HW3

Thursday, September 25, 2025 8:39 AM

1. CB 2.14 X cont., ≥ 0 Show $E(X) = \int_0^\infty 1 - F(x)\,dx$

a. $E(X) = \int_0^\infty x f(x)\,dx \Rightarrow \begin{matrix} u = x \\ dv = f(x)\,dx \end{matrix} \Rightarrow \left[x F(x)\right]_0^\infty - \int_0^\infty F(x)\,dx$

goes to 0 at $F(0)$, need to deal with $x \to \infty$

I got help on this part, recognizing we need to add a constant to the $\int f(x)\,dx$:

$\begin{matrix} u = x \\ dv = f(x)\,dx \end{matrix} \Rightarrow \begin{matrix} du = 1 \\ V = F(x) - 1 \end{matrix} \Rightarrow \left[x\left(F(x) - 1\right)\right]_0^\infty - \int_0^\infty F(x)^{-1}\,dx = \lim_{x \to \infty} x\left(F(x) - 1\right) - 0\left(F(0) - 1\right) - \int_0^\infty F(x)^{-1}\,dx$

$\Rightarrow \lim_{x \to \infty} F(x) = 1 \;\therefore\; \lim_{x \to \infty} x\left(F(x) - 1\right) = \lim_{x \to \infty} x(1 - 1) = 0 \Rightarrow = 0 - 0 - \int_0^\infty F(x) - 1\,dx = \int_0^\infty 1 - F(x)\,dx$

b. Show $E(X) = \sum_{k=0}^{\infty}\left(1 - F_X(k)\right)$

$\sum_{k=0}^{\infty}\left(1 - F(k)\right) = \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty} P(X \leq k) = \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty}\sum_{i=0}^{k} P(X = i) = \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty}\sum_{i=0}^{k} f_X(k)$

$P(X \leq k) = P(X = 0) + P(X = 1) + \ldots + P(X = k - 1) + P(X = k) = \sum_{i=0}^{k} P(X = i)$

$= \sum_{k=0}^{\infty}\left(1 - \sum_{i=0}^{k} f_X(k)\right) = \sum_{k=0}^{\infty} k\left(\frac{1}{k} - \sum_{i=0}^{k}\frac{f(k)}{k}\right) = \sum_{k=0}^{\infty} k f_X(k)$

2. CB 2.17

a. $\int_0^m 3x^2\,dx = \frac{1}{2} \Rightarrow x^3\Big|_0^m = \frac{1}{2}$ $\quad m^3 - 0^3 = \frac{1}{2}$ $\quad m^3 = \frac{1}{2}$ $\quad m = \left(\frac{1}{2}\right)^{(1/3)}$

b. $\int_m^\infty \frac{1}{\pi(1 + x^2)}\,dx = \frac{1}{2} = \frac{1}{\pi}\int_m^\infty \frac{1}{1 + x^2}\,dx = \frac{1}{2} \Rightarrow \frac{1}{\pi}\operatorname{atan}(x)\Big|_m^\infty = \frac{1}{\pi}\left(\operatorname{atan}(\infty) - \operatorname{atan}(m)\right) = \frac{1}{2}$

$\Rightarrow \frac{\pi}{2} = \frac{\pi}{2} - \operatorname{atan}(m)$ $\quad 0 = \operatorname{atan}(m)$ $\quad m = \tan(0)$ $\quad m = 0$

3. CB 2.20 $f_X = (1-p)^{x-1} p$ $\quad E(X) = \int x(1-p)^{x-1} p\,dx = \begin{matrix} u = x & du = 1 \\ dv = (1-p)^{x-1} & v = \frac{(1-p)^{x-1}}{\ln(1-p)} \end{matrix} = p\left[\frac{x(1-p)^{x-1}}{\ln(1-p)} - \int \frac{1}{\ln(1-p)}(1-p)^{x-1}\,dx\right.$

$= p\left[\frac{x(1-p)^{x-1}}{\ln(1-p)}\right]_0^\infty - p\left[\frac{(1-p)^{x-1}}{\ln(1-p)^2}\right]\Bigg|_0^\infty = \frac{1}{p}$ $\quad E(X^2) = p\int x^2(1-p)^{x-1} \Rightarrow p\left[\frac{x^2(1-p)^{x-1}}{\ln(1-p)} - \int 2x(1-p)^{x-1}\right] =$

