ESE448: Systems Lab
October 5th, 2019

Robert Goodloe

Synthesis Module 1: Simulations of a 1g Hover

Purpose: Design linear and nonlinear simulations of the Parrot Mambo drone at a 1g hover subject to Throttle, Elevator, Aileron, and Rudder inputs.

Beginning with the state equations:

$$m\dot{V} + \omega \times mV = \vec{F_g}(\phi,\theta,\psi) + \vec{F_T} + \vec{F_A}(V,\omega,\theta)$$

$$J\dot{\omega} + \omega \times J\omega = \vec{M_T} + \vec{M}_{gyro} + \vec{M_A}(V,\omega,\theta)$$

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$

$$\dot{\theta} = (q\cos\phi - r\sin\phi)$$

$$\dot{\psi} = \frac{(q\sin\phi + r\cos\phi)}{\cos\theta}$$

$$\begin{vmatrix} X\dot{n}ed \\ Y\dot{n}ed \\ Z\dot{n}ed \end{vmatrix} = R(\psi)^T R(\theta)^T R(\phi)^T \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

With the variables representing the following values:

- p, q, r: body-axis components of angular velocity ω w.r.t x,y,z axes
- u, v, w: body-axis components of velocity V w.r.t flat earth x,y,z axes
- X_A, Y_A, Z_A: body-axis components of aerodynamic forces $\vec{F_A}$
- L_A, M_A, N_A: body-axis components of aerodynamic moments $\vec{M_A}$
- X_T, Y_T, Z_T: body-axis components of propulsive forces $\vec{F_T}$
- $L_{gyro}, M_{gyro}, N_{gyro}$: body-axis components of gyroscopic moments $\vec{M_{gyro}}$
- $X_{NED}, Y_{NED}, Z_{NED}$: inertial axis components of position
- T: Thrust
- ϕ, θ, ψ: roll-pitch-yaw Euler angles

- $R(\phi), R(\theta), R(\psi)$: positive roll, pitch, yaw rotation matrices

Note assume the following:

- Aerodynamics forces $\vec{F_A}$ are zero: These forces are not in fact zero, but the drone is small, lightweight, and moving at low speeds giving negligible values.
- Aerodynamic and gyroscopic Moments $\vec{M_A} and \vec{M}_{gyro}$ are zero: The justification for this assumption follows from the same logic as those used regarding the aerodynamic forces.
- The system behaves exactly as defined by Newton's laws: The manufacturing of these drones is not perfect and leads to imperfections in the drone's hardware and body components. This yields deviation from the ideal model assumed in Newton's laws. A more accurate simulation of the drone may be acquired using system identification.
- All four motors are identical: This follows directly from the previous assumption. The drone is modeled as if input from all four drones is identical, however this is highly unlikely and at a high degree of accuracy, it is impossible.
- The parameters for mass m and intertial moments J are known: These values were supplied by the instructor. They could have been acquired with measurement for this particular drone. Furthermore, Jxy Jxz Jyz... are assumed to be zero.

The state of a system is a set of variables that give a description of the system at a given point in time. If the values of the state variables are known at time t_0 then their values, and thus the status of the system can be determined at a future time t. The proper representation of the systems input and outputs in relation to the state variables is given by the state-space model of the system. In the nonlinear case, the state-space of the system can be generalized to:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

Solving our state equation for the state variables gives:

$$\begin{vmatrix} \dot{Z}_{NED} \\ \dot{w} \\ \dot{\theta} \\ \dot{q} \\ \dot{\phi} \\ \dot{p} \\ \dot{\psi} \\ \dot{r} \\ \dot{X}_{NED} \\ \dot{u} \\ \dot{Y}_{NED} \\ \dot{v} \end{vmatrix} = \begin{vmatrix} w\cos\phi\cos\theta) - u\sin\theta + v\cos\theta)sin(\phi) \\ qu - pv - \frac{F_{T1}+F_{T2}+F_{T3}+F_{T4}}{m} + g\cos\phi\cos\theta \\ q\cos\phi - r\sin\phi \\ \frac{-b(-F_{T1}-F_{T2}+F_{T3}+F_{T4}+J_{xx}pr-J_{zz}pr)}{J_{yy}} \\ p + \tan\theta(r\cos\phi + q\sin\phi) \\ \frac{b(F_{T1}-F_{T2}-F_{T3}+F_{T4}+J_{yy}qr-J_{zz}qr)}{J_{xx}} \\ \frac{r\cos phi + q\sin phi}{\cos\theta} \\ \frac{J_{xx}pq-J_{yy}pq}{J_{zz}} \\ w(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta) - v(\cos\phi\sin\psi - \cos\psi\sin\phi sin\theta) + u\cos\psi\cos\theta \\ rv - qw - g\sin\theta \\ v(\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta) - w\cos\psi\sin\phi - \cos\phi\sin\psi\sin\theta) + u\cos\theta\sin\psi \\ pw - ru + g\cos\theta\sin\phi \end{vmatrix}$$

Where F_{Ti} is the force of thrust given by motor i defined:

$$C_t \rho w_i^2 D^4$$

- ρ: air density = 1.22495238 $\frac{kg}{m^3}$
- C_t: coefficient of thrust given as 0.75
- w_i: motor speed
- D: diameter of propeller = 0.66mm

The output of the nonlinear system is arbitrarily defined to be:

$$h = -Z_{NED}$$

$$\begin{vmatrix} acceleration_x \\ acceleration_y \\ acceleration_z \end{vmatrix} = m\dot{V} + \omega \times mV - \frac{\vec{F_G}}{m} = \frac{\vec{F_T} + \vec{F_a}}{m}$$

