

# HW10

Thursday, November 6, 2025 2:04 PM

1-3, 6-7, 10-13, 16, 18-19

1.  $E(x) = \mu$   $\text{Var}(x) = \sigma^2$  USE FIRST ORDER TAYLOR ABOUT  $\mu$ , FIND  $E(y)$ ,  $\text{Var}(y)$  FOR:

$$E[g(x)] \approx g(E[x]) + g'(E[x])(x - E[x]) \quad \text{Var}(g(x)) \approx [g'(E[x])]^2 \cdot \text{Var}(x)$$

a.  $y = e^x$   $g(x) = e^x$   $g'(x) = e^x$   $E(y) \approx e^\mu + e^\mu(x - \mu)$   $\text{Var}(y) \approx [e^\mu]^2 \cdot \sigma^2 = e^{2\mu} \sigma^2$

b.  $y = \frac{1}{x}$  ( $x \neq 0$ )  $g(x) = \frac{1}{x}$   $g'(x) = -\frac{1}{x^2}$   $E(y) \approx \frac{1}{\mu} - \frac{1}{\mu^2}(x - \mu)$   $\text{Var}(y) \approx \frac{\sigma^2}{\mu^4}$

c.  $y = \ln(x)$  ( $x > 0$ )  $g(x) = \ln x$   $g'(x) = \frac{1}{x}$   $E(y) \approx \ln(\mu) + \frac{x - \mu}{\mu}$   $\text{Var}(y) \approx \left(\frac{\sigma}{\mu}\right)^2$

2.  $X \sim \text{UNIF}(0, 1)$   $Y_n \sim \text{Exp}\left(\frac{1}{n}\right)$   $X_n = X + Y_n$  Show  $X_n \xrightarrow{L^2} X$

NEED TO SHOW  $\lim_{n \rightarrow \infty} E(|X_n - X|^2) = 0$   $E(|X_n - X|^2) = E(|X_n - X|)^2 - \text{Var}(|X_n - X|)$

$$\begin{aligned} \lim_{n \rightarrow \infty} 0 &= 0 \checkmark \\ &= E(|Y_n|)^2 - \text{Var}(|Y_n|) = E(Y_n)^2 - \text{Var}(Y_n) \\ &= \left(\frac{1}{n}\right)^2 - \left(\frac{1}{n}\right)^2 = 0 \end{aligned}$$

$\therefore X_n \xrightarrow{L^2} X$

3.  $X_n = \begin{cases} 0, & p = \frac{n-1}{n} \\ \sqrt{n}, & p = \frac{1}{n} \end{cases}$  WHAT DOES  $X_n$  CONVERGE TO? I PROPOSE CONVERGES TO 0

a.  $X_n \xrightarrow{D} 0$   $\lim_{n \rightarrow \infty} X_n \rightarrow 0$  WHEN  $n$  IS LARGE,  $n \gg X_n = \begin{cases} 0, & p = \frac{Bx_n-1}{Bx_n} = 1 \\ \sqrt{Bx_n}, & p = 0 \end{cases} = 0 \therefore X_n \xrightarrow{D} 0$

b.  $X_n \xrightarrow{P} 0$   $\lim_{n \rightarrow \infty} P(|X_n - 0| \geq \varepsilon) = 0$   $P(|X_n - 0| \geq \varepsilon) = P(X_n \geq \varepsilon) = \frac{1}{n} \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \therefore X_n \xrightarrow{P} 0$

c.  $X_n \xrightarrow{L^1} 0$   $\lim_{n \rightarrow \infty} E(|X_n - 0|) = 0$   $\lim_{n \rightarrow \infty} E(X_n) = \begin{cases} 0, & p = \frac{n-1}{n} = 1 \\ \sqrt{n}, & p = 0 \end{cases} = 0 \therefore X_n \xrightarrow{L^1} 0$

d.  $X_n \xrightarrow{a.s.} 0$   $P\left(\lim_{n \rightarrow \infty} |X_n - 0| < \varepsilon\right) \leq 1 \Rightarrow X_n \text{ WHERE } n >> 1 \Rightarrow \frac{Bx_n-1}{Bx_n} \approx 1 \Rightarrow X_n \rightarrow 0$

