies $m = \ell = 0$, so $f = g = \varphi$.

Proof *of Theorem 5.7*: $\supseteq$ is clear. So we only prove the $\subseteq$ case.

Suppose we have $\varphi \in \mathcal{O}_X(D(f))$. Then for all $p \in D(f)$, $\exists U_p \ni p$ and $\varphi \upharpoonright U_p = \frac{g_{p'}}{f_{p'}}$ for $g_{p'}, f_{p'} \in A(X)$.

Take a nonempty $D(h_p) \subseteq U_p$ and write $g_p = g_{p'} h_p$ and $f_p = f_{p'} h_p$.

Then $\varphi \upharpoonright D(f_p) = \frac{g_p}{f_p} = \frac{g_p f_p}{f_p^2}$. So assume $g_p = 0$ on $V(f_p)$.

Now we claim that $\forall p, q \in D(f)$, we have $g_p f_q = g_q f_p$ in $A(X)$.

Then $D(f) = \bigcup_p D(f_p)$. Then $V(f) = \bigcap_p V(f_p)$. Nullstellensatz says that $\sqrt{\langle f \rangle} = \sqrt{\langle f_p : p \in D(f) \rangle}$ as ideals in $A(X)$. But then, $f^m = \sum k_p f_p$. By Noetherian-ness, this is a finite sum. We claim that $g = \sum k_p f_p$.

Then $\frac{g}{f^m} = \frac{g_q}{f_q}$ on $D(f_q)$ for all $q \in D(f)$. So,

$$g f_q = \sum_p k_p g_p f_q = \sum_p k_p f_p g_q = g_q f^m.$$

■

6. Sheaves

Let $\mathcal{A}$ be a category: AbGrp, Rings, $\Bbbk$-algebras. Given a topological space X, $\mathrm{Top}(X)$ is a category where the objects are open subsets $U \subseteq X$ and morphisms are inclusions between $U \subseteq V$ open subsets.

Definition 6.1: A presheaf (with values in $\mathcal{A}$) on X is a contravariant functor $\mathcal{F} : \mathrm{Top}(X) \to \mathcal{A}$.

$\mathcal{F}$ is further a sheaf if for every open cover $\{U_i\}_i$ of any open subset $U \subseteq X$ if

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

Translation:

1) Assignment $U \mapsto \mathcal{F}(U) \in \mathrm{obj}(\mathcal{A})$ such that $\forall U \subseteq V \subseteq X$ open,

$$\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that $\mathrm{res}_{U,U} = \mathrm{id}$ and $\mathrm{res}(V,U) \circ \mathrm{res}(W,V) = \mathrm{res}(W,U)$.

2) If $(f_i)_i \in \prod_i \mathcal{F}(U_i)$ such that $\mathrm{res}_{U_i,U_i \cap U_j}(f_i) = \mathrm{res}_{U_j,U_i \cap U_j}(f_j)$, then $\exists! f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i}(f) = f_i$. Also $\mathcal{F}(\emptyset) = 0$ as a consequence.

$f \upharpoonright V := \mathrm{res}_{U,V}(f)$, $f \in \mathcal{F}(U)$. $\mathcal{F}(U)$ elements are called sections of $\mathcal{F}$ over U.

Example 6.2: Note that throughout these examples, X is a topological space and $U \subseteq X$.

1) $\mathcal{F}_{\mathrm{ct}}(U) := \{\varphi : U \to \mathbb{R}\}$. Then if $U' \subseteq U$, $\mathrm{res}_{U,U'}(f) := f \upharpoonright U'$.
2) $C(U) := \{\varphi : U \to \mathbb{R} \text{ cts}\}$.
3) $C^\infty(U) := \{\varphi : U \to \mathbb{R} \text{ smooth}\}$ ($X \subseteq \mathbb{R}^n$ open).
4) $\underline{\mathbb{R}}(U) := \{\varphi : U \to \mathbb{R}, \text{constant}\}$. This is not a sheaf. If we consider a constant function that takes the value a on U and b on U', then there is no value c such that they can be glued together to be equal on both sets.
5) $\mathcal{O}_{X(U)} := \{\varphi : U \to \Bbbk \text{ regular}\}$

Remark: A constant sheaf $A_{X(U \text{ conn})} = A$. A locally constant sheaf (locally $\mathcal{F} \upharpoonright U$ is constant). Locally constant does not imply constant.

