Sequences and Strings

4.3

Sequences

Definition: *Sequence* **[1st Attempt]**

An ordered list of items

Notation:

- Labels are lower-case letters
- Elements are subscripted: $e_1, e_2, \ldots$
- $\{e_n\} \Rightarrow e$ is an n-element sequence.

Example: Soup!

Cost sequence: $s = 2,4,6,8,10,\ldots$ ($2 per can)

Soup Saturday: Buy 3 cans of soup, get on free!

$s' = 2,4,6,6,8,10,12,12\ldots$ (Not a set!)

Rules

Recall: $\sum_{i=1}^{n} 2i$ ← **Sequence defined by the rule 2i**

Example:

$s_n = 2n$ defines the original soup price sequence

$n^2 + 1, n \geq 0$ defines the infinite sequence

$1, 2, 5, 10, 17, \ldots$

More notation:

Infinite sequences:

1. Ellipses (as in $1, 2, 5, 10, 17, \ldots$)
2. $\{d_n\}_{n=1}^{\infty}$

Sequences and Functions

Definition: *Sequence* **[Final Version]**

A sequence is the ordered range of a function from a set of integers to some set S

Example:

$o(n) = 2n - 1$ on the domain $\{1,2,3,4,5\}$ defines the sequence $1,3,5,7,9$

As a relation: $\{(1,1), (2,3), (3,5), (4,7), (5,9)\}$

Range of $\{1,3,5,7,9\}$

(Thus, the “ordered range” wording)

Arithmetic and Geometric Sequences

Definition: *Arithmetic Sequence (a.k.a. Arithmetic Progression)*

In an arithmetic sequence, the *common difference* $d = a_{n+1} - a_n$ is constant

Definition: *Geometric Sequence (a.k.a. Geometric Progression)*

In a geometric sequence, the *common ratio* $r = \dfrac{g_{n+1}}{g_n}$ is constant

Example:

In o: 1, 3, 5, 7,9 $\qquad d = 2$

In $g : 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27} \ldots,$ $\qquad r = \frac{2}{3}$

Arithmetic Series

- The sum of the terms of an arithmetic sequence (a.k.a arithmetic series): $s_n = a_1 + \ldots + a_n = \frac{1}{2}n(a_1 + a_n)$
- Here's why: First, note that $a_n = a_1 + (n-1)d$.
- Next, here are two expressions for s_n:
    - $s_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$
    - $s_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \ldots + (a_n - d) + a_n$
- Sum these expressions, and the d terms cancel, leaving:
    - $2s_n = na_1 + na_n$ or $s_n = \frac{1}{2}n(a_1 + a_n)$
    - Ex: In 1,3,5 $d = 2, a_1 = 1, a_1 + d = 3$ and $a_1 + 2d = 5$

Increasing Sequences

Definition: *Increasing Sequence*

An increasing sequence labeled i is ordered such that $i_n \leq i_{n+1}$.

Definition: *Non-Decreasing Sequence*

A non-decreasing sequence labeled i is ordered such that $i_n \leq i_{n+1}$ [Same as increasing!]

Definition: *Strictly Increasing Sequence*

A strictly increasing sequence labeled i is ordered such that $i_n < i_{n+1}$

Decreasing Sequences

Definition: *Decreasing Sequence*

A decreasing sequence labeled i is ordered such that $i_n \geq i_{n+1}$.

Definition: *Non-Increasing Sequence*

A non-increasing sequence labeled i is ordered such that $i_n \geq i_{n+1}$ [Same as decreasing!]

Definition: *Strictly Decreasing Sequence*

A strictly decreasing sequence labeled i is ordered such that $i_n > i_{n+1}$

Examples: Increasing/Decreasing Sequences

- The sequence g = 1,2,2,2,6,8,8,9 is:
  - Increasing
  - Non-Decreasing

- $h_n = \frac{1}{n},\ 4 \le n \le 7 \ \ (h = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7})$
- $\{h_n\}_{n=4}^{7}$ is:
  - Decreasing, Non-Increasing
  - Strictly Decreasing

Subsequences

Definition: *Subsequence*

Sequence x is a subsequence of sequence y when the elements of x are found within y in the same relative order

Example:

Is $\frac{1}{4}, \frac{1}{6}$ a subsequence of $\{h_n\}_{n=4}^{7}$?

