Identifying Departures from the Fully Developed Speckle Hypothesis in Intensity SAR Data with Non-Parametric Estimation of the Entropy

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***Abstract*—SAR Data are affected by speckle, a non-additive and non-gaussian interference noise-like pattern. The type of distribution these data follow is paramount for their processing and analysis. Good statistical models provide flexibility and accuracy, often at the cost of using several parameters. The $\mathcal{G}^0$ distribution is one of the most successful models for SAR data. It includes the Gamma law as a particular case which arises in the presence of fully developed speckle. Although the latter is a limit distribution of the former, using the same estimation technique for the more general model is numerically unfeasible. We propose a two-stage estimation procedure: first, we verify the hypothesis that the data are fully-developed speckle. If this assumption is rejected, we proceed to estimate the parameters that index the $\mathcal{G}^0$ distribution; otherwise, we proceed with the Gamma model. Given the uncertainty of the underlying distribution, and the negative impact that using an inadequate model has on maximum likelihood estimation, we employ a non-parametric approach to estimate entropy under the fully-developed speckle hypothesis.**

***Index Terms*—SAR; entropy estimation; non-parametric analysis; order statistics**

I. Introduction

Synthetic aperture radar (SAR) has become a fundamental technology for environmental monitoring and disaster management because of its ability to provide daytime and nighttime imagery in all weather conditions [1]. However, the utility of SAR data depends on a thorough understanding of their statistical properties. Speckle is part of SAR data because of the imaging process' coherent nature. Its non-additivity and non-Gaussianity require robust statistical models that can accurately characterize the data.

Among these models, the $\mathcal{G}^0$ distribution stands out as a powerful framework. Notably, this distribution encompasses the well-known Gamma distribution as a special case, particularly under the assumption of fully developed speckle. The interplay between these two distributions is apparent, with the Gamma distribution representing a limiting case of the more general $\mathcal{G}^0$ model.

When deciding which model is the best, practitioners face a problem. On the one hand, if they opt for the Gamma law when the data come from the $\mathcal{G}^0$ distribution, they lose all the information about the number of scatterers, which is revealed by one of the parameters of the latter model [2]. On the other hand, if they apply the $\mathcal{G}^0$ distribution under fully developed speckle, maximum likelihood estimation is tricky: bias increases making estimation unreliable [3], and the likelihood is flat, so numerical optimization may not converge [4]. The two-stage technique we propose tackles this problem by using the entropy as a proxy to decide which is the best model.

Estimating the entropy faces practical challenges, particularly when the model is unknown; non-parametric methods are utilized in such cases. Among non-parametric approaches, [5] discussed the use of spacing methods. This non-parametric strategy offers flexibility to address a wide range of models without imposing specific parametric constraints. We extend the exploration of non-parametric entropy estimators by incorporating enhanced bootstrap methodologies.

Our aim is to develop a test statistic that helps discriminating between fully-developed speckle and heterogeneous clutter. We work with non-parametric estimators of the entropy, we reduce the bias of those based on spacings using bootstrap, and select the best ones. Next, we study the empirical distribution of these estimators under the null hypothesis, followed by a study of the size and power of the proposed test. We conclude the study with applications to SAR data.

The article is structured as follows: Section II covers statistical modeling and entropy estimation for Intensity SAR data. Section III outlines hypothesis testing based on non-parametric entropy. In Section IV, we present experimental results. Finally, in Section V conclusions are exhibited.

II. Background

A. Statistical modeling of Intensity SAR data

The primary models used for intensity SAR data include the Gamma and $\mathcal{G}_I^0$ distributions [6]. The first is suitable for fully developed speckle and is a limiting case of the second, which is appealing due to its versatility in accurately representing regions with various roughness characteristics [7]. We denote $Z \sim \Gamma_{\text{SAR}}(L, \mu)$ and $Z \sim \mathcal{G}_I^0(\alpha, \gamma, L)$ to indicate that Z follows the distributions characterized by the respective probability density functions:

$$f_Z(z; L, \mu) = \frac{L^L}{\Gamma(L)\mu^L} z^{L-1} \exp\{-Lz/\mu\} \mathbb{1}_{\mathbb{R}_+}(z), \quad (1)$$

$$f_Z(z; \alpha, \gamma, L) = \frac{L^L \Gamma(L-\alpha)}{\gamma^\alpha \Gamma(-\alpha)\Gamma(L)} \cdot \frac{z^{L-1}}{(\gamma + Lz)^{L-\alpha}} \mathbb{1}_{\mathbb{R}_+}(z), \quad (2)$$

where, in (1) $\mu > 0$ is the mean; in (2) $\gamma > 0$ is the scale, $\alpha < -1$ measures the roughness, $L \geqslant 1$ is the number of looks, $\Gamma(\cdot)$ is the gamma function, and $\mathbb{1}_A(z)$ is the indicator function of the set A.

