Fórmula de cambio de variables

• En la integral simple

$$\int_c^d f(\underbrace{g(x)}_{u})\cdot \underbrace{g'(x)\,dx}_{du} = \int_{g(c)}^{g(d)} f(u)\,du$$

(regla de sustitución)

$$[c,d] \xrightarrow{\;g\;} \text{intervalo de extremos } g(c),\ g(d)$$

• $u = g(x)$ • fórmula para cambiar dx por du: $du = u'(x)\,dx$

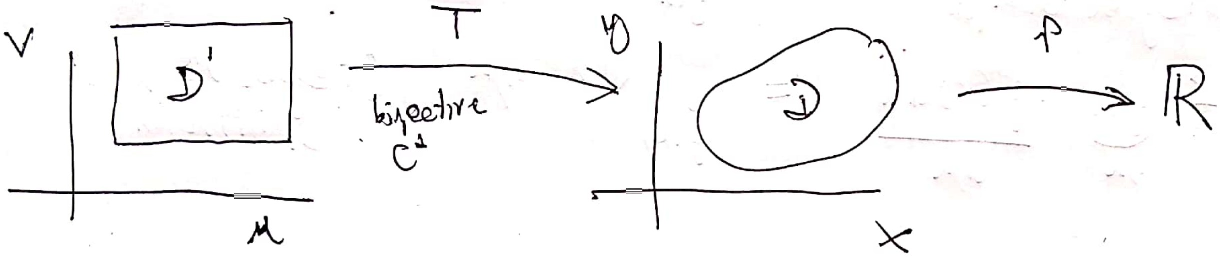
• $[c,d]$ cambia por el intervalo de extremos $g(c)$, $g(d)$

Ejemplo: $$\int_0^1 e^{1-x^2}(-2x)\,dx = \int_1^0 e^{u}\,du = -\int_0^1 e^{u}\,du = \dots$$

$u = 1 - x^2$; $x=0 \Rightarrow u=1$

$du = -2x\,dx$; $x=1 \Rightarrow u=0$

• Fórmula de cambio de variable en integrales dobles



$$T(u,v) = \big(x(u,v),\, y(u,v)\big)$$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Teorema:

$$\iint_D f(x,y)\, dA(x,y) = \iint_{D'} f\big(x(u,v), y(u,v)\big) \cdot \underbrace{\overbrace{\left| JT(u,v) \right|}^{¡\neq 0!}}_{(*)} dA(u,v)$$

(en el plano (u,v))

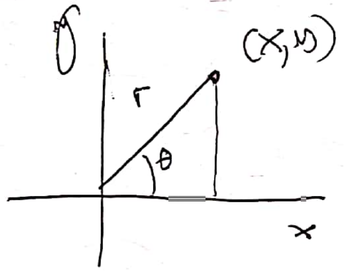
(*) "Jacobiano de T"

$$\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = JT(u,v) \quad ; \quad JT(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

$$dA(x,y) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v)$$

$$\text{"}dx\,dy\text{"} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \text{"}du\,dv\text{"}$$

Ejemplo: coordenadas polares $\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$; $r \geq 0$, $\theta \in [0, 2\pi]$



$$D = \{(x,y)\in\mathbb{R}^2 : 1 \leq x^2+y^2 \leq 4 \ , \ y \geq 0\}$$

cambiamos a polares

$$\iint_D \sqrt{9-x^2-y^2}\, dA(x,y) \stackrel{\downarrow}{=} \iint_{D'(?)} \sqrt{9-(r\cos\theta)^2-(r\,\mathrm{sen}\,\theta)^2} \cdot \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dA(r,\theta)$$

Escaneado con CamScanner

4

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{pmatrix} = r(\cos\theta)^2 + r(\text{sen}\,\theta)^2 = r$$

$$\boxed{dA(x,y) = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dA(r,\theta) = r\,dA(r,\theta)}$$

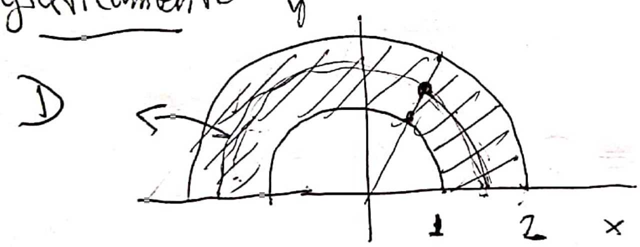
$$= \iint_{D'} \sqrt{9-r^2}\,.\,r\,.\,dA(r,\theta)$$

volvemos a la integral

$\underline{\underline{D'}}$: → traducción de D (en el plano xy) a D' en el plano (r,θ).

