

21-849: Algebraic Geometry

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I don't know what a sheave or a category is. ❤️

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1. Introduction

1.0.1. Administrivia

- Grade consists of two takehomes and one presentation/paper.
- Exercise List/Notes: Canvas
- Prerequisites: basic algebra, topology, and “multivariable calculus”.
- Textbooks: [G] Gathmann, [H1] Hartshorne, [H2] Harris
- OH: 2-4pm Wednesday, Wean 8113

1.1. Features of algebraic geometry

Consider the two functions e^z and $z^2 - 3z + 2$.

- Both are continuous in \mathbb{R} or \mathbb{C} .
- Both are holomorphic in \mathbb{C} .
- Both are analytic (power series expansion at every point).
- Both are C^∞ .

There are differences as well.

- $f(z) = a$ has no solution or infinitely many solutions for e^z , but for almost all a , 2 solutions for $z^2 - 3z + 2$.
- e^z is not definable from $\mathbb{Z} \rightarrow \mathbb{Z}$ but $z^2 - 3z + 2$ is.
- $\left(\frac{d}{dz}\right)^\ell \neq 0$ for all $\ell > 0$ for e^z but not for $z^2 - 3z + 2$.
- For nontrivial polynomials, as $z \rightarrow \infty$, $p(z)$ goes to infinity. So, it can be defined as a function from $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. But e^z can be periodic as the imaginary part tends to infinity.

This motivates the following result:

Theorem 1.1 (GAGA Theorems): Compact (projective) \mathbb{C} -manifolds are algebraic.

Here are more cool things about algebraic geometry:

1) Enumeration:

- How many solutions to $p(z)$?
- How many points in $\{f(x, y) = g(x, y) = 0\}$?
- How many lines meet a given set of 4 general lines in \mathbb{C}^3 ? The answer is 2.
- How many conics ($\{f(x, y) = 0\}$, $\deg f = 2$) are tangent to given 5 conics (in 2-space)? Obviously it's 3264...
- Now for any question of the previous flavor, the answer is coefficients of chromatic polynomials of graphs.

2) Birationality:

- Open sets are *huge*. That is, if we have X, Y and $U \subseteq X, V \subseteq Y$ such that $U \cong V$, then X and Y are closely related.

3) Arithmetic Geometry:

- Over $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, etc.
- Weil conjectures: X carved by polynomials with \mathbb{Z} -coefficients. $H^2(X_{\mathbb{C}}, \mathbb{Q})$ related to integer solutions.

2. Affine algebraic sets

2.1. Nullstellensatz

Notation: \mathbb{k} is an algebraically closed field ($\mathbb{k} = \mathbb{C}$).

Definition 2.1 (Affine space): An n -affine space $\mathbb{A}_{\mathbb{k}}^n$ is the set

$$\{(a_1, \dots, a_n) \mid a_i \in \mathbb{k}, \forall i = 1, \dots, n\} = \mathbb{k}^n.$$

An affine algebraic subset of \mathbb{A}^n is a subset $Z \subseteq \mathbb{A}^n$ such that

$$Z = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0, \forall f \in T\}$$

for some subset $T \subseteq \mathbb{k}[x_1, \dots, x_n]$. We write $Z = V(T)$.

Example 2.2 (An affine space):

- $V(x^2 - y) \subset \mathbb{A}^2$. This is a parabola.
- $V(x^2 + y^2) \subset \mathbb{A}^2$. Note that $x^2 + y^2 = (x + iy)(x - iy)$, so this is two lines.
- $V(x^2 - y, xy - z) \subseteq \mathbb{A}^3$. We actually have $V(x^2 - y, xy - z) = \{(x, x^2, x^3) \mid x \in \mathbb{k}\}$. Then note that if we project to any two dimensional plane (xy, yz, xz) , then we get another affine subset but on \mathbb{A}^2 .

This leads us to the following question:

Question: $X \subseteq \mathbb{A}^n \Rightarrow \pi(X) \subseteq \mathbb{A}^{\{n-1\}}$?

SOLUTION: Consider $V(1 - xy) \subseteq \mathbb{A}^2$. If we project this to either axis, then we will miss the origin.

