

21-849: Algebraic Geometry

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January 13, 2025

I don't know what a sheave or a category is. ❤️

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1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.

- Both are continuous in \mathbb{R} or \mathbb{C} .
- Both are holomorphic in \mathbb{C} .
- Both are analytic (power series expansion at every point).
- Both are C^∞ .

There are differences as well.

- $f(z) = a$ has no solution or infinitely many solutions for e^z , but for almost all a , 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \rightarrow \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \rightarrow \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

Theorem 1.1 (GAGA Theorems): Compact (projective) \mathbb{C} -manifolds are algebraic.

Here are more cool things about algebraic geometry:

1) Enumeration:

- How many solutions to $p(z)$?
- How many points in $\{f(x, y) = g(x, y) = 0\}$?
- How many lines meet a given set of 4 general lines in \mathbb{C}^3 ? The answer is 2.
- How many conics ($\{f(x, y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
- Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.

2) Birationality:

- Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.

3) Arithmetic Geometry:

- Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
- Weil conjectures: X carved by polynomials with \mathbb{Z} -coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: \mathbb{k} is an algebraically closed field ($\mathbb{k} = \mathbb{C}$).

Definition 2.1 (Affine space): An n -affine space $\mathbb{A}_{\mathbb{k}}^n$ is the set

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{k}, \forall i = 1, \dots, n\} = \mathbb{k}^n.$$

An affine algebraic subset of \mathbb{A}^n is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \mathbb{k}[x_1, \dots, x_n]$. We write $Z = V(T)$.

Example 2.2 (An affine space):

- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \mathbb{k}\}$. Then note that if we project to any two dimensional plane (xy, yz, xz) , then we get another affine subset but on \mathbb{A}^2 .

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

SOLUTION: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin.

■

Definition 2.3 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \mathbb{k}[x] \mid f(p) = 0, \forall p \in Z\}.$$

Example 2.4:

- 0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
- 1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
- 2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.5:

- 1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
- 2) $T \subseteq \mathbb{k}[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, \dots, f_m)$ for some f_i .
- 3) For $\mathfrak{a} \subseteq \mathbb{k}[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \mathbb{k}[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
- 4) Algebraic subsets of \mathcal{A}^n are closed under finite unions and arbitrary intersections.

PROOF: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\mathbb{k}[x]$ is Noetherian.

■

Theorem 2.6 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{k}[x]\}.$$

PROOF:

- 1) Finite type field extensions $L \supseteq F$ are finite. Remember that finite type means that $F[x_1, \dots, x_m] \twoheadrightarrow L$.
- 2) This implies that maximal ideals of $\mathbb{k}[x]$ are of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ for $a_i \in \mathbb{k}$, using the fact that \mathbb{k} is algebraically closed. So, $\mathbb{k}[x]/\mathfrak{m} \simeq \mathbb{k}$.
- 3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq \mathbb{k}[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
- 4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq \mathbb{k}[x, y]$. So for any (a_1, \dots, a_n, b) that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in \mathbb{k}[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some ℓ .

But also

$$\mathbb{k}[x, y]/\langle 1 - yf \rangle \simeq \mathbb{k}[x] \left[\frac{1}{f} \right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \dots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal \mathfrak{a} inside R .

■

Remark: We say R is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

Theorem 2.7: R Jacobson $\Rightarrow R[x]$ Jacobson.

Definition 2.8 (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\mathbb{k}[x]/I(X)$.

1) $X \xrightarrow{f} \mathbb{k}$

2) $\text{maxSpec } A(X) = \{\text{maximal ideals in } A(X)\} = X$.

3. Projective Spaces

Definition 3.1: $\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus \{0\}) / \sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \mathbb{k}$. That is, $\mathbb{P}^n = \{1\text{-subspaces of } \mathbb{k}^{\{n+1\}}\}$. For $(a_0, \dots, a_n) \in \mathbb{k}^{n+1} \setminus \{0\}$, we write $[a_0 : \dots : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \mathbb{k}^{n+1}$. $\mathbb{P}V = V \setminus \{0\} / \sim$

Definition 3.2: $f \in \mathbb{k}[\underline{x}]$ is homogeneous if $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\ell f(x_1, \dots, x_n)$.

Definition 3.3: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \dots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for T a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

Example 3.4: Let $W \subseteq \mathbb{k}^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an m -plane in \mathbb{P}^n .

Example 3.5 (Twisted cubic curve): We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2t : st^2 : t^3] \mid [s : t] \in \mathbb{P}^1\}$. Then we have that $C = V(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix A , $\text{rank}(A) \leq r \iff \text{all } (r+1) \times (r+1) \text{ minors} = 0$.

Question: Can there exist F, G such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \mathbb{k}[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of X .

Exercise 3.6:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow \{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \mathbb{k}[\underline{x}] \text{ such that } \mathfrak{a} \neq \mathbb{k}[\underline{x}] \text{ or } \langle x_0, \dots, x_n \rangle\}.$$

This last part is called the “irrelevant ideal”.

Definition 3.7 (General Position): In \mathbb{P}^n , any subset of size $\leq n + 1$ points are linearly independent.

Theorem 3.8: Every set Γ of $2n$ points in \mathbb{P}^n in general position is carved out by quadrics.

PROOF: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose q is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\text{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \text{span}(\Gamma_1)$ or $q \in \text{span}(\Gamma_2)$.

Let p_1, \dots, p_k be a minimal subset of Γ whose span $\ni q$ ($k \leq n$). Now pick any Λ such that $|\Lambda| = n - k + 1$ which does not contain any of the p_i . We claim that $q \notin \text{span}(p_2, \dots, p_k, \Lambda)$.

We then conclude that for any $|S| = n - 1$, $S \subseteq \Gamma \setminus p_1, \dots, p_k$, we have that $\text{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \text{span}(p, S)$$

is the intersection at least n many hyperplanes, each of them containing p_1, q . But the intersection of n many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ■

Definition 3.9: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X$, $\exists g \in PGL_{n+1}$.

Proposition 3.10: Let (M_0, \dots, M_3) be any \mathbb{k} -basis of

$$\mathbb{k}[s, t]_3 = \{f \in \mathbb{k}[s, t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s, t) : \dots : M_3(s, t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2t : st^2 : t^3]\}$.

Example 3.11 (Rational normal curve): Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$. Or we could map it to any basis of $\mathbb{k}[s : t]_n$.

Exercise 3.12: $I(\varphi(\mathbb{P}^1)) = ?$.

Example 3.13: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

Fact: If we pick any 3 linearly independent $M_0, M_1, M_2 \in \mathbb{k}[s, t]_3$. Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ by M_0, M_1, M_2 has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ using 4 elements from $\mathbb{k}[s, t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

Question: Are all twisted quartic curves projectively equivalent?

SOLUTION: No. In fact, there are infinitely many distinct families. ■

Question (Hartshorne's Question): Is every irreducible curve in \mathbb{P}^3 carved out by 2 equations?

