- Propiedades de integrales dobles
- Integración en $\mathbb{R}^3$ (cálculo)

Propiedades:

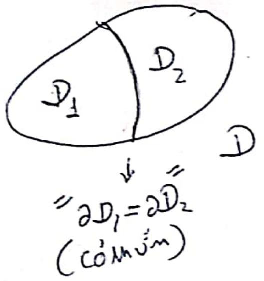
1) $f, g : D \to \mathbb{R}$ continuas, $\alpha, \beta \in \mathbb{R}$: $\iint_D \alpha f(x,y) + \beta g(x,y)\, dA =$
$D \subset \mathbb{R}^2$

$= \alpha \iint_D f(x,y)\, dA + \beta \iint_D g(x,y)\, dA.$

2) Si $f(x,y) \leq g(x,y)$ para $(x,y) \in D$ $\Rightarrow$ $\iint_D f(x,y)\, dA \leq \iint_D g(x,y)\, dA.$

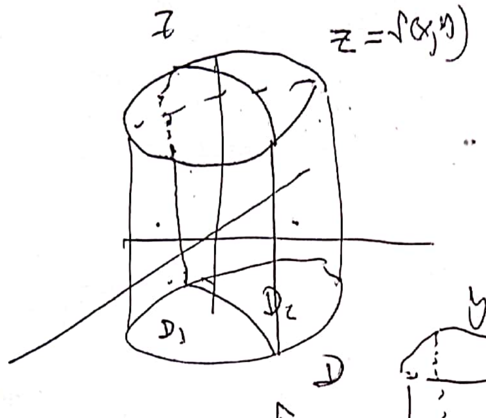
3) Supongamos que el dominio D se descompone en dos partes D_1, D_2

$D = D_1 \cup D_2$; $D_1 \cap D_2 =$ frontera común (una curva)



entonces $\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$.

$f(x,y) \geq 0$

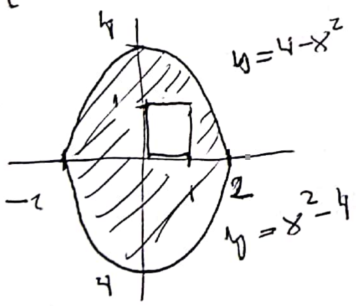


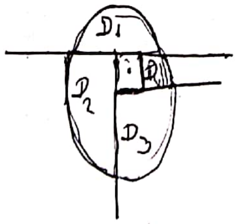
sólido bajo $z=f(x,y)$, sobre D queda partido en dos sólidos, ambos bajo $z=f(x,y)$, uno sobre D_1 el otro sobre D_2

vol sobre D_1 + vol sobre D_2 = vol sobre D

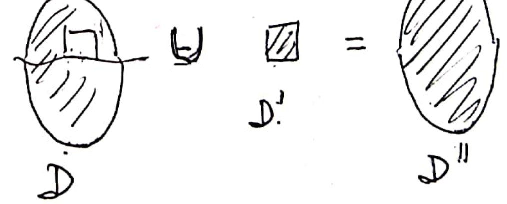
Ejemplo: Calcular $\iint_D x e^y \, dA$,

$D = \{ -2 \leq x \leq 2 \; ; \; x^2-4 \leq y \leq 4-x^2 \} - [0,1]\times[0,1]$





o bien



Usamos 3): $\iint_{D''} xe^{y}\,dA = \underbrace{\iint_{D} xe^{y}\,dA}_{\text{SE DESPEJA ?}} + \underbrace{\iint_{[0,1]\times[0,1]} xe^{y}\,dA}_{\checkmark}$,

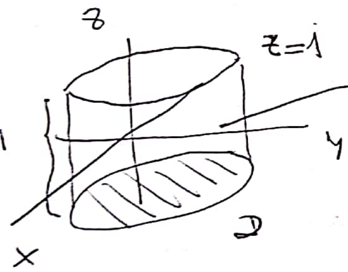
$\underbrace{\iint_{D''} xe^{y}\,dA}_{\checkmark}$

$$D'' = \{-2 \leq x \leq 2;\ x^2-4 \leq y \leq 4-x^2\}$$

$$\int_{-2}^{2}\left(\int_{x^2-4}^{4-x^2} xe^{y}\,dy\right)dx$$

$$\int_0^1\left(\int_0^1 xe^{y}\,dy\right)dx$$

4) $\iint_D 1\,dA = \text{área}(D)$.



vol. = area base × altura

= area (D) × 1 = area (D).