■

Definition 2.3 (Ideal): Let $Z \subseteq \mathbb{A}^n$ be an algebraic subset. Then

$$I(Z) = \{f \in \mathbb{k}[x] \mid f(p) = 0, \forall p \in Z\}.$$

Example 2.4:

- 0) $Z = V(x^2) \subseteq \mathbb{A}^2$, then $I(Z) = \langle x \rangle$.
- 1) If $Z = V(x^2 - y)$, then $I(Z) = \langle x^2 - y \rangle$
- 2) If $Z = V(x^2 - y, xy - z)$, then $I(Z) = \langle x^2 - y, xy - z \rangle$.

Proposition 2.5:

- 1) $I(Z)$ an ideal. $Z_1 \subseteq Z_2 \Rightarrow I(Z_1) \supseteq I(Z_2)$.
- 2) $T \subseteq \mathbb{k}[x]$. $V(T) = V(\langle T \rangle)$ AND $V(T) = V(f_1, \dots, f_m)$ for some f_i .
- 3) For $\mathfrak{a} \subseteq \mathbb{k}[x]$ ideal, $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, where $\sqrt{\mathfrak{a}} = \{f \in \mathbb{k}[x] \mid f^m \in \mathfrak{a}, \exists m > 0\}$.
- 4) Algebraic subsets of \mathcal{A}^n are closed under finite unions and arbitrary intersections.

PROOF: We prove number 2 by using the Hilbert Basis Theorem. In particular, $\mathbb{k}[x]$ is Noetherian.

■

Theorem 2.6 (Nullstellensatz): Let Z be an algebraic subset. Then $V(I(Z)) = Z$ and $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. That is,

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{k}[x]\}.$$

PROOF:

- 1) Finite type field extensions $L \supseteq F$ are finite. Remember that finite type means that $F[x_1, \dots, x_m] \twoheadrightarrow L$.
- 2) This implies that maximal ideals of $\mathbb{k}[x]$ are of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ for $a_i \in \mathbb{k}$, using the fact that \mathbb{k} is algebraically closed. So, $\mathbb{k}[x]/\mathfrak{m} \simeq \mathbb{k}$.
- 3) (Weak Nullstellensatz) $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = \langle 1 \rangle$. That is, $\mathfrak{a} \subsetneq \mathbb{k}[x], \exists \mathfrak{m} \supseteq \mathfrak{a}$.
- 4) So if $f \in I(V(\mathfrak{a}))$, then consider $\mathfrak{a} + \langle 1 - yf \rangle \subseteq \mathbb{k}[x, y]$. So for any (a_1, \dots, a_n, b) that vanishes on $\mathfrak{a} + \langle 1 - yf \rangle$, we realize that since $1 - yf = 1$, we have a unit ideal. That is, we can say $1 = g_1 h_1 + g_2(1 - yf)$ for $h_1 \in \mathfrak{a}$ and $g_1, g_2 \in \mathbb{k}[x, y]$. From here, we can conclude that $f^\ell \in \mathfrak{a}$ for some ℓ .

But also

$$\mathbb{k}[x, y]/\langle 1 - yf \rangle \simeq \mathbb{k}[x] \left[\frac{1}{f} \right] = R.$$

So,

$$\frac{1}{1} = g_1 + \frac{g_2}{f} + \frac{g_3}{f^2} + \dots + \frac{g_\ell}{f^{\ell-1}}$$

for $g_i \in$ ideal \mathfrak{a} inside R .

■

Remark: We say R is Jacobson if every radical ideal $= \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.

Theorem 2.7: R Jacobson $\Rightarrow R[x]$ Jacobson.

Definition 2.8 (Coordinate ring): The coordinate ring $A(X)$ of $X \subseteq \mathbb{A}^n$ is $\mathbb{k}[x]/I(X)$.

1) $X \xrightarrow{f} \mathbb{k}$

2) $\text{maxSpec } A(X) = \{\text{maximal ideals in } A(X)\} = X$.