4. The Zariski Topology

Definition 4.1 (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \mathbb{k}[\underline{x}]\}$ form the closed sets of a topology on \mathbb{A}^n called the Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

Example 4.2: Take \mathbb{A}^1 . Two closed subsets are \mathbb{A}^1 and \emptyset . The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

Definition 4.3: A topological space X is irreducible $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

By definition, we will also say that irreducible implies nonempty.

Remark:

- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

Proposition 4.4: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. X is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

PROOF:

- \Rightarrow : Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If X is irreducible, then at least one of them completely contains X . That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means f or $g \in I(X)$.
- \Leftarrow : Suppose for sake of contradiction that X is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■

Remark: When people say affine variety, some people mean that it is also irreducible. But for us, affine variety is the same thing as affine algebraic set.

Then a quasi-affine variety is an open subset of an affine variety.

Example 4.5:

- 1) \mathbb{A}^n is irreducible. ($\mathbb{k}[\underline{x}]$ domain)
- 2) $V(x^2 + y^2) \subset \mathbb{A}^2$ is reducible ($\text{char } \mathbb{k} \neq 2$)
- 3) Let $f \in \mathbb{k}[\underline{x}]$ be square-free ($f = f_1 \dots f_\ell$ irreducible). Then $V(f)$ is irreducible if and only if f is irreducible.
- 4) $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then $A(X) = \frac{\mathbb{k}[x, y, z]}{\langle x^2 - yz \rangle}$. This is irreducible due to Eisenstein's on f . Now if we take $f \in A(X)$ and look at $V_X(f) \subset X$ is irreducible $\Leftrightarrow f$ irreducible element in $A(X)$.

Definition 4.6: A topological space X is Noetherian if $\nexists X \supseteq Y_0 \supsetneq Y_1 \supsetneq \dots$ such that each Y_i is closed.

Proposition 4.7: An affine variety is Noetherian. (Because $A(X)$ is Noetherian).

Theorem 4.8: A Noetherian topological space X is uniquely a finite union of maximal irreducible closed subsets.

PROOF: Consider

{nonempty closed subsets of X that does not admit a decomposition into irreducible closed subsets.}.

Suppose it is nonempty. Then it has a minimal element Y . Y is not irreducible, so $Y = Y_1 \cup Y_2$ (both proper and closed). Since Y is minimal, Y_1 and Y_2 both have decompositions into irreducible closed subsets. So if we just union those decompositions, then we contradict Y 's membership in the set. As such, the original set must have actually been empty.

Uniqueness and maximality are left as an exercise. ■

Proposition 4.9:

- 1) X irreducible and $U \subseteq X$ open. Then $\overline{U} = X$.
- 2) $V \subseteq X$, V irreducible $\implies \overline{V}$ irreducible.
- 3) $f : X \rightarrow Y$ continuous. Image of irreducible set under f is irreducible. (Irreducibility is a topological property).

Example 4.10: Let's have $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^m$ by $\varphi(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x}))$ for some $f_1, \dots, f_m \in \mathbb{k}[\underline{x}]$. Then $\text{im}(\varphi)$ is irreducible. It is left to show that φ is a continuous map.

Definition 4.11: Let X be a nonempty topological space.

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

Then let $Y \subseteq X$ closed irreducible subset.

$$\operatorname{codim}_X Y := \sup\{n \mid \exists Y \subseteq Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

Example 4.12:

- 1) $\dim \mathbb{A}^1 = 1$.
- 2) $X = V(xz, yz) \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Let p be a point on the axis not touching the x - y plane. Then let q be the origin. We have that $\operatorname{codim}_X p = 1$ and $\operatorname{codim}_X q = 2$. Also $\dim p = \dim q = 0$.

Definition 4.13: Height of a prime $\mathfrak{p} \subset R$ is

$$\operatorname{ht} \mathfrak{p} := \sup\{n \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n, \text{ each } \mathfrak{p}_i \text{ prime}\}.$$

Then Krull dimension of R is

$$\dim R := \sup\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ prime}\}.$$

Definition 4.14: For an ideal I , we have that

$$\operatorname{ht} I := \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}.$$

“inf of sup”.

From these, we can basically show from definition that

$$\operatorname{ht} I + \dim R/I \leq \dim R.$$

The $<$ case is possible if R is not a domain. For example, if we have that $R = \mathbb{k}[x, y, z]/\langle xz, yz \rangle$ and then $I = \langle x, y, z - 1 \rangle$.

But the $<$ case is also possible even if R is a domain and I prime.

Before we cover the next theorem, we note that

$$\{\text{minimal primes over } I\} = \{\mathfrak{p} \text{ prime } \mathfrak{p} \supseteq I, \text{ and } \nexists \mathfrak{p} \supsetneq \mathfrak{q} \supseteq I, \text{ prime } \mathfrak{q}\}$$

Theorem 4.15 (Krull Principal Ideal Theorem / Height Theorem): Let R be a Noetherian ring and $f_1, \dots, f_c \in R$.

- 1) Minimal primes over $\langle f_1 \rangle$ have height ≤ 1 . And the height is equal to 1 if f_1 is nonzerodivisor and nonunit.
- 2) Minimal primes over $\langle f_1, \dots, f_c \rangle$ have height $\leq c$.

“We could do this proof, but it’s like proving that there exists a complete ordered field satisfying the least upper bound property.”

Theorem 4.16: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ irreducible affine varieties.

- 1) $\dim(X \times Y) = \dim X + \dim Y$.
- 2) If $Y \subseteq X$, then $\dim Y + \operatorname{codim}_X Y = \dim X$.

Remark (Noether normalization): For $X \subseteq \mathbb{A}^n$ irreducible affine variety. There exists $y_1, \dots, y_d \in A(X)$ such that $\mathbb{k}[Y_1, \dots, Y_d] \rightarrow A(X)$ with $Y_i \mapsto y_i$ which is a finite extension (injective and $A(X)$ is finitely generated $\mathbb{k}[Y]$ -module) and $d = \dim X$.

Corollary 4.17:

- 1) $\dim \mathbb{A}^n = n$.
- 2) $X \subseteq \mathbb{A}^n$ irreducible affine variety. $0 \neq f \in A(X)$ non unit. Then $V_X(f) = V(f) \cap X$ has dimension $\dim X - 1$.

Exercise 4.18: Let $U \subseteq X$ be open for X affine variety irreducible. Then $\dim U = \dim X$.

Proposition 4.19: Let R be Noetherian domain. Then R UFD \iff every ht = 1 prime is principal.

PROOF: R being a UFD implies that \mathfrak{p} has height 1. So let $f = f_1, \dots, f_\ell \in \mathfrak{p}$. Suppose $f_1 \in \mathfrak{p}$. So then $0 \neq \langle f_1 \rangle \subseteq \mathfrak{p}$. But as $\operatorname{ht} \mathfrak{p} = 1$, we have that $\langle f_1 \rangle = \mathfrak{p}$.

