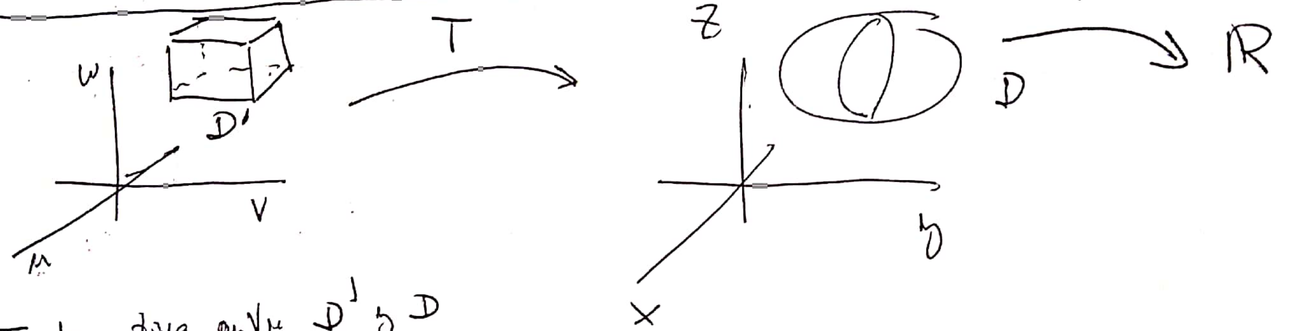
Fórmula de Cambio de Variables en $\mathbb{R}^3$



T biyectiva entre D' y D

C^1, la inversa T^{-1}, es C^1.

$$T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w))$$

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

Teorema

$$\iiint_D f(x,y,z)\, dV(x,y,z) = \iiint_{D'} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \cdot \underbrace{\left| JT(u,v,w) \right|}_{\text{Jacobiano del Cambio de variables}} dV(u,v,w)$$

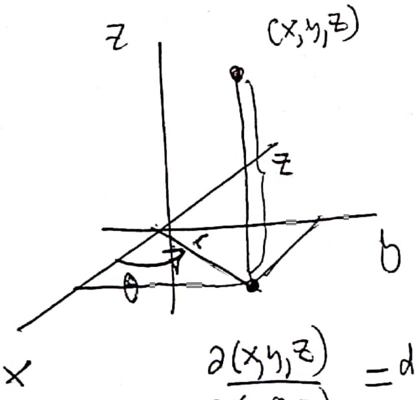
$$JT(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

Regla de cambio de diferenciales:

$$dV(x,y,z) = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \cdot dV(u,v,w)$$

Ejemplos

1) Coordenadas cilíndricas ("polares + z")



| (x,y) $\rightsquigarrow$ (r,θ) |
z igual |

$$\begin{cases} x = r\cos\theta & , \ r \geq 0 \\ y = r\,\mathrm{sen}\,\theta & \theta \in [0, 2\pi) \\ z = z & z \in \mathbb{R} \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & -r\,\mathrm{sen}\,\theta & 0 \\ \mathrm{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

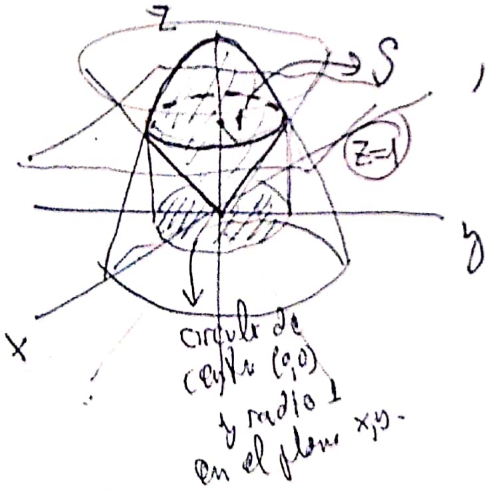
$$dV(x,y,z) = r\, dV(r,\theta,z).$$

$$|r| = r$$

Ejemplo:

1) Calcular el centro de masa del sólido S

$$S = \{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq z \leq 2 - x^2 - y^2\}$$



