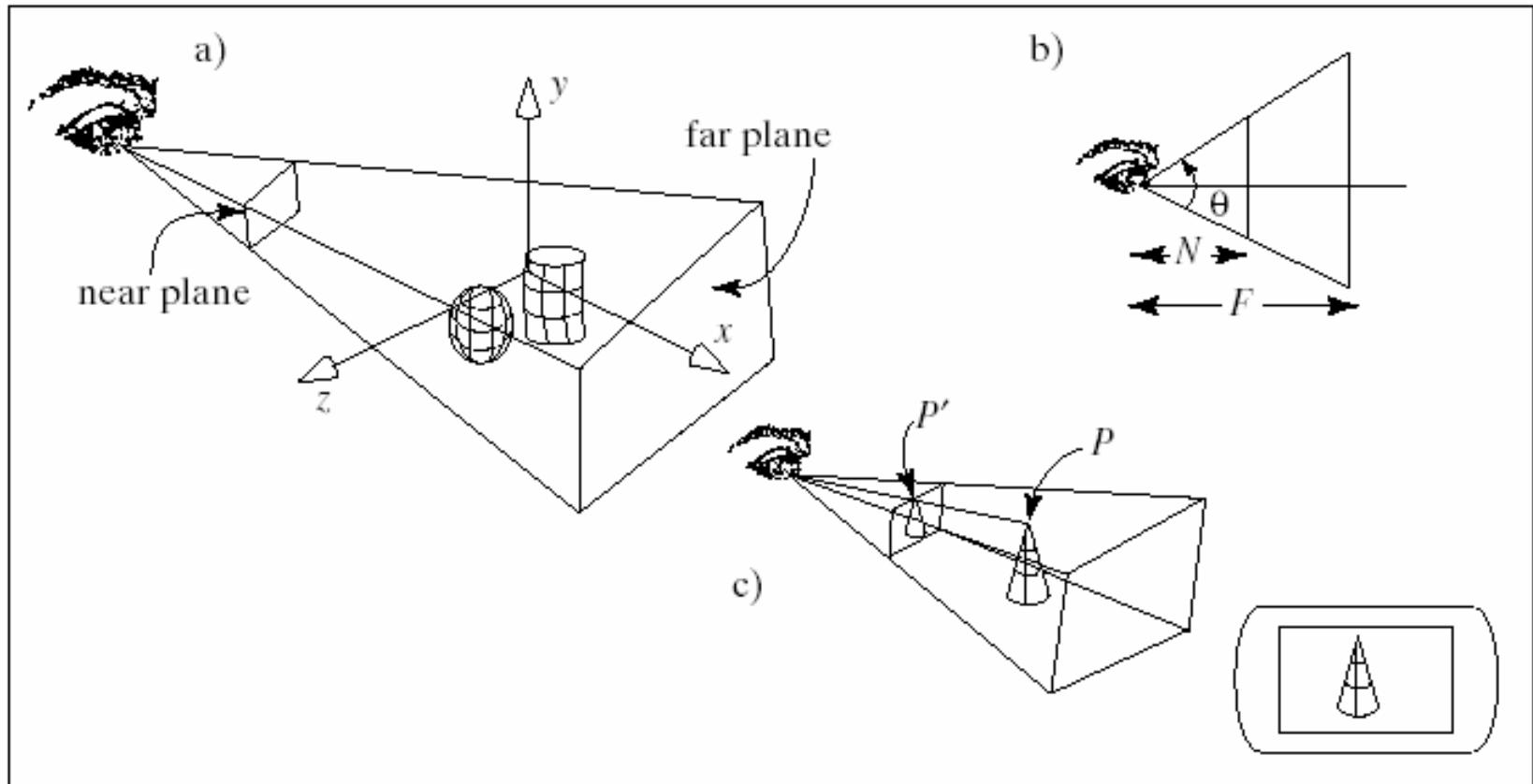


Reminder: Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\times w} \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

$$\begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} \xrightarrow{/w} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \xrightarrow{} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Projection transformations



Introduction to Projection Transformations

[Hill: 371-378, 398-404.
Foley & van Dam: p.
229-242]

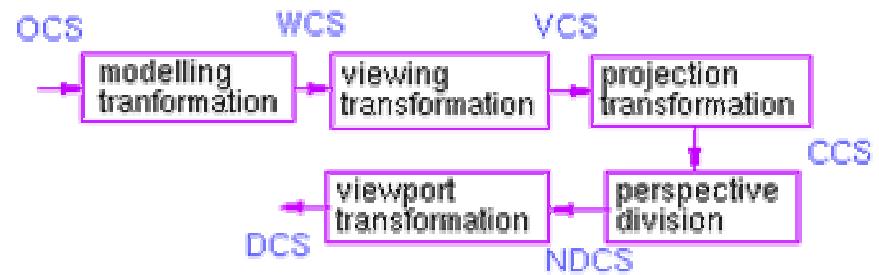
Mapping: $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Projection: $n > m$

Planar Projection: Projection on
a plane.

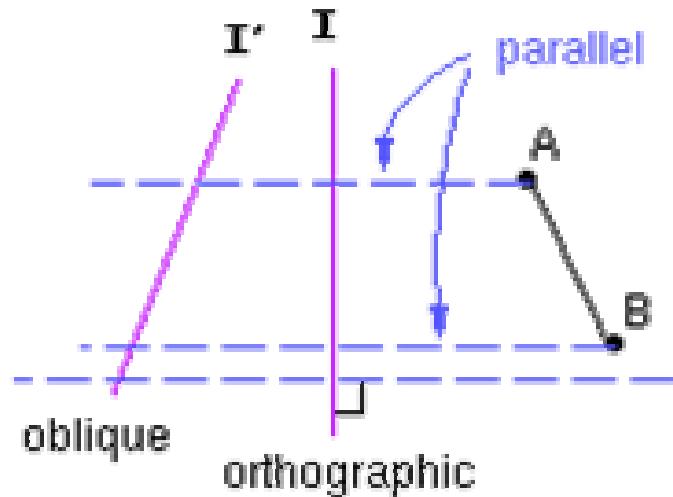
$\mathbb{R}^3 \rightarrow \mathbb{R}^2$ or

$\mathbb{R}^4 \rightarrow \mathbb{R}^3$ homogenous
coordinates.