Definition 6.3: $U \subseteq X$, $\mathcal{F}$ sheaf on X, $\mathcal{F} \upharpoonright U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 6.4: $\mathcal{F}$ sheaf on X. $p \in X$. The stalk of $\mathcal{F}$ at p, $\mathcal{F}_p := \lim_{U \ni p} \mathcal{F}(U)$. This is actually just equal to $\{(U, f) \mid U \ni p, f \in \mathcal{F}(U)/\sim\}$ where $(U_1, f_1) \sim (U_2, f_2)$ if $\exists V \ni p$ such that $f_1 \restriction V = f_2 \restriction V$.

Remember that $\mathcal{F}\mathrm{ct}_S(U) = \{f : U \to S\}$ and $C(U) = \{f : U \to R \text{ cts}\}$.

Remark: $\mathcal{F}(V) \to \mathcal{F}(U)$ by $'\mathrm{res}_{V,U}$. This map need not be surjective.

Theorem 6.5: Let X be an affine variety and $x \in X$. $\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_X}$.

Proof: Consider the ring map $A(X)_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$ where we map $\frac{f}{g} \mapsto \frac{f}{g}$.

Now if $\frac{f}{g} = \frac{f'}{g'}$ in $A(X)_{\mathfrak{m}_x}$, then we need to check that the same is true around x in $\mathcal{O}_{X,x}$.

Now if $\frac{f}{g}$ is 0 around x (in $D(h)$), we can deduce that $\frac{f}{g} = 0$ in $A(X)_{\mathfrak{m}_x}$. ■

Definition 6.6: A ringed space $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf on X with values in Ring. We call $\mathcal{O}_X$ the structure sheaf of this ringed space.

Definition 6.7: $f : X \to Y$ continuous and $\mathcal{F}$ a sheaf on X.

$$\text{pushforward } f_* \mathcal{F}(V) = \mathcal{F}(f^{-1} V)$$

where $V \supset Y$ is open.

Definition 6.8: Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. $\Phi : \mathcal{F} \to \mathcal{G}$ means that for each $U \subseteq X$, we specify $\Phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ where for $U \subseteq V$, we have the following diagram commuting:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\Phi(U)"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u, "\mathrm{res}"] \arrow[r, "\Phi(V)"] & \mathcal{G}(V) \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

where $V \supset Y$ is open.

Definition 6.9: A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $\left(f, f^{\#}\right)$ where $f : X \to Y$ is continuous and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$.

Example 6.10: $(U \subseteq \mathbb{R}^n, C^1) \to (V \subseteq \mathbb{R}^m, C^1)$.

Remark: When we say $(X, \mathcal{O}_X)$, we mean that $\mathcal{O}_X$ is a subsheaf of $\mathcal{F}\mathrm{ct}_{\Bbbk}$ for some fixed $\Bbbk$. Then $\mathcal{O}_X = \{\varphi : U \to \Bbbk \mid \varphi \text{ satisfies some condition}\}$. And, $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is always given by precomposition.

6.1. Category of quasi Affine varieties "qAffVar".

To define this cateogry, we'll have the objects be open subsets of an affine variety consdiered as a ringed space. The morphisms will be maps of ringed spaces with $f^{\#}$ being the precomposition as our convention above dictates.

Theorem 6.11: X, Y both affine varieties. $U \subseteq X$ open. Then there exists a natural bijection.

$$\mathrm{Mor}(U, Y) \simeq \mathrm{Hom}_{\Bbbk\text{-alg}}(A(Y), \mathcal{O}(U)).$$

7. Projective morphisms

7.0.1. Ok unfortunately i was forced to miss two classes so there is gap here

Proposition 7.1: Suppose X, Y are prevarieties with affine covers $\{U_i\}$ and $\{V_j\}$ respectively. Then $X \times Y$ is a product in the category of prevarieties constructed by gluing together $U_i \times V_j$ and $U_{i'} \times V_{j'}$,

$$(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

for all such pairs.

We did not prove that gluing gives you a prevariety, but we will believe it. Also note that $X \times Y$ is a prevariety by affine cover $\{U_i \times V_j\}$.

Proposition 7.2: For $Y \subseteq X$ closed where $\iota : Y \to X$ and X is a prevariety. Then for $U' \subseteq Y$ open,

$$\iota^* \mathcal{O}_X(U') = \{f : U' \to \Bbbk \mid \forall y \in U', \exists U_y \subset X \text{ with } \varphi \in \mathcal{O}_X(U_y) \text{ such that } f \restriction U_y \cap Y = \varphi\}.$$

Then $\iota^* \mathcal{O}_X$ is a sheaf and $(Y, \iota^* \mathcal{O}_X)$ is a prevariety, and $\iota : Y \to X$ is a morphism.

Proof: We will believe that $\iota^* \mathcal{O}_X$ is a sheaf. Also ι is a morphism of prevarieties.

Let $U \subseteq X$ be affine open. We claim that $(U \cap Y, \iota^* \mathcal{O}_X \restriction U \cap Y)$ is affine. We claim that $Y = V(I)$ for some $I \subseteq A(X)$.

