HW 1

Tuesday, September 9, 2025 3:37 PM

1. CB 1.2

(a) $A \setminus B \Rightarrow A = (A \cap B) \cup (A \cap B^c)$

IT'S GIVEN THAT $A \setminus B = A \setminus (A \cap B)$ (YOU WILL ONLY REMOVE ELEMENTS IN B THAT ARE ALSO IN A)

$\therefore A \setminus (A \cap B) = [(A \cap B) \cup (A \cap B^c)] \setminus (A \cap B) = \underline{A \cap B^c = A \setminus B}$

b) $(B \cap A) \cup (B \cap A^c) = ((B \cap A) \cup B) \cap ((B \cap A) \cup A^c) = B \cap ((B \cup A^c) \cap (A \cup A^c))$

$= B \cap ((B \cup A^c) \cap S') = B \cap (B \cup A^c) = (B \cap B) \cup (B \cap A^c) = B \cup (B \cap A^c)$

$B \cap A^c \in B \therefore \underline{B = (B \cap A) \cup (B \cap A^c)}$

c) $B \setminus A = [(B \cap A) \cup (B \cap A^c)] \setminus (B \cap A) = \underline{B \cap A^c}$ (USING (a), (b))

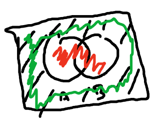
d) $A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = (A \cup (B \cap A)) \cup (A \cup (B \cap A^c))$

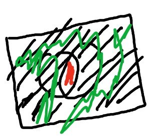
$B \cap A \in A \therefore = A \cup (A \cup (B \cap A^c)) = \underline{A \cup (B \cap A^c)}$

2. CB 1.3 b c

b) 1. $A \cup (B \cup C) = (A \cup B) \cup (A \cup C) = (A \cup B \cup A) \cup ((A \cup B) \cup C) = (A \cup B) \cup ((A \cup B) \cup C) = (A \cup B) \cup C$

2. $A \cap (B \cap C) = (A \cap B) \cap (B \cap C) = (A \cap B \cap A) \cap ((A \cap B) \cap C) = (A \cap B) \cap ((A \cap B) \cap C) = (A \cap B) \cap C$

c) 1. $(A \cup B)^c$:  : $A^c \cap B^c$

2. $(A \cap B)^c$:  : $A^c \cup B^c$

3. $A_n = \left(\frac{1}{2n}, \frac{1}{n}\right)$

$A = \bigcap_{n=1}^{\infty} A_n = \left(\frac{1}{2}, 1\right) \cap \left(\frac{1}{4}, \frac{1}{2}\right) \cap \left(\frac{1}{6}, \frac{1}{3}\right) \cap \dots = \underline{\emptyset}$

$B = \bigcup_{n=1}^{\infty} A_n = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \dots \cup \left(\frac{1}{\infty}, \frac{1}{\infty}\right) = \underline{(0, 1)}$

4. CB 1.11

a) i. $\emptyset \in \mathcal{B}$: STATED

ii. $A \in \mathcal{B} \therefore A^c \in \mathcal{B}$: $A \in S'$ & $A^c \in S'$, $S \in \mathcal{B} \therefore$ MET

iii. $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$: $\bigcup_{n=1}^{\infty} A_n \in S' \therefore \in \mathcal{B}$: MET

b) i., ii, iii CAN BE SHOWN USING EXACT SAME AS PART (a). (FOR i, $\emptyset \in S'$)

c) $\mathcal{B}_1 \cap \mathcal{B}_2$: i. $\emptyset \in \mathcal{B}_1$, $\emptyset \in \mathcal{B}_2$ ∴ $\emptyset \in \mathcal{B}_1 \cap \mathcal{B}_2$ KNOWN

ii. LET $A \in \mathcal{B}_1 \cap \mathcal{B}_2 \therefore A \in \mathcal{B}_1$ AND $A \in \mathcal{B}_2 \therefore A^c \in \mathcal{B}_1$ AND $A^c \in \mathcal{B}_2 \therefore A^c \in \mathcal{B}_1 \cap \mathcal{B}_2$

iii. SAME AS ABOVE

5. $\mathcal{B}_1 \cup \mathcal{B}_2$: i. $\emptyset \in \mathcal{B}_1$, $\emptyset \in \mathcal{B}_2 \therefore \emptyset \in \mathcal{B}_1 \cup \mathcal{B}_2$ ✓

NO ii. LET $A \in B_1 \cup B_2 \therefore A \in B_1$ OR $A \in B_2$ DOES NOT IMPLY $A \in B_1 \cup B_2$

