Polinomio de Taylor

polinomio de Taylor en 1 variable

$f : I \to \mathbb{R}$, n-veces derivable en I:
$f', f'', \ldots, f^{(n)}$.

$\uparrow$ intervalo abierto

$a \in \mathbb{R}$

Polinomio de Taylor de orden n (grado n)
de f en a:

$$P_n(x) = \boxed{f(a) + f'(a) \cdot (x-a)} + \frac{f''(a)}{2} \cdot (x-a)^2 +$$

$$+ \frac{f'''(a)}{3!} \cdot (x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(3.2) $\qquad n! = n \cdot (n-1) \cdots 2$.

$P_n(x)$ aproxima a $f(x)$ cerca de a.

$n=1 \quad P_1(x) = f(a) + f'(a) \cdot (x-a)$

Ejemplo: $f(x) = \operatorname{sen} x$, $a=0$, $n=4$

$P_4(x)$

$$P_4(x) = f(0) + f'(0) \cdot (x-0) + \frac{f''(0)}{2} \cdot (x-0)^2 +$$

$$+ \frac{f'''(0)}{6} \cdot (x-0)^3 + \frac{f^{IV}(0)}{4!} \cdot (x-0)^4.$$

Escaneado con CamScanner

$f(x) = \operatorname{sen} x$; $f'(x) = \cos x$; $f''(x) = -\operatorname{sen} x$

$f'''(x) = -\cos x$; $f^{IV}(x) = \operatorname{sen} x$.

$\underline{a=0}$: $f(0) = 0$; $f'(0) = 1$, $f''(0) = 0$; $f'''(0) = -1$;

$f^{IV}(0) = 0$

$$\underline{P_4(x)} = x - \frac{1}{6}x^3$$

McLaurin: polinomio de Taylor si $a=0$.

Propiedades del polinomio de Taylor (de orden n) en a.

1) $P_n(x)$ es el $\underline{\text{único}}$ polinomio $\varphi(x)$ de grado a lo sumo n $(\text{grado}(\varphi) \leq n)$ /

$f(a) = \varphi(a)$; $f'(a) = \varphi'(a)$, $f''(a) = \varphi''(a)$, ...

..., $f^{(n)}(a) = \varphi^{(n)}(a)$.

2) $P_n(x)$ es el $\underline{\text{único}}$ polinomio $\varphi(x)$ de grado $\leq n$ /

$$\lim_{x \to a} \frac{f(x) - \varphi(x)}{(x-a)^n} = 0.$$

Escaneado con CamScanner

$f(x) - P_n(x) = R_n(x)$ resto (o error) de orden n de f en x, en torno a "a".

la propiedad de recién:

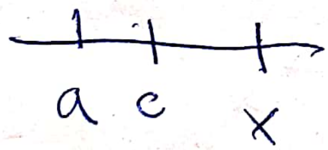
$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0.$$

3) Fórmula de Lagrange de $R_n(x)$:

$f: I \to \mathbb{R}$, $n+1$-veces derivable en I.

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1}$$

c (desconocido), está entre a y x.



Ejemplo: $f(x) = \operatorname{sen} x$, $a = 0$; $n = 4$.

$$P_4(x) = x - \frac{x^3}{6}.$$

$$R_4(x) = \frac{f^{V}(c)}{5!}(x-0)^5 = \frac{\cos(c)}{120} \cdot x^5.$$

c está entre 0 y x.

Calculamos (aproximadamente $\operatorname{sen}(\frac{1}{2})$.

$$\operatorname{sen}\left(\frac{1}{2}\right) \sim P_4\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{6} = \frac{1}{2} - \frac{1}{48} = \frac{23}{48}.$$