While the system is clearly nonlinear, the nonlinear system can be difficult to analyze. However, the system can be simplified by linearizing the system about a reference point. The reference point chosen here will be the equilibrium state held by the drone at a 1g hover. In the linear case the system becomes:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

To obtain this linear state space model, the Jacobian of the nonlinear state matrix is taken with respect to the twelve state variables. Similar to a simple derivative, the Jacobian gives a linear best approximation of the vector field about a reference point.The values describing the 1g hover will be the value at which the Jacobian is evaluated. Yielding a linear approximation of the system's behavior in a 1g hover as desired. The values of every state variables will be zero excluding Z_{NED} which will have a value of 1 describing a hover at 1 meter above the ground. Similarly, linearizing the state matrix with respect to motor speeds w_i (the inputs to the system) and evaluating at w_i such that $F_T = F_g$ gives a corresponding B matrix. Thus the linear state space model is given by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} Z_{NED} \\ w \\ \theta \\ q \\ \phi \\ p \\ \psi \\ r \\ X_{NED} \\ u \\ Y_{NED} \\ v \end{vmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{2F_{T1}}{m} & -\frac{2F_{T2}}{m} & -\frac{2F_{T3}}{m} & -\frac{2F_{T4}}{m} \\ 0 & 0 & 0 & 0 \\ \frac{2bF_{T1}}{J_{yy}} & \frac{2bF_{T2}}{J_{yy}} & -\frac{2bF_{T3}}{J_{yy}} & -\frac{2bF_{T4}}{J_{yy}} \\ 0 & 0 & 0 & 0 \\ \frac{2bF_{T1}}{J_{xx}} & -\frac{2bF_{T2}}{J_{xx}} & -\frac{2bF_{T3}}{J_{xx}} & \frac{2bF_{T4}}{J_{xx}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

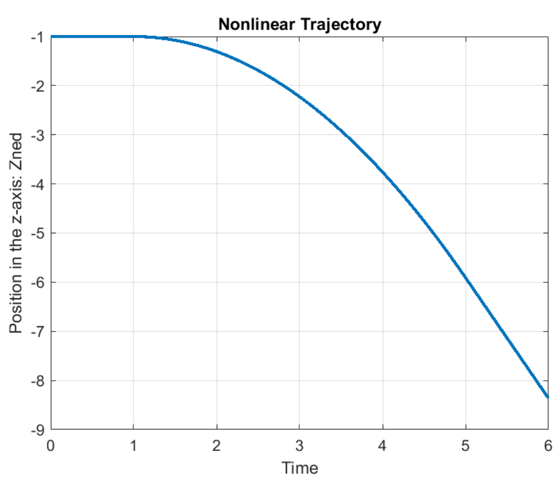
With output equation,

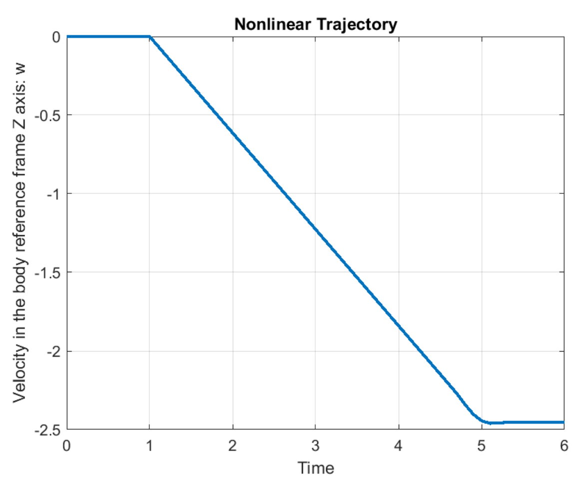
$$y = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2F_{T1}}{m} & -\frac{2F_{T2}}{m} & -\frac{2F_{T3}}{m} & -\frac{2F_{T4}}{m} \end{bmatrix} u$$

Results:
The linear and nonlinear system models were simulated using ODE45 and Simulink block diagrams for the following three scenarios:

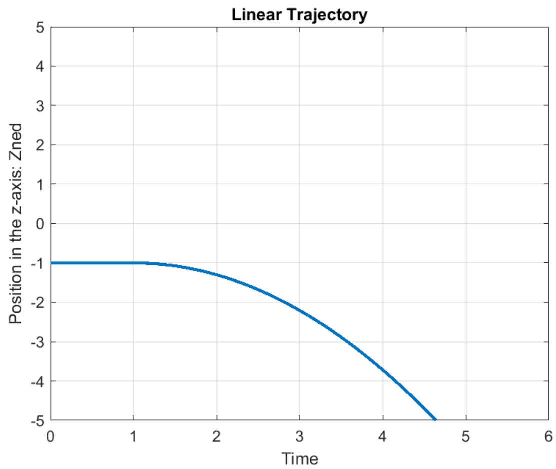
- A 6s duration simulation of all states and outputs during a throttle step input of 4s duration that engages at 1s and disengages at 5s
- A 6s simulation of all states and outputs during an elevator step input of 4s duration that engages at 1s and disengages at 5s
- A 6s simulation of all states and outputs during an aileron step input of 4s duration that engages at 1s and disengages at 5s

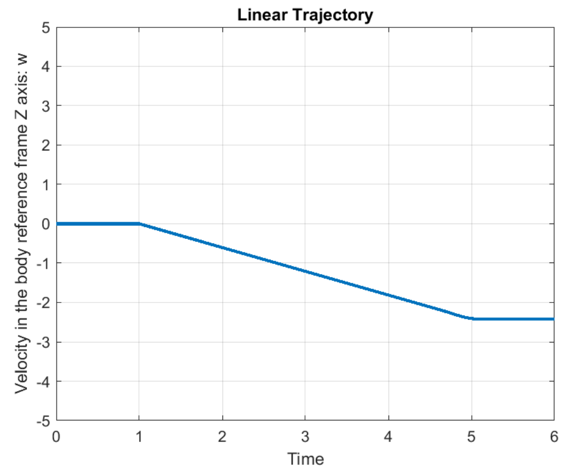
Note if a state trajectory did not exhibit any deviation from the equilibrium, its graph was not included.
The simulation of the throttle step input from scenario 1: The state variable trajectories for all twelve states are graphed below. It is clear from the trajectories of the Z_{NED} and $\dot{w}$ that the drone is rising (note the positive z axis is defined down). The other states remain zero as is expected.

