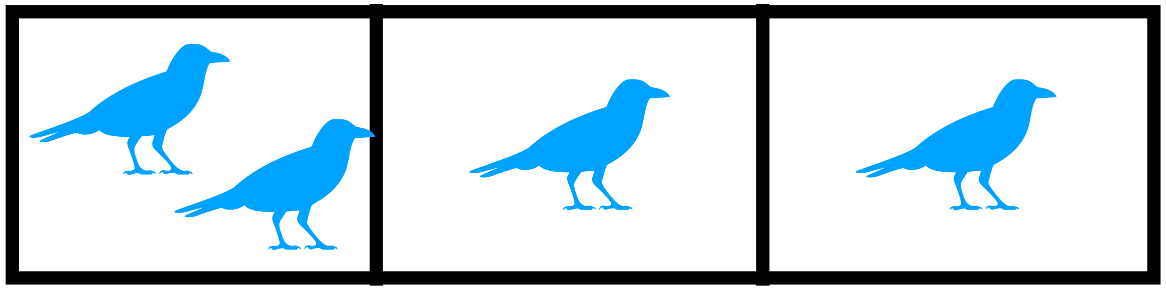
# Methods of Counting

6.1-6.6

# The Pigeonhole Principle

**Example:** **A 3-box Pigeon coop and 4 pigeons**



**Definition:** *Pigeonhole Principle*

If $n$ items are placed in $k$ boxes, at least one box contains at least $\left\lceil \frac{n}{k} \right\rceil$ items

**Definition:** *Pigeonhole Principle (w/functions)*

Let $f : X \to Y$, $|X| = n$, $|Y| = k$, and $m = \left\lceil \frac{n}{k} \right\rceil$.

There are at least $m$ values such that

$$f(a_1) = f(a_2) = \ldots = f(a_m)$$

# The Pigeonhole Principle

**Example:**

The last week of the semester has just 3 days of class meetings but you have 7 assignments due that week. By the pigeonhole principle, at least on day has at least $\lceil \frac{7}{3} \rceil$ = 3 assignments due.

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How many contacts must be in your cell phone to ensure that 2 last names begin with the same pair of letters?

***Answer:*** $26^2 + 1 = 676 + 1 = 677$

# Playposit

What is the smallest number of people that need to be in the class to guarantee that two students have the same birth month?

A. 24
B. 25
C. 12
D. 13

# The Multiplication Principle

**Example:**

How many possible 3-digit octal numbers are there?

*Answer:* <u>8</u> <u>8</u> <u>8</u> $\Rightarrow 8 \cdot 8 \cdot 8 = 8^3 = 2^{3^3} = 2^9 = 512$

**Definition:** *Multiplication Principle (a.k.a. Product Rule)*

If there are $s$ steps in an activity, with $n_x$ ways to accomplish step $x$, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_s$ ways to complete <u>all</u> $s$ steps.

For the Octal example, $s = 3$ and $n_1 = n_2 = n_3 = 8$

# The Multiplication Principle

**Example:**

Party choices: 3 to choose from on Thrusday, 6 on Friay, 5 on Saturday, and 2 on Sunday. If you attend only one party per night, how many party schedules can be created?

*Answer:* By the M.P. $3 \cdot 6 \cdot 5 \cdot 2 = 180$ schedules

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Now consider three digital octal numbers without digit reuse. How many such values are there?

*Answer:* By the M.P. $8 \cdot 7 \cdot 6 = 336$

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Note: $|P_1 \times P_2 \times \ldots \times P_s| = |P_1| \cdot |P_2| \cdot \ldots \cdot |P_s|$.

# Playposit

How many 4 digit hexadecimal numbers can we make?

A. $16 \cdot 15 \cdot 14 \cdot 13$
B. $16$
C. $64$
D. $16^4$

# The Addition Principle

**Definition:** *Addition Principle (a.k.a. Product Rule)*

If there are $t$ tasks, with $n_x$ ways to accomplish the $x^{th}$ task, there are $n_1 + n_2 + \ldots + n_t$ ways to accomplish <u>one</u> of these tasks, assuming that the tasks are non-interfering.

