HW7

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1. CB 4.32

a. $Y|A \sim \text{Pois}(A)$ $A \sim \text{Gamma}(\alpha, \beta)$ Find Y, $E(Y)$, $Var(y)$, and $Y \sim \text{Neg-Binom} \mid \alpha = 1,2,3 \dots$

$$f_Y = \int_0^\infty f_Y(y|a) f_A(a)\,da = \int_0^\infty \frac{a^y e^{-a}}{y!} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} a^{\alpha-1} e^{-a/\beta}\,da = \frac{1}{y!\,\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty a^{(\alpha+y)-1} e^{-a(1+1/\beta)}\,da}_{\text{KERNEL OF GAMMA: } \alpha = \alpha+y,\ \beta = \frac{1}{1+1/\beta}}$$

$$f_Y = \frac{\Gamma(\alpha+y)\left(\frac{1}{1+1/\beta}\right)^{\alpha+y}}{y!\,\Gamma(\alpha)\,\beta^\alpha}$$

$$E_Y = E_A\left(E_{Y|A}(Y|A)\right) = E_A(A) = \alpha\beta \qquad E_Y = \alpha\beta$$

$$Var_Y = E(Var(Y|A)) + Var(E(Y|A)) = E(A) + Var(A) = \alpha\beta + \alpha\beta^2 = Var_Y$$

If α is integer, $f_Y = \frac{(\alpha+y)!}{y!\,\alpha!}\left(\frac{\beta}{1+\beta}\right)^{\alpha+y} \cdot \frac{1}{\beta}^\alpha = \binom{y+\alpha-1}{y} \cdot \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha = \text{NegBinom}\left(\alpha, \frac{1}{1+\beta}\right)$

b. $Y|N \sim \text{Binom}(N,p)$ $N|A \sim \text{Pois}(A)$ $A \sim \text{Gamma}(\alpha,\beta)$

$$f_{Y|N} = P(Y=y|\lambda) = P(Y=y|N=n,\lambda)\,P(N=n|\lambda) = \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum \frac{1}{y!(n-y)!}\left(\frac{p}{1-p}\right)^y ((1-p)\lambda)^n e^{-\lambda} = \frac{e^{-\lambda}}{y!}\left(\frac{p}{1-p}\right)^y \sum_{m=0}^{\infty} \frac{((1-p)\lambda)^{m+y}}{m!} = \frac{e^{-\lambda}}{y!}\left(\frac{p(1-p)\lambda}{1-p}\right)^y \sum_{m=0}^{\infty}\frac{((1-p)\lambda)^m}{m!}$$

$$= \frac{e^{-\lambda}}{y!}(p\lambda)^y e^{(1-p)\lambda} = \frac{(p\lambda)^y e^{-(p\lambda)}}{y!} = \text{Pois}(p\lambda)$$

$\therefore$ $f_{Y|N} \sim \text{Pois}$ which is the same start as (a), but with $p\beta$ instead of β:

$$f_Y = \frac{(\alpha+y)!}{y!\,\Gamma(\alpha)(p\beta)^\alpha}\left(\frac{p\beta}{1+p\beta}\right)^{\alpha+y}$$

2. 4.35 $X|P \sim \text{Binom}(P)$ $P \sim \text{Beta}(\alpha,\beta)$

a. $Var(X) = E(Var(X|P)) + Var(E(X|P)) = E(nP(1-P)) + Var(nP) = n\,E(P(1-P)) + n(n-1)\,Var(P)$

$$= n\,E(P)(1-E(P)) + n(n-1)\,Var(P) = Var(X)$$

b. $Y|A = \text{Poisson}(A)$ $A \sim \text{Gamma}(\alpha,\beta)$

As shown in 4.32 (a), $Var(Y) = \alpha\beta + \alpha\beta^2$ $E(A) = \alpha\beta \Rightarrow Var(Y) = E(A) + \frac{1}{\alpha}E(A)^2 = \mu + \frac{1}{\alpha}\mu^2$ where $\mu = E(A)$