Conversely, we need to show that irreducible implies prime. That is, recall that (ACCP + irreducible = prime) implies that we have a UFD.

So let $f \in \text{irred}$. Krull’s PIT says $\langle f \rangle \subseteq \mathfrak{p}$ where \mathfrak{p} has height 1. So by definition, $\mathfrak{p} = \langle g \rangle$, but $\langle f \rangle \subseteq \langle g \rangle$ implies that $f = g$ because f is irreducible. ■

Example 4.20: Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then let $Y = V(x, y) \subseteq X \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Then $\dim Y = 1$. So can we find f such that $\langle f, x^2 - yz \rangle = I(Y)$? The answer to this is no.

But can we find f such that $\sqrt{\langle f, x^2 - yz \rangle} = \langle x, y \rangle$? Take $f = y$ and analyze $\langle y, x^2 - yz \rangle$. This is the same as $\langle y, x^2 \rangle$, whose radical is $\langle x, y \rangle$ as we desire.

Example 4.21: Now consider $X = V(xw - yz) \subseteq \mathbb{A}^4$. $\dim X = 3$ and let $Y = V(x, y)$. Now does there exist f such that $\sqrt{\langle f, xw - yz \rangle} = \langle x, y \rangle$?

This is false, but we don't have the tools to prove it.

Definition 4.22: Zariski topology on \mathbb{P}^n has projective algebraic sets as its closed subsets.

Two ways: projective varieties \rightarrow affine varieties.

1) $U_i = \{x_i \neq 0\} = \{[x_0 : \dots : x_i = 1 : \dots : x_n]\} \simeq \mathbb{A}^n$.

Proposition 4.23: $\forall i = 0, \dots, n$, say $i = 0$, $\mathbb{A}^n \rightarrow U_0, (x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$ is a homeomorphism.

PROOF:

- *Homogenization:* let $f \in \mathbb{k}[x_1, \dots, x_n]$. Then we have

$$f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{k}[x_1, \dots, x_n].$$

If $Z = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n$, $\varphi(Z) = U_0 \cap V(f_1^h, \dots, f_m^h)$ is closed.

If $Z' = V(F_1, \dots, F_\ell) \cap U_0$, then $\varphi(Z') = V(F_1(1, x_2, \dots, x_n), \dots, F_\ell(1, x_2, \dots, x_n))$.

Now $U_0 \cup \dots \cup U_n = \mathbb{P}^n$.

■

Exercise 4.24: Let $Y \subseteq \mathbb{A}^n \simeq U_0$ be an affine variety. $\overline{Y} = V(?)$. Suppose $V(f_1, \dots, f_m) = Y$. It is tempting to say $\overline{Y} = V(f_1^h, \dots, f_m^h)$.

Corollary 4.25:

- 1) $\dim \mathbb{P}^n = n$.
- 2) If $H_i = V(x_i) \subseteq \mathbb{P}^n$ does not contain any irreducible components of $Y \subseteq \mathbb{P}^n$, then $\dim Y = \dim Y \cap U_i$.

Definition 4.26: Let $Y \subseteq \mathbb{P}^n$ be a projective variety. The affine cone $\hat{Y} = C(Y)$ is

$$\theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}$$

where

$$\theta : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n.$$

Proposition 4.27:

- 1) $\hat{Y} = V(I(Y))$. In fact, $I(\hat{Y}) = I(Y)$.
- 2) $\dim \hat{Y} = \dim Y + 1$.
- 3) \hat{Y} is irreducible if and only if Y is irreducible.

Theorem 4.28: If $X, Y \subseteq \mathbb{P}^n$ are projective varieties and $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

Lemma 4.29: If $X, Y \subseteq \mathbb{A}^n$ affine varieties, then $X \cap Y = \emptyset$ or every irreducible component of $X \cap Y$ has $\dim \geq \dim X + \dim Y - n$.

PROOF: Let $\Delta = V(x_1 - y_1, \dots, x_n - y_n) \subseteq \mathbb{A}^{n+n}$. Note that

$$X \times Y \cap \Delta \simeq X \cap Y.$$

So, $\dim(X \times Y \cap \Delta) \geq \dim X + \dim Y - n$ by Krull's height theorem.

If $\underline{a} = (a_1, \dots, a_n)$ are varieties, then $I_{\underline{a}(X)} = \{f(\underline{a} \mid f \in I(X)\}$. Then,

$$A(X \cap Y) = \frac{\mathbb{k}[\underline{z}]}{\sqrt{\langle I_{z(X)} + I_{z(Y)} \rangle}}$$

and

$$A(X \times Y \cap \Delta) = \frac{\mathbb{k}[\underline{x}, \underline{y}]}{\sqrt{\langle I_{x(X)} + I_{y(Y)} + I(A) \rangle}}.$$

So this implies that $x_i = y_i$ for all i , meaning they are isomorphic rings. ■

PROOF of Theorem 4.28: X, Y irreducible implies that \hat{X} and \hat{Y} are irreducible. So, then

$$\dim(\hat{X} \cap \hat{Y}) \geq \dim X + 1 + \dim Y + 1 - (n + 1) \geq \dim X + \dim Y - n + 1.$$

$\hat{X} \cap \hat{Y}$ contains origin by construction, but it has at least one other point because dimension. ■

5. Morphisms

Definition 5.1: For $U \subseteq \mathbb{R}^n$, $U' \subseteq \mathbb{R}^m$ open, $\varphi : U \rightarrow U'$ is continuous/continuously differentiable/smooth if $f \circ \varphi$ is smooth for any smooth $f : U' \rightarrow \mathbb{R}$.

$f' : U' \rightarrow \mathbb{R}$ is smooth if f is smooth at every point $p \in U'$.

Definition 5.2: For affine variety $X \subseteq \mathbb{A}^n$ and $U \subseteq X$ open, a function $\varphi : U \rightarrow \mathbb{k} = \mathbb{A}^1$ is regular if $\forall p \in U$, $\exists U_p \ni p$ open and $f_p, g_p \in A(X)$ such that $\varphi(x) = \frac{f_p(x)}{g_p(x)}$ for all $x \in U_p$. In particular, $g_p(x) \neq 0$ for all $x \in U_p$.

$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$. This is also a \mathbb{k} -algebra.

Example 5.3: Let $U \subseteq X$, $\varphi : U \rightarrow \mathbb{A}^1$ regular $\nRightarrow \varphi = \frac{f}{g}$ globally for some $f, g \in A(X)$.

$X = V(xw - yz) \subset \mathbb{A}^4$, $U = X \setminus V(x, y)$.

$$\varphi(x, y, z, w) = \begin{cases} \frac{z}{x} & \text{if } x \neq 0 \\ \frac{w}{y} & \text{if } y \neq 0 \end{cases}$$

Lemma 5.4: $\varphi : U \rightarrow \mathbb{A}^1$ regular, then $V(\varphi) = \{x \in U \mid \varphi(x) = 0\}$ is closed in U . In particular φ is continuous.

