HW9

Thursday, November 20, 2025 8:10 AM

1. CB 3.44 FOR R.V. X WHERE $E[X^2]$ AND $E|X|$, SHOW THAT

$P(|X| \geq b) \leq \frac{E(X^2)}{b^2}$, $b > 0$

$= P(X^2 \geq b^2) \leq \frac{E|X|}{b}$

$E(g(x)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \geq \int_{x:g(x)\geq b}^{\infty} g(x) f_X(x)\,dx$

$g(x) = X^2$: $\geq \int_{x:g(x)\geq b}^{\infty} b^2 f_X(x)\,dx = b^2 P(X^2 \geq b^2) \Rightarrow P(|X| \geq b) \leq \frac{E(X^2)}{b^2}$

$g(x) = |X|$: $\geq \int_{x:g(x)\geq b}^{\infty} |b| f_X(x)\,dx = b\,P(|X| \geq b) \Rightarrow P(|X| \geq b) \leq \frac{E|X|}{b}$

IF $f_X = e^{-x}$, $x > 0$, SHOW ONE BOUND IS BETTER FOR $b = 3$, AND THE OTHER FOR $b = \sqrt{2}$

$P(|X| \geq 3) = \int_3^{\infty} |x| e^{-x}\,dx \Rightarrow \begin{matrix} u = x & dv = e^{-x} \\ du = 1 & v = -e^{-x} \end{matrix} = -x e^{-x} - \int_3^{\infty} -e^{-x}\,dx = -x e^{-x} - e^{-x} \Big|_3^{\infty} = 3e^{-3} - e^{-3} = 2e^{-3} \approx .0996$

$P(|X| \geq \sqrt{2}) = \int_{\sqrt{2}}^{\infty} \cdots = \sqrt{2}\, e^{-\sqrt{2}} - e^{-\sqrt{2}} = (\sqrt{2} - 1)\, e^{-\sqrt{2}} \approx .1007$

$E(X^2) = \int_0^{\infty} x^2 e^{-x}\,dx = \begin{matrix} u = x^2 & dv = e^{-x} \\ du = 2x & v = -e^{-x} \end{matrix} = -x^2 e^{-x} + \int 2x e^{-x}\,dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \Big|_0^{\infty} = -0 - 0 - 0 + 0 + 0 + 2 = 2$

$E(|X|) = \int_0^{\infty} x e^{-x} \ldots = -x e^{-x} - e^{-x} \Big|_0^{\infty} = -0 - 0 + 0 + 1$

$\frac{E(X^2)}{3^2} = .\overline{222}$ $\quad \frac{E(|X|)}{3} = .\overline{333}$ $\quad \frac{E(X^2)}{\sqrt{2}^2} = 1$ $\quad \frac{E(|X|)}{\sqrt{2}} = .707$

WHEN $b = 3$: $\frac{E(X^2)}{b^2}$ GIVES A TIGHTER BOUND

WHEN $b = \sqrt{2}$: $\frac{E|X|}{b}$ GIVES A TIGHTER BOUND

2. CB 3.45 a

a. X IS R.V. WITH $M_X(t)$, $-h < t < h$. PROVE $P(X \geq a) \leq e^{-at} M_X(t)$, $0 < t < h$

$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \geq \int_a^{\infty} e^{tx} f_X(x)\,dx \geq \int_a^{\infty} e^{ta} f_X(x)\,dx = e^{ta} P(X \geq a)$

$\therefore P(X \geq a) \leq e^{-at} M_X(t)$

3. CB 3.46 USE $K = .5, 1, \sqrt{2}, 1.5, \sqrt{3}, 2, 4, 6, 10$

CALCULATE $P(|X - \mu_X| \geq K\sigma_X)$ FOR $X \sim \text{UNIF}(0,1)$, $X \sim \text{Exp}(1)$. COMPARE TO BOUND FROM CHEBYCHEV'S

FOR UNIF(0,1): $\mu_X = .5$, $\sigma_X = \frac{1}{\sqrt{12}}$ $\quad P(|X - \mu_X| \geq K\sigma_X) = \int^{1-\mu_X} \frac{1}{b-a}\,dx = \int^{1.5} 1\,dx = x \Big|_{K/\sqrt{12}}^{.5} = .5 - K/\sqrt{12}$

FOR Exp(1): $\mu_X = 1 \quad \sigma_X = 1$

$$= \int_{K/1}^{\infty} 1 e^{-x-1} dx = -e^{-x-1}\Big|_{K/1}^{\infty} = e^{-K-1}$$