Then, $\iota^* \mathcal{O}_X$ are the functions that are locally restrictions of regular functions on X. Then $\mathcal{O}_{V(I)}$ are functions that are locally quotient of polynomials on $\mathbb{A}^n$. These are equal. ■

We say that ι is a closed embedding.

Example 7.3:
1) $\mathbb{A}^2 \to \mathbb{A}^2$ via $(x, y) \mapsto (x, xy)$. This maps $\mathbb{A}^2$ to itself without the y-axis, but still including the origin. Note that the image of this map is neither open nor closed in $\mathbb{A}^2$.

   Remark: $Y \subseteq X$ is locally closed if it is $U \cap V$ where U is open in X and V is closed in X.

   This example is not even locally closed.
2) Glue $\mathbb{A}^1$ and $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ via the identity. This is basically a line with two origins. Let this line be called X. Consider $g : X \to X$ via switching the origins and keeping the other points the same. This is a morphism where our open subsets are lines including only one

of the origins, and it is not hard to check that this is actually a morphism. Then, $\{g(x) = x)\} \simeq \mathbb{A}^1 \setminus \{0\}$ which is not closed in X.

Definition 7.4: A prevariety of X is a variety of the diagonal map $\Delta : X \to X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

Lemma 7.5: For checking if something is a variety, Δ being a topologically closed embedding is a sufficient condition.

Corollary 7.6: Open and closed subprevarieties of varieties are varieties.

Lemma 7.7: Let X and Y be affine varieties and $f : X \to Y$. If $f^{\#} : A(Y) \twoheadrightarrow A(X)$, then f is a closed embedding.

Proof: Let J be the kernel. And consider the surjective map

$$\frac{k\left[\underline{y}\right]}{I(Y)} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}.$$

This is also surjective onto $\frac{k\left[\underline{y}\right]}{I(Y)+J}$. This is $V_Y(J)$, so we get that $A(V_Y(J)) \simeq A(X)$ as desired. ∎

Lemma 7.8: Let X be an affine variety, then $\Delta : X \to X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

Proof: $A(X) = \frac{k[\underline{x}]}{I(X)}$, $A(X \times X) \simeq A(X) \otimes A(X)$. THen we can map

$$\frac{k\left[\underline{x}, \underline{y}\right]}{I_{x(X)} + I_{y(Y)}} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}$$

via $x_i \mapsto x_i$ and $y_i \mapsto x_j$. ∎

Proposition 7.9: X prevariety is a variety if $\Delta(X) \subseteq X \times X$ is closed.

Proof: We claim that $\forall x \in X$, take any $x \in U \subset X$ affine open. Then $\Delta \restriction U : U \to U \times U$ is a closed embedding.

Since $\Delta(X) \subseteq X \times X$ is closed (topologically closed embedding, locally closed embedding implies closed embedding. So $\mathcal{O}_X \simeq \mathcal{O}_{\Delta(X)}$.

It is important theat $\Delta \restriction U$ is closed because of the following:

$Y \subseteq X$ such that $\forall y \in Y, \exists U_y \subset X$ containing y such that $U_y \cap Y$ is closed in U_y does not imply that $Y \subseteq X$ is closed. However, under our assumption, it would be that way. ■

Corollary 7.10: qAffVar are varieties.

Corollary 7.11: $X \overset{f}{\to} Y$ morphism of varieties. The graph $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

Proof: $X \times Y \overset{f \times \mathrm{id}}{\to} Y \times Y$. Then $(f \circ \mathrm{id})^{-1}(\Delta(Y)) = \Gamma_f$. ■

Exercise 7.12: Let X be a variety and $U, U' \subseteq X$ open affine subsets. Then $U \cap U'$ is also affine open.

Definition 7.13: Let $X \subseteq \mathbb{P}^n$ be a projective algebraic subset. $\mathcal{O}_X(U) = \left\{\varphi : U \to \Bbbk \mid \text{locally encrypted } (\varphi \restriction U') = \frac{F}{G} \text{ for some homogeneous poly } F, G \text{ of same degree.}\right\}$.

Proposition 7.14: $X \subseteq \mathbb{P}^n$ projected algebraic set $\rightsquigarrow (X, \mathcal{O}_X)$ if for all $i = 0, 1, ..., n$, let $U_i = \mathbb{P}^n \setminus V(x_i) \overset{\text{homeo}}{\simeq} \mathbb{A}^n$. Then $(X \cap U_i, \mathcal{O}_X \restriction U_i)$ is isomorphic as a ringed space to $X \cap U_i$ considered as a closed subset in $\mathbb{A}^n$ under $\mathbb{A}^n \simeq U_i$ and hence an affine variety. In particular, X is a prevariety.

