HW10

Thursday, November 6, 2025 2:04 PM

1-3, 6-7, 10-13, 16, 18-19

1. $E(X) = \mu$ $VAR(X) = \sigma^2$ USE FIRST ORDER TAYLOR ABOUT μ, FIND $E(Y), VAR(Y)$ FOR:

$$E[g(x)] \approx g(E[X]) + g'(E[X])(X - E[X]) \qquad VAR(g(x)) \approx [g'(E(x))]^2 \cdot VAR(X)$$

a. $Y = e^X$ $g(x) = e^x$ $g'(x) = e^x$ $\underline{E(Y) \approx e^{\mu} + e^{\mu}(x-\mu)}$ $\underline{VAR(Y) \approx [e^{\mu}]^2 \cdot \sigma^2 = e^{2\mu}\sigma^2}$

b. $Y = \frac{1}{X}$ $(\mu \neq 0)$ $g(x) = \frac{1}{x}$ $g'(x) = -\frac{1}{x^2}$ $\underline{E(Y) \approx \frac{1}{\mu} - \frac{1}{\mu^2}(x-\mu)}$ $\underline{VAR(Y) \approx \frac{\sigma^2}{\mu^4}}$

c. $Y = \ln(X)$ $(x > 0)$ $g(x) = \ln x$ $g'(x) = \frac{1}{x}$ $\underline{E(Y) \approx \ln(\mu) + \frac{x-\mu}{\mu}}$ $\underline{VAR(Y) \approx \left(\frac{\sigma}{\mu}\right)^2}$

2. $X \sim UNIF(0,1)$ $Y_n \sim Exp\left(\frac{1}{n}\right)$ $X_n = X + Y_n$ SHOW $X_n \xrightarrow{L^2} X$

NEED TO SHOW $\lim_{n \to \infty} E(|X_n - x|^2) = 0$ $\qquad E(|X_n - X|^2) = E(|X_n - x|)^2 - VAR(|X_n - x|)$

$$= E(|Y_n|)^2 - VAR(|Y_n|) = E(Y_n)^2 - VAR(Y_n)$$

$\lim_{n \to \infty} 0 = 0$ ✓ $\qquad = \left(\frac{1}{n}\right)^2 - \left(\frac{1}{n}\right)^2 = 0$

$\therefore$ $\underline{X_n \xrightarrow{L^2} X}$

3. $X_n = \begin{cases} 0, & P = \frac{n-1}{n} \\ \sqrt{n}, & P = \frac{1}{n} \end{cases}$ WHAT DOES X_n CONVERGE TO? I PROPOSE CONVERGES TO $\underline{0}$

a. $X_n \xrightarrow{D} 0$ $\lim_{n \to \infty} X_n \to 0$ WHEN n LARGE, $n \to \infty$ $\lim X_n = \begin{cases} 0, P = \frac{BIG-1}{BIG} = 1 \\ \sqrt{BIG}, P = 0 \end{cases} = 0$ $\therefore$ $\underline{X_n \xrightarrow{D} 0}$

b. $X_n \xrightarrow{P} 0$ $\lim_{n \to \infty} P(|X_n - 0| \geq \varepsilon) = 0$ $P(|X_n - 0| \geq \varepsilon) = P(X_n \geq \varepsilon) = \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$ $\therefore$ $\underline{X_n \xrightarrow{P} 0}$

c. $X_n \xrightarrow{L^1} 0$ $\lim_{n \to \infty} E(|X_n - 0|^1) = 0$ $\lim_{n \to \infty} E(X_n) = \begin{cases} 0, P = \frac{\infty - 1}{\infty} = 1 \\ \sqrt{\infty}, P = 0 \end{cases} = 0$ $\therefore$ $\underline{X_n \xrightarrow{L^1} 0}$

