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Polinomio de Taylor - parte 2.

Una forma abreviada de presentar a $P_2(x,y)$ de $f = f(x,y)$ en $P = (a,b)$

f es C^3 en D (disco centrado en P)

Llamamos matriz Hessiana de f en P

$$Hf(P) = \begin{pmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{pmatrix}$$

Si $f = f(x,y,z)$; $P = (a,b,c)$

$$Hf(P) \begin{pmatrix} f_{xx}(P) & \boxed{f_{xy}(P)} & f_{xz}(P) \\ \boxed{f_{yx}(P)} & f_{yy}(P) & f_{yz}(P) \\ f_{zx}(P) & f_{zy}(P) & f_{zz}(P) \end{pmatrix}$$

Recordando que $\nabla f(P) = (f_x(P), f_y(P))$

(si $f = f(x,y,z)$; $\nabla f(P) = (f_x(P), f_y(P), f_z(P))$)

$$P_2(x,y) = f(P) + \nabla f(P) \cdot (x-a, y-b)$$

$$\underbrace{+ \frac{1}{2} \underbrace{(x-a \mid y-b)}_{1\times 2} \underbrace{Hf(a,b)}_{2\times 2} \underbrace{\begin{pmatrix} x-a \\ y-b \end{pmatrix}}_{2\times 1}}_{1\times 1 \text{ (número)}}$$

Si $f=f(x,y,z)$, f es C^2 en una bola de $\mathbb{R}^3$, centrada en $P=(a,b,c)$ ← punto base.

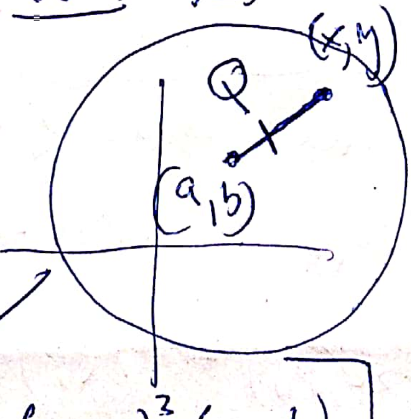
$$P_2(x,y,z) = f(a,b,c) + \nabla f(a,b,c)\cdot(x-a,\ y-b,\ z-c) +$$

$$\frac{1}{2}\begin{pmatrix}x-a & y-b & z-c\end{pmatrix} Hf(a,b,c)\begin{pmatrix}x-a\\ y-b\\ z-c\end{pmatrix}.$$

Fórmula (de Lagrange) del Resto $R_2(x,y)$

$f=f(x,y)$, $R_2(x,y) = f(x,y) - P_2(x,y)$

f es C^3 en un disco D centrado en $P=(a,b)$ (existen y son continuas en todo D todas las derivadas parciales de orden 3).



En el segmento de extremos (a,b) y (x,y) hay un punto $Q=(c,d)$ -desconocido-

$$\boxed{R_2(x,y) = \frac{f_{xxx}(Q)}{6}\cdot(x-a)^3 + \frac{f_{xxy}(Q)}{2}\cdot(x-a)^2\cdot(y-b) + \frac{f_{xyy}(Q)}{2}\cdot(x-a)(y-b)^2 + \frac{f_{yyy}(Q)}{6}\cdot(y-b)^3.}$$

fórmula de Lagrange del resto $R_2(x,y)$.

Observaciones sobre esta fórmula:

1) aparecen todas las derivadas parciales de orden 3 – escondidas:

$$f_{xxy}(\varphi) = f_{xyx}(\varphi) = f_{yxx}(\varphi)$$

lo mismo pasa con ~~f~~ $f_{yyx}(\varphi)$, $f_{yxy}(\varphi)$, $f_{xyy}(\varphi)$

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2) todos los términos de $R_2(x,y)$ tienen grado 3: $(x-a)^3$; $(x-a)^2(y-b)$; $(x-a)(y-b)^2$; $(y-b)^3$.

