HW2

Thursday, September 18, 2025 9:03 AM

1. SETUP: PROB. DISEASE : $P(D) = .01$ $P(A|D^c) = .06$

PROB. NEG. A RESULT : $P(A^c)$ $P(A^c|D) = .1$ $P(B|D^c) = .08$

FOR B: $P(B^c|D) = .05$ $\therefore P(A|D) = .9$ $P(B|D) = .95$

a) $P(D|A,B) = \dfrac{P(A,B|D)P(D)}{P(A,B)} = \dfrac{P(A|D)P(B|D)P(D)}{P(A|D)P(B|D)P(D) + P(A|D^c)P(B|D^c)P(D^c)} = \dfrac{(.9)(.95)(.01)}{(.9)(.95)(.01)+(.06)(.08)(.99)}$

$= .643$

b) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A|D)P(B|D)P(D) + P(A|D^c)P(B|D^c)P(D^c)}{P(\ |D^c)P(D^c) + P(A|D)P(D)} = \dfrac{(.9)(.95)(.01) + (.06)(.08).99}{(.06)(.99) + (.9)(.01)}$

$= .194$

c) $P(A \cup B|D) = P(A|D) + P(B|D) - \underbrace{P(A \cap B|D)}_{-P(A|D)P(B|D)} = .9 + .95 - (.9)(.95) = .995$

2. CB 1.38 PROVE EACH

a) IF $P(B)=1$, SHOW $P(A|B) = P(A)$: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ - SINCE $P(x \in B) = 1$ ANY SUBSET ALSO $\therefore$ HAS $P(x \in B) = 1$ $= \dfrac{P(A)(1)}{1} = P(A)$

b) IF $A \subset B$, THEN $P(B|A) = 1$ AND $P(A|B) = P(A)/P(B)$

$A \subset B$ MEANS $x \in A$ IMPLIES $x \in B$ FOR ALL x. $\therefore$ $P(B|A)$ MEANS GIVEN AN $x \in A$, WHAT PROB. DOES $x \in B$? ALL

$\therefore P(B|A) = 1$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ - AS SHOWN BEFORE, THIS IS JUST $P(A)$ $= \dfrac{P(A)}{P(B)}$

c) A,B MUT. EXCL. THEN $P(A|A \cup B) = \dfrac{P(A)}{P(A)+P(B)}$

$P(A|A \cup B) = \dfrac{P(A \cap (A \cup B))}{P(A \cup B)} = \dfrac{P(A) \cup P(A \cap B)}{P(A) + P(B) - P(A \cap B)}$ BECAUSE MUT. EXCL, $P(A \cap B) = 0$ $\therefore = \dfrac{P(A)}{P(A)+P(B)}$

d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$ $P(A \cap B \cap C) = P(A|B \cap C) \cdot P(B \cap C) = P(A|B \cap C) \cdot P(B|C)P(C)$

$P(A|B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)}$ $P(B|C) = \dfrac{P(B \cap C)}{P(C)}$

3. CB 1.39

a) A,B MUT. EXCL., MEANS CANNOT BE IND.

MUT. EXCL. MEANS $P(A \cap B) = 0$ IND. MEANS $P(A \cap B) = P(A)P(B)$, BUT SINCE $P(A \cap B) = 0 \neq P(A)P(B)$, BUT $P(A) > 0$ & $P(B) > 0$

$\therefore$ CANNOT WORK

b) A,B IND., CAN'T BE MUT. EXCL.

A,B IND MEANS $P(A \cap B) = P(A)P(B) > 0$ THUS $P(A \cap B) > 0$ WHICH MEANS A,B NOT MUT. EXCL.

4. CB 1.40 A,B IND. SHOW:

b) A^c,B ALSO IND. : SHOW $P(A^c \cap B) = P(A^c)P(B)$. $P(A^c)P(B) = \left(P(A^c|B)P(B) + P(A^c|B^c)P(B^c)\right)P(B)$

$= \left[\dfrac{P(A^c \cap B)P(B)}{P(B)} + \dfrac{P(A^c \cap B^c)P(B^c)}{P(B^c)}\right]P(B) = (P(A^c \cap B) + P(A^c \cap B^c))P(B) = P(A^c \underbrace{\cap B)P(B)}_{\text{JUST } P(B)} + \underbrace{P(A^c \cap B^c)P(B)}_{0} = P(A^c \cap B)$

$[\ldots]P(B^c)$ … $P(A^c|B)P(B)]P(B^c)$

c) A^c, B^c ALSO IND.: SHOW $P(A^c \cap B^c) = P(A^c)P(B^c)$. $P(A^c)P(B^c) = [P(A^c \ldots$

$$= \left[\frac{P(A^c \cap B^c)P(B^c)}{P(B^c)} + \frac{P(A^c \cap B)P(B)}{P(B)}\right] P(B^c) = P(A^c \cap B^c)P(B^c) + P(A^c \cap B)P(B^c) = \underline{P(A^c \cap B^c)}$$