$\therefore P(0 < \varepsilon) \leq 1$ , BUT,  $P(X_n = 0) \neq 1$  (COULD STILL BE 0) THUS

$X_n \xrightarrow{a.s.} 0$

6.  $X_1, X_2, \dots$  ARE i.i.d.  $X_{(n)}$  IS MAX.

a.  $X_i \sim \text{Exp}(1)$  FIND LIMITING DIST. OF  $X_{(n)} - \log(n) = Y_n$

$$F_{X_{(n)}} = [F_{X_1}(x)]^n = \left[1 - e^{-x}\right]^n \quad F_{Y_n} = \left[1 - e^{-(x + \ln(n))}\right]^n = \left[1 - e^{-\frac{x}{n}}\right]^n$$

$$\text{WANT } \lim_{n \rightarrow \infty} F_{Y_n} = \lim_{n \rightarrow \infty} \left[1 - \frac{e^{-x}}{n}\right]^n \quad \text{WE KNOW } \lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = e^k$$

$$\therefore \lim_{n \rightarrow \infty} F_{Y_n} = e^{-e^{-x}} \quad \text{THIS IS GUMBEL}(0, 1)$$

b.  $F_{X_i} = 1 - x^{-2}$ ,  $x > 1$ , 0 OTHERWISE, FIND LIMITING DIST. OF  $Y_n = n^{-1/2} X_{(n)}$   $X_{(n)} = Y_n n^{1/2}$

$$F_{X_{(n)}} = [F_{X_1}(x)]^n = \left[1 - x^{-2}\right]^n \quad F_{Y_n} = \left[1 - (n^{-1/2})^2\right]^n = \left[1 - \frac{x^{-2}}{n}\right]^n$$

$$\text{WANT } \lim_{n \rightarrow \infty} F_{Y_n} = \lim_{n \rightarrow \infty} \left[1 - \frac{x^{-2}}{n}\right]^n \quad \text{WE KNOW } \lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = e^k$$

$$\therefore \lim_{n \rightarrow \infty} F_{Y_n} = e^{x^{-2}} = e^{\left(\frac{1}{x^2}\right)}$$

7.  $X_1, X_2, \dots \stackrel{iid}{\sim} \text{Beta}(1, \beta)$ . FIND LIMITING DIST. OF  $n^{1/\beta} (1 - X_{(n)}) = Y$   $X = 1 - \frac{Y}{n^{1/\beta}}$

$$F_{X_{(n)}} = [F_{X_1}(x)]^n = \left[\frac{1}{\beta(1, \beta)} x^{1-1} (1-x)^{\beta-1}\right]^n = \left(\frac{1}{\beta(1, \beta)}\right)^n \left[(1-x)^{\beta-1}\right]^n \quad F_{Y_n} = \left(\frac{1}{\beta(1, \beta)}\right)^n \left[\left(1 - \frac{Y}{n^{1/\beta}}\right)^{\beta-1}\right]^n$$

$$F_{Y_n} = \left(\frac{1}{\beta(1, \beta)}\right)^n \left[\left(\frac{Y}{n^{1/\beta}}\right)^{\beta-1}\right]^n = \left[\frac{1}{\beta(1, \beta)}\right]^n \left[\frac{Y^{\beta-1}}{n^{1-\beta/\beta}}\right]^n$$

$$\lim_{n \rightarrow \infty} F_{Y_n} = e^{-\frac{y^{\beta-1}}{\beta(1, \beta)}}$$

10. CB S.39 b CASE OF  $X_1 = I_{[0, 1]}$   $X_2 = I_{[0, 1/2]}$   $X_3 = I_{[1/2, 1]}$   $X_4 = I_{[0, 1/3]}$  ...

b. Find a subsequence of the  $X_i$ 's that converges almost surely, that is, that converges pointwise

For this we'll pick the subset where the identity range starts at 0:  $X_1 = I_{[0, 1]}$   $X_2 = I_{[0, 1/2]}$   $X_4 = I_{[0, 1/3]}$   $X_7 = I_{[0, 1/4]}$  ...

