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$$X \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2), S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

a)

$$\text{Show that } MSE_{\sigma^2}(aS^2) = \frac{2a^2\sigma^4}{n-1} + (a-1)^2\sigma^4$$

$$\begin{aligned} MSE_{\sigma^2}(aS^2) &= Var(aS^2) + (\mathbb{E}[aS^2] - \sigma^2)^2 \text{ by (7.3.1)} \\ &= a^2 Var(S^2) + (a\mathbb{E}[S^2] - \sigma^2)^2 \end{aligned}$$

$$\text{As shown in class: } Var(S^2) = \frac{2\sigma^4}{n-1}$$

$$\begin{aligned} &= a^2 \frac{2\sigma^4}{n-1} + (a\sigma^2 - \sigma^2)^2 \\ &= \frac{2a^2\sigma^4}{n-1} + (a-1)^2\sigma^4 \end{aligned}$$

b)

$$\text{Show that } \tilde{S}^2 = \frac{\sum (x_i - \bar{x})^2}{n} \text{ satisfies } MSE_{\sigma^2}(\tilde{S}^2) \leq MSE_{\sigma^2}(S^2) \forall \sigma^2 > 0$$

$$\text{Note: } MSE_{\sigma^2}(S^2) = \frac{2\sigma^4}{n-1}, \tilde{S}^2 = \frac{n-1}{n} S^2$$

$$\begin{aligned} MSE_{\sigma^2}(\tilde{S}^2) &= Var_{\sigma^2}(\tilde{S}^2) + (\mathbb{E}_{\sigma^2}[\tilde{S}^2] - \sigma^2)^2 \\ &= Var_{\sigma^2}\left(\frac{n-1}{n} S^2\right) + (\mathbb{E}_{\sigma^2}\left[\frac{n-1}{n} S^2\right] - \sigma^2)^2 \\ &= \left(\frac{n-1}{n}\right)^2 Var_{\sigma^2}(S^2) + \left(\frac{n-1}{n} \mathbb{E}_{\sigma^2}[S^2] - \sigma^2\right)^2 \\ &= \left(\frac{n-1}{n}\right)^2 \frac{2\sigma^4}{n-1} + \left(\frac{n-1}{n} \sigma^2 - \sigma^2\right)^2 \\ &= \frac{2(n-1)\sigma^4}{n^2} + \sigma^4 \left(\frac{n-1}{n} - 1\right)^2 \\ &= \frac{2(n-1)\sigma^4}{n^2} + \sigma^4 \left(\frac{-1}{n}\right)^2 \\ &= \frac{2n\sigma^4 - 2\sigma^4}{n^2} + \frac{\sigma^4}{n^2} \\ &= \frac{2\sigma^4(n-1/2)}{n^2} \\ &= MSE_{\sigma^2}(S^2) \frac{(n-1)(n-1/2)}{n^2} \\ &= MSE_{\sigma^2}(S^2) \frac{n^2 - 3/2n + 3/2}{n^2} \end{aligned}$$

$$\text{For any } n > 1, \frac{n^2 - 3/2n + 3/2}{n^2} < 1 \text{ thus: } MSE_{\sigma^2}(\tilde{S}^2) \leq MSE_{\sigma^2}(S^2)$$

c)

Find value of a that minimizes the MSE from aS^2

$$\text{We need to minimize: } \frac{2a^2\sigma^4}{n-1} + (a-1)^2\sigma^4 = \frac{2a^2\sigma^4 + (a-1)^2(n-1)\sigma^4}{n-1} = \frac{\sigma^4}{n-1} (2a^2 + (a^2 - 2a + 1)(n-1)) \propto a^2(n+1) - 2a(n-1) + n-1$$

$$\frac{d}{da} a^2(n+1) - 2a(n-1) + n - 1 = 2a(n+1) - 2(n-1) = 0 \implies a = \frac{n-1}{n+1}$$

d)

$$I_n(\mu, \sigma^2) = \mathbb{E}_{(\mu, \sigma^2)} [\psi(x; \mu, \sigma^2) \psi(x; \mu, \sigma^2)^\top]$$

$$\psi(x; \mu, \sigma^2) = [\frac{\partial}{\partial \mu} \ln(f(x|\mu, \sigma^2)), \frac{\partial}{\partial \sigma^2} \ln(f(x|\mu, \sigma^2))]$$

