Homework 2
Math 128A 2021 Summer Session 2
Due Wednesday, 10/18/2021, 11:59 PM

1. (17 points) (T) Suppose that $x_0 = \frac{1}{4}, x_1 = \frac{1}{2}, x_2 = 1,$ and $x_3 = 2$.
   (a) Use divided differences to compute the newton form of the third degree polynomial such that $p(x_0) = 1, p(x_1) = 2, p(x_2) = 5,$ and $p(x_3) = -1$.
   (b) For the same data, write the Lagrange interpolating polynomial.
   (c) Show that these two forms are the same polynomial.

2. (27 points) Given the points (x_i, y_i) for $i = 0 \ldots n$ the Newton form of the interpolating polynomial is

$$p(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

   The textbook provides code for computing the coeffients by a divided difference table and evaluating the polynomial. I provided this code on the class website.
   (a) Compute the coefficients of the Newton form of the interpolating polynomial for the fifth degree polynomial that interpolates $f(x) = \cos(2\pi x)$ at the points $x_j = j/5$ for $j = 0, \ldots, 5$. Display the results in a table.
   (b) Make a plot of $f(x)$ and the interpolating polynomial, $p(x)$, from part (a) on the same axes for $0 \le x \le 1$. Plot the difference of f and p for $0 \le x \le 1$.
   (c) Estimate the maximum of $|f(x) - p(x)|$ on the interval $[0, 1]$ by evaluating f and p for a large number of points between 0 and 1.
   (d) Evaluate $p(1.5)$, $p(2.0)$, and $p(2.5)$ and report the results. Why are the values of p so different from the values of f at these points compared to points in $[0, 1]$?

3. (18 points) (T) Suppose you wish to find a polynomial p that satisfies the conditions

$$p(0) = 0, \quad p'(0) = 0, \quad p''(0) = 0, \quad p(1) = 1, \quad p'(1) = 0, \quad p''(1) = 0$$

   What degree polynomial satisfies these conditions? Prove your claim. (It is not necessary that you $p(x)$, but finding it would be one way to prove it.)

4. (35 points) Given the nodes

$$a = x_0 < x_1 < \cdots < x_n = b,$$

   let S be the piecewise Hermite cubic interpolant of f at these nodes. That is S interpolates f and f' at these nodes, and on each subinterval $[x_i, x_{i+1}]$, S coincides with a cubic polynomial $S_i(x)$.

(a) Let $h = \max_i(x_{i+1} - x_i)$ and assume that f has at least four continuous derivatives on $[a, b]$. Show that

$$|f(x) - S(x)| \leq \frac{h^4}{384} \max_{\xi \in [a,b]} \left| f^{(4)}(\xi) \right|$$

(b) (T) On the interval $[x_i, x_{i+1}]$ let S_i be represented as

$$S_i(x) = a_i + b(x - x_i) + c(x - x_i)^2 + d(x - x_i)^3.$$

Derive expressions for a_i, b_i, c_i, and d_i in terms of $h_i = x_{i+1} - x_i$, $f_i = f(x_i)$, $f'_i = f'(x_i)$, $f_{i+1} = f(x_{i+1})$, and $f'_{i+1} = f'(x_{i+1})$.

(c) Evaluating S requires the location of the nodes (x_j's) and the coefficients computed in the previous part. For easy evaluation, the coefficients can be stored in an $(n + 1)$ by 4 array. For example, let P be an $(n + 1)$ by 4 array with elements defined as $P_{i,1} = a_i$, $P_{i,2} = b_i$,$P_{i,3} = c_i$, and $P_{i,4} = d_i$.

Write a routine which takes as input the three arrays corresponding to the location of the nodes, the function values at these nodes, and the derivative values at the nodes, and it returns the array of coefficients, `P`.

Use your routine to compute the coefficients defining S for the function $f(x) = \cos(2\pi x)$ at the nodes $x_i = i/5$ for $i = 0, \ldots, 5$. Display the resulting coefficients in a table.