$p\left[\frac{x^2(1-p)^{x-1}}{\ln(1-p)} - \frac{2x(1-p)^{x-1}}{\ln(1-p)^2} + \frac{2(1-p)^{x-1}}{\ln(1-p)^3}\right] = \frac{1-p}{p^2} + \frac{1}{p^2}$ $\quad \operatorname{Var}(X) = \frac{1}{p^2} - \frac{1}{p} + \frac{1}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$

4. CB 2.24 Find $E(X)$ & $\operatorname{Var}(X)$

a. $f(x) = a x^{a-1}$ $\quad 0 < x < 1,\ a > 0$

$E(X) = \int_0^1 x\left(a x^{a-1}\right)dx = \int_0^1 a x^a\,dx = \frac{a}{a+1} x^{a+1}\Big|_0^1 = \frac{a}{a+1}$

$\operatorname{Var}(X) = E(X^2) - E(X)^2 = \int_0^1 x^2\left(a x^{a-1}\right)dx - \left(\frac{a}{a+1}\right)^2 = \int_0^1 a x^{a+1}\,dx - \left(\frac{a}{a+1}\right)^2 = \frac{a}{a+2} x^{a+2}\Big|_0^1 - \left(\frac{a}{a+1}\right)^2$

$= \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$

b. $f(x) = \frac{1}{n}$ $\quad x = 1, 2, \ldots, n$, $\quad n > 0$

$E(X) = \sum_{x=1}^{n} x f(x) = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n}{n} = \underbrace{\frac{1}{n} + \frac{n-1}{n}}_{1} + \underbrace{\frac{2}{n} + \frac{n-2}{n}}_{1} + \ldots + \underbrace{\frac{n}{n}}_{1} = \frac{n+1}{2}$

there will be $\frac{n+1}{2}$ of these

$1 + 4 + 9 + \ldots + n^2$ is sum of squares $= \frac{n(n+1)(2n+1)}{6}$

$\operatorname{Var}(X) = E(X^2) - E(X)^2 = \sum_{x=1}^{n} x^2 f(x) - \left(\frac{n+1}{2}\right)^2 = \frac{1}{n} + \frac{4}{n} + \frac{9}{n} + \ldots + \frac{n^2}{n} - \left(\frac{n+1}{2}\right)^2$

$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 = \frac{n+1(2n+1)}{6} - \frac{(n+1)^2}{4} = (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right] = (n+1)\left[\frac{n}{3} - \frac{n}{4} + \frac{1}{6} - \frac{1}{4}\right]$

$= (n+1)\left[\frac{n}{12} - \frac{1}{12}\right] = \frac{(n+1)(n-1)}{12} = \frac{n^2 - 1}{12}$

c. $f(x) = \frac{3}{2}(x-1)^2$ $\quad 0 < x < 2$

$E(X) = \int_0^2 x\left(\frac{3}{2}(x-1)^2\right)dx = \frac{3}{2}\int_0^2 x\left(x^2 - 2x + 1\right)dx = \frac{3}{2}\int_0^2 x^3 - 2x^2 + x\,dx = \frac{3}{2}\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\Big|_0^2\right] \frac{16}{4} - \frac{16}{3} + 2 - (0 - 0 + 0)$

$3/2 \cdot\ 4 + 2 - \frac{16}{3} = 6 - \frac{16}{3} = \frac{3}{2}\left[\frac{18}{3} - \frac{16}{3}\right] = \frac{2}{3} \cdot \frac{3}{2} = 1$ $\quad \frac{3}{2}\int_0^2 x^4 - 2x^3 - x^2\,dx - 1 = \frac{3}{2}\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]\Big|_0^2 - 1$

$$Var(X) = E(X^2) - E(X)^2 = \frac{3}{2}\int_0^2 x^2(x^2-2x+1)\,dx - (1)^2 = \frac{3}{2}\int_0 x - \dots$$

$$= \frac{3}{2}\left[\frac{32}{5} - 8 + \frac{8}{3}\right] - 1 = \frac{3}{2}\left[\frac{96}{15} + \frac{40}{15}\right] - 13 = \frac{3}{2}\left(\frac{136}{15}\right) - 13 = \frac{408}{30} - \frac{390}{30} = \frac{18}{30} = \frac{6}{10}$$

