HW8

Thursday, November 6, 2025 2:04 PM

1. CB 4.54 $X_i \overset{iid}{\sim} \text{UNIF}(0,1)$ FIND PDF OF $\prod_{i=1}^{n} X_i$

$$Z = \prod_{i=1}^{n} X_i \Rightarrow \ln(z) = \sum_{i=1}^{n} \underbrace{\ln(X_i)}_{-\text{Exp}(1)} = -\underbrace{\sum_{i=1}^{n} \text{Exp}(1)}_{\text{Gamma}(n,1)} = -\text{Gamma}(n,1)$$

FOR PDF, NEED TO WRITE $\text{Gamma}(-\ln(z) \mid n,1)$:

$$f_Z = \frac{1}{\Gamma(n)} (-\ln(z))^{n-1} e^{-(-\ln(z))} \frac{d}{dz}(-\ln(z)) = \frac{1}{\Gamma(n)} (-\ln(z))^{n-1} \frac{z}{z}$$

$$f_Z = \frac{[-\ln(z)]^{n-1}}{\Gamma(n)}$$

2. 4.55 $X_1, X_2, X_3 \sim \text{Exp}(\lambda)$ WHAT IS $\text{Max}(X_1, X_2, X_3)$? $X_1 \perp\!\!\!\perp X_2 \perp\!\!\!\perp X_3 \perp\!\!\!\perp X_1$

WE KNOW $f_{X_{(n)}} = n\left[F_X(x)\right]^{n-1} \cdot f_X(x) \therefore f_{X_{(3)}} = 3\left[1 - e^{-\lambda x}\right]^2 \cdot \lambda e^{-\lambda x}$

$$f_{X_{(3)}} = 3\left[1 - 2e^{-\lambda x} + e^{-2\lambda x}\right] \lambda e^{-\lambda x} = 3\lambda\left[e^{-\lambda x} - 2e^{-2\lambda x} + e^{-3\lambda x}\right]$$

3. $\underset{\sim}{Y} = \underset{\sim}{W} + \epsilon$ $\underset{\sim}{W} \sim N(\mu \underset{\sim}{1}, \underset{\sim}{\Sigma})$ $\epsilon \sim N(\underset{\sim}{0}, \tau^2 \underset{\sim}{I})$

a. WHAT IS f_Y? $y_i = w_i + \epsilon_i$ $w_i = N(\mu, \sigma^2)$ $\epsilon \sim N(0, \tau^2)$

WE KNOW THAT FOR $X \sim N(\mu_X, \sigma_X^2)$ $Y \sim N(\mu_Y, \sigma_Y^2)$, $X + Y = Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

THUS $Y_i \sim N(\mu, \sigma^2 + \tau^2) \Rightarrow Y \sim N(\mu \underset{\sim}{1}, \underset{\sim}{\Sigma} + \tau^2 \underset{\sim}{I})$

b. $E[e^{y_i}] = M_Y(1) = e^{[(1)\mu + (1)(\sigma^2 + \tau^2)/2]} = e^{\left[\mu + \frac{\sigma^2 + \tau^2}{2}\right]} = E[e^{y_i}]$

c. $\text{Cov}(e^{y_i}, e^{y_i}) = E[e^{y_i} e^{y_i}] - E[e^{y_i}]E[e^{y_j}] = E[e^{y_i + y_j}] - E[e^{y_i}]E[e^{y_j}]$

I saw the solution involved using the covariance matrix. I actually have no idea how to construct/manipulate that...

$$= e^{2\mu + (\Sigma_{ii} + 2\Sigma_{ij} + \Sigma_{jj} + 2\tau^2)/2} - e^{\mu + \frac{1}{2}(\Sigma_{ii} + \tau^2)} e^{\mu + \frac{1}{2}(\Sigma_{jj} + \tau^2)}$$

$$2\mu + (\Sigma_{ii} + 2\Sigma_{ij} + \Sigma_{jj} + 2\tau^2)/2 \qquad 2\mu + \tau^2 + \tfrac{1}{2}(\Sigma_{ii} + \Sigma_{jj})$$

$= e \quad\quad\quad\quad\quad\quad - e$

4. $y_i \mid \lambda_i \sim \text{Pois}(\lambda_i)$ $\quad \lambda_i \mid w_i = e^{w_i}$ $\quad\quad w_i \sim N(0, \sigma)$

a. Find $E[y_i]$ $\quad E[y_i] = E_{w_i}\left[E_{\lambda_i|w_i}\left(E_{y_i|\lambda_i}(y_i \mid \lambda_i)\right)\right] = E_{w_i}\left[E_{\lambda_i|w_i}(\lambda_i \mid w_i)\right] = E_{w_i}\left[e^{w_i}\right] = e^{\sigma^2/2}$