PROOF: Closedness is a local condition, and around any $p \in U$, $\{\varphi \upharpoonright U_p : 0\} = V(f_p) \cap U_p$. ■

Remark: If $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ for U irreducible, and $\varphi_1 \upharpoonright U' = \varphi_2 \upharpoonright U'$ for some $\emptyset \neq U' \subseteq U$, then $\varphi_1 = \varphi_2$.

Definition 5.5: $X \subseteq \mathbb{A}^n$ affine variety. A distinguished open subset U of X is an open subset of the form $X \setminus V(f)$ for some $f \in A(X)$, denoted $D(f)$, D_f , U_f , X_f . X_f is probably the most descriptive as it actually mentions X .

Remark: $\{D(f)\}_{f \in A(X)}$ form a basis for Zariski topology. What that means is that any $U \subseteq X$ is a union for $D(f)$'s.

Exercise 5.6: $D(f)$ is homeomorphic to $V(I(X) + \langle 1 - yf \rangle) \subseteq \mathbb{A}^{n+1}$.

Theorem 5.7: $\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} \mid g \in A(X), m \in \mathbb{Z}_{\geq 0} \right\}$. In fact, $\mathcal{O}_X(D(f)) = A(X)_f$.

Example 5.8: $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = A(\mathbb{A}^2) = \mathbb{k}[x, y]$. Then,

$$\mathbb{A}^2 \ni \varphi = \begin{cases} \frac{f}{x^m} \text{ for some } f \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(x) \\ \frac{g}{y^\ell} \text{ for some } g \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(y) \end{cases}.$$

Then we say $y^\ell f = x^m g$ on \mathbb{A}^2 . Because we are in a UFD, this means that $x^m \mid f$ and $y^\ell \mid g$. But this implies $m = \ell = 0$, so $f = g = \varphi$.

PROOF of Theorem 5.7: \supseteq is clear. So we only prove the \subseteq case.

Suppose we have $\varphi \in \mathcal{O}_X(D(f))$. Then for all $p \in D(f)$, $\exists U_p \ni p$ and $\varphi \upharpoonright U_p = \frac{g_{p'}}{f_{p'}}$ for $g_{p'}, f_{p'} \in A(X)$.

Take a nonempty $D(h_p) \subseteq U_p$ and write $g_p = g_{p'} h_p$ and $f_p = f_{p'} h_p$.

Then $\varphi \upharpoonright D(f_p) = \frac{g_p}{f_p} = \frac{g_{p'} f_p}{f_p^2}$. So assume $g_p = 0$ on $V(f_p)$.

Now we claim that $\forall p, q \in D(f)$, we have $g_p f_q = g_q f_p$ in $A(X)$.

Then $D(f) = \bigcup_p D(f_p)$. Then $V(f) = \bigcap_p V(f_p)$. Nullstellensatz says that $\sqrt{\langle f \rangle} = \sqrt{\langle f_p : p \in D(f) \rangle}$ as ideals in $A(X)$. But then, $f^m = \sum k_p f_p$. By Noetherian-ness, this is a finite sum. We claim that $g = \sum k_p f_p$.

Then $\frac{g}{f^m} = \frac{g_q}{f_q^m}$ on $D(f_q)$ for all $q \in D(f)$. So,

$$g f_q = \sum_p k_p g_p f_q = \sum_p k_p f_p g_q = g_q f^m.$$

■

6. Sheaves

Let \mathcal{A} be a category: AbGrp, Rings, \mathbb{k} -algebras. Given a topological space X , $\text{Top}(X)$ is a category where the objects are open subsets $U \subseteq X$ and morphisms are inclusions between $U \subseteq V$ open subsets.

Definition 6.1: A presheaf (with values in \mathcal{A}) on X is a contravariant functor $\mathcal{F} : \text{Top}(X) \rightarrow \mathcal{A}$.

\mathcal{F} is further a sheaf if for every open cover $\{U_i\}_i$ of any open subset $U \subseteq X$ if

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

Translation:

- 1) Assignment $U \mapsto \mathcal{F}(U) \in \text{obj}(\mathcal{A})$ such that $\forall U \subseteq V \subseteq X$ open,

$$\text{res}_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$$

such that $\text{res}_{U,U} = \text{id}$ and $\text{res}(V, U) \circ \text{res}(W, V) = \text{res}(W, U)$.

- 2) If $(f_i)_i \in \prod_i \mathcal{F}(U_i)$ such that $\text{res}_{U_i, U_i \cap U_j}(f_i) = \text{res}_{U_j, U_i \cap U_j}(f_j)$, then $\exists! f \in \mathcal{F}(U)$ such that $\text{res}_{U, U_i}(f) = f_i$. Also $\mathcal{F}(\emptyset) = 0$ as a consequence.

$f \upharpoonright V := \text{res}_{U,V}(f)$, $f \in \mathcal{F}(U)$. $\mathcal{F}(U)$ elements are called sections of \mathcal{F} over U .

Example 6.2: Note that throughout these examples, X is a topological space and $U \subseteq X$.

- 1) $\mathcal{F}_{\text{ct}}(U) := \{\varphi : U \rightarrow \mathbb{R}\}$. Then if $U' \subseteq U$, $\text{res}_{U,U'}(f) := f \upharpoonright U'$.
- 2) $C(U) := \{\varphi : U \rightarrow \mathbb{R} \text{ cts}\}$.
- 3) $C^\infty(U) := \{\varphi : U \rightarrow \mathbb{R} \text{ smooth}\}$ ($X \subseteq \mathbb{R}^n$ open).
- 4) $\underline{\mathbb{R}}(U) := \{\varphi : U \rightarrow \mathbb{R}, \text{ constant}\}$. This is not a sheaf. If we consider a constant function that takes the value a on U and b on U' , then there is no value c such that they can be glued together to be equal on both sets.
- 5) $\mathcal{O}_{X(U)} := \{\varphi : U \rightarrow \mathbb{k} \text{ regular}\}$

Remark: A constant sheaf $A_{X(U \text{ conn})} = A$. A locally constant sheaf (locally $\mathcal{F} \upharpoonright U$ is constant). Locally constant does not imply constant.

Definition 6.3: $U \subseteq X$, \mathcal{F} sheaf on X , $\mathcal{F} \upharpoonright U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 6.4: \mathcal{F} sheaf on X . $p \in X$. The stalk of \mathcal{F} at p , $\mathcal{F}_p := \lim_{U \ni p} \mathcal{F}(U)$. This is actually just equal to $\{(U, f) \mid U \ni p, f \in \mathcal{F}(U) / \sim\}$ where $(U_1, f_1) \sim (U_2, f_2)$ if $\exists V \ni p$ such that $f_1 \upharpoonright V = f_2 \upharpoonright V$.

Remember that $\mathcal{F}ct_S(U) = \{f : U \rightarrow S\}$ and $C(U) = \{f : U \rightarrow R \text{ cts}\}$.