5. $X \sim \text{GAMMA}(1/2, \beta)$

a. $$E(X^{1/2}) = \int_0^\infty \frac{x^{1/2}}{\Gamma(1/2)\beta^{1/2}} x^{-1/2} e^{-x/\beta}\,dx = \frac{1}{\sqrt{\pi\beta}}\int_0^\infty e^{-x/\beta}\,dx = \frac{1}{\sqrt{\pi\beta}}\left[-\beta e^{-x/\beta}\right]_0^\infty = \frac{1}{\sqrt{\pi\beta}}\left[\frac{\beta}{e^0} - \frac{\beta}{e^\infty}\right] = \sqrt{\frac{\beta}{\pi}}$$

$E(X^{1/2}) = \sqrt{\frac{\beta}{\pi}}$ $\quad E(X) = \frac{\beta}{2}$ $\quad E(X)^{1/2} = \sqrt{\frac{\beta}{2}}$ $\quad E(X)^{1/2} > E(X^{1/2})$

b. x^2 IS CONVEX. $\therefore$ BY JENSEN INEQUALITY, $E(X^2) \geq E(X)^2$

c. $E(\log(X))$ vs. $\log(E(X))$: Log(x) is concave. Thus, by Jensen's inequality: $E(\log(X)) \leq \log(E(X))$

6. $X \sim N(\mu, \sigma^2)$

$$E(e^X) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^x e^{\frac{-x^2+2\mu x-\mu^2}{2\sigma^2}}\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2} + x\left(\frac{\mu}{\sigma^2}+1\right) - \frac{\mu^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2} + \frac{(\mu+\sigma^2)^2}{2\sigma^2}}\,dx = \frac{e^{\left[\frac{(\mu+\sigma^2)^2}{2\sigma^2}\right]}}{1}\underbrace{\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}\right]}\,dx}_{\text{PDF OF } N(\mu+\sigma^2,\, \sigma^2)}$$

$$\therefore E(e^X) = e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}}$$

$$e^{E(X)} = e^\mu$$

$$E(e^X) \geq e^{E(X)}$$

7. $f(x) = 2x \quad 0 \leq x \leq 1$

a. $E(x) = \int x\,2x\,dx = \int_0^1 2x^2 = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

b. $E(x^2) = \int_0^1 x^2\,2x\,dx = \int_0^1 2x^3 = \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}$

$Var(x) = E[x^2] - E[x]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$

c. $E(4x+9) = \int_0^1 (4x+9)\,2x\,dx = \int_0^1 4x^2 + 18x\,dx = \frac{4}{3}x^3 + 9x^2\Big|_0^1 = \frac{4}{3} + 9 = \frac{31}{3}$

d. $E(x(1-x)) = \int_0^1 (x(1-x))\,2x\,dx = \int_0^1 (x-x^2)\,2x\,dx = \int_0^1 2x^2 - 2x^3\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^4\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

8. $X \sim B(\alpha, \beta) \quad 0 \leq x \leq 1 \quad \alpha, \beta > 0 \qquad B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$E(X) = \int_0^1 x\,x^{\alpha-1}(1-x)^{\beta-1}\,dx = \int_0^1 x^\alpha(1-x)^{\beta-1}\,dx = \beta(\alpha+1, \beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}$

WE KNOW $\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ $\quad \therefore \beta(\alpha+1, \beta) =$

WE KNOW $\Gamma(n) = (n-1)!$ $\quad \therefore$

$$\boxed{E(X) = \frac{\alpha!\,(\beta-1)!}{(\alpha+\beta)!}}$$

$E(X^2) = \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx = \beta(\alpha+2, \beta) = \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)} = \frac{(\alpha+1)!\,(\beta-1)!}{(\alpha+\beta+1)!}$

$$Var(X) = E(X^2) - E(X)^2 = \frac{(\alpha+1)!\,(\beta-1)!}{(\alpha+\beta+1)!} - \left[\frac{\alpha!\,(\beta-1)!}{(\alpha+\beta)!}\right]^2$$