$K/\sqrt{12}$ $\quad$ $14/\sqrt{12}$

K	CHEBY BOUND	X ~ UNIF(0,1)	X ~ EXP(1)
.5	2	.356	$e^{-1} \approx .368$
1	1	.211	$e^{-2} \approx .135$
$\sqrt{2}$	1/2	.092	$e^{-\sqrt{2}-1} \approx .089$
1.5	4/9	.067	$e^{-5/2} \approx .082$
$\sqrt{3}$	1/3	0	$e^{-\sqrt{3}-1} \approx .065$
2	1/4	0	$e^{-3} \approx .05$
4	1/16	0	$e^{-5} \approx .007$
6	1/36	0	$e^{-7} \approx .0009$
10	1/100	0	$e^{-11} \approx .000015$

4. CB 5.21 PROB THAT MAX OF X_1, X_2 EXCEEDS MEDIAN

BECAUSE iid

$$P(\text{MAX}(X_1 \ldots X_n) > m) = 1 - P(X_{(n)} \le m) = 1 - P(X_1 \le m, X_2 \le m, X_3 \le m \ldots) = 1 - [P(X_1 \le m)]^n$$

$$= 1 - \left(\tfrac{1}{2}\right)^n$$

5. CB 5.24 $X_1 \ldots X_n$, $f_X = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{o.w.} \end{cases}$ $\quad F_X = \frac{x}{\theta} \quad 0 < x < \theta$

SHOW THAT $\frac{X_{(1)}}{X_{(n)}}$ AND $X_{(n)}$ ARE INDEPENDENT R.V.S

LET $Y = X_{(1)}$ $Z = X_{(n)}$ $\quad f_{Y,Z} = n(n-1) f_Y(y) f_Z(z) \left[F_Z - F_Y\right]^{n-2} = n(n-1)\left(\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)\left[\frac{z}{\theta} - \frac{y}{\theta}\right]^{n-2}$

$$= \frac{n(n-1)}{\theta^2}\left[\frac{z-y}{\theta}\right]^{n-2} = \frac{n(n-1)}{\theta^2 \theta^{n-2}}(z-y)^{n-2} = \frac{n(n-1)}{\theta^n}(z-y)^{n-2} \quad 0 < y < z < \theta$$

$A = \frac{Y}{Z}$ $\quad B = Z$ $\quad Y = AB$ $\quad \frac{\partial y}{\partial A} = B$ $\quad \frac{\partial y}{\partial B} = A$ $\quad \frac{\partial z}{\partial A} = 0$ $\quad \frac{\partial z}{\partial B} = 1$ $\quad J = b$

$$f_{A,B}(a,b) = f_{Y,Z}(ab, b) \cdot b = \frac{n(n-1)}{\theta^n}(b - ab)^{n-2} \cdot b = \frac{n(n-1)}{\theta^n} b^{n-2}(1-a)^{n-2} \cdot b = \underbrace{\frac{n(n-1)}{\theta^n} b^{n-1}}_{f(b)} \underbrace{(1-a)^{n-2}}_{g(a)}$$

THIS MEANS $A \perp\!\!\!\perp B \therefore \frac{Y}{Z} \perp\!\!\!\perp Z \quad \therefore \frac{X_{(1)}}{X_{(n)}} \perp\!\!\!\perp X_{(n)}$

6. $X_1 \ldots X_n$, $f_X = 2x \quad 0 < x < 1$ $\quad F_X = x^2$

FIND $E[X_{(n)}]$ $\quad F_{X_{(n)}} = P(X_{(n)} \le x) = P(X_1 \le x, X_2 \le x \ldots) = P(X_1 \le x)^n = F_X^n$

$$f_{X_{(n)}} = n\left[F_X(x)\right]^{n-1} \cdot f_X(x) = n\left[x^2\right]^{n-1} \cdot 2x$$

$$E[X_{(n)}] = \int_0^1 g(x) \cdot f(x)\, dx = \int_0^1 n x^{2n-2} \cdot 2x \cdot 2x\, dx = 4n \int_0^1 x^{2n} dx = 4n\left[\frac{1}{2n+1} x^{2n+1}\right]_0^1 = \frac{4n}{2n+1}$$

$$E[X_{(n)}] = \frac{4n}{2n+1}$$

7. $X_1 \dots X_n \sim UNIF(0,1)$ WHAT IS f_R WHERE $R = X_{(n)} - X_{(1)}$

$R = X_{(n)} - X_{(1)}$ $\quad W = X_{(1)}$ $\quad X_{(n)} = R + W$ $\quad \frac{\partial X_{(1)}}{\partial R} = 0$ $\quad \frac{\partial X_{(1)}}{\partial W} = 1$ $\quad \frac{\partial X_{(n)}}{\partial R} = 1$ $\quad \frac{\partial X_{(n)}}{\partial W} = 1$ $\quad J = |-1| = 1$

$$f_{R,W}(r,w) = f_{X_{(1)},X_{(n)}}(w, r+w) = n(n-1)\, f_X(w) f_X(r+w) \left[F_X(r+w) - F_X(w)\right]^{n-2} = n(n-1)(1)(1)\left[r+w-w\right]^{n-2}$$