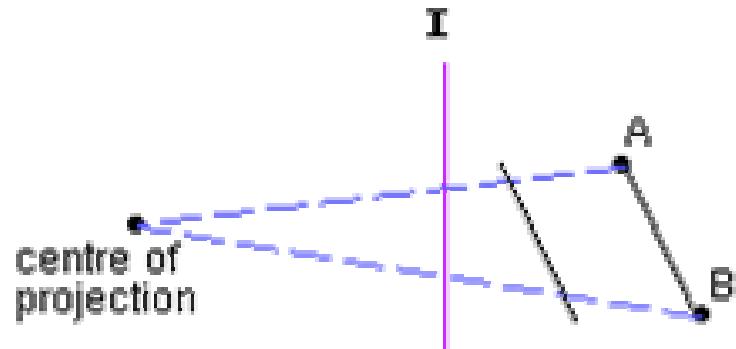


Basic projections

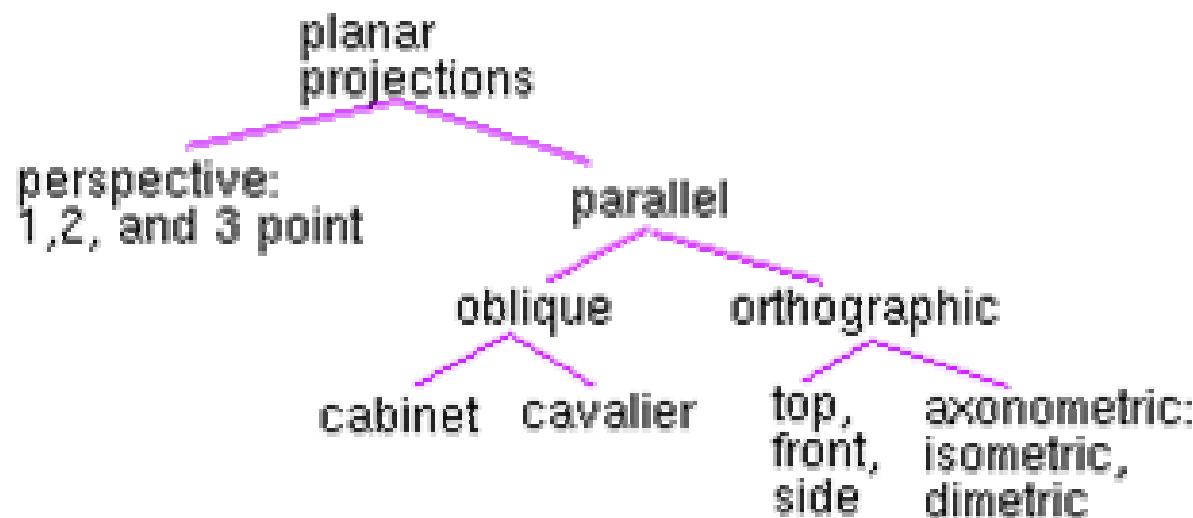
Parallel



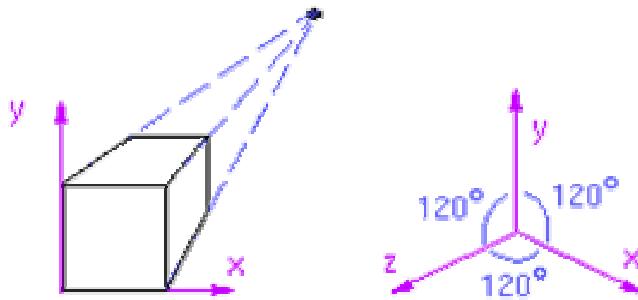
Perspective



Taxonomy

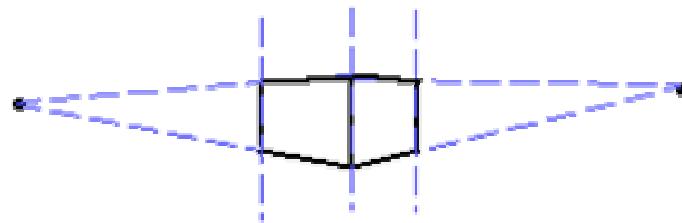


Examples

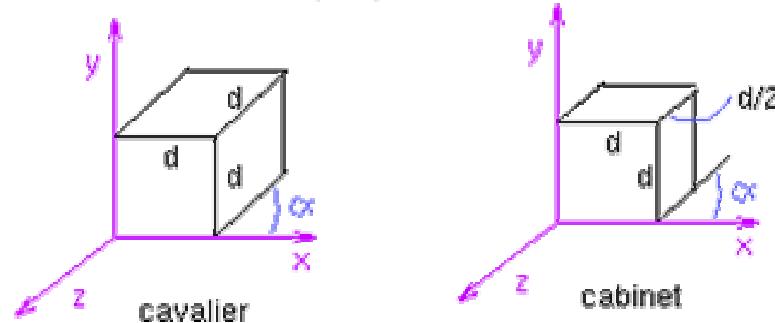


one-point
perspective

isometric



two-point
perspective



cavalier

cabinet

A basic orthographic projection

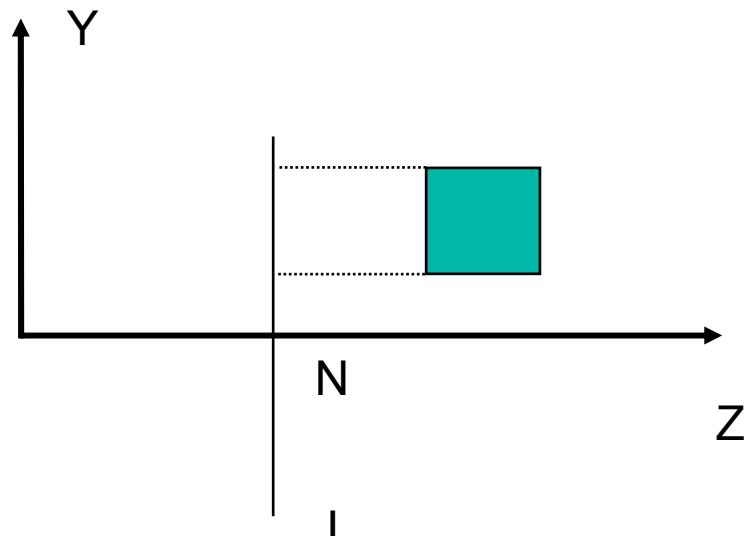
$$x' = x$$

$$y' = y$$

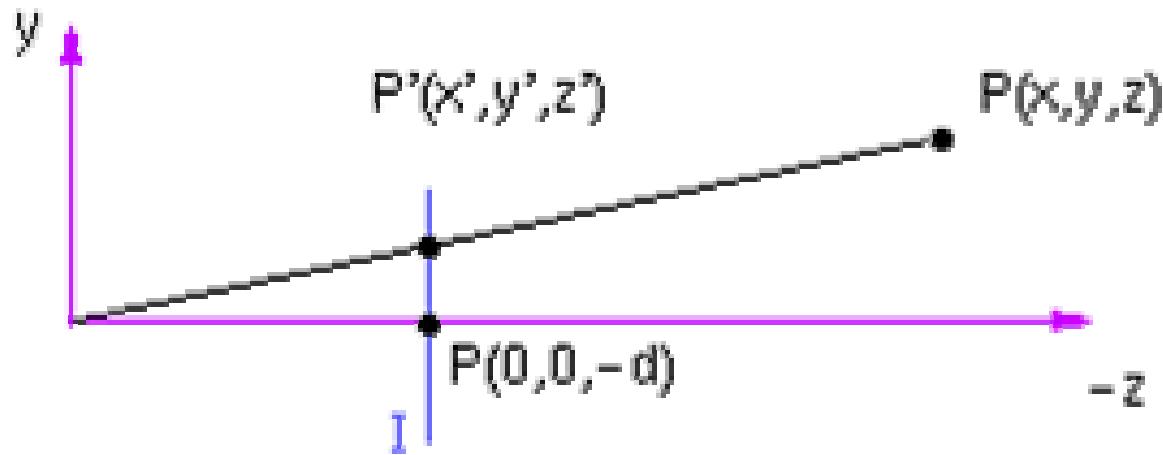
$$z' = N$$

Matrix Form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ N \\ 1 \end{bmatrix}$$



A basic perspective projection



Similar triangles

$$x'/d = x/(-z)$$

$$y'/d = y/(-z)$$

$$z' = -d$$

In matrix form

Matrix form of
 $x'/d = x/(-z)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -z/d \end{bmatrix}$$

$y'/d = y/(-z)$

$$\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} xd \\ yd \\ zd \\ -z \end{bmatrix}$$

$z' = -d$

Moving from 4D to 3D

$$\begin{bmatrix} x \\ y \\ z \\ -z/d \end{bmatrix} \xrightarrow{h=-z/d} \begin{bmatrix} x/h \\ y/h \\ z/h \\ 1 \end{bmatrix} \xrightarrow{} \begin{bmatrix} xd/(-z) \\ yd/(-z) \\ -d \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

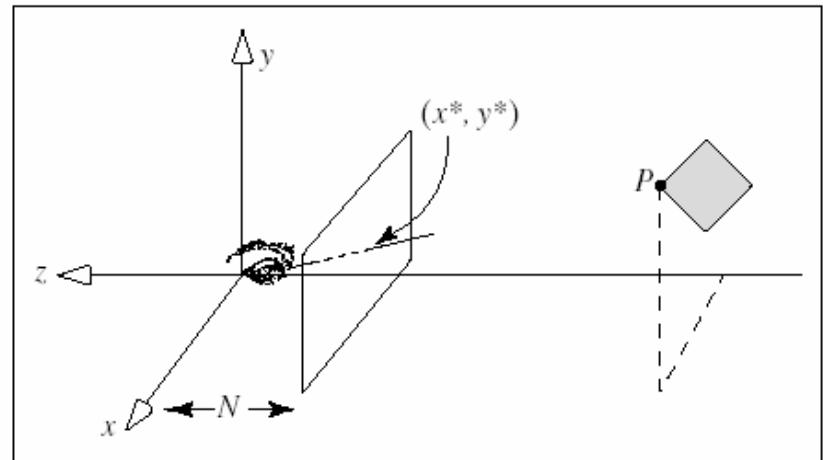
Camera coordinate system

Image plane = near plane

Camera at $(0,0,0)$

Looking at $-z$

Image plane at $z = -N$

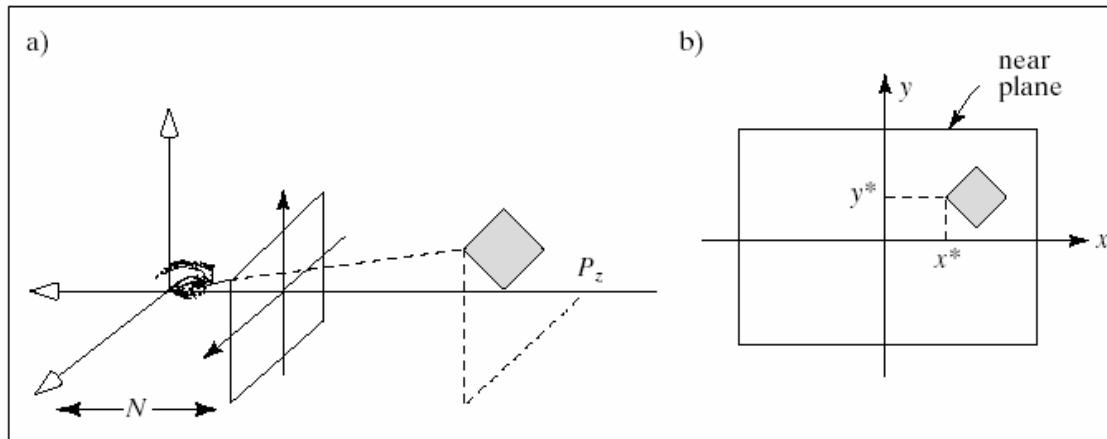


Perspective projection of a point

In eye coordinates $P = [Px, Py, Pz, 1]^T$

$$x/Px = N/(-Pz) \Rightarrow x = NPx/(-Pz)$$

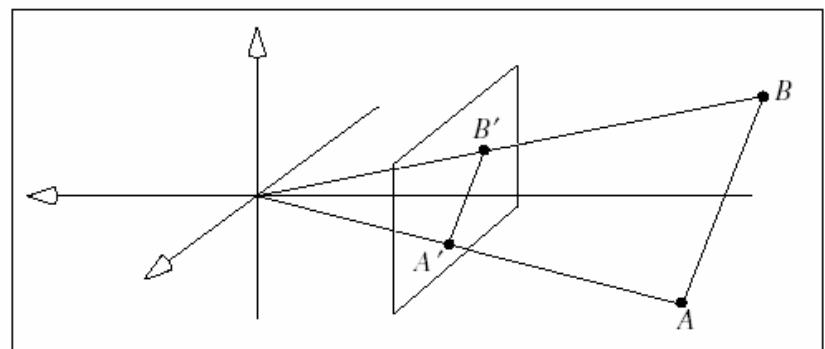
$$y/Px = N/(-Pz) \Rightarrow y = NPy/(-Pz)$$



Observations

- Perspective foreshortening
- Denominator becomes undefined for $z = 0$
- If P is behind the eye P_z changes sign
- Near plane just scales the picture
- Straight line \rightarrow straight line

$$x' = -N \frac{P_x}{P_z}$$
$$y' = -N \frac{P_y}{P_z}$$
$$z' = -N$$



Perspective projection of a line

$$L(t) = \mathbf{A} + \vec{\mathbf{c}}t = \begin{bmatrix} A_x \\ A_y \\ A_z \\ 1 \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \\ 0 \end{bmatrix} t$$

$$\tilde{L}(t) = \mathbf{M}L(t) = \mathbf{M}(\mathbf{A} + \vec{\mathbf{c}}t) = \mathbf{M} \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \\ 1 \end{bmatrix} = \begin{bmatrix} N(A_x + c_x t) \\ N(A_y + c_y t) \\ N(A_z + c_z t) \\ -(A_z + c_z t) \end{bmatrix}$$

*Perspective Division,
drop fourth coordinate*



$$L'(t) = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

Is it a line?

$$\text{Original: } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected: } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

$$x' = -N(A_x + c_x t)/(A_z + c_z t) \Rightarrow x'(A_z + c_z t) = -N(A_x + c_x t) \Rightarrow$$
$$x' A_z + x' c_z t = -N A_x - N c_x t \Rightarrow \begin{cases} x' A_z + N A_x = -(x' c_z + N c_x) t \\ \text{and similarly for y} \\ y' A_z + N A_y = -(y' c_z + N c_y) t \end{cases}$$

Cont'd next slide

Is it a line? (cont'd)

$$\begin{array}{l} x' A_z + N A_x = -(x' c_z + N c_x) t \\ y' A_z + N A_y = -(y' c_z + N c_y) t \end{array} \Rightarrow \begin{array}{l} x' A_z + N A_x = -(x' c_z + N c_x) t \\ -(y' c_z + N c_y) t = y' A_z + N A_y \end{array} \Rightarrow$$

$$(x' A_z + N A_x)(y' c_z + N c_y) = (x' c_z + N c_x)(y' A_z + N A_y) \Rightarrow$$

$$x' A_z y' c_z + x' A_z N c_y + N A_x y' c_z + N^2 A_x c_y = x' c_z y' A_z + x' c_z N A_y + N c_x y' A_z + N^2 A_y c_x \Rightarrow$$

$$(A_z N c_y - c_z N A_y) x' + (N A_x c_z + N c_x A_z) y' + N^2 (A_x c_y + A_y c_x) = 0 \Rightarrow$$

\Rightarrow $ax' + by' + c = 0$ which is the equation of a line.

So is there a difference?

$$\text{Original: } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected: } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

So is there a difference?

