Algorithms

3.1, 5.3, 5.4

Algorithms

Definition: *Algorithm*

A finite set of instructions for performing a task

Example:

Is Binary Search an algorithm? **Yes!**

Is the Division Algorithm an algorithm? **No!**

(It’s not a set of instructions)

The Framework

1. **Computable** - means that the solution can be described by an algorithm

   (a) **Tractable** - the algorithm is efficient

   (b) **Intractable** - no efficient solutions

2. **Non-computable** - no algorithm will ever describe the solution.

Algorithm Characteristics

1. **Input -** Data is provided from outside of the algorithm

2. **Output -** Information produced by the algorithm

3. **Generality -** The instructions can solve a collection of similar problems

Algorithm Characteristics

4. **Definiteness -** (a.ka. Precision, Uniqueness) The instructions are not open to interpretation.

5. **Correctness -** The output is the accepted answer for the given input.

6. **Finiteness -** The complete output is produced by the execution of a finite quantity of instructions

Tooth-brushing Algorithm

1. Grab the toothpaste
2. Uncap the toothpaste
3. Grab your toothbrush
4. Squeeze toothpaste onto your toothbrush
5. Brush your teeth

Some problems with this algorithm:

What if the tube is empty? (Input)
Does this algorithm solve related problems? (Generality)
Brushing technique? (Definiteness)
When do we stop? (Finiteness)

Some Sample Iterative Algorithms

Example: Decimal to Base X Conversion

Input:	`n`	**Base 10 value to be converted**
	`base`	**Destination number system**
Output:	`digit()`	**digit(0) holds LSD of result**

```
quotient <-- n
i <-- 0
while quotient does not equal 0:
    digit(i) <-- quotient modulo base
    quotient <-- the floor of quotient/base
    increment i by 1
end while
```

Some Sample Iterative Algorithms

What is the cost to evaluate $f(x) = 2x^3 - 4x^2 + 3x + 6$?

Naive evaluation:

$$f(x) = 2 \cdot x \cdot x \cdot x - 4 \cdot x \cdot x + 3 \cdot x + 6$$

1 2 3 1 4 5 2 6 3

3+'s, 6 ·'s

But can we do better?

$$f(x) = x(2x^2 - 4x + 3) + 6$$

$$= x(x(2x - 4) + 3) + 6$$

$$= x(x(x(2) - 4) + 3) + 6$$

3 2 1 1 2 3

3+'s, 3 ·'s

Some Sample Iterative Algorithms

Example: Horner's Algorithm for Polynomial Evaluation

Input:	x	**Value used to evaluate the polynomial**
	n	**Largest Exponent**
	a(0).. a(n)	**Coefficients of** $x^0 \ldots x^n$
Output:	result	**Evaluation of the polynomial**

```
result <-- a(n)
index <-- n-1
while index>=0:
    result <-- x * result + a(index)
    decrement index by 1
end while
output result
```

Recursive Definitions

Definition: *Recursive Definition*

A complete recursive definition has three parts:

(a) The **basis clause** determines how trivial cases are to be handled

(b) The **inductive clause** describes complex problem instances in terms of simpler instances

(c) The **extremal clause** provides bounds on the definition

Recursive Definitions

Example:

Consider the sequence $S : 13,10,7,4,1$

Basis: $S_1 = 13$
Recurrence: $S_n = S_{n-1} - 3$
Extremal: $1 \leq n \leq 5$

Consider the non-negative integers (Z^*)

Basis: $1 \in \mathbb{Z}$
Recurrence: if $n \in \mathbb{Z}$, then $n + 1 \in \mathbb{Z}$
Extremal: N/A

Consider general trees

Basis: Empty tree (0 nodes)
Recurrence: The root has >= 0 subtrees that are general trees
Extremal: N/A

Recursive Algorithms

Definition: *Recursive Definition*

A recursive algorithm express the solution to a task in terms of a simpler case of the same problem.

Aside: Control Structures in Programming Languages

1. Sequence
2. Selection
3. Iteration…or Recursion!

Example: Factorials

Definition: *Factorial*

The factorial of $n \in \mathbb{Z}^*$, denoted $n!$, is the product of all integers 1 through n, where $0! = 1$.