$f_{R,W} = n(n-1)\, r^{n-2}$ $\quad w>0 \quad r>0 \quad w+r<1$

$$f_R = \int_0^{1-r} n(n-1)\, r^{n-2}\, dw = n(n-1)\, r^{n-2}\, w \Big|_0^{1-r} = n(n-1)\, r^{n-2}(1-r) = f_R$$

8. $X_1 \dots X_n \sim$ GEOM, $f_X = p^x(1-p)$, $x = 0,1,2,\dots \infty$

FIND $f_{X_{(1)}}$: GIVEN IN CLASS: $P(X_{(j)} \le x_i) = \sum_{k=j}^{n} \binom{n}{k} P_j^k (1-P_j)^{n-k}$

$P(X_{(1)} \le x_i) = \sum_{k=1}^{n} \binom{n}{k} P_i^k (1-P_i)^{n-k}$ $\qquad P_i = P(X=x) = p^x(1-p)$

$$= \sum_{k=1}^{n} \binom{n}{k} \left(p^x(1-p)\right)^k \left(1-p^x(1-p)\right)^{n-k} = \sum_{k=1}^{n} \binom{n}{k} p^{kx}(1-p)^k \left(1-p^x(1-p)\right)^{n-k} = f_X$$

9. WE HAVE $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}$ WHERE $f_X = 2x$ $\quad F_X = x^2$

a. FIND JOINT OF $X_{(3)}, X_{(4)}$: $f_{X_{(3)}X_{(4)}}(a,b) = \frac{4!}{(3-1)!(4-1-3)!(4-4)!} f_X(a) f_X(b) \left[F_X(a)\right]^{3-1} \left[F_X(b) - F_X(a)\right]^{4-1-3} \left[F_X(b)\right]^{4-4}$

$$= \frac{4!}{2!0!0!}(2a)(2b)\left[a^2\right]^2 \left[b-a\right]^0 \left[b^2\right]^0 = 12(4)ab\,a^4 = 48a^5 b$$

$f_{X_{(3)},X_{(4)}}(a,b) = 48a^5 b$ $\quad 0<a<b<1$

b. FIND $X_{(3)} | X_{(4)} = x_4$ $\qquad X_{(3)} | X_{(4)} = \frac{f_{X_{(3)}X_{(4)}}}{f_{X_{(4)}}}$ $\qquad f_{X_{(4)}} = \frac{4!}{(4-1)!(4-4)!} f_X(x)\left[F_X(x)\right]^{4-1}\left[1-F_X(x)\right]^{4-4}$

$f_{X_{(4)}} = \frac{4!}{3!} \cdot 2x\left[x^2\right]^3 \left[1-x^2\right]^0 = 8x \cdot x^6 = 8x^7$ $\quad (x \Rightarrow b)$

$X_{(3)} | X_{(4)} = \frac{48a^5 b}{8b^7} = 6a^5 b^{-6}$

$f_{X_{(3)}|X_{(4)}=x_4}(a) = 6a^5 x_4^{-6}$ $\quad 0<a<x_4$

c. FIND $E\left[X_{(3)} | X_{(4)} = x_4\right] = \int_0^{x_4} a\, 6a^5 x_4^{-6}\, da = 6x_4^{-6} \int_0^{x_4} a^6\, da = 6x_4^{-6}\left[\frac{1}{7}a^7\right]_0^{x_4} = \frac{6}{7} x_4^{-6} x_4^7$

$E\left[X_{(3)} | X_{(4)} = x_4\right] = \frac{6}{7} x_4$

10. $X_1 \ldots X_n \overset{iid}{\sim} \text{Exp}(1) \quad f_X = e^{-x} \quad F_X = -e^{-x}$

a. Give the PDF of the time to failure if the components are connected in series. If they are connected in series, the system fails when the any component fails for the first time, meaning we are looking for the min fail time

$$f_{X_{(1)}} = n\left[1 - F_X(x)\right]^{n-1} \cdot f_X(x) = n\left[1 + e^{-x}\right]^{n-1} \cdot e^{-x}$$

b. Give the PDF of the time to failure if the components are connected in parallel. If they are connected in parallel, the system fails when the last component fails, meaning we are looking for the max fail time

$$f_{X_{(n)}} = n\left[F_X(x)\right]^{n-1} \cdot f_X(x) = n\left[-e^{-x}\right]^{n-1} \cdot e^{-x} = n(-1)^{n-1}\left[e^{-x}\right]^n$$

c. Give the PDF of the time to failure if 5 components are connected, and the system fails when at least 3 of them fail.