5. A, B, C MUT. IND., SHOW $A \cap B$, C IND. & $A \cup B$, C IND.

a) $A \cap B$, C IND MEANS $P(A \cap B \cap C) = P(A \cap B)P(C)$. $P(A \cap B \cap C) = \overbrace{P(A)P(B)P(C)}^{\text{MUT. IND.}} = \underline{P(A \cap B)P(C)}$

b) $A \cup B$, C IND MEANS $P((A \cup B) \cap C) = P(A \cup B)P(C)$. $P((A \cup B) \cap C) = P(A \cap C) \cup P(B \cap C) = P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$

$= P(A)P(C) + P(B)P(C) - P(A \cap B \cap C) = [P(A) + P(B) - P(A)P(B)]P(C) = [P(A) + P(B) - P(A \cap B)]P(C) = \underline{P(A \cup B)P(C)}$

6. CB 1.47 b, d, e CDF MEANS 1) $\lim_{x \to -\infty} = 0$, $\lim_{x \to \infty} = 1$ 2) FOR $x_1 \le x_2$, $F_X(x_1) \le F_X(x_2)$ 3. $\lim_{x \downarrow x_0} F_X(x) = F_X(x_0)$

b) $(1 + e^{-x})^{-1}$

i. $\lim_{x \to -\infty} \frac{1}{1+e^{-x}} \Rightarrow \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$ $\lim_{x \to \infty} \frac{1}{1+e^{-x}} \Rightarrow \frac{1}{1+e^{0}} = \frac{1}{1} = 1$

ii. BASICALLY, SHOW DERIVATIVE IS ALWAYS POSITIVE (SLOPE ALWAYS >0 MEANING INCREASING

$\frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2}$ e^{-x} - ALWAYS ≥ 0, $(1+e^{-x})^2$ - ALWAYS ≥ 0 $\therefore$ ALWAYS INCREASING

iii. FOLLOWING BOOK, $\lim_{x \downarrow x_0} F_X(x + \epsilon) = F_X(x)$ FOR $\epsilon > 0$ AND SMALL (CONTINUOUS)

d) $1 - e^{-x}$ $x \in (0, \infty)$

i. $\lim_{x \to -\infty} = \lim_{x \to 0} 1 - e^{-x} = 1 - e^{0} = 1 - 1 = 0$ $\lim_{x \to \infty} = 1 - e^{-\infty} = 1 - 0 = 1$

ii. $\frac{d}{dx}(1 - e^{-x}) = e^{-x} \ge 0$

iii. CONTINUOUS $\lim_{x \to \infty} \epsilon + \frac{1-\epsilon}{1+0} = \epsilon + 1 - \epsilon = 1$

e) $\frac{1-\epsilon}{1+e^{-y}}$ $y < 0$; $\epsilon + \frac{1-\epsilon}{1+e^{-y}}$ $y \ge 0$

i. $\lim_{x \to -\infty} \frac{1-\epsilon}{1+e^{\infty}} \Rightarrow 0$

ii $\frac{d}{dy}\left(\frac{1-\epsilon}{1+e^{-y}}\right) = \frac{(1-\epsilon)e^{-y}}{(1+e^{-y})^2} \ge 0$

iii. CONTINUOUS OVER EACH STEP, $\ge$ ON THE JUNCTION

7. CB 1.53 $F_Y(y) = 1 - \frac{1}{y^2}$ $1 \le y < \infty$

a) CDF: i. $\lim_{y \to -\infty} = 1 - \frac{1}{1^2} = 1 - 1 = 0$ $\lim_{y \to \infty} = 1 - \frac{1}{\infty^2} = 1 - 0 = 1$

ii. $\frac{d}{dy}(1 - y^{-2}) = 2y^{-3} \ge 0$

iii. CONTINUOUS

b) PDF $= \frac{d}{dy}$ CDF $= \frac{d}{dy}(1 - y^{-2}) = \underline{2y^{-3}}$

c) $Z = 10(y-1)$ $\frac{Z}{10} = y - 1$ $y = 1 + \frac{Z}{10}$ $\therefore$ $\underline{F_Z(z) = 1 - \frac{1}{(1+\frac{z}{10})^2} \text{ ON } 0 \le Z < \infty}$

