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Ex. Proving a CDF is valid. Toss until a head appears
S = {H, TH, TTH, TTTH, ...}

$\mathcal{B} = \mathcal{P}(S)$ POWERSET: ALL POSSIBLE SUBSETS

X is # Ts until H = {1, 2, 3, 4, ...} = $\mathcal{X}$

$\mathcal{B} = \mathcal{P}(\mathcal{X})$

REMEMBER: $\sum_{i=0}^{x-1} ar^i = a\left(\frac{1-r^x}{1-r}\right) \quad |r|<1$

$P(X=x) = (1-p)^{x-1}p \quad x \in \{1,2,3,\ldots\}$

$F_X(x) = P(X \le x) = \sum_{i=1}^{x} (1-p)^{i-1} p = p\sum_{i=0}^{x-1}(1-p)^{i-1} = p\left(\frac{1-(1-p)^x}{1-(1-p)}\right)$

$= 1-(1-p)^x$

$F_X(x) = 1-(1-p)^x$

Show this is a valid CDF:

a) $\lim_{x \to -\infty} F_X(x) = 0$ TRUE (BOUNDED BY $x \ge 1$)

$\lim_{x\to\infty} F_X(x) = 1 \Rightarrow \lim_{x\to\infty} 1-(1-p)^x \quad \therefore \quad 1-0=1$

$1-p<1$

b) LET $x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$

$F(x_1) = 1-(1-p)^{x_1}$ $\qquad (1-p)^{x_2} \le (1-p)^{x_1} \quad \therefore \quad 1-(1-p)^{x_2} \ge 1-(1-p)^{x_1}$

$F(x_2) = 1-(1-p)^{x_2}$

c) For a discrete distribution, focus on discontinuity points

$x \in \{1,2,3,\ldots\} \quad \lim_{\varepsilon \downarrow 0} F_X(x+\varepsilon) = \lim_{\varepsilon \downarrow 0} \sum_{i=1}^{x+\varepsilon} p(1-p)^{i-1} = \sum_{i=1}^{x} p(1-p)^{i-1} = F_X(x)$

Ex. PROVE $F(x) = \frac{1}{1+e^{-x}}$ IS VALID CDF

a) $\lim_{x\to-\infty} \frac{1}{1+e^{-x}} \Rightarrow \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0 \qquad \lim_{x\to\infty} \frac{1}{1+e^{-x}} \Rightarrow \frac{1}{1+0} = 1$

b) $\frac{d}{dx}\frac{1}{1+e^{-x}} = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

c) CONTINUOUS

A r.v X is continuous if $F_X(x)$ Is a continuous function of x.

A r.v X is discrete if $F_X(x)$ Is a step function

$$F_X(x) = \begin{cases} \frac{1-\varepsilon}{1+e^{-x}} & x < 0 \\ \varepsilon + \frac{1-\varepsilon}{1+e^{-x}} & x \geq 0 \end{cases}$$

Neither continuous not discrete. A mixture of both

Two random variables X, Y are identically distributed if for every $A \in \mathcal{B}$,

$P(X \in A) = P(Y \in A)$ For continuous, this is the borrel sets (the powerset of non-countable/discrete)

Ex. 3 coin toss. X = # heads. Y = # tails. S = {HHH ,HHT, HTH, HTT, THH, THT, TTH, TTT}

K	0	1	2	3
X	1/8	3/8	3/8	1/8
Y	1/8	3/8	3/8	1/8

Thus, X,Y are identically distributed: $Y \sim X$

Theorem: The following statements are equivalent -

a) $X \sim Y$

b) $F_X(x) = F_Y(x)$ ALL $x \in \mathbb{R}$

PDF (density) and PMF (mass)

Remember, these are induced distributions of a sample space

Def: Probability Mass Function (PMF) (same reasoning applies for PDFs, intuition is the same - math is different)
PMF of a discrete r.v X is given by $f_X(x) = P(X = x)$

EX. Geometric Distribution. X is # of trials until first success:

$$f(x) = P(X = x) = \begin{cases} p(1-p)^{x-1} & x \in \{1,2,3,\ldots\} \\ 0 & \text{OTHERWISE} \end{cases}$$

$$f(5) = p(1-p)^4$$

PROB. BETWEEN a & b: $\sum_{x=a}^{b} f(x) = \sum_{x=a}^{b} p(1-p)^{x-1}$

Probability Density Function (PDF) of a continuous r.v. X is the function that satisfies the following:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \text{FOR ALL } x \quad \therefore \frac{dF(x)}{dx} = f(x)$$

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx = P(a < x < b) = P(a \leq x < b) \ldots = F(b) - F(a)$$

$$f(x) = 2x I_x = \begin{cases} 2x & 0 \leq x < 1 \\ \vdots \end{cases} \qquad I_A(x) = \begin{cases} 1 & \text{IF } x \in A \\ 0 & \text{IF } x \notin A \end{cases}$$

Ex. $x \in [0,1)$, ...

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 2t\,I(t)\,dt = \int_{0}^{x} 2t\,dt = t^2 \Big|_0^x = \begin{cases} x^2 & 0 \le x < 1 \\ 0 & x < 0 \\ 1 & x \ge 1 \end{cases}$$

Ex. $F(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-\theta x} & x \ge 0 \end{cases} \quad \therefore\ f(x) = \frac{d}{dx} 1-e^{-\theta x} = \begin{cases} \theta e^{-\theta x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

Theorem 1.6.5 in book:
A function f(.) is a valid PMF/PDF of r.v. X iff:

a) $f(x) \ge 0 \quad \forall x$

b) $\sum_{x \in X} f(x) = 1$ OR $\int_X f(x)\,dx = 1$

Ex. $X = \{1, 2, \ldots 10\}$ $f(x) = c\,x\,I_X(x)$ WHAT IS c?

$$1 = \sum_{x=1}^{10} cx = c\,55 \quad \therefore\ c = \frac{1}{55}$$

Ex. $X : x \in (0,2)$, $f(x) = cx^3\,I_X(x)$

$$1 = \int_0^2 cx^3 = \frac{c}{4}x^4 \Big|_0^2 = \frac{16c}{4} = 4c \quad \therefore\ c = \frac{1}{4}$$