**Example:**

You need to enroll in a literature class. 4 English Lit, 3 Poetry, and 5 World Lit classes fit your schedule.

By the A.P, there are $4 + 3 + 5 = 12$ possible ways for you to enroll in a Lit class.

# The Addition Principle

**Example:**

Grade Sheet Identifiers 4-8 printable ASCII characters

How many IDs of 4 letters are there?

How many ID's of 5 letters are there?

But you can choose one of any of the 5 legal lengths.

By the A.P., there are

$$\sum_{i=4}^{8} 95^i = 6{,}704{,}780{,}953{,}650{,}625$$

(6.7 quadrillion!) possible grade sheet identifiers

# Playposit

Given that the animal shelter has 4 pitbulls, 5 shitzus, and 6 mix breed dogs, how many possible ways are there to adopt a dog?

A. 20
B. 120
C. 15
D. 3

# The Principle of Inclusion-Exclusion

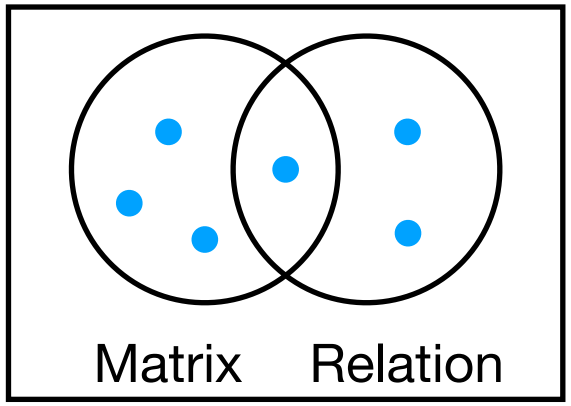
- A problem with the Addition Principle:
  - "Non-interfering" - no overlapping of tasks may occur!

**Example:**

I need a quiz question. I have four questions about matrices and three about relations. But if one is about matrix representation of relations, it is a member of both groups.

$\Rightarrow$ The Addition Principle does not apply!

(It reports $4 + 3 = 7$, but there are only 6 questions - the intersecting question is being counted twice.)



# The Principle of Inclusion-Exclusion

**Definition:** *Principle of Inclusion-Exclusion for Two Sets*

The cardinality of the union of sets $M$ and $N$ is the sum of their individual cardinalities, excluding the cardinality of their intersection

That is: $|M \cup N| = |M| + |N| = |M \cap N|$

$$\begin{aligned}
|\textbf{Matrix} \cup \textbf{Relation}| &= |\textbf{Matrix}| + |\textbf{Relation}| - |\textbf{Matrix} \cap \textbf{Relation}| \\
&= 4 + 3 - 1 \\
&= 6
\end{aligned}$$