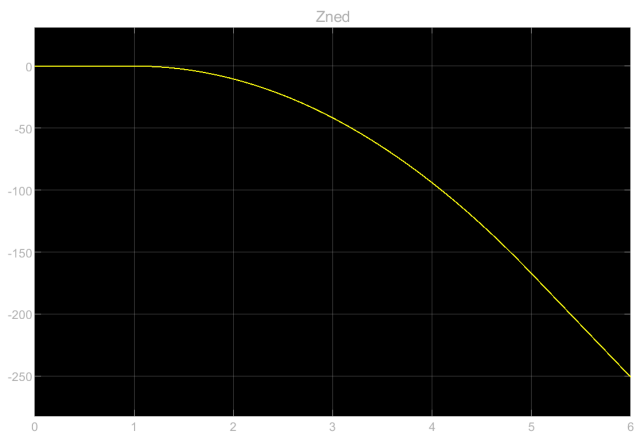


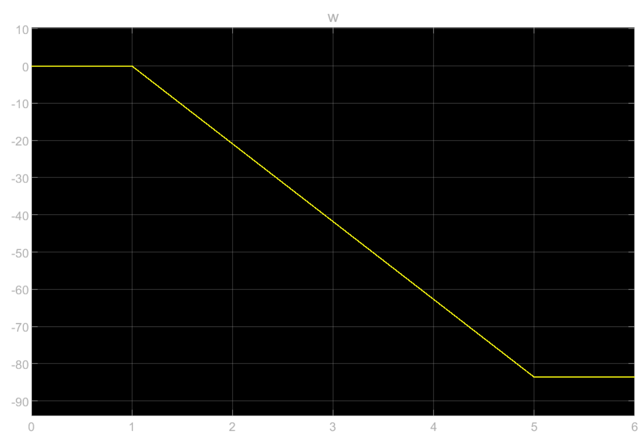


In the linear simulation similar results were found. The graphs that exhibited behavior other than remaining at equilibrium are included below. Note these are the same state variables that changed on the nonlinear simulation, concluding that the linear simulation is an accurate approximation. Similarly, the nonlinear and linear Simulink block diagrams accurately depicted the behavior seen in the systems when solved using ODE45.

