HW4

Thursday, October 2, 2025 7:58 AM

1. CB 2.1 FIND PDF $f_Y = f_X(y^{-1}) \frac{dx}{dy}$

a. $Y = X^3$ $f_X = 42x^5(1-x)$ $x = y^{1/3}$ $\frac{dx}{dy} = \frac{1}{3y^{2/3}}$ $f_Y = 42(y^{1/3})^5(1-y^{1/3}) \cdot \frac{1}{3y^{2/3}}$

$f_Y = 14y(1-y^{1/3})$ $0<y<1$

b. $Y = 4x+3$ $f_X = 7e^{-7x}$ $0<x<\infty$ $x = \frac{y-3}{4}$ $\frac{dx}{dy} = \frac{1}{4}$ $f_Y = 7e^{-\frac{7(y-3)}{4}} \cdot \frac{1}{4}$

$f_Y = \frac{7}{4}e^{\frac{-7(y-3)}{4}}$ $3<y<\infty$

c. $Y = x^2$ $f_X = 30x^2(1-x)^2$ $0<x<1$ $x = \sqrt{y}$ $\frac{dx}{dy} = \frac{1}{2}y^{-1/2}$ $f_Y = 30\left(\frac{1}{2}y^{-1/2}\right)^2\left(1-\frac{1}{2}y^{-1/2}\right)^2 \cdot \frac{1}{2}y^{-1/2}$

$f_Y = \frac{15}{4}y^{-3/2}\left(1-\frac{1}{2}y^{-1/2}\right)^2$ $0<y<1$

2. CB 2.2 FIND PDF

a. $Y = x^2$ $f_X = 1$ $0<x<1$ $x = y^{1/2}$ $\frac{dx}{dy} = \frac{1}{2}y^{-1/2}$ $f_Y = 1 \cdot \frac{1}{2}y^{-1/2}$ $f_Y = \frac{1}{2}y^{-1/2}$ $0<y<1$

b. $Y = -\log x$ $f_X = \frac{(n+m+1)!}{n!m!}x^n(1-x)^m$ $x = e^{-y}$ $\frac{dx}{dy} = -e^{-y}$ $f_Y = \frac{(n+m+1)!}{n!m!}(e^{-y})^{n+1}(1-e^{-y})^m$ $-\infty<y<0$

c. $Y = e^X$ $0<x<\infty$ $f_X = \frac{1}{\sigma^2}xe^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$ $x = \ln(y)$ $\frac{dx}{dy} = \frac{1}{y}$ $f_Y = \frac{1}{\sigma^2}(\ln y)e^{-\frac{1}{2\sigma^2}(\ln y)^2} \cdot \frac{1}{y}$

$f_Y = \frac{1}{y\sigma^2}(\ln(y))e^{-\frac{1}{2}\left[\frac{\ln(y)}{\sigma}\right]^2}$ $1<y<\infty$

3. CB 2.3

x	0	1	2	3	4
y	0	1/2	2/3	3/4	4/5

$Y = \frac{X}{X+1}$ $f_X = \frac{1}{3}\left(\frac{2}{3}\right)^x$ $x = 0,1,2,\ldots$ $xy+y=x$ $y = x-xy = x(1-y)$ $x = \frac{y}{1-y}$

$f_Y = \frac{1}{3}\left(\frac{2}{3}\right)^{\left(\frac{y}{1-y}\right)}$ $y = 0, \frac{1}{2}, 2/3, 3/4, \ldots$

4. CB 2.6 FIND PDF OF Y

a. $Y = |X|^3$ $f_X = \frac{1}{2}e^{-|x|}$ $|x| = y^{1/3}$ Y NOW BOUNDED BY $0,\infty$ $x = y^{1/3}$ $\frac{dx}{dy} = \frac{1}{3}y^{-2/3}$ $f_Y = \frac{1}{2}e^{-|y^{1/3}|} \cdot \frac{1}{3}y^{-2/3}$

$f_Y = \frac{1}{6}y^{-2/3}e^{-y^{1/3}}$ $0<y<\infty$

b. $Y = 1-x^2$ $f_X = \frac{3}{8}(x+1)^2$ $-1<x<1$ $x = (1-y)^{1/2}$ $\frac{dx}{dy} = -\frac{1}{2}(1-y)^{-1/2}$ $1-(-1)^2 = 0$ $1-(1)^2 = 0$ $1-(0)^2 = 1$ $0<y<1$