D. $X_n \xrightarrow{a.s.} 0$ $P\left(\lim_{n \to \infty} |X_n - 0| < \varepsilon\right) < 1$ $\Rightarrow$ X_n WHERE $n >> 1$ $\Rightarrow$ $\frac{BIG-1}{BIG} \approx 1$ $\Rightarrow X_n \to 0$

$\therefore = P(0 < \varepsilon) \leq 1$, BUT, $P(X_n = 0) \neq 1$ (COULD STILL BE $\sqrt{n}$) THUS

$\underline{X_n \xrightarrow{a.s.} 0}$

6. $X_1, X_2 \ldots$ ARE iid. $X_{(n)}$ IS MAX.

a. $X_i \sim \text{Exp}(1)$ FIND LIMITING DIST. OF $X_{(n)} - \log(n) = Y_n$

$F_{X_{(n)}} = [F_{X_1}(x)]^n = [1 - e^{-x}]^n \qquad F_{Y_n} = [1 - e^{-(x+\ln(n))}]^n = \left[1 - \frac{e^{-x}}{n}\right]^n$

WANT $\lim_{n\to\infty} F_{Y_n} = \lim_{n\to\infty} \left[1 - \frac{e^{-x}}{n}\right]^n$ WE KNOW $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$

$\therefore \lim_{n\to\infty} F_{Y_n} = e^{-e^{-x}}$ THIS IS GUMBEL (0,1)

b. $F_{X_i} = 1 - x^{-2}$, $x > 1$, 0 OTHERWISE. FIND LIMITING DIST. OF $Y_n = n^{-1/2} X_{(n)}$ $\quad X_{(n)} = Y_n n^{1/2}$

$F_{X_{(n)}} = [F_{X_1}(x)]^n = [1 - x^{-2}]^n \qquad F_{Y_n} = [1 - (y n^{1/2})^{-2}]^n = \left[1 - \frac{y^{-2}}{n}\right]^n$

WANT $\lim_{n\to\infty} F_{Y_n} = \lim_{n\to\infty} \left[1 - \frac{x^{-2}}{n}\right]^n$ WE KNOW $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$

$\therefore \lim_{n\to\infty} F_{Y_n} = e^{x^{-2}} = e^{\left(\frac{1}{x^2}\right)}$

7. $X_1, X_2, \ldots \overset{iid}{\sim} \text{BETA}(1, \beta)$. FIND LIMITING DIST. OF $n^{1/\beta}(1 - X_{(n)}) = Y$ $\quad X = 1 - \frac{Y}{n^{1/\beta}}$

$F_{X_{(n)}} = [F_{X_1}(x)]^n = \left[\frac{1}{\beta(1,\beta)} x^{1-1} (1-x)^{\beta-1}\right]^n = \left(\frac{1}{\beta(1,\beta)}\right)^n [(1 - x)^{\beta-1}]^n \qquad F_{Y_n} = \left(\frac{1}{\beta(1,\beta)}\right)^n \left[\left(1 - 1 + \frac{y}{n^{1/\beta}}\right)^{\beta-1}\right]^n$

$F_{Y_n} = \left(\frac{1}{\beta(1,\beta)}\right)^n \left[\left(\frac{y}{n^{1/\beta}}\right)^{\beta-1}\right]^n = \left[\frac{1}{\beta(1,\beta)}\right]^n \left[\frac{y^{\beta-1}}{n^{1-1/\beta}}\right]^n$

$\lim_{n\to\infty} F_{Y_n} = e^{\left[\frac{y^{\beta-1}}{B(1,\beta)}\right]}$

10. CB 5.39 b $\qquad$ CASE OF $X_1 = I_{[0,1]}$ $X_2 = I_{[0,1/2]}$ $X_3 = I_{[1/2,1]}$ $X_4 = I_{[0,1/3]} \ldots$

b. Find a subsequence of the X_i s that converges almost surely, that is, that converges pointwise

