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# Synthesis Module 2: Simulations of a 1g Hover with Signal Processing

Purpose: Redesign linear and nonlinear simulations of the Parrot Mambo drone at a 1g hover subject to Throttle, Elevator, Aileron, and Rudder inputs. Introduce yaw and gyroscopic moments to both the nonlinear and linear systems. Interface with the drone's sensors to retrieve signals for frequency domain analysis. Redesigning the $B$ matrix of the linear system to respond to $\delta T, \delta E, \delta A, \delta R$ inputs. Decouple the linear, 12-state, 4-output, 4-input, multi-input, multi-output (MIMO) linear system into 4 single-input, multi-output (SIMO)linear systems. Perform stability, controllability, and observability analysis on these new linear systems.

Beginning with the state equations as in Synthesis Module 1:

$$m\dot{V} + \omega \times mV = \vec{F_g}(\phi, \theta, \psi) + \vec{F_T} + \vec{F_A}(V, \omega, \theta)$$

$$J\dot{\omega} + \omega \times J\omega = \vec{M_T} + \vec{M}_{gyro} + \vec{M_A}(V, \omega, \theta)$$

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$

$$\dot{\theta} = (q\cos\phi - r\sin\phi)$$

$$\dot{\psi} = \frac{(q\sin\phi + r\cos\phi)}{\cos\theta}$$

$$\begin{vmatrix} X\dot{n}ed \\ Y\dot{n}ed \\ Z\dot{n}ed \end{vmatrix} = R(\psi)^T R(\theta)^T R(\phi)^T \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

With the variables representing the following values:

- $p, q, r$: body-axis components of angular velocity $\omega$ w.r.t x,y,z axes
- $u, v, w$: body-axis components of velocity $V$ w.r.t flat earth x,y,z axes
- $X_A, Y_A, Z_A$: body-axis components of aerodynamic forces $\vec{F_A}$
- $L_A, M_A, N_A$: body-axis components of aerodynamic moments $\vec{M_A}$
- $X_T, Y_T, Z_T$: body-axis components of propulsive forces $\vec{F_T}$

- $L_{gyro}, M_{gyro}, N_{gyro}$: body-axis components of gyroscopic moments $\vec{M_{gyro}}$
- $X_{NED}, Y_{NED}, Z_{NED}$: inertial axis components of position
- $T$: Thrust
- $\phi, \theta, \psi$: roll-pitch-yaw Euler angles
- $R(\phi), R(\theta), R(\psi)$: positive roll, pitch, yaw rotation matrices

Note assume the following:

- Aerodynamics forces $\vec{F_A}$ are zero: These forces are not in fact zero, but the drone is small, lightweight, and moving at low speeds giving negligible values.
- ~~Aerodynamic and gyroscopic Moments $\vec{M_A}$ and $\vec{M}_{gyro}$ are zero: The justification for this assumption follows from the same logic as those used regarding the aerodynamic forces.~~
- The system behaves exactly as defined by Newton's laws: The manufacturing of these drones is not perfect and leads to imperfections in the drone's hardware and body components. This yields deviation from the ideal model assumed in Newton's laws. A more accurate simulation of the drone may be acquired using system identification.
- All four motors are identical: This follows directly from the previous assumption. The drone is modeled as if input from all four drones is identical, however this is highly unlikely and at a high degree of accuracy, it is impossible.
- The parameters for mass $m$ and intertial moments $J$ are known: These values were supplied by the instructor. They could have been acquired with measurement for this particular drone. Furthermore, Jxy Jxz Jyz... are assumed to be zero. This is a valid assumption given the symmetry of the UAV drone body.

Disregarding the previous assumption to ignore the aerodynamic and gyroscopic moments, introduces a yaw moment into the dynamics of the system. Newton's laws state the angular momentum of the UAV body $l$ and its four rotors $l_1, l_2, l_3$ and $l_4$ must be conserved. It is assumed propellers produce thrust only in the z-axis, and therefore the z-component of thrust is zero in the moment matrix. Aerodynamic forces from the propellers are assumed to be negligible except that it is assumed there is a drag torque applied opposite to the rotation of the propellers. Accounting for these dynamics, the sum of all moments $\mathbb{M}$ about the center of gravity of the UAV in body-axis moment matrix becomes

$$\mathbb{M} = \vec{M_T} + \vec{M_{gyro}} + \vec{M_A} = \begin{bmatrix} L_T(\omega 1) - L_T(\omega 2) - L_T(\omega 3) + L_T(\omega 4) \\ M_T(\omega 1) + M_T(\omega 2) - M_T(\omega 3) - M_T(\omega 4) \\ -N_A(\omega 1) + N_A(\omega 2) - N_A(\omega 3) + N_A(\omega 4) \end{bmatrix}$$

this notation allows a concise way to write Newton's laws:

$$\mathbb{M} = \frac{\partial d}{\partial dt}(l + \sum_{i=1}^{4} l_i)$$

Assuming the propeller has only moment of inertia in the z-axis, $Jp$, the right-hand side (RHS) can be expanded to:

$$(l + \sum_{i=1}^{4} l_i) = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \begin{vmatrix} p \\ q \\ r \end{vmatrix} + J_p \begin{bmatrix} 0 \\ 0 \\ \dot{\omega}_1 - \dot{\omega}_2 + \dot{\omega}_3 - \dot{\omega}_4 \end{bmatrix}$$