3. X, Y have finite variances. Prove:

a) $E(XY) = E(X\,E(Y|X))$ $\quad E(xy) = \iint xy\, f_{xy}\, dx\, dy = \iint xy\, f_{y|x} f_x\, dx\, dy$

$$= \int x f_x \left[\int y\, f_{y|x}\, dy\right] dx = \int x f_x\, E(y|x)\, dx = E[X\,E(Y|X)]$$

b) $Cov(X,Y) = Cov(X, E(Y|X))$ $\quad Cov(x,Y) = E(xy) - E(x)E(y)$

$$Cov(X, E(Y|X)) = E(X\,E(Y|X)) - E(x)\,E(E(Y|X)) = E(xy) - E(x)E(y)$$

c) $X,\ Y - E(Y|X)$ ARE UNCORRELATED: $\quad Cov(X, Y - E(Y|X)) = E(xy - xE(y|x)) - E(x)\,E(Y - E(Y|X))$

$$= E(xy) - E(xy) - E(x)E(y) + E(x)E(E(y|x)) = 0 - E(x)E(y) + E(x)E(y) = 0$$

4. FIND $E(X)$

a) $X|Y \sim BINOM(n, Y)$ $\quad Y \sim BETA(a,b)$ $\quad E(X) = E_Y(E_{X|Y}(X|Y)) = E_Y(nY) = nE_Y(Y) = \frac{na}{a+b}$

b) $X|Y \sim EXP(Y)$ $\quad Y \sim GAMMA(a,b)$ $\quad E(X) = E_Y(E_{X|Y}(X|Y)) = E_Y(Y) = ab$

5. FIND $VAR(X)$ FOR 4(a)

$$VAR(X) = E(VAR(X|Y)) + VAR(E(X|Y)) = E(nY(1-Y)) + VAR(nY) = nE(Y - Y^2) + n(n-1)\,VAR(Y)$$

$$= \frac{na}{a+b} + \frac{n(n-1)ab}{(a+b)^2(a+b+1)} - nE(Y^2)$$

$\hookrightarrow = VAR(Y) + E(Y)^2 = \frac{a(a+1)}{(a+b)(a+b+1)}$

$$= \frac{na}{a+b}\left[1 + \frac{(n-1)b}{(a+b)(a+b+1)} - \frac{a+1}{a+b+1}\right]$$

6. $P(Y=1) = .3$ $\quad P(Y=2) = .7$ $\quad X|Y = POISSON(Y)$ $\quad$ FIND $M_X(t)$

$X \sim .3\,POIS(1) + .7\,POIS(2)$ $\quad M_X(t) = M_{POIS(1)}(.3t) \cdot M_{POIS(2)}(.7t) = e^{(e^{.3t}-1)} \cdot e^{2(e^{.7t}-1)}$

$$= e^{e^{.3t} + 2e^{.7t} - 3} = M_X(t)$$

7. CB 4.41 SHOW ANY R.V. IS UNCORRELATED WITH A CONSTANT

NEED TO SHOW: $Cov(X,c) = 0$ $\quad Cov(X,c) = E(Xc) - E(X)E(c) = cE(X) - cE(X) = 0 = Cov(x,c)$

$\therefore Corr = \frac{Cov}{\sigma\sigma} = 0$

8. CB 4.42 $\quad X \perp\!\!\!\perp Y,\ \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ $\quad$ EXPRESS $CORR(XY, Y)$ IN THOSE TERMS

$Corr(XY,Y) = \frac{Cov(XY,Y)}{\sigma\ \sigma}$ $\quad Cov(XY,Y) = E(XYY) - E(XY)E(Y)$

$\sigma_{xy}\ \sigma_Y$

$Var(XY) = E(X^2Y^2) - E(XY)^2 = E(X^2)E(Y^2) - E(X)^2E(Y)^2 = (Var(X) + \mu_X^2)(Var(Y) + \mu_Y^2) - \mu_X^2\mu_Y^2$

$E(X^2) = Var(X) + E(X)^2$

$Var(XY) = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y + \sigma_Y^2\mu_X + \mu_X^2\mu_Y^2 - \mu_X^2\mu_Y^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y + \sigma_Y^2\mu_X$

$E(XY) = E(X)E(Y) = \mu_X\mu_Y \quad \cdot E(Y) = \mu_X\mu_Y^2$

$E(XY^2) = E(X)E(Y^2) = E(X)(Var(Y) + E(Y)^2) = \mu_X(\sigma_Y^2 + \mu_Y^2)$

$Cov(XY, Y) = \mu_X\sigma_Y^2 + \mu_X\mu_Y^2 - \mu_X\mu_Y^2 = \mu_X\sigma_Y^2$

$$Corr(XY, Y) = \frac{\mu_X\sigma_Y^2}{\sigma_Y(\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y + \sigma_Y^2\mu_X)^{1/2}} = \frac{\mu_X\sigma_Y}{(\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y + \sigma_Y^2\mu_X)^{1/2}}$$