An iterative factorial algorithm is easy to create:

```
product <-- 1
while n is larger than 1:
    product <-- product * n
    n<--n-1
end while
output product
```

Example: Factorials

Factorials can be easily computed recursively:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3!$$

But what are the Basis, Inductive, and Extremal clauses?

Basis: $0! = 1$

Inductive: $n! = n \cdot (n-1)!$

Extremal: $n!$ **is defined** $\forall n \in \mathbb{Z}^*$

Example: Factorials

Recursive pseudocode algorithm:

```
subprogram factorial (given: n) returns: n!
    if n is 0
        return 1
    else
        answer <-- n * factorial(n-1)
    end if
end subprogram
```

(Basis)

(Inductive)

Extremal? Assumed!

Can We Prove Our Algorithm?

<u>Conjecture</u>: **`factorial(n)` returns** $n!$

Proof (structural induction):

<u>Basis:</u> Let $n = 0$. The algorithm returns 1, and by definition, $0! = 1$. Ok!

<u>Inductive Step</u>: If $\texttt{factorial(n)}$ returns $n!$, then $\texttt{factorial(n+1)}$ returns $(n+1)!$.

When the input is $(n+1)$, the algorithm will compute $(n+1)!$ to be $(n+1) * \texttt{factorial(n)}$

(Continues …)

Can We Prove Our Algorithm?

By the Inductive Hypothesis, we know that

`factorial(n)` computes $n!$. And, from the recursive definition of factorial, we know that

$n! * (n+1) = (n+1)!$.

Therefore, `factorial(n)` computes $n!$

Another Structural Induction Proof

Conjecture: **In a binary tree, the number of null references equals one more than the number of nodes in the tree, for all non-empty binary trees.**

Proof (structural induction):

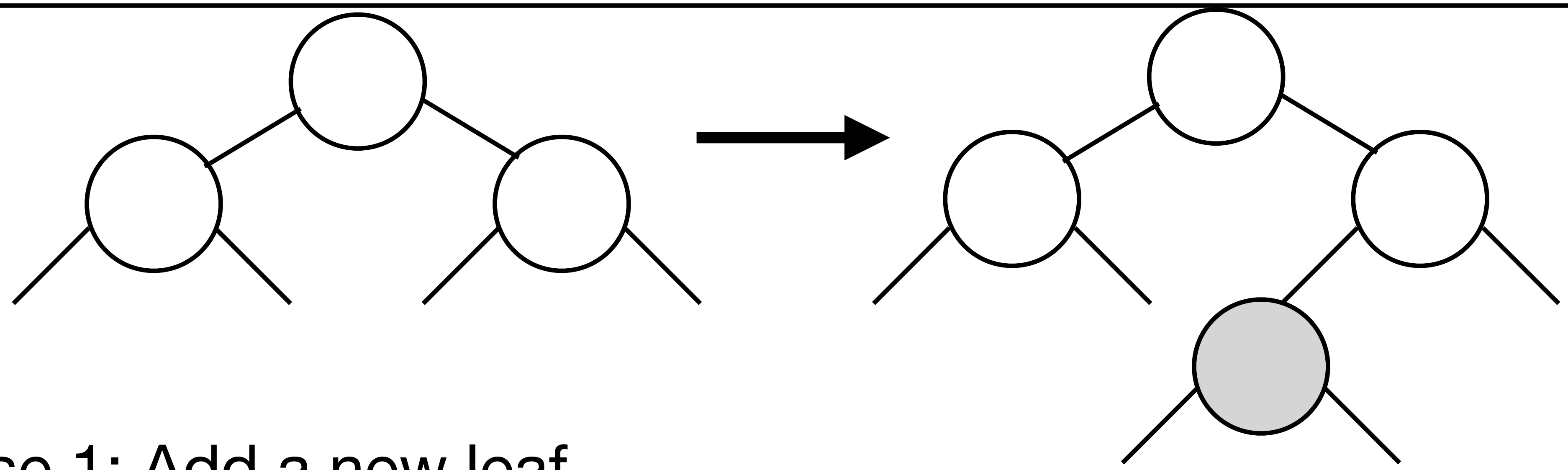
Basis: A binary tree with one node has 2 nulls. Ok!