# The Principle of Inclusion-Exclusion

**Definition:** *Principle of Inclusion-Exclusion for Three Sets*

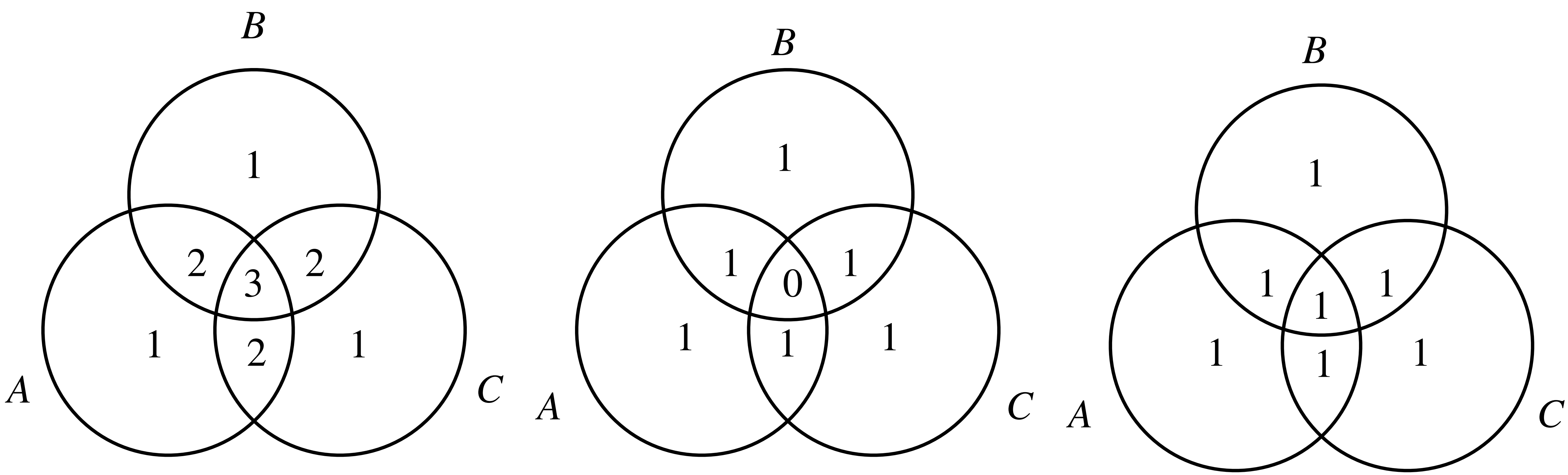
The cardinality of the union of sets $M$, $N$, and $O$ is the sum of their individual cardinalities, excluding the sum of the cardinalities of their pairwise intersections but including the cardinality of their intersection

That is: $$\begin{aligned} |M \cup N \cup O| &= |M| + |N| + |O| \\ &\quad - |M \cap N| - |M \cap O| - |N \cap O| \\ &\quad + |M \cap N \cap O| \end{aligned}$$

**And, of course, this can be extended beyond 3 sets.**

# The Principle of Inclusion-Exclusion

- Why so complex?




$$|A| + |B| + |C| \qquad - |A \cap B| - |A \cap C| - |B \cap C| \qquad + |A \cap B \cap C|$$

# The Principle of Inclusion-Exclusion

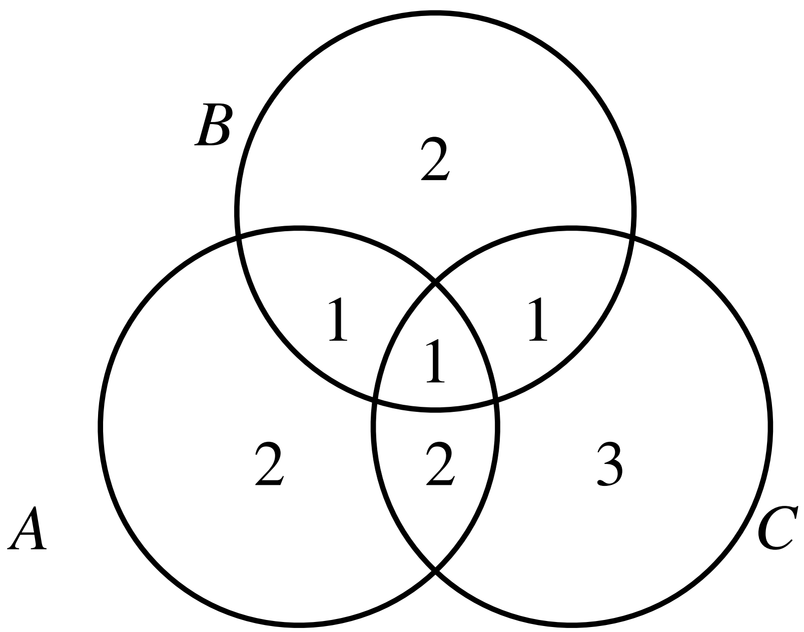
**Example:**

Let $|A| = 6$, $|B| = 5$, $|C| = 7$, $|A \cap B| = 2$, $|A \cap C| = 3$, $|B \cap C| = 2$ and $|A \cap B \cap C| = 1$.

What is $|A \cup B \cup C|$?

$$|A \cup B \cup C| = 6 + 5 + 7 - (2 + 3 + 2) + 1 = 12 \text{ items}$$

Hint: Fill the Venn diagram from the center and work outward.