Proof: $X = V(F_1, ..., F_m)$. Then $X \cap U_i \subset \mathbb{A}^n$ is $V(F_1(x_0, ..., x_i = i, ..., x_n), ..., F_m)$.

If we have $\frac{F}{G} \in (X \cap U_i, \mathcal{O}_X \restriction U_i)$, then we can just dehomogenize to get $\frac{f}{g} \in \left(X \cap U_i, \mathcal{O}_{X \cap U_i \subseteq \mathbb{A}^n}\right)$. ■

Lemma 7.15: $X \subset \mathbb{P}^n$ a projective variety. Let $F_0, ..., F_m$ be homogeneous polynomials on $\mathbb{P}^n$ of same degree. Then $F : X \setminus V(F_0, ..., F_m) \to \mathbb{P}^m$ by $x \mapsto (F_0(x), ..., F_m(x))$ is a morphism.

Proof: As a set map, this is well-defined. We will verify that $\forall j = 0, ..., m$, the distinguished open subset $U_j = \left\{[\underline{Y}] \in \mathbb{P}^m \mid Y_j \neq 0\right\}$. We have

$$F^{-1}(U_j) = \mathbb{P}^n \setminus V(F_j).$$

Then $F^{-1}(U_j) \to U_j$ where $U_j \simeq \mathbb{A}^m$. We'll call the coordinates of $\mathbb{A}^m$ by $\left(\frac{Y_0}{Y_j}, ..., \frac{Y_m}{Y_j}\right)$. So if we have a point $x \in F^{-1}(U_j)$, then the associated point in $\mathbb{A}^m$ would attained by sending

$$x \mapsto \left(\frac{F_0(x)}{F_j(x)}, ..., \frac{F_m(x)}{F_j(x)}\right).$$

We showed that if we had a map from a quasi affine variety $W \to X$ where X is an affine variety, we just had to map $A(X) \to \mathcal{O}(W)$. In an exercise, we showed that you can replace "quasi affine variety" with "prevariety" and get the same result. ∎

Example 7.16:

1) $\mathbb{P}^1 \to \mathbb{P}^n$ where we map $[s : t] \mapsto [s^n : s^{n-1}t : \cdots : t^n]$. We know this will be a morphism by our lemma as long as we verify that it is full of zeroes only when $s = t = 0$, but this is clear.

   We can also map $\mathbb{P}^1 \to \mathbb{P}^2$ by $[s : t] \mapsto [s^3 : s^2t : t^3]$. This is because it maps nicely as above to $[s^3 : s^2t : st^2 : t^3]$, then we can project to drop the third coordinate to get the map we are describing. We are left to show that projections are morphisms
2) Projections: $\mathbb{P}^n \setminus \{[1 : 0 : \cdots : 0]\} \to \mathbb{P}^{n-1}$ by mapping

$$[x_0 : \cdots x_n] \mapsto [x_1 : \cdots : x_n].$$

   More formally, we can consider $\mathbb{P}V \setminus \{[v]\} \to \mathbb{P}\left(\frac{V}{\text{span}(v)}\right)$, or $\mathbb{P}V \setminus \mathbb{P}W \to \mathbb{P}(V/W)$ where $W \subset V$.

   So, the second example above becomes $\mathbb{P}^3 \setminus [0 : 0 : 1 : 0] \to \mathbb{P}^2$.
3) Veronese embedding.

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

   by $[x] \mapsto$ [every monomial of x of degree d].

   Exercise 7.17: ν_d is a closed embedding.

4) Segre embedding:

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)}$$

   by $([\underline{X}], [\underline{Y}]) \mapsto \begin{pmatrix} x_0y_0 & \cdots & x_0y_m \\ \vdots & & \vdots \\ x_ny_0 & \cdots & x_ny_m \end{pmatrix}$ where this really should just be one long vector, but it is easier to represent as such. We will prove that this is a closed embedding.

