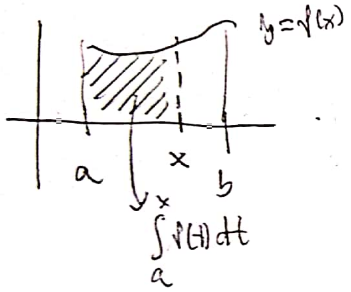
ANEXO: Teoremas sobre integración

1. Teorema (Fundamental del Cálculo)

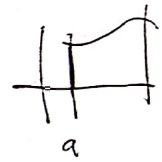
$f:[a,b]\to\mathbb{R}$ continua, $x\in[a,b]$, $F(x)=\int_a^x f(t)\,dt$



Entonces $F:[a,b]\to\mathbb{R}$ es continua y es derivable a (a,b)

$F'(x)=f(x)$.

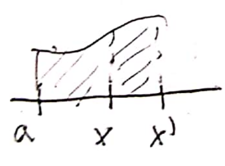
demostración: observación: si $x=a$ $\quad F(a)=\int_a^a f(t)\,dt=0$.



Continuidad de F: $x,x'\in[a,b]$, sup. $x\leq x'$

$$F(x')-F(x)=\int_a^{x'} f(t)\,dt-\int_a^{x} f(t)\,dt=\int_x^{x'} f(t)\,dt.$$

Propiedad:



: $\int_a^x f+\int_x^{x'} f=\int_a^{x'} f$

$$|F(x')-F(x)| = \left|\int_x^{x'} f(t)\,dt\right| \le \int_x^{x'} |f(t)|\,dt \le \int_x^{x'} M\,dt = M.(x'-x)$$

Propiedad: g continua $\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt$.

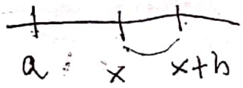
f continua a $[a,b]$ $\Rightarrow$ $|f(t)| \le M$ cte

$$g(t) \le |g(t)| \;\wedge\; -g(t) \le |g(t)| \Rightarrow \pm\int_a^b g(t)\,dt \le \int_a^b |g(t)|\,dt$$

es decir $|F(x')-F(x)| \le M.|x'-x|$ $\Rightarrow$ F es continua a $[a,b]$

$$F(x+h)-F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

$h>0$



Consideremos $f:[x,x+h] \to \mathbb{R}$, f alcanza máximo y mínimo (absolutos) a $[x,x+h]$: es decir hay $y_h, z_h \in [x,x+h]$ tales que $f(y_h) \le f(t) \le f(z_h)$, para todo $t \in [x,x+h]$

$$\Rightarrow f(y_h)\cdot h = \int_x^{x+h} f(y_h)\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} f(z_h)\,dt = f(z_h)\cdot h$$

$$f(y_h)\cdot h \le F(x+h) - F(x) \le f(z_h)\cdot h$$

dividimos todo por h ($h>0$): queda

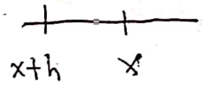
$$\underbrace{f(y_h)}_{\to f(x)} \le \frac{F(x+h)-F(x)}{h} \le \underbrace{f(z_h)}_{\to f(x)}$$

. Tomamos límite $h \to 0^+$.

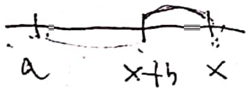
$y_h, z_h \in [x, x+h]$, $h \to 0$

$$\Rightarrow \lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x).$$

Si $h<0$



$$F(x+h) - F(x) = -\Big(\underbrace{F(x) - F(x+h)}_{\int_{x+h}^{x} f(t)\,dt}\Big) = -\int_{x+h}^{x} f(t)\,dt$$



en $[x+h, x]$, f alcanza mínimo $f(y_h)$ y un máximo $f(z_h)$

$$f(y_h) \le f(t) \le f(z_h) \Rightarrow f(y_h)(-h) = \int_{x+h}^{x} f(y_h)\,dt \le \int_{x+h}^{x} f(t)\,dt \le \int_{x+h}^{x} f(z_h)\,dt = f(z_h)\cdot(-h)$$

$$-h\cdot f(y_h) \le \int_{x+h}^{x} f(t)\,dt \le -h\cdot f(z_h)$$

$$-h \cdot f(y_h) \leq -[F(x+h) - F(x)] \leq -f(z_h) \cdot h$$ 4

dividimos todo por $\underline{-h > 0}$

$$f(y_h) \leq \frac{F(x+h) - F(x)}{h} \leq f(z_h) \quad ; \quad h \to 0^- \ \ldots$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) : \quad F'(x) = f(x).$$