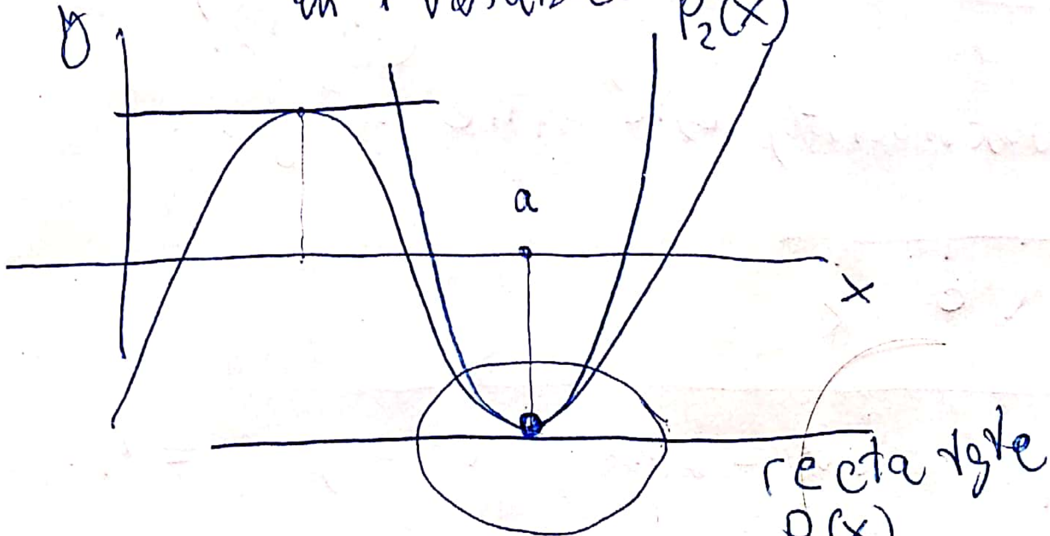
$$\left|R_4\left(\frac{1}{2}\right)\right| = \left|\frac{\cos(c)}{120}\cdot\left(\frac{1}{2}\right)^5\right| = \frac{\overbrace{|\cos c|}^{\leq 1}}{120}\cdot\frac{1}{32} \leq \frac{1}{120\cdot 32}$$

$0 < c < \frac{1}{2}$.

Polinomio de Taylor de $f = f(x,y)$,

de orden $n = 2$.

idea en 1 variable.



$f = f(x,y)$, $f: D \to \mathbb{R}$, f es C^2 en D:

disco abierto centrado en $P = (a,b)$

existen $f_x, f_y, f_{xx}, f_{xy} = f_{yx}, f_{yy}$ en D, y son todas continuas en D.

$P=(a,b)$ punto base:

Polinomio de Taylor de orden 2 de f en torno a P:

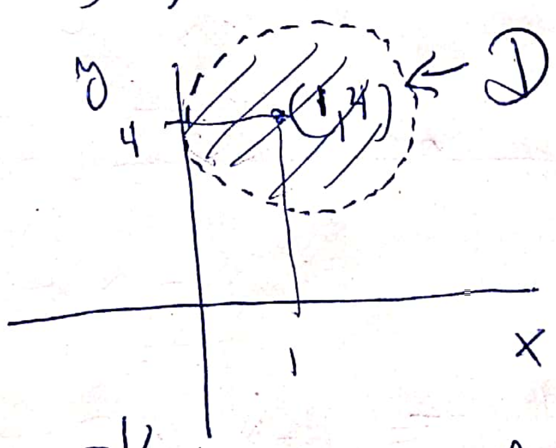
$$P_2(x,y) = f(a,b) + f_x(a,b)\cdot(x-a) + f_y(a,b)\cdot(y-b)$$

$$+ \frac{1}{2} f_{xx}(a,b)\cdot(x-a)^2 + f_{xy}(a,b)\cdot(x-a)(y-b) +$$

$$+ \frac{1}{2} f_{yy}(a,b)\cdot(y-b)^2 .$$

Ejemplo: $f(x,y) = x^{1/2} + xy + y^{1/2}$.

$P=(1,4)$; $n=2$.



$f: D \to \mathbb{R}$

$$f_x = \frac{1}{2}\cdot x^{-1/2} + y \;;\; f_y = x + \frac{1}{2} y^{-1/2}$$

$$= \frac{1}{2}\,\frac{1}{x^{1/2}} + y$$

Escaneado con CamScanner

$$f_{xx} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-3/2} \quad , \quad f_{xy} = 1 = f_{yx}$$ 6

$$f_{yy} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) y^{-3/2}.$$

son todas continuas en D. $\quad P=(1,4)$

$$f(1,4) = 1^{1/2} + 1.4 + 4^{1/2} = 7$$

$$f_x(1,4) = \frac{1}{2}\cancel{x}^{-1/2} + \cancel{y} = \frac{1}{2} + 4 = \frac{9}{2}$$

$$f_y(1,4) = 1 + \frac{1}{2}\cdot(4)^{-1/2} = \frac{5}{4}$$

$$f_{xx}(1,4) = -\frac{1}{4}\cdot 1^{-3/2} = -\frac{1}{4}$$

$$f_{xy}(1,4) = 1$$

$$f_{yy}(1,4) = -\frac{1}{4}\cdot(4)^{-3/2} = -\frac{1}{32}$$

$$P_2(x,y) = 7 + \frac{9}{2}(x-1) + \frac{5}{4}\cdot(y-4) + \frac{1}{2}\cdot\left(-\frac{1}{4}\right)\cdot(x-1)^2$$

$$+ \frac{1}{2}\left(-\frac{1}{32}\right)(y-4)^2 + 1\cdot(x-1)(y-4)$$

$(x,y) = (\underbrace{1.1, 3.8}_{\text{próximos a } (1,4)})$

Escaneado con CamScanner

$P_2(1.1, 3.8) = 7.178135$

$f(1.1, 3.8) = 7.17816....$

Propsedad (de $P_2(x,y)$)