# Playposit

We have one room for one more course in our schedule. We can choose from the following courses: 10 CS courses, 12 Math courses, 8 ISTA courses. However, 4 of the CS courses are also Math courses, 2 of the CS courses are also ISTA courses, 3 of the Math courses are ISTA courses and 2 of the courses are cross listed in CS, Math, and ISTA.  How many different ways can we choose a course?

A. 21
B. 23
C. 30
D. 18

# Permutations

**Definition:** *Permutation*

An ordering of $n \geq 0$ distinct elements.

**Example:**

Consider a golf tournament with a 5-way playoff between players A, B, C, D, and E.  To determine the order of play they draw #s from a hat.

This generates a permutation of the players …

… but how many possible permutations are there?

# Permutations

**Conjecture: There are $n!$ possible permutations of $n$ elements**

**<u>Proof</u>** (direct):

There are $n$ ways to select the 1st element.

$n-1$ ways to select the 2nd, etc.

By the multiplication principle, the number of possible orderings is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$

Therefore, there are $n!$ possible permutations of $n$ elements.

# Playposit

Given a class with 10 students, how many different way are there to place the students in a line?

A. $10$
B. $10!$
C. $10^{10}$

# $r$-Permutations

**Definition:** *r-Permutation* $P(n, r)$

An ordering of an $r$-element subset of $n$ distinct elements is called an $r$-permutation.

**Conjecture:** **The number of $r$- permutations of $n$ elements denoted $P(n, r)$, is $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$, $r \leq n$**

**Proof Outline:**

**1st** **2nd** **r-th**

$$n \cdot (n-1) \cdot \ldots \cdot (n-(r-1))$$

$$\ldots \cdot (n-r+1)$$

# $r$-Permutations

**Observation:**

$$n \cdot (n-1) \cdot \ldots \cdot (n-r+1) \cdot \boxed{(n-r) \cdot \ldots \cdot 2 \cdot 1} = n!$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

**Example:**

How many 3-permutations can be formed from 5 elements?

$n - r + 1 = 5 - 3 + 1 = 3$ and $5 * 4 * 3 = 60$

Or: $P(5,3) = \dfrac{n!}{n-r}! = \dfrac{5!}{2!} = \dfrac{5*4*3*2*1}{2*1} = 60$

# $r$-Permutations

**Example:**

16 countries are competing for medals (gold, silver, and bronze) in Team Discrete math at the Olympics. In how many was can medals be awarded?

*Answer:* $P(16,3) = \dfrac{16!}{13!} = 16 \cdot 15 \cdot 14 = 3360$

# Playposit

How many different 3 digit numbers can we create without repeating digits?

A. $10!$
B. $720$
C. $27$
D. $3!$

# $r$- Combination

**Definition:** *$r$-Combination*

An $r$-Combination of an $n$-element set $X$ is an $r$-element subset of $X$. The quantity of $r$- element subsets is denoted $C(n, r)$ or $\binom{n}{r}$, and is read "$n$ choose $r$

Other Notations: ${}_nC_r$ $\quad C_{r,n}$

**Example:**

In how many ways my 2-element subsets be chosen from $\{A, B, C\}$?

*Answer:* Order does not matter in sets, so $\binom{3}{2} = 3$

The sets: $\{A, B\}$, $\{A, C\}$, and $\{B, C\}$.

Note that $P(3,2) = 6$.

# $r$- Combination

**The $r$-Permutation - $r$-Combination Connection:**

When order matters, the # of choices grows

Example: $\{A, B\}$ vs. $(A, B)$ and $(B, A)$.

But … grows by how much? There are $r!$ possible arrangements, so:

$$P(n, r) = \binom{n}{r} \cdot r!, \text{ or } \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

**Example:**

$$\binom{5}{3} = \frac{P(5,3)}{3!} = \frac{60}{6} = 10$$

Or: $\dfrac{5!}{3!(5-3)!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10$

# $r$- Combination

**Example:**

From a Chess Club of 12 members, how many ‘traveling squads’ of 6 can be formed?