6. CB 1.4

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

b) $P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$

c) $P((A^c \cap B^c)^c) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

d) $P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$

7. a) $P(B) = P(B \cap A) + P(B \cap A^c) \therefore \quad P(B \cap A^c) = P(B) - P(B \cap A)$

b) $P(A \cup B) = P(A \cap B) + P(A \cap B^c) + P(B \cap A^c) = P(A \cap B) + (P(A) - P(A \cap B)) + (P(B) - P(A \cap B))$

$= P(A \cap B) + P(A) + P(B) - 2P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

c) SINCE $A \subset B$, $P(B) = P(B \cap A) + P(B \cap A^c) \therefore P(B \cap A) = P(A) = P(B) - P(B \cap A^c) \therefore P(A) \le P(B)$

DEPENDS ON $P(B \cap A^c)$

8. CB 1.13

$P(A) = \frac{1}{3}$, $P(B^c) = \frac{1}{4}$

SINCE $P(B^c) < P(A)$, THERE IS NO WAY FOR A TO BE CONTAINED ENTIRELY IN B^c

NOT DISJOINT

9. CB 1.7

WALL AREA: A

a) $P(i \text{ POINTS} \mid \text{BOARD IS HIT}) = \frac{(6-i)^2 - (5-i)^2}{5^2}$ $\quad \therefore P(i \text{ POINTS}) = \frac{\pi r^2}{A} \cdot \frac{(6-i)^2 - (5-i)^2}{5^2}$

$P(\text{BOARD IS HIT}) = \frac{\pi r^2}{A}$

b) I mean, I assumed that for part A...not sure what I need to show. It's basically just a scalar transform of the original form (Assuming A and r are constant), thus the conditional probability remains the same

10. Breakouts:

Think of each minute slot as a 'bucket', and the bankers are balls to put in the buckets

Each ball could be first (2 possbilities)

What's the chance that the second banker is in a bucket within 10 to the right?

- Ball1 has a 1/60 chance of going in any bucket
  - For Ball1, for any bucket 1-50 it goes in, there are 10 possible buckets that are a success(ie. Ball 2 is in the 10 buckets to the right) (10/60) = 1/6
  - For Ball1, if it goes in bucket 51 there are 9 possible options that are a success: 9/60
  - For Ball1, if it goes in bucket 52, there are 8 possible options that are a success: 8/60...
- 50/60 *1/6 + 1/60 * (9/60) + 1/60*(8/60)+....= 5/36 + 10/60* (9+8+7+...)/60) = 5/36 + 1/6*(45/60) = 0.2639

Now, multiply by 2:

.2639 * 2 = 0.5278

11. CB 1.18 Big thanks to Nathan and Brett to thinking this through with me!

Since we have n balls to n cells, if we are looking for exactly 1 empty cell, that means that we have 1 cell with 2 balls. So, we have n possible cells that have no balls, and (n-1) possible cells that have 2 balls. Giving n(n-1) possibliites (but we also need to divide by 2 because it doesn't matter the order of the 2

Giving n(n-1) possibilites (but we also need to divide by 2 because it doesn't matter the order of the 2 balls in the cell with 2 balls, giving n(n-1)/2

For the remaining n balls going into n cells (only 1 ball per cell), there are n! ways to partition them out

We have n distinct balls going to n distinct cells, so total possible combinations is n^n

THIS GIVES TOTAL POSSIBILITES OF $\dfrac{\frac{n(n-1)}{2} n!}{n^n} = \dfrac{\binom{n}{2} n!}{n^n}$

12. CB 1.22

a) TOTAL DRAWS: $\binom{366}{180}$

Evenly distributed I'm assuming to mean each day has 180/12 = 15 days picked (as opposed to proportional to days in month). Thus, Jan has (31 choose 15) options, Feb has (29 choose 15), etc. 7months have 31 days, 4 months have 30 days, and 1 month has 29 days. So, total math is:

$$\frac{7\binom{31}{15} \times 4\binom{30}{15} \times \binom{29}{15}}{\binom{366}{180}} = .167 \times 10^{-8}$$

b) Removing the 30 days from September means we have 336 remaining days available to pick for the first 30 days (ie. 336 choose 366)

$$\therefore \frac{\binom{336}{30}}{\binom{366}{30}}$$

13. Partitions:
Pick first number (13) (3 of a kind)
Pick second number (12)
Pick suits for first number (4 choose 3)
Pick suits for second number (4 choose 2)