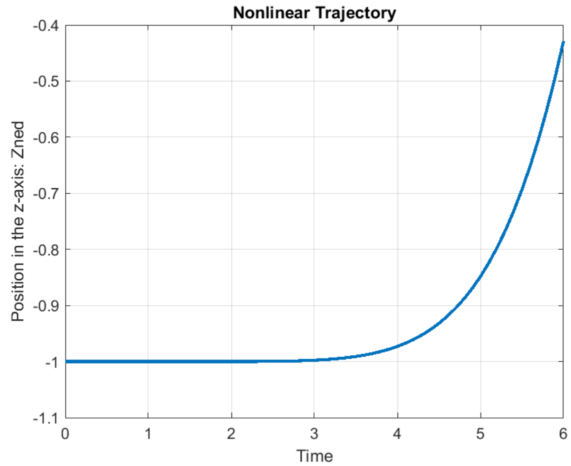


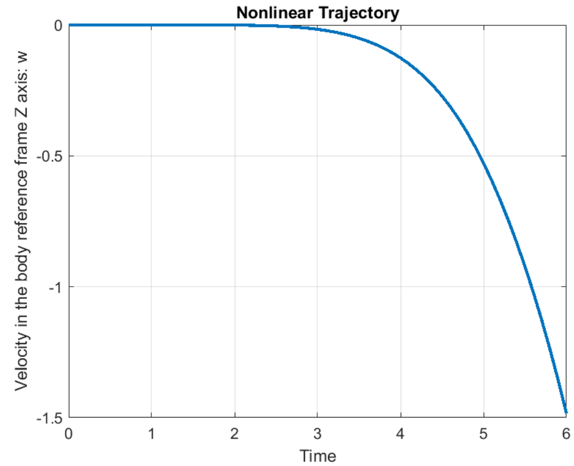


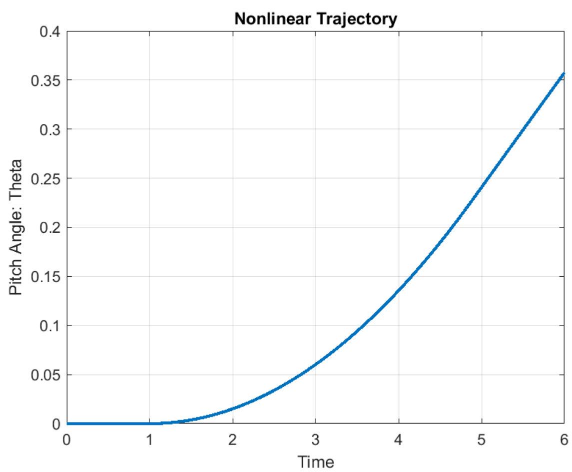


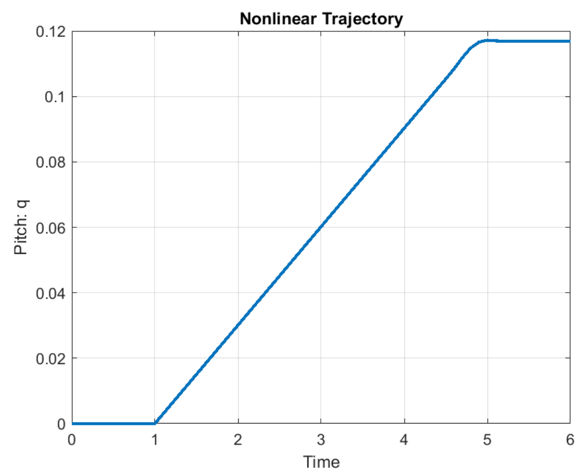


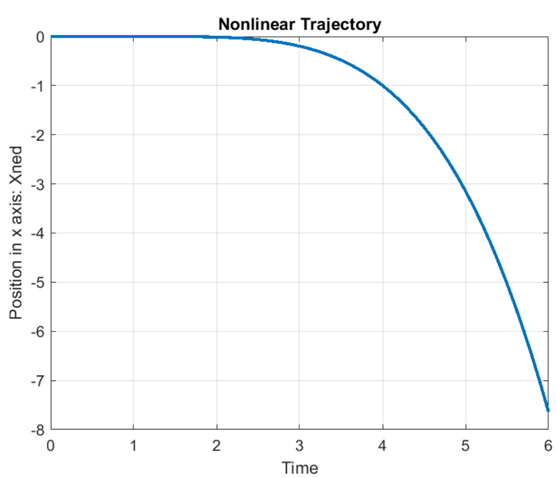
The simulation of the elevator step input from scenario 2: The state variable trajectories for all twelve states are graphed below. It is clear from the trajectories of the Z_{NED} and $\dot{w}$ that the drone is rising slightly. The drone is no longer at a 1g hover, but it is rising quite slowly. The pitch increases as the pitch rate holds a steady positive value. This causes the drone to drift in the negative x-direction which is captured in the trajectories of $\dot{X_{NED}}$ and $\dot{u}$. The other state variables remain in equilibrium. This is clearly consistent with the expected response of an elevator input.

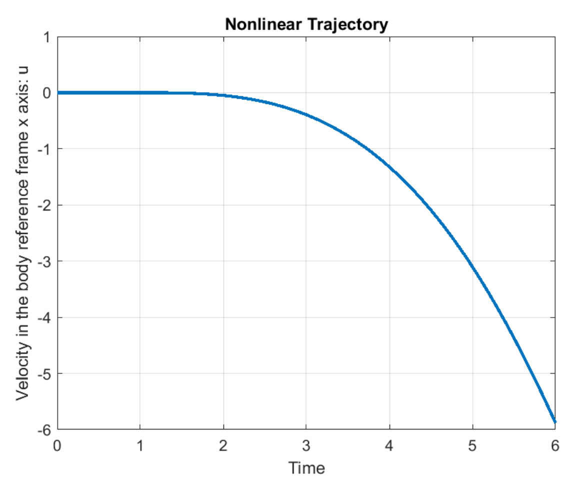

Nonlinear Trajectory
Position in the z-axis: Zned
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
-1.1
0 1 2 3 4 5 6
Time


Nonlinear Trajectory
Velocity in the body reference frame Z axis: w
0
-0.5
-1
-1.5
0 1 2 3 4 5 6
Time

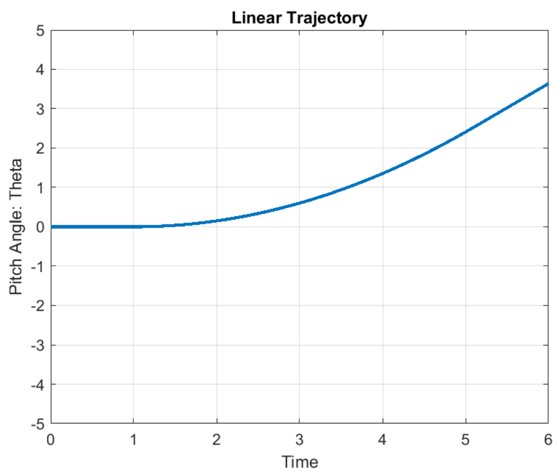

Nonlinear Trajectory
Pitch Angle: Theta
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0 1 2 3 4 5 6
Time