$E[y_i] = e^{\sigma^2/2}$

b. Using solution from 3(c) $= e^{(\sigma_{ii}+\sigma_{jj})/2}\left(e^{\sigma_{ij}} - 1\right)$

5. CB 5.1 $\quad p = 1\%$ $\quad$ How large n s.t. $P(X \ge 1) \ge .95$ $\quad$ where $X \sim \text{Binom}(n, p)$

Same as $P(X=0) \le .05 \Rightarrow \binom{n}{0}(.01)^0(1-.01)^{n-0} = (1)(1)(.99)^n \le .05 \Rightarrow$

$n \ge \ln_{\left(\frac{99}{100}\right)}\left(\frac{1}{20}\right) \approx 298.07$

$\therefore\ n \ge 299$

6. CB 5.2 $\quad X_1, X_2 \ldots$ cont., have F_X . X_i = Annual rainfall at location

a. Find dist. of years until first year is exceeded. (Let this be Y)

$Y_1: X_2 > X_1 \quad\quad Y_2: X_2 < X_1,\ X_3 > X_1$ etc.

$P(Y=1) = P(X_2 > X_1) = 1 - P(X_2 \le X_1) = 1 - F_X$

$P(Y=2) = P(X_3 > X_1) \cdot P(X_2 < X_1) = (1 - F_X) \cdot F_X$

$P(Y=3) = P(X_4 > X_1) \cdot P(X_3 < X_1)\, P(X_2 < X_1) = (1 - F_X) \cdot F_X \cdot F_X$

$\ldots = (1 - F_X) \cdot F_X^{n-1} = F_Y \quad\quad f_Y = \frac{d}{dy} F_Y = -f_X F_X^{n-1} \cdot (n-1) f_X (1 - F_X) F_X^{n-2}$

$= f_X F_X^{n-2}\left[(n-1) f_X (1 - F_X) - F_X\right) = f_Y$

b. Show $E(Y) = \infty$

Y is geometric: $E(Y) = \frac{1}{p}$ $\quad$ In this case, p is just F_X

$0 \le F_X \le 1$ $\therefore$ we will get a distribution in denom. which will result in $\frac{1}{\text{very small}} \Rightarrow \infty$

Thus $E(Y) = \infty$

7. CB 5.6 $X \perp\!\!\!\perp Y$, f_X, f_Y

GET FORMULAS SIMILAR TO: $Z = X+Y \therefore f_Z = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\,dw$

a. $Z = X - Y$: $W = X$ $\frac{\partial X}{\partial w} = 1$ $\frac{\partial X}{\partial z} = 0$ $\frac{\partial Y}{\partial w} = 1$ $\frac{\partial Y}{\partial z} = -1$ $J = 1$

$Y = W - Z$

$f_{Z,W} = f_{X,Y}(w, w-z) = f_X(w) f_Y(w-z) \therefore \underline{f_Z = \int_{-\infty}^{\infty} f_X(w) f_Y(w-z)\,dw}$

b. $Z = XY$: $W = X$ $Y = \frac{Z}{W}$ $\frac{\partial x}{\partial z} = 0$ $\frac{\partial X}{\partial w} = 1$ $\frac{\partial y}{\partial z} = \frac{1}{w}$ $\frac{\partial Y}{\partial w} = \text{BLAH} \Rightarrow J = \left|-\frac{1}{w}\right|$

$f_{Z,W} = f_{X,Y}\left(w, \frac{z}{w}\right) \cdot \frac{-1}{w} = f_X(w) f_Y\left(\frac{z}{w}\right)\left|\frac{-1}{w}\right| \Rightarrow \underline{f_Z = \int_{-\infty}^{\infty} \left|-\frac{1}{w}\right| f_X(w) f_Y\left(\frac{z}{w}\right) dw}$

C. $Z = \frac{X}{Y}$: $W = X$ $Y = \frac{W}{Z} \Rightarrow \frac{\partial x}{\partial z} = 0$ $\frac{\partial x}{\partial w} = 1$ $\frac{\partial Y}{\partial z} = -\frac{w}{z^2}$ $\frac{\partial Y}{\partial w} = \frac{1}{z} \Rightarrow J = \frac{w}{z^2}$