From (2), the rth moment of Z is expressed as:

$$\mathbb{E}_{\mathcal{G}_I^0}(Z^r) = \left(\frac{\gamma}{L}\right)^r \frac{\Gamma(-\alpha - r)}{\Gamma(-\alpha)} \cdot \frac{\Gamma(L+r)}{\gamma(L)}, \quad \alpha < -r. \quad (3)$$

Even though the $\mathcal{G}_I^0$ distribution is defined by the parameters α and γ, SAR literature commonly utilizes the texture α and the mean μ [8]. In this way, we compute the expected value μ using the expression in (3), and we reparametrize (2) using μ, α, and L. Then

$$\mu = \left(\frac{\gamma}{L}\right) \frac{\Gamma(-\alpha - 1)}{\Gamma(-\alpha)} \cdot \frac{\Gamma(L+1)}{\gamma(L)} = -\frac{\gamma}{\alpha + 1}.$$

Thus, the probability density function that characterize the $G_I^0(\mu, \alpha, L)$ law is

$$f_Z(z; \mu, \alpha, L) = \frac{L^L \Gamma(L-\alpha)}{\big(-\mu(\alpha+1)\big)^\alpha \Gamma(-\alpha)\Gamma(L)} \frac{z^{L-1}}{\big(-\mu(\alpha+1) + Lz\big)^{L-\alpha}}. \quad (4)$$

B. The Shannon Entropy

The parametric representation of Shannon entropy for a system described by a continuous random variable is:

$$H(Z) = -\int_{-\infty}^{\infty} f(z) \ln f(z)\, \mathrm{d}z, \quad (5)$$

here, $f(\cdot)$ is the probability density function that characterizes the distribution of the real-valued random variable Z.

Using (5), we can express the Shannon entropy of Γ_{SAR}in (1) and G_I^0in (4) based on [7] and [9]:

$$H_{\Gamma_{\text{SAR}}}(L, \mu) = L - \ln L + \ln \Gamma(L) + (1-L)\psi^{(0)}(L) + \ln \mu, \quad (6)$$

$$\begin{aligned} H_{G_I^0}(\mu, \alpha, L) = {} & L - \ln L + \ln \Gamma(L) + (1-L)\psi^{(0)}(L) + \ln \mu \\ & - \ln \Gamma(L-\alpha) + (L-\alpha)\psi^{(0)}(L-\alpha) \\ & - (1-\alpha)\psi^{(0)}(-\alpha) + \ln(-1-\alpha) + \ln \Gamma(-\alpha) - L, \end{aligned} \quad (7)$$

where $\psi^{(0)}(\cdot)$ is the digamma function.

In Fig. 1 we see how the Entropy of the Gamma SAR distribution changes with μ for various values of L.

Fig. 2 illustrates the entropy of G_I^0 distribution as a function of three key parameters: μ, α, and L.

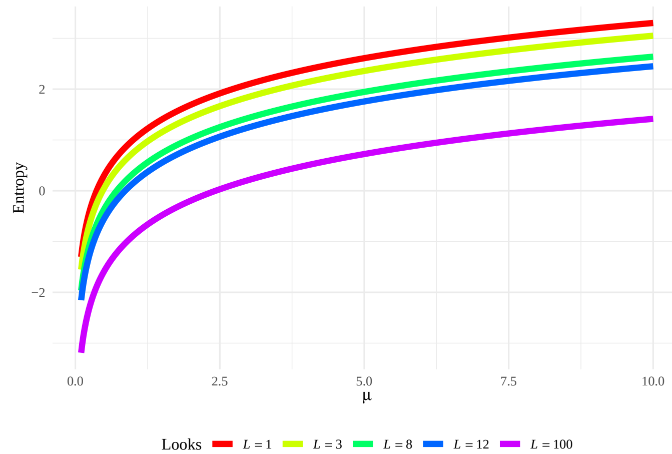

Figure 1. The Shannon Entropy under the Gamma SAR model.