THEN LET  $X_j(s) = s + X_n$  AS  $s \rightarrow \infty$ ,  $X_n \rightarrow 0$  SO  $s + X_n \rightarrow s$  WHEN  $s > 0$

IF  $s = 0$ , WE'LL DEFINE  $X_0 = 1$  SO  $X_0 \geq s+1 = 1$

$$\therefore X_j(s) \xrightarrow{a.s.} \begin{cases} s & s > 0 \\ s+1 & s = 0 \end{cases}$$

11. CB S.44  $X_1, X_2, \dots \stackrel{iid}{\sim} \text{BERN}(p)$   $Y_n = \frac{1}{n} \sum_{i=1}^n X_i$

a. SHOW  $\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$

By CLT we know  $\sqrt{n}(\hat{p} - p) \xrightarrow{D} N(0, p(1-p))$

$$\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i = Y_n \quad \text{thus by CLT} \quad \sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$$

b. Show, for  $p \neq 1/2$ ,  $\hat{\sigma}_n^2 = Y_n(1-Y_n)$  satisfies  $\sqrt{n}[Y_n(1-Y_n) - p(1-p)] \xrightarrow{D} N(0, (1-2p)^2 p(1-p))$

$$\text{By CLT, know } \sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p)) \quad g(Y_n) = Y_n(1-Y_n) \quad g'(Y_n) = (1-Y_n) - Y_n = 1-2Y_n$$

$$\text{By Delta: } \sqrt{n}(Y_n(1-Y_n) - p(1-p)) \xrightarrow{D} N(0, \sigma^2(1-2p)^2) = N(0, p(1-p)(1-2p)^2)$$

c. Show, for  $p = 1/2$ ,  $n[Y_n(1-Y_n) - \frac{1}{4}] \xrightarrow{D} \frac{1}{4} \chi_1^2$   $g(p) = p(1-p) \quad g'(p) = 1-2p = 0 \quad \text{at } p = \frac{1}{2} \quad g''(p) =$

$$\text{use 2nd order Delta: } n[g(Y_n) - g(p)] \xrightarrow{D} g''(p) \frac{\sigma^2}{2} \chi_1^2, \quad \therefore n[Y_n(1-Y_n) - \frac{1}{4}(1-\frac{1}{2})] \xrightarrow{D} \frac{1}{2} \chi_1^2$$

$$\therefore n[Y_n(1-Y_n) - \frac{1}{4}] \xrightarrow{D} \frac{1}{4} \chi_1^2$$

12.  $a \in \mathbb{R}$ , constant  $X_1, X_2, \dots$  R.V.s.

a. Sketch CDF of  $a$ . What does  $X_n \xrightarrow{D} a$  mean?



$X_n$  converging in distribution to  $a$  means that as  $n$  gets large, the sequence of random variables approaches  $a$ .

b. Show that, if  $X_n \xrightarrow{D} a$  then  $X_n \xrightarrow{P} a$

This is from theorem in class which states that a sequence of random variables converges in probability to a constant if and only if that sequence converges in distribution to that constant. By that theorem, the above statement is true.

$$13. n=40 \quad X_i \sim N(5, 2) \quad Y = \sum_{i=1}^{40} X_i \sim N(40 \cdot 5, \sqrt{40 \cdot 2}) = N(200, 4\sqrt{10}) \quad \text{CALC IN R}$$

$$\text{a. FIND } P\left(\frac{1}{n} \sum_{i=1}^{40} X_i > 5.33\right) = P\left(\frac{1}{n} \sum_{i=1}^{40} X_i > 213.33\right) = P(N(200, 4\sqrt{10}) > 213.33) = .019$$

CALC IN R  
↓

$$\text{b. FIND } P\left(\sum_{i=1}^{40} X_i < 150\right) = P(N(200, 4\sqrt{10}) < 150) = .0000129$$

MEAN DAILY OZONE

16.  $X_i \stackrel{iid}{\sim} \text{Gamma}(5, 10)$  LET  $r = e^{\bar{X}} - \text{MEAN OF 31 DAILY MAXIMA}$

WHAT IS APPROX TEST. OF  $r^2$ ?