3. Projective Spaces

Definition 3.1: $\mathbb{P}^n = (\mathbb{k}^{n+1} \setminus \{0\}) / \sim$. That is, $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \mathbb{k}$. That is, $\mathbb{P}^n = \{1\text{-subspaces of } \mathbb{k}^{\{n+1\}}\}$. For $(a_0, \dots, a_n) \in \mathbb{k}^{n+1} \setminus \{0\}$, we write $[a_0 : \dots : a_n] \in \mathbb{P}^n$.

Remark: $V \simeq \mathbb{k}^{n+1}$. $\mathbb{P}V = V \setminus \{0\} / \sim$

Definition 3.2: $f \in \mathbb{k}[\underline{x}]$ is homogeneous if $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\ell f(x_1, \dots, x_n)$.

Definition 3.3: A projective algebraic set, $X \subseteq \mathbb{P}^n$ is

$$V(T) = \{[x_0 : \dots : x_n] \mid f(x) = 0, \forall f \in T\}$$

for T a set of homogeneous polynomials.

We have that $\mathbb{P}^n \supset U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0, x_i = 1\}$. So then

$$\mathbb{P}^n = (U_i = \mathbb{A}^n) \sqcup \mathbb{P}^{n-1}.$$

Example 3.4: Let $W \subseteq \mathbb{k}^{n+1}$ of $\dim_k W = m + 1$. Then $\mathbb{P}W \subseteq \mathbb{P}^n$ is a projective algebraic subset which is an m -plane in \mathbb{P}^n .

Example 3.5 (Twisted cubic curve): We have $\mathbb{P}^3 \supset C = \{[s^3 : s^2t : st^2 : t^3] \mid [s : t] \in \mathbb{P}^1\}$. Then we have that $C = V(x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$. Then $U_0 \cap C = \{[1 : t : t^2 : t^3]\}$. Additionally, we have $C \setminus U_0 = \{[0 : 0 : 0 : 1]\}$. Another way we can view this is

$$V\left(2 \text{ by } 2 \text{ minors of } \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}\right).$$

Now note that for a matrix A , $\text{rank}(A) \leq r \iff \text{all } (r+1) \times (r+1) \text{ minors} = 0$.

Question: Can there exist F, G such that $V(F, G) = C$? (Answer is yes)

For $X \subseteq \mathbb{P}^n$, algebraic subset, let

$$I(X) = \{\text{homogeneous } f \in \mathbb{k}[\underline{x}] \mid f(p) = 0, \forall p \in X\}$$

be the homogeneous ideal of X .

Exercise 3.6:

$$\{\emptyset \neq X \subseteq \mathbb{P}^n \text{ algebraic subsets}\} \longleftrightarrow \{\text{homogeneous radical ideals } \mathfrak{a} \subseteq \mathbb{k}[\underline{x}] \text{ such that } \mathfrak{a} \neq \mathbb{k}[\underline{x}] \text{ or } \langle x_0, \dots, x_n \rangle\}.$$

This last part is called the “irrelevant ideal”.

Definition 3.7 (General Position): In \mathbb{P}^n , any subset of size $\leq n + 1$ points are linearly independent.

Theorem 3.8: Every set Γ of $2n$ points in \mathbb{P}^n in general position is carved out by quadrics.

PROOF: We want to show that if $q \in V(\{\text{all quadrics vanishing on } \Gamma\})$, then $q \in \Gamma$. Suppose q is given. For any partition of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $|\Gamma_i| = n$, $\text{span}(\Gamma_1)$ is a hyperplane. Then for every such equi-partition, $q \in \text{span}(\Gamma_1)$ or $q \in \text{span}(\Gamma_2)$.

Let p_1, \dots, p_k be a minimal subset of Γ whose span $\ni q$ ($k \leq n$). Now pick any Λ such that $|\Lambda| = n - k + 1$ which does not contain any of the p_i . We claim that $q \notin \text{span}(p_2, \dots, p_k, \Lambda)$.

We then conclude that for any $|S| = n - 1$, $S \subseteq \Gamma \setminus p_1, \dots, p_k$, we have that $\text{span}(p_1, S) \ni q$. Because then

$$\bigcap_S \text{span}(p, S)$$

is the intersection at least n many hyperplanes, each of them containing p_1, q . But the intersection of n many hyperplanes is a point, so $q = p_1$. This also concludes that in fact $k = 1$. ■

Definition 3.9: Two sets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if $X' = g \cdot X$, $\exists g \in PGL_{n+1}$.