Apply the equation of Coriolis to the RHS to yield a final expression for the angular velocity state equations:

$$\mathbb{M} = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \begin{vmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{vmatrix} + J_p \begin{bmatrix} 0 \\ 0 \\ \dot{\omega}_1 - \dot{\omega}_2 + \dot{\omega}_3 - \dot{\omega}_4 \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} (l + \sum_{i=1}^{4} l_i)$$

Now, when exposed to rudder input, the elements in the x and y axes become zero.$\dot{r} also becomes zero.$ It is assumed that at equilibrium, the moment from a 1 hz rudder increase affects approximately six-times as much angular acceleration as its corresponding gyroscopic moments, hence $J_p$ is scaled by six. In the z-axis the equation becomes:

$$-N_A(\omega 1) + N_A(\omega 2) - N_A(\omega 3) + N_A(\omega 4) = 6J_p(\dot{\omega}_1 - \dot{\omega}_2 + \dot{\omega}_3 - \dot{\omega}_4)$$

The magnitude of the yawing moment from a single propeller's aerodynamics, $N_A$, is defined:

$$N_A = \rho C_N {\omega_i}^2 D^4 d$$

- $\rho$: air density = 1.22495238 $\frac{kg}{m^3}$
- $C_N$: coefficient of yawing moment
- $w_i$: motor speed
- $D$: diameter of propeller = 0.66mm
- $d$: distance from the rotor to the vehicle's center 4.5mm

Modeling the propeller as a slender rod, rotating about its center of mass, the moment of inertia of a single propeller, weighing 1g, is estimated to be:

$$I = \frac{1}{12} m D^2 = 3.63 \times 10^{-7} \; kg \; m^2$$

Applying parallel axis theorem to estimate the moment of inertia of the propellers about the UAV's center of mass

$$J_p = I = I_{\text{cm}} + md^2 = 4.8993 \times 10^{-6}\ kg\ m^2$$

Now, solving

$$\rho C_N D^4 d((2\omega_e - 1)^2 + 2(\omega_e + 1)^2) = 6 \times 4.8993 \times 10^{-6}$$

for $C_N$:

$$C_N = 0.0072$$

The nonlinear system of the form:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

when updated becomes:

$$\begin{vmatrix} \dot{Z}_{NED} \\ \dot{w} \\ \dot{\theta} \\ \dot{q} \\ \dot{\phi} \\ \dot{p} \\ \dot{\psi} \\ \dot{r} \\ \dot{X}_{NED} \\ \dot{u} \\ \dot{Y}_{NED} \\ \dot{v} \end{vmatrix} = \begin{vmatrix} w\cos\phi\cos\theta) - u\sin\theta + v\cos\theta)sin(\phi) \\ qu - pv - \frac{F_T(\omega_1)+F_T(\omega_2)+F_T(\omega_3)+F_T(\omega_4)}{m} + g\cos\phi\cos\theta \\ q\cos\phi - r\sin\phi \\ \frac{M_T(\omega_1)+M_T(\omega_2)-M_T(\omega_3)-M_T(\omega_4)+J_{xx}pr-J_{zz}pr}{J_{yy}} \\ p + \tan\theta(r\cos\phi + q\sin\phi) \\ \frac{L_T(\omega_1)-L_T(\omega_2)-L_T(\omega_3)+L_T(\omega_4)+J_{yy}qr-J_{zz}qr}{J_{xx}} \\ \frac{r\cos phi+q\sin phi}{\cos\theta} \\ \frac{-N_A(\omega_1)+N_A(\omega_2)-N_A(\omega_3)+N_A(\omega_4)+J_{xx}pq-J_{yy}pq}{J_{zz}} \\ w(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta) - v(\cos\phi\sin\psi - \cos\psi\sin\phi sin\theta) + u\cos\psi\cos\theta \\ rv - qw - g\sin\theta \\ v(\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta) - w\cos\psi\sin\phi - \cos\phi\sin\psi\sin\theta) + u\cos\theta\sin\psi \\ pw - ru + g\cos\theta\sin\phi \end{vmatrix}$$

Where $F_T(\omega_i)$ is the force of thrust given by motor $i$ defined:

$$C_t \rho w_i^2 D^4$$

- $\rho$: air density = 1.22495238 $\frac{kg}{m^3}$
- $C_t$: coefficient of thrust given as 0.75
- $w_i$: motor speed
- $D$: diameter of propeller = 0.66mm

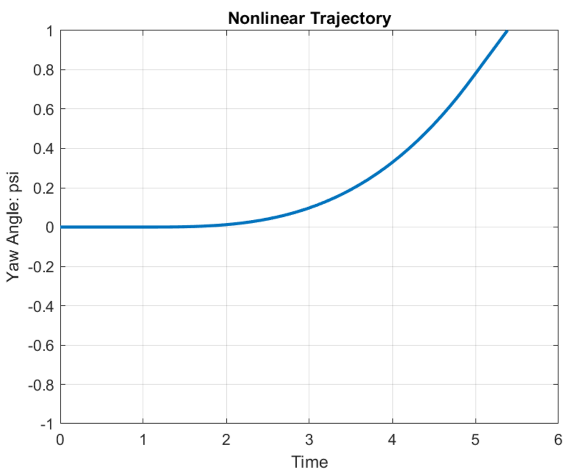
and $L_T = M_T = \frac{F_T \times b}{\omega_i}$. With b equal to 0.047625 the distance from the center of mass to the x and y axes in meters.

