HW6

Tuesday, October 21, 2025 8:22 AM

1. CB 4.15 $X \sim POIS(\theta)$ $Y \sim POIS(\lambda)$ $X+Y \sim POIS(\theta+\lambda)$ $f_{XY} = \frac{e^{-\theta}\theta^{x}}{x!} \cdot \frac{e^{-\lambda}\lambda^{y}}{y!}$

A. SHOW $X|X+Y = BIN(n, \frac{\theta}{\theta+\lambda})$ $A = X+Y$ $B = X$ $X|X+Y = \frac{f_{AB}}{f_B}$

$X = B$ $A = B+Y$ $Y = A-B$ $X = 0,1,2,\ldots$ $Y = 0,1,2,\ldots$ $\Rightarrow$ $B = 0,1,2,\ldots$ $A = B+0, B+1, B+2, \ldots$

$f_{AB} = f_{XY}(X,Y) = \sum \frac{e^{-\theta}\theta^{B}}{B!} \cdot \frac{e^{-\lambda}\lambda^{A-B}}{(A-B)!} = e^{-\theta}e^{-\lambda} \sum \frac{1}{B!(A-B)!}\theta^{B}\lambda^{A-B} = e^{-\theta}e^{-\lambda} \sum \frac{(\theta+\lambda)^{A}}{B!(A-B)!}\left(\frac{\theta}{\theta+\lambda}\right)^{B}\left(1-\frac{\theta}{\theta+\lambda}\right)^{A-B}$

$\left(\frac{\theta}{\theta+\lambda}\right)^{B} \cdot \left(1-\frac{\theta}{\theta+\lambda}\right)^{A-B} = \left(\frac{\theta}{\theta+\lambda}\right)^{B} \cdot \left(\frac{\lambda}{\theta+\lambda}\right)^{A-B} = \theta^{B}\lambda^{A-B}(\theta+\lambda)^{-B} \cdot (\theta+\lambda)^{B-A} = \theta^{B}\lambda^{A-B}(\theta+\lambda)^{-A}$

$e^{-\theta}e^{-\lambda} \sum_{B=0}^{n} \sum_{A=B+0}^{B+n} = e^{-\theta}e^{-\lambda}\binom{A}{B}\left(\frac{\theta}{\theta+\lambda}\right)^{B}\left(1-\frac{\theta}{\theta+\lambda}\right)^{A-B} = BINOM\left(A, \frac{\theta}{\theta+\lambda}\right)$

B. WHAT IS $Y|X+Y$?

$Y|X+Y$: JUST SWITCH PARAMETERS : $BINOM\left(B, \frac{\lambda}{\lambda+\theta}\right)$

2. CB 4.17 $X \sim EXP(1)$ $Y = X+1$, $Y = i+1$ IFF $i \le X < i+1$, $i = 0,1,2\ldots$

a. $P(Y=K) = P(X+1=K) = P(X=K-1)$ (INTEGER) $X_{INT} = K-1$ ANY TIME $K-1 \le X < K$ $\therefore P(X_{INT}=K-1) = P(K-1 \le X < K)$

$P(K-1 \le X < K) = P(X<K) - P(X \le K-1) = F_X(K) - F_X(K-1) = e^{-K} - e^{-(K-1)} = e^{-K} - e^{-K}e = e^{-K}(1-e)$ $= e^{-K} - e^{-(K+1)}$

$\left(1-\frac{1}{e}\right)\left(1-\left[1-\frac{1}{e}\right]\right)^{K-1} = \left(1-\frac{1}{e}\right)\left(\frac{1}{e}\right)^{K-1} = \left(1-\frac{1}{e}\right)e^{-(K-1)} = e^{(K-1)} - e^{-K}$

$Y \sim GEOM\left(P = 1-\frac{1}{e}\right)$

b. FIND $X-4|Y \ge 5$ $A = Y$ $B = X-4$ $X = B+4$ $Y = A$ $f_{XY} = e^{-X} \cdot e^{-(X+1)} \cdot I_{X=0,1,2,\ldots}$