By CLT:  $\sqrt{n}(\bar{X} - \alpha\beta) \sim N(0, \alpha\beta^2)$   $g(x) = e^x$   $g'(x) = e^x$

$$\therefore \sqrt{n}(e^{\bar{X}} - e^{\alpha\beta}) \xrightarrow{D} N\left(0, \alpha\beta^2 \left[e^{\alpha\beta}\right]^2\right) = N\left(0, \alpha\beta^2 e^{\alpha\beta^2}\right)$$

I did this problem on a previous homework before it was removed, so even though we didn't need to do it for this homework, I already had it ready haha, figured I would just include it

17. SHOW THESE ARE EXPONENTIAL FAMILIES: Show  $f(x|\theta) = h(x)c(\theta)e^{\sum_{i=1}^k w_i(\theta)t_i(x)}$

a.  $N(\mu, \sigma^2)$  M+S KNOWN:  $f(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

LET  $h(x) = \frac{1}{\sqrt{2\pi}}, c(\sigma) = \frac{1}{\sigma}, w_1(\sigma) = -\frac{1}{2\sigma^2}, t_1(x) = (x-\mu)^2$

$\therefore f(x|\sigma) = h(x)c(\sigma)$

b.  $N(\mu, \tilde{\sigma})$  M IS KNOWN:  $f(x|\mu) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-\frac{1}{2\tilde{\sigma}^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-\frac{1}{2\tilde{\sigma}^2}(x^2 - 2x\mu + \mu^2)}$

$$= \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-\frac{\mu^2}{2\tilde{\sigma}^2}} \cdot e^{-\frac{x^2}{2\tilde{\sigma}^2} + \frac{2x\mu}{\tilde{\sigma}^2}}$$

$h(x) = 1$   $c(\mu) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-\frac{\mu^2}{2\tilde{\sigma}^2}}$   $w_1 = -\frac{1}{2\tilde{\sigma}^2}$   $t_1 = x$   $w_2 = \frac{\mu}{\tilde{\sigma}^2}$   $t_2 = x$

c. Gamma( $\alpha, \beta$ )  $\alpha$  IS KNOWN:  $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$h(x) = x^{\alpha-1}$   $c(\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$   $t_1(x) = -x$   $w(\beta) = \frac{1}{\beta}$

d. Gamma( $\alpha, \beta$ )  $\beta$  IS KNOWN:  $\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$

$h(x) = e^{-x/\beta}$   $c(\alpha) = \frac{1}{\Gamma(\alpha)} \beta^\alpha$   $x^{\alpha-1} = e^{(\alpha-1)\ln(x)}$   $w_1(x) = \alpha-1$   $t_1(x) = \ln(x)$

e. Gamma( $\alpha, \beta$ ) BOTH UNKNOWN:  $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$h(x) = 1$   $c(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$   $\frac{(\alpha-1)\ln(x) - x/\beta}{e}$   $w_1(\alpha, \beta) = \alpha-1$   $t_1(x) = \ln(x)$

$$w_2(\alpha, \beta) = -\frac{1}{\beta} \quad t_2(x) = x$$

F.  $\text{BETA}(\alpha, \beta)$   $\alpha$  IS KNOWN :  $\frac{1}{B(\alpha, \beta)} x^{\alpha-1} \underbrace{(1-x)^{\beta-1}}$

$$h(x) = x^{\alpha-1} \quad c(\beta) = \frac{1}{B(\alpha, \beta)}$$

$$e^{(\beta-1)\ln(1-x)} \quad w_1(\beta) = \beta-1 \quad t_1(x) = \ln(1-x)$$

G.  $\text{BETA}(\alpha, \beta)$   $\beta$  IS KNOWN :  $\underbrace{\frac{1}{B(\alpha, \beta)}}_{c(\alpha)} x^{\alpha-1} \underbrace{(1-x)^{\beta-1}}_{h(x)}$