$(x-a)$, $(y-b)$

$$\frac{R_2(x,y)}{\|(x,y)-(a,b)\|^2} = \frac{\text{suma de términos de grado 3}}{(x-a)^2+(y-b)^2}$$

con técnicas básicas de límites $\lim\limits_{(x,y)\to(a,b)} \dfrac{R_2(x,y)}{\|(x,y)-(a,b)\|^2} = 0$

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3) $\underset{(c,d)}{\overset{\parallel}{\varphi}}$ entre (x,y) y (a,b) es el análogo del valor intermedio c entre a y x en la fórmula de Lagrange del resto para funciones de 1 variable.

Ejemplo: Hallar $P_2(x,y)$ y la fórmula de Lagrange del resto $R_2(x,y)$ para

$$f(x,y) = x^{7/2}\cdot y^{1/3} \quad ; \quad P=(1,1).$$

$$f(x,y) = x^{7/2}\cdot y^{1/3} \quad ; \quad f(1,1) = 1$$

$$f_x = \frac{7}{2}\cdot x^{5/2}\cdot y^{1/3} \quad ; \quad f_x(1,1) = \frac{7}{2}$$

$$f_y = x^{7/2}\cdot\frac{1}{3}\cdot y^{-2/3} \quad ; \quad f_y(1,1) = \frac{1}{3}$$

$$f_{xx} = \frac{7}{2}\cdot\frac{5}{2}\cdot x^{3/2}\cdot y^{1/3} \quad ; \quad f_{xx}(1,1) = \frac{35}{4}$$

$$f_{xy} = \frac{7}{2}\cdot x^{5/2}\cdot\frac{1}{3}\, y^{-2/3} \quad ; \quad f_{xy}(1,1) = \frac{7}{6}$$

$$f_{yy} = x^{7/2}\cdot\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot y^{-5/3} \quad ; \quad f_{yy}(1,1) = -\frac{2}{9}$$

$$P_2(x,y) = 1 + \frac{7}{2}\cdot(x-1) + \frac{1}{3}\cdot(y-1) + \frac{1}{2}\cdot\frac{35}{4}\cdot(x-1)^2$$
$$+ \frac{1}{3}\cdot(x-1)(y-1) + \frac{1}{2}\left(-\frac{2}{9}\right)\cdot(y-1)^2$$

$$f_{xxx} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot x^{1/2}\cdot y^{1/3}$$

$$f_{xxy} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{1}{3}\, x^{3/2}\cdot y^{-2/3}$$

$$f_{xyy} = \frac{7}{2}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)\cdot x^{5/2}\cdot y^{-5/3}$$

$$f_{yyy} = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{5}{3}\right)\cdot x^{7/2}\cdot y^{-8/3}$$

en D continuas: f es C^3 en D.

$$\varphi = (c,d)$$

$$R_2(x,y) = \frac{105}{8}\cdot\frac{1}{6}\cdot c^{1/2} d^{1/3}\cdot(x-1)^3 + \frac{35}{12}\cdot\frac{1}{2}\, c^{3/2} d^{-2/3}(x-1)^2(y-1)$$

$$- \frac{7}{9}\cdot\frac{1}{2}\, c^{5/2}\cdot d^{-5/3}\cdot(x-1)^1\cdot(y-1)^2 + \frac{10}{27}\cdot\frac{1}{6}\cdot c^{7/2}\cdot d^{-8/3}\cdot(y-1)^3$$

Calculemos aproximadamente $(0.9)^{7/2}\cdot(1.2)^{1/3}$

$$= \underbrace{f(0.9, 1.2)}_{\text{próximo a } P=(1,1)} \sim P_2(\underbrace{0.9}_{x}, \underbrace{1.2}_{y}) =$$

$$= 1 + \frac{7}{2}\cdot(-0.1) + \frac{1}{3}\cdot(0.2) + \frac{35}{8}\cdot(-0.1)^2 +$$