Proposition 3.10: Let (M_0, \dots, M_3) be any \mathbb{k} -basis of

$$\mathbb{k}[s, t]_3 = \{f \in \mathbb{k}[s, t] \text{ homog degree } 3\} \cup \{0\}.$$

Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ by $\varphi : [s : t] \mapsto [M_0(s, t) : \dots : M_3(s, t)]$. Also, $\varphi(\mathbb{P}^1)$ is projectively equivalent to $C = \{[s^3 : s^2t : st^2 : t^3]\}$.

Example 3.11 (Rational normal curve): Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ via $\varphi : [s : t] \mapsto [s^n : s^{n-1}t : \dots : t^n]$. Or we could map it to any basis of $\mathbb{k}[s : t]_n$.

Exercise 3.12: $I(\varphi(\mathbb{P}^1)) = ?$.

Example 3.13: $[s^3 : s^2t : t^3]$ is the same as $V(y^3 - x^2z)$. Also take $[st^2 - s^3 : t^3 - s^2t : s^3]$. This is carved out by $V(y^2z - x^3 - x^2z)$.

Fact: If we pick any 3 linearly independent $M_0, M_1, M_2 \in \mathbb{k}[s, t]_3$. Then $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ by M_0, M_1, M_2 has image projectively equivalent to one of the two curves above.

Now consider $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ using 4 elements from $\mathbb{k}[s, t]_4$. We consider $P \simeq C = \{[s^4 : s^3t : st^3 : t^4]\}$. This is called the twisted quartic curve.

Question: Are all twisted quartic curves projectively equivalent?

SOLUTION: No. In fact, there are infinitely many distinct families. ■

Question (Hartshorne's Question): Is every irreducible curve in \mathbb{P}^3 carved out by 2 equations?

4. The Zariski Topology

Definition 4.1 (Zariski topology): The sets $\{V(I) \subseteq \mathbb{A}^n \mid I \subseteq \mathbb{k}[\underline{x}]\}$ form the closed sets of a topology on \mathbb{A}^n called the Zariski topology.

Given $X \subseteq \mathbb{A}^n$, give it the subspace topology.

Example 4.2: Take \mathbb{A}^1 . Two closed subsets are \mathbb{A}^1 and \emptyset . The other closed subsets are collections of finitely many points. As such, the open subsets are the complements of finitely many points.

Definition 4.3: A topological space X is irreducible $X = Y_1 \cup Y_2$ (each closed) implies that $X = Y_1$ or $X = Y_2$.

By definition, we will also say that irreducible implies nonempty.

Remark:

- Irreducible implies connected
- Connected does not imply irreducible
- Irreducible is useless in Hausdorff setting.

Proposition 4.4: Let $X \subseteq \mathbb{A}^n$ be a nonempty algebraic subset. X is irreducible if and only if $I(X)$ is prime if and only if $A(X)$ is a domain.

PROOF:

- \Rightarrow : Suppose $fg \in I(X)$. This means $V(f) \cup V(g) \supseteq X$. If X is irreducible, then at least one of them completely contains X . That is, $V(f) \supseteq X$ or $V(g) \supseteq X$. But this exactly means f or $g \in I(X)$.
- \Leftarrow : Suppose for sake of contradiction that X is not irreducible. We have $X = Y_1 \cup Y_2$ (both proper), then $I(Y_2) \supsetneq I(X)$. Take $f_i \in I(Y_i) \setminus I(X)$. Now analyze $f_1 f_2$. $V(f_1 f_2) \supset Y_1 \cup Y_2 = X$. Therefore, $f_1 f_2 \in I(X)$. But this is a contradiction, so we are done.

■

Remark: When people say affine variety, some people mean that it is also irreducible. But for us, affine variety is the same thing as affine algebraic set.

Then a quasi-affine variety is an open subset of an affine variety.

