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Enumerating equally likely outcomes

- Assume a large but finite sample space $S, |S| = N$

WANT $P(A)$ WHERE $A \subset S$ & $A \in \mathcal{B}$

$$P(A) = \frac{\text{\# THINGS IN } A}{N}$$

Product Rule of counting:

- If a job consists of k separate experiments, the ith of which can be done in n_i ways, then the job can be done in n_1 * n_2 * ... * n_k ways

SAMPLING

	W/O REP.	W/ REPLACEMENT
Ordered PERM.	$\frac{n!}{(n-1)!}$	n^r
Not ordered COMB.	$\frac{n!}{(n-r)!\,r!}$	$\binom{n+r-1}{r}$

$\binom{n}{r}$

Example: How many ways to get exactly one pair in a five card poker hand?
Split problem into simpler tasks, then combine them afterrwards

1) Select denomination of the pair (1, 2, … Queen, King)
2) Select suits associated with pair
3) Select denomination of other 3 cards
4) Select suits of other 3 cards

1) Only 13 possible numbers $\binom{13}{1}$

2) Pick 2 of the 4 suits for those cards (order doesn't matter) $\binom{4}{2}$

3) 12 remaining numbers, can't have any pairs there (without replacement) $\binom{12}{3}$

4) Remaining cards can be any of the suits (4 suits, 3 cards) : 4*4*4

Ans. $\binom{13}{1}\binom{7}{2}\binom{[illegible]}{3} 4$

Rule of Sum: eleta

Example: How many ways I can fly between SLC and NYC

1) Southwest: connects through Denver. 4 Routes between NYC and Denver. 5 routes between Denver and SLC
2) Delta: 2 direct NYC:SLC
3) United: NYC:Chicago (8), Chicago:SLC (3)

Find the possibilities for each airline, then add together:

1) 4*5 = 20
2) 2 total
3) 8*3 = 24

Ans. 20+2+24 = 46

Inclusion/Exclusion $|S'| = N$

Want to enumerate elements in A $\quad$ i.e. $N_A = |A|$

Sometimes easier to enumerate: $\quad N_{A^c} = |A^c| \quad \therefore N_A = N - N_{A^c}$

Uniform Probability over continuous domain

Consider: $S' \subset \mathbb{R}^d$

Assume events occur with uniform probability.

Then for $A \subseteq S'$, $\quad P(A) = \dfrac{\int_A ds}{\int_{S'} ds}$

Conditional Probability

If $A, B \subseteq S'$ and $P(B) > 0$,

Cond. Prob. A given B: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Adding conditions redefines sample space. Sample space goes from S to B

Reframing this in conditional forms:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Often use law of total probability

$$P(B) = \sum_{i=1}^{n} P(B|C_i)\,P(C_i)$$

$$C_i \cap C_j = \emptyset \qquad \bigcup_{i=1}^{n} C_i = S$$

Example: 3 prisoner example
3 prisoners: A,B,C
All sentenced to death, but one will be selected to be pardoned
Warden knows who will be selected, A asks warden to tell him one of the people who is being executed
If B is pardoned, say C. If C is pardoned, say A. If A is pardoned, say B or C (50/50)

Warden says B is being executed. A is happy because thinks now his probability is 1/2. Prisoner C thinks now C has a 2/3 chance of being pardoned

Event A: A is pardoned (same for B & C)
Event W: Warden says B will be executed

PRIS. PARDONED	PROB. PARDON	WARDEN SAYS	P (W|X PARDONED)
A	1/3	B, C	$P(W\|A) = \frac{1}{2}$
B	1/3	B	$P(W\|B) = 0$
C	1/3	C	$P(W\|C) = 1$

$$P(A|W) = \frac{P(A \cap W)}{P(W)} = \frac{P(W|A)\,P(A)}{P(W|A)\,P(A) + P(W|B)\,P(B) + P(W|C)\,P(C)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 0 + 1\left(\frac{1}{3}\right)} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P(C|W) = \ldots \frac{2}{3}$$

Independent Events

Two events A, B are statistically independent if: $P(A \cap B) = P(A)\,P(B)$ DENOTED $A \perp\!\!\!\perp B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

Theorem: If A and B are independent events the following pairs are also independent:

$$A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp B^c, \quad A^c \perp\!\!\!\perp B, \quad A^c \perp\!\!\!\perp B^c$$

PROOF: $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B))$
$= P(A)P(B^c)$

Mutual Independence

A collection of events A1, …, An is mutually independent if any subcollection Ai1, …, Aik we have:

$$P\left(\bigcap_{j=1}^{K} A_{i_j}\right) = \prod_{j=1}^{K} P(A_{i_j})$$