, la densidad de S: $\boxed{\delta(x,y,z) = z + \sqrt{x^2+y^2}}$

buscamos $\begin{cases} z = \sqrt{x^2+y^2} \rightarrow z^2 = x^2+y^2 \\ z = 2 - x^2 - y^2 \rightarrow z = 2 - z^2 \\ \quad 2 - (x^2+y^2) \end{cases}$

$$z = \sqrt{x^2+y^2} \ ; \ z = 2 - x^2 - y^2$$

$z^2 + z - 2 = 0 \rightarrow 1$; reemplazando $z=1$ en la 1era

$\cancel{-2}$

$1 = \sqrt{x^2+y^2}$; $\boxed{1 = x^2+y^2}$ (con $z=1$)

traducimos S a cilíndricos

$\boxed{\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}}$ z? $\boxed{r \leq z \leq 2 - r^2}$

centro de masa:

$$\text{masa}(S) = \iiint_S \delta(x,y,z)\, dV(x,y,z) \underset{\Uparrow}{=} \iiint \delta(r\cos\theta, r\,\text{sen}\,\theta, z) \cdot r\, dV(r,\theta,z)$$

pasamos a S' cilíndricos $\{(r,\theta,z): 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ r \leq z \leq 2-r^2\}$

$$= \int_0^{2\pi}\left(\int_0^1\left(\int_r^{2-r^2}(z+r)\cdot r\, dz\right)dr\right)d\theta = \int_0^{2\pi}\left(\int_0^1\left(\frac{z^2}{2}r + r^2 z\Big|_r^{2-r^2}\right)dr\right)d\theta$$

$\delta(x,y,z) = z + \sqrt{x^2+y^2}$

$\delta(r\cos\theta, r\,\text{sen}\,\theta, z) = z + r$

$$= \int_0^{2\pi} \left(\int_0^1 \frac{(2-r^2)^2 r}{2} + r^2(2-r^2) - \frac{r^3}{2} + r^3 \, dr \right) d\theta = \ldots$$

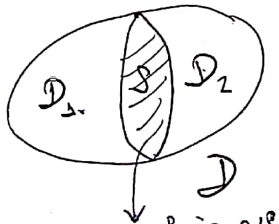
centro de masa $(\bar{x}, \bar{y}, \bar{z})$: $\bar{x} = \dfrac{\iiint_S x\,\delta(x,y,z)\,dV(x,y,z)}{\text{masa}(S)}$

$$\bar{y} = \frac{\iiint_S y\,\delta(x,y,z)\,dV(x,y,z)}{\text{masa}(S)} \; ; \quad \bar{z} = \frac{\iiint_S z\,\delta(x,y,z)\,dV(x,y,z)}{\text{masa}(S)}$$

calcularlo en cilíndricas.

Propiedades (de $\iiint$, similares a $\iint$)

1)



$S =$ superficie que separa D_1 de D_2

$D_1 \cup D_2 = D$

$D_1 \cap D_2 = S$ (superficie)

$$\iiint_D f(x,y,z)\,dV(x,y,z) = \iiint_{D_1} f(x,y,z)\,dV(x,y,z) + \iiint_{D_2} f(x,y,z)\,dV(x,y,z)$$

2) $\iiint_D 1 \, dV(x,y,z) = \text{vol}(D)$ 16

Recordar: $\dfrac{\iiint_D f(x,y,z)\,dV(x,y,z)}{\text{vol}(D)} = f_{\text{promedio}}$; en este caso $f(x,y,z)=1$, $f_{\text{promedio}}=1$.

Ejemplo: Calcular el volumen de S:

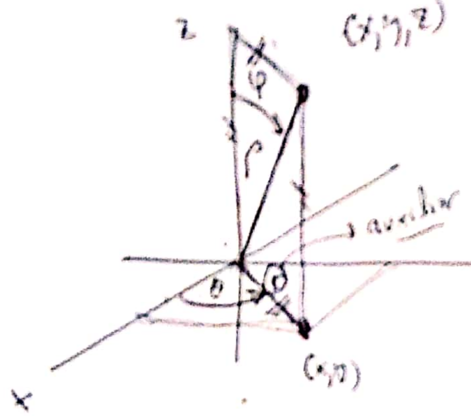
$S = \{ \sqrt{x^2+y^2} \leq z \leq 2-x^2-y^2 \}$

en cilíndricas: $S' = \{ 0 \leq r \leq 1 , 0 \leq \theta \leq 2\pi ; \; r \leq z \leq 2-r^2 \}$

$$\text{vol}(S) = \iiint_S 1 \, dV(x,y,z) = \iiint_{S'} 1 \cdot r \, dV(r,\theta,z)$$

$$= \int_0^{2\pi} \left(\int_0^1 \left(\int_r^{2-r^2} 1 \cdot r \, dz \right) dr \right) d\theta = \dots$$

2) Coordenadas esféricas.



