HW2

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1. $X_1 \dots X_n \sim \text{Gamma}(\alpha, \beta)$

a. $\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} = e^{\ln x^{\alpha-1}} e^{-1/\beta} e^{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{(\alpha-1)\ln(x) - \frac{1}{\beta}x} = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\alpha\ln(x) - \frac{1}{\beta}x - \ln(x)}$

$\eta_1 = \alpha \qquad \alpha = \eta_1$

$\eta_2 = -\frac{1}{\beta} \qquad \beta = -\frac{1}{\eta_2} \quad \Rightarrow \quad \frac{\eta_2^{\eta_1}}{\Gamma(\eta_1)} e^{\eta_1 \ln(x) + \eta_2 x - \ln(x)}$

b. $\ell(\eta_1, \eta_2 | \underset{\sim}{x}) = \sum_{i=1}^{n} \left[\eta_1 \ln(\eta_2) - \ln(\Gamma(\eta_1)) + \eta_1 \ln(x_i) - \eta_2 x_i - \ln(x_i)\right] = n\eta_1 \ln(-\eta_2) - n\ln(\Gamma(\eta_1)) + \eta_1 \sum \ln(x_i) + \eta_2 \sum x_i - \sum \ln(x_i)$

$\frac{\partial}{\partial \eta_1} = n\ln(-\eta_2) - n\frac{\Gamma'(\eta_1)}{\Gamma(\eta_1)} + \sum \ln(x_i) = 0 \Rightarrow \frac{1}{n}\sum \ln(x_i) = \frac{\Gamma'(\eta_1)}{\Gamma(\eta_1)} - \ln(-\eta_2)$

$\frac{\partial}{\partial \eta_2} = \frac{n\eta_1}{-\eta_2} - \sum x_i = 0 \Rightarrow \frac{1}{n}\sum x_i = \frac{-\eta_1}{\eta_2}$

c. $\frac{1}{n}\sum \ln(x_i) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \ln(\beta)$

$\frac{1}{n}\sum x_i = \frac{\alpha}{\beta}$

2. $X_1 \dots X_n \sim \text{Pois}(\lambda), \lambda > 0$

a. $\pi(\lambda) = \text{Gamma}(\alpha, \beta), \alpha, \beta > 0$

This is conjugate because the Poisson has the parameter^ something * e^(paramter), as does the Gamma, so they combine nicely into another thing that matches the Gamma Kernel

b. $f(\underset{\sim}{y}|\lambda) = \frac{\lambda^{y} e^{-\lambda}}{y!} \Rightarrow \pi(\lambda|y_i) = \prod_{i=1}^{n}\left[\frac{\lambda^{y_i} e^{-\lambda}}{y_i!}\right] \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$= \frac{\beta^{\alpha}}{\Gamma(\alpha)\, y_i!^{n}} \lambda^{\alpha + \sum y_i - 1} e^{-\lambda(1+\beta)}$

$= \frac{\beta^{\alpha}}{\Gamma(\alpha)\, y_i!^{n}} \frac{\Gamma(\alpha + \sum y_i)}{(1+\beta)^{\alpha + \sum y_i}} \text{Gamma}(\alpha + \sum y_i, 1+\beta)$

c. Mean: • $(\alpha + \sum y_i)(1+\beta)^{-1}$

Var: • $(\alpha + \sum y_i)(1+\beta)^{-2}$

3. $f(x|\theta) = \theta(1-\theta)^{x} \quad \theta \in (0,1) \quad x = 0, 1, 2$

a. Find MLE of θ: $\ell(\theta, x) = \sum_{i=1}^{n} \ln(\theta) + x_i \ln(1-\theta) = n\ln(\theta) + \sum x_i \ln(1-\theta)$

$\frac{d}{\,} \quad \frac{n}{\,} \quad \sum x_i$

$$d\theta = \frac{}{\theta} - \frac{}{1-\theta} = 0 \quad \Rightarrow \quad n - n\theta - \theta \sum x_i = 0 \qquad n = \theta(n + \sum x_i) \qquad \theta = \frac{}{n + \sum x_i} = \frac{}{\frac{1}{n}n + \frac{1}{n}\sum x_i}$$

$$= \frac{1}{1 + \frac{1}{n}\sum x_i}$$

$$\frac{d^2}{d\theta^2} = -\frac{n}{\theta^2} + \frac{\sum x_i}{(1-\theta)^2}$$

$$\hat{\theta} = \frac{n}{n + \sum x_i}$$

b. $\eta = \frac{1-\theta}{\theta} \qquad \theta\eta = 1-\theta \qquad \theta(\eta+1) = 1 \qquad \theta = \frac{1}{\eta+1}$