Integrales en $\mathbb{R}^3$:

$$Q = [a,b] \times [c,d] \times [e,f] \qquad f : Q \to \mathbb{R}$$

Def

$$\iiint_{Q} f(x,y,z)\, dV = \lim \sum_{k=1}^{k} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_n^k, y_{j\ell}^k, z_\ell^k) \cdot \frac{b-a}{n} \cdot \frac{d-c}{m} \cdot \frac{f-e}{k}$$

(límite de sumas de Riemann

$$\begin{cases} n \to \infty \\ m \to \infty \\ k \to \infty \end{cases}$$

Cómo se calculan?

Teorema (Fubini en $\mathbb{R}^3$): $f : Q \to \mathbb{R}$ continua

$$\iiint_{Q} f(x,y,z)\, dV = \iint_{[a,b]\times[c,d]} \left(\int_{e}^{f} f(\underbrace{x, y}_{\text{ctes}}, \overset{\text{variable}}{z})\, dz \right) dA = \iint_{[a,b]\times[e,f]} \left(\int_{c}^{d} f(\underbrace{x, \overset{\text{var.}}{y}, z}_{\text{ctes.}})\, dy \right) dA$$

$$= \iint_{[c,d]\times[e,f]} \left(\int_{a}^{b} f(\overset{\text{var.}}{x}, \underbrace{y, z}_{\text{ctes}})\, dx \right) dA$$

Ejemplo:

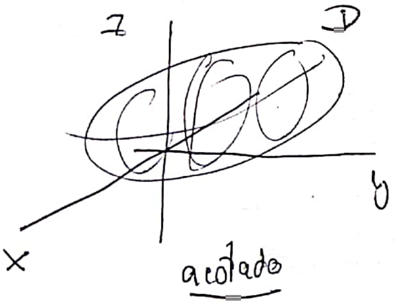
Calcular $\displaystyle\iiint_{[-1,1]\times[0,1]\times[2,3]} x y^2 e^{z^3}\, dV$

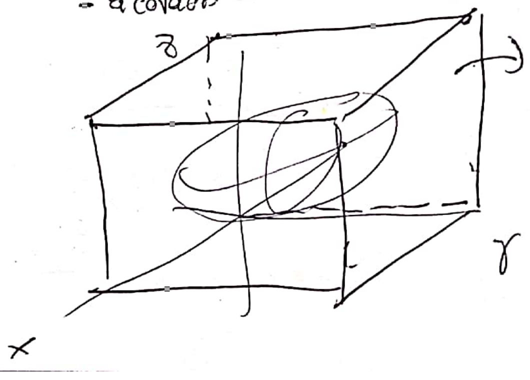
$$= \iint_{[0,1]\times[2,3]} \left(\int_{-1}^{1} x y^2 e^{z^3}\, dx \right) dA = \iint_{[0,1]\times[2,3]} \left(\frac{x^2}{2} y^2 e^{z^3} \Big|_{x=-1}^{x=1} \right) dA$$

$$= \iint_{[0,1]\times[2,3]} \Big(\underbrace{\tfrac{1}{2} y^2 e^{z^3} - \tfrac{1}{2} y^2 e^{z^3}}_{=0} \Big) dA = \iint_{[0,1]\times[2,3]} 0 \, dA = 0.$$

Integrales en dominios más generales de $\mathbb{R}^3$ (DEFINICIÓN)