The speed of the lines if cz is not 0

$$\text{Original : } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix} \Rightarrow \boxed{\frac{\partial L(t)}{\partial t} = \vec{c}}$$

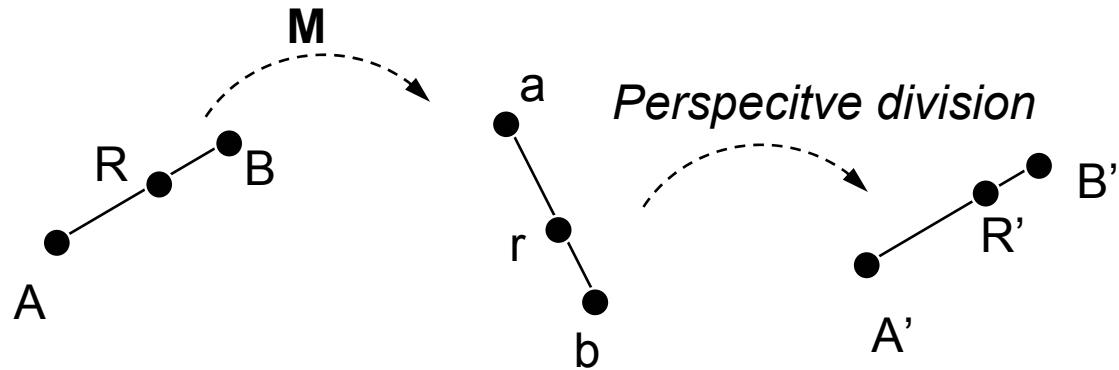
$$\text{Projected : } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix} \Rightarrow$$

$$\frac{\partial x'}{\partial t} = -N \frac{\partial}{\partial t} \left((A_x + c_x t)/(A_z + c_z t) \right) = -N \frac{c_x (A_z + c_z t) - (A_x + c_x t)c_z}{(A_z + c_z t)^2} = -N \frac{c_x A_z - A_x c_z}{(A_z + c_z t)^2} \Rightarrow$$

$$\frac{\partial L'(t)}{\partial t} = \frac{-N}{(A_z + c_z t)^2} \begin{bmatrix} c_x A_z - A_x c_z \\ c_y A_z - A_y c_z \end{bmatrix}$$

Inbetween points

How do points on lines transform?



$$R(g) = (1-g)A + gB$$

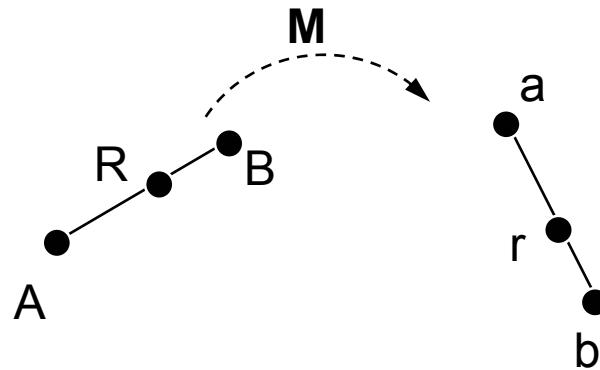
$$r = MR$$

$$R'(f) = (1-f)a' + fb' \text{ in cartesian coordinates}$$

What is the relationship between g and f ?

First step

World to homogeneous space (4D)



$$R = (1 - g)A + gB$$

$$r = MR = M[(1 - g)A + gB] = (1 - g)MA + gMB \Rightarrow$$

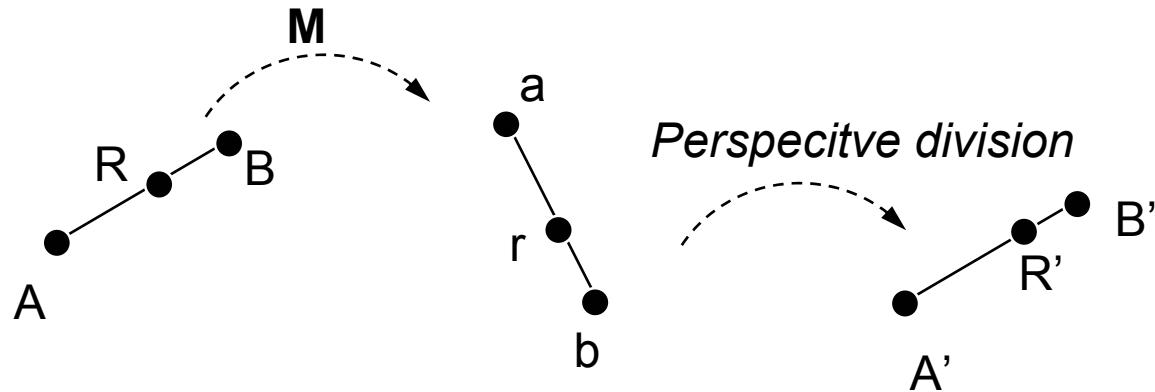
$$r = (1 - g)a + gb$$

$$a = MA = (a_1, a_2, a_3, a_4)$$

$$b = MB = (b_1, b_2, b_3, b_4)$$

Second step

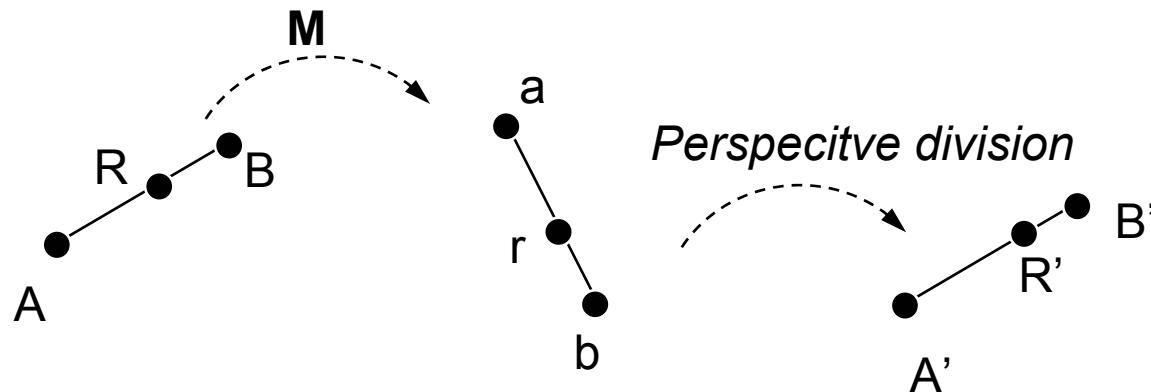
Perspective division



$$\left\{ \begin{array}{l} r = (1 - g)a + gb \\ a = (a_1, a_2, a_3, a_4) \\ b = (b_1, b_2, b_3, b_4) \end{array} \right\} \Rightarrow$$

$$R'_1 = \frac{r_1}{r_4} = \frac{(1 - g)a_1 + gb_1}{(1 - g)a_4 + gb_4}$$

Putting all together



$$R'_1 = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4} = \frac{\text{lerp}(a_1, b_1, g)}{\text{lerp}(a_4, b_4, g)}$$

At the same time :

$$R' = (1-f)A' + fB' \Rightarrow R'_1 = (1-f)A'_1 + fB'_1$$

$$R'_1 = (1-f)\frac{a_1}{a_4} + f\frac{b_1}{b_4} = \text{lerp}\left(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f\right)$$

}

Relation between the fractions

$$\left. \begin{array}{l} R'_1(f) = \frac{\text{lerp}(a_1, b_1, g)}{\text{lerp}(a_4, b_4, g)} \\ R'_1(f) = \text{lerp}\left(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f\right) \end{array} \right\} \Rightarrow g = \frac{f}{\text{lerp}\left(\frac{b_4}{a_4}, 1, f\right)}$$

substituting this in $R(g) = (1 - g)A + gB$ yields

$$R_1 = \frac{\text{lerp}\left(\frac{A_1}{a_4}, \frac{B_1}{b_4}, f\right)}{\text{lerp}\left(\frac{1}{a_4}, \frac{1}{b_4}, f\right)}$$

THAT MEANS: For a given f in **screen space** and A, B in **world space** we can find the corresponding R (or g) in **world space** using the above formula.

“ A ” can be texture coordinates, position, color, normal etc.

Effect of perspective projection on lines [Hill 375]

Equations

$$\text{Original : } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected : } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

What happens to parallel lines?

Effect of perspective projection on lines [Hill 375]

Parallel lines

$$\text{Original : } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix}$$

$$\text{Projected : } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

If parallel to view plane then:

$$c_z = 0 \rightarrow L'(t) = -\frac{N}{A_z}(A_x + c_x t, A_y + c_y t)$$

$$\text{slope} = \frac{c_y}{c_x}$$

Effect of perspective projection on lines [Hill 375]

Parallel lines

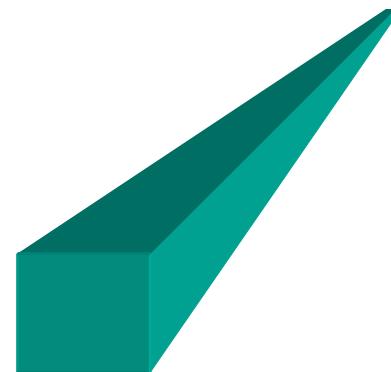
$$\text{Original : } L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A_x + c_x t \\ A_y + c_y t \\ A_z + c_z t \end{bmatrix} \Rightarrow \frac{\partial L(t)}{\partial t} = \vec{c}$$

$$\text{Projected : } L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(A_x + c_x t)/(A_z + c_z t) \\ -N(A_y + c_y t)/(A_z + c_z t) \\ -N \end{bmatrix}$$

If not parallel to view plane then:

$$c_z \neq 0 \rightarrow \lim_{t \rightarrow \infty} L'(t) = -\frac{N}{c_z}(c_x, c_y)$$

Vanishing point!