Inductive Step: If a binary tree of n nodes has $n + 1$ nulls, then a binary tree of $n + 1$ nodes has $n + 2$ nulls.

There are three possible insertion situations

(Proof Continues …)

Another Structural Induction Proof



Case 1: Add a new leaf.

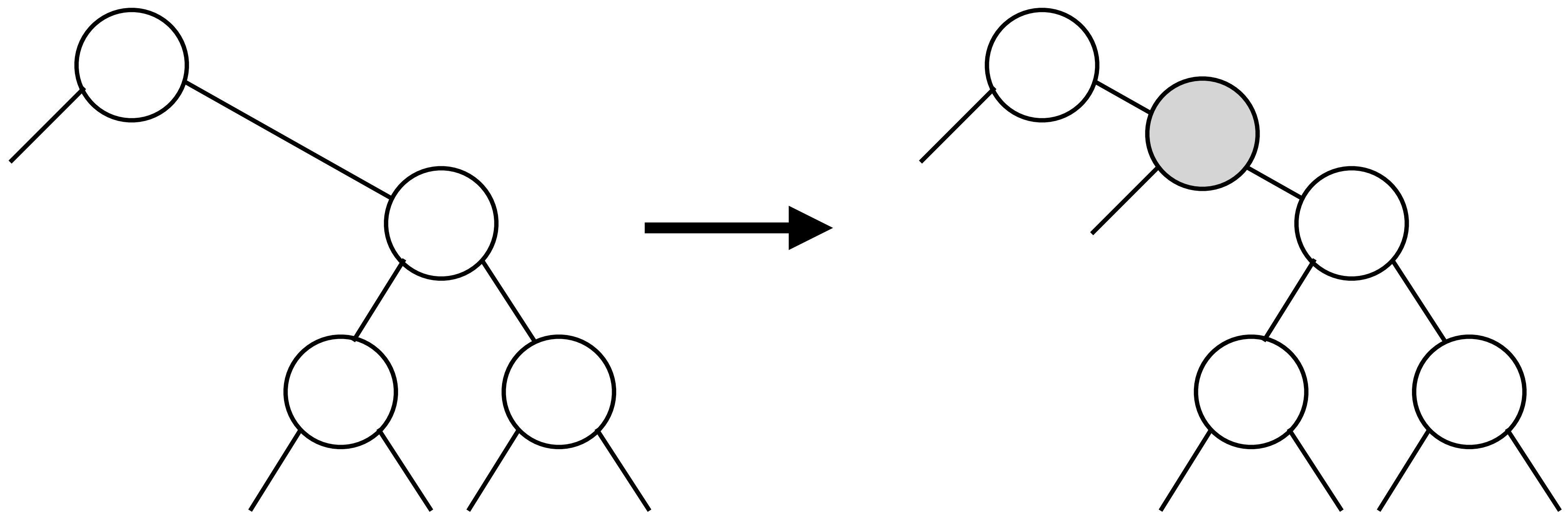
By the Inductive Hypothesis, we have n nodes and $n + 1$ nulls in our tree.

Adding a leaf adds one node and two nulls, but occupies (removes) an existing null.

This is a net gain of one node and one null, giving a total of $n + 1$ nodes and $n + 2$ nulls, as desired.

(Proof Continues)

