Rényi Entropy-Based Heterogeneity Detection in SAR Data

APPENDIX

Derivation of the Rényi Entropy for the $\Gamma_{\mathbf{SAR}}(L,\mu)$ Distribution.

The Rényi entropy of order λ for a continuous random variable Z with density $f_Z(z)$ is given by

$$H_\lambda(Z) = \frac{1}{1-\lambda}\,\ln\Big(\int_0^\infty [f_Z(z)]^\lambda\,dz\Big), \quad \lambda>0,\ \lambda\neq 1. \tag{1}$$

Let $Z\sim\Gamma_{\text{SAR}}(L,\mu)$ with pdf

$$f_{\Gamma_{\text{SAR}}}(z;L,\mu) = \frac{L^L}{\Gamma(L)\,\mu^L}\,z^{\,L-1}\exp\Big(-\tfrac{Lz}{\mu}\Big)\mathbb{1}_{\mathbb{R}_+}(z).$$

Define

$$I = \int_0^\infty \big[f_{\Gamma_{\text{SAR}}}(z;L,\mu)\big]^\lambda\,dz = \left(\frac{L^L}{\Gamma(L)\,\mu^L}\right)^\lambda \int_0^\infty z^{\,\lambda(L-1)}\exp\Big(-\tfrac{\lambda L}{\mu}\,z\Big)\,dz.$$

Using the Gamma integral $\displaystyle\int_0^\infty x^{p-1}e^{-qx}\,dx = \frac{\Gamma(p)}{q^p}$, with $p=\lambda L-\lambda+1$ and $q=\frac{\lambda L}{\mu}$, it follows that

$$I = \left(\frac{L^L}{\Gamma(L)\,\mu^L}\right)^\lambda \frac{\Gamma(\lambda L-\lambda+1)}{\left(\frac{\lambda L}{\mu}\right)^{\lambda L-\lambda+1}}.$$

Taking the natural logarithm,

$$\ln I = \lambda\Big(L\ln L - L\ln\mu - \ln\Gamma(L)\Big) \;+\; \ln\Gamma\big(\lambda L-\lambda+1\big) \;-\; \big(\lambda L-\lambda+1\big)\Big(\ln(\lambda L)-\ln\mu\Big). \tag{2}$$

By expanding (2) and collecting terms in $\ln L$ and $\ln\mu$,

$$\ln I = (1-\lambda)\big(\ln\mu-\ln L\big) \;-\; \lambda\,\ln\Gamma(L) \;+\; \ln\Gamma\big(\lambda(L-1)+1\big) \;-\; \big(\lambda(L-1)+1\big)\,\ln\lambda. \tag{3}$$

Substituting (3) into (1) and simplifying,

$$H_\lambda\big(\Gamma_{\text{SAR}}(L,\mu)\big) = \ln\mu - \ln L + \frac{1}{1-\lambda}\Big[-\lambda\,\ln\Gamma(L) + \ln\Gamma\big(\lambda\,(L-1)+1\big) - \big(\lambda\,(L-1)+1\big)\,\ln\lambda\Big]. \tag{4}$$

This completes the derivation.

Derivation of the Rényi Entropy for the $\mathcal{G}_I^0$ Distribution

Let $Z\sim\mathcal{G}_I^0(\alpha,\gamma,L)$ with pdf

$$f_{\mathcal{G}_I^0}(z;\alpha,\gamma,L) = \frac{L^L\,\Gamma(L-\alpha)}{\gamma^\alpha\,\Gamma(-\alpha)\,\Gamma(L)}\,\frac{z^{\,L-1}}{\big(\gamma+L\,z\big)^{\,L-\alpha}}\mathbb{1}_{\mathbb{R}_+}(z).$$

In particular, this parameterization is consistent with $\gamma = -\mu(\alpha+1)$, so the final expression can be rewritten in terms of μ.

Define

$$I = \int_0^\infty \left[f_{\mathcal{G}_I^0}(z;\alpha,\gamma,L)\right]^\lambda dz = C^\lambda \int_0^\infty \frac{z^{\,\lambda(L-1)}}{\left(\gamma + L\,z\right)^{\,\lambda(L-\alpha)}}\,dz,$$

where

$$C = \frac{L^L\,\Gamma(L-\alpha)}{\gamma^\alpha\,\Gamma(-\alpha)\,\Gamma(L)}.$$

Using the change of variables $t = \frac{Lz}{\gamma}$, $z = \frac{\gamma\,t}{L}$, and $dz = \frac{\gamma}{L}\,dt$, we obtain

$$\begin{aligned} I &= C^\lambda \int_0^\infty \left(\tfrac{\gamma\,t}{L}\right)^{\lambda(L-1)} \left(\gamma + L\,\tfrac{\gamma\,t}{L}\right)^{-\lambda(L-\alpha)} \tfrac{\gamma}{L}\,dt \\ &= C^\lambda\,\frac{\gamma^{\,1+\lambda(\alpha-1)}}{L^{\,1+\lambda(L-1)}} \int_0^\infty \frac{t^{\,\lambda(L-1)}}{(1+t)^{\,\lambda(L-\alpha)}}\,dt. \end{aligned}$$

