9 Sep 2025

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COLLECTION OF SUBSETS OF S IS σ-ALGEBRA $(\mathcal{B})$ IFF:

a. $\emptyset \in \mathcal{B}$

b. $A \in \mathcal{B}$ THEN $A^c \in \mathcal{B}$

c. IF $A_1, A_2 \ldots \in \mathcal{B}$ THEN $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$

PROVE: σ-ALG'S ARE CLOSED UNDER COUNTABLE INTERSECTION:

LET $A_1, A_2 \ldots \in \mathcal{B}$ THEN $A_1^c, A_2^c \ldots \in \mathcal{B}$ (b)

THEN (c) $\bigcup_{n=1}^{\infty} A_n^c = \mathcal{B}$ $\therefore \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{B}$ (b)

$\therefore$ DE MORGAN'S: $\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c = \bigcap_{n=1}^{\infty} (A_n^c)^c = \bigcap_{n=1} A_n \in \mathcal{B}$

CONSTRUCTING:

EASY IF S IS FINITE/COUNTABLE INFINITE : σ-ALG = $\mathcal{P}(S)$ = POWER SET OF S

= ALL POSSIBLE SUBSETS (INCL. S)

Ex. $S = \{H, T\}$ $\mathcal{B} = \{\{H\}, \{T\}, \{H,T\}, \emptyset\}$

S IS UNCOUNTABLE $(S = [0,1])$: USE BOREL SETS

$\mathcal{B}$ = COLLECTION (a,b), $[a,b)$, $[a,b]$ FOR $a<b$ AND ALL COUNTABLE INTERSECTIONS ; UNIONS OF THOSE

PROBABILITY FUNCTION/AXIOMS:

DEF: GIVEN S ; σ-ALG., PROB. FUNC. P WITH DOMAIN $\mathcal{B}$ SATISFIES:

1) $P(A) \geq 0$ FOR $A \in \mathcal{B}$

2) $P(S) = 1$

3) IF $A_1, A_2 \ldots \in \mathcal{B}$ ARE PAIRWISE DISJOINT $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

Ex. $S = \{s_1, \ldots, s_n\}$, $\mathcal{B}$ IS POWERSET OF S

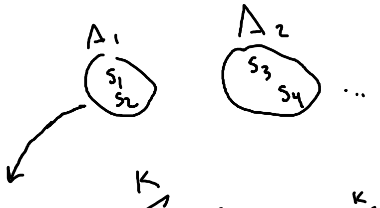
LET $p_1, \ldots p_n$ NONNEGATIVE S.T. $\sum_{n=1}^{\infty} p_i = 1$

FOR ANY $A \in \mathcal{B}$, $P(A) = \sum_{i: s_i \in A} p_i$

IS P A PROB. FUNC.?

1) FOR ANY A, $P(A) = \sum_{i: s_i \in A} p_i \geq 0$ BECAUSE $p_i \geq 0$

2) $P(S) = \sum_{i: s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$



3) LET $A_1, \ldots, A_k$ BE P-WISE DISJ. : $P\left(\bigcup_{i=1} A_i\right) = \sum_{j : s_j \in \bigcup_{i=1} A_i} p_j = \sum_{i=1} \sum_{j : s_j \in A_i} p_j = \sum_{i=1} P(A_i)$

PROBABILITY TRIPLE : $\left(\mathcal{S}, \overset{\text{COMES FROM } \mathcal{S}}{\mathcal{B}}, \overset{\text{MAPS TO } [0,1]}{P}\right)$

PROPERTIES OF PROB. FUNCS .:

1) $\underline{P(\emptyset) = 0}$: $P(\mathcal{S}) = 1 = P(\mathcal{S} \cup \emptyset) = P(\mathcal{S}) + P(\emptyset) = 1 + P(\emptyset) \therefore P(\emptyset) = 1 - 1 = 0$

2) $A \subset \mathcal{S}$ THEN $\underline{P(A) \leq 1}$: $1 = P(\mathcal{S}) = P(A \cup A^c) = P(A) + P(A^c) \Rightarrow P(A) = 1 - P(A^c) \therefore P(A) \leq 1$

3) $\underline{P(A^c) = 1 - P(A)}$: SEE PREVIOUS

4) $\underline{P(B \cap A^c) = P(B) - P(A \cap B)}$: HW

5) $\underline{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$: USE (4)

6) $A \subseteq B$ THEN $\underline{P(A) \leq P(B)}$: HW

7) LET $C_1, C_2, \ldots$ BE PARTITION OF $\mathcal{S}$ (ie. $C_i \cap C_j = \emptyset$ $i \neq j$) $\left(\bigcup_{i=1}^{\infty} C_i = \mathcal{S}\right)$

$\therefore \underline{P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)}$: $P(A) = P(A \cap \mathcal{S}) = P\left(A \cap \left(\bigcup_{i=1}^{\infty} C_i\right)\right)$

$= P\left(\bigcup_{i=1}^{\infty} (A \cap C_i)\right) = \sum_{i=1}^{\infty} P(A \cap C_i)$

8) FOR ANY $A_1, A_2 \ldots$ $\underline{P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)}$ (YOU HAVE TO REMOVE INTERSECTIONS FOR UNIONS)

: DEFINE DISJOINT SETS: $A_1^* = A_1$ $A_2^* = A_2 - A_1$ $A_3^* = A_3 - (A_1 \cup A_2)$ … $A_i^* = A_i - \bigcup_{j=1}^{i-1} A_j$

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i^*\right)$ $A_1 \cup A_2 \overset{?}{=} A_1^* \cup A_2^*$ CAN DEDUCE THIS IS TRUE

$A_i^* \subseteq A_i \Rightarrow P(A_i^*) \leq P(A_i)$

$= \sum_{i=1}^{\infty} P(A_i^*) \leq \sum_{i=1}^{\infty} P(A_i)$

Enumerating equally likely outcomes

- Assume a large but finite sample space $\mathcal{S}$, $|\mathcal{S}| = N$

WANT $P(A)$ WHERE $A \subset \mathcal{S}$ & $A \in \mathcal{B}$

$$P(A) = \frac{\text{\# THINGS IN } A}{N}$$

Product Rule of counting:

- If a job consists of k separate experiments, the ith of which can be done in n_i ways, then the job can be done in n_1 * n_2 * … * n_k ways