   Proof: Fix some $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we have

$$\begin{aligned} U_{ij} &\simeq \mathbb{A}^{mn+m+n} \\ &= \left\{[z_{ab}] \in \mathbb{P}^{(n+1)(m+1)-1} \mid z_{ij} \neq 0\right\}. \end{aligned}$$

   Then $S^{-1}(U_{ij}) = U_i \times U_j$ where $U_i \subset \mathbb{P}^n$ and $U_j \subset \mathbb{P}^m$.The coordinates are $\frac{x_a}{x_i}$'s and $\frac{y_b}{y_j}$'s. This maps $\mathbb{A}^{n+m} \to \mathbb{A}^{n+m+nm}$ where the coordinates are $\frac{z_{ab}}{z_{ij}}$'s. We could map

$$\frac{z_{ab}}{z_{ij}} \mapsto \frac{x_a y_b}{x_i y_j}.$$

We claim that this is surjective. This is clear, as for example $\frac{z_{aj}}{z_{ij}} \mapsto \frac{x_a}{x_i}$. ■

5) $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ maps $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$. The matrix is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then the image is the same as $V(xw - yz)$ where $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.
6) $X \subset \mathbb{P}^n$ and $V(F_0, ..., F_m) \cap X = \emptyset$. Then $F : X \to \mathbb{P}^m$ is a well-defined morphism. The question is: do all maps from $X \to \mathbb{P}^m$ arise in this way?

   Well the answer is no because $P^1 \times P^1 \to \mathbb{P}^3$ as defined in the last example works. We can project $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. And the counterexample arises because there is no F_0, F_1 of the same degree such that there is no map $\mathbb{P}^3 \setminus V(F_0, F_1) \to \mathbb{P}^1$ that makes the diagram commute.

   Let $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ and let $Q = \operatorname{im}(S)$. We want $Q \cap V(F_0, F_1) = \emptyset$. However, $V(Q, F_0, F_1)$ has codimension at most 3, so dimension at least 0, in particular non-empty. This comes from Krull's height theorem.
7) If we are given four random lines in $\mathbb{P}^3$, how many meets all 4? The answer is 2.

   As an exercise, consider 3 random lines in $\mathbb{P}^3$, we can consider the union of all lines that touch all 3 and show that it is a projective variety.

7.0.2. I skipped class again oops

8. Rational maps

For today and the rest of the week, we assume that every variety is irreducible.

Warm-up: Let $f, g : X \to Y$ be maps of varieties such that $f \upharpoonright U = g \upharpoonright U, \exists\, \emptyset \neq U \overset{\text{open}}{\subseteq} X$, then $f = g$.

Proof: Let $X \to X \times X$ by the diagonal map Δ. Then let $X \times X \to Y \times Y$ by $f \times g$. The inverse image of $\Delta(Y)$ is $\{x \mid f(x) = g(x)\}$. Since they agree on an open subset and it is dense, they are actually equal. ■

Definition 8.1: A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, φ_U) where we have that $\emptyset \neq U \subset X$ is open and $\varphi_U : U \to Y$ is a morphism. Then we have $(U, \varphi_U) \sim (V, \psi_V)$ if we have that $\varphi_U \upharpoonright U \cap V = \psi_V \upharpoonright U \cap V$.

Definition 8.2: $\varphi : X \dashrightarrow Y$ is dominant if $\varphi(U)$ is dense in Y or some/every rep (U, φ_U).

φ is birational if $\exists \psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.

Two varieties are birational if there exists a birational map between them.

Remark: In general, you cannot compose rational maps.

Example 8.3:
1) $\mathbb{P}^{n+m}$ and $\mathbb{P}^n \times \mathbb{P}^m$ are birational. This is because there is a copy of $\mathbb{A}^{n+m}$ in both of them.
2) $\mathbb{A}^1$ and $V(x^3 - y^2)$ are birational. Consider $t \mapsto (t^2, t^3)$, or rather $(x, y) \mapsto \frac{y}{x}$ in the opposite direction.
3) $\mathbb{P}^1$ and $V(y^2 z - x^3 - x^2 z) \subset \mathbb{P}^2$. Take $[x : y] \mapsto \left[x : y : \frac{x^3}{y^2 - x^2}\right]$.

Remark: A variety X is rational if it is birational to $\mathbb{A}^n$ for some n.

Question: Is there a non-rational variety? (Yes.)

Is $\varphi : X \dashrightarrow Y$ dominant and injective in a nonempty open subset, is it birational? This is true for characteristic zero, but false for characteristic > 0.

Remark: X is unirational if $\exists$ dominant $\mathbb{A}^n \dashrightarrow X$. Rational and unirational are not equivalent. There are also non unirational varieties.

Definition 8.4: A rational function on X is a rational map from $X \dashrightarrow \mathbb{A}^1$. We denote

$$K(X) := \{\text{rational functions on } X\}$$

and call it the (rational) function field.

Theorem 8.5:

$$\{\text{dominant rational maps } X \dashrightarrow Y\} \longleftrightarrow \{k\text{-alg extensions } K(Y) \subseteq K(X)\}$$

by the map $f \mapsto (\varphi \mapsto \varphi \circ f)$.

