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1.1 Point Estimation

Definition: A point estimator is any scalar (or vector) -valued function of the sample. $(x_1, ...x_n) \sim f(x|\theta)$

A point estimator for $\tau(\theta)$ is a statistic $T(x)$ with the purpose of approximating $\tau(\theta)$

1.2 Method of Moments

The k-th moment of a r.v. X is $\mu_k(\theta) = \mathbb{E}_\theta(X^k) = \int\limits_{\mathbb{X}} x^k f(x|\theta)dx$

Given an iid sample $x_1...x_n \overset{\text{iid}}{\sim} f(x|\theta)$ we have sample moments: $\hat{\mu}_k = \frac{1}{n}\sum\limits_{i=1}^{n} x_i^k$

Suppose $\theta \in \Theta \in \mathbb{R}^P$, and that $\mu_k(\theta)$ exists and is fininte for k = 1, ..., p.

Definition: The method of moments estimator of θ is the solution to the system of equations:

$$\mu_1(\theta) = \hat{\mu}_1 \tag{1}$$
$$. \tag{2}$$
$$. \tag{3}$$
$$\mu_p(\theta) = \hat{\mu}_p \tag{4}$$

We call it $\hat{\theta}_{MM}$

Example: $X_i \overset{\text{iid}}{\sim} Beta(\alpha, \beta), \theta = (\alpha, \beta)$

$\mu_1(\theta) = \frac{\alpha}{\alpha+\beta}$

$\mu_2(\theta) = Var_\theta(x_1) + \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2}$

$\frac{\alpha}{\alpha+\beta} = \hat{\mu}_1, \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2} = \hat{\mu}_2$

$\beta = \left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)\alpha$ from the first equation. Plug this into the second expression and solve:

$\hat{\alpha}_{MM} = \hat{\mu}_1\left[\frac{\hat{\mu}_1(1-\hat{\mu}_1)}{\hat{\mu}_2-\hat{\mu}_1^2} - 1\right]$

$\implies \hat{\beta}_{MM} = \frac{(1-\hat{\mu}_1)}{\hat{\mu}_1}\hat{\alpha}_{MM}$

Example: $x_i \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$

$\mu_1(\theta) = \hat{\mu}_1, \mu_1(\theta) = \mu, \implies \hat{\mu}_{MM} = \mu$

$\mu_2(\theta) = Var_\theta(x_1) + \mu_1^2 = \sigma^2 + \mu^2 = \hat{\mu}_2$

$\sigma^2_{MM} = \hat{\mu}_2 - \hat{\mu}_1^2 = \frac{n-1}{n}s^2$

Note: If μ, σ^2 are the mean/variance of any family, their MM estimators are $\overline{X}, \frac{n-1}{n}s^2$.

For families where parameters are not just the mean and variance, you can find it via two ways: use mean/variance in MM, then calculate parameters, or use parameters in MM then calculate mean/variance. Both yield same results.

Fact: MM estimators are invariant of re-parameterizations.

Let $\eta = \eta(\theta)$ be a 1:1 mapping (intervtible). Then, $\hat{\eta}_{MM} = \eta(\hat{\theta}_{MM})$

Let's say $x_i \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ and we want to estimate $\tau(\theta) = \frac{\mu}{\sigma}$. This isn't 1:1. What can we do? We can do the 'Transformations' method and create a second value τ_2, etc. We can also just plug in the estimators.

<u>Definition</u>: The MM estimator for a parametric function $\tau(\theta)$ is just $\hat{\tau}_{MM}(\theta) = \tau(\hat{(\theta)}_{MM})$

<u>Properties</u>:

- MM equations may have a unique solution, no solution, or many solutions
- Often, MM estimators are used as initial values for another estimation technique (ie. a root finding method)
- Why should it work? Let's say θ^* is the true value of θ. Then Law of Large Numbers says: $\hat{\mu}_k \xrightarrow{P} \mu_k(\theta*)$. Then we are solving $\mu_k(\theta) = \hat{\mu}_k \approx \mu_k(\theta^*)$