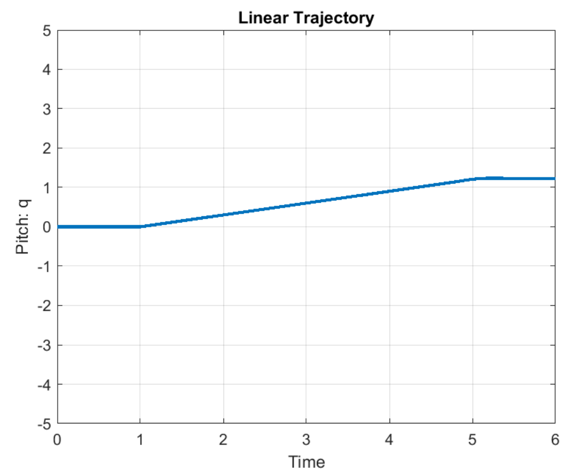

Nonlinear Trajectory
Pitch: q
0.12
0.1
0.08
0.06
0.04
0.02
0
0 1 2 3 4 5 6
Time

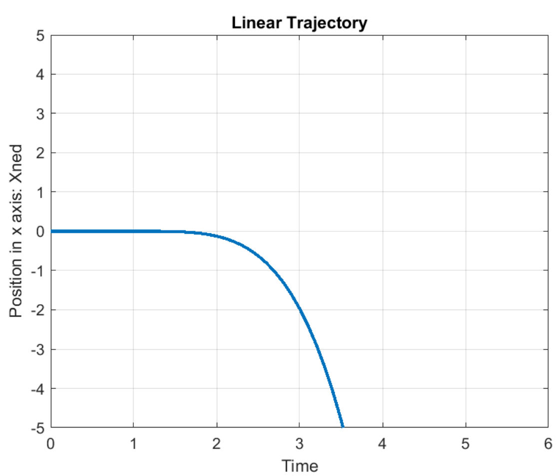

Nonlinear Trajectory
Position in x axis: Xned
0
-1
-2
-3
-4
-5
-6
-7
-8
0 1 2 3 4 5 6
Time

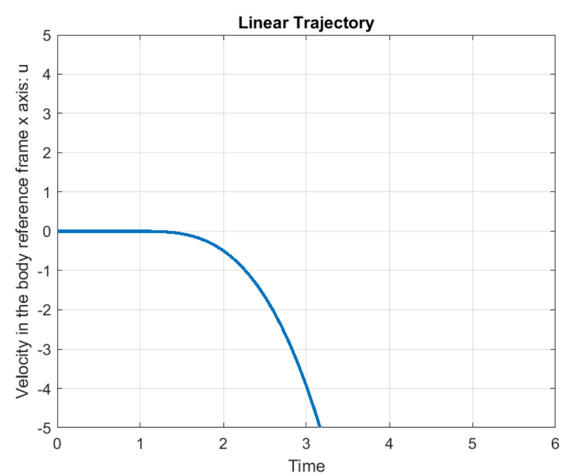

Nonlinear Trajectory
Velocity in the body reference frame x axis: u
1
0
-1
-2
-3
-4
-5
-6
0 1 2 3 4 5 6
Time

In the linear simulation similar results were found. The graphs that exhibited behavior other than remaining at equilibrium are included below. Note these are the same state variables that changed on the nonlinear simulation, concluding that the linear simulation is an accurate approximation.

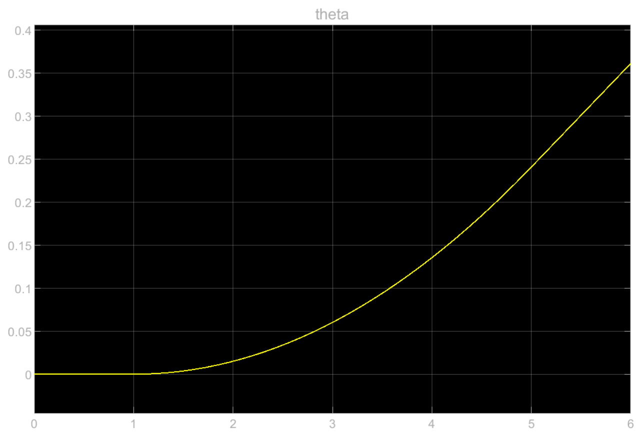


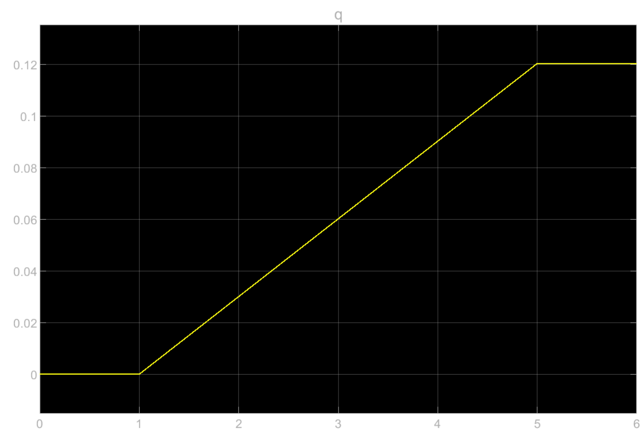


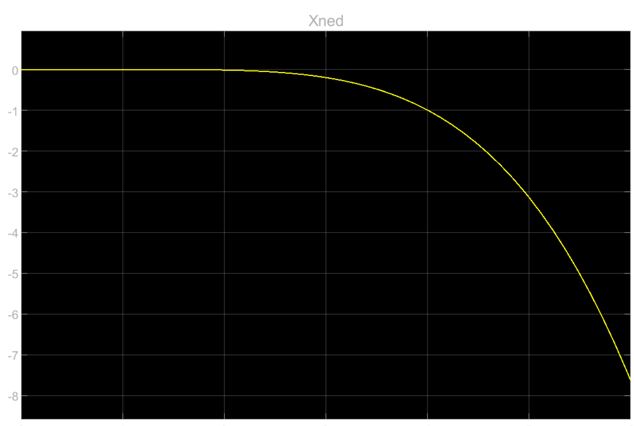


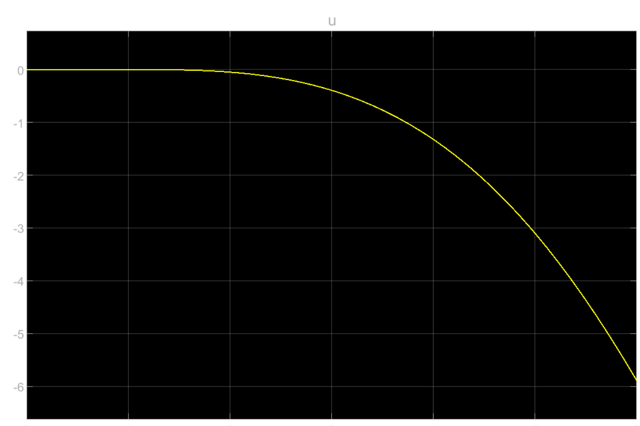


Similarly, the nonlinear and linear Simulink block diagrams accurately depicted the behavior seen in the systems when solved using ODE45.

