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Mutual Independence (review):

Box of tickets:

111	111
010	010
100	100
001	001

A1 = First digit is a 1
A2 = Second digit is a 1
A3 = Third digit is a 1

P(A1) = 4/8
P(A2) = 4/8
P(A3) = 4/8

$$P(A_1 \cap A_2) = \frac{2}{8} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2})$$
$$P(A_1 \cap A_3) = \ldots$$
$$P(A_2 \cap A_3) = \ldots$$
$$P(A_1 \cap A_2 \cap A_3) = \frac{2}{8} \neq \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

So, these events are all pairwise independent but not mutually independent

Conditional Independence

Two events A and B are conditionally independent given C if:

$$P([A \cap B] \mid C) = P(A \mid C) P(B \mid C)$$

If A,B are conditionally independent, does that mean A,B are independent. Not necessarily (A and B given C is a subset of A and B, what applies for the smaller space might not apply for the whole)

Ex: (standard Bayes' stuff to start)

- 1% of women aged 40-50 have breast cancer.
- Women with breast cancer have 90% chance of testing positive from mammogram
- Women without breast cancer have 10% chance of testing positive from mammogram (false positive)
- B = Has Breast Cancer, T = Positive Test

WANT: $P(B \mid T)$

$P(B) = 1/100$

$P(T \mid B) = \frac{9}{10}$ $\quad P(T \mid B^c) = \frac{1}{10}$

$$P(B \mid T) = \frac{P(T \mid B) P(B)}{P(T \mid B) P(B) + P(T \mid B^c) P(B^c)}$$
$$= \frac{.9(.01)}{.9(.01) + .1(.99)} = \frac{9}{108}$$

Now we have a second test.

INDEPENDENCE

Now we have a second test.

T1 = positive first test, T2 = positive second test

$$P(B \mid T_1, T_2) = \frac{P(T_1, T_2 \mid B)\, P(B)}{P(T_1, T_2)}$$

IF WE ASSUME CONDITIONAL INDEP.

$$= \frac{\text{'' \quad ''}}{P(T_1 \mid B) P(T_2 \mid B) P(B) + P(T_1 \mid B^c) P(T_2 \mid B^c) P(B^c)}$$

$$= \frac{9}{20}$$

-OR-

$$P(B \mid T_1, T_2) = \frac{P(T_2 \mid B)\, P(B \mid T_1)}{P(T_2 \mid B)\, P(B \mid T_1) + P(T_2 \mid B^c)\, P(B^c \mid T_1)}$$

Basically, use the probability from test 1 in the calculation for test 2

Ex: Flip two fair coins. Flip each once

A = first coin is heads. B = second coin is heads. C = Both coins are same

A & B are independent: $P(A \cap B) = P(A)P(B) = \frac{1}{4}$

$P(A \cap B \mid C) = \frac{1}{2}$ (COUNT OUT ALL SAMPLES)

$P(A \mid C) = \frac{1}{2}$ $\quad P(B \mid C) = 1/2$

Therefore, not conditionally independent, but are marginally independent (independent from each other on some level)

Random Variable

Random variable: function, mapping. Mapping from sample space into new 'image'

Def: a random variable (vector) is a function from the sample space to the real numbers

Formally: $X : \mathcal{S} \Rightarrow \mathbb{R}$ $\quad \underset{\sim}{X} : \mathcal{S} \Rightarrow \mathbb{R}$

Example:

Experiment	Sample Space	Random Variable
Toss two dice	{(1, 1), (1, 2)…(6, 6)}	X = Sum, Y = Max
Toss coin 25 times	{H,T}^25	X = # heads

$$\mathcal{S} \xRightarrow{X} \boxed{\mathcal{X}}$$

Ex: Let S = {s1, …, sn} with probability function P(.). Let new subspace X_fancy = {x1, …, xn}.

Since X = xi iff sj in S maps to xi (ie. X(sj) = xi) then the new probability (induced probability) function on X_fancy is: $P(X = x_i) = P(\{s_j \in \mathcal{S} : X(s_j) = x_i\})$

Ex. Toss 3 fair coins, consider the r.v. X = # of heads

- X(hhh) = 3, X(hht) = 2, …, X(ttt) = 0

X	0	1	2	3

P(X)	1/8	3/8	3/8	1/8

- This is the induced Probability Mass Function

Cumulative Distribution Function

The Cumulative Distribution Function (CDF) of a r.v. is denoted $F_X(x)$
And is defined by $P_X(X \le x)$ FOR ALL $x \in \mathbb{R}$

Ex. Tossing 3 fair coins
$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/8 & 0 \le x < 1 \\ 4/8 & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

1. $\lim_{x \to -\infty} F_X(x) = 0 \quad \lim_{x \to \infty} F_X(x) = 1$
2. $F_X(x)$ IS NON-DECREASING ie. FOR $x_1 \le x_2$ $F_X(x_1) \le F_X(x_2)$
3. $F_X(x)$ IS RIGHT CONTINUOUS $\lim_{x \downarrow x_0} F_X(x) = F_X(x_0)$