- acotado -





$\to \beta = [a,b]\times[c,d]\times[e,f]$

$\beta \supset D$.

definimos: $$\hat{f}(x,y,z) = \begin{cases} f(x,y,z) & , \text{ si } (x,y,z) \in D \\ 0 & ; \text{ si } (x,y,z) \in Q - D \end{cases}$$ 6

ponemos (por definición): $$\iiint_D f(x,y,z)\, dV = \underbrace{\iiint_{[a,b]\times[c,d]\times[e,f]} \hat{f}(x,y,z)\, dV}_{\text{usamos Fubini}} .$$

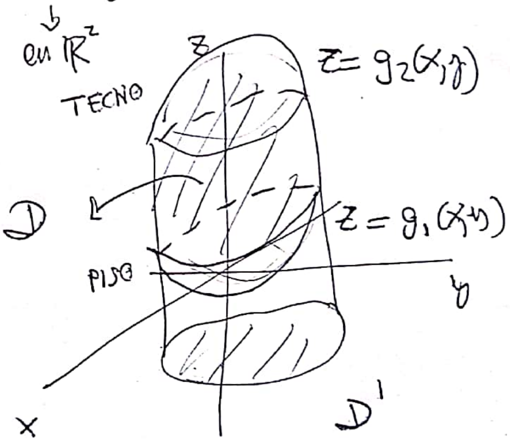
Dominios elementales de $\mathbb{R}^3$: (3 tipos)

Tipo I: $D = \{ (x,y,z) \in \mathbb{R}^3 ;\ (x,y) \in D' ,\ g_1(x,y) \leq z \leq g_2(x,y) \}$

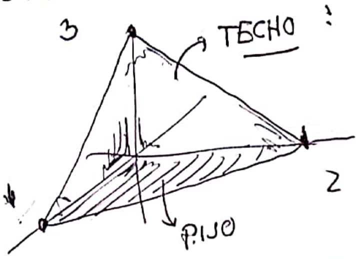
en $\mathbb{R}^2$

$g_1, g_2 : D' \to \mathbb{R}$; $g_1(x,y) \leq g_2(x,y)$

continuas



Ejemplo: consideremos la pirámide

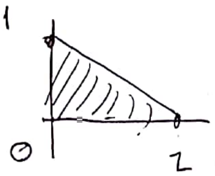


$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

$$z = \underbrace{3 - 3x - \frac{3}{2}y}_{g_2(x,y)}$$

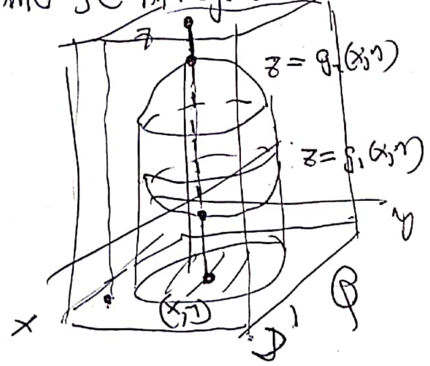
$$z = \underbrace{0}_{g_1(x,y)}$$

$D' =$



$$D = \left\{ (x,y) \in D' \ ; \ 0 \leq z \leq 3 - 3x - \frac{3}{2}y \right\}$$

Cómo se integra en un dominio de tipo I.



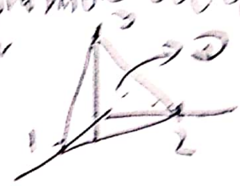
idea: integramos primero respecto de z

$$\iiint_D f(x,y,z)\,dV = \iiint_{\Omega} f(x,y,z)\,dV \underset{\text{Fubini}}{=}$$

$$= \iint_{[a,b]\times[c,d]} \left(\int_e^f f(x,y,z)\,dz \right) dA \Rightarrow$$

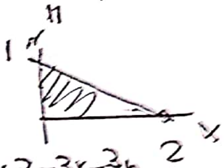
$$= \iint_{D'} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right) dA .$$

