Anexo: Diferenciación

(La teoría rigurosa detrás de las cuentas)

un punto de $\mathbb{R}^n$ lo notaremos

$x \in \mathbb{R}^n, \quad x = (x_1, \dots, x_n)$

con $x_j \in \mathbb{R}$, $j = 1 \dots n$.

Se define el **producto interno** entre puntos de $\mathbb{R}^n$ como

$x, y \in \mathbb{R}^n; \quad x = (x_1, \dots, x_n)$
$\qquad\qquad\qquad y = (y_1, \dots, y_n)$

$$x \cdot y = \sum_{j=1}^{n} x_j y_j$$

Props. Elementales

1 - $x \cdot y = y \cdot x$

2 - $x \cdot (y + z) = x \cdot y + x \cdot z$

3 - Si definimos

$$|x| = (x \cdot x)^{1/2} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

se verifica

3.1) $|x| \geq 0$ y $|x| = 0 \iff x = 0$

3.2) Si $\lambda \in \mathbb{R}$, $|\lambda x| = |\lambda| \cdot |x|$

Lema (desigualdad de Cauchy-Schwarz)

Si $x, y \in \mathbb{R}^n$, entonces

$$|x \cdot y| \leq |x|\,|y|$$

Dem

(1) Si $a, b \in \mathbb{R}$, $a, b \geq 0$

$\Rightarrow \quad a \cdot b \leq \frac{a^2}{2} + \frac{b^2}{2}$ desigualdad de Young.

$0 \leq (a-b)^2 = a^2 - 2ab + b^2$

$\Rightarrow 2ab \leq a^2 + b^2 \Rightarrow ab \leq \frac{a^2 + b^2}{2}$.

(2) $|x \cdot y| \leq \dfrac{|x|^2}{2} + \dfrac{|y|^2}{2}$

D/

$$|x \cdot y| = \left|\sum_{j=1}^{m} x_j \cdot y_j\right| \leq$$

$$\leq \sum_{j=1}^{m} |x_j \cdot y_j| = \sum_{j=1}^{m} |x_j|\,|y_j| \overset{\text{Young.}}{\leq}$$

$$\leq \sum_{j=1}^{m} \left(\frac{|x_j|^2}{2} + \frac{|y_j|^2}{2} \right) =$$

$$= \frac{1}{2} \underbrace{\sum_{j=1}^{m} |x_j|^2}_{|x|^2} + \frac{1}{2} \underbrace{\sum_{j=1}^{m} |y_j|^2}_{|y|^2}$$

$$= \frac{|x|^2}{2} + \frac{|y|^2}{2}$$

(3) Fin de la prueba.

- Si $x=0$ o $y=0$ no hay nada que probar.
- Sup. que $|x| = 1 = |y|$

Aplico (2) y obtengo

$$|x \cdot y| \leq \frac{\overbrace{|x|^2}^{1}}{2} + \frac{\overbrace{|y|^2}^{1}}{2} = \frac{1}{2} + \frac{1}{2} = 1 = |x| \cdot |y|$$

- Si $x \neq 0$ e $y \neq 0$.

Sea $\bar{x} = \dfrac{x}{|x|}$, $\bar{y} = \dfrac{y}{|y|}$

luego, $|\bar{x}| = 1 = |\bar{y}|$

$\Rightarrow |\bar{x} \cdot \bar{y}| \leq 1$

pero $|\bar{x} \cdot \bar{y}| = \left| \dfrac{x}{|x|} \cdot \dfrac{y}{|y|} \right| = \dfrac{|x \cdot y|}{|x|\,|y|}$

$\Rightarrow \dfrac{|x \cdot y|}{|x|\,|y|} \leq 1 \Rightarrow |x \cdot y| \leq |x|\,|y|$ ☒

COROLARIO (desigualdad triangular)

Si $x, y \in \mathbb{R}^n \Rightarrow |x+y| \le |x| + |y|$

D/

$$|x+y|^2 = (x+y)\cdot(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y$$

$$= |x|^2 + 2\, x\cdot y + |y|^2 \le |x|^2 + 2\,|x\cdot y| + |y|^2$$

$$\underset{\text{C-S}}{\le} |x|^2 + 2\,|x|\,|y| + |y|^2 = (|x|+|y|)^2 \quad \blacksquare$$

Notación $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$

U es el dominio de f.

Si $x \in U \subset \mathbb{R}^n \Rightarrow f(x) \in \mathbb{R}^k$

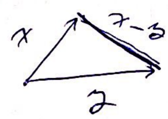
$f(x) = (f_1(x), \dots, f_k(x))$

a $f_i : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ se la llama función coordenada $i = 1 \dots k$.

