HW1

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1. $X_1, \ldots X_n \overset{iid}{\sim}$ $\theta_1 = E(X_i)$ $\theta_2 = Var(X_i)$

Show $\hat{\theta}_{2\,MM} = \bar{\mu}_2 - \bar{\mu}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$m_1 = \bar{X}$ $m_2 = \frac{1}{n}\sum X_i^2$

$\bar{\mu}_1 = E(X_i) = \theta_1 \Rightarrow \hat{\theta}_1 = \bar{\mu}_1$

$\bar{\mu}_2 = E(X_i^2) = \underbrace{Var(X_i)}_{\theta_2} + \underbrace{E(X_i)^2}_{\bar{\mu}_1^2}$

$\hat{\theta}_2 = \bar{\mu}_2 - \bar{\mu}_1^2$

$m_1 = \mu_1 \Rightarrow \bar{X} = \bar{\mu}_1$ $m_2 = \bar{\mu}_2 \Rightarrow \frac{1}{n}\sum X_i^2 = \hat{\mu}_2$

$\hat{\theta}_2 = \bar{\mu}_2 - \bar{\mu}_1^2 = \frac{1}{n}\sum X_i^2 - \bar{X}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$

$\hat{\theta}_2 = \frac{1}{n}\sum(X_i - \bar{X})^2$

2. CB 7.6 b, c

$X_1 \ldots X_n \overset{iid}{\sim} f(x|\theta) = \theta x^{-2}$ $0 < \theta \le x < \infty$

b. Find MLE of θ

$L(\theta|X) = \prod_{i=1}^{n} \theta x_i^{-2} I_{(\theta \le x_i < \infty)} = \theta^n \prod_{i=1}^{n} x_i^{-2} I_{(\theta \le x_1 \le \infty)}$

To maximize this in theta, we need to make it as big as possible. However, it is bounded due to the indicator function which says that theta must be smaller than all the x's. Thus:

$\hat{\theta} = X_{(1)}$

c. Find $\hat{\theta}_{MM}$ $\mu_1 \overset{=\hat{\theta}_{MM}}{=} E(X) = \int_{\theta}^{\infty} \theta x^{-2} \cdot x\, dx = \int_{\theta}^{\infty} \theta x^{-1}\, dx = \theta \ln(x)\Big|_{\theta}^{\infty} = \infty$

$\hat{\theta}_{MM}$ DOES NOT EXIST

3. CB 7.7 $X_1, \ldots X_n \overset{iid}{\sim}$ $\begin{cases} \theta = 0: & f(x|\theta) = 1 \quad 0 < x < 1 \\ \theta = 1: & f(x|\theta) = \frac{1}{2\sqrt{x}} \quad 0 < x < 1 \end{cases}$

Find MLE of θ

Two cases: $L(0|x) = 1$ $0 < x < 1$ $L(0|x) \ge L(1|x)$ when $1 \ge \prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}}$

$$L(1 \mid x) = \prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}}$$

AND $L(1 \mid x) > L(0 \mid x)$ OTHERWISE

$$\therefore \hat{\theta} = \begin{cases} 1 & 1 \geq \prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} \\ 0 & \text{OTHERWISE} \end{cases}$$

4. CB 7.9 $\quad X_1 \ldots X_n \overset{iid}{\sim} f(x \mid \theta) = \frac{1}{\theta} \quad 0 \leq x \leq \theta \quad 0 < \theta$

FIND $\hat{\theta}_{MM}$ & $\hat{\theta}_{MLE}$

MM: $m_1 = \bar{x} \quad \mu_1 = E(X) = \int_0^\theta \frac{x}{\theta}\,dx = \frac{1}{2\theta} x^2 \Big|_0^\theta = \frac{\tilde{\theta}}{2} \qquad m_1 = \mu_1 \Rightarrow \bar{X} = \frac{\tilde{\theta}}{2} \qquad \tilde{\theta} = 2\bar{X}$

$$E(\tilde{\theta}) = E(2\bar{x}) = 2E(X) = 2\frac{\theta}{2} = \theta$$

$$VAR(\tilde{\theta}) = VAR(2\bar{X}) = 4\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

BECAUSE UNIFORM

$$\hat{\theta}_{MM} = 2\bar{x} \quad E(\hat{\theta}) = \theta \quad VAR(\hat{\theta}) = \frac{\theta^2}{3n}$$