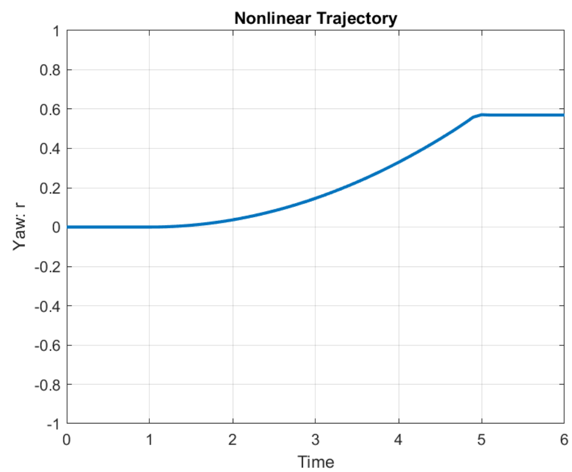
The output of the nonlinear system is arbitrarily defined to be:

$$h = -Z_{NED}$$

$$\begin{vmatrix} acceleration_x \\ acceleration_y \\ acceleration_z \end{vmatrix} = m\dot{V} + \omega \times mV - \frac{\vec{F_G}}{m} = \frac{\vec{F_T} + \vec{F_a}}{m}$$

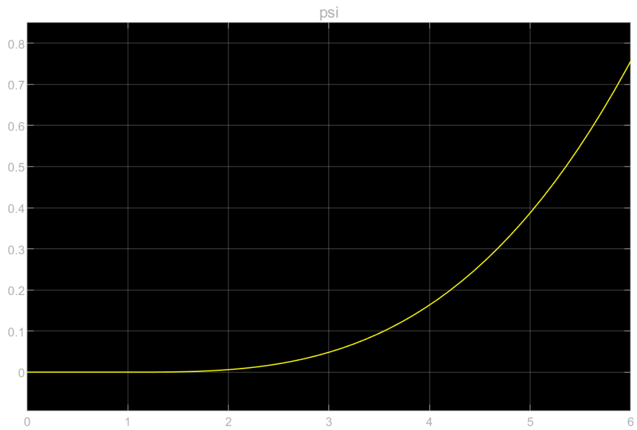
Altering the MATLAB ODE45 script to plot the trajectories of the states in response to a rudder ramp input with constant slope 0.25 $Hz$, the behavior is consistent with expectation:

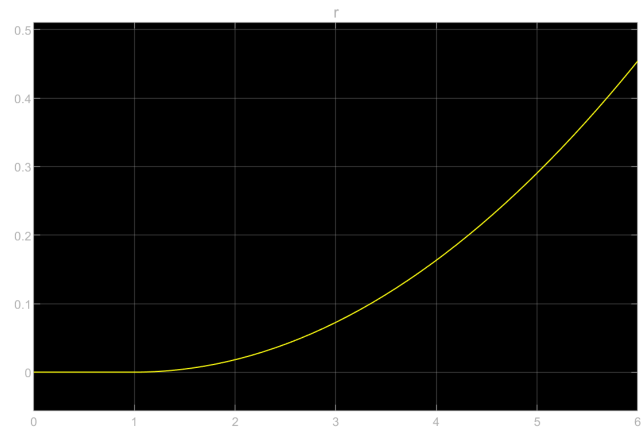


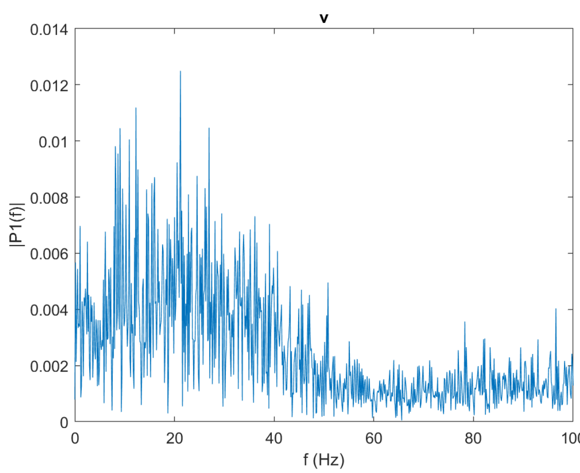



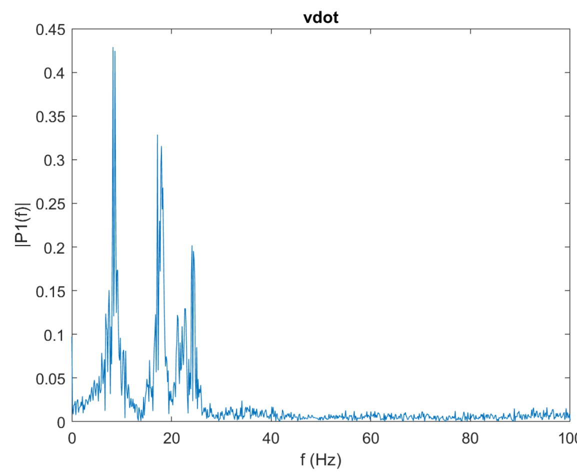

The system showed positive yaw while all other states remained at equilibrium. Similiarly, when updating the Simulink model:

Basic Signal Processing:
In the Simulink model of the Parrot Mambo drone's flight control system, the signals $p, q, r, \dot{u}, \dot{v}, \dot{w}, \phi, \theta, \psi, u,$ were collected. While the system does not explicitly offer the signals $\phi, \theta, \psi$ they can be found by integrating the signals $p, q, r$. To achieve this, the signals were passed through a Tustin transform of the *Laplace* transfer function $\frac{1}{s}$.
Taking a fast finite Fourier transform of the signals illuminates noise that can be eliminated with the use of a filter. The fast Fourier transforms graphs below show that excess noise needed to be removed from the signals $v, \dot{v}, u, \dot{u}$ below 50 $Hz$ using a high pass filter, and the signal $w$ above 60 $Hz$. While other signals appear to be clean enough to forego filtering, The plot of the fast Fourier transform of  has been included for reference.




These high and low pass filters were constructed using the same method as the integration done previously. Applying the Tustin transform to the filter transfer functions:

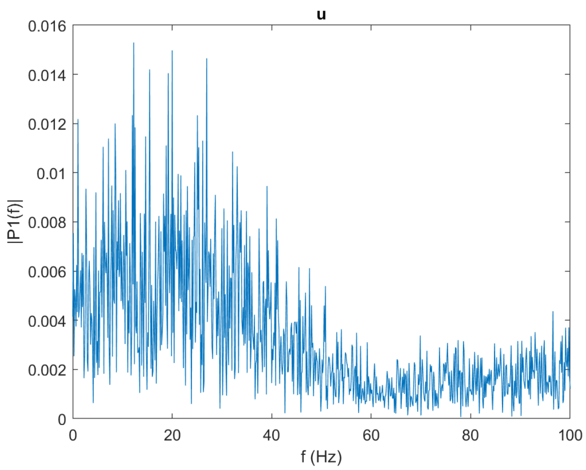
$$filter_{highpass} = \frac{s}{1 + \frac{s}{\omega}}$$

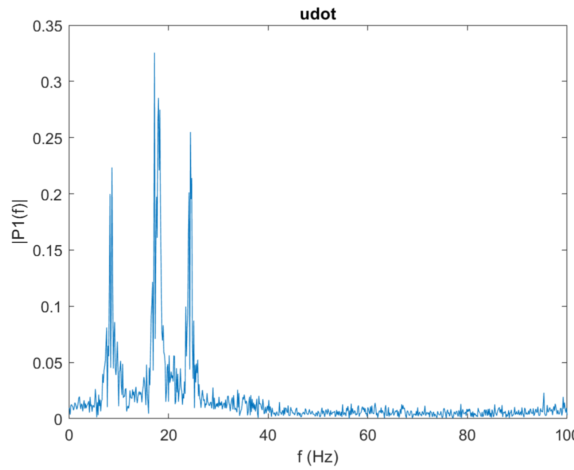
$$filter_{lowpass} = \frac{1}{1 + \frac{s}{\omega}}$$

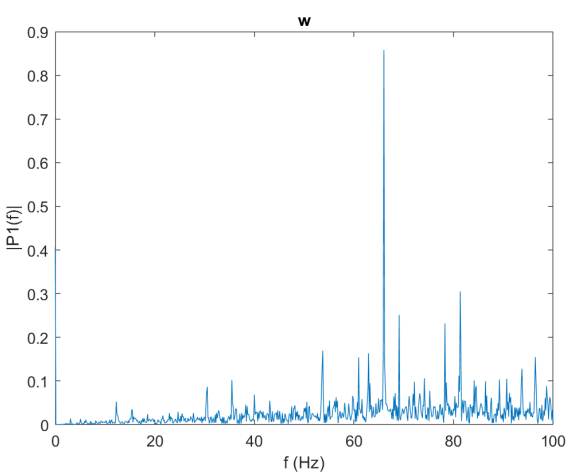
Where $\omega$ is equal to 50 and 60 $Hz$ respectively. This yielding transfer functions:

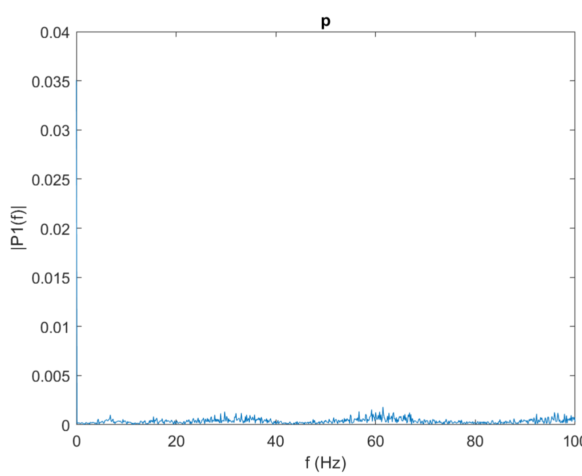
$$filter_{highpass} = \frac{0.5z^4 - 0.5z^2}{0.01125z^4 + 0.0025z^3 - 0.00875z^2}$$

$$filter_{lowpass} = \frac{0.3z^2 + 0.3z}{2.3z^2 - 1.7z}$$

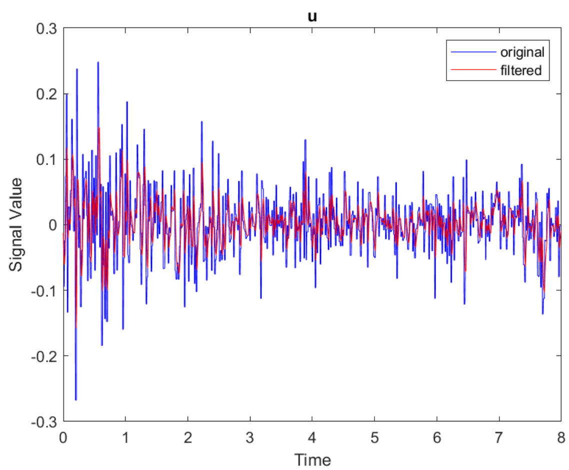

u
|P1(f)|
f (Hz)
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
20
40
60
80
100

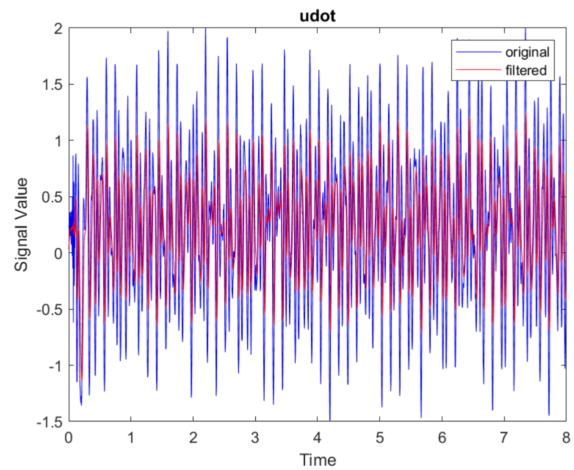

udot
|P1(f)|
f (Hz)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100

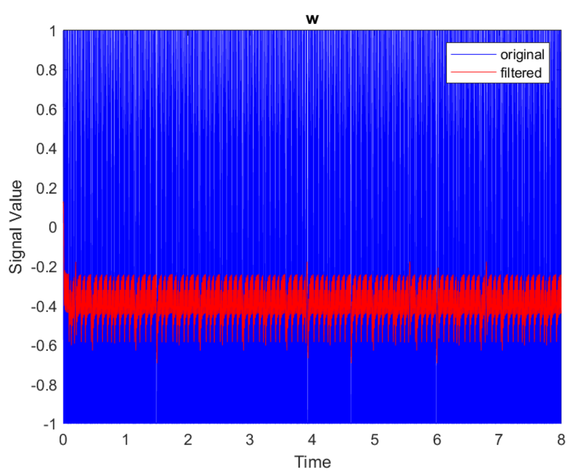

w
|P1(f)|
f (Hz)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100


p
|P1(f)|
f (Hz)
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
20
40
60
80
100

After filtering the signals:

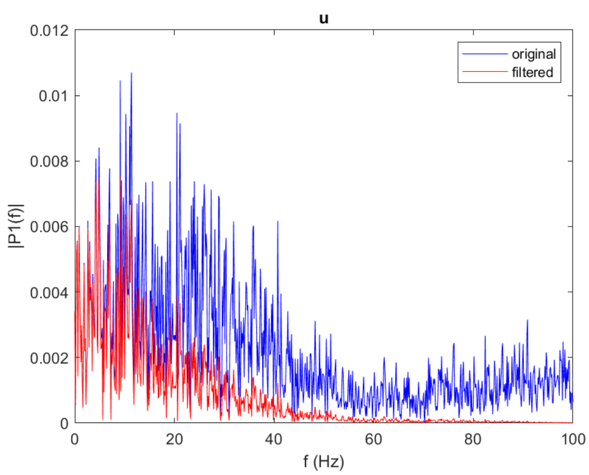


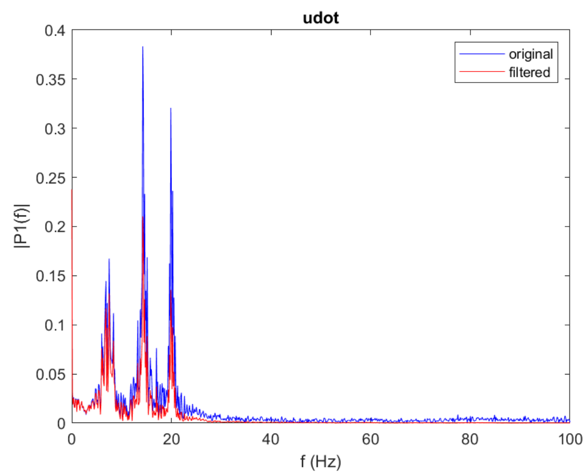



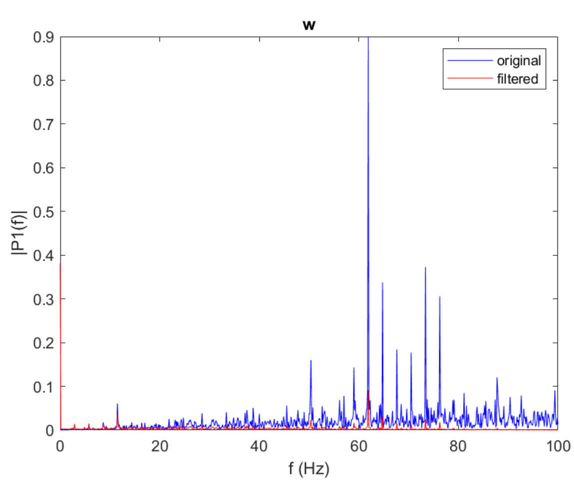




The updated fast fourier transforms:

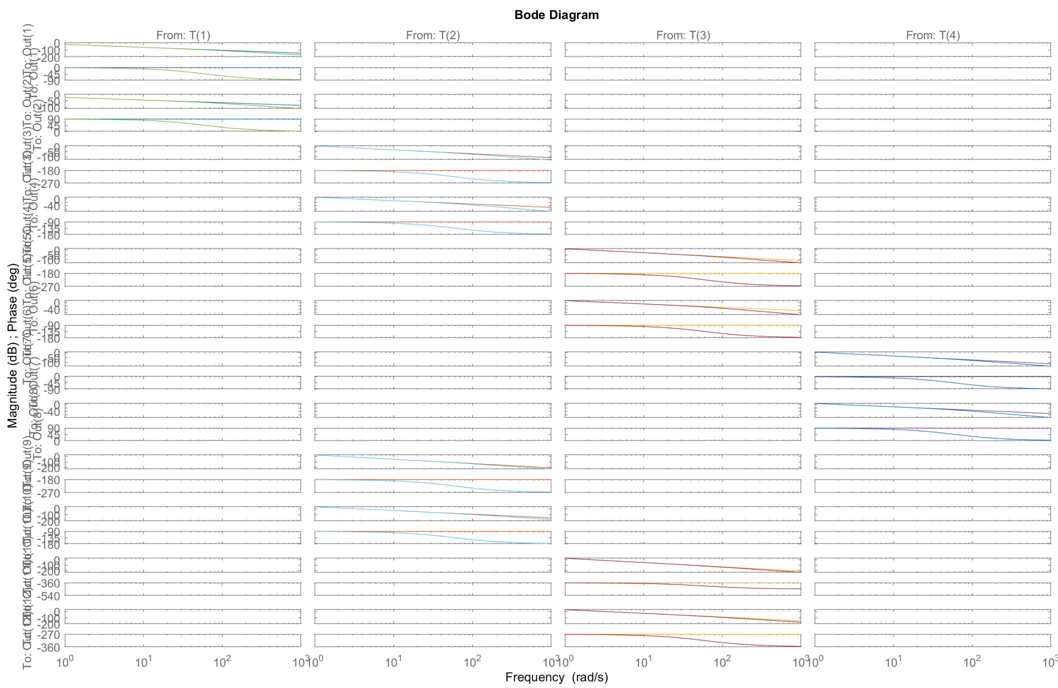
Clearly, there were issues with the high pass filtering, to be solved at a later date.


u
original
filtered
|P1(f)|
f (Hz)