Def (distancia entre puntos)

Si $x, y \in \mathbb{R}^n$, se define

$$\operatorname{dist}(x,y) = |x-y|$$



Def $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$ y $x_0 \in U$.

Decimos que f es cont. en x_0 si $\forall \varepsilon > 0$, existe $\delta > 0$ tal que

$$\underbrace{\operatorname{dist}(f(x), f(x_0))}_{|f(x)-f(x_0)|} < \varepsilon \text{ si } \underbrace{\operatorname{dist}(x, x_0)}_{|x-x_0|} < \delta$$

Escaneado con CamScanner

Lema $f(x) = (f_1(x), \ldots, f_k(x))$

es cont. en $x_0 \iff f_i$ es cont. en x_0 $\forall\, i = 1 \ldots k$.

Dem

$(\Rightarrow)$ q.v.q ~~tiene~~ $\forall i=1\ldots k$ dado $\varepsilon > 0$, existe $\delta > 0$ tal que

$$|f_i(x) - f_i(x_0)| < \varepsilon \text{ si } |x - x_0| < \delta.$$

Pero como f es cont., existe $\delta > 0$ tq $|f(x) - f(x_0)| < \varepsilon$ si $|x - x_0| < \delta$

Ahora

$$|f_i(x) - f_i(x_0)| = \sqrt{(f_i(x) - f_i(x_0))^2} \leq$$

$$\leq \sqrt{\sum_{l=1}^{k} (f_l(x) - f_l(x_0))^2} = |f(x) - f(x_0)| < \varepsilon$$

$$\Rightarrow |f_i(x) - f_i(x_0)| < \varepsilon.$$

$(\Leftarrow)$ Dado $\varepsilon > 0$, para cada $i = 1 \ldots k$ $\exists\, \delta_i > 0$ tal que

$$|f_i(x) - f_i(x_0)| < \frac{\varepsilon}{\sqrt{k}} \text{ si } |x - x_0| < \delta_i$$

Sea $\delta = \min\{\delta_1 \ldots \delta_k\}$

$$|f(x) - f(x_0)| = \sqrt{\sum_{i=1}^{k} (f_i(x) - f_i(x_0))^2} \leq$$

$$\leq \sqrt{k \max_{1 \leq i \leq k} (|f_i(x) - f_i(x_0)|^2)} =$$

$$= \sqrt{k} \cdot \max_{1 \leq i \leq k} (|f_i(x) - f_i(x_0)|) <$$

$$< \cancel{\sqrt{k}} \cdot \frac{\varepsilon}{\cancel{\sqrt{k}}}$$

si $|x - x_0| < \delta < \delta_i$ $\forall i$

Ej – $f : \mathbb{R}^2 \text{ o } \mathbb{R}^3 \longrightarrow \mathbb{R}$

– $r : \mathbb{R} \longrightarrow \mathbb{R}^3$

$r(t) = (x(t), y(t), z(t))$

Ej coord. polares

$x = r\cos\theta$
$y = r \operatorname{sen}\theta$

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\phi(r,\theta) = (x,y)$
$= (r\cos\theta, r\operatorname{sen}\theta)$

Def $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$

$f(x) = (f_1(x), \dots, f_k(x))$

Se define las derivadas parciales de f como:

$$\boxed{D_j f_i(x_0) = \frac{\partial f_i}{\partial x_j}(x_0) = \lim_{h \to 0} \frac{f_i(x_0 + h e_j) - f_i(x_0)}{h}}$$

$e_j = (0 \dots \underbrace{1}_{j} \dots 0)$ el j-ésimo canónico de $\mathbb{R}^n$.

si el límite existe.

Def

$$Df(x_0) = \begin{bmatrix} D_1 f_1(x_0) & D_2 f_1(x_0) & \dots & D_m f_1(x_0) \\ D_1 f_2(x_0) & D_2 f_2(x_0) & \dots & D_m f_2(x_0) \\ \vdots & & & \\ D_1 f_k(x_0) & D_2 f_k(x_0) & \dots & D_m f_k(x_0) \end{bmatrix}_{k \times m}$$

Esta se llama la matriz diferencial de f en x_0.

Obs: - Sup $f : \mathbb{R}^2 \text{ ó } \mathbb{R}^3 \longrightarrow \mathbb{R}$

$Df = \nabla f$.

- $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$

$\mathbf{r}(t) = (x(t), y(t), z(t))$

$D\mathbf{r} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (\mathbf{r}')^t$

Ej: $\phi(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\,\text{sen}\,\theta}_{y})$

$$D\phi = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{bmatrix}$$

~~Lema~~

Def $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $x_0 \in U$.

