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$\nu : \mathcal{B} \to \mathbb{R}$ IS A PROB. FUNC.

DECOMPOSE ν INTO: $\nu = \nu_{cont.} + \underset{\underset{\text{DISCRETE}}{\uparrow}}{\nu_{PP}} + \underset{\underset{\text{SINGULAR / MEASURE ZERO}}{\uparrow}}{\nu_{SING}}$

$\nu_{cont} \left(\{ -\infty, x \} \right)$

$\nu_{PP} \left(\{ 1, 2, 3, ... \} \right)$

Ex. X IS A r.v. WHERE $P[x=0] = \frac{1}{4}$ $P(x=1) = \frac{1}{4}$

$P(x \in (a,b)) = \frac{b-a}{2}$ $0 < a < b < 1$

$P\left(x \notin [0,1] \right) = 0$

Kernel: Any non-negative function with a finite integral or sum can be made into a PDF or PMF

- $h(x) \geq 0 \; \forall x$

- $\int_{x \in X} h(x) dx = K \quad 0 < K < \infty$

- $f_x(x) = \frac{1}{K} h(x) I_x(x)$

Ex. $\int_{-\infty}^{\infty} \underbrace{e^{-1/2 x^2}}_{\text{KERNEL OF NORMAL}} dx = \sqrt{2\pi}$

COMMON PDFs / PMFs

DISCRETE	CONTINUOUS	
BERNOULLI	BETA	CAUCHY
BINOMIAL	χ^2	T-DIST.
UNIFORM	EXPONENTIAL	F
GEOMETRIC	GAMMA	NORMAL
HYPERGEOMETRIC	LOG NORMAL	WEIBULL
NEGATIVE BINOM.	PARETO	

- SURVIVAL FUNCTION: $S_X(x) = P(X > x) = 1 - F_X(x)$

- HAZARD FUNCTION: $H_X(x) = \dfrac{f_X(x)}{S_X(x)}$

- GAMMA FUNCTION: $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1} e^{-t}\, dt$

  IF α IS INT: $\Gamma(\alpha) = (\alpha-1)!$

  FOR GENERAL α: $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

EXPECTED VALUE OF A r.v. , $g(x)$

$$E[g(x)] = \begin{cases} \displaystyle\int_{-\infty}^\infty g(x)\, f_X(x)\, dx & \text{CONTINUOUS} \\[2em] \displaystyle\sum_{x \in X} g(x)\, f_X(x) & \text{DISCRETE} \end{cases}$$

- $E[g(x)] = \infty$, THEN E.V. DOES NOT EXIST

- NOTE: IS X IS AN R.V. AND $Y = g(X)$,

  $$E[g(x)] = \int_{-\infty}^\infty g(x)\, f_X(x)\, dx$$

  $$= E(Y) = \int_{-\infty}^\infty y\, f_Y(y)\, dy$$

Law of unconscious statistician

Ex. $Y \sim$ INVERSE GAMMA (α, β)

$$f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\, y^{-(\alpha+1)}\, e^{-\frac{1}{y\beta}} \qquad \begin{array}{l} y > 0 \\ \alpha > 0 \\ \beta > 0 \end{array}$$

$$E(Y) = \int_{-\infty}^\infty y\, f_Y(y)\, dy = \int_0^\infty y\, \frac{1}{\Gamma(\alpha)\beta^\alpha}\, y^{-(\alpha+1)}\, e^{-\frac{1}{y\beta}}\, dy$$

$$\underline{\qquad} \quad \int_0^\infty y^{-\alpha}\, e^{-\frac{1}{y\beta}}\, dy \qquad \dots\,?$$

$$= \Gamma(a)\beta^a \Big|_0^{?} \underbrace{\qquad\qquad}_{\text{KERNEL OF}}$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot \Gamma(\alpha-1)\beta^{\alpha-1}$$

$$= \frac{\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} \beta^{a-1-\alpha}$$

$$= \frac{1}{(\alpha-1)\beta}$$

Ex. $X \sim \text{GAMMA}(a,b)$, $Y = \frac{1}{X} \sim \text{INVERSE GAMMA}(\alpha,\beta)$

$$E(Y) = E\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx$$

$$= \int_0^{\infty} \frac{1}{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{(\alpha-1)\beta}$$

You can express probability as n expectation

$\underset{\text{CONT.}}{\text{LET}}$ X BE AN R.V.

$$P(X \in A) = \int_A f_X(x)\, dx \;=\; \int_{-\infty}^{\infty} I_A(x) f_X(x)\, dx \;=\; E\left[I_A(x)\right]$$

Derives idea behind Monte Carlo estimate of prob.

Properties of Expectations

Let X be an r.v with a,b,c constants. Assume E[g(x)], E[g1(x)], E[g2(x)] all exist.

1. $E(ax+b) = aE(x)+b$

  $E(a g_1(x) + b g_2(x)) = a E[g_1(x)] + b E[g_2(x)]$

2. IF $g(x) \geq 0 \;\forall\, x \in X$, THEN $E[g(x)] \geq 0$

3. IF $g_1(x) \geq g_2(x) \;\forall\, x \in X$, THEN $E[g_1(x)] \geq E[g_2(x)]$

$\cdots$ $\leq E[g(x)] \leq b$