(d) Write a routine which takes as input the location of the nodes, the array of coefficients, and an evaluation point, x, and returns the value of $S(x)$.

Let $f(x) = \cos(2\pi x)$ and let $x_i = i/n$ for $i = 0, 1, \ldots, n$ be the $n+1$ equally spaced points on $[0, 1]$. For a range of n values use your code to estimate $\max_{x \in [0,1]} |f(x) - S(x)|$ by evaluating f and S on a large set of values in $[0, 1]$. Display your results in a table and/or graph (use log-log plot). Discuss how your results demonstrate $\mathcal{O}(h^4)$ convergence.

5. (35 points) At the end of this assignment, code is given that takes as input vectors x and y, and returns an array, P, which contains the coefficients of the natural cubic spline through the points (x_i, y_i). The cubic spline is defined as $S(x) = S_i(x)$ for $x_i \leq x \leq x_{i+1}$, and

$$S_i(x) = P_{i1} + P_{i2}(x - x_i) + P_{i3}(x - x_i)^2 + P_{i4}(x - x_i)^3.$$

Note that you can used your code Problem 4c to evaluate the spline.

(a) (T) In the absence of derivatives at the endpoints, one commonly uses the *not-a-knot* boundary condition rather than natural boundary conditions. If the points are indexed from $j = 0 \ldots n$, then the not-a-knot condition requires that $S'''(x)$ be continuous at x_1 and x_{n-1}. For $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$, our textbook derives the system of linear equations for the values of c_i:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1})$$

for $i = 1 \ldots n - 1$, where $h_i = x_{i+1} - x_i$. We need two more equations to be able to solve for the c values. Derive the two equations that come from the not-a-knot conditions involving only c values and h values.

(b) Write a routine to compute the coefficients of a cubic spline with not-a-knot boundary conditions. This requires only minor changes from the natural spline code provided.

(c) Plot the natural and not-a-knot splines that interpolate $f(x) = \cos(2\pi x)$ at the points $x_j = j/5$ for $j = 0, \ldots, 5$. Which spline looks more like the cosine function?

(d) For each spline, estimate the maximum error in using the spline to approximate $f(x)$ between $x = 0$ and $x = 1$. Do this by using your code to evaluate f and each spline at a large number of points on $[0, 1]$ compute the maximum absolute value of of the difference on these evaluation

points.

6. (a) (35 points) Use $N = 11$ equally spaced points between $x = -1$ and $x = 1$, and plot the $N-1$-degree polynomial that interpolates

$$g(x) = \frac{1}{1 + 25x^2}$$

at these points for $-1 \leq x \leq 1$. Repeat for $N = 21$. What is the maximum error in using these interpolating polynomials to approximate $g(x)$.

(b) Repeat part (a), but use a piecewise Hermite cubic polynomial.

(c) Repeat part (a), but use a not-a-knot spline.

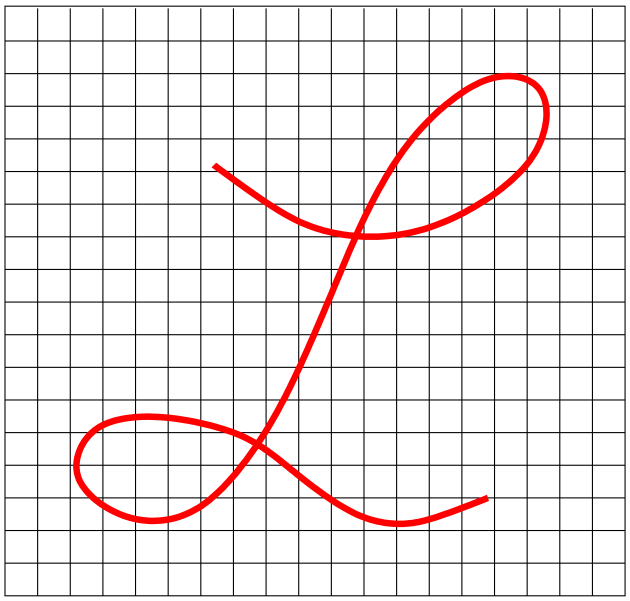
(d) Repeat part (a) using the points

$$x_j = \cos\left(\frac{j}{n}, \pi\right), \quad j = 0 \ldots n,$$

instead of equally spaced points.