$$\hat{\theta} = \frac{n}{n+\sum x_i} \Rightarrow \frac{1}{\eta+1} = \frac{n}{n+\sum x_i} \qquad \eta + 1 = \frac{n+\sum x_i}{n} = 1 + \frac{\sum x_i}{n}$$

$$\Rightarrow \hat{\eta} = \frac{\sum x_i}{n}$$

c. SHOWN ABOVE: $f(x \mid \eta) = \frac{1}{\eta+1}\left(1 - \frac{1}{\eta+1}\right)^x = \frac{1}{\eta+1}\left(\frac{\eta}{\eta+1}\right)^x = \frac{\eta^x}{(\eta+1)^{x+1}}$

$\theta = \frac{1}{\eta+1}$

d. $\pi(\eta) \propto \eta^{\alpha-1}(1+\eta)^{-\beta} \quad \eta > 0,\ 0 < \alpha < \beta$

$$\pi(\eta \mid x_i) \propto \prod_{i=1}^{n} \frac{\eta^{x_i}}{(\eta+1)^{x_i+1}} \eta^{\alpha-1}(1+\eta)^{-\beta} = \frac{\eta^{\sum x_i}\eta^{\alpha-1}}{(\eta+1)^{\sum x_i + n}(1+\eta)^{\beta}} = \eta^{\alpha+\sum x_i - 1}(\eta+1)^{-(\beta+\sum x_i+n)}$$

SAME FORM AS PRIOR

$$\int_0^\infty \eta^{\alpha+\sum x_i - 1}(\eta+1)^{-(\beta+\sum x_i+n)}\,d\eta = 1$$

BETA PRIME: $a: \alpha + \sum x_i \qquad$ NEED $-a-b$: $\quad \alpha + \sum x_i + b = \beta + \sum x_i + n$

$b = \beta + \alpha + n$

$BP(\alpha + \sum x_i,\ \beta + \alpha + n)$

$$c = \frac{\Gamma(\alpha + \sum x_i)\,\Gamma(\beta+\alpha+n)}{\Gamma(2\alpha + \beta + \sum x_i + n)}$$

e. WE JUST SHOWED THE POSTER FOR IS A $BP(\alpha + \sum x_i,\ \beta + \alpha + n)$

$$\delta_\pi = E(\pi(\theta \mid x)) = \frac{\alpha + \sum x_i}{\beta + \alpha + n - 1}$$

f. $$\int \frac{(t-\eta)^2}{\eta^{\eta}(1+\eta)} \eta^{\alpha+\sum x_i - 1}(1+\eta)^{-(\beta+\sum x_i+n)}\,d\eta = \int (t-\eta)^2 \eta^{\alpha+\sum x_i - 1 - 1}(1+\eta)^{-(\beta+\sum x_i+n+1)}$$

$a = \alpha + \sum x_i - 1 \qquad -\beta - \sum x_i - n - 1 = -\alpha - \sum x_i + 1 - b \Rightarrow b = \beta + n - \alpha + 2$

$$\delta_\pi = E(\tilde{\pi}(\theta \mid x)) = \frac{\alpha + \sum x_i - 1}{\beta + n - \alpha + 1}$$

In terms of parts e and f where we're supposed to plot the risk functions, I'm not really sure what we're supposed to be plotting. The parameter does not show up in the risk so I'm not sure what the variates are supposed to be...

4. $X_i \mid \theta \sim \text{Bern}(\theta) \quad \theta \in (0,1) \qquad \pi(\theta) \sim \text{Unif}(0,1)$

$\rho(\theta) = \frac{(t-\theta)^2}{}$ (Loss)

$\theta(1-\theta)$

FIND BAYES RULE FOR ESTIMATING θ

$$\pi(\theta|x) \propto \text{BERN}(\theta)\ \text{UNIF}(0,1) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \cdot \frac{1}{1-0} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$r_\pi = \int \frac{(t-\theta)^2}{\theta(1-\theta)}\ \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\, d\theta = \int (t-\theta)^2\ \underbrace{\theta^{\sum x_i - 1}(1-\theta)^{n-\sum x_i - 1}}_{\text{BETA}(\sum x_i,\ n-\sum x_i)}\, d\theta$$

$$\Rightarrow \tilde{\pi}(\theta|x) = \text{BETA}\left(\sum x_i,\ n-\sum x_i\right)$$

DUE TO LOSS OF $(t-\theta)^2$:

$$r_\pi = E(\tilde{\pi}(\theta|x)) = E(\text{BETA}) = \frac{\sum x_i}{\sum x_i + n - \sum x_i} = \frac{\sum x_i}{n} = \underline{\bar{x}}$$