$Y \ge 5 = A \ge 5$

$Y = X+1 \Rightarrow X \ge 5 \Rightarrow B+4 \ge 5$ $B \ge 1$

$f_{AB} = e^{-(B+4)}e^{-(B+4+1)} = e^{-(B+4+B+4+1)} = e^{-(2B+9)} = e^{-9}e^{-2B} = \frac{2e^{-9}}{2}e^{-2B} = \frac{e^{-9}}{2} \cdot \underbrace{2e^{-2B}}_{EXP\left(\frac{1}{2}\right)}$

$X-4|Y \ge 5 = \frac{e^{-9}}{2} \cdot EXP\left(\frac{1}{2}\right)$

3. CB 4.19 X_1, X_2 IND. $\sim N(0,1)$

a. FIND $\frac{(X_1-X_2)^2}{2}$ USING 4.3.4 WE KNOW $X_1 - X_2 \sim N(0,2)$ $\therefore$ LET $U = X_1 - X_2 = N(0,2)$

$= \frac{U^2}{2}$ LET $Y = \frac{U^2}{2}$ USING 2.19 WE KNOW $f_Y = \frac{1}{\sqrt{2\pi}\,2}e^{-(-\sqrt{2y})^2/8}\left|-\frac{\sqrt{2}}{2\sqrt{y}}\right| + \frac{1}{\sqrt{2\pi}\,2}e^{-(-\sqrt{2y})^2/8}\left|\frac{\sqrt{2}}{2\sqrt{y}}\right|$

$\sqrt{2}y^{1/2} = U$ $U' = -\frac{\sqrt{2}}{2}y^{-1/2}$ $= \frac{1}{\sqrt{2\pi}\,2}\frac{\sqrt{2}}{\sqrt{y}}e^{-2y/8} = \frac{1}{\sqrt{2\pi}}\frac{1}{2^{1/2}}\frac{1}{y^{1/2}}e^{-y/4}$

b. X_1, X_2 IND. $\sim GAMMA(\alpha_i, 1)$ FIND MARG. DISTS. OF $\frac{X_1}{X_1+X_2}$ AND X_1+X_2

$A = \frac{X_1}{X_1+X_2}$ $B = X_1+X_2$ $X_1 = A(X_1+X_2)$ $X_1 - AX_1 = AX_2$ $X_2 = X_1\frac{1-A}{A}$ $X_1 = B - X_2$ $X_2 = (B-X_2)\frac{1-A}{A}$

$X_2 + X_2\left(\frac{1-A}{A}\right) = B\left(\frac{1-A}{A}\right)$ $X_2\left(1+\frac{1-A}{A}\right) = B\left(\frac{1-A}{A}\right) = X_2\left(\frac{A+1-A}{A}\right) = B\left(\frac{1-A}{A}\right) = X_2\frac{1}{A} \Rightarrow X_2 = B(1-A) = B - BA$

$X_1 = B - X_2 = B - B(1-A) = B(1-1+A) = AB$ $X_1 > 0$ $X_2 > 0$ $AB > 0$ $B(1-A) > 0$ FOR $B>0$: $A>0$, $A<1$

FOR $B<0$: $A<0$, NOT POSSIBLE

$X_1 = AB \quad X_2 = B(1-A)$

$f_{X_1} = \frac{1}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-x_1} \quad f_{X_2} = \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-x_2} \quad f_{X_1,X_2} = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-(x_1+x_2)} \quad \therefore B>0,\ 0<A<1$

$\frac{\partial x_1}{\partial A} = B \quad \frac{\partial x_1}{\partial B} = A \quad \frac{\partial x_2}{\partial A} = -B \quad \frac{\partial x_2}{\partial B} = 1-A \quad J = |B(1-A) - (A)(-B)| = |B - AB + AB| = B$