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2. <u>Teorema</u> (Regla de Barrow)

$f : [a,b] \to \mathbb{R}$ continua. Sea $G : [a,b] \to \mathbb{R}$, continua ~~y~~ en $[a,b]$, derivable en (a,b) / $G'(x) = f(x)$. (G es <u>una primitiva</u> de f)

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

<u>demostración</u>: el T.F.C. $F(x) = \int_a^x f(t)\,dt$ es <u>otra</u> primitiva de f.

con la característica $F(a) = 0$. F, G son dos primitiva de f :

5

$(F - G)' = f' - f' = 0 \Rightarrow F - G = cte$ (Ejercicio $h' = 0$ en $(a,b) \Rightarrow h = cte$)
T. Valor medio de Lagrange.

en particular $x = a$ $\quad \underbrace{F(a)}_{=0} - G(a) = cte \Rightarrow -G(a) = cte$

$F(x) - G(x) = -G(a)$. Pongamos $\underline{x=b}$
$\forall x \in [a,b]$ queda $F(b) - G(b) = -G(a)$

$$\int_a^b f(t)\,dt = \underbrace{F(b)} = G(b) - G(a)$$

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3. Teorema (Fórmula de cambio de variable en la integral simple)

$$\int_{g(a)}^{g(b)} f(t)\,dt = \int_a^b f(g(u)) \cdot g'(u)\,du$$

f continua, $\underbrace{g \text{ derivable con } g' \text{ continua}}_{g \text{ es } C^1}$.

$t = g(u) \rightarrow dt = g'(u)\,du$
$u = a \rightarrow t = g(a)$
$u = b \rightarrow t = g(b)$

demostración: $H(x) = \int_{g(a)}^{g(x)} f(t)\,dt$; $K(x) = \int_{a}^{x} f(g(u))\cdot g'(u)\,du$

6

Calculamos H', K':

$\boxed{K'(x) = f(g(x))\cdot g'(x)}$ ← TFC.

H': H es una composición.

$L(x) = \int_{g(a)}^{x} f(t)\,dt$; $H(x) = L(g(x))$

$\boxed{H'(x)} \underset{\text{cadena}}{=} L'(g(x))\cdot g'(x) = \boxed{f(g(x))\cdot g'(x)}$

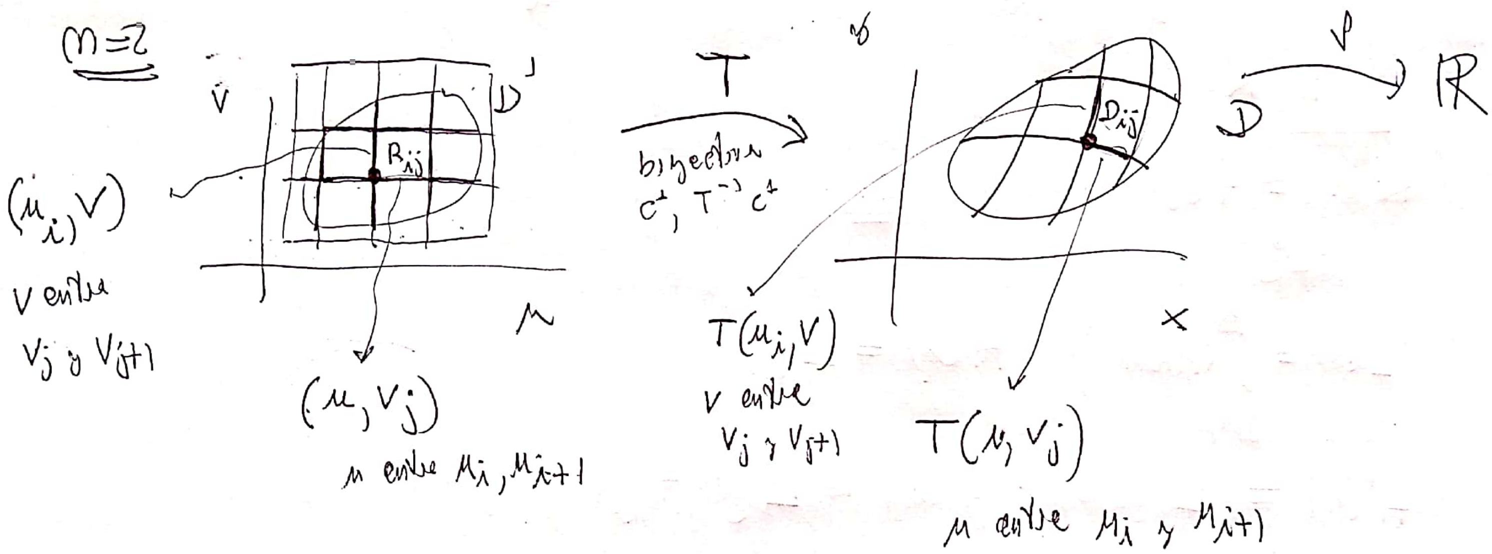
$L'(x) \underset{\text{TFC}}{=} f(x)$

$\Rightarrow K' = H' \Rightarrow K - H = \underline{cte}$. Para $x=a$: $H(a) = 0$; $K(a) = 0$