Remark: $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ by $'\text{res}_{V,U}$. This map need not be surjective.

Theorem 6.5: Let X be an affine variety and $x \in X$. $\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_X}$.

PROOF: Consider the ring map $A(X)_{\mathfrak{m}_x} \rightarrow \mathcal{O}_{X,x}$ where we map $\frac{f}{g} \mapsto \frac{f}{g}$.

Now if $\frac{f}{g} = \frac{f'}{g'}$ in $A(X)_{\mathfrak{m}_x}$, then we need to check that the same is true around x in $\mathcal{O}_{X,x}$.

Now if $\frac{f}{g}$ is 0 around x (in $D(h)$), we can deduce that $\frac{f}{g} = 0$ in $A(X)_{\mathfrak{m}_x}$. ■

Definition 6.6: A ringed space (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf on X with values in Ring. We call \mathcal{O}_X the structure sheaf of this ringed space.

Definition 6.7: $f : X \rightarrow Y$ continuous and \mathcal{F} a sheaf on X .

$$\text{pushforward } f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

where $V \supset Y$ is open.

Definition 6.8: Let \mathcal{F} and \mathcal{G} be sheaves on X . $\Phi : \mathcal{F} \rightarrow \mathcal{G}$ means that for each $U \subseteq X$, we specify $\Phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ where for $U \subseteq V$, we have the following diagram commuting:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi(U)} & \mathcal{G}(U) \\ \uparrow \text{res} & & \uparrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{\Phi(V)} & \mathcal{G}(V) \end{array}$$

where $V \supset U$ is open.

Definition 6.9: A morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \rightarrow Y$ is continuous and $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$.

Example 6.10: $(U \subseteq \mathbb{R}^n, C^1) \rightarrow (V \subseteq \mathbb{R}^m, C^1)$.

Remark: When we say (X, \mathcal{O}_X) , we mean that \mathcal{O}_X is a subsheaf of $\mathcal{Fct}_{\mathbb{k}}$ for some fixed \mathbb{k} . Then $\mathcal{O}_X = \{\varphi : U \rightarrow \mathbb{k} \mid \varphi \text{ satisfies some condition}\}$. And, $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is always given by precomposition.

6.1. Category of quasi Affine varieties “qAffVar”.

To define this category, we'll have the objects be open subsets of an affine variety considered as a ringed space. The morphisms will be maps of ringed spaces with $f^\#$ being the precomposition as our convention above dictates.

Theorem 6.11: X, Y both affine varieties. $U \subseteq X$ open. Then there exists a natural bijection.

$$\text{Mor}(U, Y) \simeq \text{Hom}_{\mathbb{k}\text{-alg}}(A(Y), \mathcal{O}(U)).$$

7. Projective morphisms

7.0.1. Ok unfortunately i was forced to miss two classes so there is gap here

Proposition 7.1: Suppose X, Y are prevarieties with affine covers $\{U_i\}$ and $\{V_j\}$ respectively. Then $X \times Y$ is a product in the category of prevarieties constructed by gluing together $U_i \times V_j$ and $U_{i'} \times V_{j'}$,

$$(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$$

for all such pairs.

We did not prove that gluing gives you a prevariety, but we will believe it. Also note that $X \times Y$ is a prevariety by affine cover $\{U_i \times V_j\}$.

Proposition 7.2: For $Y \subseteq X$ closed where $\iota : Y \rightarrow X$ and X is a prevariety. Then for $U' \subseteq Y$ open,

$$\iota^* \mathcal{O}_X(U') = \{f : U' \rightarrow \mathbb{k} \mid \forall y \in U', \exists U_y \subset X \text{ with } \varphi \in \mathcal{O}_X(U_y) \text{ such that } f|_{U_y \cap Y} = \varphi\}.$$

Then $\iota^* \mathcal{O}_X$ is a sheaf and $(Y, \iota^* \mathcal{O}_X)$ is a prevariety, and $\iota : Y \rightarrow X$ is a morphism.

PROOF: We will believe that $\iota^* \mathcal{O}_X$ is a sheaf. Also ι is a morphism of prevarieties.

Let $U \subseteq X$ be affine open. We claim that $(U \cap Y, \iota^* \mathcal{O}_X|_{U \cap Y})$ is affine. We claim that $Y = V(I)$ for some $I \subseteq A(X)$.

Then, $\iota^* \mathcal{O}_X$ are the functions that are locally restrictions of regular functions on X . Then $\mathcal{O}_{V(I)}$ are functions that are locally quotient of polynomials on \mathbb{A}^n . These are equal. ■

We say that ι is a closed embedding.

Example 7.3:

- 1) $\mathbb{A}^2 \rightarrow \mathbb{A}^2$ via $(x, y) \mapsto (x, xy)$. This maps \mathbb{A}^2 to itself without the y -axis, but still including the origin. Note that the image of this map is neither open nor closed in \mathbb{A}^2 .

Remark: $Y \subseteq X$ is locally closed if it is $U \cap V$ where U is open in X and V is closed in X .

This example is not even locally closed.

- 2) Glue \mathbb{A}^1 and \mathbb{A}^1 along $\mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus \{0\}$ via the identity. This is basically a line with two origins. Let this line be called X . Consider $g : X \rightarrow X$ via switching the origins and keeping the other points the same. This is a morphism where our open subsets are lines including only one

of the origins, and it is not hard to check that this is actually a morphism. Then, $\{g(x) = x\} \simeq \mathbb{A}^1 \setminus \{0\}$ which is not closed in X .

Definition 7.4: A prevariety of X is a variety of the diagonal map $\Delta : X \rightarrow X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

Lemma 7.5: For checking if something is a variety, Δ being a topologically closed embedding is a sufficient condition.

Corollary 7.6: Open and closed subprevarieties of varieties are varieties.

Lemma 7.7: Let X and Y be affine varieties and $f : X \rightarrow Y$. If $f^\# : A(Y) \twoheadrightarrow A(X)$, then f is a closed embedding.

PROOF: Let J be the kernel. And consider the surjective map

$$\frac{k[\underline{y}]}{I(Y)} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}.$$

This is also surjective onto $\frac{k[\underline{y}]}{I(Y)+J}$. This is $V_Y(J)$, so we get that $A(V_Y(J)) \simeq A(X)$ as desired. ■

Lemma 7.8: Let X be an affine variety, then $\Delta : X \rightarrow X \times X$ defined by $x \mapsto (x, x)$ is a closed embedding.

PROOF: $A(X) = \frac{k[\underline{x}]}{I(X)}$, $A(X \times X) \simeq A(X) \otimes A(X)$. Then we can map

$$\frac{k[\underline{x}, \underline{y}]}{I_{x(X)} + I_{y(Y)}} \twoheadrightarrow \frac{k[\underline{x}]}{I(X)}$$

via $x_i \mapsto x_i$ and $y_i \mapsto x_j$. ■

Proposition 7.9: X prevariety is a variety if $\Delta(X) \subseteq X \times X$ is closed.