For this we'll pick the subset where the Identity range starts at 0: $X_1 = I_{[0,1]}$ $X_2 = I_{[0,1/2]}$ $X_4 = I_{[0,1/3]}$ $X_7 = I_{[0,1/4]} \ldots$

THEN LET $X_j(s) = s + X_n$ AS $s \to \infty$, $X_n \to 0$ SO $s + X_n \to s$ WHEN $s > 0$

IF $s = 0$, WE'LL DEFINE $X_0 = 1$ SO $X_j(s) \to s + 1 = 1$

$\therefore X_j(s) \overset{a.s.}{\to} \begin{cases} s & s > 0 \\ s+1, & s = 0 \end{cases}$

11. CB 5.44 $\quad X_1, X_2 \ldots \overset{iid}{\sim} \text{BERN}(p) \quad Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

a. SHOW $\sqrt{n}(Y_n - p) \overset{D}{\to} N(0, p(1-p))$

By CLT we know $\sqrt{n}(\hat{p}-p) \xrightarrow{D} N(0, p(1-p))$

$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i = Y_n$ Thus by CLT $\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$

b. Show, for $p \neq 1/2$, $\hat{\text{Var}} = Y_n(1-Y_n)$ satisfies $\sqrt{n}\left[Y_n(1-Y_n) - p(1-p)\right] \xrightarrow{D} N\left[0, (1-2p)^2 p(1-p)\right]$

By CLT, know $\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$ $g(Y_n) = Y_n(1-Y_n)$ $g'(Y_n) = (1-Y_n) - Y_n = 1 - 2Y_n$

By Delta: $\sqrt{n}(Y_n(1-Y_n) - p(1-p)) \xrightarrow{D} N(0, \sigma^2(1-2p)^2) = N(0, p(1-p)(1-2p)^2)$

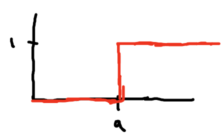
c. Show, for $p = 1/2$, $n\left[Y_n(1-Y_n) - \frac{1}{4}\right] \xrightarrow{D} \frac{1}{4}\chi_1^2$ $g = p(1-p)$ $g'(p) = 1-2Y_n$

Assume $n[\hat{p} - p] \xrightarrow{D} p^2\chi_1^2$ $\therefore$ $p = 1/2 \Rightarrow$ $n\left[Y_n(1-Y_n) - (\frac{1}{2})(1-\frac{1}{2})\right] \xrightarrow{D} \left(\frac{1}{2}\right)^2 \chi_1^2$

$\therefore n\left[Y_n(1-Y_n) - \frac{1}{4}\right] \xrightarrow{D} \frac{1}{4}\chi_1^2$

12. $a \in \mathbb{R}$, constant $X_1, X_2 \ldots$ R.V.s.

a. Sketch CDF of a. What does $X_n \xrightarrow{D} a$ mean?



X_n converging in distribution to a means that as n gets large, the sequence of random variables approaches a.

b. Show that, if $X_n \xrightarrow{D} a$ then $X_n \xrightarrow{P} a$

This is from theorem in class which states that a sequence of random variables converges in probability to a constant if and only if that sequence converges in distribution to that constant. By that theorem, the above statement is true.

13. $n = 40$ $X_i \sim N(5, 2)$ $Y = \sum_{n=1}^{40} X_i \sim N(40 \cdot 5, \sqrt{40(4)}) = N(200, 4\sqrt{10})$ Calc in R

a. Find $P\left(\frac{1}{n}\sum_{n=1}^{40} X_i > 5.33\right) = P\left(\sum_{n=1}^{40} X_i > 213.33\right) = P(N(200, 4\sqrt{10}) > 213.33) = .018$

Calc in R

b. Find $P\left(\sum_{n=1}^{40} X_i < 150\right) = P(N(200, 4\sqrt{10}) < 150) = .0000128$

- MAX. DAILY OZONE

16. $X_i \overset{iid}{\sim}$ GAMMA $(5, 10)$ LET $r = e^{\bar{X}}$ - MEAN OF 31 DAILY MAXIMA

WHAT IS APPROX DIST. OF r?