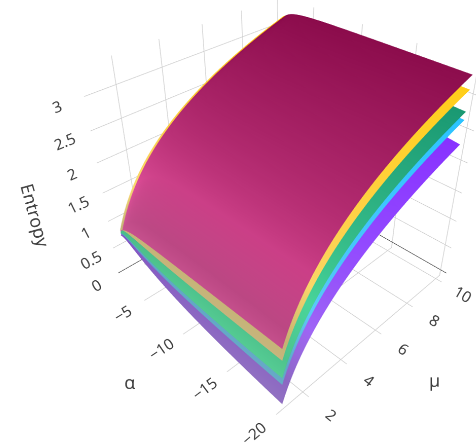

Figure 2. The Shannon Entropy under G_I^0 models.

As we explore the 3D plot, we can observe how changes in μ, α, and L collectively influence the entropy of the G_I^0 distribution. We can identify regions where entropy is high or low, providing insights into the predictability and structure of the distribution in various regions of the parameter space.

C. Estimation of the Shannon Entropy

One of the earliest non-parametric estimators relying on spacings was introduced by [10]. Assuming that $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$ is a random sample from the distribution $F(z)$, the estimator is defined as:

$$\widehat{H}_{\text{V}}(\boldsymbol{Z}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left[\frac{n}{2m} \left(Z_{(i+m)} - Z_{(i-m)} \right) \right],$$

where $m < n/2$ is a positive integer, $Z_{(i+m)} - Z_{(i-m)}$ is the m-spacing and $Z_{(1)} \leqslant Z_{(2)} \leqslant \ldots \leqslant Z_{(n)}$ are the order statistics and $Z_{(i)} = Z_{(1)}$ if $i < 1$, $Z_{(i)} = Z_{(n)}$ if $i > n$.

Several authors have explored adaptations to Vasicek's estimator. In this work, we consider three entropy estimators known for their superior performance:

- [11]: $\widehat{H}_{\text{C}}$.

- [12]: $\widehat{H}_{\mathrm{E}}$.
- [13]: $\widehat{H}_{\mathrm{AO}}$.

These estimators, along with others, are described and studied in [7].

D. Enhanced Bootstrap Technique

We employ the bootstrap technique to refine the precision of existing non-parametric entropy estimators. This approach involves generating new datasets through resampling with repetition from an existing one.

Let's assume that non-parametric entropy estimators $\widehat{H} = \widehat{\theta}(\boldsymbol{Z})$ are inherently biased, that is:

$$\operatorname{Bias}\big(\widehat{\theta}(\boldsymbol{Z})\big) = E\big[\widehat{\theta}(\boldsymbol{Z})\big] - \theta. \tag{8}$$

Our objective is to devise unbiased estimators with reduced variance. To achieve this, we introduce an "ideal estimator" $\breve{\theta}(\boldsymbol{Z})$ using the bias information:

$$\breve{\theta}(\boldsymbol{Z}) = \widehat{\theta}(\boldsymbol{Z}) - \operatorname{Bias}\big(\widehat{\theta}(\boldsymbol{Z})\big). \tag{9}$$

However, $\breve{\theta}(\boldsymbol{Z})$ is not an estimator, because it depends on the true parameter θ, prompting the formulation of a new estimator $\widetilde{H}$. From (8) and (9) we have:

$$\widetilde{H} = 2\widehat{\theta}(\boldsymbol{Z}) - \frac{1}{B}\sum_{b=1}^{B}\widehat{\theta}_b(\boldsymbol{Z}^{(b)}),$$

where B is the number of replications in the bootstrap technique. Applying this methodology, the original estimators by Correa, Ebrahimi, and Al-Omari are now denoted as the proposed bootstrap-enhanced versions: $\widetilde{H}_{\mathrm{C}}$, $\widetilde{H}_{\mathrm{E}}$, and $\widetilde{H}_{\mathrm{AO}}$, respectively.