$\dots = \frac{1}{ct}\left[e^{tc} - 1\right]$

9. CB 2.30 FIND MGFS

a. $f(x) = \frac{1}{c}, \quad 0 < x < c \qquad M_x = E(e^{tx}) = \int_0^c e^{tx}\frac{1}{c}\,dx = \frac{1}{c}\left[\frac{1}{t}e^{tx}\right]_0^c$

b. $f(x) = \frac{2x}{c^2}, \quad 0 < x < c \qquad M_x = \int_0^c e^{tx}\frac{2x}{c^2} \qquad \begin{matrix} u = x & du = 1 \\ dv = e^{tx} & v = \frac{1}{t}e^{tx}\end{matrix} \Rightarrow \frac{2}{c^2}\left[\frac{x}{t}e^{tx} - \int_0^c \frac{1}{t}e^{tx}\,dx\right] = \frac{2}{c^2}\left[\frac{x}{t}e^{tx} - \frac{1}{t^2}e^{tx}\right]\Big|_0^c$

$= \frac{2}{c^2}\left[\frac{c}{t}e^{tc} - \frac{1}{t^2}e^{tc} + \frac{1}{t^2}\right]$

$I\begin{cases} -1 & -\infty < x < \alpha \\ 1 & \alpha < x < \infty\end{cases}$

c. $f(x) = \frac{1}{2\beta}e^{-|x-\alpha|/\beta} \quad -\infty < x < \infty \qquad M_x = \frac{1}{2\beta}\int_{-\infty}^{\infty} e^{tx}e^{\frac{-|x-\alpha|}{\beta}}\,dx = \frac{1}{2\beta}\int_{-\infty}^{\infty} e^{\frac{-|x-\alpha|}{\beta}+tx}\,dx = \frac{-\frac{I}{\beta}+t}{2\beta}e^{\frac{-|x-\alpha|}{\beta}+tx}\Big|_{-\infty}^{\infty}$

$= \frac{\frac{1}{\beta}+t}{2\beta}e^{-\infty} - \frac{\frac{1}{\beta}+t}{2\beta}e^{t\alpha} - \left[\frac{-\frac{1}{\beta}+t}{2\beta}e^{t\alpha} - \frac{-\frac{1}{\beta}+t}{2\beta}e^{-\infty}\right] = \frac{\frac{1}{\beta}+t}{2\beta}\left[0 + e^{t\alpha} + e^{t\alpha} - 0\right] = \frac{2e^{t\alpha}\left(\frac{1}{\beta}+t\right)}{2\beta} = \frac{e^{t\alpha}}{1-(\beta t)^2}$

Full transparency: I'm very bad at series. As in, I barely learned them 9 years ago.
I used online sources to help walk me through this series.

d. $P(x) = \binom{r+x-1}{x}p^r(1-p)^x \quad x \geq 0, r > 0 \quad 0 < p < 1 \qquad M_x = \sum e^{tx}\binom{r+x-1}{x}p^r(1-p)^x$

$q = 1-p: \quad M_x = p^r\sum\binom{r+x-1}{x}(qe^t)^x$ = NEGATIVE BINOMIAL SERIES

$= p^r\frac{1}{(1-qe^t)^r} = \left[\frac{p}{(1-qe^t)}\right]^r$

$= \frac{p^r}{(1-(1-p)e^t)^r}$

10. CB 2.33

a. $p(x) = \frac{e^{-\lambda}\lambda^x}{x} \qquad M_x = e^{\lambda(e^t-1)} \qquad E(x) = \sum x\frac{e^{-\lambda}\lambda^x}{x} = e^{-\lambda}\lambda^x\Big|_{x\in X} = \lambda$

$= M'(0) = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda e^{\lambda(1-1)} = \lambda$

$E(x) = \lambda \qquad E(x^2) = M''(0) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t}e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda + \lambda^2$

$\mathrm{VAR}(x) = E(x^2) - E(x)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

b. $P(x) = p(1-p)^x \qquad M_x = \frac{p}{1-(1-p)e^t} \qquad E(x) = M'(0) = \frac{pe^t(1-p)}{(1-(1-p)e^t)^2}\Big|_{t=0} = \frac{p(1-p)}{(1-(1-p))^2} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$