$f_Y = \frac{3}{8}\left((1-y)^{1/2}+1\right)^2 \cdot \left|-\frac{1}{2}(1-y)^{-1/2}\right|$ $f_Y = \frac{3}{16(1-y)^{1/2}}\left[(1-y)^{1/2}+1\right]^2$ $0<y<1$

c. ↑ SAME BUT $X \le 0$, AND FOR $X>0$, $Y = 1-X$, $X = 1-Y$ $\frac{dx}{dy} = -1$ $f_Y = \frac{3}{8}\left[(1-y)+1\right]^2 = \frac{3}{8}\left[2-y\right]^2$

$X \le 0: \; Y \in [0,1] \quad f_Y = \frac{}{16(1-y)^{1/2}} \left[(1-y)^{\cdot} + 1 \right]$

$X > 0: \; Y \in [0,1) \quad f_Y = -\frac{3}{8}\left[2 - y\right]^2$

5. CB 2.7a

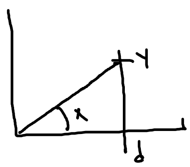
a. $Y = X^2 \quad f_X = \frac{2}{9}(x+1) \quad -1 \le x \le 2$ BREAK INTO $-1 \le x < 0 \Rightarrow 0 \le y \le 1$, $0 \le x \le 2 = 0 \le y \le 4$ $\quad x = \pm y^{1/2} \quad \frac{dx}{dy} = \pm\frac{1}{2} y^{-1/2}$

$F_Y = P(X^2 \le Y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X \, dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+1)\, dx = \frac{2}{9}\left[\frac{1}{2}x^2 + x\right]\Big|_{-\sqrt{y}}^{\sqrt{y}}$

$= \frac{2}{9}\left[\frac{1}{2}y + \sqrt{y} - \left[\frac{1}{2}y - \sqrt{y}\right]\right] = \frac{2}{9}\left[2\sqrt{y}\right] = \frac{4}{9}y^{1/2}$

$f_Y = F_Y' = \frac{2}{9}y^{-1/2} \quad 0 < y \le 1, \quad \frac{1}{2\sqrt{y}} \cdot \frac{2}{9}(\sqrt{y}+1) \quad 1 < y \le 4$

6. CB 2.12



$\tan(x) = \frac{y}{d} \quad y = d\tan(x) \quad x = \tan^{-1}\left(\frac{y}{d}\right) \quad \frac{dy}{dx} = \frac{1}{d}\frac{1}{y^2+1}$

$0 < x < \pi/2 \quad f_X = \frac{2}{\pi} \quad f_Y = \frac{2}{\pi} \cdot \frac{1}{d} \cdot \frac{1}{y^2+1}$

$f_Y = \frac{2}{\pi d(y^2+1)} \quad 0 < y < d$

LET $z = y^2+1 \quad dz = 2y\,dy$

$E(Y) = \int_0^d \frac{2y}{\pi d(y^2+1)} dy = \frac{2}{\pi d}\int_0^d \frac{y}{y^2+1} dy \quad \therefore \quad = \frac{2}{\pi d}\int_{0_y}^{d_y} \frac{1}{2z} dz = \frac{1}{\pi d}\left[\ln(z)\right]_{0_y}^{d_y} = \frac{1}{\pi d}\left[\ln(y^2+1)\right]_0^d$

$= \frac{1}{\pi d}\left[\ln(d^2+1) - \ln(1)\right]$

$E(Y) = \frac{\ln(d^2+1)}{\pi d}$

7. CB 2.23 $\quad Y = X^2 \quad f_X = \frac{1}{2}(1+x) \quad -1 \le x \le 1 \quad -1 \le x \le 0: Y \in [0,1] \quad 0 \le x \le 1: Y \in [0,1]$

a. $x = y^{1/2} \quad \frac{dx}{dy} = \frac{1}{2}y^{-1/2} \quad F_Y = P(X^2 \le Y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} + \frac{1}{2}x\,dx = \frac{1}{2}x + \frac{1}{4}x^2\Big|_{-\sqrt{y}}^{\sqrt{y}}$

$= \frac{1}{2}\sqrt{y} + \frac{1}{4}y - \left[\frac{1}{2}(-\sqrt{y}) + \frac{1}{4}y\right] = \sqrt{y} \quad f_Y = \frac{1}{2}y^{-1/2}$

$f_Y = \frac{1}{2}y^{-1/2} \quad 0 \le y \le 1$

b. $E(Y) = \int_0^1 y^{3/2}\,dy = \frac{2}{5}y^{5/2}\Big|_0^1 = \frac{2}{5} \qquad E(Y^2) = \int_0^1 y^{5/2}\,dy = \frac{2}{7}y^{7/2}\Big|_0^1 = \frac{2}{7}$

$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{2}{7} - \frac{4}{25} = \frac{50-28}{175} = \frac{22}{175}$

