Hypothesis testing based on nonparametric entropies

October 2, 2023

Consider the problem of planning hypothesis tests based on nonparametric entropies from the functional (**?**) and (**?**): Let $f(\cdot)$ be a density,

$$T_{\mathscr{D}}(f) = \int_{-\infty}^{\infty} f(x)\Phi(f(x))\mathrm{d}x = \int_{-\infty}^{\infty} f(x)h\Big(\frac{1}{f(x)}\Big)\mathrm{d}x, \tag{1}$$

where $\Phi(x)$ and $h(x)$ are real-valued functions on $[0,\infty)$ satisfying certain regularity conditions. As an example of (**??**), the definition of $\Phi(x) = h(1/x) = -\log(x)$ becomes $T_{\mathscr{D}}(f)$ the Shannon entropy formula of a distribution on $(-\infty,\infty)$. From now on, we use $T_{\mathscr{D}} := T_{\mathscr{D}}(f)$ as a notation for a possible or well-defined entropy (such as Shannon, Rényi, and Tsallis).

A consistent estimator for $T_{\mathscr{D}}$ is given for the next class of statistics: Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the ordered sample from an n-point random sample and m be an integer such that $1 \le m \le n$,

$$T_{n,m} = \frac{1}{2}\frac{1}{n-m}\sum_{j=1}^{n-m}\Phi\Big(\frac{m}{n+1}\Big[X_{j+m:n} - X_{j:n}\Big]^{-1}\Big) = \frac{1}{n}\sum_{j=1}^{n-m}h\Big(\frac{n}{m}\Big[X_{j+m:n} - X_{j:n}\Big]\Big).$$

? have derived general asymptotic results for functions of spacings; while **?** have developed a correction for the case of Shannon entropy.]After **?** and **?**, the next result applies.

Lemma 0.1 *Suppose that $f(\cdot)$ is a bounded density bounded away from zero and satisfies a Lipschitz condition on its support. Then the next asymptotic results follow: If $m, n \to \infty$ and $m = o(n^{1/2})$, then:*

(i)

$$\sqrt{n}\Big(V^*_{m,n} + \int_{-\infty}^{\infty} f(x)\log f(x)\mathrm{d}x\Big) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \mathrm{Var}(\log f(X))),$$

where $o(\cdot)$ represents the little-o and

$$V^*_{m,n} = \frac{1}{n-m}\sum_{j=1}^{n-m}\log\Big(\frac{n+1}{m}\Big[X_{j+m:n} - X_{j:n}\Big]\Big) + \sum_{k=m}^{n}\frac{1}{k} + \log m - \log(n+1).$$

(ii)

$$\sigma^{-1}\sqrt{n}\Big(T^*_{m,n} - \int_{-\infty}^{\infty} h(1/f(x))f(x)\mathrm{d}x\Big) \xrightarrow{\mathscr{D}} \mathscr{N}(0,1),$$

where $h(1/f(x)) = \Phi(f(x))$ provides a unification between the results of **?** *and* **?**, *$F(x)$ means the cumulative distribution function (cdf) with respect to $f(x)$,*

$$T^*_{n,m} = \frac{1}{n}\sum_{j=1}^{n-m}\Bigg[h\Big(\frac{n}{m}\Big[X_{j+m:n} - X_{j:n}\Big]\Big) + \frac{[(n/m)(X_{j+m:n} - X_{j:n})]^2}{2m}h''\Big(\frac{n}{m}\Big[X_{j+m:n} - X_{j:n}\Big]\Big)\Bigg]$$

and

$$\sigma^2 = \int_{-\infty}^{\infty} h'\Big(\frac{1}{f(x)}\Big)\frac{1}{f(x)}\mathrm{d}x + \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{y}\Big(\frac{1}{f(x)} - \frac{F(x)f'(x)}{f^3(x)}\Big)h'\Big(\frac{1}{f(x)}\Big)f(x)\mathrm{d}x\Big]^2\frac{f(y)}{F^2(y)}\mathrm{d}x$$

for $h'(x)$ and $h''(x)$ as first and second order derivatives, respectively.

Let us now consider, starting from the previous lemma, the test of the null hypothesis $\mathscr{H}_0 : T_{\mathscr{D}} = D_0$ as opposed to one of the other three:

$$(i)\mathscr{H}_1 : T_{\mathscr{D}} \neq D_0, \quad (ii)\mathscr{H}_1 : T_{\mathscr{D}} > D_0, \quad \text{or} \quad (iii)\mathscr{H}_1 : T_{\mathscr{D}} < D_0.$$

For this purpose, we can use the test statistics:

$$Z_{m,n} = \begin{cases} \frac{\sqrt{n}(V^*_{m,n} - D_0)}{\sqrt{\mathrm{Var}(\log f(X))}}, & \text{for } \Phi(x) = \log(x), \\ \frac{\sqrt{n}(T^*_{m,n} - D_0)}{\sigma}, & \text{for any } h(x) \text{ possible.} \end{cases}$$

So the null hypothesis should be rejected if (i) $Z_{n,m} > z_{\alpha/2}$ or $Z_{n,m} < -z_{\alpha/2}$ for $\Phi_{\mathscr{N}}(z_{\alpha/2}) = 1 - \alpha/2$ and $\Phi_{\mathscr{N}}$ being the standard normal cdf, (ii) $Z_{n,m} > z_{\alpha/2}$ or (iii) $Z_{n,m} < -z_{\alpha/2}$.

The power function for case (i) (two-sided test) at $t \neq D_0$ is given by

$$\pi_{m,n}(t) = 1 - \Phi_{m,n}\Big(z_{\alpha/2} - \frac{\sqrt{n}(Z_{m,n} - D_0)}{\sigma}\Big) + \Phi_{m,n}\Big(-z_{\alpha/2} - \frac{\sqrt{n}(Z_{m,n} - D_0)}{\sigma}\Big),$$

for a sequence of cdfs $\Phi_{m,n}(x)$ which tends uniformly to $\Phi_{\mathscr{N}}(x)$.