Proof: Let $\Theta : K(Y) \hookrightarrow K(X)$. We may assume Y is affine, $Y \subseteq \mathbb{A}^n$. Now look at the functions $\Theta(y_1), ..., \Theta(y_m)$, where $A(Y) = \frac{\Bbbk[y]}{I}$ for some ideal I. All of the functions listed are regular on some open $U \subset X$.

So we have made a map from $A(Y) \to \mathcal{O}_X(U)$ by $y_i \mapsto \Theta(y_i)$, which defines a morphism (there is some theorem that says having a map from a coordinate ring to the structure sheaf defines a morphism). ■

Corollary 8.6: This bijection is an equivalence of categories:

$$\{\text{vars and rational dominant maps}\} \leftrightarrow \{\text{finitely generated field extensions over } \Bbbk\}.$$

Corollary 8.7: X and Y varieties. The following are equivalent:

1) X and Y are birational.
2) $\exists \emptyset \neq U \overset{\text{open}}{\subseteq} X, V \overset{\text{open}}{\subseteq} Y$ such that $U \simeq V$ isomorphic.
3) $K(X) \simeq K(Y)$.

1 to 2 can be verified. 2 to 3 uses the theorem above.

Theorem 8.8: Let $f : X \dashrightarrow Y$ be a dominant map. Then f is generically finite (i.e. for any representative $f : U \to Y$, general fiber is finite) if and only if $K(Y) \subseteq K(X)$ is a finite extension. Further, if gen. fin. and char $k = 0$, then general fiber has exactly $[K(X) : K(Y)]$.

Corollary 8.9: In characteristic zero, a rational dominant map that is generically one to one is birational. This is very false in positive characteristic.

Proof *of Theorem 8.8*: Reduce to X, Y affine, $X \overset{f}{\to} Y$ where $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ where $m \leq n$. Reduce to $m = n - 1$, where this map is now $(z_1, ..., z_n) \mapsto (z_1, ..., z_{n-1})$.

Now we split into cases:

1) $z_n \in K(X) = \Bbbk[z_1, ..., z_n]/I$ is algebraic over $K(Y)$. By definition, there is a minimal polynomial $G = a_d(z_1, ..., z_{n-1})z_n^d + \cdots + a_1(z_1, ..., z_{n-1})z_n + a_o(z_1, ..., z_{n-1}) \in K(Y)[z_n]$. We may assume that $G \in A(Y)[z_n]$.

   $D_Y(a_d) \neq \emptyset$ open in Y, f is finite over $D(a_d)$. The discrimnant Δ of G will be nonzero on Y. In other words, on $D_{y(a_d \cdot \Delta)}$, $|\text{fiber}| = d$.

■

8.1. Wasn't here for first part of blowups

Definition 8.10: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $I = \langle f_0, ..., f_m \rangle \subset A(X)$. The blowup, which we define as $\tilde{X} = B\ell_I X$, is the subvariety of $X \times \mathbb{P}^m (\subset \mathbb{A}^n \times \mathbb{P}^m)$ given by u-homogeneous elements of $\ker(\Bbbk[\underline{x}][\underline{u}] \twoheadrightarrow A(X)[tI] \subseteq A(X)[t])$. $\pi : \tilde{X} \to X$ the "blow-down" map.

Proposition 8.11: $B\ell_I X$ is independent of the choice of generators $f_0, ..., f_m$.

Proof: $B\ell_{\langle f_0, ..., f_m \rangle} X \simeq B\ell_{\langle f_0, ..., f_m, g \rangle} X$. ■

Proposition 8.12: $I = \langle f_0, ..., f_m \rangle \subset A(X)$. Then $\tilde{X} = B\ell_I X \simeq$ closure in $X \times \mathbb{P}^m$ of the image of

$$(X \setminus V(I)) \to X \times \mathbb{P}^m$$

given by

$$x \mapsto \left(x, \left[f_0(x), ..., f_{m(x)}\right]\right).$$

PROOF:

1) $\overline{X} \subseteq \tilde{X}$:

2) $\tilde{X} \subseteq \overline{X}$:

■

8.2. im retarded

9. Smoothness/Nonsingularness

What is a tangent vector? Rather, for $0 \in X \subseteq \mathbb{A}^n$, how can we find a tangent vector to X?

1) Something in the tangent space?

Example 9.1: Suppose char $k \neq 2$, $X = V\big((x-1)^2 + (y-1)^2 - 2\big)$. Then we can say that

$$T_0 X = V(x+y).$$

But how do we get this? We can see that

$$(x-1)^2 + (y-1)^2 - 2 = x^2 - 2x + y^2 - 2y.$$

And we see $-2(x+y)$ is the gradient of this function or something idk.