Another Structural Induction Proof



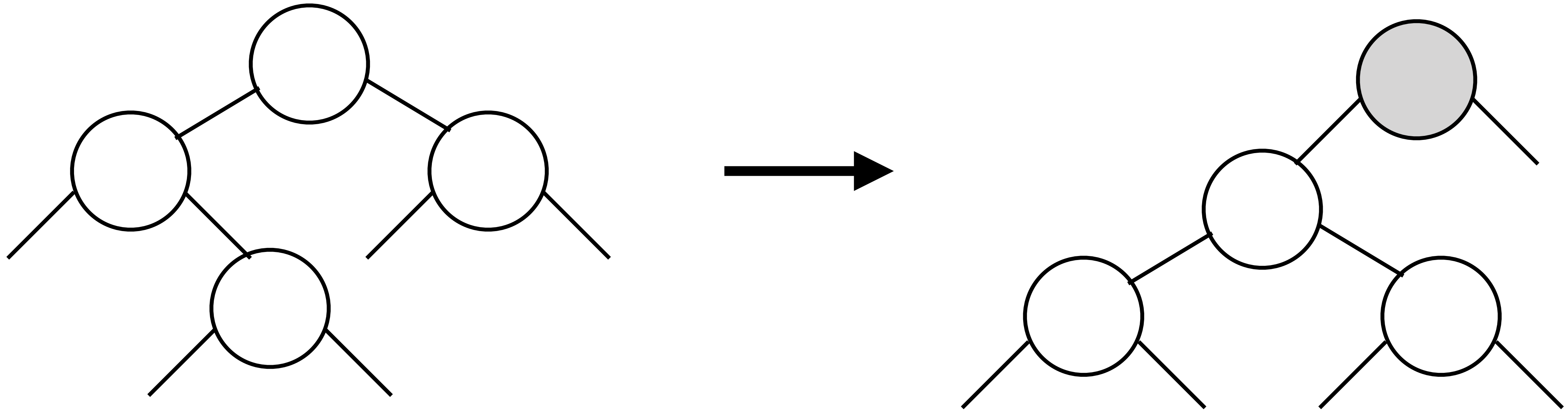
Case 2: Insert between nodes.

We add a node, occupy an existing null, and use one of its children, leaving one extra new null.

As before this is a gain of one node and one null.

(Proof Continues)

Another Structural Induction Proof



Case 3: Insert a new root.

We add a node and occupy of its nulls in referencing the old root. Again, a net gain of one node and one null.

Therefore, #-nulls = 1+ # nodes, for all non-empty binary trees

Example: Fibonacci Sequence

Definition: *Fibonacci Sequence*

The n^{th} term of the Fibonacci sequence is the sum of terms $n-1$ and $n-2$, where $F(0)=0$ and $F(1)=1$

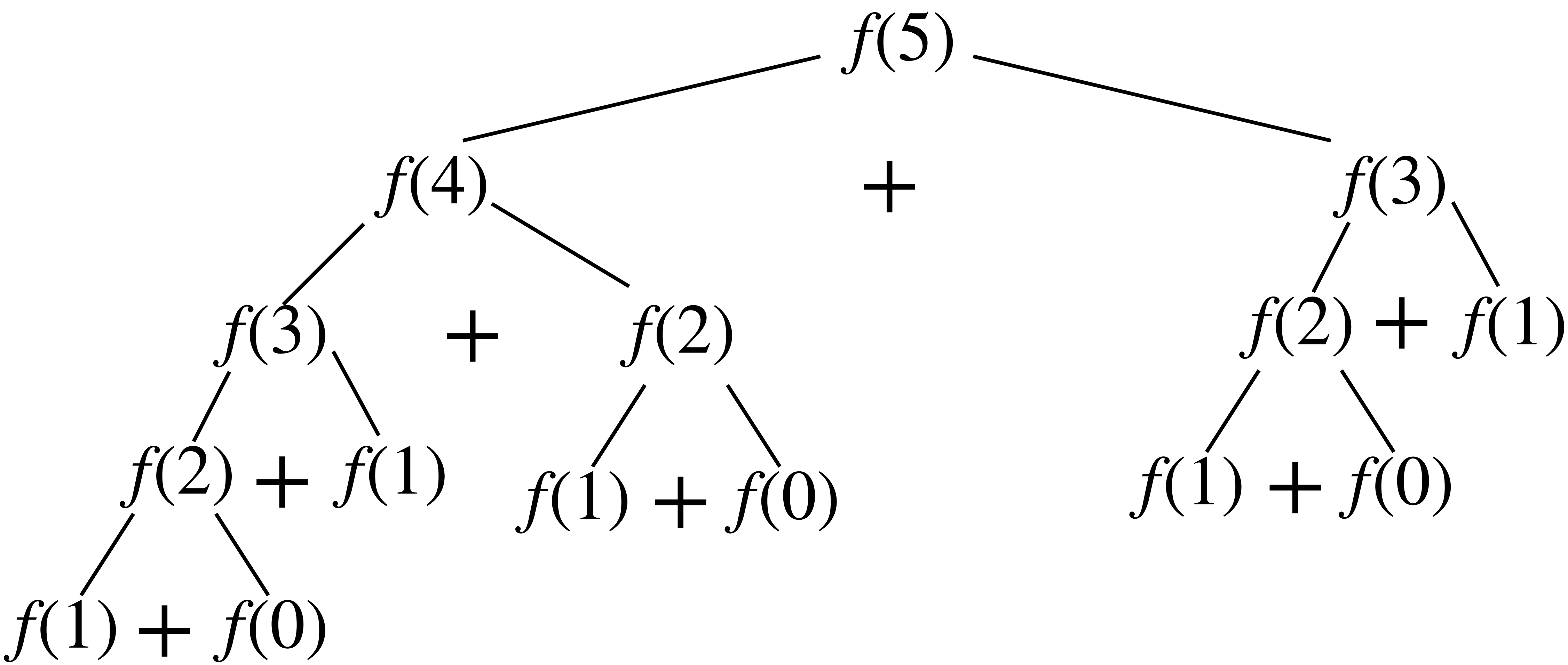
Recursively generating terms of the sequence is easy…

```
subprogram fibonacci (given: n) returns: nth term
    if n is 0 or 1
        return n
    else
        return fibonacci(n-1) + fibonacci(n-2)
    end if
end subprogram
```