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - P_2(x,y)}{\|(x,y) - (a,b)\|^2} = 0.$$

Vinculado a:

$n=1$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - \overbrace{P_1(x,y)}^{\text{plano tangente.}}}{\|(x,y) - (a,b)\|^1} = 0$$

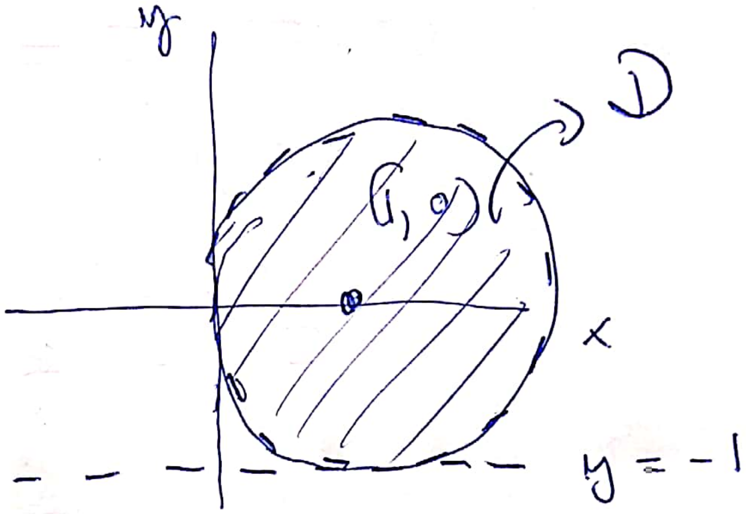
f es diferenciable en (a,b)

Ejemple: comprobar que

$$\lim_{(x,y)\to(1,0)} \frac{\overbrace{x\ln(1+y)}^{f(x,y)} - xy + \frac{1}{2}y^2}{\underbrace{(x-1)^2 + y^2}_{\|(x,y) - (1,0)\|^2}} = 0$$

$$f(x,y) = x\ln(1+y)$$ debe ser $y \geq -1$.

$$P = (1,0)$$



$$f(1,0) = 1\cdot\ln(1+0) = 0$$

$$f_x = \ln(1+y)$$

$$f_y = x\cdot\frac{1}{1+y}\cdot 1 = \frac{x}{1+y} = x\cdot(1+y)^{-1}$$

$$f_{xx} = 0$$

$$f_{xy} = \frac{1}{1+y}$$

$$f_{yy} = x\cdot(-1)(1+y)^{-2} = \frac{-x}{(1+y)^2}$$

en $(1,0)$

$$f_x(1,0) = 0$$
$$f_y(1,0) = 1$$
$$f_{xx}(1,0) = 0$$
$$f_{xy}(1,0) = 1$$
$$f_{yy}(1,0) = -1$$

$$P_2(x,y) = f(1,0) + f_x(1,0)\cdot(x-1) +$$
$$+ f_y(1,0)\cdot(y-0) + \frac{1}{2}f_{xx}(1,0)\cdot(x-1)^2 + f_{xy}(1,0)(x-1)y$$
$$+ \frac{1}{2}f_{yy}(1,0)\cdot(y-0)^2.$$

9

$$P_2(x,y) = 1\cdot y + 1\cdot(x-1)\cdot y + \frac{1}{2}\cdot(-1)y^2$$

$$= y + (x-1)y - \frac{1}{2}y^2$$

Propiedad (vale!)

$$0 \overset{\checkmark}{=} \lim_{(x,y)\to(1,0)} \frac{x\ln(1+y) - \{y + \overbrace{(x-1)y}^{xy-y} - \frac{1}{2}y^2\}}{\|(x,y)-(1,0)\|^2}$$

qué tiene que ver con

$$\lim_{(x,y)\to(1,0)} \frac{x\ln(1+y) - xy + \frac{1}{2}y^2}{\underbrace{\|(x,y)-(1,0)\|^2}_{(x-1)^2+y^2}} = 0 \; ?$$