Question 1:
If there are 3 u's, there are 2 spots left with 4 letters, so there are 4 choose 2 = 4!/(2!*(4-2)!)=4!/(2!*2!)=6 combinations with 3 u's.
If there are 2 u's, there are 3 spots left with 4 letters, so there are 4 choose 3 = 4 combinations with 2 u's.
If there is 1 u, then there are 4 spots left with 4 letters, so there is 4 choose 4 = 1 combination with 1 u.
There are no combinations with no u's.
6+4+1=11

11 unique subsets exist.

3 u's: 6 * 5! / 3! = 120
2 u's: 4 * 5! / 2! = 240
1 u: 1 * 5! / 1! = 120

480 different strings could be made.

Question 2:
NB: The question is unclear on what a "pair" is, so I'm assuming that a "pair" refers to two cards of the same rank (for example, an ace of spades and an ace of diamonds would constiute a pair)

First card has 52 options
Second card has 3 options (since it has to match the first card)
Third card has 48 options
Fourth card has 3 options
Fifth card has 11 options
52*3*48*3*11=247104
Each pair has two orders, and the two pairs can be swapped, so divide the result by eight
247104/8=30888

There are 30888 different ways.

Question 3:

Case 1: Fighting couple gets 0 songs:
16 songs among 6 couples:
Stars and bars, 21 choose 5 = 20349 options

Case 2: Fighting couple gets 1 song:
15 songs among 6 couples:
Stars and bars, 20 choose 5 = 15504 options
Total = 20349+15504=35853

There are 35853 different ways.

Question 4:
Let f(n) equal the number of BSTs with n nodes.
f(0) = 1 (the empty tree)
A tree can be broken down into the root node + two subtrees.
The number of nodes of both of the subtrees must add up to the number of nodes in the parent tree minus 1 (because of the root node)
Therefore, f(n)=sum from k=0 to n - 1 of f(k)*f(n-1-k)

We need to figure out how many 2-node trees

f(0)=1
f(1)=f(0)*f(0)=0*0=1
f(2)=f(0)*f(1)+f(1)*f(0)=1*1+1*1=2
f(3)=f(0)*f(2)+f(1)*f(1)+f(2)*f(0)=1*2+1*1+2*1=5
f(4)=f(0)*f(3)+f(1)*f(2)+f(2)*f(1)+f(3)*f(0)=1*5+1*2+2*1+5*1=14
f(5)=f(0)*f(4)+f(1)*f(3)+f(2)*f(2)+f(3)*f(1)+f(4)*f(0)=1*14+1*5+2*2+5*1+14*1=42

The answer to the problem is f(2)*f(5)*f(3)=2*5*42=420
There are 420 possible trees. (Insert obligatory joke here)

Question 5:
If the nurse is on break:
1,1,1
2,1,1
3,1,1
4,1,1
5,1,1
6,1,1
7,1,1
8,1,1
2,2,1
3,2,1
4,2,1
5,2,1
6,2,1
7,2,1
3,3,1
4,3,1
5,3,1
6,3,1
4,4,1
5,4,1
2,2,2
3,2,2
4,2,2
5,2,2
6,2,2
3,3,2
4,3,2
5,3,2
4,4,2
3,3,3
4,3,3

31 options

If the nurse is not on break:
1,1,1,1
2,1,1,1
3,1,1,1
4,1,1,1
5,1,1,1
6,1,1,1
2,2,1,1
3,2,1,1
4,2,1,1

3,3,1,1
4,3,1,1
5,3,1,1
4,4,1,1
2,2,2,1
3,2,2,1
4,2,2,1
5,2,2,1
3,3,2,1
4,3,2,1
3,3,3,1
2,2,2,2
3,2,2,2
4,2,2,2
3,3,2,2

24 options

There are 55 options total.