$f_{Z,W} = f_{X,Y}\left(w, \frac{x}{z}\right) \cdot \frac{w}{z^2} = f_X(w) \cdot f_Y\left(\frac{x}{z}\right) \cdot \frac{w}{z^2} \Rightarrow \underline{f_Z = \int_{-\infty}^{\infty} \left|\frac{w}{z^2}\right| f_X(w) f_Y\left(\frac{w}{z}\right) dw}$

8. CB 5.11 $\bar{X}$ & S^2 COME FROM $X_1, X_2, \ldots$ WITH VAR σ^2 . WE KNOW $E(S^2) = \sigma^2$

SHOW THAT $E(S) \le \sigma$, AND IF $\sigma^2 > 0$, $E(S) < \sigma$

S^2: CONVEX: THUS, $E(S^2) \ge E(S)^2 \Rightarrow \sigma^2 \ge E(S)^2 \Rightarrow \underline{\sigma \ge E(S)}$

THIS CAN ONLY BE $E(S^2) = E(S)^2$ WHEN $E(S^2)$ IS LINEAR, WHICH IS ONLY AT SET POINTS

THUS, $E(S^2) > E(S)^2$ EXCEPT WHERE $P(S = E(S)) = 1$, WHICH ONLY OCCURS AT $\sigma = 0$, THUS $\underline{E(S) < \sigma}$ WHEN $\sigma > 0$

9. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ FIND $\mathrm{VAR}(S^2)$ WHERE $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$\mathrm{VAR}(S^2) = \frac{\sigma^2}{n-1} \mathrm{VAR}\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{\sigma^2}{n-1} \mathrm{VAR}\left(\chi^2_{n-1}\right) = \frac{\sigma^2}{n-1} 2(n-1) = 2\sigma^2$

$\underline{\mathrm{VAR}(S^2) = 2\sigma^2}$

10. $U \sim N(0,1)$ $V \sim \chi^2_{(p)}$ $U \perp\!\!\!\perp V$

a. Find $f_{X,Y}$ where $X = \frac{U}{\sqrt{V/p}}$ $Y = V$ $U = x\left(\frac{y}{p}\right)^{1/2}$

$\frac{\partial u}{\partial x} = \sqrt{\frac{y}{p}}$ $\frac{\partial u}{\partial y} = \frac{1}{2p} x \left(\frac{y}{p}\right)^{-1/2}$ $\frac{\partial y}{\partial x} = 0$ $\frac{\partial v}{\partial y} = 1$

$J = \left|\sqrt{\frac{y}{p}}\right|$

$$f_{X,Y} = f_{U,V}\left(x\left(\frac{y}{p}\right)^{1/2}, y\right) \cdot \left|\sqrt{\frac{y}{p}}\right| = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x\left(\frac{y}{p}\right)^{1/2}\right)^2} \cdot \frac{1}{\Gamma(p/2)\, 2^{p/2}} y^{p/2-1} e^{-y/2} \cdot \left(\frac{y}{p}\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(p/2)\, 2^{p/2}\, p^{1/2}} e^{-\frac{1}{2}y\left(\frac{x^2}{p}+1\right)} \cdot y^{\frac{1}{2}(p-1)} = f_{X,Y}$$

b. Find f_X. Show it's $\sim t_p$ $X = \frac{U}{\sqrt{V/p}} = \frac{N(0,1)}{\sqrt{\frac{\chi^2_p}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{(n-1)s^2}{\sigma^2}}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{(n-1)\sigma^2}}} = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_p$

11. $U \sim \chi^2_q$ $V \sim \chi^2_p$ $U \perp\!\!\!\perp V$

a. Find $f_{X,Y}$: $X = \frac{U/p}{V/q}$ $Y = V$ $U = \frac{p}{q}xy$ $\frac{\partial u}{\partial x} = \frac{p}{q}y$ $\frac{\partial u}{\partial y} = \frac{p}{q}x$ $\frac{\partial v}{\partial x} = 0$ $\frac{\partial y}{\partial y} = 1$ $J = \left|\frac{p}{q}y\right|$

$$f_{X,Y} = f_{U,V}\left(\frac{p}{q}xy, y\right) \cdot \frac{p}{q}y = \frac{p}{q\,\Gamma(p/2)\, 2^{p/2}\, \Gamma(q/2)\, 2^{q/2}} \cdot \left(\frac{p}{q}xy\right)^{p/2-1} \cdot e^{-\frac{p}{2q}xy} \cdot y^{q/2-1} \cdot e^{-y/2}$$