Summary

Forshortening

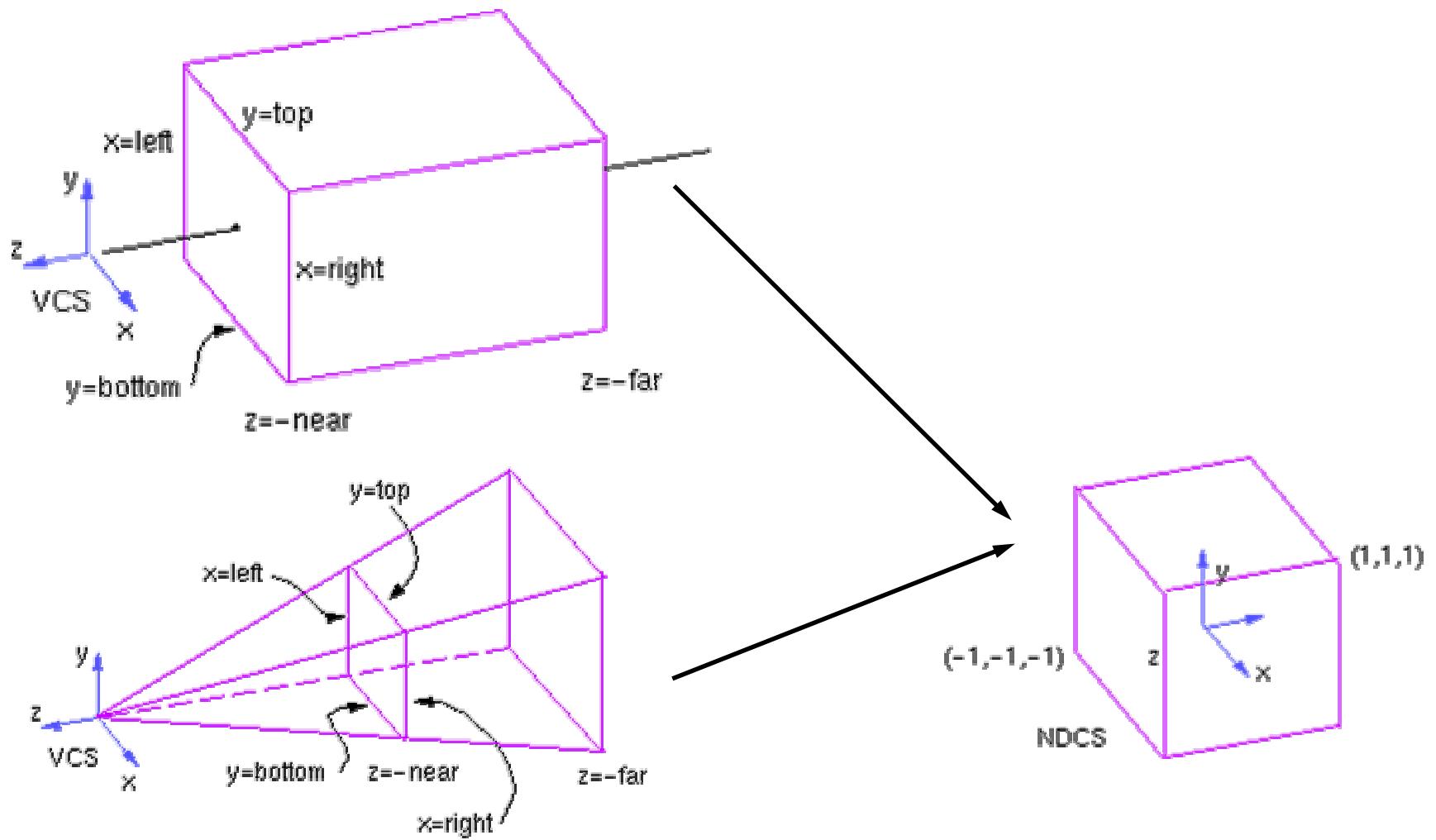
Non-linear

Lines go to lines

Parallel lines either intersect or remain parallel

Inbetweeness

View volumes

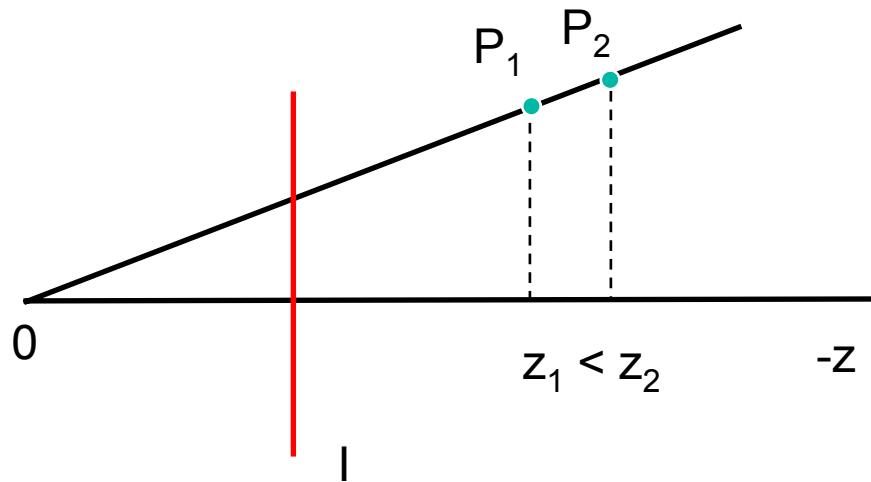


Transformation vs Projection

We want to keep z

Why?

- Pseudodepth



Derivation of the orthographic transformation

Scaling and Translation

$$y_{VCS} \rightarrow y$$

$$y_{NDCS} \rightarrow y'$$

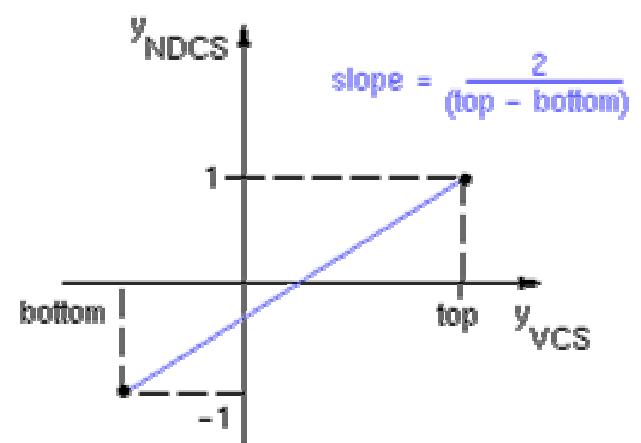
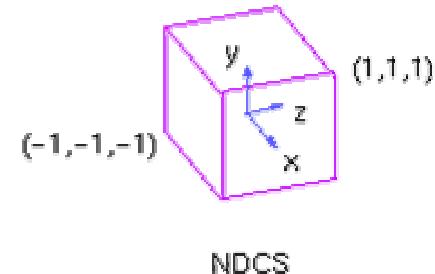
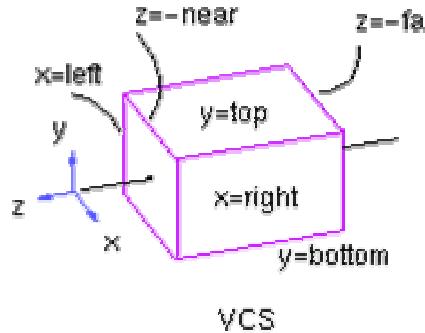
$$(y_b, y'_b) = (bottom, -1) \quad \text{and}$$

$$(y_t, y'_t) = (top, 1)$$

Line equation $\frac{y' - y'_b}{y - y_b} = \frac{y'_t - y'_b}{y_t - y_b}$

$$\frac{y' - (-1)}{y - bottom} = \frac{1 - (-1)}{top - bottom} \rightarrow$$

$$y' = \frac{2}{top - bottom}y - \frac{top + bottom}{top - bottom}$$



All three coordinates

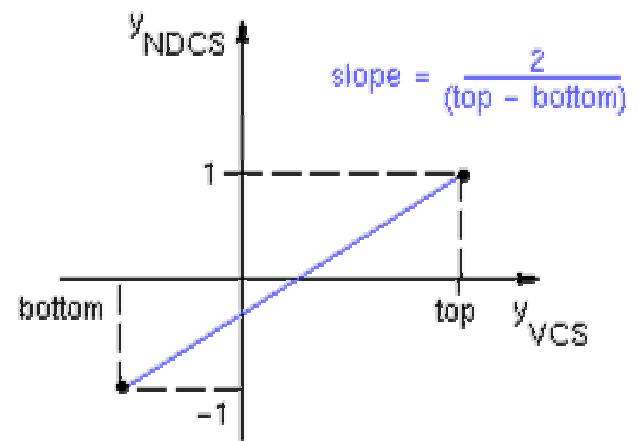
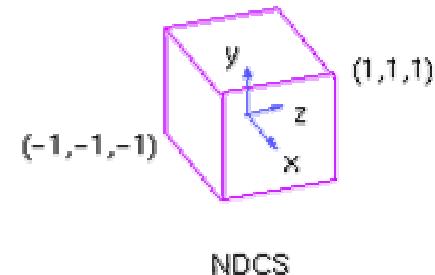
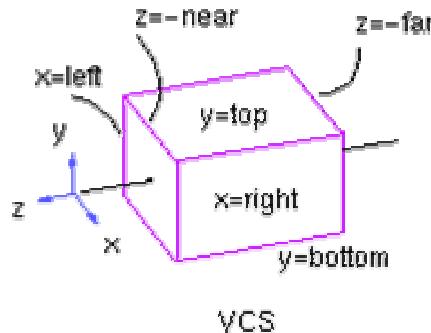
Scaling and Translation

Similarly,

$$x' = \frac{2}{right - left}x - \frac{right + left}{right - left}$$

$$y' = \frac{2}{top - bottom}y - \frac{top + bottom}{top - bottom}$$

$$z' = \frac{-2}{far - near}z - \frac{far + near}{far - near}$$



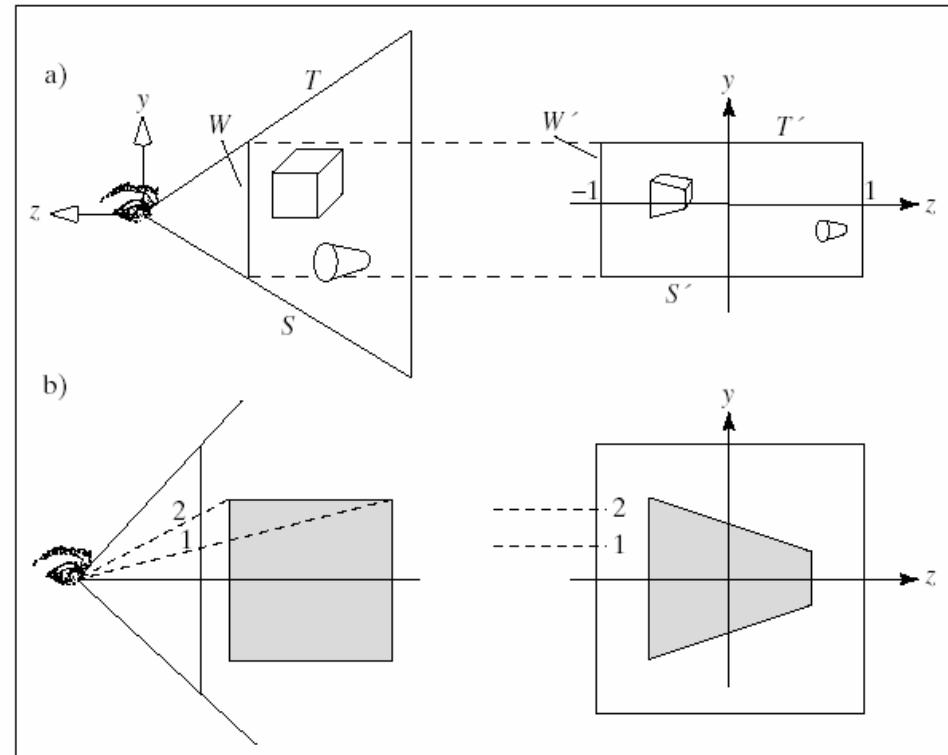
Matrix form

$$P' = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix} P$$

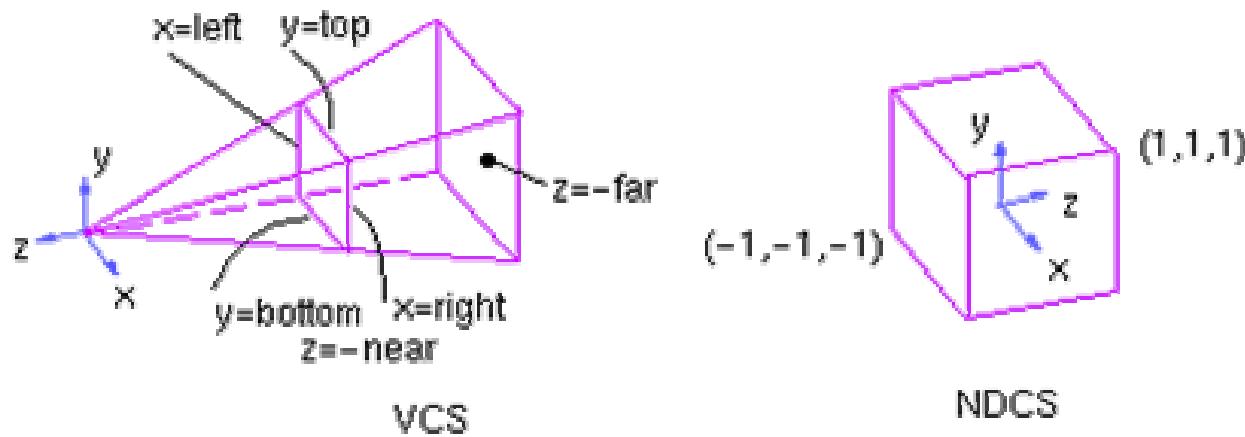
Perspective transformation

*Wraps the view volume
and the objects in it.*

- Keeping z



Derivation of the perspective transformation

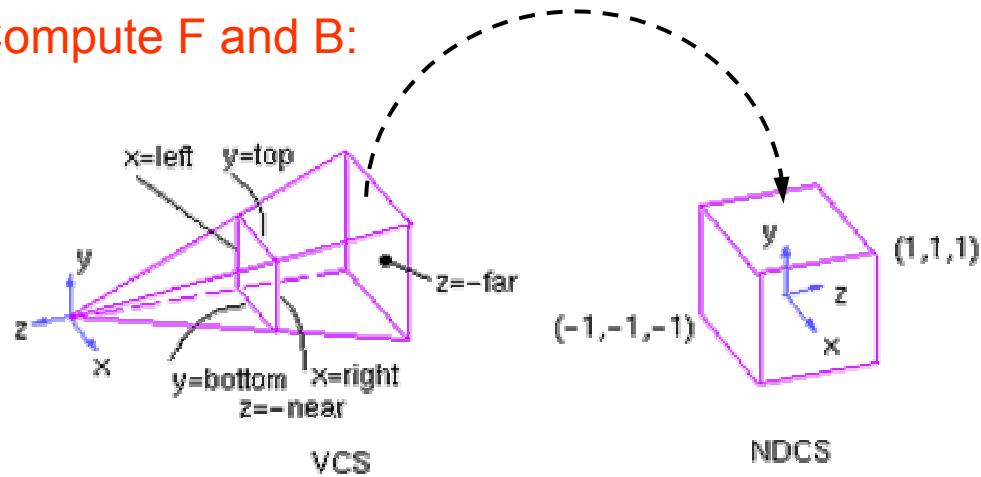


It is basically a mapping of planes

Normalized view volume is a left handed system!