$f_{AB} = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (AB)^{\alpha_1-1} (B(1-A))^{\alpha_2-1} e^{-B} \cdot B = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} A^{\alpha_1-1}(1-A)^{\alpha_2-1} B^{\alpha_1-1} B^{\alpha_2-1} B e^{-B} = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} A^{\alpha_1-1} (1-A)^{\alpha_2-1} B^{(\alpha_1+\alpha_2)-1} e^{-B}$

$$f_B = \frac{B^{(\alpha_1+\alpha_2-1)} e^{-B}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{\int_0^1 A^{\alpha_1-1}(1-A)^{\alpha_2-1}\, dA}_{\text{BETA } \int = 1} \Rightarrow \frac{\beta(\alpha_1, \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B^{(\alpha_1+\alpha_2)-1} e^{-B}$$

$$f_A = \frac{A^{\alpha_1-1} A^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{\int_0^\infty B^{(\alpha_1+\alpha_2)-1} e^{-B}\, dB}_{\text{GAMMA}: \ \alpha = \alpha_1+\alpha_2 \ \ \beta = 1} \Rightarrow \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} A^{\alpha_1-1}(1-A)^{\alpha_2-1}$$

4. CB 4.20 $X_1, X_2 \sim N(0, \sigma^2) \quad Y_1 = x_1^2 + x_2^2 \quad Y_2 = \frac{X_1}{\sqrt{Y_1}} \Rightarrow X_1 = Y_2 Y_1^{1/2}$

$Y_1 = Y_2^2 Y_1 + X_2^2$

$X_2 = [Y_1(1-Y_2^2)]^{1/2}$

$Y_2 = X_1(x_1^2 + x_2^2)^{-1/2}$

a. FIND $f_{Y_1 Y_2}$ $\quad \frac{\partial x_1}{\partial y_1} = \frac{1}{2} Y_2 Y_1^{-1/2} \quad \frac{\partial x_1}{\partial y_2} = Y_1^{1/2} \quad \frac{\partial x_2}{\partial y_1} = \frac{1}{2}[Y_1(1-Y_2^2)]^{-1/2} \quad \frac{\partial x_2}{\partial y_2} = -Y_2[Y_1(1-Y_2^2)]^{-1/2}$

$|J| = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_2}{\partial y_1}\frac{\partial x_1}{\partial y_2} = \left| -\frac{Y_2^2}{2Y_1}(1-Y_2^2)^{-1/2} - \frac{1}{2}(1-Y_2^2)^{-1/2} \right|$

$= \frac{1}{2}(1-Y_2^2)^{-1/2}$

$f_{Y_1 Y_2} = f_{X_1 X_2}(y_1, y_2)|J| = \frac{1}{2\pi\sigma^2} e^{-\frac{x_1^2}{2\sigma^2} - \frac{x_2^2}{2\sigma^2}} = \frac{1}{\pi\sqrt{1-Y_2^2}\,\sigma^2} \underbrace{e^{-\frac{Y_2^2 Y_1}{2\sigma^2} - \frac{Y_1(1-Y_2^2)}{2\sigma^2}}}$ $\quad -\frac{1}{2\sigma^2}[Y_1(Y_2^2 - Y_2^2 + 1)] = -\frac{1}{2\sigma^2}(Y_1)$

$$f_{Y_1 Y_2} = \frac{1}{2\pi\sqrt{1-y_2^2}\,\sigma^2} \cdot e^{-\frac{y_1}{2\sigma^2}}$$

b. Y_1 & Y_2 ARE IND. BECAUSE $f_{Y_1 Y_2} = g(Y_1) \cdot g(Y_2)$ $\quad g(Y_1) = e^{-\frac{y_1}{2\sigma^2}} \quad g(Y_2) = \frac{1}{2\pi\sqrt{1-y_2^2}\,\sigma^2}$

5. CB 4.30 b $\quad Y|X=x \sim N(x, x^2) \quad f_x = \text{UN}(0,1) \quad f_{xy} = \text{UNIF}(0,1) \cdot N(x, x^2)$

b. PROVE Y/X AND X ARE IND. $\quad U = Y/X \quad V = X \quad X = V \quad Y = UX = UV$