(e) Comment on your results.

7. (33 points) Use cubic splines to represent a curve $(x(s), y(s))$ that is in the shape of the script letter L below. List the points you used to make the splines, and make a plot of your curve. You may use the provided grid to estimate the nodes by hand, or use software to help identify points on the curve. For example, in MATLAB the commands `imread, image`, and `ginput` could be used to aid in selecting your points.



```
% Natural spline coefficients
%  Input:   x - n by 1 vector of the nodes
%           y - n by 1 vector with the corresponding function values
%  Output:  P - n-1 by 4 matrix
%           the elements of the ith row of P give the coefficients
%           of the cubic between [x(i),x(i+1)] as
%           Si(x) =  P(i,1) + P(i,2)*(x-x(i))
%                  + P(i,3)*(x-x(i))^2 + P(i,4)*(x-x(i))^3
%
function P=naturalspline(x,y);
  n = length(x);

  % force x and y to be column vectors
  %
  x = shiftdim(x); y = shiftdim(y);

  % compute the distances between the points
  %
  h = x(2:n) - x(1:n-1);

  % set up the tridiagonal linear system to solve
  %
  d0 = 2*(h(1:n-2)+h(2:n-1));    % diagonal
  dm1 = h(2:n-1);                % subdiagional, note dm1(end) not used below
  d1  = h(1:n-2);                % superdiagional, note d1(1) not used below

  % form the matrix and rhs
  %
  A = spdiags([dm1 d0 d1],-1:1,n-2,n-2);
  b = 3*(y(3:n)-y(2:n-1))./h(2:n-1) - 3*(y(2:n-1)-y(1:n-2))./h(1:n-2);

  % solve the linear system
  %
  z = A\b;

  % append the natural boundary conditions
  %
  z = [0; z; 0];

  % compute the coefficients
  %
  P = zeros(n-1,4);
  P(:,1) =  y(1:n-1);
  P(:,2) = (y(2:n) -y(1:n-1))./h - h.*(z(2:n)+2*z(1:n-1))/3;
  P(:,3) =  z(1:n-1);
```

```
P(:,4) = (z(2:n)-z(1:n-1))./(3*h);
```

```
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#           of the cubic between [x(i),x(i+1)] as
#           Si(x) =  P(i,1) + P(i,2)*(x-x(i))
#                  + P(i,3)*(x-x(i))^2 + P(i,4)*(x-x(i))^3
#
import numpy as np
from scipy.linalg import solve_banded
def naturalspline(x,y):
    n=len(x)

    # spacing between points
    #
    h = x[1:n] - x[0:n-1]

    # main diagonal, super, and sub
    #  note note first element not used in super, last element not used in sub
    #
    d0 =2.0*(h[0:n-2]+h[1:n-1])  # main diag
    d1 =  h[0:n-2];              # superdiag
    dm1 =  h[1:n-1];             # subdiag

    # solve the linear system
    #
    b   = 3.0*(y[2:n]-y[1:n-1])/h[1:n-1] - 3.0*(y[1:n-1]-y[0:n-2])/h[0:n-2]
    A   = np.matrix([d1,d0,dm1])
    sol = solve_banded((1, 1), A, b)

    # append the natural boundary conditions
    #
    z = np.zeros(n)
    z[1:n-1]=sol

    # compute the coeffients
    #
    P = np.zeros([n-1,4])
    P[:,0] = y[0:n-1]
    P[:,1] = (y[1:n]-y[0:n-1])/h - h*(z[1:n]+2*z[0:n-1])/3.0
    P[:,2] = z[0:n-1]
    P[:,3] = (z[1:n]-z[0:n-1])/(3.0*h);

    return P
```