ANEXO DE LAS TEÓRICAS 11 y 12

Def $f = f(x,y)$ se dice diferenciable

en (x_0, y_0) si

(1) $\frac{\partial f}{\partial x}(x_0, y_0)$ y $\frac{\partial f}{\partial y}(x_0, y_0)$ existen

(2) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \left[f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \right]}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Teorema 1 Si f es diferenciable en (x_0, y_0), entonces f es continua en (x_0, y_0).

Dem Q. v. q. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

equivalentemente, $\lim_{P\to P_0} f(P) = f(P_0)$

$P=(x,y)$, $P_0=(x_0,y_0)$

$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)=\underbrace{\left(\frac{\partial f}{\partial x}(x_0,y_0),\frac{\partial f}{\partial y}(x_0,y_0)\right)}_{\nabla f(x_0,y_0)}\cdot(x-x_0,y-y_0)$$

$$=\nabla f(P_0)\cdot(P-P_0)$$

Luego, f es diferenciable si

$$\lim_{P\to P_0}\frac{f(P)-[f(P_0)+\nabla f(P_0)\cdot(P-P_0)]}{|P-P_0|}=0$$

$$f(P) = \underbrace{\left(\frac{f(P) - f(P_0) - \nabla f(P_0)\cdot(P-P_0)}{|P-P_0|}\right)}_{\substack{\xrightarrow[P\to P_0]{} 0 \text{ pues} \\ f \text{ es dif.}}} \underbrace{|P-P_0|}_{\xrightarrow[P\to P_0]{} 0} + f(P_0) + \underbrace{\nabla f(P_0)\cdot(P-P_0)}_{\xrightarrow[P\to P_0]{} 0}$$

Luego $\lim\limits_{P\to P_0} f(P) = f(P_0)$ como queríamos ver. $\blacksquare$

Ej

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{si } (x,y)\neq(0,0) \\ 0 & \text{si } (x,y)=(0,0) \end{cases}$$

Calculemos $\frac{\partial f}{\partial x}(0,0)$ y $\frac{\partial f}{\partial y}(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\overset{=0}{f(h,0)} - \overset{=0}{f(0,0)}}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{\overset{=0}{f(0,k)} - \overset{=0}{f(0,0)}}{k} = 0$$

Veamos que f no es cont.

Tomo $y=x$

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Luego f no es cont. y por el Teo, f no es dif. en el $(0,0)$.

Teorema 2

Si $f(x,y)$ verifica que
$\frac{\partial f}{\partial x}(x,y)$ y $\frac{\partial f}{\partial y}(x,y)$ son continuas
en un entorno de (x_0,y_0),
entonces f es diferenciable en (x_0,y_0)

Def: Una función se dice de clase C^1 si es continua y sus derivadas parciales también son continuas.

Obs El Teo 2 dice que

$$C^1 \Rightarrow \text{diferenciable}$$

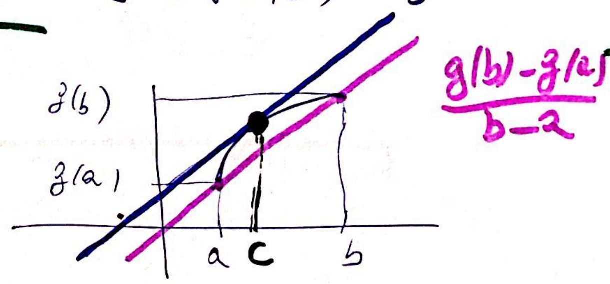
Demostración

Para la demostración usaremos el Teorema del valor medio de Lagrange.