$\frac{\partial x}{\partial u} = -YX^{-2} \quad \frac{\partial y}{\partial u} = x^{-1} \quad \frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial v} = 0 \quad = Yx^{-1} \quad |\ | = -Yx^{-2} \cdot 0 - x^{-1} \cdot 1 = -x^{-1}$

$\cdot \frac{dy}{dy} x^{-1}$

$f_{UV} = \text{UNIF}(0,1) \cdot N(V, V^2)$ WHERE $V = X$ IT CAN BE WRITTEN AS $f_{UV} = g(U) \cdot g(V)$ WHERE $g(U) = \text{UNIF}(0,1)$, $g(V) = N(V, V^2)$

$\therefore U \perp\!\!\!\perp V \ \therefore \ \frac{Y}{X} \perp\!\!\!\perp X$

6. CB 4.45 $\quad (X,Y) \sim \text{BIV. NORM.}(\mu_x, \mu_y, x, \sigma_y^2, \rho) \quad f_{xy} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right]$

a. SHOW $f_x = N(\mu_x, \sigma_x^2)$, $f_y = N(\mu_y, \sigma_y^2)$

$$f_X = \int_{-\infty}^{\infty} f_{xy}\,dy \Rightarrow \quad z = \frac{y-\mu_y}{\sigma_y} \quad dy = \sigma_y dz \quad w = \frac{x-\mu_x}{\sigma_x} \Rightarrow \underbrace{\frac{e^{-\frac{w^2}{2(1-\rho^2)}}}{2\pi\sigma_x\sqrt{1-\rho^2}}}_{C} \int_{-\infty}^{\infty} \text{Exp}\left[\frac{-1}{2(1-\rho^2)}\left[(z^2 - 2\rho w z + \rho^2 w^2) - \rho^2 w^2\right]\right] dz$$

Complete the square

$$= C e^{\frac{\rho^2 w^2}{2(1-\rho^2)}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{(z-\rho w)^2}{2(1-\rho^2)}} dz}_{n(\rho w,\, 1-\rho^2)} = \frac{e^{-w^2/2}}{2\pi\sigma_x\sqrt{1-\rho^2}} \cdot \sqrt{2\pi}\sqrt{1-\rho^2} = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)} = N(\mu_x, \sigma_x)$$

b. Show $Y|X = N\left(\mu_y + \rho\left(\frac{\sigma_y}{\sigma_x}\right)(x-\mu_x),\ \sigma_y^2(1-\rho)^2\right)$

$$Y|X = \frac{f_{xy}}{f_x} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \text{Exp}\left[\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right] \cdot \sqrt{2\pi}\,\sigma_x\, \text{Exp}\left[\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \text{Exp}\left[\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - (1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right) - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \text{Exp}\left[\frac{-1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \text{Exp}\left[\frac{-1}{2(1-\rho^2)}\left(\frac{y-\mu_y}{\sigma_y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \text{Exp}\left[-\frac{1}{2(1-\rho^2)\sigma_y^2}\left(y - \left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)\right)^2\right]$$

$$= N\left(\mu_y + \rho\left(\frac{\sigma_y}{\sigma_x}\right)(x-\mu_x),\ \sigma_y^2(1-\rho^2)\right)$$

c. Show $aX + bY = N\left(a\mu_x + b\mu_y,\ a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y\right)$

Solution shows it requires Covariance. Dr. Page said skip everything that needed covariance. Thus, this solution is left to the grader haha
Also did confirm in class with Dr. Page that we shouldn't do this one

7. CB. 4.52 $\quad B_1 = (X_1, Y_1) \quad B_2 = (X_2, Y_2) \quad D = \left[(Y_2 - Y_1)^2 + (X_2 - X_1)^2\right]^{1/2} \quad X_1, X_2, Y_1, Y_2 \overset{\text{IND.}}{\sim} N(0,1)$

$\underbrace{Y_2 - Y_1}_{A} \Rightarrow N(0, \sqrt{2}) \quad \underbrace{X_2 - X_1}_{B} \Rightarrow N(0, \sqrt{2})$ (Sum of Normals)