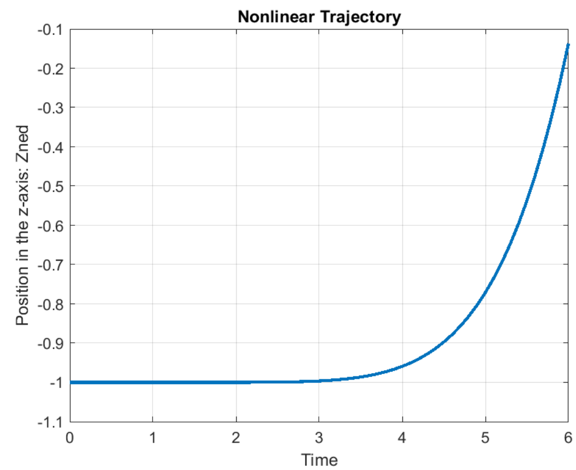

theta

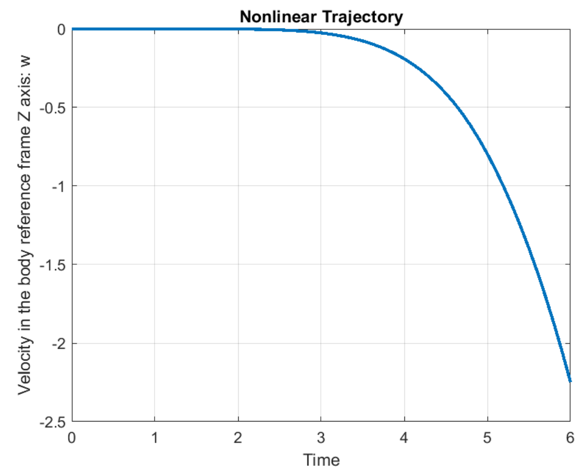

q

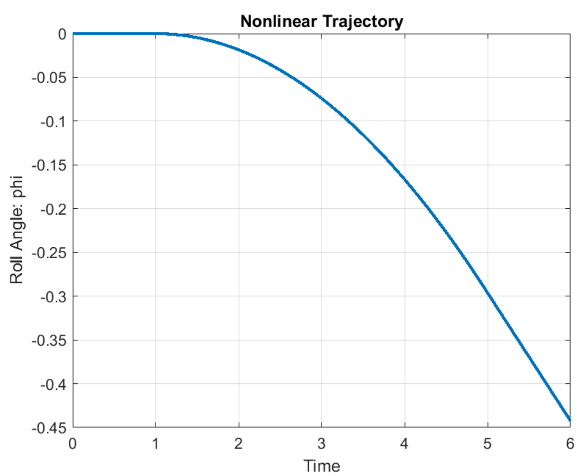

Xned

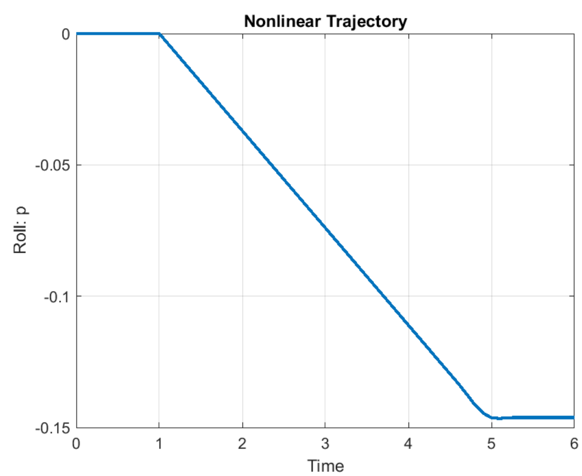

u

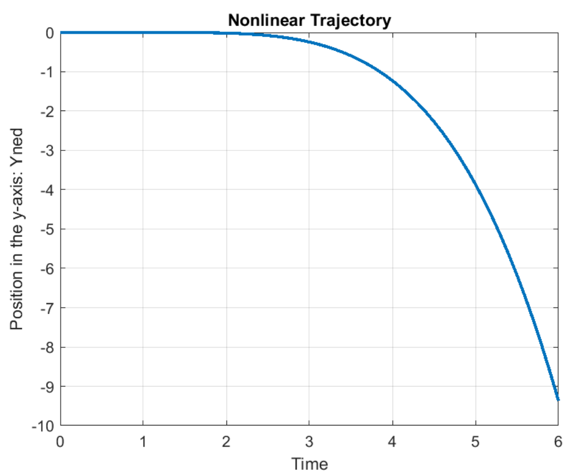
The simulation of the aileron step input from scenario 3: The state variable trajectories for all twelve states are graphed below. It is clear from the trajectories of the Z_{NED} and $\dot{w}$ that the drone is rising slightly. The drone is no longer at a 1g hover, but it is rising quite slowly. The roll increases as the roll rate holds a steady positive value. This causes the drone to drift in the positive y-direction which is captured in the trajectories of $\dot{Y_{NED}}$ and $\dot{v}$. The other state variables remain in equilibrium. This is clearly consistent with the expected response of an aileron input.