Ejemplo: Calcular $\iiint_{D} 2x + 4z \, dV$, D = pirámide del ejemplo anterior



$$\iiint_{D} 2x+4z \, dV = \odot$$

$$D = \left\{ (x,y) \in D' \, ; \, 0 \leq z \leq 3 - 3x - \tfrac{3}{2} y \right\}$$



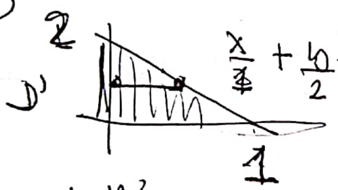
$$\odot = \iint_{D'} \left(\int_{0}^{3-3x-\frac{3}{2}y} 2x + 4z \, dz \right) dA = \iint_{D'} \left(2xz + 2z^2 \Big|_{z=0}^{z=3-3x-\frac{3}{2}y} \right) dA$$

$$= \iint_{D'} 2x\left(3-3x-\frac{3}{2}y\right) + 2\left(3-3x-\frac{3}{2}y\right)^2 - 0 \, dA$$

$$\left(9+9x^2+\frac{9}{4}y^2-18x-9y+9xy\right)$$

$$= \iint_{D'} 6x - 6x^2 - 3xy + 18 + 18x^2 + \frac{9}{2}y^2 - 36x - 18y + 18xy \, dA$$

$$= \iint_{D'} -30x - 12x^2 + 15xy + 18 + \frac{9}{2}y^2 \, dA$$



$\frac{x}{1}+\frac{y}{2}=1 \to x = 1-\frac{y}{2}$

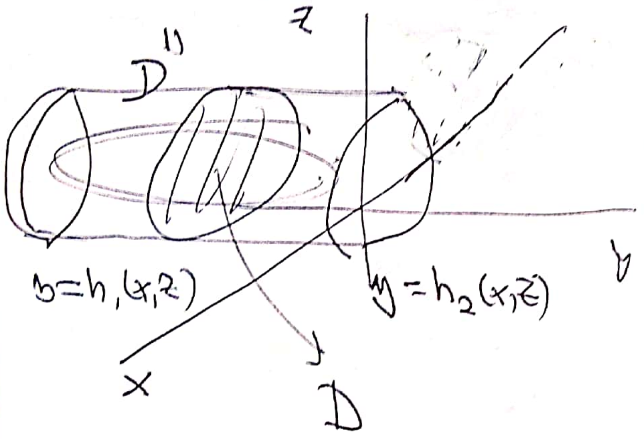
$0 \le y \le 2$

$0 \le x \le 1-\frac{y}{2}$

$$= \int_0^2 \left(\int_0^{1-y/2} -30x-12x^2+15xy+18+\frac{9}{2}y^2 \, dx \right) dy = \dots$$

Dominios de tipo II:

$$D = \{ h_1(x,z) \leq y \leq h_2(x,z) \; ; \; (x,z) \in D'' \text{ del plano } xz \}$$



: fórmula para integrar en tipo II

$$\iiint_D f(x,y,z)\, dV =$$

$$= \iint_{D''} \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dy \right) dA$$

Dominio de Tipo III

$$D = \{ k_1(y,z) \leq x \leq k_2(y,z) \; ; \; (y,z) \in D''' \text{ del plano } yz \}$$

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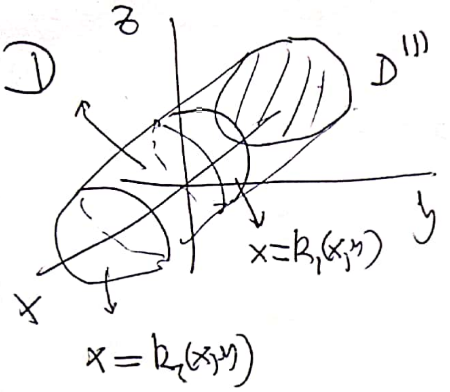