8. $Y = X(2-X) \quad f_X = \frac{x}{2} \quad 0 \le x \le 2$ SPLIT AT $x=1$: $0 \le x \le 1: Y \in [0,1] \quad 1 \le x \le 2: Y \in [0,1]$

$-x^2 + 2x = y \quad -x^2 + 2x - y = 0 \quad x^2 - 2x + y = 0 \quad (x - \sqrt{y})^2 = 0 \quad x = \sqrt{y}$ RANGES OVERLAP ON $[0, \sqrt{y}]$

$F_Y = \int_0^{\sqrt{y}} \frac{x}{2}\,dx = \frac{1}{4}x^2\Big|_0^{\sqrt{y}} = \frac{1}{4}y \quad f_Y = \frac{1}{4} \quad 0 \le y \le 1$

$(4-y)^{1/3} \quad \left|\frac{dx}{dy}\right| = \frac{1}{3}(4-y)^{-2/3}$

9. $Y = 4 - X^3 \quad f_X = \frac{x}{2} \quad 0 \le x \le 2 \qquad x^3 = 4 - y \qquad x = (4-y)^{1/3} \qquad \left|\frac{dx}{dy}\right|$

$$f_Y = \frac{1}{2}(4-y)^{1/3} \cdot \left(\frac{1}{3}(4-y)^{-2/3}\right) = \frac{1}{6}\frac{(4-y)^{1/3}}{(4-y)^{2/3}} \qquad f_Y = \frac{1}{6}(4-y)^{-1/3} \quad -4 \le y \le 4$$

10. $Y = |X| \quad f_X = \frac{1}{2n+1} \quad x = -n, -n+1, \dots, 0, \dots, n-1, n \qquad Y \in 0, 1, \dots, n-1, n \qquad X: \dots, -\frac{1}{5}, -\frac{1}{3}, -1, 1, \frac{1}{3}, \frac{1}{5}, \dots$

$Y: \dots, \frac{1}{5}, \frac{1}{3}, 1, 1, \frac{1}{3}, \frac{1}{5}$

Looking at Y, you get each of the positive values of X, twice.

$$f_Y = \frac{2}{2n+1} \quad Y \in 0, 1, \dots, n-1, n$$

11. CB 3.16 VERIFY:

a. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}\,dt \qquad u = t^\alpha \quad du = \alpha t^{\alpha-1} \quad dv = e^{-t} \quad v = -e^{-t}$

$$= -t^\alpha e^{-t}\Big|_0^\infty + \int_0^\infty \alpha t^{\alpha-1} e^{-t}\,dt = 0 + 0 + \alpha\int_0^\infty t^{\alpha-1} e^{-t}\,dt$$

$\alpha \qquad \Gamma(\alpha)$

$$\alpha\Gamma(\alpha) = \alpha\int_0^\infty t^{\alpha-1} e^{-t}\,dt$$

b. $\Gamma(1/2) = \sqrt{\pi} \qquad \Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,dt \quad$ LET $t = u^2 \quad dt = 2u\,du \Rightarrow \int_0^\infty (u^2)^{-1/2} e^{-u^2} 2u\,du = \int_0^\infty \frac{2u}{u} e^{-u^2}\,du = \underbrace{\int_0^\infty 2e^{-u^2}\,du}_{\text{HALF OF A GAUSSIAN}}$

$$= 2 \cdot \int_0^\infty 2e^{-u^2}\,du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$\sqrt{\pi}/2$

12. LEMMA 2.3.14: $\lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a$

a. $$\lim_{n\to\infty}\left(\frac{\theta_n}{a+\theta_n}\right)^{\theta_n} = \lim_{\theta_n\to\infty}\left(\frac{1}{1+\frac{a}{\theta_n}}\right)^{\theta_n} = \frac{\lim_{\theta_n\to\infty}(1)^{\theta_n}}{\lim_{\theta_n\to\infty}\left(1+\frac{a}{\theta_n}\right)^{\theta_n}} = \frac{1}{e^a} = e^{-a}$$

b. $$\lim_{n\to\infty}\left(\frac{N-X}{N}\right)^N = \lim_{n\to\infty}\left(1 + \frac{-X}{N}\right)^N = e^{-X}$$

SEE (a)

c. $$\lim_{(n-m)\to\infty}\left(\frac{n-m}{(n-m)-K+X}\right)^{n-m} \; : n-m = z \quad Y = (-K+X) \quad \lim_{z\to\infty}\left(\frac{z}{z+Y}\right)^z = \lim_{z\to\infty}\left(\frac{1}{1+\frac{Y}{z}}\right)^z = \frac{1}{e^Y} = e^{-Y}$$

$Y = (-K+X) \quad \therefore = e^{K-X}$