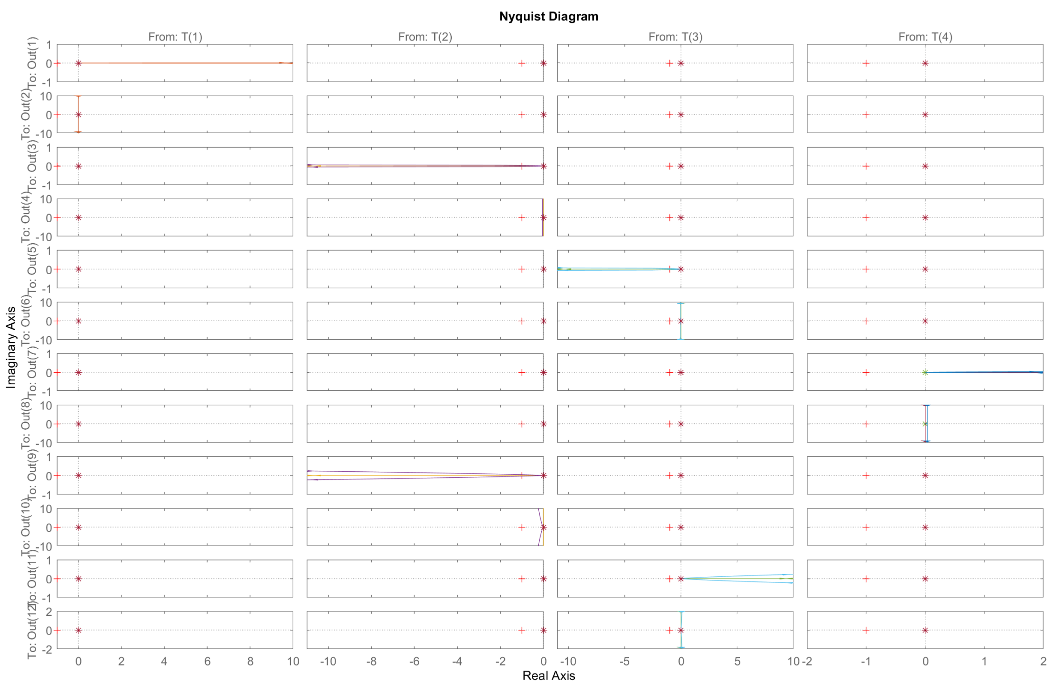

udot
original
filtered
|P1(f)|
f (Hz)


w
original
filtered
|P1(f)|
f (Hz)

The bode response for the system responding to $\delta T, \delta E, \delta A, \delta R$ inputs:




The nyquist response for the system responding to $\delta T, \delta E, \delta A, \delta R$ inputs:



Nyquist Diagram
From: T(1)
From: T(2)
From: T(3)
From: T(4)
To: Out(1)
To: Out(2)
To: Out(3)
To: Out(4)
To: Out(5)
To: Out(6)
To: Out(7)
To: Out(8)
To: Out(9)
To: Out(10)
To: Out(11)
To: Out(12)
Imaginary Axis
Real Axis

In the linear case the system becomes:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

The linear state space model found in Synthesis Model 1, updated with the introduction of the yaw and gyroscopic moments is given by:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} Z_{NED} \\ w \\ \theta \\ q \\ \phi \\ p \\ \psi \\ r \\ X_{NED} \\ u \\ Y_{NED} \\ v \end{vmatrix} + 2\rho\omega_e C_t D^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} \\ 0 & 0 & 0 & 0 \\ \frac{b}{J_{yy}} & \frac{b}{J_{yy}} & -\frac{b}{J_{yy}} & -\frac{b}{J_{yy}} \\ 0 & 0 & 0 & 0 \\ \frac{b}{J_{xx}} & -\frac{b}{J_{xx}} & -\frac{b}{J_{xx}} & \frac{b}{J_{xx}} \\ 0 & 0 & 0 & 0 \\ -\frac{d}{J_{zz}} & \frac{d}{J_{zz}} & -\frac{d}{J_{zz}} & \frac{d}{J_{zz}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \omega 1 \\ \omega 2 \\ \omega 3 \\ \omega 4 \end{vmatrix}$$

With output equation,

$$y = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + 2\rho\omega_e C_t D^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} \end{bmatrix} v$$

Note: $\omega_e$ is the equilibrium motor speed that generates a force of thrust $\vec{F_T} = \vec{F_G}$.

The original linear system defined inputs as the motor speeds. These can now be transformed to virtual inputs representing Throttle, Elevator, Aileron, Roll:

$$\vec{v} = \begin{vmatrix} \omega 1 \\ \omega 2 \\ \omega 3 \\ \omega 4 \end{vmatrix} \Rightarrow \vec{u} = \begin{vmatrix} \delta T \\ \delta E \\ \delta A \\ \delta R \end{vmatrix}$$

This is accomplished by defining a motor mixing matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Now,

$$\vec{v} = M\vec{u} + \vec{u_0}$$

where $u_0$ is a bias vector that sets the motors to their equilibrium value. Applying the chain rule,

$$\frac{\partial v}{\partial u} = M$$

Given that B was defined by taking the Jacobian w.r.t $v$ for equilibrium values $x_0, v_0$:

$$\bar{B} := \frac{\partial}{\partial v} f(x_0, v_0)$$

Then by the chain-rule, the new B matrix corresponding to the virtual inputs is:

$$B := \frac{\partial f}{\partial u} = (\frac{\partial f}{\partial v})(\frac{\partial v}{\partial u}) = \bar{B}M$$

Using this formulation for the D matrix as well, the linear system becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} Z_{NED} \\ w \\ \theta \\ q \\ \phi \\ p \\ \psi \\ r \\ X_{NED} \\ u \\ Y_{NED} \\ v \end{vmatrix} + 8\rho\omega_e C_t D^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{b}{J_{yy}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{J_{xx}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{J_{zz}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \delta T \\ \delta E \\ \delta A \\ \delta R \end{vmatrix}$$