Total possibilites = 52 choose 5

Final equation is: $\dfrac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}} = \dfrac{13 \cdot 12 \cdot 4 \cdot 6}{2598960} = 0.00144$

14. CB 1.34

$P(B|1) = \frac{2}{3}$ $P(G|1) = \frac{1}{3}$ $P(B|2) = \frac{3}{5}$ $P(G|2) = \frac{2}{5}$

a) $P(B) = P(B|1)P(1) + P(B|2)P(2) = \frac{2}{3}\left(\frac{1}{2}\right) + \frac{3}{5}\left(\frac{1}{2}\right) = \frac{1}{3} + \frac{3}{10} = .633$ ie. $\frac{19}{30}$

b) $P(1|B) = \dfrac{P(1 \cap B)}{P(B)} = \dfrac{P(B|1)P(1)}{P(B)} = \dfrac{\frac{2}{3}\left(\frac{1}{2}\right)}{19/30} = \frac{1}{3} \cdot \frac{30}{19} = \frac{10}{19}$

15. CB 1.35

i. $P(\cdot|B) \geq 0$ WE KNOW $P(B) > 0$, THUS $P(\cdot|B) = \dfrac{P(\cdot \cap B)}{P(B)} \geq 0$ ← INTERSECTIONS CAN'T BE NEGATIVE
-> > 0, CAN'T BE NEGATIVE

ii. P(AT LEAST ONE EVENT) = 1 TOTAL RANGE IS $P(\cdot|B) + P(\cdot^c|B) = 1$, AND BOTH ARE NON-ZERO (See i)
THUS THIS IS MET

iii. $P(\bigcup A_n) = \sum P(A_n)$ $P(\cdot|B)$ IS A SUBSET OF $P(\cdot)$, THUS ESTABLISHED ADDITIVITY FOR $P(\cdot)$ ALSO APPLIES TO $P(\cdot|B)$

16. CB 1.36

i. This is easier to approach it as 1 - Probability of Only 1 Hit - Probability of no hits:
Probability of only 1 hit: Hit: 1/5, 9 misses at (4/5), 10 possible ways to do this:

- 1/5 * (4/5)^9 * 10

Probability of no hits: (4/5)^10
Thus:

$$P(\geq 2 \text{ HITS}) = 1 - \frac{10}{5}\left(\frac{4}{5}\right)^9 - \left(\frac{4}{5}\right)^{10} = \underline{.624}$$

ii Probability that it is hit at least twice, given it's hit once, we can use the same strategy as above, where it's just, of the 9 remaining shots, do 1 - Probability that it's not hit any more times
Probability no more hits: (4/5)^9
Thus:

$$P(\geq 2 \text{ HITS} \mid 1 \text{ HIT}) = 1 - \left(\frac{4}{5}\right)^9 = \underline{.893}$$

17. CB 1.37

PARDONED	PROB	W SAYS	PROB	TOTAL PROB
B	1/3	C	1	1/3
C	1/3	B	1	1/3
A	1/3	C	$1-\gamma$	$\frac{1-\gamma}{3}$
A	1/3	B	γ	$\gamma/3$

a) $$P(A \mid W) = \frac{P(A \cap W)}{P(W)} = \frac{\gamma/3}{1/3 + \frac{\gamma}{3}} = \frac{\gamma/3}{\frac{\gamma+1}{3}} = \frac{\gamma}{\gamma+1}$$

IF $\gamma = \frac{1}{2}$, THEN $P(A \mid W) = \frac{1/2}{1/2+1} = \frac{1/2}{3/2} = \frac{1}{3}$ $\therefore$ $\underline{P(A \mid W) = \frac{1}{3} \text{ IF } \gamma = \frac{1}{2}}$

IF $\gamma > \frac{1}{2}$ (IE. $2/3$), $P(A \mid W) = \frac{2/3}{2/3+1} = \frac{2/3}{5/3} = \frac{2}{5}$ $\therefore$ $\underline{P(A \mid W) > \frac{1}{3} \text{ IF } \gamma > 1/2}$

IF $\gamma < \frac{1}{2}$ $P(A \mid W) = \frac{1/3}{1/3+1} = \frac{1/3}{4/3} = 1/4$ $\therefore$ $\underline{P(A \mid W) < \frac{1}{2} \text{ IF } \gamma < \frac{1}{2}}$

b) In this case we are basically just looking for P(C|W), because that is the risk/probability that A will now be assuming. That value is 1 - P(A|W) (which as shown previously) is 1-1/3, or <u>2/3</u>