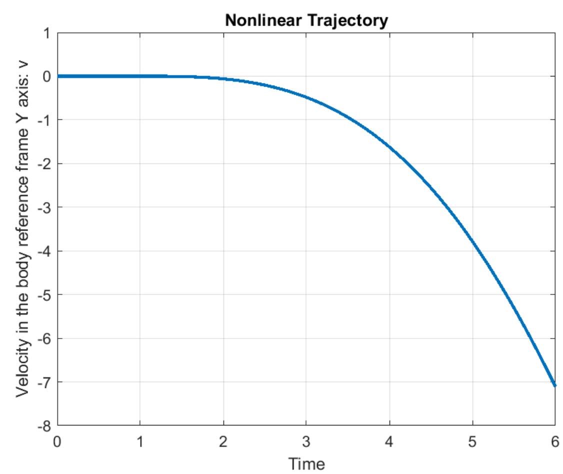




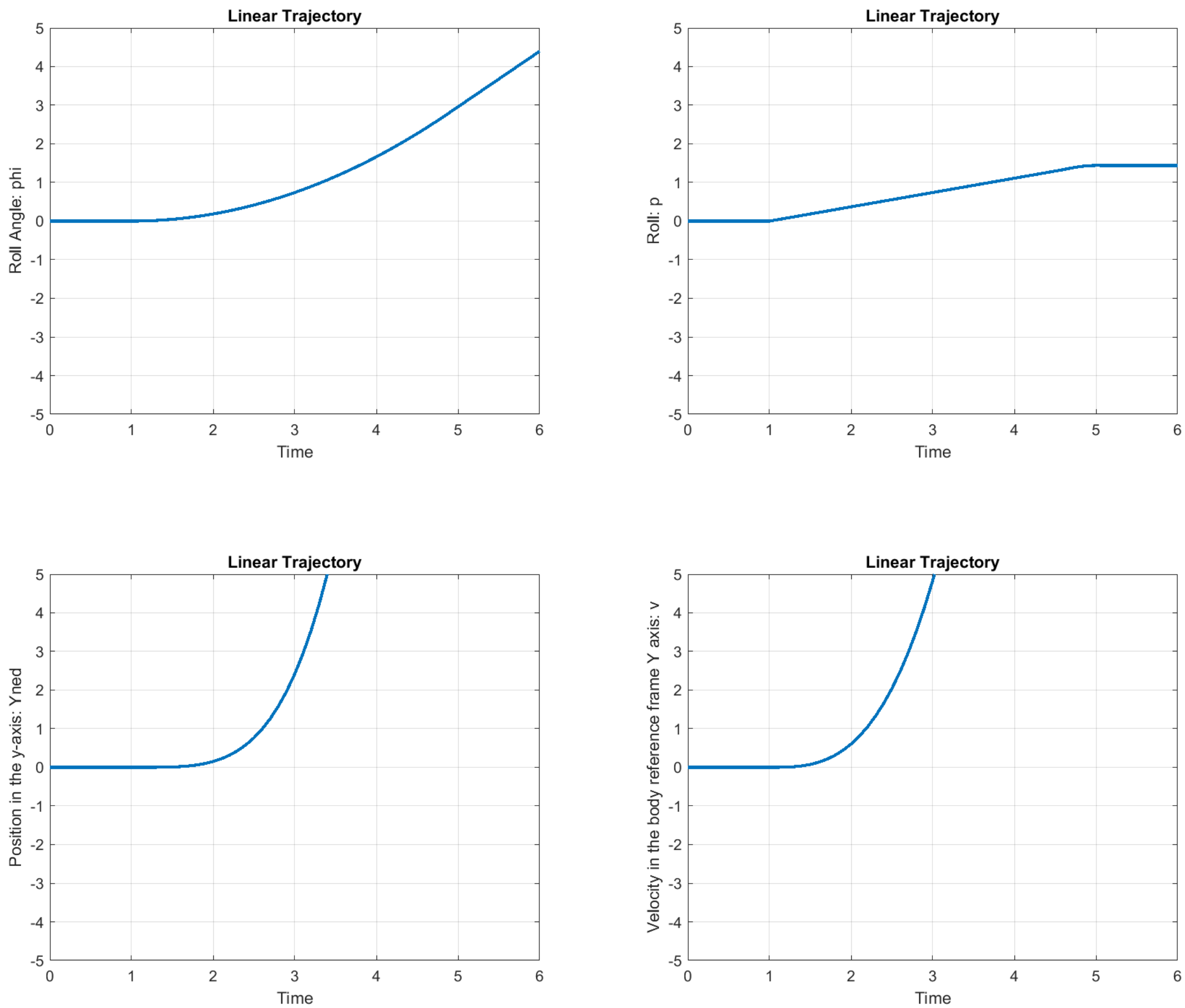




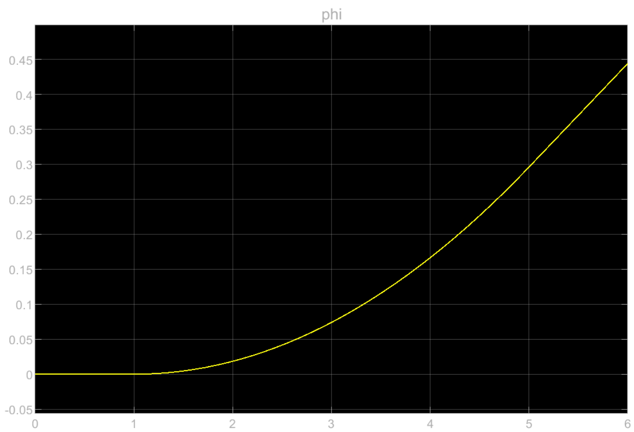

Nonlinear Trajectory
Position in the y-axis: Yned
Time
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0 1 2 3 4 5 6