$$\frac{\partial}{\partial \mu} \ln(f(x|\mu, \sigma^2)) = \frac{\sum (x - \mu)}{\sigma^2}$$

$$\frac{\partial}{\partial \sigma^2} \ln(f(x|\mu, \sigma^2)) = \frac{\sum (x - \mu)^2}{(\sigma^2)^2}$$

$$\psi(x; \mu, \sigma^2) = \begin{pmatrix} \frac{2x - 2\mu}{\sigma^2} \\ \frac{x^2 - 2x\mu + \mu^2}{\sigma^4} \end{pmatrix}$$

Taking the expected value of this, the $(x - \mu)$ terms disappear (leaving a factor of n) $\implies I_n = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$

e)

$\tau_a(\sigma^2) = a\sigma^2$. Find the Cramer-Rao Lower Bound. Any values of a for which it obtains the lower bound?

$$Var_a(\sigma^2) \geq \frac{(\tau'(\sigma^2))^2}{I_n(\sigma^2)} = \frac{a}{\frac{n}{2\sigma^4}} = \frac{2a\sigma^4}{n}$$

No, there are no values where the estimator reaches the Cramer-Rao lower bound, because the estimator has an $(n-1)$ in the denominator, so the closest it can get is $a = 1; \frac{2\sigma^4}{n-1}$ while the lower bound would be $\frac{2\sigma^4}{n-1}$

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$Y_i = \beta x_i + \epsilon_i, \epsilon \sim N(0, \sigma^2), x_i$ is known constants.

a) CB2 7.20

i) Show that $\sum Y_i / \sum x_i$ is an unbiased estimator for β

$$\mathbb{E}[\sum Y_i / \sum x_i] = \frac{\mathbb{E}[\sum \beta x_i + \epsilon_i]}{\sum x_i} = \frac{\sum \mathbb{E}[\beta x_i + \epsilon_i]}{\sum x_i} = \frac{\sum \beta x_i}{\sum x_i} = \frac{\beta \sum x_i}{\sum x_i} = \beta$$

ii) Calculate the variance of $\sum Y_i / \sum x_i$ and compare to the variance of the MLE

$$Var(\sum Y_i / \sum x_i) = \frac{Var(\sum Y_i)}{(\sum x_i)^2} = \frac{\sum Var(Y_i)}{(\sum x_i)^2} = \frac{\sum \sigma^2}{(\sum x_i)^2} = \frac{n\sigma^2}{(\sum x_i)^2}$$

$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum (x_i^2)}; \sum (x_i^2) < (\sum x_i)^2 / n \implies \frac{\sigma^2}{\sum (x_i^2)} \leq \frac{n\sigma^2}{(\sum x_i)^2} \implies Var(\hat{\beta}) \leq Var(\sum Y_i / \sum x_i)$$

b) CB2 7.21

i) Show that $\sum [Y_i / x_i] / n$ is also an unbiased estimator of β

$$\mathbb{E}[\sum [Y_i / x_i] / n] = \frac{1}{n} \sum \mathbb{E}[(\beta x_i + \epsilon_i) / x_i] = \frac{1}{n} \sum \mathbb{E}[\beta] = \frac{n\beta}{n} = \beta$$

ii) Calculate the variance of $\sum [Y_i / x_i] / n$ and compare to the previous variances

$$Var(\sum [Y_i / x_i] / n) = \frac{1}{n^2} Var(\sum [Y_i / x_i]) = \frac{1}{n^2} \sum \frac{Var Y_i}{x_i^2} = \frac{\sigma^2}{n^2} \sum \frac{1}{x_i^2}$$

By Jensen's inequality: $\frac{1}{n^2(\sum x_i)^2} \leq \frac{1}{n} \sum \frac{1}{x_i^2} \implies \underline{Var(\sum [Y_i/x_i]/n) \geq Var(\sum Y_i / \sum x_i) \geq Var(\hat{\beta})}$

c) Compute the CR Lower Bound for estimating β . Is the MLE the MVUE for this model?