MLE: $L(\theta \mid x) = \prod_{i=1}^{n} \frac{1}{\theta} I_{(0 \leq x_i \leq \theta)} = \frac{1}{\theta^n} I_{(0 \leq x_i \leq \theta)}$: $\frac{1}{\theta^n}$ GETS SMALLER AS θ INCREASES: WANT TO MINIMIZE θ. θ LIMITED TO BE LARGER THAN ALL x_i, SO $\hat{\theta}_{MLE} = X_{(n)}$

$$\hat{\theta}_{MLE} = X_{(n)} \qquad f_{X_{(n)}} = n\left[F_{X_i}\right]^{n-1} \cdot f_{X_i} = n\left(\frac{x}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} = n\frac{x^{n-1}}{\theta^n}$$

$$\therefore E(\hat{\theta}_{MLE}) = \int_0^\theta n\frac{x^n}{\theta^n}\,dx = \frac{n}{\theta^n}\frac{1}{n+1}x^{n+1}\Big|_0^\theta = \frac{n}{n+1}\frac{\theta^{n+1}}{\theta^n} = \frac{n\theta}{n+1}$$

$$E(\hat{\theta}_{MLE}^2) = \int_0^\theta n\frac{x^{n+1}}{\theta^n}\,dx = \frac{n}{\theta^n}\frac{1}{n+2}x^{n+2}\Big|_0^\theta = \frac{n}{n+2}\theta^2$$

$$VAR(\hat{\theta}_{MLE}) = E(\hat{\theta}_{MLE}^2) - E(\hat{\theta}_{MLE})^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \theta^2\left(\frac{n(n+1)^2 - n^2(n+2)}{n+2\,(n+1)^2}\right) =$$

$$\theta^2\left(\frac{n^3 + 2n^2 + n - n^3 - 2n^2}{n+2(n+1)^2}\right) = \theta^2\left(\frac{n}{n+2(n+1)^2}\right)$$

$$\hat{\theta}_{MLE} = X_{(n)} \quad E(\hat{\theta}) = \frac{n\theta}{n+1} \quad VAR(\hat{\theta}) = \theta^2\left(\frac{n}{(n+2)(n+1)^2}\right)$$

Method of moments gives us an unbiased estimator for parameter theta. The bias on the MLE is small when n is large, and when n is large, the variance of the MLE is less than that of the MM. Thus, for large n we would probably prefer the MLE

5. CB 7.11 $\quad X_1 \ldots X_n \overset{iid}{\sim} f(x \mid \theta) = \theta x^{\theta - 1} \quad 0 \leq x \leq 1 \quad 0 < \theta < \infty$

a. FIND $\hat{\theta}_{MLE}$. SHOW AS $n \to \infty$, VAR $\to 0$: $\ell(\theta \mid x) = \sum_{i=1}^{n} \ln(\theta x_i^{\theta-1})\, I_{(0 \leq x \leq 1)} = \sum_{i=1}^{n} \ln(\theta) + (\theta - 1)\ln(x_i)\, I_{(0 \leq x \leq 1)}$

$= n\ln(\theta) + (\theta - 1)\sum \ln(x_i)$

$\frac{d}{d\theta}\ell(\theta|x) = \frac{n}{\theta} + \sum \ln(x_i) = 0 \qquad \sum \ln(x_i) = -\frac{n}{\theta} \qquad \hat{\theta} = \frac{-n}{\sum \ln(x_i)}$

$\frac{d^2}{d\theta^2}\ell(\theta|x) = -\frac{n}{\theta^2} < 0$ THUS MLE

INVERTED GAMMA

At this point, I used the solution manual to help me identify that: $-\sum \ln(x_i) \sim \text{GAMMA}(n, \frac{1}{\theta})$ SO $\hat{\theta} = \frac{n}{\text{GAMMA}(n, \frac{1}{\theta})}$

$\text{VAR}(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)^2(n-2)}$ Just looking at the variance, it's pretty obvious that as n increases, the denominator increases meaning the variance gets smaller, approaching 0

b. FIND $\hat{\theta}_{MM}$:

$m_1 = \bar{x} \quad \mu_1 = E(X) = \int_0^1 \theta x^{\theta} dx = \frac{\theta}{\theta+1} x^{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1}$

$\frac{\theta}{\theta+1} = \bar{x} \qquad \theta = \bar{x}(\theta+1) \qquad \theta = \theta\bar{x} + \bar{x} \qquad \theta - \theta\bar{x} = \bar{x} \qquad \theta = \frac{\bar{x}}{1-\bar{x}}$

$\hat{\theta}_{MM} = \frac{\bar{x}}{1-\bar{x}}$

6. CB 7.12 a $X_1 \dots X_n \overset{iid}{\sim} f(x|\theta) = \theta^x(1-\theta)^{1-x} \quad x = 0 \text{ OR } 1 \quad 0 \le \theta \le \frac{1}{2}$

a. FIND $\hat{\theta}_{MLE}$ & $\hat{\theta}_{MM}$

$\hat{\theta}_{MLE}: \ell(\theta|x) = \sum_{i=1}^{n} \ln(\theta^{x_i}(1-\theta)^{1-x_i}) = \sum_{i=1}^{n}\left[x_i\ln(\theta) + \ln(1-\theta) - x_i\ln(1-\theta)\right]$

$= \ln(\theta)\sum x_i + n\ln(1-\theta) - \ln(1-\theta)\sum x_i = n\ln(1-\theta) + \sum x_i\left[\ln(\theta) - \ln(1-\theta)\right]$