de dos maneras:

gráficamente



$\underline{\underline{D'}}$: $\boxed{\begin{matrix} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \end{matrix}}$

$$= \iint_{[1,2]\times[0,\pi]} \sqrt{9-r^2}\,r\,dA(r,\theta) = \int_0^{\pi}\left(\int_1^2 \sqrt{9-r^2}\,.\,r\,.\,dr\right)d\theta = \int_0^{\pi}\left(-\frac{(9-r^2)^{3/2}}{3}\Bigg|_1^2\right)d\theta$$

$$= \int_0^{\pi} -\left(\frac{9-4}{3}\right)^{3/2} + \left(\frac{9-1}{3}\right)^{3/2} d\theta = \left(\frac{8^{3/2}}{3} - \frac{5^{3/2}}{3}\right)\pi .$$

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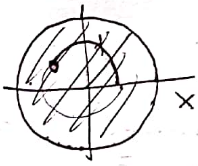
2) Area del circulo $C_d = \{x^2 + y^2 \le d^2\}$ centro $(0,0)$ radio d

$$\text{area}(C_d) = \iint_{C_d} 1 \cdot dA(x,y) \underset{\uparrow}{=} \iint_{C'_d} 1 \cdot r \cdot dA(r,\theta) = \iint_{[0,d]\times[0,2\pi]} r \, dA(r,\theta)$$

$\boxed{\text{pasamos a polares}}$

C'_d



C_d

en polares: $0 \le r \le d$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \left(\int_0^d r \, dr \right) d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} \Big|_0^d \right) d\theta = \int_0^{2\pi} \frac{d^2}{2} \, d\theta = \frac{d^2}{2} \cdot 2\pi = \pi d^2 .$$

Otro modo (no geométrico, más mecánico) en ambos ejemplos

en el segundo $\quad C_d = \{ x^2 + y^2 \le d^2 \} \to r^2\cos^2\theta + r^2\,\text{sen}^2\theta \le d^2$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \qquad r^2(\cos^2\theta + \text{sen}^2\theta) \le d^2$$

$$r^2 \le d^2$$

y θ? $\boxed{0 \le \theta \le 2\pi}$ $\qquad \boxed{0 \le r \le d}$

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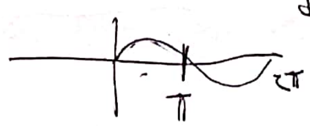
en el primer ejemplo

$D = \{ 1 \le x^2 + y^2 \le 4 \ ; \ y \ge 0 \}$ traducimos estas ecuaciones a r, θ:

$1 \le r^2 \le 4 \longleftrightarrow \boxed{1 \le r \le 2}$

$y \ge 0: \ r\,\text{sen}\,\theta \ge 0 \longleftrightarrow \text{sen}\,\theta \ge 0 \longleftrightarrow \boxed{0 \le \theta \le \pi}$



Otro ejemplo $\iint_D (x-y)\, e^{x+y}\, dA(x,y) = (*)$

D limitado por las rectas $y=x$; $y=x+1$; $y=-x$; $y=-x+2$.

cambio de variables: (LINEAL) $\begin{cases} u = x-y \\ v = x+y \end{cases}$; $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$?

despejo

sumando ambos: $u+v=2x$

restando " $v-u=2y$

$$\begin{cases} x = \frac{u}{2} + \frac{v}{2} \\ y = -\frac{u}{2} + \frac{v}{2} \end{cases}$$

$$dA(x,y) = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| \cdot dA(u,v) = \left|\det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\right| dA(u,v) = \frac{1}{2} dA(u,v).$$

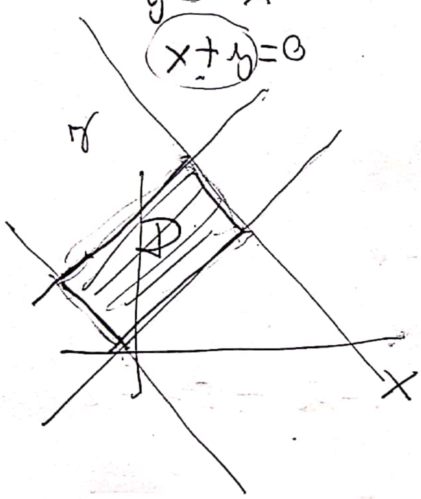
$(*) = \iint_{D'} u \cdot e^{v} \cdot \frac{1}{2}\, dA(u,v)$. ¿Quién es D'?