*Answer:* Order doesn't matter, so: $\binom{12}{6} = 924$

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The University is forming a committee with 5 (of 9 available) faculty and (of 8) staff members.  In how many ways can the committee be formed?

*Answer:* By combinations and the Multiplication Principle:

$$\binom{9}{5} \cdot \binom{8}{4} = 126 \cdot 70 = 8820$$

# Playposit

Given a group of 15 friends, each person has a car that holds 5 people. The group will need 3 cars to drive all of them. How many ways are there to select the 3 drivers?

A. $\binom{15}{3}$

B. $15 \cdot 14 \cdot 13$

C. $15!$

D. $\binom{15}{5}$

# Repetition and Permutations

- We've already seen this!
  - **but we haven't been allowing repetition recently**

**Example:**

Recall: 3 digit octal numbers:

With repetition: $8 \cdot 8 \cdot 8$

Without repetition: $8 \cdot 7 \cdot 6$

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- In General: When object repetition is permitted, the number of $r$-permutations of a set on $n$ objects is $n^r$

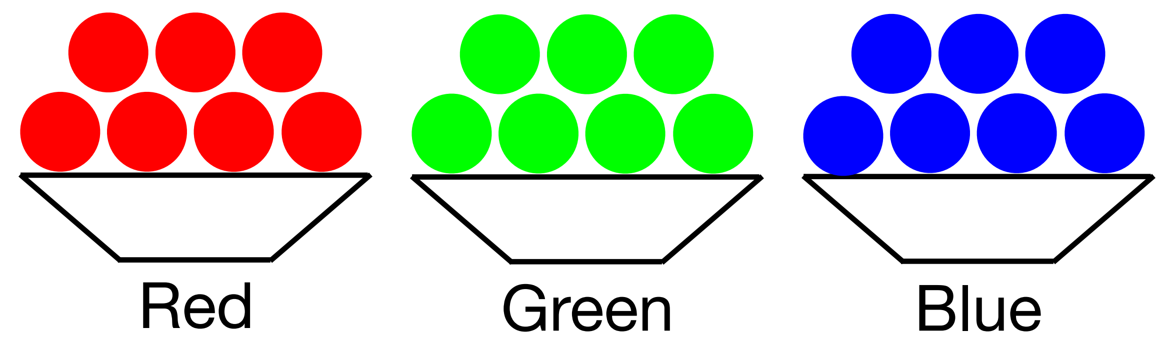
**Here:** $8^3$

# Repetition and Combinations

**Example:** ‘Experienced’ Golf Balls

**Pips & Pipes**

In how many ways can a golfer select two balls




*Answer:* 6 (RR,GG,BB,RG,RB,GB)

Imagine a ball tray - only the balls and dividers matter!

$$\begin{array}{ll} 200 \rightarrow \cdot\cdot\,|| & 110 \rightarrow \cdot|\cdot\,| \\ 020 \rightarrow |\cdot\cdot & 101 \rightarrow \cdot||\cdot \\ 002 \rightarrow ||\cdot\cdot & 011 \rightarrow |\cdot|\cdot \end{array}$$

We have 4 positions for 2 balls ($\binom{4}{2}$) and 2 remaining positives for dividers ($\binom{2}{2}$). By M.P.: $\binom{4}{2}\binom{2}{2} = 6$

# Repetition and Combinations

**Example:** At a cafeteria, how many ways exist to select 4 utensils from bins of forks, spoons, knives, & soup spoons?

*Answer:* 4 bins $\Rightarrow$ 3 dividers, and

3 dividers + 4 utensils = 7 items

$$\therefore \binom{7}{4} = \binom{7}{3} = \frac{7!}{4!3!} = 35$$

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- In General: When repetition is allowed, the number of $r$-combinations of a set on $n$ objects is

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$ here $r = 4$ utensils and $n = 4$ bins

# Repetition and Combinations

- A Small Extension:

**Example:** Consider a pot-luck with 5 platters of food.A child must have one serving from each platter but may have 3 more servings of anything. In how many ways can the child form 8 total servings?