Example 4.5:

- 1) \mathbb{A}^n is irreducible. ($\mathbb{k}[\underline{x}]$ domain)
- 2) $V(x^2 + y^2) \subset \mathbb{A}^2$ is reducible ($\text{char } \mathbb{k} \neq 2$)
- 3) Let $f \in \mathbb{k}[\underline{x}]$ be square-free ($f = f_1 \dots f_\ell$ irreducible). Then $V(f)$ is irreducible if and only if f is irreducible.
- 4) $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then $A(X) = \frac{\mathbb{k}[x, y, z]}{\langle x^2 - yz \rangle}$. This is irreducible due to Eisenstein's on f . Now if we take $f \in A(X)$ and look at $V_X(f) \subset X$ is irreducible $\Leftrightarrow f$ irreducible element in $A(X)$.

Definition 4.6: A topological space X is Noetherian if $\nexists X \supseteq Y_0 \supsetneq Y_1 \supsetneq \dots$ such that each Y_i is closed.

Proposition 4.7: An affine variety is Noetherian. (Because $A(X)$ is Noetherian).

Theorem 4.8: A Noetherian topological space X is uniquely a finite union of maximal irreducible closed subsets.

PROOF: Consider

{nonempty closed subsets of X that does not admit a decomposition into irreducible closed subsets.}.

Suppose it is nonempty. Then it has a minimal element Y . Y is not irreducible, so $Y = Y_1 \cup Y_2$ (both proper and closed). Since Y is minimal, Y_1 and Y_2 both have decompositions into irreducible closed subsets. So if we just union those decompositions, then we contradict Y 's membership in the set. As such, the original set must have actually been empty.

Uniqueness and maximality are left as an exercise. ■

Proposition 4.9:

- 1) X irreducible and $U \subseteq X$ open. Then $\overline{U} = X$.
- 2) $V \subseteq X$, V irreducible $\implies \overline{V}$ irreducible.
- 3) $f : X \rightarrow Y$ continuous. Image of irreducible set under f is irreducible. (Irreducibility is a topological property).

Example 4.10: Let's have $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^m$ by $\varphi(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x}))$ for some $f_1, \dots, f_m \in \mathbb{k}[\underline{x}]$. Then $\text{im}(\varphi)$ is irreducible. It is left to show that φ is a continuous map.

Definition 4.11: Let X be a nonempty topological space.

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

Then let $Y \subseteq X$ closed irreducible subset.

$$\operatorname{codim}_X Y := \sup\{n \mid \exists Y \subseteq Y_0 \subsetneq \cdots \subsetneq Y_n, \text{ each } Y_i \text{ irreducible and closed}\}.$$

Example 4.12:

1) $\dim \mathbb{A}^1 = 1$.

2) $X = V(xz, yz) \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Let p be a point on the axis not touching the x - y plane. Then let q be the origin. We have that $\operatorname{codim}_X p = 1$ and $\operatorname{codim}_X q = 2$. Also $\dim p = \dim q = 0$.

Definition 4.13: Height of a prime $\mathfrak{p} \subset R$ is

$$\operatorname{ht} \mathfrak{p} := \sup\{n \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_n, \text{ each } \mathfrak{p}_i \text{ prime}\}.$$

Then Krull dimension of R is

$$\dim R := \sup\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ prime}\}.$$

Definition 4.14: For an ideal I , we have that

$$\operatorname{ht} I := \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}.$$

“inf of sup”.

From these, we can basically show from definition that

$$\operatorname{ht} I + \dim R/I \leq \dim R.$$

The $<$ case is possible if R is not a domain. For example, if we have that $R = \mathbb{k}[x, y, z]/\langle xz, yz \rangle$ and then $I = \langle x, y, z - 1 \rangle$.

But the $<$ case is also possible even if R is a domain and I prime.

Before we cover the next theorem, we note that

$$\{\text{minimal primes over } I\} = \{\mathfrak{p} \text{ prime } \mathfrak{p} \supseteq I, \text{ and } \nexists \mathfrak{p} \supsetneq \mathfrak{q} \supseteq I, \text{ prime } \mathfrak{q}\}$$

Theorem 4.15 (Krull Principal Ideal Theorem / Height Theorem): Let R be a Noetherian ring and $f_1, \dots, f_c \in R$.