$$+ \frac{1}{2}\cdot(-0.1)\cdot(0.2) - \frac{1}{9}(0.2)^2 = 0{,}736194$$

Estimemos el error ~~$R_2(x,y)$~~ $R_2(0.9, 1.2)$

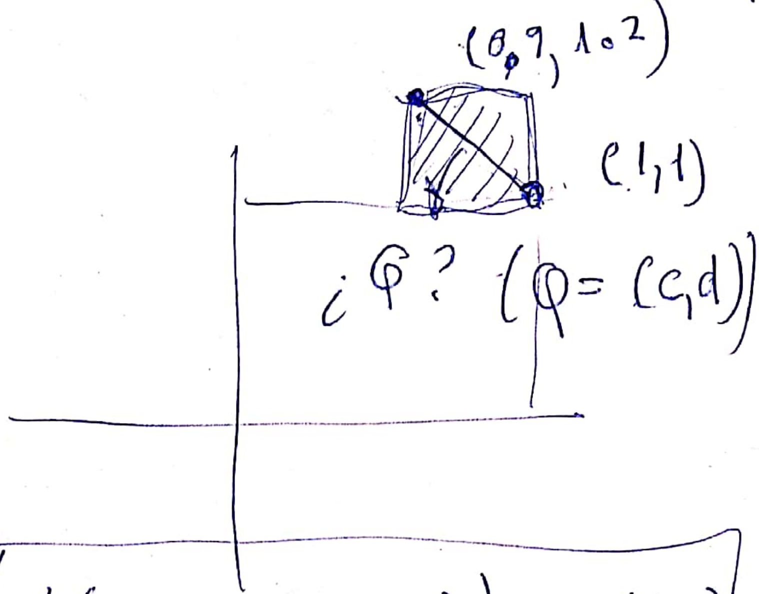
$$|R_2(0.9, 1.2)| = \left|\frac{35}{16}\cdot c^{1/2} d^{1/3}\cdot(-0.1)^3 + \frac{35}{24}\, c^{3/2}\cdot d^{-2/3}\cdot(-0.1)^2(0.2)\right.$$

$$\left. - \frac{7}{18}\cdot c^{5/2} d^{-5/3}\cdot(-0.1)\cdot(0.2)^2 + \frac{5}{81}\cdot c^{7/2} d^{-8/3}\cdot(0.2)^3\right|$$

$$|R_2(0.9, 1.2)| \leq \frac{35}{16} \cdot c^{1/2} \cdot d^{1/3} \cdot \frac{1}{1000} + \frac{35}{24} \cdot c^{3/2} d^{-2/3} \cdot \frac{1}{100} \cdot \frac{2}{10}$$

desig. triang.

$$+ \frac{7}{18} \cdot c^{5/2} \cdot d^{-5/3} \cdot \frac{4}{1000} + \frac{5}{81} \cdot c^{7/2} \cdot d^{-8/3} \cdot \frac{8}{1000}$$



$$\boxed{\varphi = t\big((0.9, 1.2) - (1,1)\big) + (1,1)}, \quad t \in [0,1]$$

$\varphi = (c,d)$ está entre $(1,1)$ y $(0.9, 1.2)$

$$0.9 \leq c \leq 1 \rightarrow c^{1/2};\ c^{3/2};\ c^{5/2};\ c^{7/2} \leq 1$$

$$1 \leq d \leq 1.2 \rightarrow d^{1/3} \leq (1.2)^{1/3} \leq 2 \quad (\sqrt[3]{8})$$

$$1 \leq d \Rightarrow d^{-2/3} \leq 1^{-2/3} = 1$$

$$d^{-5/3};\ d^{-8/3} \leq 1$$

$$|R_2(0.9, 1.2)| \leq \frac{35}{16} \cdot 1 \cdot 2 \cdot \frac{1}{1000} + \frac{35}{24} \cdot 1 \cdot 1 \cdot \frac{2}{1000}$$

$$+ \frac{7}{18} \cdot 1 \cdot 1 \cdot \frac{4}{1000} + \frac{5}{81} \cdot 1 \cdot 1 \cdot \frac{8}{1000} \sim 0.00671\ldots$$