$$h(x) = (1-x)^{\beta-1} \quad c(\alpha) = \frac{1}{B(\alpha, \beta)} \quad w_1(\alpha) = (\alpha-1) \quad t_1(x) = \ln(x)$$

$$H. \text{POISSON}(\lambda) : \frac{e^{-\lambda} \lambda^x}{x!} \quad h(x) = x! \quad c(\lambda) = 1 \quad e^{-\lambda} \lambda^x = e^{-\lambda} e^{x \ln(\lambda)} = e^{\lambda \ln(x) + \lambda}$$

$$h(x) = x! \quad c(\lambda) = 1 \quad w_1(\lambda) = \ln(\lambda) \quad w_2(\lambda) = \lambda \quad t_1(x) = x \quad t_2(x) = 1$$

18. SHOW THESE ARE EXPONENTIAL FAMILIES :  $f(x|\theta) = h(x) c(\theta) \exp \left[ \sum_{i=1}^k w_i(\theta) t_i(x) \right]$

a. NEG. BINOM. ( $r, p$ )  $r$  IS KNOWN :  $f(x|p) = \binom{r+x-1}{x} p^r \underbrace{(1-p)^x}_{e^{\ln((1-p)^x)}} = e^{x \ln(1-p)}$

$$h(x) = \binom{r+x-1}{x} \quad c(p) = p^r \quad w_1(p) = (1-p) \quad t_1(x) = x$$

b. PARETO( $\alpha, \beta$ )  $\alpha$  IS KNOWN :  $f(x, \beta) = \beta^\alpha \underbrace{x^{-(\beta+1)}}_{e^{\ln(x^{-(\beta+1)})}} = e^{-(\beta+1) \ln(x)}$

$$h(x) = 1 \quad c(\beta) = \beta^\alpha \quad w_1(\beta) = -(\beta+1) \quad t_1(x) = \ln(x)$$

c.  $\text{BETA}(\alpha, \beta)$  BOTH UNKNOWN :  $f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} \underbrace{(1-x)^{\beta-1}}_{e^{\ln(x^{\alpha-1} (1-x)^{\beta-1})}} = e^{(\alpha-1) \ln(x) + (\beta-1) \ln(1-x)}$

$$h(x) = 1 \quad c(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \quad w_1(\alpha, \beta) = (\alpha-1) \quad w_2(\alpha, \beta) = (\beta-1) \quad t_1(x) = \ln(x) \quad t_2(x) = \ln(1-x)$$

19. WRITE PDF IN NATURAL PARAM., FIND MEAN & VAR OF X

a. GAMMA ( $\alpha, \beta$ ):  $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-x/\beta}$  USE  $f(x|n) = h(x) c^*(n) \exp \left[ \sum_{i=1}^n \eta_i t_i(x) \right]$

From 11(c) we know:  $h(x)=1$   $c(\alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^{-\alpha}$   $\omega_1 = (\alpha-1)$   $\omega_2 = \beta^{-1}$   $t_1 = \ln(x)$   $t_2 = -x$

$$h(x)=1 \quad c^* = \frac{1}{\Gamma(n_1+1)} n_1^\alpha \quad \eta_1 = \alpha-1 \quad \eta_2 = \frac{1}{\beta} \quad \alpha = n_1 + 1 \quad \beta = n_2^{-1}$$

$$\begin{aligned} E(t_1(x)) &= -\frac{\partial}{\partial n_1} \ln(c^*(n)) = -\frac{\partial}{\partial n_1} \ln\left(\frac{n_1^\alpha}{\Gamma(n_1+1)}\right) = -\frac{\partial}{\partial n_1} \ln(n_1^\alpha) + \frac{\partial}{\partial n_1} \ln(\Gamma(n_1+1)) \\ &= -\frac{\partial}{\partial n_1} \ln\left(\frac{n_1^\alpha}{\Gamma(n_1+1)}\right) = -\frac{\partial}{\partial n_1} \ln(n_1^\alpha) + \frac{\partial}{\partial n_1} \ln(\Gamma(n_1+1)) = -\frac{\alpha n_1^{\alpha-1}}{n_1^\alpha} = -\frac{\alpha}{n_1} = \alpha \beta \end{aligned}$$