PROOF: We claim that $\forall x \in X$, take any $x \in U \subset X$ affine open. Then $\Delta \upharpoonright U : U \rightarrow U \times U$ is a closed embedding.

Since $\Delta(X) \subseteq X \times X$ is closed (topologically closed embedding, locally closed embedding implies closed embedding). So $\mathcal{O}_X \simeq \mathcal{O}_{\Delta(X)}$.

It is important that $\Delta \upharpoonright U$ is closed because of the following:

$Y \subseteq X$ such that $\forall y \in Y, \exists U_y \subset X$ containing y such that $U_y \cap Y$ is closed in U_y does not imply that $Y \subseteq X$ is closed. However, under our assumption, it would be that way. ■

Corollary 7.10: qAffVar are varieties.

Corollary 7.11: $X \xrightarrow{f} Y$ morphism of varieties. The graph $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

PROOF: $X \times Y \xrightarrow{f \times \text{id}} Y \times Y$. Then $(f \circ \text{id})^{-1}(\Delta(Y)) = \Gamma_f$. ■

Exercise 7.12: Let X be a variety and $U, U' \subseteq X$ open affine subsets. Then $U \cap U'$ is also affine open.

Definition 7.13: Let $X \subseteq \mathbb{P}^n$ be a projective algebraic subset. $\mathcal{O}_X(U) = \{\varphi : U \rightarrow \mathbb{k} \mid \text{locally encrypted } (\varphi \upharpoonright U') = \frac{F}{G} \text{ for some homogeneous poly } F, G \text{ of same degree.}\}$.

Proposition 7.14: $X \subseteq \mathbb{P}^n$ projected algebraic set $\leadsto (X, \mathcal{O}_X)$ if for all $i = 0, 1, \dots, n$, let $U_i = \mathbb{P}^n \setminus V(x_i) \xrightarrow{\text{homeo}} \mathbb{A}^n$. Then $(X \cap U_i, \mathcal{O}_X \upharpoonright U_i)$ is isomorphic as a ringed space to $X \cap U_i$ considered as a closed subset in \mathbb{A}^n under $\mathbb{A}^n \simeq U_i$ and hence an affine variety. In particular, X is a prevariety.

PROOF: $X = V(F_1, \dots, F_m)$. Then $X \cap U_i \subset \mathbb{A}^n$ is $V(F_1(x_0, \dots, x_i = 1, \dots, x_n), \dots, F_m)$.

If we have $\frac{F}{G} \in (X \cap U_i, \mathcal{O}_X \upharpoonright U_i)$, then we can just dehomogenize to get $\frac{f}{g} \in (X \cap U_i, \mathcal{O}_{X \cap U_i \subseteq \mathbb{A}^n})$. ■

Lemma 7.15: $X \subset \mathbb{P}^n$ a projective variety. Let F_0, \dots, F_m be homogeneous polynomials on \mathbb{P}^n of same degree. Then $F : X \setminus V(F_0, \dots, F_m) \rightarrow \mathbb{P}^m$ by $x \mapsto (F_0(x), \dots, F_m(x))$ is a morphism.

PROOF: As a set map, this is well-defined. We will verify that $\forall j = 0, \dots, m$, the distinguished open subset $U_j = \{[\underline{Y}] \in \mathbb{P}^m \mid Y_j \neq 0\}$. We have

$$F^{-1}(U_j) = \mathbb{P}^n \setminus V(F_j).$$

Then $F^{-1}(U_j) \rightarrow U_j$ where $U_j \simeq \mathbb{A}^m$. We'll call the coordinates of \mathbb{A}^m by $\left(\frac{Y_0}{Y_j}, \dots, \frac{Y_m}{Y_j}\right)$. So if we have a point $x \in F^{-1}(U_j)$, then the associated point in \mathbb{A}^m would be attained by sending

$$x \mapsto \left(\frac{F_0(x)}{F_j(x)}, \dots, \frac{F_m(x)}{F_j(x)}\right).$$

We showed that if we had a map from a quasi affine variety $W \rightarrow X$ where X is an affine variety, we just had to map $A(X) \rightarrow \mathcal{O}(W)$. In an exercise, we showed that you can replace “quasi affine variety” with “prevariety” and get the same result. ■

Example 7.16:

- 1) $\mathbb{P}^1 \rightarrow \mathbb{P}^n$ where we map $[s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$. We know this will be a morphism by our lemma as long as we verify that it is full of zeroes only when $s = t = 0$, but this is clear.

We can also map $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ by $[s : t] \mapsto [s^3 : s^2t : t^3]$. This is because it maps nicely as above to $[s^3 : s^2t : st^2 : t^3]$, then we can project to drop the third coordinate to get the map we are describing. We are left to show that projections are morphisms

- 2) Projections: $\mathbb{P}^n \setminus \{[1 : 0 : \dots : 0]\} \rightarrow \mathbb{P}^{n-1}$ by mapping

$$[x_0 : \dots : x_n] \mapsto [x_1 : \dots : x_n].$$

More formally, we can consider $\mathbb{P}V \setminus \{[v]\} \rightarrow \mathbb{P}\left(\frac{V}{\text{span}(v)}\right)$, or $\mathbb{P}V \setminus \mathbb{P}W \rightarrow \mathbb{P}(V/W)$ where $W \subset V$.

So, the second example above becomes $\mathbb{P}^3 \setminus [0 : 0 : 1 : 0] \rightarrow \mathbb{P}^2$.

- 3) Veronese embedding.

$$\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$

by $[x] \mapsto [\text{every monomial of } x \text{ of degree } d]$.

Exercise 7.17: ν_d is a closed embedding.

- 4) Segre embedding:

$$\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)}$$

by $([X], [Y]) \mapsto \begin{pmatrix} x_0y_0 & \dots & x_0y_m \\ \vdots & & \vdots \\ x_ny_0 & \dots & x_ny_m \end{pmatrix}$ where this really should just be one long vector, but it is easier to represent as such. We will prove that this is a closed embedding.

PROOF: Fix some $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we have

$$\begin{aligned} U_{ij} &\simeq \mathbb{A}^{mn+m+n} \\ &= \{[z_{ab}] \in \mathbb{P}^{(n+1)(m+1)-1} \mid z_{ij} \neq 0\}. \end{aligned}$$

Then $S^{-1}(U_{ij}) = U_i \times U_j$ where $U_i \subset \mathbb{P}^n$ and $U_j \subset \mathbb{P}^m$. The coordinates are $\frac{x_a}{x_i}$'s and $\frac{y_b}{y_j}$'s. This maps $\mathbb{A}^{n+m} \rightarrow \mathbb{A}^{n+m+nm}$ where the coordinates are $\frac{z_{ab}}{z_{ij}}$'s. We could map

$$\frac{z_{ab}}{z_{ij}} \mapsto \frac{x_a y_b}{x_i y_j}.$$

We claim that this is surjective. This is clear, as for example $\frac{z_{aj}}{z_{ij}} \mapsto \frac{x_a}{x_i}$. ■

- 5) $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ maps $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$. The matrix is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then the image is the same as $V(xw - yz)$ where $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.
- 6) $X \subset \mathbb{P}^n$ and $V(F_0, \dots, F_m) \cap X = \emptyset$. Then $F : X \rightarrow \mathbb{P}^m$ is a well-defined morphism. The question is: do all maps from $X \rightarrow \mathbb{P}^m$ arise in this way?