$= \sum x p(1-p)^x = p(1-p) + 2p(1-p)^2 + \ldots = \frac{1-p}{p}$

$E(x) = \frac{1-p}{p} \qquad E(x^2) = M''(0) = \frac{p(1-p)e^t}{(1-(1-p)e^t)^2} + \frac{-2p(1-p)e^t\left[(1-p)e^t\right]}{(1-(1-p)e^t)^3}\Big|_{t=0} = \frac{p(1-p)}{p^2} - \frac{2p(1-p)(1-p)}{p^3} = \frac{1-p}{p} - \frac{2(1-p)^2}{p^2}$

$\mathrm{VAR}(x) = \left[\frac{1-p}{p} - \frac{2(1-p)^2}{p^2}\right] - \left[\frac{1-p}{p}\right]^2 = \frac{1-p}{p} - \frac{3(1-p)^2}{p^2}$

c. $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad M_x = e^{\mu t + \sigma^2t^2/2} \qquad E(x) = M'(0) = (\sigma^2t + \mu)e^{\mu t + \frac{\sigma^2t^2}{2}}\Big|_{t=0} = (\sigma^2(0) + \mu)e^0 = \mu$

$= \frac{1}{\sqrt{2\pi}\sigma}\int x e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \begin{matrix} u = x & du = 1 \\ dv = e^{-\frac{(x-\mu)^2}{2\sigma^2}} & v = \frac{1}{2}\ldots\end{matrix} = \mu$

$E(x) = \mu \qquad E(x^2) = M''(0) = \sigma^2 e^{\mu t + \frac{\sigma^2t^2}{2}} + (\sigma^2 t + \mu)^2 e^{\mu t + \frac{\sigma^2t^2}{2}}\Big|_{t=0} = \sigma^2 + \mu^2$

$\mathrm{VAR}(x) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

11. CB 2.38

a. FIND MGF:

$\mathrm{MGF} = \frac{p^r}{[1-(1-p)e^t]^r}$

b. $Y = 2pX \Rightarrow \lim_{p\to 0} M_Y = \left(\frac{1}{1-2t}\right)^r \quad |t| \leq \frac{1}{2} \qquad M_Y = E\left[e^{2ptx}\right] = \frac{p^r}{(1-(1-p)e^{2pt})^r}$

$M_Y = \frac{p^r}{[1-(1-p)e^{2pt}]^r} \qquad \lim_{p\to 0} M_Y \Rightarrow \left[\frac{p}{(1-2t)p}\right]^r = \left(\frac{1}{1-2t}\right)^r$

12. $Z \sim N(0,1) \quad M_Z = e^{t^2/2} \qquad X = \mu + \sigma Z \qquad X \sim N(\mu, \sigma^2)$

a. $M_X = M_{\mu+\sigma Z} = E[e^{tX}] = E\left[e^{t(\mu+\sigma Z)}\right] = E\left[e^{\mu t}e^{\sigma Z t}\right] = e^{\mu t}E\left[e^{\sigma Z t}\right] = e^{\mu t}e^{(\sigma t)^2/2}$

$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$

b. $Y = \frac{X}{n}$ $M_Y = E[e^{tX/n}] = e^{\frac{\mu}{n} t + \frac{1}{2}\left(\frac{\sigma t}{\sqrt{n}}\right)^2}$

13. $f(x) = \begin{cases} 1-p & x=0 \\ pe^{-x} & x>0 \\ 0 & \end{cases}$ $0<p<1$

a. $E(X) = (1-p)(0) + \int_0^\infty p e^{-x} dx = -pe^{-x}\Big|_0^\infty = 0 + p = p$

b. $M_X = E[e^{tX}] = 0 + p\int_0^\infty e^{tx-x} dx = p\int_0^\infty e^{x(t-1)} dx = \frac{p}{t-1} e^{(t-1)x}\Big|_0^\infty$ NO VALUE WHEN $t>1$ WHEN $t<1$, $x \to \infty \Rightarrow 0$

$= -\frac{p}{t-1} e^0$

$M_X = -\frac{p}{t-1}$