Si $g(x)$ es una función de 1 variable que es derivable, entonces

$$g(b)-g(a) = g'(c)\cdot(b-a)$$

para algún punto intermedio $c \in (a,b)$



$$f(x,y) - f(x_0,y_0) = \underbrace{f(x,y) - f(x_0,y)}_{\frac{\partial f}{\partial x}(\xi,y)(x-x_0)} + \underbrace{f(x_0,y) - f(x_0,y_0)}_{\frac{\partial f}{\partial y}(x_0,\eta)(y-y_0)}$$

- $g(x) = f(x,y)$

$$g(x) - g(x_0) = g'(\xi)\cdot(x-x_0) \qquad x_0 < \xi < x \text{ o } x < \xi < x_0$$

$$\underbrace{f(x,y) - f(x_0,y) = \frac{\partial f}{\partial x}(\xi,y)(x-x_0)}\qquad \Rightarrow \xi \longrightarrow x_0 \text{ si } x \to x_0.$$

- $h(y) = f(x_0,y)$ $\qquad y_0 < \eta < y$ o $y < \eta < y_0$

$$h(y) - h(y_0) = h'(\eta)\cdot(y-y_0) \qquad \Rightarrow \eta \longrightarrow y_0 \text{ si } y \to y_0$$

$$\underbrace{f(x_0,y) - f(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,\eta)\cdot(y-y_0)}$$

luego

$$\overline{f(x,y)-f(x_0,y_0)} - \underbrace{\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)} = \overline{\underbrace{\frac{\partial f}{\partial x}(\xi,y)(x-x_0) + \frac{\partial f}{\partial y}(x_0,\eta)(y-y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}}$$

$$= \left[\frac{\partial f}{\partial x}(\xi, y) - \frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, \eta) - \frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

Ahora,

$$\frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} =$$

$$= \left[\frac{\partial f}{\partial x}(\xi, y) - \frac{\partial f}{\partial x}(x_0,y_0)\right]\frac{x - x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} + \left[\frac{\partial f}{\partial y}(x_0,\eta) - \frac{\partial f}{\partial y}(x_0,y_0)\right]\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$\xi \downarrow x_0$, $y \downarrow y_0$ $\quad \downarrow \frac{\partial f}{\partial x}(x_0, y_0)$

$\left| \quad \right| \leq 1$

$\eta \downarrow y_0$ $\quad \downarrow \frac{\partial f}{\partial y}(x_0, y_0)$

$\left| \quad \right| \leq 1$

$$\xrightarrow[(x,y) \to (x_0, y_0)]{} 0 \qquad \boxed{///}$$

Teorema 3 de (Clairaut–Shwarz)

Si f_{xy} y f_{yx} son continuas en un entorno de (x_0, y_0)

$\Longrightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Dem

Consideremos $g(x) = f(x,y) - f(x,y_0)$

$h(y) = f(x,y) - f(x_0,y)$

Observemos que $\underbrace{g(x) - g(x_0)}_{g'(\xi)(x-x_0)} = \underbrace{h(y) - h(y_0)}_{h'(\eta)(y-y_0)}$

$$g'(\xi) = \frac{\partial f}{\partial x}(\xi, y) - \frac{\partial f}{\partial x}(\xi, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(\xi, \bar{\eta})\,(y - y_0)$$

Lagrange

$$h'(\eta) = \frac{\partial f}{\partial y}(x, \eta) - \frac{\partial f}{\partial y}(x_0, \eta) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(\bar{\xi}, \eta)\,(x - x_0)$$

$$\Rightarrow \frac{\partial^2 f}{\partial y\,\partial x}(\xi, \bar{\eta})\cancel{(y-y_0)}\cancel{(x-x_0)} = \frac{\partial^2 f}{\partial x\,\partial y}(\bar{\xi}, \eta)\cancel{(x-x_0)}\cancel{(y-y_0)}$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(\xi, \bar{\eta}) = \frac{\partial^2 f}{\partial x\,\partial y}(\bar{\xi}, \eta)$$

$\eta, \bar{\eta} \xrightarrow[y \to y_0]{} y_0$

$\xi, \bar{\xi} \xrightarrow[x \to x_0]{} x_0$

Pasando al límite, como las derivadas son continuas concluyo lo deseado $\blacksquare$