1 1

$X_i \overset{\text{iid}}{\sim} Pois(\lambda), \lambda > 0$. Use only the definition of sufficiency to prove that $T(X) = \sum_{i=1}^{n} X_i$ is sufficient for λ

Class Definition: A statistic T is sufficient if the distribution of $X|T$ does not depend on θ.

Because the X_is are independent, the sum is also Poisson: $T(X) \sim Pois(n\lambda)$

$P(\underset{\sim}{\boldsymbol{X}} = \underset{\sim}{\boldsymbol{x}}|T(X)) = \frac{P(\underset{\sim}{\boldsymbol{X}}=\underset{\sim}{\boldsymbol{x}})}{P(T(X)=t)} = \frac{\prod \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}}{\frac{e^{-n\lambda}(n\lambda)^t}{t!}} = \frac{e^{-n\lambda}\lambda^{\sum x_i} t!}{\prod x_i! e^{-n\lambda} n^t \lambda^t}$

We defined above that in this case, $t = \sum x_i$, otherwise the above is zero/doesn't mean anything:

$\therefore = \frac{\lambda^{\sum x_i} t!}{\prod x_i! n^t \lambda^{\sum x_i}} = \underline{\frac{t!}{\prod x_i! n^{\sum x_i}}}$

That final expression does not depend on λ, thus $T(X) = \sum x_i$ is a sufficient statistic.

2 CB2 6.1

X is one observation from $N(0, \sigma^2)$. Is $|X|$ a sufficient statistic?

WE can ignore the front coefficient that only depends on σ^2:

$X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ with $\mu = 0 \implies X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

Because the mean is 0, that leaves only x^2 in the numerator of the exponent, which we can also rewrite as $|x|^2$, thus:

$X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}} = g(T(X), \sigma^2)h(x), h(x) = 1$

Because we can factor it, that means $T(X) = |X|$ is a sufficient statistic (though I think that only holds when $\mu = 0$).

3 CB2 6.2

$f_{X_i}(x|\theta) = \begin{cases} e^{i\theta - x} & x \geq i\theta \\ 0x < i\theta \end{cases}$

Prove that $T = min_i(\frac{X_i}{i})$ is a suffucient statistic for θ

$f(\underset{\sim}{\boldsymbol{x}}|\theta) = \prod e^{i\theta - x} I_{(x_i \geq i\theta)}$

For the indicator to hold, we need all $x_i \geq i\theta$, which means we just need the $min(x_i) \geq \theta$, so we can plug in $min(x_i)$ in that portion:

$= e^{i\theta - \sum x_i} I_{x_{i_{(1)}} \geq i\theta} = \underline{[e^{i\theta} I_{x_{i_{(1)}} \geq i\theta}] e^{-\sum x_i}} = g(T(X), \theta)h(x)$

4 CB2 6.6

$X \sim Gamma(\alpha, \beta)$. Find a 2D sufficient statistic

Gamma is an exponential family, so we can just put it into exponential family form:

$L(\alpha, \beta|X) \propto \prod x_i^{\alpha-1} e^{-\beta x_i} = \prod e^{(\alpha-1)ln(x_i) - \beta x_i} = e^{(\alpha-1)\sum ln(x_i) - \beta \sum x_i}$

The sufficient statistics are all the $T_i(X)$ in the exponent, so the sufficient statistic is:

$T(X) = (\sum x_i, \sum ln(x_i))$

5 CB2 6.13

$X_1, X_2 \overset{\text{iid}}{\sim} \alpha x^{\alpha-1} e^{-x^\alpha}, x > 0, \alpha > 0$. Show that $\frac{ln(X_1)}{ln(X_2)}$ is ancillary.

Let $y = ln(x) \implies f(y|\alpha) = \alpha(e^y)^{\alpha-1} e^{-(e^y)^\alpha} e^y = \alpha e^{\alpha y - y + y - e^{-y\alpha}} = \alpha e^{\alpha y - e^{-y\alpha}}$

This can be rewritten as a scale-family: $\frac{1}{1/\alpha} e^{\frac{y}{1/\alpha} - e^{(-\frac{y}{1/\alpha})}}$

This is a scale transformation of some variable Z which is not dependent on α (ie. $Z \sim e^{z - e^{-z}}$)

Thus: $\frac{ln(X_1)}{ln(X_2)} = \frac{Y_1}{Y_2} = \frac{(1/\alpha)Z_1}{(1/\alpha)Z_2} = \frac{Z_1}{Z_2}$. This does not depend on α, thus $\frac{ln(X_1)}{ln(X_2)}$ is an ancillary statistic.

6 6

$X \sim Pois(\lambda)$. Cannot use 6.2.25, the theorem of completeness for exponential families.

i) Show that $T(X) = \sum_{i=1}^{n} X_i$ is a complete statistic when $\lambda > 0$

FINISH MEEEEE

ii) Show that for any integer $r > 0$: $\mathbb{E}[X_i(X_i - 1)(X_i - 2)...(X_i - r + 1)] = \lambda^r$

$\mathbb{E}[X_i(X_i - 1)(X_i - 2)...(X_i - r + 1)] = \mathbb{E}[X_i]\,\mathbb{E}[X_i - 1]\,\mathbb{E}[X_1 - 2]...\,\mathbb{E}[X_i - r + 1] = \mathbb{E}[X_i](\mathbb{E}[X_i] - 1)(\mathbb{E}[X_i] - 2)...(\mathbb{E}[X_i] - r + 1) = \lambda(\lambda - 1)(\lambda - 2)...(\lambda - r + 1) = \frac{\lambda!}{(\lambda - r)!}$

FINISH MEEEEEE

iii) Restrict $\lambda \in 1, 3$. Show that the $T(X)$ from part (i) is not a complete statistic.

FINISH MEEEEE

7 CB2 6.15

FINISH MEEEEE

8 CB2 6.22

$f(x|\theta) = \theta x^{\theta-1}$

a) Is $\sum X_i$ a sufficient statistic?

$L(\theta|x) = \prod \theta x_i^{\theta-1} = \theta^n \prod e^{(\theta-1) ln(x_i)} = \theta^n e^{(\theta-1)\sum ln(x_i)}$

$\sum ln(x_i)$ is sufficient, but there is no direct linear transformation to $\sum x_i$, meaning $\sum x_i$ is not sufficient.

b) FINISH MEEEEE