Traducimos las ecuaciones x, y a ecuaciones con u, v 18

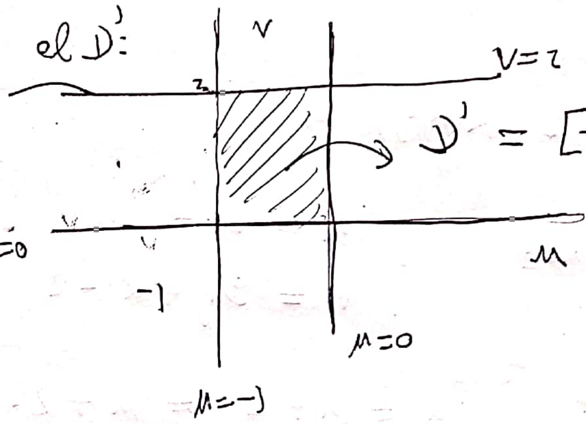
$$y = x \rightarrow -\frac{u}{2} + \frac{v}{2} = \frac{u}{2} + \frac{v}{2} \; ; \; \boxed{0 = u}$$

$$y = x+1 \rightarrow -\frac{u}{2} + \frac{v}{2} = \frac{u}{2} + \frac{v}{2} + 1 \; ; \; \boxed{-1 = u}$$

$$y = -x, \quad x+y=0 \rightarrow \boxed{v = 0} \; ; \quad y = -x+2, \quad \underbrace{x+y}_{v} = 2 \quad \boxed{v = 2}$$



el D':

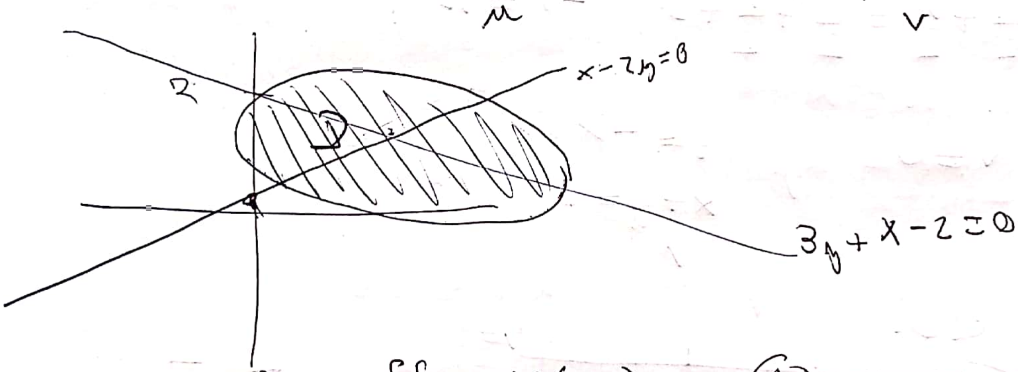


$$D' = [-1,0] \times [0,2]$$

$$= (*) = \iint_{[-1,0]\times[0,2]} u e^{v} \frac{1}{2} \, dA(u,v) = \ldots$$

Ejemplo: Calcular el área del dominio encerrado por la curva:

$$\underbrace{(x-2y)}_{u}^2 + \underbrace{(3y+x-2)}_{v}^2 = 1$$



$$\text{area}(D) = \iint_D 1 \, dA(x,y) = (*)$$

cambio de variable: (LINEAL)

$$\begin{cases} u = x - 2y \\ v = 3y + x - 2 \end{cases} \quad ; \quad v - u = 5y - 2$$

$$\boxed{y = \frac{v-u+2}{5}} \qquad \boxed{x = u + 2y = u + 2\cdot\left(\frac{v-u+2}{5}\right)}$$

$$dA(x,y) = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| dA(u,v)$$

$$= \left|\det\begin{pmatrix} 3/5 & 2/5 \\ -\frac{1}{5} & 1/5 \end{pmatrix}\right| dA(u,v) = \frac{1}{5}\, dA(u,v).$$

$$\text{Area}(D) = \iint_{D'} 1 \cdot \frac{1}{5}\, dA(u,v) = \frac{1}{5}\iint_{u^2+v^2\leq 1} 1\, dA(u,v) = \frac{\pi}{5}.$$

D' : traducimos de x,y a u,v : $u^2+v^2=1$

(dominio ilustrado)