$\rho = \sqrt{x^2+y^2+z^2}$

$\rho \geq 0$

$\theta \in [0,2\pi)$

$\varphi \in [0,\pi]$

$x = r\cos\theta = \rho\cos\theta\,\mathrm{sen}\,\varphi$

$\rho\,\mathrm{sen}\,\varphi$

$y = r\,\mathrm{sen}\,\theta = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi$

$z = \rho\cos\varphi$

$$\begin{cases} x = \rho\cos\theta\,\mathrm{sen}\,\varphi \\ y = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi \\ z = \rho\cos\varphi \end{cases} \qquad \begin{array}{l} \rho \geq 0 \\ \theta \in [0,2\pi) \\ \varphi \in [0,\pi] \end{array} \quad ; \quad \rho^2 = x^2+y^2+z^2.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{pmatrix} \cos\theta \operatorname{sen}\varphi & -\rho \operatorname{sen}\theta \operatorname{sen}\varphi & \rho \cos\theta \cos\varphi \\ \operatorname{sen}\theta \operatorname{sen}\varphi & \rho\cos\theta \operatorname{sen}\varphi & \rho \operatorname{sen}\theta \cos\varphi \\ \cos\varphi & 0 & -\rho \operatorname{sen}\varphi \end{pmatrix}$$

$$= -\rho^2 \boxed{\cos^2\theta \operatorname{sen}^3\varphi} - \rho^2 \operatorname{sen}^2\theta \cos^2\varphi \operatorname{sen}\varphi - \rho^2 \cos^2\theta \cos^2\varphi \operatorname{sen}\varphi - \rho^2 \boxed{\operatorname{sen}^2\theta \operatorname{sen}^3\varphi}$$

$$= -\rho^2 \cdot \left\{ \operatorname{sen}^3\varphi \cdot \left(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}\right) + \cos^2\varphi \operatorname{sen}\varphi \left(\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{=1}\right) \right\}$$

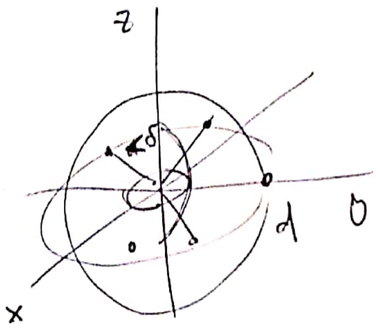
$$-\rho^2 \cdot \left\{ \operatorname{sen}^3\varphi + \cos^2\varphi \operatorname{sen}\varphi \right\} = -\rho^2 \operatorname{sen}\varphi \left(\operatorname{sen}^2\varphi + \cos^2\varphi\right) = -\rho^2 \operatorname{sen}\varphi.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = -\rho^2 \operatorname{sen}\varphi \; ; \qquad dV(x,y,z) = \left| -\rho^2 \operatorname{sen}\varphi \right| dV(\rho,\theta,\varphi)$$

$\varphi \in [0,\pi] \Rightarrow \operatorname{sen}\varphi \geq 0$

$$\boxed{dV(x,y,z) = \rho^2 \operatorname{sen}\varphi \; dV(\rho,\theta,\varphi)}$$

Ejemplo: Calcular el volumen de $B_d = \{ \underbrace{x^2+y^2+z^2}_{\rho^2} \leq d^2 \}$ 9



la inecuación de B_d: $\rho^2 \leq d^2$; $\rho \leq d$

$\boxed{0 \leq \rho \leq d}$ $\boxed{0 \leq \theta \leq 2\pi}$ $\boxed{0 \leq \varphi \leq \pi}$

$$\iiint_{B_d} 1 \, dV(x,y,z) \underset{\text{esféricas}}{\overset{\Uparrow}{=}} \iiint_{B'_d} 1 \cdot \rho^2 \operatorname{sen}\varphi \, dV(\rho,\theta,\varphi)$$

$0 \leq \rho \leq d$
$0 \leq \theta \leq 2\pi$
$0 \leq \varphi \leq \pi$

$$= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^d \rho^2 \operatorname{sen}\varphi \, d\rho \right) d\varphi \right) d\theta = \int_0^{2\pi} \left(\underbrace{\int_0^{\pi} \frac{d^3}{3} \operatorname{sen}\varphi \, d\varphi}_{\frac{d^3}{3} \cdot 2} \right) d\theta = \frac{4\pi d^3}{3}.$$