Decimos que f es diferenciable en x_0 si existen sus derivadas parciales y

$$f(x) = f(x_0) + \overbrace{Df(x_0)}^{k\times n} \cdot \overbrace{(x-x_0)}^{n\times 1} + \varepsilon_f(x,x_0)$$

donde $\displaystyle\lim_{x\to x_0} \frac{\varepsilon_f(x,x_0)}{|x-x_0|} = 0$.

Ejercicio

Sup. que existe una matriz A, $A \in \mathbb{R}^{k\times n}$ tal que

$$\frac{f(x) - f(x_0) - A\cdot(x-x_0)}{|x-x_0|} \xrightarrow[x\to x_0]{} 0$$

$\Rightarrow$ f es dif. y $A = Df(x_0)$

Lema Si f es diferenciable en x_0 $\Rightarrow$ f es cont. en x_0.

Dem ✓

Lema $f(x) = (f_1(x), \dots, f_k(x))$ es dif. en $x_0 \iff f_i$ es dif. en x_0 $\forall\, i = 1 \dots k$.

D/

$$f(x) = f(x_0) + Df(x_0)\cdot(x - x_0) + \varepsilon_f(x, x_0)$$

$$\Updownarrow$$

$$f_1(x) = f_1(x_0) + \nabla f_1(x_0)\cdot(x - x_0) + \varepsilon_{f_1}(x, x_0)$$

$$\vdots$$

$$f_k(x) = f_k(x_0) + \nabla f_k(x_0)\cdot(x - x_0) + \varepsilon_{f_k}(x, x_0)$$

$$\varepsilon_f(x, x_0) = (\varepsilon_{f_1}(x, x_0), \dots, \varepsilon_{f_k}(x, x_0))$$

$$\frac{\varepsilon_f(x, x_0)}{|x - x_0|} = \left(\frac{\varepsilon_{f_1}(x, x_0)}{|x - x_0|}, \dots, \frac{\varepsilon_{f_k}(x, x_0)}{|x - x_0|} \right)$$

Luego, $\dfrac{\varepsilon_f(x, x_0)}{|x - x_0|} \xrightarrow[x \to x_0]{} 0 \iff \dfrac{\varepsilon_{f_i}(x, x_0)}{|x - x_0|} \xrightarrow[x \to x_0]{} 0$ $\forall\, i = 1 \dots k$

$A \cdot X$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{k1} & \cdots & a_{km} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ k \end{array}$$

$$A_i = (a_{i1} \dots a_{im})$$

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix}$$

$$A \cdot X = \begin{pmatrix} A_1 \cdot X \\ \vdots \\ A_k \cdot X \end{pmatrix}$$

Corolario Si $D_j f_i$ es cont. en x_0 $\forall\, i = 1 \dots k$, $\forall\, j = 1 \dots m$ $\Rightarrow f$ resulta diferenciable en x_0.

Def $A \in \mathbb{R}^{k\times m}$

$$A = (a_{ij})_{\substack{1\le i\le k\\ 1\le j\le m}} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{k1} & \dots & a_{km} \end{bmatrix}$$

Se define $\|A\|_2 = \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{m} a_{ij}^2}$

Obs

Si llamamos $A_i = (a_{i1}, \dots, a_{in}) \in \mathbb{R}^n$

$$\Rightarrow \|A\|_2 = \sqrt{\sum_{i=1}^{k} |A_i|^2}$$

Lema $A \in \mathbb{R}^{k\times n}$ y $x \in \mathbb{R}^n$

$$|Ax| \le \|A\|_2 \cdot |x|$$

Dem

$$A\cdot x = \begin{pmatrix} A_1\cdot x \\ \vdots \\ A_k \cdot x \end{pmatrix}$$

$$\Rightarrow |Ax|^2 = \sum_{i=1}^{k} (A_i\cdot x)^2 \le$$

$$\le \sum_{i=1}^{k} |A_i|^2 |x|^2 =$$

$$= \underbrace{\left(\sum_{i=1}^{k} |A_i|^2\right)}_{\|A\|_2^2} |x|^2$$

▨

Lema Si f es diferenciable en x_0, entonces existe $R>0$ y $L>0$ tal que

$$|f(x) - f(x_0)| \le L\,|x - x_0|$$

si $|x - x_0| < R$.