$D = [A^2 + B^2]^{1/2} \quad A^2 \Rightarrow \underbrace{\sqrt{2}\chi_1^2}_{C} \quad B^2 \Rightarrow \underbrace{\sqrt{2}\chi_1^2}_{E}$ (Square of Normals)

$D = (C+E)^{1/2} \quad C + E \Rightarrow \underbrace{\sqrt{2}\chi_2^2}_{F}$ (Sum of Chi-Squared)

$D = F^{1/2} = 2^{1/4} F^{1/2} = 2^{1/4}(\chi_2^2)^{1/2} = 2^{1/4}\underbrace{\chi_2}_{\text{Rayleigh Dist. w/ } \sigma=1}$

$D = 2^{1/4}\chi_2 \leftarrow$ Rayleigh, $\sigma = 1$

8. CB. 4.53 $\quad A, B, C \overset{\text{IND.}}{\sim} \text{Unif}(0,1) \quad$ What is Prob. $Ax^2 + Bx + C$ has real roots?

Need $B^2 - 4AC$ to be ≥ 0, and need $A \neq 0$.

$B^2 \geq 4AC \qquad \underbrace{-2\ln(B)}_{D} \geq -\ln(4) - \underbrace{\ln(A)}_{E} - \underbrace{\ln(C)}_{F} = -\ln(4) + (\underbrace{-\ln(A)}_{E} + \underbrace{-\ln(C)}_{F})$

As stated in problem, D & E are ... Exponential(1). $E + F = \text{Gamma}(2,1)$

... STATED IN THEOREM ... ARE NOW ...

$\therefore$ $\text{Exp}(2) \geq -\ln(4) + \text{GAMMA}(2,1) \Rightarrow -\ln(4) \leq \underset{Y}{\text{GAMMA}(2,1)} - \underset{X}{\text{Exp}(2)} \Rightarrow X \leq Y - \ln(4)$

$$P(X < Y - \ln(4)) = \int_{\ln(4)}^{\infty} P(X \leq y - \ln 4)\, f_Y\, dy = \int_{\ln 4}^{\infty}\int_0^{y-\ln 4} \frac{1}{2} e^{-x/2} \cdot \frac{1}{\underset{1}{\Gamma(2)} 1^2}\, y^{2-1} e^{-y/1}\, dx\, dy = \int_{\ln 4}^{\infty}\int_0^{y-\ln 4} \frac{1}{2} e^{-x/2}\, y e^{-y}\, dx\, dy$$

$$= \int_{\ln 4}^{\infty} \frac{1}{2} y e^{-y} dy \left(-e^{-x/2}\right)\Big|_0^{y-\ln 4} = \int_{\ln 4}^{\infty} \frac{1}{2} y e^{-y} dy \left(1 - e^{-y/2} e^{\ln 4/2}\right) = \frac{1}{2}\int_{\ln 4}^{\infty} y e^{-y} - e^{\frac{\ln 4}{2}}\, y e^{-3/2 y}\, dy$$

$$= \frac{1}{4}(1 + \ln 4) - \frac{1}{24}\left(\frac{2}{3} + \ln 4\right) = \underline{.511}$$

9. $X, Y \overset{IND.}{\sim} \text{GEOM}(P)$ $\quad f_{XY} = P(1-P)^{x-1} \cdot P(1-P)^{y-1}$ $\quad \frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = 0 \quad \frac{\partial y}{\partial u} = -1 \quad \frac{\partial y}{\partial v} = 1$