- 1) Minimal primes over $\langle f_1 \rangle$ have height ≤ 1 . And the height is equal to 1 if f_1 is nonzerodivisor and nonunit.
- 2) Minimal primes over $\langle f_1, \dots, f_c \rangle$ have height $\leq c$.

“We could do this proof, but it’s like proving that there exists a complete ordered field satisfying the least upper bound property.”

Theorem 4.16: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ irreducible affine varieties.

- 1) $\dim(X \times Y) = \dim X + \dim Y$.
- 2) If $Y \subseteq X$, then $\dim Y + \operatorname{codim}_X Y = \dim X$.

Remark (Noether normalization): For $X \subseteq \mathbb{A}^n$ irreducible affine variety. There exists $y_1, \dots, y_d \in A(X)$ such that $\mathbb{k}[Y_1, \dots, Y_d] \rightarrow A(X)$ with $Y_i \mapsto y_i$ which is a finite extension (injective and $A(X)$ is finitely generated $\mathbb{k}[Y]$ -module) and $d = \dim X$.

Corollary 4.17:

- 1) $\dim \mathbb{A}^n = n$.
- 2) $X \subseteq \mathbb{A}^n$ irreducible affine variety. $0 \neq f \in A(X)$ non unit. Then $V_X(f) = V(f) \cap X$ has dimension $\dim X - 1$.

Exercise 4.18: Let $U \subseteq X$ be open for X affine variety irreducible. Then $\dim U = \dim X$.

Proposition 4.19: Let R be Noetherian domain. Then R UFD \iff every ht = 1 prime is principal.

PROOF: R being a UFD implies that \mathfrak{p} has height 1. So let $f = f_1, \dots, f_\ell \in \mathfrak{p}$. Suppose $f_1 \in \mathfrak{p}$. So then $0 \neq \langle f_1 \rangle \subseteq \mathfrak{p}$. But as $\operatorname{ht} \mathfrak{p} = 1$, we have that $\langle f_1 \rangle = \mathfrak{p}$.

Conversely, we need to show that irreducible implies prime. That is, recall that (ACCP + irreducible = prime) implies that we have a UFD.

So let $f \in \text{irred}$. Krull’s PIT says $\langle f \rangle \subseteq \mathfrak{p}$ where \mathfrak{p} has height 1. So by definition, $\mathfrak{p} = \langle g \rangle$, but $\langle f \rangle \subseteq \langle g \rangle$ implies that $f = g$ because f is irreducible. ■

Example 4.20: Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$. Then let $Y = V(x, y) \subseteq X \subseteq \mathbb{A}^3$. Then $\dim X = 2$. Then $\dim Y = 1$. So can we find f such that $\langle f, x^2 - yz \rangle = I(Y)$? The answer to this is no.

But can we find f such that $\sqrt{\langle f, x^2 - yz \rangle} = \langle x, y \rangle$? Take $f = y$ and analyze $\langle y, x^2 - yz \rangle$. This is the same as $\langle y, x^2 \rangle$, whose radical is $\langle x, y \rangle$ as we desire.

Example 4.21: Now consider $X = V(xw - yz) \subseteq \mathbb{A}^4$. $\dim X = 3$ and let $Y = V(x, y)$. Now does there exist f such that $\sqrt{\langle f, xw - yz \rangle} = \langle x, y \rangle$?

This is false, but we don't have the tools to prove it.

Definition 4.22: Zariski topology on \mathbb{P}^n has projective algebraic sets as its closed subsets.

Two ways: projective varieties \rightarrow affine varieties.

1) $U_i = \{x_i \neq 0\} = \{[x_0 : \dots : x_i = 1 : \dots : x_n]\} \simeq \mathbb{A}^n$.

Proposition 4.23: $\forall i = 0, \dots, n$, say $i = 0$, $\mathbb{A}^n \rightarrow U_0, (x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$ is a homeomorphism.

PROOF:

• *Homogenization:* let $f \in \mathbb{k}[x_1, \dots, x_n]$. Then we have

$$f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{k}[x_1, \dots, x_n].$$

If $Z = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n$, $\varphi(Z) = U_0 \cap V(f_1^h, \dots, f_m^h)$ is closed.