BY CLT: $\sqrt{n}(\bar{X} - \alpha\beta) \sim N(0, \alpha\beta^2)$ $g(x) = e^x$ $g'(x) = e^x$

$\therefore \sqrt{n}(e^{\bar{x}} - e^{\alpha\beta}) \xrightarrow{D} N(0, \alpha\beta^2[e^{\alpha\beta^2}]^2) = N(0, \alpha\beta^2 e^{\alpha^2\beta^4})$

I did this problem on a previous homework before it was removed, so even though we didn't need to do it for this homework, I already had it ready haha, figured I would just include it

17. SHOW THESE ARE EXPONENTIAL FAMILIES: SHOW $f(x|\theta) = h(x)\, c(\theta)\, e^{\sum_{i=1}^{k} w_i(\theta) t_i(x)}$

a. $N(\mu, \sigma^2)$ μ IS KNOWN : $f(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

LET $h(x) = \frac{1}{\sqrt{2\pi}}$, $c(\sigma) = \frac{1}{\sigma}$, $w_1(\sigma) = -\frac{1}{2\sigma^2}$ $t_1(x) = (x-\mu)^2$

$\therefore f(x|\sigma) = f(x|\theta)$

b. $N(\mu, \sigma^2)$ σ IS KNOWN : $f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)}$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2}}$

$h(x) = 1$ $c(\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mu^2}{2\sigma^2}}$ $w_1 = -\frac{1}{2\sigma^2}$ $t_1 = x^2$ $w_2 = \frac{\mu}{\sigma^2}$ $t_2 = x$

C. GAMMA (α, β) α IS KNOWN : $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$h(x) = x^{\alpha-1}$ $c(\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$ $t_1(x) = -x$ $w(\beta) = \frac{1}{\beta}$

D. GAMMA (α, β) β IS KNOWN : $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$h(x) = e^{-x/\beta}$ $c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$ $x^{\alpha-1} = e^{\ln(x^{\alpha-1})} = e^{(\alpha-1)\ln(x)}$ $w_1(\alpha) = \alpha - 1$ $t_1(x) = \ln(x)$

E. GAMMA (α, β) BOTH UNKNOWN: $\frac{1}{\Gamma(\alpha)\beta^\alpha} \underbrace{x^{\alpha-1} e^{-x/\beta}}$

$h(x) = 1$ $c(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$ $e^{(\alpha-1)\ln(x) - x/\beta}$ $w_1(\alpha, \beta) = \alpha - 1$ $t_1(x) = \ln(x)$

$w_2(\alpha,\beta) = -\frac{1}{\beta}$ $\quad t_2(x) = x$

F. BETA(α,β) α IS KNOWN : $\frac{1}{B(\alpha,\beta)} x^{\alpha-1} \underbrace{(1-x)^{\beta-1}}$

$e^{(\beta-1)\ln(1-x)}$

$h(x) = x^{\alpha-1}$ $\quad c(\beta) = \frac{1}{B(\alpha,\beta)}$ $\quad w_1(\beta) = \beta-1$ $\quad t_1(x) = \ln(1-x)$

G. BETA(α,β) β IS KNOWN : $\underbrace{\frac{1}{B(\alpha,\beta)}}_{c(\alpha)} x^{\alpha-1} \underbrace{(1-x)^{\beta-1}}_{h(x)}$

$x^{\alpha-1} \to e^{(\alpha-1)\ln(x)}$

$h(x) = (1-x)^{\beta-1}$ $\quad c(\alpha) = \frac{1}{B(\alpha,\beta)}$ $\quad w_1(\alpha) = (\alpha-1)$ $\quad t_1(x) = \ln(x)$