By the Beta-function identity

$$\int_0^\infty \frac{t^{\,a-1}}{(1+t)^{\,a+b}}\,dt = B(a,b),$$

where

$$a = \lambda(L-1)+1, \quad b = \lambda(-\alpha+1)-1,$$

it follows that

$$I = C^\lambda\,\frac{\gamma^{\,1+\lambda(\alpha-1)}}{L^{\,1+\lambda(L-1)}}\,B(a,b).$$

Next, we note that $\gamma^{\,1+\lambda(\alpha-1)} = \gamma^{\,1-\lambda+\lambda\alpha}$ and $L^{\,1+\lambda(L-1)} = L^{\,\lambda L+1-\lambda}$. Since

$$C^\lambda = \left(\tfrac{L^L}{\gamma^\alpha\,\Gamma(-\alpha)\,\Gamma(L)}\,\Gamma(L-\alpha)\right)^\lambda = L^{\lambda L}\,\gamma^{-\alpha\lambda}\left(\tfrac{\Gamma(L-\alpha)}{\Gamma(-\alpha)\,\Gamma(L)}\right)^\lambda,$$

we obtain

$$I = \gamma^{\,1-\lambda}\,L^{\,\lambda-1}\left(\tfrac{\Gamma(L-\alpha)}{\Gamma(-\alpha)\,\Gamma(L)}\right)^\lambda B(a,b).$$

By (1), the Rényi entropy, is given by:

$$H_\lambda(Z) = \frac{1}{1-\lambda}\,\ln I.$$

Hence,

$$H_\lambda(Z) = \frac{1}{1-\lambda}\,\ln\!\left[\gamma^{\,1-\lambda}\,L^{\,\lambda-1}\left(\tfrac{\Gamma(L-\alpha)}{\Gamma(-\alpha)\,\Gamma(L)}\right)^\lambda B(a,b)\right].$$

Thus, for $Z \sim \mathcal{G}_I^0(\alpha,\gamma,L)$,

$$H_\lambda\big(\mathcal{G}_I^0(\alpha,\gamma,L)\big) = \ln\!\left(\tfrac{\gamma}{L}\right) + \frac{1}{1-\lambda}\left[\lambda\big(\ln\Gamma(L-\alpha) - \ln\Gamma(-\alpha) - \ln\Gamma(L)\big) + \ln B(a,b)\right].$$

Using the property
$$\ln B(a,b) = \ln\Gamma(a) + \ln\Gamma(b) - \ln\Gamma(a+b),$$
where $a+b = \lambda(L-\alpha)$, we have
$$H_\lambda\big(\mathcal{G}_I^0(\alpha,\gamma,L)\big) = \ln\Big(\tfrac{\gamma}{L}\Big) + \frac{1}{1-\lambda}\Big[\lambda\big(\ln\Gamma(L-\alpha) - \ln\Gamma(-\alpha) - \ln\Gamma(L)\big) + \ln\Gamma(a) + \ln\Gamma(b) - \ln\Gamma\big(\lambda(L-\alpha)\big)\Big]. \tag{5}$$

Finally, noting that
$$\mu = -\tfrac{\gamma}{\alpha+1} \quad\Longrightarrow\quad \gamma = -\mu(\alpha+1),$$
and substituting γ into (5), we obtain
$$\begin{aligned} H_\lambda\big(\mathcal{G}_I^0(\alpha,\mu,L)\big) = \ln\mu - \ln L + \ln(-1-\alpha) + \frac{1}{1-\lambda}\Big[\lambda\Big(\ln\Gamma(L-\alpha) - \ln\Gamma(-\alpha) - \ln\Gamma(L)\Big) \\ + \ln\Gamma\big(\lambda(L-1)+1\big) + \ln\Gamma\big(\lambda(-\alpha+1)-1\big) - \ln\Gamma\big(\lambda(L-\alpha)\big)\Big], \end{aligned} \tag{6}$$
which completes the derivation.