$\Rightarrow K(x) = H(x)$ $\forall x$, en particular $x = b$: $K(b) = H(b)$ (✓)

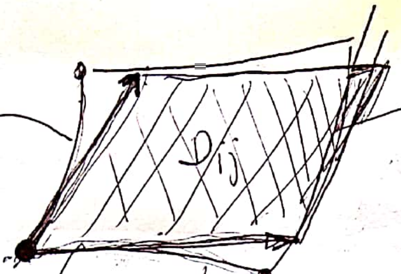
4. Fórmula de cambio de variables en integrales múltiples.

$\underline{n=2}$



$$\iint_D f(x,y)\, dA(x,y) = \text{límite de sumas de } \boxed{\text{"Riemann } \underline{\text{curvadas}}\text{"}} \sum_{j=1}^{m}\sum_{i=1}^{n} f(T(u_i,v_j)) \cdot \text{área}(D_{ij})$$

estudiamos cómo $\underline{\text{aproximar}}$ área(D_{ij})



$T(u_i, v)$ v entre v_j y v_{j+1}

velocidad x tiempo

$T_v(u_i, v_j) \cdot \Delta v$

P_{ij} paralelogramo tangente a D_{ij}

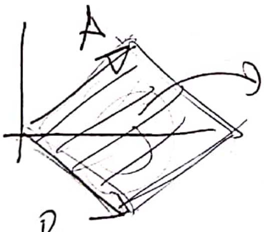
$\text{area}(D_{ij}) \approx \text{area}(P_{ij})$

$T(u, v_j)$ u entre u_i y u_{i+1}

vector velocidad x tiempo transcurrido

$T_u(u_i, v_j) \cdot \Delta u$

$\text{area}(P_{ij})$ = area del paralelogramo de los $T_u(u_i, v_j) \cdot \Delta u$ y $T_v(u_i, v_j) \cdot \Delta v$



$\text{area} = \|A \times B\| = \left| \det \begin{pmatrix} A \\ B \end{pmatrix} \right|$

Ejercicio

$$\text{area}(P_{ij}) = \left| \det \begin{pmatrix} T_u(u_i,v_j)\cdot \Delta u \\ T_v(u_i,v_j)\cdot \Delta v \end{pmatrix} \right| = \Delta u \Delta v \left| JT(u_i,v_j) \right|$$

$$\iint_D f(x,y)\, dA(x,y) = \text{límite de sumas de Riemann curvadas} = \sum_{j=1}^{m} \sum_{i=1}^{n} f(T(u_i,v_j))\cdot \underbrace{\text{area}(D_{ij})}_{\approx\, \text{area}(P_{ij})}$$

$$\approx$$

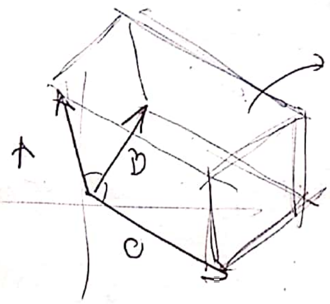
$$\underset{\text{límite}}{=} \underbrace{\sum_{j=1}^{m} \sum_{i=1}^{n} f(T(u_i,v_j))\cdot \left| JT(u_i,v_j) \right| \cdot \Delta u \cdot \Delta v}_{\text{suma de Riemann de la función } f(T(u,v))\cdot |JT(u,v)| \text{ en la región de } D'}$$

$$= \iint_{D'} f(T(u,v))\cdot \left| JT(u,v) \right| dA(u,v)$$

Escaneado con CamScanner

10

en $\mathbb{R}^3$: mismo ~~A~~ idea



P_{ijk} : $\mathrm{vol}(P_{ijk}) = |\langle A, B\times C\rangle|$

$= |A \cdot (B\times C)|$

$= \left|\det\begin{pmatrix} A \\ \hline B \\ \hline C \end{pmatrix}\right|$

Ej