If $Z' = V(F_1, \dots, F_\ell) \cap U_0$, then $\varphi(Z') = V(F_1(1, x_2, \dots, x_n), \dots, F_\ell(1, x_2, \dots, x_n))$.

Now $U_0 \cup \dots \cup U_n = \mathbb{P}^n$.

■

Exercise 4.24: Let $Y \subseteq \mathbb{A}^n \simeq U_0$ be an affine variety. $\overline{Y} = V(?)$. Suppose $V(f_1, \dots, f_m) = Y$. It is tempting to say $\overline{Y} = V(f_1^h, \dots, f_m^h)$.

Corollary 4.25:

- 1) $\dim \mathbb{P}^n = n$.
- 2) If $H_i = V(x_i) \subseteq \mathbb{P}^n$ does not contain any irreducible components of $Y \subseteq \mathbb{P}^n$, then $\dim Y = \dim Y \cap U_i$.

Definition 4.26: Let $Y \subseteq \mathbb{P}^n$ be a projective variety. The affine cone $\hat{Y} = C(Y)$ is

$$\theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}$$

where

$$\theta : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n.$$

Proposition 4.27:

- 1) $\hat{Y} = V(I(Y))$. In fact, $I(\hat{Y}) = I(Y)$.
- 2) $\dim \hat{Y} = \dim Y + 1$.
- 3) \hat{Y} is irreducible if and only if Y is irreducible.

Theorem 4.28: If $X, Y \subseteq \mathbb{P}^n$ are projective varieties and $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

Lemma 4.29: If $X, Y \subseteq \mathbb{A}^n$ affine varieties, then $X \cap Y = \emptyset$ or every irreducible component of $X \cap Y$ has $\dim \geq \dim X + \dim Y - n$.

PROOF: Let $\Delta = V(x_1 - y_1, \dots, x_n - y_n) \subseteq \mathbb{A}^{n+n}$. Note that

$$X \times Y \cap \Delta \simeq X \cap Y.$$

So, $\dim(X \times Y \cap \Delta) \geq \dim X + \dim Y - n$ by Krull's height theorem.

If $\underline{a} = (a_1, \dots, a_n)$ are varieties, then $I_{\underline{a}(X)} = \{f(\underline{a} \mid f \in I(X)\}$. Then,

$$A(X \cap Y) = \frac{\mathbb{k}[\underline{z}]}{\sqrt{\langle I_{z(X)} + I_{z(Y)} \rangle}}$$

and

$$A(X \times Y \cap \Delta) = \frac{\mathbb{k}[\underline{x}, \underline{y}]}{\sqrt{\langle I_{x(X)} + I_{y(Y)} + I(A) \rangle}}.$$

So this implies that $x_i = y_i$ for all i , meaning they are isomorphic rings. ■

PROOF of Theorem 4.28: X, Y irreducible implies that \hat{X} and \hat{Y} are irreducible. So, then

$$\dim(\hat{X} \cap \hat{Y}) \geq \dim X + 1 + \dim Y + 1 - (n + 1) \geq \dim X + \dim Y - n + 1.$$

$\hat{X} \cap \hat{Y}$ contains origin by construction, but it has at least one other point because dimension. ■

5. Morphisms

Definition 5.1: For $U \subseteq \mathbb{R}^n$, $U' \subseteq \mathbb{R}^m$ open, $\varphi : U \rightarrow U'$ is continuous/continuously differentiable/smooth if $f \circ \varphi$ is smooth for any smooth $f : U' \rightarrow \mathbb{R}$.

$f' : U' \rightarrow \mathbb{R}$ is smooth if f is smooth at every point $p \in U'$.

Definition 5.2: For affine variety $X \subseteq \mathbb{A}^n$ and $U \subseteq X$ open, a function $\varphi : U \rightarrow \mathbb{k} = \mathbb{A}^1$ is regular if $\forall p \in U$, $\exists U_p \ni p$ open and $f_p, g_p \in A(X)$ such that $\varphi(x) = \frac{f_p(x)}{g_p(x)}$ for all $x \in U_p$. In particular, $g_{p(x)} \neq 0$ for all $x \in U_p$.

$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$. This is also a \mathbb{k} -algebra.