Relation to the Γ_{SAR} Distribution

The Rényi entropy of the $\mathcal{G}_I^0(\alpha,\mu,L)$ distribution can be expressed in terms of the Rényi entropy of the $\Gamma_{\text{SAR}}(L,\mu)$ distribution, plus additional terms involving α and the Gamma function. Specifically, we can write:
$$\begin{aligned} H_\lambda\big(\mathcal{G}_I^0(\alpha,\mu,L)\big) = \underbrace{\Big[\ln\mu \;-\; \ln L \;+\; \frac{1}{1-\lambda}\Big(-\lambda\,\ln\Gamma(L) \;+\; \ln\Gamma\big(\lambda(L-1)+1\big) \;-\; \big(\lambda(L-1)+1\big)\,\ln\lambda\Big)\Big]}_{H_\lambda\left(\Gamma_{\text{SAR}}(L,\mu)\right)} \\ + \ln\big(-1-\alpha\big) + \frac{1}{1-\lambda}\Big[\lambda\big(\ln\Gamma(L-\alpha) - \ln\Gamma(-\alpha)\big) + \ln\Gamma\big(\lambda(-\alpha+1)-1\big) - \ln\Gamma\big(\lambda(L-\alpha)\big) + \big(\lambda(L-1)+1\big)\,\ln(\lambda)\Big]. \end{aligned} \tag{7}$$

From (7), the bracketed expression on the first line matches $H_\lambda\big(\Gamma_{\text{SAR}}(L,\mu)\big)$, while the remaining terms account for the parameter α through additional Gamma functions and logarithmic corrections. This decomposition highlights the close relationship between the Rényi entropies of the $\mathcal{G}_I^0$ and Γ_{SAR} distributions.

Limit Behavior of $H_\lambda(\mathcal{G}_I^0)$ as $\alpha \to -\infty$

Proof. We want to show that
$$\lim_{\alpha\to-\infty} H_\lambda\big(\mathcal{G}_I^0\big)(\mu,\alpha,L) = H_\lambda\big(\Gamma_{\text{SAR}}\big)(\mu,L).$$

We can express (7) as follows:
$$\begin{aligned} H_\lambda\big(\mathcal{G}_I^0\big)(\mu,\alpha,L) = H_\lambda\big(\Gamma_{\text{SAR}}\big)(\mu,L) \;+\; \ln\big(-1-\alpha\big) \\ + \frac{1}{1-\lambda}\ln\left[\frac{\Gamma(L-\alpha)^\lambda\,\Gamma\big(\lambda(-\alpha+1)-1\big)\,\lambda^{\lambda(L-1)+1}}{\Gamma(-\alpha)^\lambda\,\Gamma\big(\lambda(L-\alpha)\big)}\right]. \end{aligned}$$

Set

$$\Delta(\alpha) = H_\lambda\big(\mathcal{G}_I^0\big)(\mu, \alpha, L) - H_\lambda\big(\Gamma_{\text{SAR}}\big)(\mu, L).$$

Then

$$\Delta(\alpha) = \ln(-1-\alpha) + \frac{1}{1-\lambda} \ln\left[\frac{\Gamma(L-\alpha)^\lambda\, \Gamma\big(\lambda(-\alpha+1)-1\big)\, \lambda^{\lambda(L-1)+1}}{\Gamma(-\alpha)^\lambda\, \Gamma\big(\lambda(L-\alpha)\big)}\right]. \tag{8}$$

As $\alpha \to -\infty$, we have $-1-\alpha \approx |\alpha|$, so

$$\ln(-1-\alpha) \sim \ln|\alpha|.$$

Note that for large $|\alpha|$, we can the asymptotic relation $\Gamma(x+a)/\Gamma(x+b) \sim x^{a-b}$. Specifically:

$$\Gamma(L-\alpha)/\Gamma(-\alpha) \;\sim\; |\alpha|^L, \quad \Gamma\big(\lambda(-\alpha+1)-1\big)/\Gamma\big(\lambda(L-\alpha)\big) \;\sim\; \big(\lambda|\alpha|\big)^{(\lambda-1)-\lambda L}.$$

Thus, inside the logarithm in (8),

$$\frac{\Gamma(L-\alpha)^\lambda\, \Gamma\big(\lambda(-\alpha+1)-1\big)}{\Gamma(-\alpha)^\lambda\, \Gamma\big(\lambda(L-\alpha)\big)} \;\sim\; |\alpha|^{\lambda L} \times |\alpha|^{(\lambda-1)-\lambda L} = |\alpha|^{\lambda-1}.$$

Since $\lambda^{\lambda(L-1)+1}$ does not depend on α, multiplying by this constant factor does not alter the asymptotic behavior in α. Therefore,

$$\frac{1}{1-\lambda} \ln\big[\dots\big] \;\sim\; \frac{1}{1-\lambda} \ln\big(|\alpha|^{\lambda-1}\big) = \frac{\lambda-1}{1-\lambda} \ln|\alpha| = -\ln|\alpha|.$$

Hence

$$\Delta(\alpha) \;\sim\; \ln|\alpha| - \ln|\alpha| = 0 \quad \text{as}\; \alpha \to -\infty.$$

This shows $\Delta(\alpha) \to 0$, and consequently

$$\lim_{\alpha \to -\infty} H_\lambda\big(\mathcal{G}_I^0\big)(\mu, \alpha, L) = H_\lambda\big(\Gamma_{\text{SAR}}\big)(\mu, L).$$

$\square$