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E(X) is known as the First Moment

Moments: For each integer n, the nth moment of an r.v. X is: $E(X^n)$

The nth central moment is: $E(X - E(X))^n$

Variance is 2nd central moment: $Var(X) = E(X - E(X))^2$

PROPERTIES OF VAR:

- $Var(aX+b) = a^2 Var(X)$
- $Var(X) = E(X^2) - [E(X)]^2$

Ex. $X \sim$ INV. GAMMA (α, β) $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{-(\alpha+1)} e^{-\frac{1}{\beta x}}$

FROM LAST CLASS: $E(X) = \frac{1}{(\alpha-1)\beta}$

$E(X^2) = \int_0^\infty x^2 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{-(\alpha+1)} e^{-\frac{1}{\beta x}} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{-(\alpha-2+1)} e^{-1/\beta x} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \Gamma(\alpha-2)\beta^{\alpha-2}$

WITH $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\therefore = \frac{1}{(\alpha-2)(\alpha-1)\beta^2}$

$\therefore Var(X) = E(X^2) - E(X)^2 = \frac{1}{(\alpha-2)(\alpha-1)\beta^2} - \left(\frac{1}{(\alpha-1)\beta}\right)^2 = \frac{1}{(\alpha-2)(\alpha-1)^2\beta^2}$

Let's say instead of E(X) we are interested in E(Y) where Y = g(X). Want to know: E(g(x)) = g(E(X))
This holds under very restrictive conditions, and often those are not met. However, we can usually compare

Jensen's inequality: a function g(x) is convex if $g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y)$
$\lambda \in [0,1]$, $x, y \in \mathbb{R}$

$g(x)$ IS CONCAVE IF $-g(x)$ IS CONVEX

IF $\lambda = \frac{1}{2}$: $g\left(\frac{x+y}{2}\right) \le \frac{1}{2}(g(x) + g(y))$

HOW DETERMINE IF CONCAVE

- DRAW IT
- SECOND DERIVATIVE: IF $g''(x) > 0$, CONVEX

THEOREM: 4.7.7

IF $g(x)$ IS CONVEX, $E(g(x)) \ge g(E(x))$

IF $g(x)$ IS CONCAVE, $E(g(x)) \le g(E(x))$

Ex. $g(x) = x^2$: CONVEX $\therefore E(x^2) \ge E(x)^2$

$g(x) = e^x$: CONVEX $\therefore E(e^x) \ge e^{E(x)}$

$g(x) = \ln(x)$: CONCAVE $\therefore E(\ln(x)) \le \ln(E(x))$

EX. DIFF MEANS OF POS. NUMS.

$a_A = \frac{1}{n}(a_1 + a_2 + \dots + a_n)$

$a_G = (a_1 + a_2 + \dots + a_n)^{\frac{1}{n}}$

$a_H = \dfrac{1}{1/n(a_1 + a_2 + \dots + a_n)}$ PROVE: $a_H \le a_G \le a_A$

X IS R.V. WITH SUPPORT $(a_1, a_2, \dots, a_n)$; $P(X = a_i) = \frac{1}{n}$

$E(X) = \sum_{i=1}^{n} a_i P(X = a_i) = \frac{1}{n}\sum_{i=1}^{n} a_i$

$\log(a_G) = \frac{1}{n}\sum_{i=1}^{n} \log(a_i) = E(\log(X)) \le \log(E(X)) = \log\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = \log(a_n) \Rightarrow a_G \le a_n$

$\log\left(\frac{1}{a_H}\right) = \log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) = \log\left(E\left[\frac{1}{X}\right]\right) \ge E\left(\log\left(\frac{1}{X}\right)\right) = -E(\log(X)) = -\log(a_G) = \log\left(\frac{1}{a_H}\right)$

$\therefore \frac{1}{a_H} \ge \frac{1}{a_G} \therefore a_H \le a_G$

Moment Generating Function (MGF)

X IS R.V. WITH CDF $F_X(x)$, THE MGF OF X IS:

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx & X \text{ IS CONT.} \\ \sum_{x \in X} e^{tx} f_X(x)\,dx & \text{DISCRETE} \end{cases}$$

This holds if the expectation exists for t in the neighborhood of 0. That is, there exists an h > 0 such that E(e^(tx)) exists for all -h < t < h

THM: IF X HAS MGF: $M_X(t)$, THEN $E(X^n) = M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0}$

Proof: (start with assumption that we can bring the derivative inside the integral, see section 2.4 in CB)

$$\frac{d^n}{dt^n} M_X(t)\Big|_{t=0} = \left[\frac{d^n}{dt^n}\int e^{tx} f_X(x)\,dx\right]\Big|_{t=0} = \left[\int \underbrace{\frac{d^n}{dt^n} e^{tx}}_{x^n e^{tx}} f(x)\,dx\right]_{t=0}$$

$$= \left[\int x^n e^{tx} f(x)\,dx\right]_{t=0} = E(x^n e^{tx})\Big|_{t=0} = E(X^n)$$

EX. X IS EXP. R.V. W/ $f(x) = \frac{1}{\theta} e^{-x/\theta}$ $x > 0$, $\theta > 0$

$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}\frac{1}{\theta} e^{-x/\theta}\,dx = \frac{1}{\theta}\int_0^{\infty} e^{-x\left(\frac{1}{\theta} - t\right)}\,dx = \frac{1}{\theta}\int_0^{\infty} e^{-x\left(\frac{1-t\theta}{\theta}\right)}\,dx$

$= \frac{1}{\theta}\,\frac{1}{\frac{1-t\theta}{\theta}} = \frac{1}{1-t\theta}$

$E(X) = M_X'(0) \Rightarrow \frac{d}{dt}\frac{1}{1-t\theta} = \frac{d}{dt}(1-t\theta)^{-1} = \frac{\theta}{(1-t\theta)^2}$ $\therefore E(X) = M_X'(0) = \frac{\theta}{(1-0)^2} = \theta$

$E(X^2) = M_X''(0)$ $\frac{d^2}{dt^2}\frac{1}{1-t\theta} = \frac{2\theta^2}{(1-t\theta)^3}$ $\therefore E(X^2) = 2\theta^2$

$\therefore VAR(X) = E(X^2) - E(X)^2 = 2\theta^2 - \theta^2 = \theta^2$