Well the answer is no because $P^1 \times P^1 \rightarrow \mathbb{P}^3$ as defined in the last example works. We can project $\mathbb{P}^1 \times \mathbb{P}^1$ to \mathbb{P}^1 . And the counterexample arises because there is no F_0, F_1 of the same degree such that there is no map $\mathbb{P}^3 \setminus V(F_0, F_1) \rightarrow \mathbb{P}^1$ that makes the diagram commute.

Let $S : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ and let $Q = \text{im}(S)$. We want $Q \cap V(F_0, F_1) = \emptyset$. However, $V(Q, F_0, F_1)$ has codimension at most 3, so dimension at least 0, in particular non-empty. This comes from Krull's height theorem.

- 7) If we are given four random lines in \mathbb{P}^3 , how many meets all 4? The answer is 2.

As an exercise, consider 3 random lines in \mathbb{P}^3 , we can consider the union of all lines that touch all 3 and show that it is a projective variety.

7.0.2. I skipped class again oops

8. Rational maps

For today and the rest of the week, we assume that every variety is irreducible.

Warm-up: Let $f, g : X \rightarrow Y$ be maps of varieties such that $f \upharpoonright U = g \upharpoonright U$, $\exists \emptyset \neq U \overset{\text{open}}{\subseteq} X$, then $f = g$.

PROOF: Let $X \rightarrow X \times X$ by the diagonal map Δ . Then let $X \times X \rightarrow Y \times Y$ by $f \times g$. The inverse image of $\Delta(Y)$ is $\{x \mid f(x) = g(x)\}$. Since they agree on an open subset and it is dense, they are actually equal. ■

Definition 8.1: A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, φ_U) where we have that $\emptyset \neq U \subset X$ is open and $\varphi_U : U \rightarrow Y$ is a morphism. Then we have $(U, \varphi_U) \sim (V, \psi_V)$ if we have that $\varphi_U \upharpoonright U \cap V = \psi_V \upharpoonright U \cap V$.

Definition 8.2: $\varphi : X \dashrightarrow Y$ is dominant if $\varphi(U)$ is dense in Y or some/every rep (U, φ_U) .

φ is birational if $\exists \psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \text{id}_Y$ and $\psi \circ \varphi = \text{id}_X$.

Two varieties are birational if there exists a birational map between them.

Remark: In general, you cannot compose rational maps.

Example 8.3:

- 1) \mathbb{P}^{n+m} and $\mathbb{P}^n \times \mathbb{P}^m$ are birational. This is because there is a copy of \mathbb{A}^{n+m} in both of them.
- 2) \mathbb{A}^1 and $V(x^3 - y^2)$ are birational. Consider $t \mapsto (t^2, t^3)$, or rather $(x, y) \mapsto \frac{y}{x}$ in the opposite direction.
- 3) \mathbb{P}^1 and $V(y^2z - x^3 - x^2z) \subset \mathbb{P}^2$. Take $[x : y] \mapsto [x : y : \frac{x^3}{y^2 - x^2}]$.

Remark: A variety X is rational if it is birational to \mathbb{A}^n for some n .

Question: Is there a non-rational variety? (Yes.)

Is $\varphi : X \dashrightarrow Y$ dominant and injective in a nonempty open subset, is it birational? This is true for characteristic zero, but false for characteristic > 0 .

Remark: X is unirational if \exists dominant $\mathbb{A}^n \dashrightarrow X$. Rational and unirational are not equivalent. There are also non unirational varieties.

Definition 8.4: A rational function on X is a rational map from $X \dashrightarrow \mathbb{A}^1$. We denote

$$K(X) := \{\text{rational functions on } X\}$$

and call it the (rational) function field.

Theorem 8.5:

$$\{\text{dominant rational maps } X \dashrightarrow Y\} \longleftrightarrow \{k\text{-alg extensions } K(Y) \subseteq K(X)\}$$

by the map $f \mapsto (\varphi \mapsto \varphi \circ f)$.

PROOF: Let $\Theta : K(Y) \hookrightarrow K(X)$. We may assume Y is affine, $Y \subseteq \mathbb{A}^n$. Now look at the functions $\Theta(y_1), \dots, \Theta(y_m)$, where $A(Y) = \frac{\mathbb{k}[y]}{I}$ for some ideal I . All of the functions listed are regular on some open $U \subset X$.

So we have made a map from $A(Y) \rightarrow \mathcal{O}_X(U)$ by $y_i \mapsto \Theta(y_i)$, which defines a morphism (there is some theorem that says having a map from a coordinate ring to the structure sheaf defines a morphism). ■

Corollary 8.6: This bijection is an equivalence of categories:

$$\{\text{vars and rational dominant maps}\} \leftrightarrow \{\text{finitely generated field extensions over } \mathbb{k}\}.$$

Corollary 8.7: X and Y varieties. The following are equivalent:

- 1) X and Y are birational.
- 2) $\exists \emptyset \neq U \subseteq_{\text{open}} X, V \subseteq_{\text{open}} Y$ such that $U \simeq V$ isomorphic.
- 3) $K(X) \simeq K(Y)$.

1 to 2 can be verified. 2 to 3 uses the theorem above.

Theorem 8.8: Let $f : X \dashrightarrow Y$ be a dominant map. Then f is generically finite (i.e. for any representative $f : U \rightarrow Y$, general fiber is finite) if and only if $K(Y) \subseteq K(X)$ is a finite extension. Further, if gen. fin. and char $k = 0$, then general fiber has exactly $[K(X) : K(Y)]$.

Corollary 8.9: In characteristic zero, a rational dominant map that is generically one to one is birational. This is very false in positive characteristic.

PROOF of Theorem 8.8: Reduce to X, Y affine, $X \xrightarrow{f} Y$ where $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ where $m \leq n$. Reduce to $m = n - 1$, where this map is now $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1})$.

Now we split into cases:

1) $z_n \in K(X) = \mathbb{k}[z_1, \dots, z_n]/I$ is algebraic over $K(Y)$. By definition, there is a minimal polynomial $G = a_d(z_1, \dots, z_{n-1})z_n^d + \dots + a_1(z_1, \dots, z_{n-1})z_n + a_0(z_1, \dots, z_{n-1}) \in K(Y)[z_n]$. We may assume that $G \in A(Y)[z_n]$.