Is $8,6,2$ a subsequence of $g = 1,2,2,2,6,8,8,9$?

Need to Identify a Sequence?

A great resource for sequences:

The Online Encyclopedia of Integer Sequences

(http://oeis.org/)

Example:

Let's try it! 2,3,5,7,11,13,17

Strings

Definition: *String*

A string is a contiguous finite sequence of 0 or more elements drawn from a set called the *alphabet*

Example:

A sequence of DNA nucleotides (e.g. ATTGACCT) is called a string.

A Java `String` also qualifies (alphabet: UNICODE values)

Strings

- Notation:
  - Lambda (λ) represents the empty (null) string
  - xy means strings x and y are concatenated
  - Superscripts denote repetition of concatenation
  - $|x|$ represents the length of string x
  - A^* is the set of strings that can bw formed using elements of an alphabet A
    - A^* is an infinite set
    - $\lambda \in A^*$

Set Cardinality Revisited

An observation about set cardinality:

Two sets A and B have the same cardinality **iff** there is a bijection from A to B

Definition: *Finite*

A set S is finite if there exists a bijective mapping between it and a set of cardinality $|S|$

Set Cardinality Revisited

Definition: *Countably Infinite (a.k.a. Denumerably Infinite)*

A set is countably infinite if there exists a bijective mapping between the set and either $\mathbb{Z}^*$ or $\mathbb{Z}^+$

Definition: *Countable*

A set is countable if it is finite or countably infinite. (Otherwise, it is *uncountable.*)

Set Cardinality Revisited

Example:

Is the set of digits in the 'house number' of the Gould-Simpson building countable?

1040 E. 4th St.. $\Rightarrow \{0,1,4\}$

This set is finite, so, yes!

Is the set of positive multiples of 5 countable?

$\mathbb{Z}^+$: 1 2 3 4 … $5z$ y

5 10 15 20 … z $\frac{y}{5}$

Invertable → Bijection → Countable! (So, yes!)

Set Cardinality Revisited

Question: Are the positive rational numbers countable?

$$
\begin{array}{ccccc}
^{(1)}\frac{1}{1} & \frac{2}{1}^{(2)} & \overset{(5)}{\frac{3}{1}} & \frac{4}{1}^{(6)} & \cdots \\[6pt]
^{(3)}\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\[6pt]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\[6pt]
^{(4)}\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\[6pt]
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Just skip duplicates

$$R = \{(1,\frac{1}{1}),(2,\frac{2}{1}),(3,\frac{1}{2}),(4,\frac{1}{3}),(5,\frac{3}{1}),\ldots\}$$

$$R^{-1} = \{(\frac{1}{1},1),(\frac{2}{1},2),(\frac{1}{2},3),(\frac{1}{3},4),(\frac{3}{1},5),\ldots\}$$

It is invertable, therefore it is a bijection.

Yes: (This is an application of a 'pairing function which invertible maps after duplicates are removed.) (With duplicates, the function is not invertable!)

(This is an example of a *boustrophedonic* path.)

Set Cardinality Revisited

Conjecture: A pairing function for $\mathbb{R}$ cannot exist

Proof (Contradiction): Assume that a pairing function for the reals does exist. Given the set of real numbers, form a new number (not in the pairing) by changing the d^{th} digit of the d^{th} number. For example:

$$\begin{array}{cccccccc}
0 & . & \boxed{8} & 9 & 4 & 6 & \dots \\
0 & . & 1 & \boxed{2} & 0 & 4 & \dots \\
0 & . & 7 & 7 & \boxed{7} & 6 & \dots \\
0 & . & 4 & 5 & 6 & \boxed{7} & \dots \\
\vdots & . & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & . & \boxed{5} & \boxed{4} & \boxed{9} & \boxed{1} & \dots
\end{array}$$

$0.5491\ldots$ **won't be in pairing, because it has 1 digit different than every number in the pairing**

The result must be a value not in the set of reals, yet it is a real. This is a contradiction (This is called *Cantor's Diagonal Argument*).

Therefore, a pairing function for $\mathbb{R}$ cannot exist