9. CB 4.43 X_1, X_2, X_3 UNCORRELATED, μ, σ^2 FIND $Cov(X_1 + X_2, X_2 + X_3)$; $Cov(X_1 + X_2, X_1 - X_2)$

a. $Cov(X_1 + X_2, X_2 + X_3) = E[(X_1 + X_2)(X_2 + X_3)] - E(X_1 + X_2)E(X_2 + X_3) = E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) - (E(X_1) + E(X_2))(E(X_2) + E(X_3))$

$= E(X_1X_2) + E(X_1X_3) + E(X_2^2) + E(X_2X_3) - (\mu + \mu)(\mu + \mu) = E(X_1)E(X_2) + E(X_1)E(X_3) + Var(X_2) + E(X_2)^2 + E(X_2)E(X_3) - 4\mu^2$

$= \mu^2 + \mu^2 + \sigma^2 + \mu^2 + \mu^2 - 4\mu^2 = \sigma^2$

$Cov(X_1 + X_2, X_2 + X_3) = \sigma^2$

b. $Cov(X_1 + X_2, X_1 - X_2) = E(X_1^2 - X_1X_2 + X_1X_2 - X_2^2) - (E[X_1] + E[X_2])(E[X_1] - E[X_2]) = E(X_1^2) - E(X_2^2) - (\mu + \mu)(\mu - \mu)$

$= Var(X_1) + E(X_1)^2 - Var(X_2) - E(X_2)^2 - (2\mu)(0) = \sigma^2 + \mu^2 - \sigma^2 - \mu^2 - 0 = 0$

$Cov(X_1 + X_2, X_1 - X_2) = 0$

10. CB 4.65 PROVE COV. INEQUALITY BY GENERALIZING: $E(g(X)(X - \mu)) \geq 0$

PROVE: LET $X \perp\!\!\!\perp Y$ $\quad q = (g(x) - g(y))(h(x) - h(y))$

a) FOR $g(x)$ MONO + ; $h(x)$ MONO - : $E(g(X)h(X)) \leq E(g(X))E(h(X))$

$E(q) = E[(g(x) - g(y))(h(x) - h(y))] = [E(g(x)) - E(g(y))][E(h(x)) - E(h(y))]$

WHEN $Y = X$: $E(q) = 0$

$Y > X$: $E(g(y)) \geq E(g(x))$, $E(h(y)) \leq E(h(x))$: $E(q) = [\text{NEGATIVE}][\text{POSITIVE}] \therefore E(q) \leq 0$

$Y < X$: $E(g(y)) \leq E(g(x))$, $E(h(y)) \geq E(h(x))$: $E(q) = [\text{POSITIVE}][\text{NEGATIVE}] \therefore E(q) \leq 0$

$E(q) \leq 0 \therefore E[(g(x) - g(y))(h(x) - h(y))] \leq 0 \quad E[g(x)h(x) - g(x)h(y) - g(y)h(x) + g(y)h(y)] \leq 0$

$E[g(x)h(x)] \le E[g(x)h(y)] + E[g(y)h(x)] - E[g(y)h(y)] = E(g(x))E(h(x))$

$\therefore \; \underline{E[g(x)h(x)] \le E[g(x)]\, E[h(x)]}$

b) FOR $g(x), h(x)$ BOTH MONO + OR $-$: $E(g(x)h(x)) \ge E(g(x))\,E(h(x))$

WHEN $Y = X$: $E(q) = 0$

MONO+: $Y > X$: $E(g(y)) \ge E(g(x))$, $E(h(y)) \ge E(h(x))$: $E(q) = $ [NEGATIVE][NEGATIVE] $\therefore E(q) \ge 0$

$Y < X$: $E(g(y)) \le E(g(x))$, $E(h(y)) \le E(h(x))$: $E(q) = $ [POSITIVE][POSITIVE] $\therefore E(q) \ge 0$

MONO $-$: $Y < X$: $E\,g(y) \ge E(g(x))$, $E(h(y)) \ge E(h(x))$: $E(q) = $ [NEGATIVE][NEGATIVE] $\therefore E(q) \ge 0$

$Y > X$: $E(g(y)) \le E(g(x))$, $E(h(y)) \le E(h(x))$: $E(q = $ POSITIVE][POSITIVE $\therefore E(q) \ge 0$

WE CAN DO THE SAME EXPECTATION MOVEMENT FROM (a) TO SHOW

$\underline{E[g(x)h(x)] \ge E[g(x)]\,E[h(x)]}$