$$= \frac{p^{p/2}\, q^{q/2}}{\Gamma(p/2)\Gamma(\frac{q}{2})\, 2^{\frac{1}{2}(p+q)}} (q+xp)^{-\frac{1}{2}(q+p)}\, x^{p/2-1}\, y^{\frac{1}{2}(p+q)-1}\, e^{-y/2} = f_{X,Y}$$

b. $$f_X = \int_0^\infty \underbrace{\frac{p^{p/2}\, q^{q/2}}{\Gamma(p/2)\Gamma(q/2)\, 2^{\frac{1}{2}(p+q)}} \cdot (q+xp)^{-\frac{1}{2}(q+p)}\, x^{p/2-1}}_{\text{All comes out} = \text{COEF}}\, y^{\frac{1}{2}(p+q)-1}\, e^{-y/2}\, dy$$

$$= \text{COEF} \int_0^\infty \underbrace{y^{\frac{1}{2}(p+q)-1} e^{-y/2}}_{\text{Gamma}\left(\frac{1}{2}(p+q),\, 2\right)} dy = \underbrace{\frac{p^{p/2}\, q^{q/2}\, \Gamma\left(\frac{1}{2}(p+q)\right) \cancel{2^{1/2(p+q)}}}{\Gamma(p/2)\, \Gamma(q/2)\, \cancel{2^{1/2(p+q)}}} \cdot (q+xp)^{-1/2(q+p)}\, x^{p/2-1}}_{\text{PDF of } F_{p,q}}$$

$f_X = F_{p,q}$

12. CB 5.16 $i = 1,2,3$ $X_i \sim N(i, i^2)$. Use X_i's to construct:

a. χ^2_3 $\frac{X_i - i}{i^2} \sim N(0,1)$ Let $Z_1 = \frac{X_1 - 1}{1}$ $Z_2 = \frac{X_2 - 2}{2}$ $Z_3 = \frac{X_3 - 3}{3}$

Then $Z_1^2, Z_2^2, Z_3^2 \sim \chi^2_1$ $\therefore$ $Z_1^2 + Z_2^2 + Z_3^2 = \chi^2_3$

THEN

b. t_2 LET $U = \frac{X_1 - 1}{1} \sim N(0,1)$ AND $V = \sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2 \sim \chi^2_2$

THEN $t_2 = \frac{U}{\sqrt{V/2}}$

c. $F_{1,2}$ LET $U = \left(\frac{X_1 - 1}{1}\right)^2$ $V = \sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2 \sim \chi^2_2$

THEN $F_{1,2} = \frac{U/1}{V/2}$

13. CB 5.17 bcd

b. FIND MEAN OF $F_{p,q}$

$F_{p,q} = \frac{U/p}{V/q}$ $E(F) = E\left(\frac{U/p}{V/q}\right) = E\left(\frac{U}{p}\right)E\left(\frac{q}{V}\right) = \frac{q}{p}E(U)\,E\left(\frac{1}{V}\right)$ $E(U) = p$

$U \sim \chi^2_p$ $V \sim \chi^2_q$

$$= q\,E\left(\frac{1}{V}\right) \quad E\left(\frac{1}{V}\right) = \int_0^\infty \frac{1}{v}\,\frac{1}{\Gamma(q/2)\,2^{q/2}}\,v^{p/2-1}e^{-v/2}\,dv = \frac{1}{\Gamma(q/2)\,2^{q/2}}\int_0^\infty \underbrace{v^{\frac{q-2}{2}-1}e^{-v/2}\,dv}_{\text{GAMMA KERNEL}}$$

$$= \frac{1}{\Gamma(q/2)\,2^{q/2}}\cdot\Gamma\left(\frac{q-2}{2}\right)2^{\left(\frac{q-2}{2}\right)} = \frac{\Gamma\left(\frac{q}{2}-1\right)2^{\frac{q}{2}-1}}{\Gamma\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-1\right)2^{q/2}} = \frac{1}{2\left(\frac{q-2}{2}\right)} = \frac{1}{q-2} = E(F)$$

c. SHOW $\frac{1}{X}$ HAS AN $F_{q,p}$ DIST: $X = F_{p,q} = \frac{\chi^2_p/p}{\chi^2_q/q}$ $\frac{1}{X} = \frac{\chi^2_q/q}{\chi^2_p/p} = F_{q,p}$