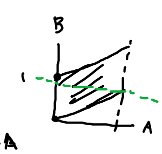
a. FIND f_{UV}, $U = X$, $V = X + Y$ $\quad \therefore X = U \quad Y = V - X = V - U$ $\quad J = |1(1) - 0| = |1| = 1$

$$f_{UV} = f_{XY}(U, V-U) \cdot 1 = P(1-P)^{U-1} \cdot P(1-P)^{(U-V)-1} = P^2(1-P)^{U-1+U-V-1}$$

$$\underline{f_{UV} = P^2(1-P)^{2U-V-2}} \qquad \begin{matrix} U = 1, 2, 3, \ldots \\ V = U+1, U+2, U+3, \ldots \end{matrix}$$

b. FIND f_V USING TRANSFORMATION

$$f_V = \int_0^n P^2(1-P)^{2U-V-2}\, dU = \left(\frac{P}{1-P}\right)^2 \cdot \frac{(1-P)^{2U-V}}{2\ln(1-P)}\Bigg|_0^n = \underline{\left(\frac{P}{1-P}\right)^2 \cdot \frac{1}{2\ln(1-P)} \cdot \left[(1-P)^{2n-V} - (1-P)^{-V}\right]}$$



10. $f_{XY} = x + y$ $\quad \begin{matrix} 0 < x < 1 \\ 0 < y < 1 \end{matrix}$ $\quad A = 2X \quad B = X + Y$ $\quad$ FIND f_{AB}, f_A, f_B $\quad \Rightarrow \begin{matrix} 0 < \frac{1}{2}A < 1 & 0 < A < 2 \\ 0 < B - \frac{1}{2}A < 1 & \frac{1}{2}A < B < 1 + \frac{1}{2}A \end{matrix}$

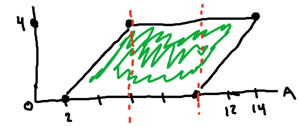
$X = \frac{1}{2}A \quad B = \frac{1}{2}A + Y \quad Y = B - \frac{1}{2}A$ $\quad \frac{\partial x}{\partial A} = \frac{1}{2} \quad \frac{\partial x}{\partial B} = 0 \quad \frac{\partial y}{\partial A} = -\frac{1}{2} \quad \frac{\partial y}{\partial B} = 1$ $\quad J = \left|\frac{1}{2}(1) - 0\left(-\frac{1}{2}\right)\right| = \frac{1}{2}$

$$f_{AB} = f_{XY}(X, Y) \cdot J = f_{XY}\left(\tfrac{1}{2}A, B - \tfrac{1}{2}A\right) \cdot \tfrac{1}{2} = \tfrac{1}{2}\left(\tfrac{1}{2}A + B - \tfrac{1}{2}A\right) = \tfrac{1}{2}B \qquad \underline{f_{AB} = \tfrac{1}{2}B} \quad \begin{matrix} 0 < A < 2 \\ \frac{1}{2}A < B < 1 + \frac{1}{2}A \end{matrix}$$

$$f_A = \int_{1/2 A}^{1 + 1/2 A} \frac{1}{2}B\, dB = \frac{1}{4}B^2\Big|_{\frac{1}{2}A}^{1+1/2 A} = \frac{1}{4}\left[\left(1 + \tfrac{1}{2}A\right)^2 - \tfrac{1}{4}A^2\right] = \frac{1}{4}\left(1 + A + \tfrac{1}{4}A^2 - \tfrac{1}{4}A^2\right) = \underline{\frac{1}{4} + \frac{1}{4}A = f_A \quad 0 < A < 2}$$

$$f_B = \begin{cases} \int_0^{2B} \frac{1}{2}B\, dA & 0 < B < 1 \\ \int_{2B-2}^{2} \frac{1}{2}B\, dA & 1 < B < 2 \end{cases} = \begin{cases} \frac{1}{2}BA\Big|_0^{2B} \\ \frac{1}{2}BA\Big|_{2B-2}^{2} \end{cases} = \begin{cases} B^2 \\ B - B^2 + B \end{cases} = \underline{f_B = \begin{cases} B^2 & 0 < B < 1 \\ 2B - B^2 & 1 < B < 2 \end{cases}}$$

11. $f_{XY} = \frac{1}{96}xy$ $\quad 0<x<4$, $1<y<5$ $\quad A = X+2Y$ $\quad$ FIND f_A $\quad$ LET $B = X$ $\quad X=B$ $\quad A=B+2Y$ $\quad Y=\frac{1}{2}(A-B)$