*Answer:* Ignore the first 5 servings, there's just one way to select them. Then: 5 platters ⇒4 dividers, plus 3 servings = 7 items.

So, $\binom{7}{3} = 35$

- In General: When repetition is allowed, the number of $r$-combinations of a set on $n$ elements when one of each is included in $r$ is

$$\binom{r-1}{r-n} = \binom{r-1}{n-1}$$ here $r = 8$ servings and $n = 5$ platters

# Playposit

The store sells 4 colors of socks: blue, yellow, red and white socks. How many ways can you choose two pairs socks to buy?

# Another View of Repetition and Combinations

- Consider: An integer variable can represent the quantity of items selected with repetition

**Example:** The Golf Ball Problem (again!)

Let $r, b, g$ be the numbers of red, blue, and green balls the customer selects. Clearly $r, b, g \in \mathbb{Z}*$.

We need solutions of $r + b + g = 2$ where $r, b, g$ are $\geq 0$.

Or we need 2 pips (the sum) and 2 pipes (the plus signs).

Again, $\binom{4}{2} = 6$ ways to buy 2 golf balls of the 3 colors

# Another View of Repetition and Combinations

**Example:** The Pot-luck Dinner Problem (again!)

Here, our equation is $x_1 + x_2 + x_3 + x_4 + x_5 = 8$ where $x_i \geq 1$. ( $\geq 1$ b/c we need $\geq 1$serving each.)

Pips and pipes needs each term to be $\geq 0$ To achieve this, let $y_i = x_i - 1$. This transforms the equation to:

$$y_1 + y_2 + y_3 + y_4 + y_5 = 3 \text{ where } y_i \geq 0$$

Or we need 3 pips (the sum) and 4 pipes (the plus signs).

As before, $\binom{7}{3} = \binom{7}{4} = 35$ ways to get 3 servings.

# Generalized Permutations

- Idea: What if some elements are indistinguishable?

**Example:**

Review: How many arrangements of the letters A-F are possible?

*Answer:* $6! = 720 = P(6,6)$

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How many arrangements of A, A, and B are possible?

*Answer:* 3: AAB, ABA, BAA because the A's are indistinguishable. Otherwise, it's a simple permutation: $3! = 6$. The difference: There are $2! = 2$ ways to order the A's in each of the three arrangements, but here those orderings don't matter. Thus, $\frac{3!}{2!} = 3$

# Generalized Permutations

- What if we have indistinguishable copies of multiple elements?

**Example:**

How many distinguishable arrangements of the letters in the word TATTOO are possible?

*Answer:* $\frac{6!}{3!2!} = 60$. There are 6! letter arrangements possible, but 3! arrangements of the T's and the $2!$ arrangements of the O's don't matter.

In general: If we have *n* objects of *t* different types, and there are $i_k$ indistinguishable objects of type *k*, then the number of distinct arrangements is $P(n; i_1, i_2, \ldots, i_t) = \frac{n!}{i_1!i_2!\ldots i_t!}$

# Playposit

How many permutations of the word **BOTTOM** exists?

# Generalized Permutations

- We can view $P(n; i_1, i_2, \ldots, i_t)$ in terms of combinations

**Example:** Consider TATTOO again

There are $\binom{6}{3} = 20$ ways to place the T's, leaving 3 empty spaces. There are $\binom{3}{2} = 3$ ways to place the O's and $\binom{1}{1} = 1$ way to place the A. By the multiplication Principle: $\binom{6}{3}\binom{3}{1}\binom{1}{1} = 20 \cdot 3 \cdot 1 = 60.$

In General:

$$P(n; i_1, i_2, \ldots, i_t) = \binom{n}{i_1}\binom{n - i_1}{i_2}\binom{n - i_1 - i_2}{i_3}\ldots\binom{n - \ldots - i_{t-1}}{i_t}$$

# More Fun With Combinations

- What if we created a table of $\binom{n}{k}$ values?