$D_Y(a_d) \neq \emptyset$ open in Y , f is finite over $D(a_d)$. The discriminant Δ of G will be nonzero on Y . In other words, on $D_{Y(a_d, \Delta)}$, $|\text{fiber}| = d$.

■

8.1. Wasn't here for first part of blowups

Definition 8.10: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $I = \langle f_0, \dots, f_m \rangle \subset A(X)$. The blowup, which we define as $\tilde{X} = \text{Bl}_I X$, is the subvariety of $X \times \mathbb{P}^m (\subset \mathbb{A}^n \times \mathbb{P}^m)$ given by u -homogeneous elements of $\ker(\mathbb{k}[\underline{x}][\underline{u}] \rightarrow A(X)[tI] \subseteq A(X)[t])$. $\pi : \tilde{X} \rightarrow X$ the “blow-down” map.

Proposition 8.11: $\text{Bl}_I X$ is independent of the choice of generators f_0, \dots, f_m .

PROOF: $\text{Bl}_{\langle f_0, \dots, f_m \rangle} X \simeq \text{Bl}_{\langle f_0, \dots, f_m, g \rangle} X$.

■

Proposition 8.12: $I = \langle f_0, \dots, f_m \rangle \subset A(X)$. Then $\tilde{X} = \text{Bl}_I X \simeq$ closure in $X \times \mathbb{P}^m$ of the image of

$$(X \setminus V(I)) \rightarrow X \times \mathbb{P}^m$$

given by

$$x \mapsto \left(x, [f_0(x), \dots, f_m(x)] \right).$$

PROOF:

1) $\overline{X} \subseteq \tilde{X}$:

2) $\tilde{X} \subseteq \overline{X}$:

■

8.2. im retarded

9. Smoothness/Nonsingularity

What is a tangent vector? Rather, for $0 \in X \subseteq \mathbb{A}^n$, how can we find a tangent vector to X ?

1) Something in the tangent space?

Example 9.1: Suppose $\text{char } k \neq 2$, $X = V((x-1)^2 + (y-1)^2 - 2)$. Then we can say that

$$T_0X = V(x+y).$$

But how do we get this? We can see that

$$(x-1)^2 + (y-1)^2 - 2 = x^2 - 2x + y^2 - 2y.$$

And we see $-2(x+y)$ is the gradient of this function or something idk.

Definition 9.2: For $0 \in X \subseteq \mathbb{A}^n$,

$$T_0X := V(f^{\text{linear}} \mid f \in I(X)).$$

For $p \in X \subseteq \mathbb{A}^n$,

$$T_pX := \ker \text{Jac} \left[\frac{\partial f_i}{\partial x_j}(p) \right]_{i,j}$$

for any generating set f_1, \dots, f_m of $I(X)$.

Definition 9.3: Let A be a k -alg and M an A -module. Then

$$\text{Der}_k(A, M) := \{\delta \in \text{Hom}_k(A, M) \mid \forall f, g \in A, \delta(fg) = f\delta(g) + g\delta(f)\}.$$

Example 9.4: Let $A = k[x, y]$ and $M = k[x, y]/\langle x-a, y-b \rangle$ for $a, b \in k$. Then this is isomorphism by k with the group action of acting by a on x .

Then

$$\text{Der}_k(A, M) = k \left[\frac{\partial}{\partial x} \Big|_{x=a, y=b}, \frac{\partial}{\partial y} \Big|_{x=a, y=b} \right].$$

So for $\delta \in A$, we have $\delta(x^n) = nx^{n-1} \cdot \delta(x) = na^{n-1}\delta(x)$.

Definition 9.5: Zariski cotangent space of $p \in X$ is $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes A(X)/\mathfrak{m}$.

Proposition 9.6: Let $0 \in X \subseteq \mathbb{A}^n$. Then

$$T_0 X \simeq \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \simeq \text{Der}_k(A(X), A(X)/\mathfrak{m} \simeq k)$$

where $\mathfrak{m} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$. We can accomplish this with the maps

$$v \mapsto (\bar{f} \mapsto f^{\text{linear}}(v))$$

from first to second and

$$\delta \mapsto (\bar{f} \mapsto \delta(f))$$

for third to second.

PROOF: Let $\varphi(\delta) = \bar{f} \mapsto \delta(f)$. Then $\delta(\mathfrak{m}^2) = 0$ implies that $\delta = 0$. Also $\delta(1) = 0$, so it is injective.

I'm too lazy to write this whole thing out ■

Remark: Note that $\dim T_p X \geq \text{codim}_X p$.

Definition 9.7: X variety is nonsingular at a point $p \in X$ if

$$\dim T_p X = \text{codim}_X p.$$

Proposition 9.8: $p \in X$ is nonsingular if and only if $\dim T_p X \leq \text{codim}_X p$. Also if and only if $T_p X = TC_p X$. Also if and only if $\dim(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = \dim \mathcal{O}_{X,p}$. Also if and only if $\text{rank Jac} \left[\left(\frac{\partial f_i}{\partial x_j}(p) \right) \right]_{i,j} = n - \text{codim}_X p$ when $X \subseteq \mathbb{A}^n$.

Example 9.9: $X = V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$. $TC_0 X = V(x^2 - y^2)$. $T_0 X = \mathbb{A}^2$.

Remark: A Noetherian ring (R, \mathfrak{m}) is a regular local ring if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Theorem 9.10: Regular local rings are UFDs.

PROOF: Something something Nakayama. ■

Proposition 9.11: Let $p \in X \subseteq \mathbb{A}^n$ is nonsingular if $\text{rank Jac}_p(f_1, \dots, f_m) \geq n - \text{codim}_X p$ for any f_1, \dots, f_m such that $V(f_1, \dots, f_m) = X$.

Exercise 9.12: Suppose $Y \subset \mathbb{P}^n$, $\langle F_1, \dots, F_m \rangle = I(Y)$. To test $X \subset \mathbb{A}^n$ is nonsingular,

$$\sum_{i=0}^n \frac{\partial F}{\partial x_i} = (\deg F)F.$$

Theorem 9.13: Let X be a variety. Then the nonsingular loci is open and nonempty in X .

PROOF: Reduce to X being affine. X is irreducible, $X \subseteq \mathbb{A}^n$, $V(f_1, \dots, f_m) = X$. Then

$$\text{rank}[\text{Jac}_p(f_1, \dots, f_m)] \geq n - \dim X$$

so $X_{\text{sing}} = V(\text{codim } X + 1 \text{ minors of Jac}) \cap X$. ■

Lemma 9.14: Any irreducible variety X is birational to an irreducible hypersurface $V(f) \subset \mathbb{A}^n$.

PROOF: Take $K(X)/k$ which is separable and finitely generated and separably generated. So there exists $x_1, \dots, x_d \in K(X)$ such that $K(X) \supset k(x_1, \dots, x_d)$ where $d = \dim X$. Then

$$K(X) \simeq \frac{k(x_1, \dots, x_d)[y]}{f(y)}$$

where the coefficients of f are in x_1, \dots, x_d . ■