Definition 9.2: For $0 \in X \subseteq \mathbb{A}^n$,

$$T_0 X := V\big(f^{\text{linear}} \mid f \in I(X)\big).$$

For $p \in X \subseteq \mathbb{A}^n$,

$$T_p K := \ker \operatorname{Jac} \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{i,j}$$

for any generating set $f_1, ..., f_m$ of $I(X)$.

Definition 9.3: Let A be a k-alg and M an A-module. Then

$$\operatorname{Der}_k(A, M) := \{\delta \in \operatorname{Hom}_k(A, M) \mid \forall f, g \in A, \delta(fg) = f\delta(g) + g\delta(f)\}.$$

Example 9.4: Let $A = k[x, y]$ and $M = k[x, y]/\langle x - a, y - b\rangle$ for $a, b \in k$. Then this is isomorphism by k with the group action of acting by a on x.

Then

$$\operatorname{Der}_k(A, M) = k\left[\frac{\partial}{\partial x}\Big|_{\substack{x=a \\ y=b}}, \frac{\partial}{\partial y}\Big|_{\substack{x=a \\ y=b}}\right].$$

So for $\delta \in A$, we have $\delta(x^n) = nx^{n-1} \cdot \delta(x) = na^{n-1}\delta(x)$.

Definition 9.5: Zariski cotangent space of $p \in X$ is $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes A(X)/\mathfrak{m}$.

Proposition 9.6: Let $0 \in X \subseteq \mathbb{A}^n$. Then

$$T_0 X \simeq \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \simeq \operatorname{Der}_k(A(X), A(X)/\mathfrak{m} \simeq k)$$

where $\mathfrak{m} = \langle \overline{x}_1, ..., \overline{x}_n \rangle$. We can accomplish this with the maps

$$v \mapsto \left(\overline{f} \mapsto f^{\text{linear}}(v)\right)$$

from first to second and

$$\delta \mapsto \left(\overline{f} \mapsto \delta(f)\right)$$

for third to second.

Proof: Let $\varphi(\delta) = \overline{f} \mapsto \delta(f)$. Then $\delta(\mathfrak{m}^2) = 0$ implies that $\delta = 0$. Also $\delta(1) = 0$, so it is injective.

I'm too lazy to write this whole thing out ∎

Remark: Note that $\dim T_p X \geq \operatorname{codim}_X p$.

Definition 9.7: X variety is nonsingular at a point $p \in X$ if

$$\dim T_p X = \operatorname{codim}_X p.$$

Proposition 9.8: $p \in X$ is nonsingular if and only if $\dim T_p X \leq \operatorname{codim}_X p$. Also if and only if $T_p X = TC_p X$. Also if and only if $\dim\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \dim \mathcal{O}_{X,p}$. Also if and only if $\operatorname{rank} \operatorname{Jac}\left[\left(\frac{\partial f_i}{\partial x_j}(p)\right)\right]_{i,j} = n - \operatorname{codim}_X p$ when $X \subseteq \mathbb{A}^n$.

Example 9.9: $X = V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$. $TC_0 X = V(x^2 - y^2)$. $T_0 X = \mathbb{A}^2$.

Remark: A Noetherian ring $(R, \mathfrak{m})$ is a regular local ring if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Theorem 9.10: Regular local rings are UFDs.

Proof: Something something Nakayama. ∎

Proposition 9.11: Let $p \in X \subseteq \mathbb{A}^n$ is nonsingular if $\operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_m) \geq n - \operatorname{codim}_X p$ for any $f_1, ..., f_m$ such that $V(f_1, ..., f_m) = X$.

Exercise 9.12: Suppose $Y \subset \mathbb{P}^n$, $\langle F_1, ..., F_m \rangle = I(Y)$. To test $X \subset \mathbb{A}^n$ is nonsingular,

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} = (\deg F)F.$$

Theorem 9.13: Let X be a variety. Then the nonsingular loci is open and nonempty in X.

Proof: Reduce to X being affine. X is irreducible, $X \subseteq \mathbb{A}^n$, $V(f_1, ..., f_m) = X$. Then

$$\operatorname{rank}\left[\operatorname{Jac}_p(f_1, ..., f_m)\right] \geq n - \dim X$$

so $X_{\text{sing}} = V(\operatorname{codim} X + 1 \text{ minors of } \operatorname{Jac}) \cap X$. ■

Lemma 9.14: Any irreducible variety X is birational to an irreducible hypersurface $V(f) \subset \mathbb{A}^n$.