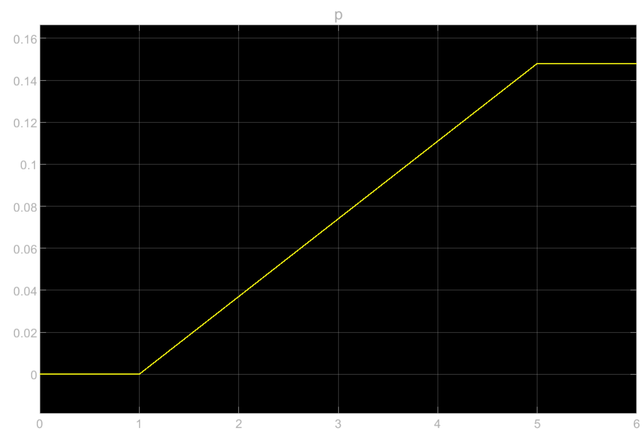

Nonlinear Trajectory
Velocity in the body reference frame Y axis: v
Time
1
0
-1
-2
-3
-4
-5
-6
-7
-8
0 1 2 3 4 5 6

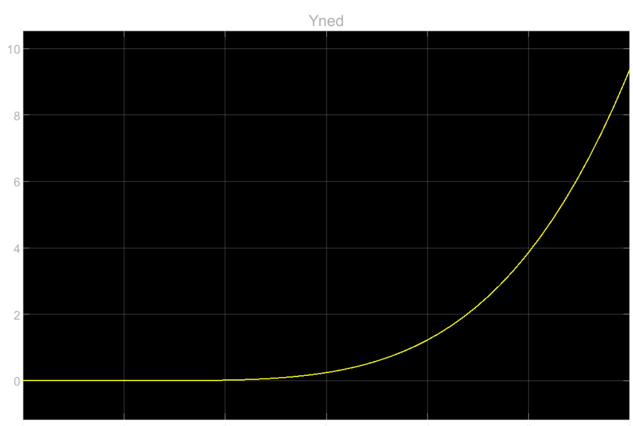
In the linear simulation similar results were found. The graphs that exhibited behavior other than remaining at equilibrium are included below. Note these are the same state variables that changed on the nonlinear simulation, concluding that the linear simulation is an accurate approximation.

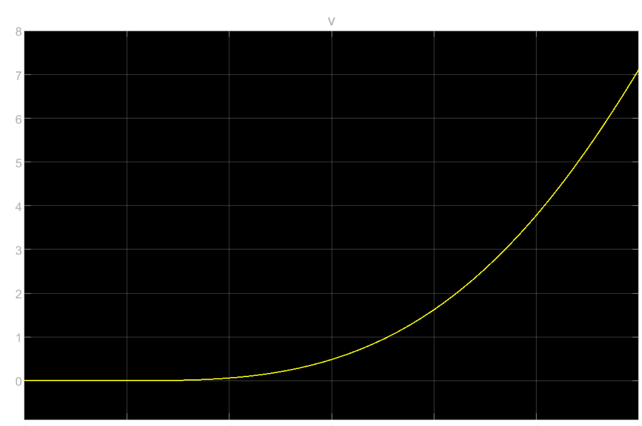


Similarly, the nonlinear and linear Simulink block diagrams accurately depicted the behavior seen in the systems when solved using ODE45.


phi
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
1
2
3
4
5
6


p
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
1
2
3
4
5
6


Yned
10
8
6
4
2
0
0
1
2
3
4
5
6


v
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6

Stability:

The unstable state variables in the system are the Euler angles phi, theta and psi. It is interesting to note that these are also the states most sensitive to input. The position and velocity state variables were more stable within the bounds of the Euler angles. That is so long as the drone did not rotate more than 90 degrees along any axis.

Coupled States:

In the real world, the simulations will lack accuracy because they were derived from Newton's laws and uniform, supplied parameters rather than from the actual measurements of the Parrot Mambo drone. This this potential error could be resolved using system identification. Furthermore, the parameters used were not precise, meaning in plain English the values could have had more values after the decimal. This would have also increased the accuracy of the simulations. The linear simulations are certainly less accurate than the nonlinear ones, because they are an approximation of a system that is in fact nonlinear. The states that are coupled can easily be seen in the linear system. The states regarding angular motion roll angle and roll, the pitch angle and pitch, the yaw angle and yaw and the states involving translational motion, u,v,w and Xned, Yned, and Zned are all coupled. This is evident from the integral dependence from one to another in the linear system. This also makes sense intuitively because the rotational velocity of the system will be affected by the angle of rotation and the same follows for the translational states.

Conclusion:

The nonlinear simulation of the MIMO system derived from Newton's law models the flight of the Parrot Mambo drone when exposed the Thrust, Elevator, Aileron, and Rudder input. The system has independent responses for each of these inputs, with each altering specific state variables. When exposed to thrust input the system acts strictly in the Z-axis. When exposed the elevator input, the system rotates in the y-axis plane and moves along the x-axis of the body reference frame. When exposed the aileron input, the system rotates in the x-axis plane and moves along the y-axis of the body reference frame. This coupling of states made the system difficult to analyze in the nonlinear case. However, the relationships between the state variables were illuminated with the linearization of the system. It is important to note that the linear system is not robust, and is in fact limited to the behavior surrounding a 1g hover. Straying from this equilibrium point will stray from the linear approximation and may not accurately model the behavior of the real system. Lastly, the system is currently unstable. The next step will be to design a feedback controller to stabilize the system.