Example 5.3: Let $U \subseteq X$, $\varphi : U \rightarrow \mathbb{A}^1$ regular $\nRightarrow \varphi = \frac{f}{g}$ globally for some $f, g \in A(X)$.

$X = V(xw - yz) \subset \mathbb{A}^4$, $U = X \setminus V(x, y)$.

$$\varphi(x, y, z, w) = \begin{cases} \frac{z}{x} & \text{if } x \neq 0 \\ \frac{w}{y} & \text{if } y \neq 0 \end{cases}$$

Lemma 5.4: $\varphi : U \rightarrow \mathbb{A}^1$ regular, then $V(\varphi) = \{x \in U \mid \varphi(x) = 0\}$ is closed in U . In particular φ is continuous.

PROOF: Closedness is a local condition, and around any $p \in U$, $\{\varphi \upharpoonright U_p : 0\} = V(f_p) \cap U_p$. ■

Remark: If $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ for U irreducible, and $\varphi_1 \upharpoonright U' = \varphi_2 \upharpoonright U'$ for some $\emptyset \neq U' \subseteq U$, then $\varphi_1 = \varphi_2$.

Definition 5.5: $X \subseteq \mathbb{A}^n$ affine variety. A distinguished open subset U of X is an open subset of the form $X \setminus V(f)$ for some $f \in A(X)$, denoted $D(f)$, D_f , U_f , X_f . X_f is probably the most descriptive as it actually mentions X .

Remark: $\{D(f)\}_{f \in A(X)}$ form a basis for Zariski topology. What that means is that any $U \subseteq X$ is a union for $D(f)$'s.

Exercise 5.6: $D(f)$ is homeomorphic to $V(I(X) + \langle 1 - yf \rangle) \subseteq \mathbb{A}^{n+1}$.

Theorem 5.7: $\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} \mid g \in A(X), m \in \mathbb{Z}_{\geq 0} \right\}$. In fact, $\mathcal{O}_X(D(f)) = A(X)_f$.

Example 5.8: $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = A(\mathbb{A}^2) = \mathbb{k}[x, y]$. Then,

$$\mathbb{A}^2 \ni \varphi = \begin{cases} \frac{f}{x^m} \text{ for some } f \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(x) \\ \frac{g}{y^\ell} \text{ for some } g \in \mathbb{k}[x, y] \text{ on } \mathbb{A}^2 \setminus V(y) \end{cases}.$$

Then we say $y^\ell f = x^m g$ on \mathbb{A}^2 . Because we are in a UFD, this means that $x^m \mid f$ and $y^\ell \mid g$. But this implies $m = \ell = 0$, so $f = g = \varphi$.

PROOF of Theorem 5.7: \supseteq is clear. So we only prove the \subseteq case.

Suppose we have $\varphi \in \mathcal{O}_X(D(f))$. Then for all $p \in D(f)$, $\exists U_p \ni p$ and $\varphi \upharpoonright U_p = \frac{g_{p'}}{f_{p'}}$ for $g_{p'}, f_{p'} \in A(X)$.

Take a nonempty $D(h_p) \subseteq U_p$ and write $g_p = g_{p'} h_p$ and $f_p = f_{p'} h_p$.

Then $\varphi \upharpoonright D(f_p) = \frac{g_p}{f_p} = \frac{g_{p'} f_{p'}}{f_{p'}^2}$. So assume $g_p = 0$ on $V(f_p)$.

Now we claim that $\forall p, q \in D(f)$, we have $g_p f_q = g_q f_p$ in $A(X)$.

Then $D(f) = \bigcup_p D(f_p)$. Then $V(f) = \bigcap_p V(f_p)$. Nullstellensatz says that $\sqrt{\langle f \rangle} = \sqrt{\langle f_p : p \in D(f) \rangle}$ as ideals in $A(X)$. But then, $f^m = \sum k_p f_p$. By Noetherian-ness, this is a finite sum. We claim that $g = \sum k_p f_p$.

Then $\frac{g}{f^m} = \frac{g_q}{f_q^m}$ on $D(f_q)$ for all $q \in D(f)$. So,

$$g f_q = \sum_p k_p g_p f_q = \sum_p k_p f_p g_q = g_q f^m.$$

■