H. POIS(λ) : $\frac{e^{-\lambda}\lambda^{x}}{x!}$ $\quad h(x) = x!$ $\quad c(\lambda) = 1$ $\quad e^{-\lambda}\lambda^{x} = e^{-\lambda} e^{x\ln(\lambda)} = e^{x\ln(\lambda)+\lambda}$

$h(x) = x!$ $\quad c(\lambda) = 1$ $\quad w_1(\lambda) = \ln(\lambda)$ $\quad w_2(\lambda) = \lambda$ $\quad t_1(x) = x$ $\quad t_2(x) = 1$

18. SHOW THESE ARE EXPONENTIAL FAMILIES : $f(x|\theta) = h(x)c(\theta)\,\mathrm{EXP}\left[\sum_{i=1}^{k} w_i(\theta)t_i(x)\right]$

a. NEG. BINOM. (r,p) r IS KNOWN : $f(x|p) = \binom{r+x-1}{x} p^{r} \underbrace{(1-p)^{x}}$

$\quad \llcorner\ e^{\ln((1-p)^{x})} = e^{x\ln(1-p)}$

$h(x) = \binom{r+x-1}{x}$ $\quad c(p) = p^{r}$ $\quad w_1(p) = (1-p)$ $\quad t_1(x) = x$

b. PARETO(α,β) α IS KNOWN : $f(x,\beta) = \beta\alpha^{\beta} \underbrace{x^{-(\beta+1)}}$

$\quad \llcorner\ e^{\ln(x^{-(\beta+1)})} = e^{-(\beta+1)\ln(x)}$

$h(x) = 1$ $\quad c(\beta) = \beta\alpha^{\beta}$ $\quad w_1(\beta) = -(\beta+1)$ $\quad t_1(x) = \ln(x)$

c. BETA(α,β) BOTH UNKNOWN : $f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} \underbrace{x^{\alpha-1}(1-x)^{\beta-1}}$

$= e^{\ln(x^{\alpha-1}(1-x)^{\beta-1})} = e^{(\alpha-1)\ln(x)+(\beta-1)\ln(1-x)}$

$h(x) = 1$ $\quad c(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}$ $\quad w_1(\alpha,\beta) = (\alpha-1)$ $\quad w_2(\alpha,\beta) = (\beta-1)$ $\quad t_1(x) = \ln(x)$ $\quad t_2(x) = \ln(1-x)$

17. WRITE PDF IN NATURAL PARAM., FIND MEAN & VAR OF X

a. GAMMA(α, β) : $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ USE $f(x|\eta) = h(x)c^*(\eta) \operatorname{Exp}\left[\sum_{i=1}^{k} \eta_i t_i(x)\right]$

FROM 11(e) WE KNOW: $h(x)=1$ $C(\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\beta^{-\alpha}$ $w_1 = (\alpha-1)$ $w_2 = \beta^{-1}$ $t_1 = \ln(x)$ $t_2 = -x$

$h(x)=1$ $c^* = \frac{1}{\Gamma(\eta_1+1)}\eta_2^\alpha$ $\eta_1 = \alpha-1$ $\eta_2 = \frac{1}{\beta}$ $\alpha = \eta_1+1$ $\beta = \eta_2^{-1}$

$E(t_j(x)) = -\frac{\partial}{\partial \eta_j}\ln(c^*(\eta)) = -\frac{\partial}{\partial\eta_1}\ln\left(\frac{\eta_2^\alpha}{\Gamma(\eta_1+1)}\right) = -\frac{\partial}{\partial\eta_1}\ln(\eta_2^\alpha) + \frac{\partial}{\partial\eta_1}\ln($

$= -\frac{\partial}{\partial\eta_2}\ln\left(\frac{\eta_2^\alpha}{\Gamma(\eta_2+1)}\right) = -\frac{\partial}{\partial\eta_2}\ln(\eta_2^\alpha) + \frac{\partial}{\partial\eta_2}(\Gamma(\eta_1+1)) = -\frac{\alpha\eta_2^{\alpha-1}}{\eta_2^\alpha} = -\frac{\alpha}{\eta_2} = \alpha\beta$