$$f_{X_{(3)}}(x) = \frac{5!}{(3-1)!\,(5-3)!} \cdot e^{-x}\left[-e^{-x}\right]^{3-1}\left[1 + e^{-x}\right]^{5-3} = \frac{5!}{2!\,2!} e^{-x}\left[-e^{-x}\right]^2\left[1 + e^{-x}\right]^2 = \frac{5 \cdot 4 \cdot 3}{2 \cdot 1} e^{-x} e^{-2x}\left[1 + 2e^{-x} + e^{-2x}\right]$$

$$= 30\left[e^{-3x} + 2e^{-4x} + e^{-5x}\right]$$

d. The components are no longer iid, and each has their own failure time. Give the pdf of the time to failure if all components are connected in series.

$X_i \sim \text{Exp}(\theta_i) \quad f_{X_i} = \frac{1}{\theta_i} e^{-x/\theta_i} \quad F_{X_i} = 1 - e^{-x/\theta_i}$

STILL WANT $X_{(1)}$ $\quad P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, X_2 > x, \ldots X_n > x)$

$= 1 - P(X_1 > x)P(X_2 > x) \ldots P(X_n > x) \qquad P(X_i > x) = 1 - F_{X_i} = 1 - 1 + e^{-x/\theta_i} = e^{-x/\theta_i}$

$$F_{X_{(1)}} = 1 - \prod_{i=1}^{n} e^{-x/\theta_i} = 1 - e^{-x\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} + \ldots\right)}$$

$$f_{X_{(1)}} = \left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right] e^{-x \sum_{i=1}^{n} \frac{1}{\theta_i}}$$

11. $X_1 \ldots X_n \overset{iid}{\sim} \text{UNIF}(0,1)$ $\quad$ PROVE $E\left[X_{(MED)}\right] = .5$ $\quad$ ($E[M]$, POPULATION MEDIAN) $\quad f_x = 1 \quad F_x = x \quad$ LET $r = m+1$, $n = 2m+1$

n IS ODD: $$E\left[X_{(m+1)}\right] = \int_0^1 x \cdot \frac{(2m+1)!}{(2m+1-m-1)!\,(m+1-1)!} x^{(m+1-1)} (1-x)^{(2m+1-m-1)}\, dx$$

$$= \int_0^1 \frac{(2m+1)!}{m!\,m!} \underbrace{x^{m+1}(1-x)^m\, dx}_{\text{BETA}(m+2,\, m+1)} = \frac{(2m+1)!}{m!\,m!} \cdot \frac{\Gamma(m+2)\Gamma(m+1)}{\Gamma(m+2+m+1)} = \frac{(2m+1)!}{m!\,m!} \cdot \frac{(m+1)\Gamma(m+1)\,(m!)}{\Gamma(2m+3)}$$

$$= \frac{(2m+1)!}{m!\,m!} \cdot \frac{(m+1)\ m!\ \cdots}{(2m+2)(2m+1)!} = \frac{\cdots}{2m+2} = \frac{1}{2}\,\frac{\cdots}{m+1} = \frac{\cdots}{2}$$

WHEN n IS ODD: $E[M] = 1/2$

WHEN n IS EVEN: LET $n=2m$

MEDIAN IS HALF-WAY BETWEEN MIDDLE VALUES: $E\left[\frac{X_{(m)} + X_{(m+1)}}{2}\right] = \frac{1}{2}\left\{E[X_{(m)}] + E[X_{(m+1)}]\right\} = E[M]$

$$EX_{(m)} = \int_0^1 x \frac{(2m)!}{(m-1)!\,m!} x^{m-1}(1-x)^m\,dx = \frac{(2m)!}{(m-1)!\,m!}\,\frac{\Gamma(m+1)\Gamma(m+1)}{\Gamma(2m+2)} = \frac{(2m)!}{(m-1)!\,m!}\,\frac{m!\,m!}{(2m+1)!} = \frac{m}{2m+1}$$

$$EX_{(m+1)} = \int_0^1 x \frac{(2m)!}{m!(m-1)!} x^m(1-x)^{m-1}\,dx = \frac{(2m)!}{m!(m-1)!}\,\frac{\Gamma(m+2)\Gamma(m)}{\Gamma(2m+2)} = \frac{(2m)!}{m!\,(m-1)!}\cdot\frac{(m+1)!\,m!}{(2m+1)!} = \frac{m+1}{2m+1}$$

$$E[M] = \frac{1}{2}\left[E[X_{(m)}] + E[X_{(m+1)}]\right] = \frac{1}{2}\left[\frac{m}{2m+1} + \frac{m+1}{2m+1}\right] = \frac{1}{2}\left[\frac{2m+1}{2m+1}\right] = .5$$