Example: Fibonacci Sequence

… but inefficient!

Consider this tree of invocations resulting from `fibonacci(5)`:



Note the three $f(2)$ trees and the two $f(3)$ trees

⇒ Repeated (and therefore wasted) effort!

Extra Slides

Example: Euclidean Algorithm for GCDs

Theorem: GCD(a,b) = GCD(b,a%b)

Proof: See Rosen 8/e p. 283

Recursive pseudocode algorithm:

```
subprogram GCD (given: a,b) returns: gcd(a,b)
    if a is 0, return b endif
    if b is 0, return a endif
    answer <-- GCD(b, a%b)
    return answer
end subprogram
```

<u>Question</u>: Is this more or less efficient than the iterative algorithm presented earlier?

Example: Sums of Odd Positive Integers

$$\mathbb{Z}^+ : 1 \quad 2 \quad 3 \quad 4 \quad \ldots \qquad n \qquad\qquad \frac{(m+1)}{2}$$

$$o : 1 \quad 3 \quad 5 \quad 7 \quad \ldots \quad 2n-1 \qquad\qquad m$$

Let `oddsum(term)` represent the sum of $o(1)$ through $o(\text{term})$.

Base: `oddsum(1)` = 1

General: `oddsum(term)` =

`oddsum(term-1) + 2*term -1`

Example: Sums of Odd Positive Integers

Recursive implementation, using pseudocode:

```
subprogram oddsum (given: term)
        returns: sum from 1 through term of (2i-1)

    if term is 1, return 1
    otherwise
        answer <-- oddsum(term-1)+2*term-1
        return answer
    end if

end subprogram
```

Proving oddsum()

Conjecture: **`oddsum(t)` produces** $\sum_{i=1}^{t}(2i-1), \forall t \geq 1$

Proof (structural induction):

Basis: Let $t = 1$. The algorithm returns 1, and $\sum_{i=1}^{1}(2i-1) = 1$. Ok!

Inductive Step: If `oddsum(t)` returns $\sum_{i=1}^{t}(2i-1)$,

then `oddsum(t+1)` returns $\sum_{i=1}^{t+1}(2i-1)$.

(Continues …)

Proving oddsum()

When given $t+1$, $\texttt{oddsum()}$ returns

$$\texttt{oddsum(t)} + [2(t+1)-1] = \texttt{oddsum(t)} + (2t+1)$$

By the Inductive Hypothesis, $\texttt{oddsum(t)}$ = $\sum_{i=1}^{t}(2i-1)$.

Substituting, $\texttt{oddsum(t+1)}$ returns $\sum_{i=1}^{t}(2i-1)+(2t+1)$.

$2t+1$ is the $(t+1)^{st}$ term of the sequence; thus

$$\sum_{i=1}^{t}(2i-1)+(2t+1)=\sum_{i=1}^{t+1}(2i-1).$$

Therefore, $\texttt{oddsum(t)}$ produces $\sum_{i=1}^{t}(2i-1),\ \forall t \geq 1$