Proof: Take $K(X)/k$ which is separable and finitely generated and separably generated. So there exists $x_1, ..., x_d \in K(X)$ such that $K(X) \supset k(x_1, ..., x_d)$ where $d = \dim X$. Then

$$K(X) \simeq \frac{k(x_1, ..., x_d)[y]}{f(y)}$$

where the coefficients of f are in $x_1, ..., x_d$. ■

Definition 9.15: $\mathbb{P}$ irreducible smooth variety, X, Y irreducible subvarieties of $\mathbb{P}$, and $X \cap Y \supset Z$ an irreducible comp. We say X and Y intersect transversally at $p \in Z$ if

1) p is smooth on X and Y.
2) $\operatorname{codim}_{T_p\mathbb{P}} T_pX + \operatorname{codim}_{T_p\mathbb{P}} T_pY = \operatorname{codim}_{T_p\mathbb{P}}(T_pX \cap T_pY)$.

Lemma 9.16: If $X \pitchfork Y$ at $p \in Z$, then p is nonsingular on Z and $\operatorname{codim} Z = \operatorname{codim} X + \operatorname{codim} Y$.

Proof: Reduce to $\mathbb{P} = \mathbb{A}^n$. Then $I(X) = \langle f_1, ..., f_k \rangle$ and $I(Y) = \langle g_1, ..., g_\ell \rangle$. Then

$$\operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_k, g_1, ..., g_\ell) \leq \operatorname{codim} T_p Z = \operatorname{rank} \operatorname{Jac}_p(f_1, ..., f_k) + \operatorname{rank} \operatorname{Jac}_p(g_1, ..., g_\ell)$$
$$\operatorname{codim} X + \operatorname{codim} Y \leq \operatorname{codim} T_P Z \leq \operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

Therefore they are all equal. ■

Theorem 9.17 (Bertini): Fix $X \subseteq \mathbb{P}^n$. Then a general hyperplane $H \subseteq \mathbb{P}^n$ intersects X transversally at all nonsingular points in X.

Proof: Let $\Gamma = \{(x, H) \mid H \text{ not transversally intersecting } X \text{ at } x\} \subseteq X_{\mathrm{sm}} \times (\mathbb{P}^n)^\curlyvee$. We claim this is closed.

Fact 1: Let $X \twoheadrightarrow Y$ and general fiber has $\dim X - \dim Y$. Then

$$\dim \Gamma = \dim X + \operatorname{codim} X_1 = n - 1.$$

So then

$$\dim \pi_2(\Gamma) \leq n - 1$$

which shows us that $\pi_2(\Gamma) \subsetneq (\mathbb{P}^n)^\curlyvee$. Then another fact is that image of a morphism of varieties is constructible. ■

Remark: Let $f : X \to \mathbb{P}^n$ and X smooth. For general $H \subseteq \mathbb{P}^n$, is $f^{-1}(H)$ smooth? This is true for $\operatorname{char} = 0$ and false otherwise.

Remark: If $X \subseteq \mathbb{P}^n$ is smooth and irreducible with $\dim X > 1$. Then for general $H \subseteq \mathbb{P}^n$, $X \cap H$ is smooth, but is it irreducible?

This is true but very hard to prove.

10. Hilbert functions

Given $X \subseteq \mathbb{P}^n$, what is $\deg(X \subseteq \mathbb{P}^n)$?

Requirements:
1) $\deg(X \text{ finite}) = \# \, |X|$.
2) $\deg(V(F)) = \deg(F)$.
5) $(\deg X)(\deg Y) = \deg(X \cap Y)$ when X transversally intersects Y.
6) $\deg(I(X) + I(Y)) = \sum_{z \text{ irred comp } X \cap Y} \text{mult}_z(\deg z)$.

idk i cant read the rest of the board

Definition 10.1: Let $S = k[x_0, ..., x_n]$ and I a homogeneous ideal. The Hilbert function of the ideal I is defined as $h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ where

$$h_{I(d)} := \dim_k (S/I)_d.$$

For $X \subseteq \mathbb{P}^n$, $h_{X(d)} := h_{I(X)}(d)$.

Example 10.2:
1) $h_{\mathbb{P}^n}(d) = \binom{d+n}{n}$.
2) $h_{\text{pt}}(d) = 1$.
3) Suppose $V(I) = \emptyset$. Then $h_I(d) = 0$ for $d \gg 0$.
4) Let $I = \langle x_0^3, x_0^2 x_1 \rangle \subseteq k[x_0, x_1]$.

Lemma 10.3: Let $I, J \subseteq S$. Then $h_{I \cap J} + h_{I+J} = h_I + h_J$.

Proof: $0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I + J) \to 0$. The first map is $f \mapsto (f, f)$ and the second is $(f, g) \mapsto f - g$. ■

Lemma 10.4: Let $I \subseteq S$. Let be homogeneous at nonzero divisors on $(S/I)_d$ for $d \gg 0$. Then $h_{I+\langle f \rangle} = h_I(d) - h_I(d - \deg f)$.