$\frac{\partial x}{\partial A}=0 \quad \frac{\partial x}{\partial B}=1 \quad \frac{\partial y}{\partial A}=\frac{1}{2} \quad \frac{\partial y}{\partial B}=-\frac{1}{2} \qquad J = \left|0(-\tfrac{1}{2})-(1)(\tfrac{1}{2})\right| = \left|-\tfrac{1}{2}\right| = \frac{1}{2}$

$0<B<4$

$1<\frac{1}{2}(A-B)<5 \Rightarrow 2<A-B<10$

$2+B<A<10+B$

$$f_{AB} = f_{XY}\left(B,\tfrac{1}{2}(A-B)\right)\cdot\frac{1}{2} = \frac{1}{384}(AB-B^2) \quad \begin{matrix}0<B<4\\ 2+B<A<10+B\end{matrix}$$

$\frac{1}{2}A-\frac{1}{3}A+\frac{2}{3} = \frac{1}{6}A+\frac{2}{3}$

$$f_A = \frac{1}{384}\begin{cases}\int_0^{A-2} AB-B^2\,dB & 2<A<6\\ \int_0^4 AB-B^2\,dB & 6<A<10\\ \int_{A-10}^4 AB-B^2\,dB & 10<A<14\end{cases} = \frac{1}{384}\begin{cases}\frac{1}{2}AB^2-\frac{1}{3}B^3\Big|_0^{A-2}\\ \frac{1}{2}AB^2-\frac{1}{3}B^3\Big|_0^4\\ \frac{1}{2}B^2A-\frac{1}{3}B^3\Big|_{A-10}^4\end{cases} = \frac{1}{384}\begin{cases}(A-2)^2\left[\frac{1}{2}A-\frac{1}{3}(A-2)\right]\\ 8A-\frac{64}{3}\\ 8A-\frac{64}{3}-(A-10)^2\left(\frac{1}{2}A-\frac{1}{3}(A-10)\right)\end{cases}$$

$\Rightarrow \frac{1}{6}A+\frac{10}{3}$

$$f_A = \frac{1}{384}\begin{cases}(A-2)^2\left(\frac{1}{6}A+\frac{2}{3}\right) & 2<A<6\\ 8A-\frac{64}{3} & 6\le A<10\\ 8A-\frac{64}{3}-(A-10)^2\left(\frac{1}{6}A+\frac{2}{3}\right) & 10<A<14\end{cases}$$

---

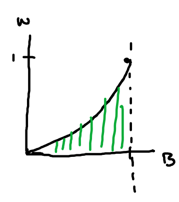
12. $f_X = 2x \quad 0\le x\le 1 \quad Y\sim \text{UNIF}(0,1) \Rightarrow f_Y = 1 \quad 0<y<1 \qquad f_{XY}=2x \qquad W = X^2Y$ $\quad$ LET $B=X$ $\quad X=B \quad W=B^2Y \quad Y=\frac{W}{B^2}$

$\frac{\partial x}{\partial w}=0 \quad \frac{\partial x}{\partial B}=1 \quad \frac{\partial y}{\partial w}=\frac{1}{B^2} \quad \frac{\partial y}{\partial B}=-\frac{1}{3}\frac{W}{B^3} \qquad J=\left|0\left(-\tfrac{1}{3}\tfrac{w}{B^3}\right)-(1)\tfrac{1}{B}\right| = \frac{1}{B^2}$

$0<B<1$

$0<\frac{w}{B^2}<1 \qquad 0<w<B^2$

$$f_{WB} = f_{XY}\left(B,\tfrac{w}{B^2}\right)\cdot\frac{1}{B^2} = \frac{2B}{B^2} = \frac{2}{B}$$



$$f_W = \int_{\sqrt{w}}^1 \frac{2}{B}\,dB = 2\ln(x)\Big|_{\sqrt{w}}^1 = -2\ln(\sqrt{w}) = -\ln(\sqrt{w}^2) = -\ln(w)$$

$$f_W = -\ln(w) \quad 0<w<1$$