$k$

| $n$ | | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | - | - | - | - | - |
| | 1 | 1 | 1 | - | - | - | - |
| $n$ | 2 | 1 | 2 | 1 | - | - | - |
| | 3 | 1 | 3 | 3 | 1 | - | - |
| | 4 | 1 | 4 | 6 | 4 | 1 | - |
| | 5 | 1 | 5 | 10 | 10 | 5 | 1 |

**This should look familiar…**

# Pascal's Triangle

… is just the centered rows of the $\binom{n}{k}$ table:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

$\binom{n-1}{k-1}$

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | - | - |
| 1 | 1 | 1 | - |
| 2 | 1 | 2 | 1 |
| 3 | 1 | 3 | 3 |

$\binom{n-1}{k}$

$\binom{n}{k}$

Observations:

1. Each row is palindromic: $\binom{n}{k} = \binom{n}{n-k}$

2. "Pascal's Identity" (Inverted Triangles): $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

# More Fun With Combinations

**Conjecture**: $\binom{n}{k} = \binom{n}{n-k}$, **where** $0 \leq k \leq n$

Proof (direct, algebraic):

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$ **[By definition]**

$$= \frac{n!}{(n-k)!k!}$$ **[Simplified]**

$$= \binom{n}{k}$$ **[By definition]**

**Therefore,** $\binom{n}{k} = \binom{n}{n-k}, 0 \leq k \leq n$

# Pascal's Identity (Combinatorial Argument Example)

**Conjecture**: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, **where** $1 \leq k \leq n$

Consider $S = \{W, X, Y, Z\}$. $|S| = 4 = n + 1$. Let $k = 2$.

There are $\binom{n+1}{k} = \binom{4}{2} = 6$ subsets of $S$ of size 2:

$\{\{W, X\}, \{W, Y\}, \{W, Z\}, \{X, Y\}, \{X, Z\}, \{Y, Z\}\}$

Consider element $W$. Either a subset contains $W$ or it does not.

If $W$ is included, to compete the subset we need one more item from the remaining three. There are $\binom{3}{1}$ such subsets.

If $W$ is not included, to compete the subset we need two more items to make the subset, but again we have just three items to choose from: $\binom{3}{2}$

Thus the number of subsets is $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ $(6 = 3 + 3)$

# Pascal's Identity (Combinatorial Proof)

**Definition:** *Combinatorial Proof*

An argument based on the principles of counting

**Conjecture**: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, **where** $1 \leq k \leq n$

Proof (direct, combinatorial ("double counting")):

Let $d \in D$, and $|D| = n + 1$. Because sets are unordered, there are $\binom{n+1}{k}$ subsets of $D$ of size $k$.

Some of these subsets include $d$, and the rest do not.

(Continued….)

# Pascal's Identity (Combinatorial Proof)

Case 1: Subsets that include $d$. Differences are due to the other $k-1$ elements. We need to select those elements from the remaining (that is, non-$d$) values of $D$.

There are $\binom{n}{k-1}$ ways to do this.

Case 2: Subsets not including $d$. We need to select $k$ more elements from $D$, again not counting $d$. There are $\binom{n}{k}$ ways to do this.

Together this is the total quantity of subsets.

Therefore, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ where $1 \leq k \leq n$

# The Binomial Theorem

The values of Pascal's triangle appear in numerous places.

For instance:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= 1a + 1b \\
(a+b)^2 &= 1a^2 + 2ab + 1b^2 \\
(a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3
\end{aligned}$$

Generalize this, and you’ve got the Binomial Theorem.

# The Binomial Theorem

**Theorem**: $(a+b)^n = \sum_{k=0}^{n} \left[ \binom{n}{k} \cdot a^{n-k} \cdot b^k \right]$

Proof: See Rosen Sect 6.4 p 437-8. (Combinatorial!)

**Example:** Find the coefficient of $x^5y^3$ in the expansion of $(x+y)^8$.

By the above theorem: $k=3,\ n=8$, and so the coefficient is $\binom{8}{3} = 56$

# Playposit

What is the coefficient of the term $x^4y^6$ in $(x + y)^{10}$?