D. SHOW $\frac{\left(\frac{p}{q}\right)X}{1+\left(\frac{p}{q}\right)X} \sim BETA(p/2, q/2)$ $Y = \frac{\frac{p}{q}X}{1+\frac{p}{q}X} = \frac{pX}{q+pX}$ $X = \frac{qY}{p(1-Y)}$ $\left|\frac{dx}{dy}\right| = \frac{q}{p}(1-y)^{-2}$

$$f_Y = \underbrace{\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}}_{\text{COEF}}\left(\frac{p}{q}\right)^{p/2}\frac{\left(\frac{qy}{p(1-y)}\right)^{\left(\frac{p-2}{2}\right)}}{\left(1+\frac{p}{q}\left(\frac{qy}{p(1-y)}\right)\right)^{(p+q)/2}}\cdot\frac{q}{p(1-y)^2}$$

$$= \text{COEF}\cdot\left(\frac{p}{q}\right)^{p/2}\cdot\left(\frac{q}{p}\right)^{\frac{p-2}{2}}\cdot\frac{q}{p}\cdot\frac{\left(\frac{y}{1-y}\right)^{\frac{p-2}{2}}}{\left(1+\frac{y}{1-y}\right)^{\frac{p+q}{2}}}\cdot\frac{1}{(1-y)^2} = \text{COEF}\cdot\left(\frac{p}{q}\right)^{p/2}\cdot\left(\frac{p}{q}\right)^{-p/2+1}\cdot\left(\frac{p}{q}\right)^{-1}\cdot\frac{y^{\frac{p-2}{2}}}{\left(1+\frac{y}{1-y}\right)^{\frac{p+q}{2}}\cdot(1-y)^2\cdot(1-y)^{\frac{p}{2}-1}}$$

$= \left(\overline{1-y}\right) \cdot (1-y)^2 \cdot (1-y) = (1-y)$ $\qquad = (1-y)$

$$= \left[\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \right] \cdot y^{p/2 - 1} \cdot (1-y)^{\frac{q}{2}-1}$$

$$\uparrow \frac{1}{\beta\left(\frac{p}{2}, \frac{q}{2}\right)}$$

$$= \text{BETA}\left(p/2, q/2\right)$$

14. CB 5.18 a, b $\quad X \sim t_p$

a. FIND $E(X)$ & $VAR(X)$

$t_p = \dfrac{U}{\sqrt{V/p}} \quad U \sim N(0,1) \quad V \sim \chi^2_p \quad U \perp\!\!\!\perp V \quad$ so $E(t_p) = E(U)\,E\left(\dfrac{1}{\sqrt{V/p}}\right) = 0 \cdot \dots = 0$

$E(X) = 0$

$$VAR(X) = E(X^2) - \overset{0}{E(X)^2} = E(X^2) = E\left(\frac{U^2}{V/p}\right) = E(U^2)\,E\left(p/V\right) = \left[VAR(U) + E(U)^2\right] \cdot p\,E\left(\frac{1}{V}\right)$$

$$= (1+0)\, p \cdot \frac{1}{p-2} = \frac{p}{p-2}$$

$$VAR(X) = \frac{p}{p-2}$$

b. SHOW $X^2 \sim F_{1,p}$ $\quad X^2 = \dfrac{U^2}{V/p} = \dfrac{\chi^2_1/1}{\chi^2_p/p} = F_{1,p} = X^2$

15. CB 5.20 a

$$\int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{v}} \cdot \frac{1}{\Gamma(v/2)\, 2^{v/2}}}_{\text{COEF}} \cdot x^{1/2} \cdot x^{\frac{v}{2}-1} \cdot e^{-\frac{t^2 x}{2v}} \cdot e^{-vx/2}\, dx = \text{COEF} \int_0^\infty \underbrace{x^{\frac{v+1}{2}-1}\, e^{-\frac{x}{2}\left(\frac{t^2}{v}+v\right)}\, dx}_{\text{GAMMA}\left(\frac{v+1}{2},\ \frac{2}{v+t^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{v}} \cdot \frac{1}{\Gamma\left(\frac{v}{2}\right) 2^{v/2}} \cdot \Gamma\left(\frac{v+1}{2}\right) \cdot \left(\frac{2}{v+t^2}\right)^{\frac{v+1}{2}} = \frac{1}{\sqrt{\pi v}} \cdot \cancel{\frac{2^{\frac{v+1}{2}}}{2^{\frac{v}{2}+\frac{1}{2}}}} \cdot \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma(v/2)} \cdot \frac{1}{(v+t^2)^{\frac{v+1}{2}}}$$

$$= \frac{1}{\sqrt{\pi v}} \cdot \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \cdot \frac{1}{(v+t^2)^{v+1/2}} \sim t_v$$