III. Hypothesis Testing Based on Non-Parametric Entropy

General asymptotic results for functions of spacings are detailed [14], while [15] developed a correction for the case of Shannon entropy. Following the work of these authors, the next result applies:

Lemma 1: Suppose that $f(\cdot)$ is a bounded density bounded away from zero and satisfies a Lipschitz condition on its support. Then, if $m, n \to \infty$ and $m = o(n^{1/2})$, holds that:

$$\sqrt{n}\left(\widetilde{H}_i + \int_{-\infty}^{\infty} f(z)\ln f(z)\mathrm{d}z\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \operatorname{Var}(\ln f(Z))).$$

Consider, starting from the previous lemma, the test of the null hypothesis $\mathcal{H}_0 : T_{\mathcal{D}} = D_0$ as opposed to one of the other three:

$$\begin{aligned}&\mathcal{H}_1 : T_{\mathcal{D}} \neq D_0,\\ &\mathcal{H}_1 : T_{\mathcal{D}} > D_0, \text{ or}\\ &\mathcal{H}_1 : T_{\mathcal{D}} < D_0.\end{aligned}$$

For this purpose, we can use the test statistics:

$$Z_{m,n} = \frac{\sqrt{n}(\widetilde{H} - D_0)}{\sqrt{\operatorname{Var}\big(\ln f(Z)\big)}},$$

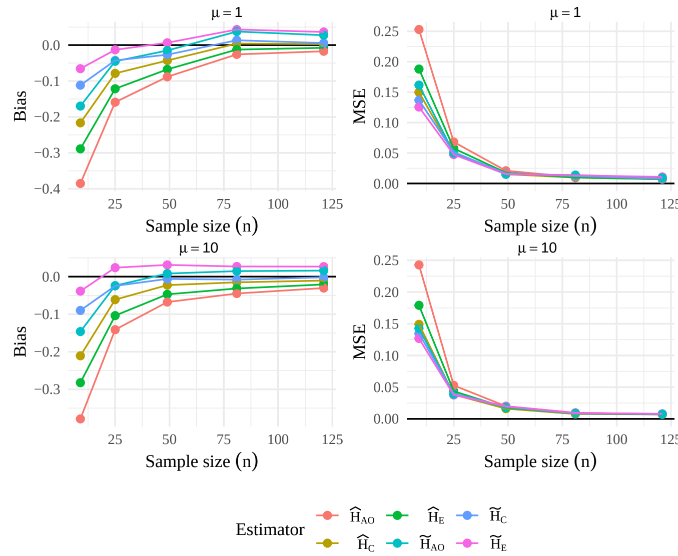

Figure 3. Bias and MSE of entropy estimators for G_I^0, $L = 8$, $\alpha = -20$.

so the null hypothesis should be rejected if (i) $Z_{n,m} > z_{\alpha/2}$ or $Z_{n,m} < -z_{\alpha/2}$ for $\Phi_{\mathcal{N}}(z_{\alpha/2}) = 1 - \alpha/2$ and $\Phi_{\mathcal{N}}$ being the standard normal cumulative distribution function, (ii) $Z_{n,m} > z_{\alpha/2}$ or (iii) $Z_{n,m} < -z_{\alpha/2}$.

The power function for case (i) (two-sided test) at $t \neq D_0$ is given by

$$\begin{aligned}\pi_{m,n}(t) = 1 - \Phi_{m,n}\Big(z_{\alpha/2} - \frac{\sqrt{n}(Z_{m,n} - D_0)}{\sigma}\Big)\\ + \Phi_{m,n}\Big(-z_{\alpha/2} - \frac{\sqrt{n}(Z_{m,n} - D_0)}{\sigma}\Big),\end{aligned}$$

for a sequence of cumulative distribution functions $\Phi_{m,n}(x)$ which tends uniformly to $\Phi_{\mathcal{N}}(z)$.

IV. Results

Simulations are conducted using G_I^0 distribution, with 200 simulated samples of size $n \in \{9, 25, 49, 81, 121\}$, with parameters $\mu \in \{1, 10\}$, $\alpha = -20$, and $L = 8$. In the case of the bootstrap technique, each sample is replicated 100 times with replacement. We choose to use the following heuristic formula for spacing, $m = [\sqrt{n} + 0.5]$.

In Fig. 3 we depict comparisons of bias and mean squared error (MSE) between the original non-parametric entropy estimators and their respective bootstrap-enhanced versions. The use of the bootstrap technique exhibits more precision, reduced bias and MSE, and improved convergence.

The results of simulation are exhibited in Table I. The precision of estimators, as evidenced by bias and MSE comparisons, benefits significantly from the bootstrap technique, particularly for sample sizes below 81.