Dem

$$|f(x)-f(x_0)| = |Df(x_0)\cdot(x-x_0)+\varepsilon_f(x,x_0)|$$

$$\leq \underbrace{|Df(x_0)\cdot(x-x_0)|}_{\mathrm{I}} + \underbrace{|\varepsilon_f(x,x_0)|}_{\mathrm{II}}$$

$\mathrm{I} \leq \|Df(x_0)\|_2 \cdot |x-x_0|$

II: Tomo $\varepsilon = 1 \Rightarrow \exists R>0$ tal que

si $|x-x_0|<R \Rightarrow \left|\dfrac{\varepsilon_f(x,x_0)}{|x-x_0|}\right| < 1$

$$|\varepsilon_f(x,x_0)| = \underbrace{\frac{|\varepsilon_f(x,x_0)|}{|x-x_0|}}_{<1} \cdot |x-x_0|$$

si $|x-x_0|<R$

Luego

$$\mathrm{I}+\mathrm{II} \leq \|Df(x_0)\|_2 |x-x_0| + |x-x_0|$$

$$= \underbrace{(\|Df(x_0)\|_2+1)}_{L} |x-x_0|$$

▨

Teorema (Regla de la cadena)

$f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$

$g: V \subset \mathbb{R}^k \longrightarrow \mathbb{R}^m$

f es diferenciable en $x_0 \in U$, $y_0 = f(x_0) \in V$ y g es diferenciable en y_0

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^k \xrightarrow{g} \mathbb{R}^m$$

Luego $(g\circ f)$ resulta diferenciable en x_0 y $\underset{m\times n}{D(g\circ f)(x_0)} = \underset{m\times k}{Dg(y_0)} \cdot \underset{k\times n}{Df(x_0)}$

Dem

$$f(x) = f(x_0) + Df(x_0)\cdot(x-x_0) + \varepsilon_f(x,x_0)$$

$$g(y) = g(y_0) + Dg(y_0)\cdot(y-y_0) + \varepsilon_g(y,y_0)$$

con $\dfrac{\varepsilon_f(x,x_0)}{|x-x_0|} \xrightarrow[x\to x_0]{} 0$ y $\dfrac{\varepsilon_g(y,y_0)}{|y-y_0|} \xrightarrow[y\to y_0]{} 0$

luego,

$$g(f(x)) = g(f(x_0)) + Dg(f(x_0))(f(x)-f(x_0)) + \varepsilon_g(f(x),f(x_0))$$

$$= g(f(x_0)) + Dg(f(x_0))\left[Df(x_0)(x-x_0) + \varepsilon_f(x,x_0)\right] + \varepsilon_g(f(x),f(x_0))$$

$$= g(f(x_0)) + \underbrace{Dg(f(x_0))\cdot Df(x_0)}_{¿D(g\circ f)(x_0)?}(x-x_0) + \underbrace{Dg(f(x_0))\cdot\varepsilon_f(x,x_0) + \varepsilon_g(f(x),f(x_0))}_{¿\varepsilon_{g\circ f}(x,x_0)?}$$

Debo ver entonces que

$$\frac{Dg(f(x_0))\cdot\varepsilon_f(x,x_0) + \varepsilon_g(f(x),f(x_0))}{|x-x_0|} \xrightarrow[x\to x_0]{} 0$$

Pero

$$\frac{Dg(f(x_0))\cdot\varepsilon_f(x,x_0)}{|x-x_0|} = \mathrm{I}$$

$$\frac{\varepsilon_g(f(x),f(x_0))}{|x-x_0|} = \mathrm{II}$$

$$|\mathrm{I}| := \frac{|Dg(f(x_0)) \circ \varepsilon_f(x,x_0)|}{|x-x_0|} \leq$$

$$\leq \|Dg(f(x_0))\|_2 \cdot \underbrace{\frac{|\varepsilon_f(x,x_0)|}{|x-x_0|}}_{\xrightarrow[x\to x_0]{} 0} \xrightarrow[x\to x_0]{} 0$$

$$|\mathrm{II}| = \frac{|\varepsilon_g(f(x), f(x_0))|}{|x-x_0|} =$$

$$= \frac{|\varepsilon_g(f(x), f(x_0))|}{|f(x)-f(x_0)|} \cdot \frac{|f(x)-f(x_0)|}{|x-x_0|}$$

cuando $x \to x_0$, $f(x) \to f(x_0)$

$$\Rightarrow \frac{|\varepsilon_g(f(x), f(x_0))|}{|f(x)-f(x_0)|} \longrightarrow 0$$

y $\frac{|f(x)-f(x_0)|}{|x-x_0|} \leq L$

si $|x-x_0| < R$

Luego, $\frac{|f(x)-f(x_0)|}{|x-x_0|}$ es acot.

$$\Rightarrow \mathrm{II} \xrightarrow[x\to x_0]{} 0 \qquad \boxed{/\!/}$$