WHEN n IS EVEN: $E[M] = \frac{1}{2}$

12. $X_1 \ldots X_n \sim \text{BERN}(p)$ PROVE $\bar{x}_n \xrightarrow{P} p$ $\quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

CHEBYCHEV'S

$$P(|\bar{X}_n - p| \geq \epsilon) = P(|\bar{X}_n - p|^2 \geq \epsilon^2) \leq \frac{E(\bar{X}_n - p)}{\epsilon^2} = \frac{\text{Var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

$$\lim_{n\to\infty} P(|\bar{X}_n - p| \geq \epsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$$

$\therefore \bar{X}_n \xrightarrow{P} p$

13. $X_1 \ldots X_n \sim \text{UNIF}(0,1)$ WHAT DOES $X_{(n)}$ CONVERGE TO?

CONCEPTUALLY, $X_{(n)} \Rightarrow 1$

$$\lim_{n\to\infty} P(|X_{(n)} - 1| < \epsilon) = 1 \qquad P(|X_{(n)} - 1| < \epsilon) = P(X_{(n)} - 1 > -\epsilon) = 1 - P(X_{(n)} < -\epsilon + 1)$$

$$= 1 - [F(-\epsilon+1)]^n = 1 - (-\epsilon+1)^n$$

$$\lim_{n\to\infty} P(|X_{(n)} - 1| < \epsilon) = \lim_{n\to\infty}\left[1 - (-\epsilon+1)^n\right] = 1 - 0 = 1 \qquad \therefore X_{(n)} \xrightarrow{P} 1$$

ALWAYS $0 < -\epsilon+1 < 1$
THIS PIECE GETS REALLY SMALL

14. $X_1 \ldots X_n$, $f_X = e^{-(x-\mu)}$ $\quad x > \mu \quad F_X = -e^{-(x-\mu)}$

$P(X_{(1)} - k > -\epsilon) = 1 - P(X_{(1)} > -\epsilon + k)$

$P(X_{(1)} - k < \epsilon) = P(X_{(1)} < \epsilon + k)$

a. SHOW $X_{(1)} \xrightarrow{P} k$ ... GUESSING $K=0$: $P(|X_{(1)} - k| < \epsilon) =$

n. FIND

$$\Rightarrow \begin{matrix} 1-\left[1-F(-\epsilon+k)\right]^n \\ \left[1-F(\epsilon+k)\right]^n \end{matrix} = \begin{matrix} 1-\left[1+e^{-(\epsilon+k-\mu)}\right]^n \\ \left[1+e^{-(\epsilon+k-\mu)}\right]^n \end{matrix}$$ THIS ONLY CONVERGES WHEN $k=\mu$

$$\lim_{n\to\infty} P(|X_{(1)}-\mu|<\epsilon) = \lim_{n\to\infty}\left[1+e^{-(\epsilon+\mu-\mu)}\right]^n = \left[1+0\right]^n = 1$$

$\therefore$ $X_{(1)} \xrightarrow{P} \mu$

b. DOES $Y=\frac{1}{X_{(1)}}$ CONVERGE? $P\left(\left|\frac{1}{X_{(1)}}-k\right|<\epsilon\right) = P\left(\frac{1}{X_{(1)}}-k<\epsilon\right) = P\left(\frac{1}{X_{(1)}}<\epsilon+k\right) = P\left(X_{(1)}>\frac{1}{\epsilon+k}\right)$

$$= 1-P\left(X_{(1)}<\frac{1}{\epsilon+k}\right) = 1-\left[1-F\left(\frac{1}{\epsilon+k}\right)\right]^n = 1-\left[1+e^{-\left(\frac{1}{\epsilon+k}-\mu\right)}\right]^n$$

THIS WOULD ONLY CONVERGE IF $k=\frac{1}{\mu}$

HOWEVER, $\frac{1}{\mu}<\mu$ WHICH IS NOT WITHIN THE SUPPORT OF $x>\mu$

$\therefore$ Y DOES NOT CONVERGE