Deriving the Matrix

Compute F and B:



$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} E & 0 & A & 0 \\ 0 & F & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Top plane:

$$\text{Before projection } y = -zt/n$$

$$\text{After projection and division } \tilde{y}/\tilde{h} = 1$$

From the matrix multiplication:

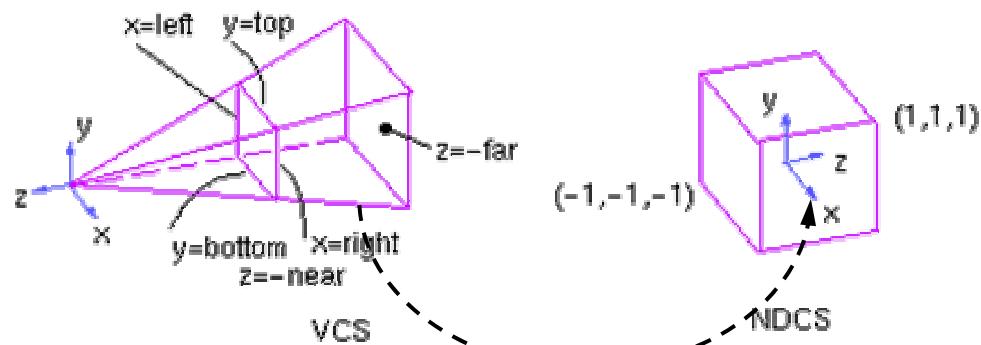
$$\tilde{y} = Fy + Bz \rightarrow (\text{perspective division})$$

$$\tilde{y}/\tilde{h} = (Fy + Bz)/\tilde{h} \rightarrow 1 = (Fy + Bz)/(-z) \rightarrow$$

$$Ft/n - B = 1 \quad (1)$$

Forming the second equation

From bottom plane



$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} E & 0 & A & 0 \\ 0 & F & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Bottom plane:

$$\text{Before projection } y = -zb/n$$

$$\text{After projection and division } \tilde{y}/\tilde{h} = -1$$

From the matrix multiplication:

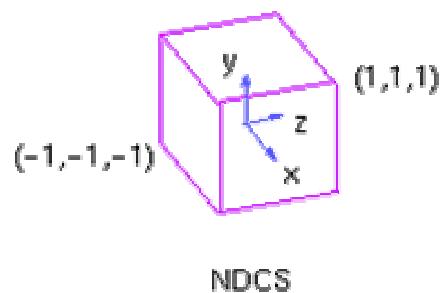
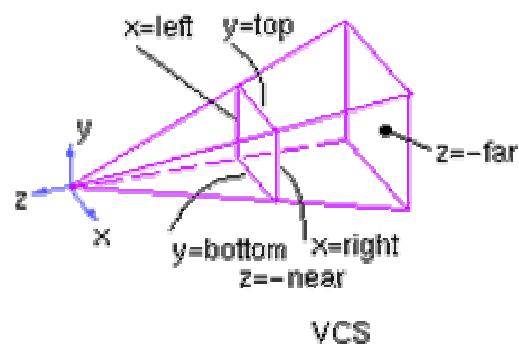
$$\tilde{y} = Fy + Bz \rightarrow (\text{perspective division})$$

$$\tilde{y}/\tilde{h} = (Fy + Bz)/\tilde{h} \rightarrow -1 = (Fy + Bz)/(-z) \rightarrow$$

$$Fb/n - B = -1 \quad (2)$$

Solving the 2x2 system

Compute F and B:

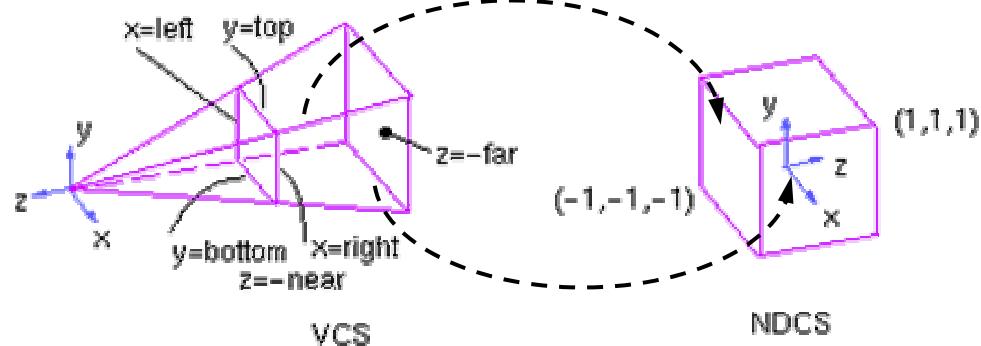


$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} E & 0 & A & 0 \\ 0 & F & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \frac{t}{n}F - B = 1 \\ \frac{b}{n}F - B = -1 \end{array} \right\} \Rightarrow \begin{array}{l} F = \frac{2n}{t-b} \\ B = \frac{t+b}{t-b} \end{array}$$

Similarly for x

Compute E and A:



$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} E & 0 & A & 0 \\ 0 & F & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \frac{r}{n}E - A = 1 \\ \frac{l}{n}E - A = -1 \end{array} \right\} \Rightarrow \begin{array}{l} E = \frac{2n}{r-l} \\ A = \frac{r+l}{r-l} \end{array}$$

Similarly z

Compute C and D

$$\tilde{z} = Cz + D \quad (1)$$

$$\tilde{h} = -z \quad (2)$$

Near plane :

$$z = -n \rightarrow \frac{\tilde{z}}{\tilde{h}} = \frac{\tilde{z}}{n} = -1 \quad (3)$$

$$(1), (2), (3) \rightarrow C(-n)/n + D/n = -1 \quad (4)$$

Similarly for far plane :

$$z = f \rightarrow \frac{\tilde{z}}{\tilde{h}} = 1 \quad (5)$$

$$(1), (2), (5) \rightarrow C(-f)/f + D/f = 1 \quad (6)$$

$$\text{From (4), (6)} \quad C = -(f+n)(f-n), \quad D = -2fn/(f-n)$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} E & 0 & A & 0 \\ 0 & F & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Putting everything together

$$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Putting everything together

$$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Same as orthographic except the scaling and the position of the x,y translation parameters

Orthographic projection in OpenGL

`glMatrixMode(GL_PROJECTION) ;`

`glLoadIdentity() ;`

Followed by one of:

`glOrtho(left,right,bottom,top,near,far) ;`

near plane at $z = -\text{near}$

far plane at $z = -\text{far}$

`gluOrtho2D(left,right,bottom,top) ;`

assumes $\text{near} = 0$ $\text{far} = 1$

Perspective projection in OpenGL

`glMatrixMode(GL_PROJECTION) ;`

`glLoadIdentity() ;`

Followed by one of:

`glFrustum(left,right,bottom,top,near,far) ;`

near plane at $z = -\text{near}$

far plane at $z = -\text{far}$

`gluPerspective(fovy, aspect,bottom,top) ;`

fov measured in degrees and center at 0

Matrix pre-multiplies Modelview matrix

$$M = Mproj^* CM$$

Important

Projection parameters are given in CAMERA Coordinate system (Viewing).

So if camera is at $z = 50$ and you give $\text{near} = 10$ where is the near plane with respect to the world?

Important

Projection parameters are given in CAMERA Coordinate system (Viewing).

So if camera at $z = 50$ and you give $\text{near} = 10$ where is the near plane with respect to the world?

Transformed by inverse(Mvcs)

Nonlinearity of perspective transformation

Tracks:

Left: $x = -1, y = -1$

Right: $x = 1, y = -1$

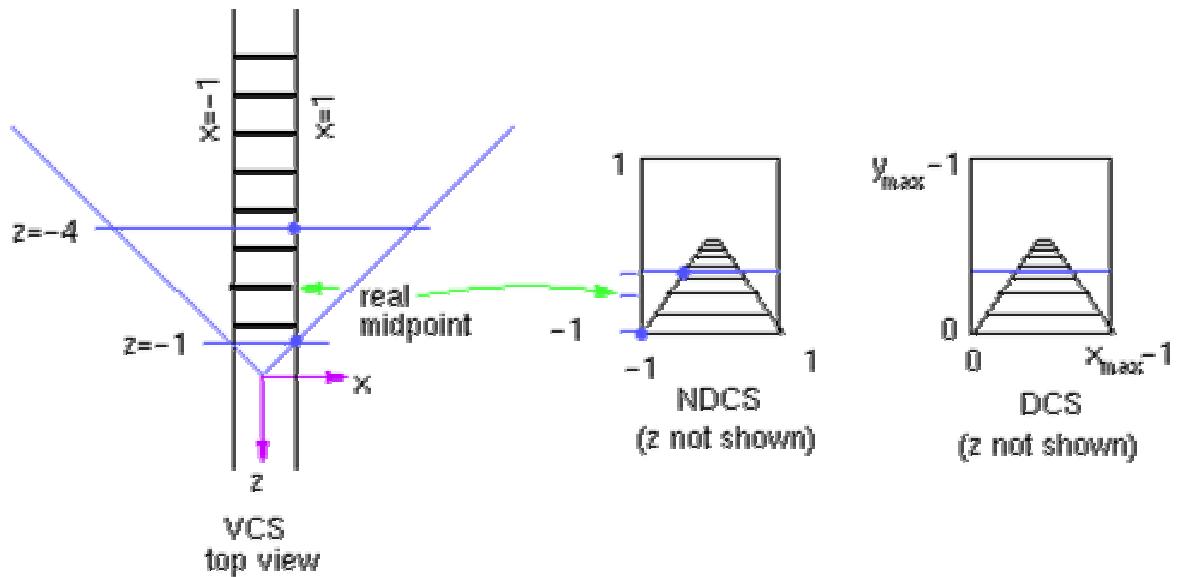
$Z = -\infty, \infty$

View volume:

Left = -1, right = 1

Bot = -1, top = 1

Near = 1, far = 4



Comparison of cs's

VCS

CCS

NDCS

X

$$X' = EX + AZ$$

$$X'' = X'/W'$$

Y

$$Y' = FY + BZ$$

$$Y'' = Y'/W'$$

Z

$$Z' = CZ + D$$

$$Z'' = Z'/W'$$

Point

Point'

Point''

1

$$X' = X = 1$$

$$X'' = -1/Z$$

-1

$$Y' = Y = -1$$

$$Y'' = 1/Z$$

Z

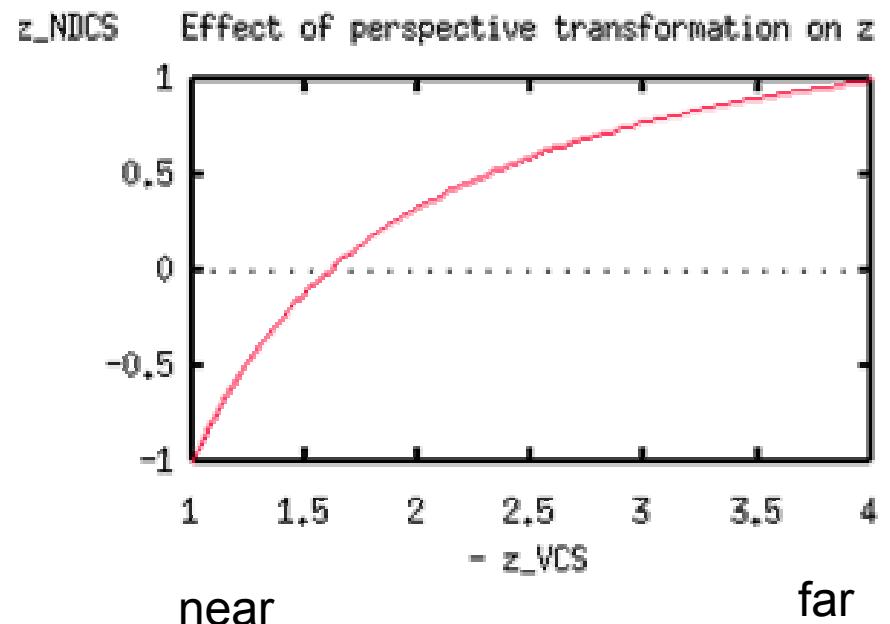
$$Z' = -5Z/3 - 8/3$$

$$Z'' = 5/3 + 8/(3Z)$$

$$W' = -Z$$

Z in NDCS vs -Z in VCS

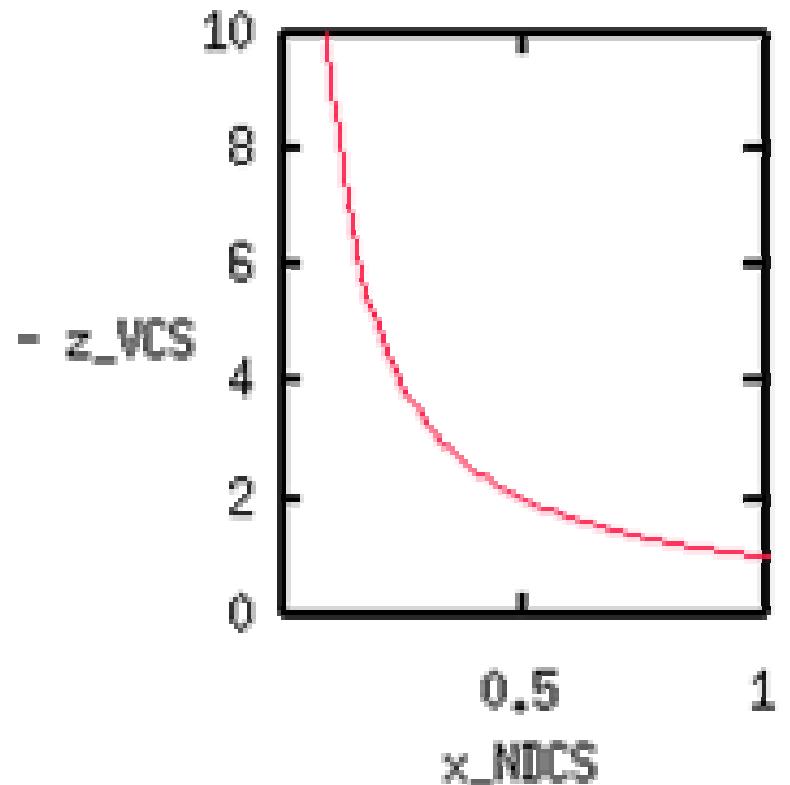
$$Z'' = 5/3 + 8/(3Z)$$



Other comparisons

$$X'' = -1/Z \rightarrow Z = -1/X''$$

z_{VCS} as a function of x_{NDCS}

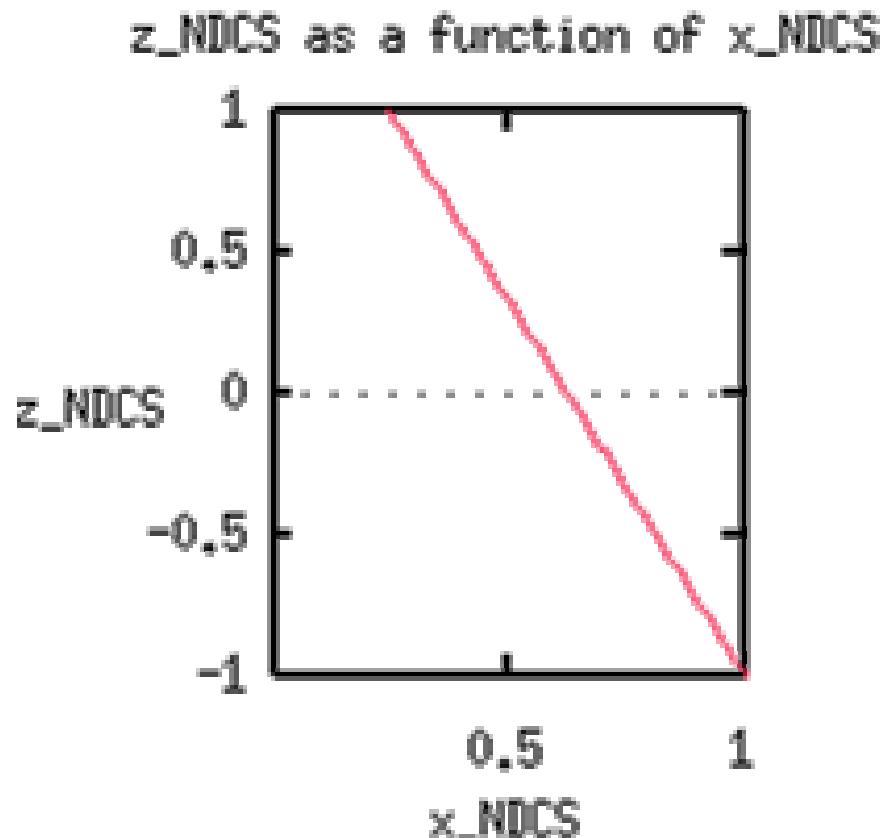


Other comparisons

$$X'' = -1/Z$$

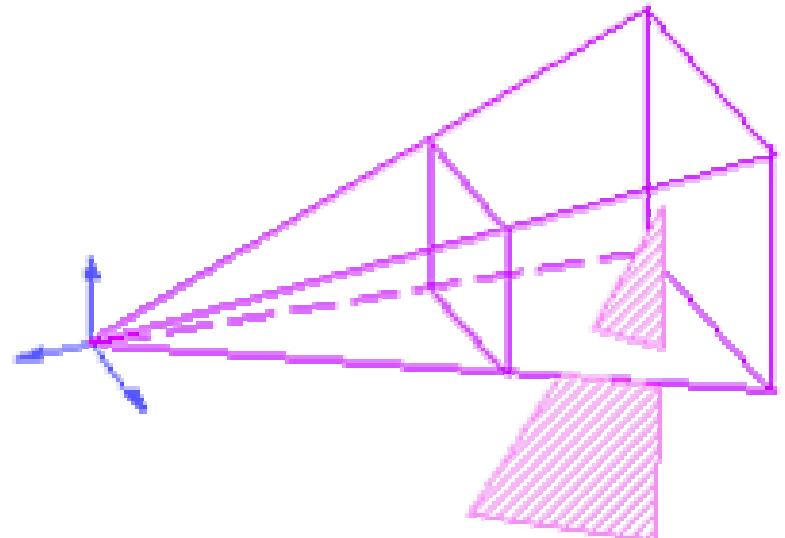
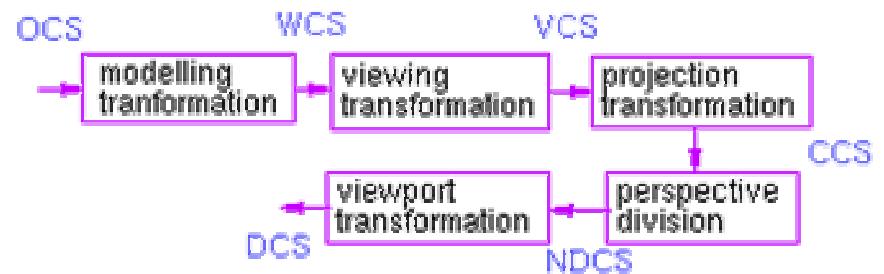
$$Z'' = 5/3 + 8/(3Z)$$

$$\rightarrow Z'' = 5/3 - (8/3)X''$$



3D Clipping

Keep what is visible



Background (reminder)

Plane equations

Implicit

$$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet \mathbf{P} + D$$

Points on Plane $F(x, y, z) = 0$

Parametric

$$\text{Plane}(s, t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$$

P_0, P_1, P_2 not colinear

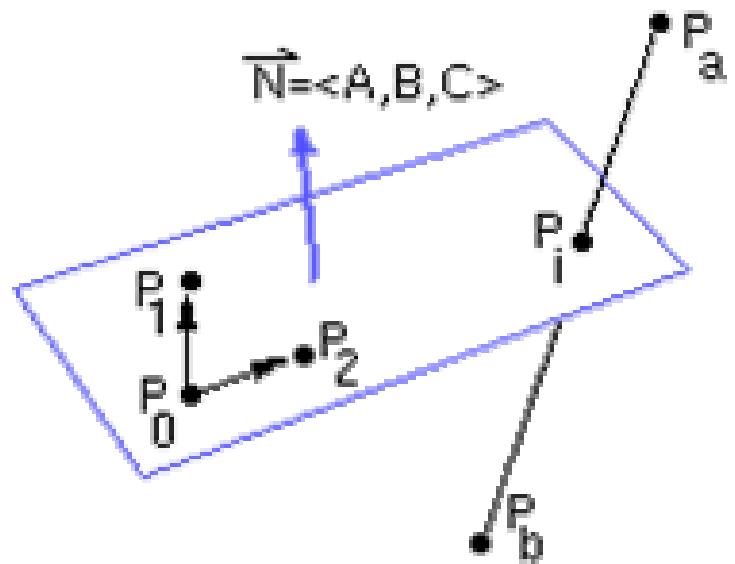
or

$$\text{Plane}(s, t) = (1 - s - t)P_0 + sP_1 + tP_2$$

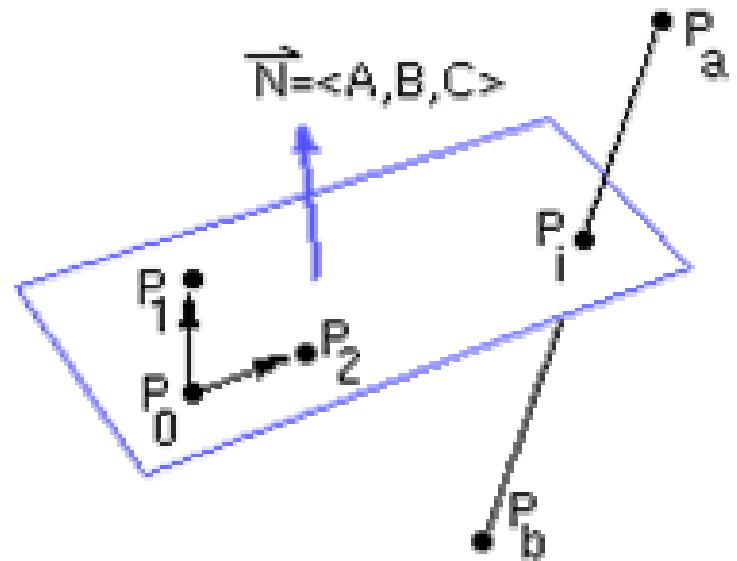
$\text{Plane}(s, t) = P_0 + sV_1 + tV_2$ where V_1, V_2 basis vectors