EXTREMES WILL BE WHEN $\sum x_i = 0$ OR $\sum x_i = n$

$\sum x_i = 0 \Rightarrow n\ln(1-\theta)$

$\sum x_i = n \Rightarrow n\ln(1-\theta) + n\ln(\theta) - n\ln(1-\theta) = n\ln(\theta)$

$\frac{d}{d\theta}\ell(\theta|x) = -\frac{n}{1-\theta} + \frac{\sum x_i}{\theta} + \frac{\sum x_i}{1-\theta} = 0 \qquad \frac{-n\theta + (1-\theta)\sum x_i + \theta\sum x_i}{\theta(1-\theta)} = 0 \qquad \sum x_i - \theta\sum x_i + \theta\sum x_i - n\theta = 0$

$\sum x_i = n\theta \qquad \hat{\theta} = \frac{\sum x_i}{n}$

$\hat{\theta}_{MLE} = \frac{\sum x_i}{n}$, BUT BOUNDED BY $\frac{1}{2}$, THUS WHEN $\sum x_i > \frac{1}{2}$, $\hat{\theta} = \frac{1}{2}$

$\hat{\theta}_{MLE} = \text{MIN}(\bar{x}, \frac{1}{2})$

$\hat{\theta}_{MM}: m_1 = \bar{x} \quad \mu = E(\text{BERNOULLI } \theta) = \theta \qquad \hat{\theta}_{MM} = \bar{x}$ (ALSO BOUNDED BY $\frac{1}{2}$)

7. CB 7.14

$f(x|\lambda) = \frac{1}{\lambda}e^{-x/\lambda}, \quad x > 0 \qquad f(y|\mu) = \frac{1}{\mu}e^{-y/\mu}, \quad y > 0$

$z = \text{MIN}(X, Y) \qquad W = \begin{cases} 1 & z = x \\ \dots \end{cases}$

From 4.26: $f(z,w) = \frac{\mu}{\mu+\lambda}\left[1 - e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}\right]$

I used this resource to find an alternate version of the Joint which we can actually use:
https://homepage.cs.uiowa.edu/~luke/classes/194/homework.pdf

$$f(z,w) = \frac{1}{\lambda^{w}\mu^{1-w}} e^{-z\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)}$$

Find MLE of μ, λ

$$f(z,w \mid \mu,\lambda) = \frac{1}{\lambda^{\sum w_i}\mu^{n-\sum w_i}} e^{-\sum z_i\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)} = \frac{1}{\lambda^{\sum w_i}} e^{-\frac{\sum z_i}{\lambda}} \cdot \frac{1}{\mu^{n-\sum w_i}} e^{-\frac{\sum z_i}{\mu}}$$

$$\ln(f) = -\sum w_i \ln(\lambda) - \frac{\sum z_i}{\lambda} - \left(n - \sum w_i\right)\ln(\mu) - \frac{\sum z_i}{\mu}$$

$$\frac{\partial}{\partial \lambda} = -\frac{\sum w_i}{\lambda} + \frac{\sum z_i}{\lambda^2} = 0 \qquad \sum w_i = \frac{\sum z_i}{\lambda} \qquad \hat{\lambda} = \frac{\sum z_i}{\sum w_i}$$

$$\frac{\partial}{\partial \mu} = \frac{-n + \sum w_i}{\mu} + \frac{\sum z_i}{\mu^2} = 0 \qquad n - \sum w_i = \frac{\sum z_i}{\mu} \qquad \mu = \frac{\sum z_i}{n - \sum w_i}$$

8. $X_1 \ldots X_n$ FIXED, KNOWN. $Y_1 \ldots Y_n \sim \beta x_i + \epsilon_i$ β UNKNOWN, $\epsilon_i \overset{iid}{\sim} N(0,\sigma^2)$

FIND MLE OF β AND ITS SAMPLING DISTRIBUTION

$$L(\beta,\sigma \mid Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2} = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{\frac{-\beta^2 \sum x_i^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum y_i^2 + \frac{\beta}{\sigma^2}\sum x_i y_i}$$

$$\ell(\beta,\sigma \mid Y) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum y_i^2 + \frac{\beta}{\sigma^2}\sum x_i y_i - \frac{\beta^2}{\sigma^2}\sum x_i^2$$

$$\frac{\partial}{\partial \beta}\ell(\beta,\sigma \mid Y) = \frac{1}{\sigma^2}\left(\sum x_i y_i - \beta \sum x_i^2\right) = 0 \qquad \sum x_i y_i = \beta \sum x_i^2$$

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

x_i IS CONSTANT, SO $\frac{x_i}{\sum x_i^2}$ IS CONSTANT $\Rightarrow$ $\hat{\beta} = \sum c_i Y_i$

4.6.10: $Z = \sum a_i X_i \sim N\left(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2\right)$ $\quad \mathrm{VAR}\,\beta = \sum c_i^2 \,\mathrm{VAR}(Y_i) = \sum\left(\frac{x_i}{\sum x_i^2}\right)^2 \sigma^2 = \frac{\sum x_i^2}{\left(\sum x_i^2\right)^2}\sigma^2$

$$\text{SO, } \hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right)$$