A hypothesis testing was conducted using non-parametric entropy estimators as test statistics. The results in Table II show that data from the G_I^0 distribution exhibit fully developed speckle behavior, specifically in the limit case with parameters $\mu = 5$, $L = 2$, and $\alpha = -1000$. The true entropy of Γ_{SAR} was

Table I
BIAS AND MSE OF BOOTSTRAP ESTIMATORS FOR G_I^0, $L=8$, $\alpha=-20$.

μ	n	Bias $\widetilde{H}_{\text{C}}$	Bias $\widetilde{H}_{\text{E}}$	Bias $\widetilde{H}_{\text{AO}}$	MSE $\widetilde{H}_{\text{C}}$	MSE $\widetilde{H}_{\text{E}}$	MSE $\widetilde{H}_{\text{AO}}$
1	9	-0.1117	-0.0657	-0.1698	0.1367	0.1255	0.1616
	25	-0.0428	-0.0131	-0.0451	0.0515	0.0474	0.0499
	49	-0.0267	0.0065	-0.0152	0.0165	0.0154	0.0151
	81	0.0134	0.0434	0.0377	0.0121	0.0133	0.0138
	121	0.0055	0.0364	0.0275	0.0078	0.0108	0.0092
10	9	-0.0900	-0.0386	-0.1463	0.1345	0.1268	0.1427
	25	-0.0247	0.0239	-0.0239	0.0399	0.0378	0.0388
	49	-0.0060	0.0312	0.0082	0.0194	0.0198	0.0183
	81	-0.0080	0.0270	0.0147	0.0088	0.0095	0.0088
	121	-0.0019	0.0267	0.0160	0.0080	0.0078	0.0075

Table II
HYPOTHESIS TESTING FOR G_I^0, $\mu=5$, $L=2$, $\alpha=-1000$, $H_{\Gamma_{\text{SAR}}}=2.493$.

n	Estimator	Mean Entropy	Z Statistic	p Value
9	$\widetilde{H}_C$	2.4689	-0.6699	0.5029
	$\widetilde{H}_E$	2.5080	0.3764	0.7066
	$\widetilde{H}_{AO}$	2.4119	-2.2774	0.0228
25	$\widetilde{H}_C$	2.4713	-1.1379	0.2552
	$\widetilde{H}_E$	2.5000	0.3392	0.7344
	$\widetilde{H}_{AO}$	2.4551	-1.9487	0.0513
49	$\widetilde{H}_C$	2.4515	-3.1650	0.0015
	$\widetilde{H}_E$	2.4882	-0.4299	0.6673
	$\widetilde{H}_{AO}$	2.4726	-1.5996	0.1097
81	$\widetilde{H}_C$	2.4937	0.0159	0.9873
	$\widetilde{H}_E$	2.5262	2.8534	0.0043
	$\widetilde{H}_{AO}$	2.4974	0.3594	0.7193
121	$\widetilde{H}_C$	2.4922	-0.1442	0.8853
	$\widetilde{H}_E$	2.5169	2.5684	0.0102
	$\widetilde{H}_{AO}$	2.5052	1.3283	0.1841

set at $H_{\Gamma_{\text{SAR}}}=2.493$.
It is observed that, for different sample sizes, entropy values converge towards the true value of the Gamma SAR distribution. Hypothesis test results were conducted with a $95\,\%$ confidence level. The Z statistic measures the discrepancy between the estimated entropy and the true entropy. The p-values associated with the Z statistic are predominantly greater than 0.05, for sample sizes above 25, suggesting that the data are consistent with the null hypothesis.

V. CONCLUSION

In this study, three estimators renowned for their robust performance across diverse distributions were chosen for evaluation. We examine the impact of applying bootstrap techniques on their precision by comparing each estimator's performance with its bootstrap-enhanced counterpart, utilizing metrics such as bias and MSE. The effectiveness of bootstrap-enhanced non-parametric entropy estimators was observed, demonstrating efficacy in most instances. Nonetheless, it is essential to recognize that the applicability of this technique may not be universal across all estimation methods.

It is worth noting that this analysis represents an initial exploration of SAR Intensity data, and future work will include a more in-depth analysis of effect size and statistical power. Additionally, exploring the impact of the α parameter on estimates and conducting more extensive analyses to assess the generalization of results to different parameter configurations is suggested.

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