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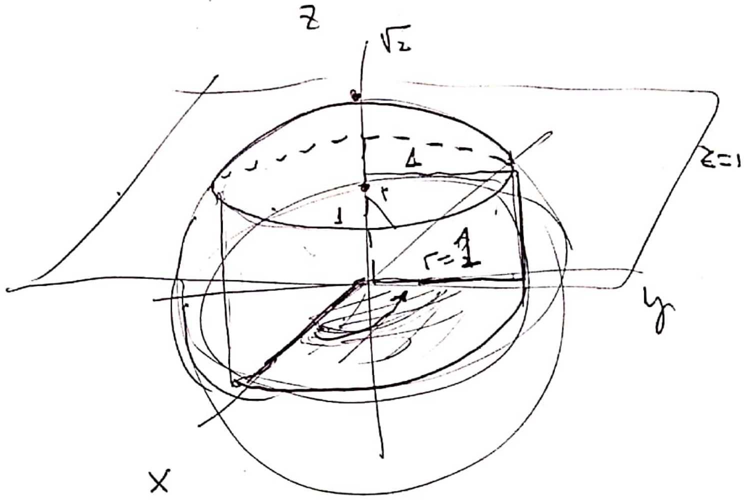
Ejemplo: ~~(0,0,0)~~ $\notin D = \left\{ \boxed{x^2+y^2+z^2 \le 2},\ \boxed{z \ge 1},\ x, y \ge 0 \right\}$

$\overbrace{x^2+y^2+z^2}^{\rho^2}$

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}$$

$$\iiint_D \frac{1}{x^2+y^2+z^2}\, dV(x,y,z).$$



$$\begin{cases} x^2+y^2+z^2 = 2 \\ z = 1 \end{cases}$$

$$\begin{cases} x^2+y^2+1 = 2 \\ z=1 \end{cases} \rightarrow \begin{cases} \boxed{x^2+y^2=1} \\ z=1 \end{cases}$$

traducimos D a esféricas: $\quad \rho \le \sqrt{2}$; $\quad 0 \le \theta \le \frac{\pi}{2}$

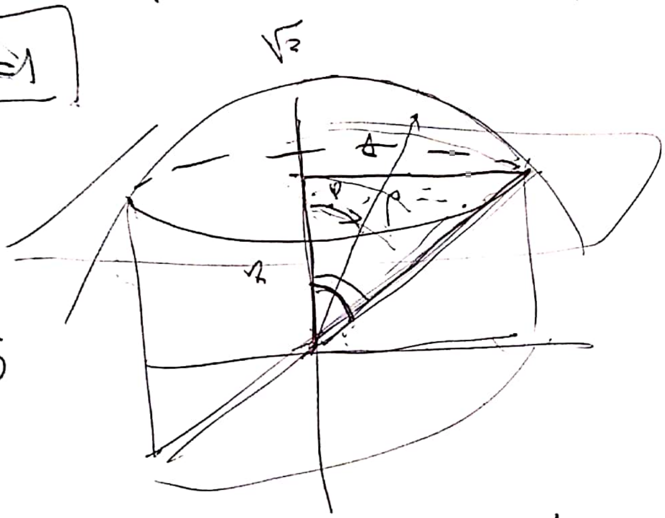
otro modo $\quad x \ge 0$: $\rho \cos\theta \sin\varphi \ge 0 \rightarrow \cos\theta \ge 0$

$\quad y \ge 0$: $\rho \sin\theta \sin\varphi \ge 0 \rightarrow \sin\theta \ge 0 \quad \Longleftrightarrow$

$z \geq 1$; $\rho \cos\varphi \geq 1$: la igualdad

$\boxed{\rho \cos\varphi = 1}$

$\rho = \dfrac{1}{\cos\varphi}$



$z \geq 1$

$\rho \cos\varphi \geq 1$

$\rho \geq \dfrac{1}{\cos\varphi}$

$0 \leq \theta \leq \dfrac{\pi}{2}$; $0 \leq \varphi \leq \dfrac{\pi}{4}$; $\dfrac{1}{\cos\varphi} \leq \rho \leq \sqrt{2}$

$$\iiint_D \frac{1}{\underbrace{x^2+y^2+z^2}_{\rho^2}} dV(x,y,z) = \int_0^{\pi/2} \left(\int_0^{\pi/4} \left(\int_{\frac{1}{\cos\varphi}}^{\sqrt{2}} \frac{1}{\cancel{\rho^2}} \cdot \cancel{\rho^2} \operatorname{sen}\varphi \, d\rho \right) d\varphi \right) d\theta$$

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$$= \int_0^{\pi/2} \Bigg(\underbrace{\int_0^{\pi/4} \operatorname{sen}\varphi \left(\sqrt{2} - \frac{1}{\cos\varphi} \right) d\varphi}_{\displaystyle \int_0^{\pi/4} \operatorname{sen}\varphi \cdot \sqrt{2} - \frac{\operatorname{sen}\varphi}{\cos\varphi} \, d\varphi} \Bigg) d\theta = \ldots$$