fórmula para integrar en dominios de tipo III:

$$\iint_{D'''}\left(\int_{k_1(y,z)}^{k_2(y,z)} f(x,y,z)\,dx\right)dA.$$

Ejemplo: Calcular la masa del sólido D

$$D=\{x\le 1-y^2-z^2;\ x\ge 0,\ y\ge 0,\ z\ge 0\}$$

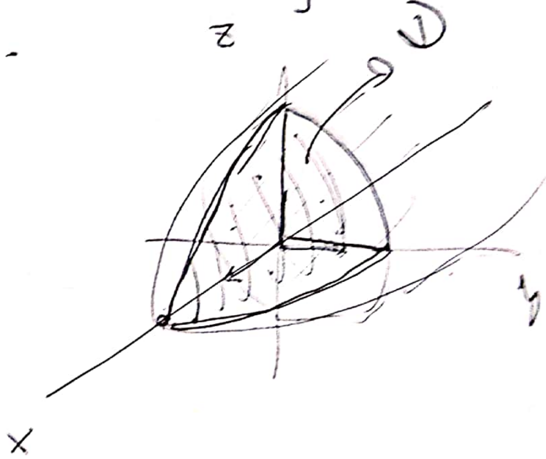
cuya densidad es $\delta(x,y,z)=yz$.

$$\text{masa}(D)=\iiint_D \delta(x,y,z)\,dV$$

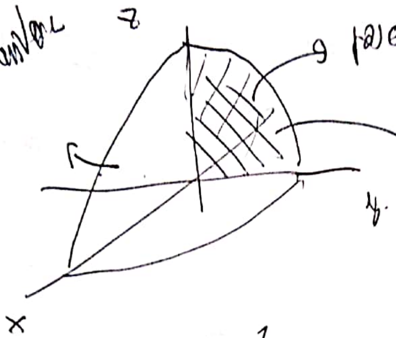
$$\iiint_D yz\,dV.$$

tomamos la igualdad

$x=1-y^2-z^2$



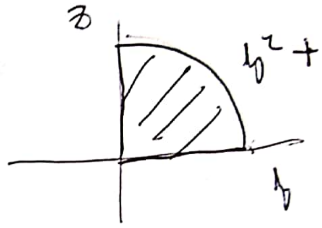
pared delantera



→ pared trasera: $x = 0$
$h_1(y,z)$

$x = \underbrace{1 - y^2 - z^2}_{h_2(y,z)}$

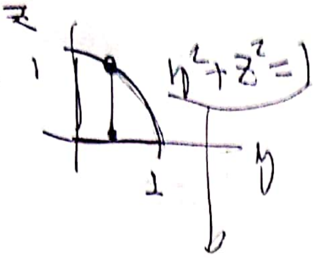
$D''':$



$$\iiint_D yz\, dV = \iint_{D'''} \left(\int_0^{1-y^2-z^2} yz\, dx \right) dA =$$

$$D = \left\{ 0 \le x \le 1 - y^2 - z^2 \;;\; (y,z) \in D''' \right\}$$

$$= \iint_{D'''} \left(xyz \Big|_{x=0}^{x=1-y^2-z^2} \right) dA = \iint_{D'''} (1 - y^2 - z^2)yz - 0\, dA = \iint_{D'''} yz - y^3 z - yz^3\, dA$$



$$0 \le y \le 1 \quad ; \quad 0 \le z \le \sqrt{1-y^2}$$

$$z = +\sqrt{1-y^2}$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-y^2}} yz - y^3 z - yz^3 \, dz \right) dy$$

$$= \int_0^1 \left(y\frac{z^2}{2} - y^3\frac{z^2}{2} - y\frac{z^4}{4} \Big|_{z=0}^{z=\sqrt{1-y^2}} \right) dy = \int_0^1 y(1-y^2) - y^3\frac{(1-y^2)}{2} - y\frac{(1-y^2)^2}{4} \, dy$$

$$= \dots$$