$E(x) = \alpha\beta$

$VAR(t_j(x)) = -\frac{\partial^2}{\partial^2\eta_2}c^* = -\frac{\partial^2}{\partial^2\eta_2}\ln\left(\frac{\eta_2^\alpha}{\Gamma(\eta_1+1)}\right) = -\frac{\partial}{\partial\eta_2}\left(-\frac{\alpha}{\eta_2}\right) = \frac{\alpha}{\eta_2^2} = \alpha\beta^2$

$VAR(x) = \alpha\beta^2$

b. POIS(λ): $h(x) = x!$ $c(\lambda)=1$ $w_1 = \eta_1 = \ln(\lambda)$ $w_2 = \eta_2 = \lambda$ $t_1 = x$ $t_2 = 1$

$\lambda = e^{\eta_1}$ $\lambda = \eta_2$

$h(x) = x!$ $c^*(\eta) = \frac{e^{\eta_1}}{\eta_2}$

$E(x) = -\frac{\partial}{\partial\eta_1}\ln\left(\frac{e^{\eta_1}}{\eta_2}\right) - \frac{\partial}{\partial\eta_2}\ln\left(\frac{e^{\eta_1}}{\eta_2}\right) = -\frac{\partial}{\partial\eta_1}\eta_1 + -\frac{\partial}{\partial\eta_2}\ln\left(\frac{1}{\eta_2}\right) = 1, -\frac{1}{\eta_2} = \lambda$

$E(x) = \lambda$

$Var(x) = -\frac{\partial^2}{\partial^2\eta_2}\ln\left(\frac{1}{\eta_2}\right) = \frac{1}{\eta_2^2} = \lambda$

$VAR(x) = \lambda$

c. BINOM(n,p): $f(x|n,p) = \binom{n}{x}p^x(1-p)^{n-x} = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \frac{p^x}{(1-p)^x x!(n-x)!} n!(1-p)^n$

Had to look up what the transformation was:

$\eta = \ln\left(\frac{p}{1-p}\right) \Rightarrow f(x|\eta) = h(x)\operatorname{Exp}\left[\eta x - n\ln(1+e^\eta)\right]$ $h(x) = \binom{n}{x}$ $t(x) = x$

$c^* = \frac{1}{(1+e^\eta)^n}$

$E(x) = -\frac{\partial}{\partial\eta}\ln c^* = -\frac{\partial}{\partial\eta}\ln\left(\frac{1}{(1+e^\eta)^2}\right) = -\frac{\partial}{\partial\eta}\left(1+e^\eta\right)^{-n}\cdot(1+e^\eta)^n = -n(1+e^\eta)^{-n-1}(1+e^\eta)^n\cdot e^\eta$

$= \frac{ne^\eta}{\dots} = \frac{n\left(\frac{p}{1-p}\right)}{\dots} \quad np \quad np$

$1+e^{-} \qquad 1+\frac{p}{1-p} = (1-p)\left(1+\frac{p}{1-p}\right) = \overline{1-p+p} = np \qquad E(X) = np$

$$Var(X) = -\frac{\partial^2}{\partial \eta^2} \ln(c^*) = -\frac{\partial}{\partial \eta} \frac{-n e^{\eta}}{1+e^{\eta}} = n \frac{e^{\eta}}{(1+e^{\eta})^2} = \frac{n\left(\frac{p}{1-p}\right)}{\left(1+\frac{p}{1-p}\right)^2} = \frac{np}{(1-p)\left(1+\frac{2p}{1-p}+\frac{p^2}{(1-p)^2}\right)} = \frac{np}{1-p+2p+\frac{p^2}{1-p}}$$

$$= \frac{np}{1+p+\frac{p^2}{1-p}} = \frac{np(1-p)}{1-p+p-p^2+p^2} = np(1-p)$$

$Var(X) = np(1-p)$