Explicit

$$z = -(A/C)x - (B/C)y - D/C, \quad C \neq 0$$



Intersection of line and plane



$$\vec{N} \cdot (P_a + t(P_b - P_a)) + D = 0$$

$$t = \frac{-D - \vec{N} \cdot P_a}{\vec{N} \cdot P_b - \vec{N} \cdot P_a} = \frac{-F(P_a)}{F(P_b) - F(P_a)}$$

Orthographic view volume

Planes

Normals pointing inside

left: $x - \text{left} = 0$

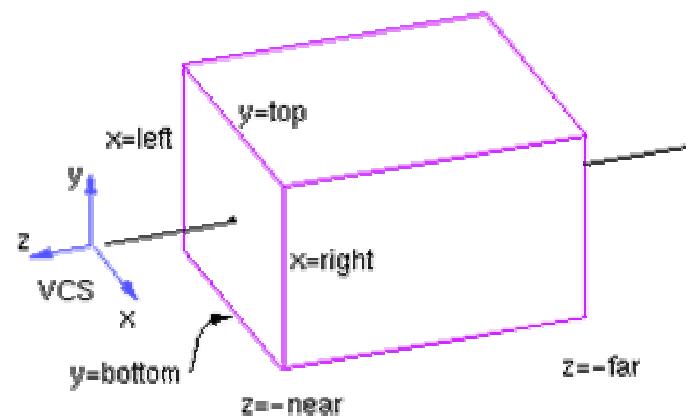
right: $-x + \text{right} = 0$

bottom: $y - \text{bottom} = 0$

top: $-y + \text{top} = 0$

front: $-z - \text{near} = 0$

back: $z + \text{far} = 0$



Perspective View volume

Planes

Normals pointing inside

$$\text{left: } x + \text{left} * z / \text{near} = 0$$

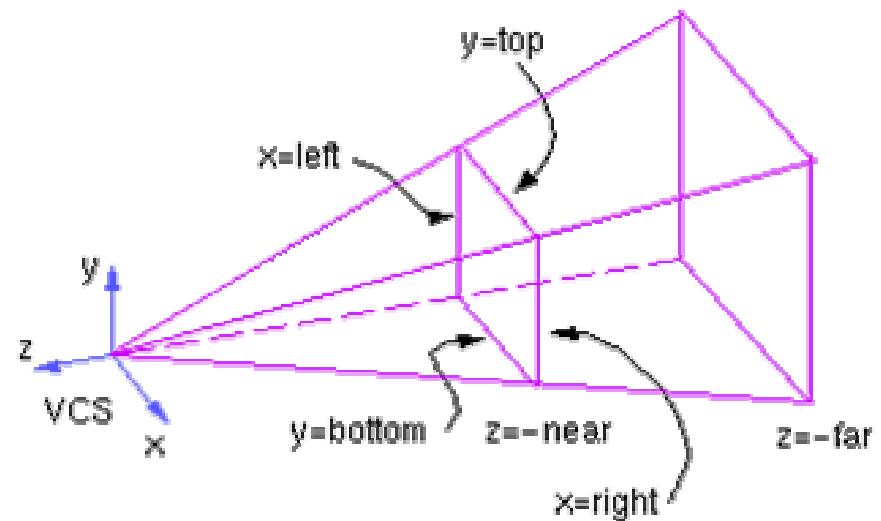
$$\text{right: } -x - \text{right} * z / \text{near} = 0$$

$$\text{top: } -y - \text{top} * z / \text{near} = 0$$

$$\text{bottom: } y + \text{bottom} * z / \text{near} = 0$$

$$\text{front: } -z - \text{near} = 0$$

$$\text{back: } z + \text{far} = 0$$



Clipping in NDCS

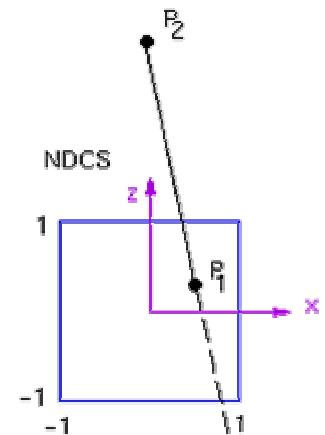
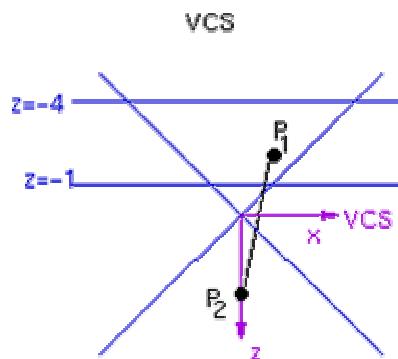
Normalized view volume

- Constant planes
- Lines in VCS lines NDCS

Problem

- Z coordinate loses its sign

	P_1	P_2
VCS	(1, 0, -2)	(0, 0, 2)
CCS	(1, 0, 2/3, 2)	(0, 0, -6, -2)
NDCS	(1/2, 0, 1/3)	(0, 0, 3)



Clipping in CCS

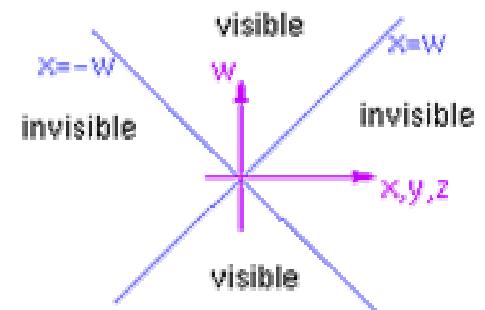
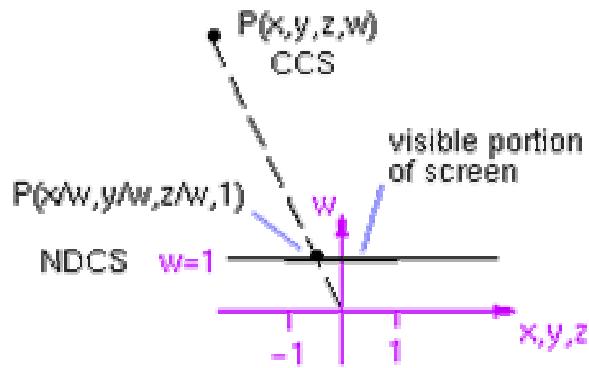
We'll define the clipping region in CCS by first looking at the clipping region in NDCS:

$$-1 \leq x/w \leq 1$$

This means that in CCS, we have:

$$-w \leq x \leq w$$

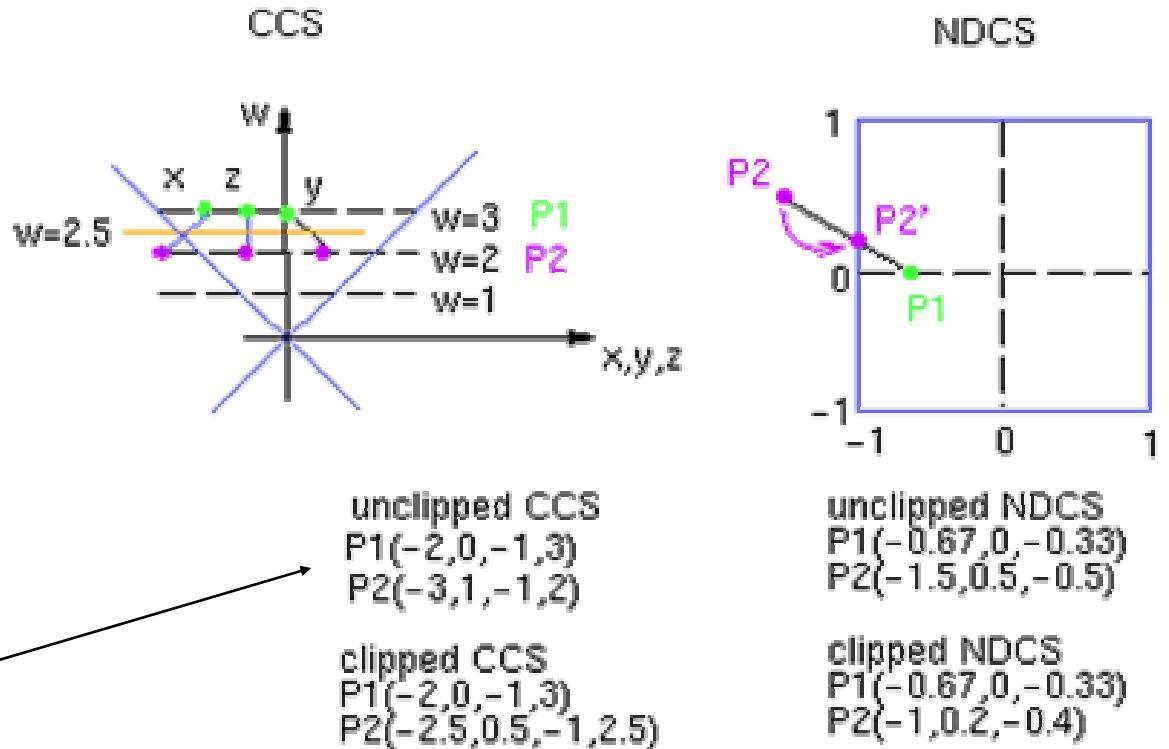
Similarly for y,z



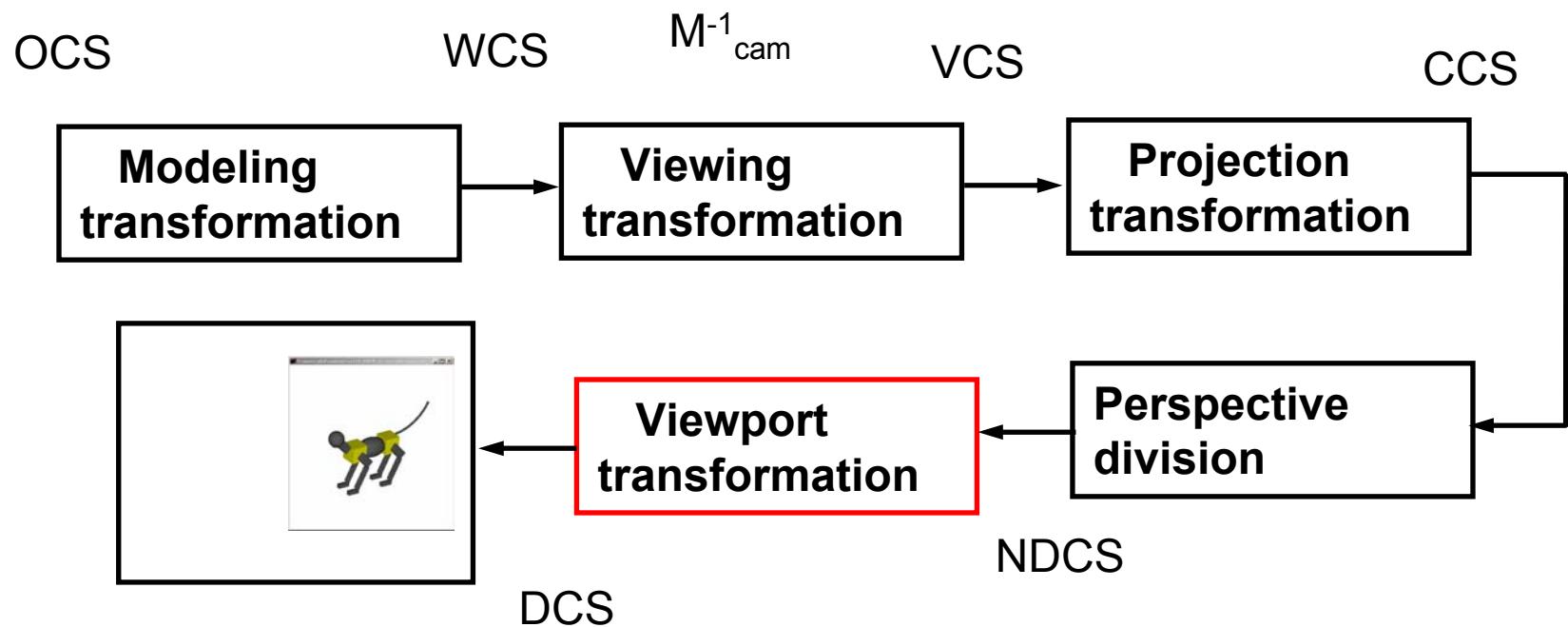
Example

The perspective transformation creates

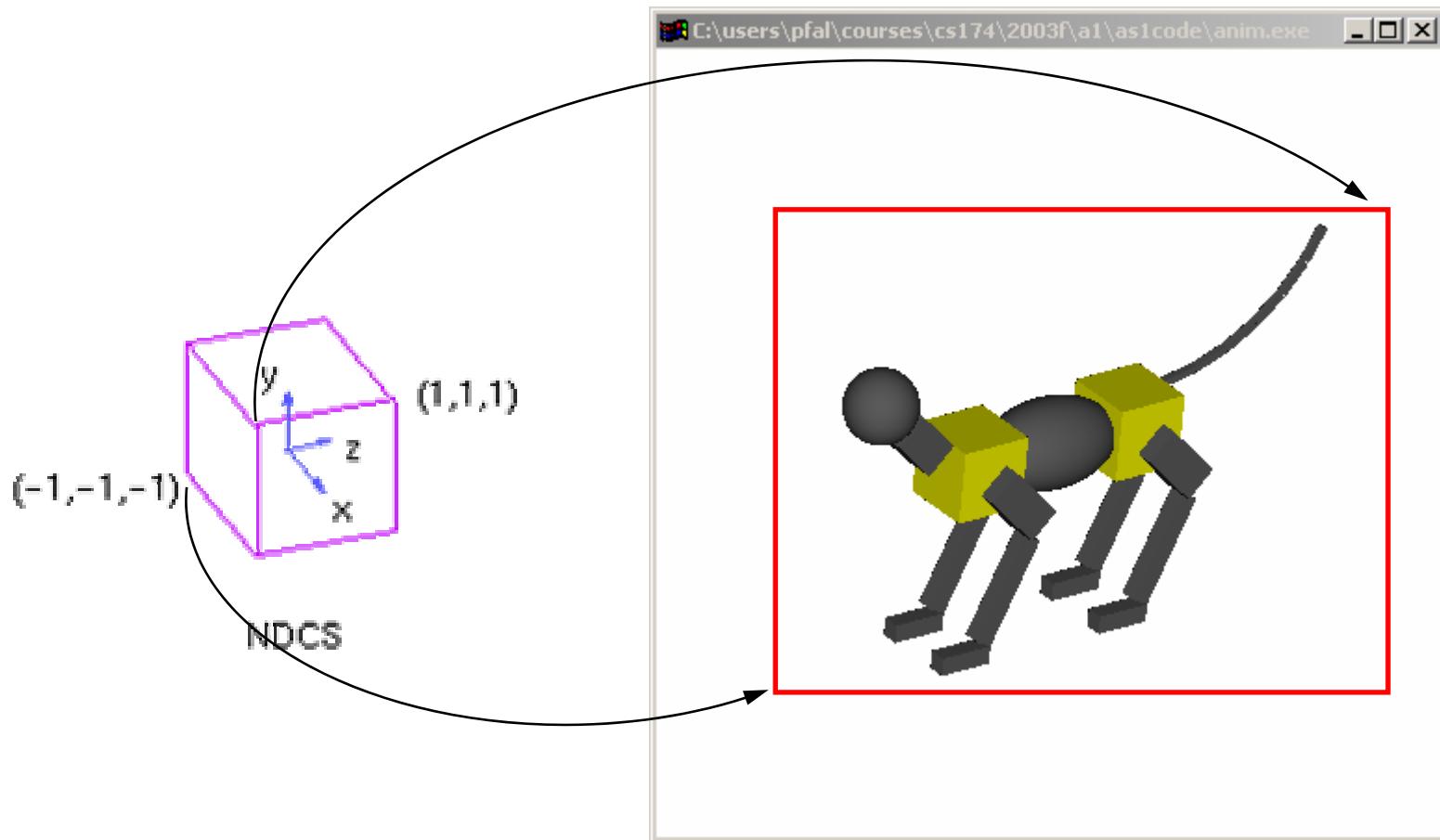
$W = -z$



Viewport transformation



Viewport



Viewport matrix

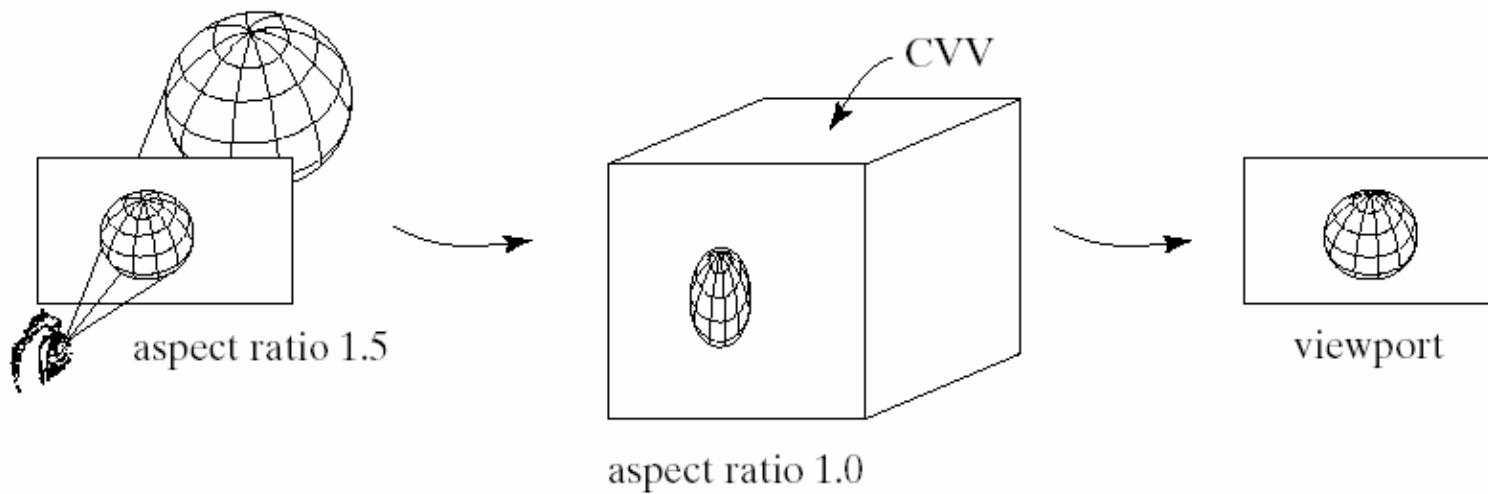
Scales the x,y to the dimensions of the viewport

Scales z to be in [0,1]

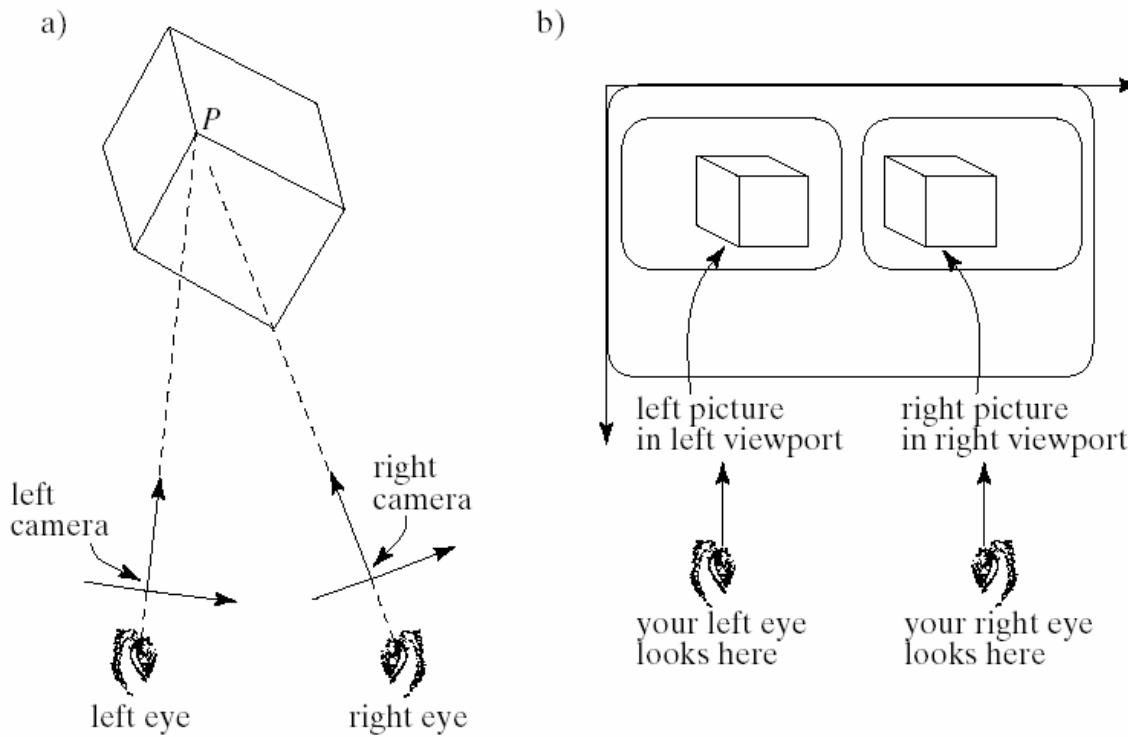
Matrix form left as an exercise.

Why viewports?

Undo the distortion of the projection transformation



Stereo views



Viewport in OpenGL

*glViewport(GLint x, GLint y, GLsizei width,
GLsizei height) ;*

x,y: lower left corner of viewport rectangle in pixels

width, height: width and height of viewport.

Transformations in the pipeline