8. PDF $= \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \therefore \end{cases}$ CDF $= \int$ PDF $= \begin{cases} \frac{1}{2}x^2 & 0 < x < 1 \\ 2x - \frac{1}{2}x^2 & 1 \le x < 2 \\ 0 & \therefore \end{cases}$

i. $\lim_{x \to -\infty} = \lim_{x \to 0} = \frac{1}{2}(0)^2 = 0$ $\lim_{x \to \infty} = \lim_{x \to 2} = 2(2) - \frac{1}{2}(2)^2 = 4 - 2 = 2 - (2(1) - \frac{1}{2}(1)^2) + \frac{1}{2}(1)^2 = 2 - 2 + \frac{1}{2} + \frac{1}{2} = 1$

ii. PDF: $x \ge 0$ ON $0 < x < 1$ $2 - x \ge 0$ ON $1 \le x < 2$

iii. CONTINUOUS ON EACH STEP, $1 \le x$ MEETS REQUIREMENTS

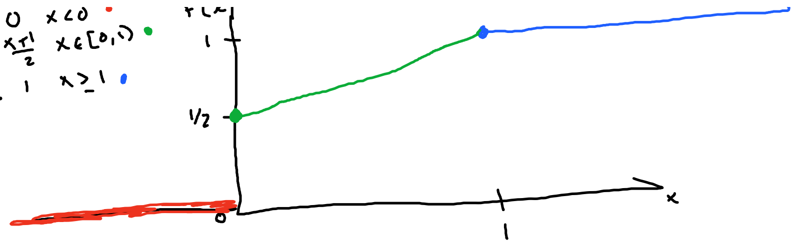
9. PDF $= \begin{cases} \frac{x}{4} & 0 < x \le 2 \\ e^{-2(x-2)} & x > 2 \\ 0 & \therefore \end{cases}$ CDF $= \begin{cases} \frac{1}{8}x^2 & 0 < x \le 2 \\ -\frac{1}{2}e^{-2(x-2)} & x > 2 \\ 0 & \therefore \end{cases}$

i. $\lim_{x \to -\infty} = \lim_{x \to 0} = \frac{1}{8}(0)^2 = 0$ $\lim_{x \to \infty} = -\frac{1}{2}e^{-2(\infty)} + \frac{1}{8}(2)^2 - (-\frac{1}{2}e^{-2(0)}) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\frac{1}{e^{\infty}} = \frac{1}{2} + \frac{1}{2} = 1$

ii. $x/4 \ge 0$ ON $0 < x \le 2$ $e^{-2(x-2)} \ge 0$

iii. CONTINUOUS ON EACH STEP

10. $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x+1}{2} & x \in [0,1) \\ 1 & x \geq 1 \end{cases}$



This is neither discrete nor continuous. It is locally continuous (for x >= 0) but the boundary down to x < 0 is discrete

11. i. $f_X(x) \geq 0$ ii $\sum f(x) = 1$, $\int f(x) = 1$

a) $f(x) = \frac{c}{2^x}$ $\quad \sum f(x) = \sum \frac{c}{2^x} = c \sum \frac{1}{2^x}$ Using internet, I see that the sum of 1/(2^x) is a geometric series which converges to 1

$= c(1) = 1 \quad \therefore \underline{c = 1}$

b) $f(x) = cx^2 \quad \int_0^1 cx^2\,dx = \frac{c}{3}x^3 \Big|_0^1 = \frac{c}{3} \Rightarrow \frac{c}{3} = 1 \quad \underline{c = 3}$

12. $f(x) = cx^2 \quad 0 \leq x < 1$

a) SHOWN IN 11.b : $\underline{c = 3}$

b) CDF: $\int 3x^2 = \underline{x^3}$

c) $P(.1 \leq x \leq .5) = \text{CDF}\Big|_{.1}^{.5} = (.5)^3 - (.1)^3 = \underline{.124}$

13. Exp. Func., $\lambda = 1 \quad f(x) = \lambda e^{-\lambda x} \quad x \geq 0 \Rightarrow f(x) = e^{-x} \quad F(x) = -e^{-x}$

a) $P(x < 10) = -e^{-x}\Big|_0^{10} = -e^{-10} - (-e^0) = -e^{-10} + 1 = \underline{1 - e^{-10} = .99995}$

b) $P(5 < x < 15) = -e^{-x}\Big|_5^{15} = -e^{-15} - (-e^{-5}) = \underline{e^{-5} - e^{-15} = 0.00674}$

c) $P(t < x) = .01 = -e^{-x}\Big|_t^{\infty} = e^{-t} - e^{-\infty} = e^{-t} = .01 \quad -t = \ln(.01) \quad \underline{t = -\ln(.01) = 4.605}$

14. $f(x) = \lambda e^{-\lambda x} \quad P(T < 1) = .05 \quad \therefore \; F(x) = -e^{-\lambda x} \quad P(T<1) = -e^{-\lambda x}\Big|_0^1 = e^{-\lambda(0)} - e^{-\lambda(1)} = e^0 - e^{-\lambda} = 1 - e^{-\lambda} = .05$

$e^{-\lambda} = .95 \quad -\lambda = \ln(.95) \quad \underline{\lambda = -\ln(.95) = 0.0513}$