$$I_n(\beta) = \mathbb{E}_\beta[\psi^2(x; \beta)] = \mathbb{E}_\beta\left[\frac{\partial}{\partial \beta} \ln(x_i \beta + \epsilon)\right] = \mathbb{E}_\beta\left[\frac{x_i}{x_i \beta + \epsilon}\right] = \frac{1}{\beta}$$

$$Var_\beta(W) \geq \frac{(\tau'(\beta))^2}{I_n(\beta)} = \frac{1}{1/\beta} = \beta$$

Yes, the MLE is the MVUE for this case

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CB2 7.38

Is there a $g(\theta)$ for which there exists an unbiased estimator whose variance attains the CRLB?

a) $f(x|\theta) = \theta x^{\theta-1}$

$$\frac{\partial}{\partial \theta} \sum \ln(\theta x^{\theta-1}) = \frac{\partial}{\partial \theta} \sum \ln(\theta) + (\theta-1) \ln(x_i) = \sum \frac{1}{\theta} + \ln(x_i) = \frac{n}{\theta} + \sum \ln(x_i) = -n\left[-\frac{1}{\theta} - \sum \frac{\ln(x_i)}{n}\right]$$

By 7.3.15, this means $-\sum \frac{\ln(x_i)}{n}$ is the MVUE of $1/\theta$

b) $f(x|\theta) = \frac{\ln(\theta)}{\theta-1} \theta^x$

$$\frac{\partial}{\partial \theta} \ln(L(x|\theta)) = \frac{\partial}{\partial \theta} \sum \ln(\ln(\theta)) - \ln(\theta-1) + x_i \ln(\theta) = \sum \frac{1}{\theta \ln(\theta)} - \frac{1}{\theta-1} + \frac{x_i}{\theta} = \frac{n}{\theta \ln(\theta)} - \frac{n}{\theta-1} + \frac{\sum x_i}{\theta} = -\frac{n}{\theta} \left[\frac{\theta}{\theta-1} - \frac{1}{\ln(\theta)} - \frac{\sum x_i}{n\theta} \right]$$

Thus, $\frac{\sum x_i}{n}$ is the MVUE for $\frac{\theta}{\theta-1} - \frac{1}{\ln(\theta)}$

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CB2 7.40

$X \sim \text{Bern}(p)$, show that \bar{X} attains the CRLB.

$$\begin{aligned}
-n \mathbb{E}_p \left[\frac{\partial^2}{\partial p^2} \ln(f(x|p)) \right] &= -n \mathbb{E}_p \left[\frac{\partial^2}{\partial p^2} \ln(p^x (1-p)^{1-x}) \right] \\
&= -n \mathbb{E}_p \left[\frac{\partial^2}{\partial p^2} x \ln(p) + (1-x) \ln(1-p) \right] \\
&= -n \mathbb{E}_p \left[\frac{\partial}{\partial p} \frac{x}{p} - \frac{1-x}{1-p} \right] \\
&= -n \mathbb{E}_p \left[-\frac{x}{p^2} - \frac{1-x}{(1-p)^2} \right] \\
&= -n \left[-\frac{p}{p^2} - \frac{1-p}{(1-p)^2} \right] \\
&= -n \left[-\frac{1}{p} - \frac{1}{(1-p)} \right] \\
&= -n \left[-\frac{1-p}{p(1-p)} - \frac{p}{p(1-p)} \right] \\
&= -n \left[\frac{-1}{p(1-p)} \right] \\
&= \frac{n}{p(1-p)}
\end{aligned}$$

$$\tau(p) = p, \tau'(p) = 1 \implies CRLB = \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n}$$

This is the variance of \bar{X} , indicating that \bar{X} attains the CRLB and thus is the MVUE.

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$X_i \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$. Find CRLB for $\tau(\sigma^2) = \sigma^2$, and confirm that $W = \frac{1}{n} \sum X_i^2$ is the MVUE

As shown in Problem 1(d), the information matrix element for σ^2 is $\frac{n}{2\sigma^4}$, when μ is unknown: with μ known the 2 disappears.

$\tau'(\sigma^2) = \frac{d}{d\sigma^2} \sigma^2 = 1 \implies CRLB = \frac{1}{\frac{n}{2\sigma^4}} = \frac{(\sigma^2)^2}{n}$. This is the same format as $W = \frac{\sum X_i^2}{n}$ meaning that W is the MVUE.