$E(x) = \alpha \beta$

$$\sqrt{Var(t_1(x))} = -\frac{\partial^2}{\partial n_1^2} c^* = -\frac{\partial^2}{\partial n_1^2} \ln\left(\frac{n_1^\alpha}{\Gamma(n_1+1)}\right) = -\frac{\partial}{\partial n_1} \left(-\frac{\alpha}{n_1}\right) = \frac{\alpha}{n_1^2} = \alpha \beta^2$$

$Var(x) = \alpha \beta^2$

b. POIS( $\lambda$ ):  $h(x) = x!$   $c(\lambda) = 1$   $\omega_1 = n_1 = \ln(\lambda)$   $\omega_2 = h_2 = \lambda$   $t_1 = x$   $t_2 = 1$

$$\lambda = e^{n_1} \quad \lambda = h_2$$

$$h(x) = x! \quad c^*(n) = \frac{e^{n_1}}{n_2}$$

$$E(x) = -\frac{\partial}{\partial n_1} \ln\left(\frac{e^{n_1}}{n_2}\right) - \frac{\partial}{\partial n_1} \ln\left(\frac{e^{n_1}}{n_2}\right) = -\frac{\partial}{\partial n_1} n_1 + -\frac{\partial}{\partial n_2} \ln\left(\frac{1}{n_2}\right) = 1 - \frac{1}{n_2} = \lambda$$

$E(x) = \lambda$

$$\sqrt{Var(x)} = -\frac{\partial^2}{\partial n_2^2} \ln\left(\frac{1}{n_2}\right) = \frac{2}{n_2^2} = \lambda$$

$\sqrt{Var(x)} = \lambda$

c. BINOM( $n, p$ ):  $f(x|n, p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} = \frac{p^x}{(1-p)^x x!(n-x)!} n! (1-p)^n$

Had to look up what the transformation was:

$$l = \ln\left(\frac{p}{1-p}\right) \Rightarrow f(x|n) = h(x) \exp\left[nx - n \ln(1+e^l)\right] \quad h(x) = \binom{n}{x} + (x) = x$$

$$c^* = \frac{1}{(1+e^l)^n}$$

$$\begin{aligned} E(x) &= -\frac{\partial}{\partial n} \ln c^* = -\frac{\partial}{\partial n} \ln\left(\frac{1}{(1+e^l)^n}\right) = -\frac{\partial}{\partial n} \left(1+e^l\right)^{-n} \cdot (1+e^l)^n = -n (1+e^l)^{n-1} (1+e^l)^n \cdot e^n \\ &= \frac{ne^n}{n} - \frac{n(\frac{p}{1-p})}{n} - \frac{np}{np} \end{aligned}$$

$$1 + e^{-\frac{p}{1-p}} = (1-p)(1 + \frac{p}{1-p}) = \overline{\frac{1-p+p}{1-p+p}} = \overline{n} p \quad \underline{\mathbf{E(X) = np}}$$

$$\sqrt{AR(X)} = \frac{-\frac{p^2}{e^2 n} \ln(e^n)}{1 + n \frac{e^n}{(1+e^n)^2}} = n \frac{\frac{p^2}{1-p}}{\left(1 + \frac{p}{1-p}\right)^2} = \frac{np}{(1-p)\left(1 + \frac{2p}{1-p} + \frac{p^2}{(1-p)^2}\right)} = \frac{np}{1-p+2p+\frac{p^2}{1-p}}$$

$$= \frac{np}{1+p+\frac{p^2}{1-p}} = \frac{np(1-p)}{1-p+p-p^2+p^2} = np(1-p)$$

$$\underline{\mathbf{\sqrt{AR(X)} = np(1-p)}}$$