With output equation,

$$y = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + 8\rho\omega_e C_t D^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{m} & 0 & 0 & 0 \end{bmatrix} u$$

The linear system can now be decoupled, representing the 12-state, multi-input, multi-output (MIMO) system as 4 single-input, multi-output (SIMO) system each corresponding to the virtual inputs $\delta T, \delta E, \delta A, \delta R$. These linear SIMO systems are defined:

For Throttle input:

$$\begin{vmatrix} \dot{\delta Z_{NED}} \\ \delta \dot{w} \end{vmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{vmatrix} \delta Z_{NED} \\ \delta w \end{vmatrix} + \begin{bmatrix} 0 \\ -\frac{8\rho\omega_e C_t D^4}{m} \end{bmatrix} \delta T$$

$$\begin{vmatrix} \delta Z_{NED} \\ \delta w \\ \delta h \\ \delta accel_z \end{vmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{vmatrix} \delta Z_{NED} \\ \delta w \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{8\rho\omega_e C_t D^4}{m} \end{bmatrix} \delta T$$

The eigenvalues and eigenvectors of the system:

$$\lambda_1 = \lambda_2 = 0$$

$$\vec{v_1} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \vec{v_2} = \begin{vmatrix} t \\ 1 \end{vmatrix}$$

The matrix exponential:

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

The modes:

$$e^{At}\vec{v_1} = e^{\lambda_1 t}\vec{v_1} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$e^{At}\vec{v_2} = e^{\lambda_2 t}\vec{v_2} = \begin{vmatrix} t \\ 1 \end{vmatrix}$$

Given that the eigenvalues $\lambda_1 = \lambda_2 \not< 0$ the system is not stable.

Determining the controllability of the states by verifying the rank of controllability matrix $W$ is full

$$W = \begin{bmatrix} B & AB & A^2B & ... & A^{n-1}B \end{bmatrix}$$

This yields:

$$W = 8\rho\omega_e C_t D^4 \begin{bmatrix} 0 & -\frac{1}{m} \\ -\frac{1}{m} & 0 \end{bmatrix}$$

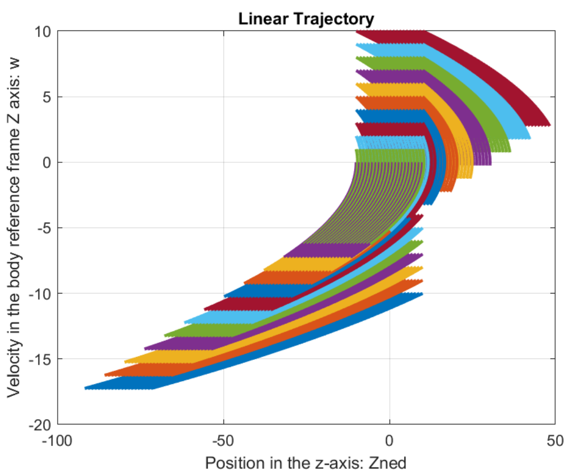
Which has full rank, thus the states $\delta Z_{NED}$ and $\delta w$ are controllable. Determining the observability of the states by verifying the rank of observability matrix $Q$ is full

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{bmatrix}$$

This yields

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Which has full rank, thus the states $\delta Z_{NED}$ and $\delta w$ are observable
The phase portrait for states $\delta Z_{NED}$ and $\delta w$:




For Elevator input:

$$\begin{vmatrix} \delta\dot{\theta} \\ \delta\dot{q} \\ \delta\dot{X}_{NED} \\ \delta\dot{u} \end{vmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \delta\theta \\ \delta q \\ \delta X_{NED} \\ \delta u \end{vmatrix} + \begin{bmatrix} 0 \\ \frac{8b\rho\omega_e C_t D^4}{Jyy} \\ 0 \\ 0 \end{bmatrix} \delta E$$

$$\begin{vmatrix} \delta\theta \\ \delta q \\ \delta X_{NED} \\ \delta u \\ \delta accel_x \end{vmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \delta\theta \\ \delta q \\ \delta X_{NED} \\ \delta u \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta E$$

The eigenvalues and eigenvectors of the system:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\vec{v_1} = \begin{vmatrix} 0 \\ 0 \\ -g \\ 0 \end{vmatrix}, \vec{v_2} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ -g \end{vmatrix}, \vec{v_3} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \vec{v_4} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix},$$

The matrix exponential:

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The modes:

$$e^{At}\vec{v_1} = e^{\lambda_1 t}\vec{v_1} = \begin{vmatrix} -\frac{gt^2}{2} \\ -gt \\ -g \\ 0 \end{vmatrix}, e^{At}\vec{v_2} = e^{\lambda_2 t}\vec{v_2} = \begin{vmatrix} -\frac{gt^3}{6} \\ -\frac{gt^2}{2} \\ -gt \\ -g \end{vmatrix}$$

$$e^{At}\vec{v_3} = e^{\lambda_3 t}\vec{v_3} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, e^{At}\vec{v_4} = e^{\lambda_4 t}\vec{v_4} = \begin{vmatrix} t \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

Given that the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \nless 0$ the system is not stable.

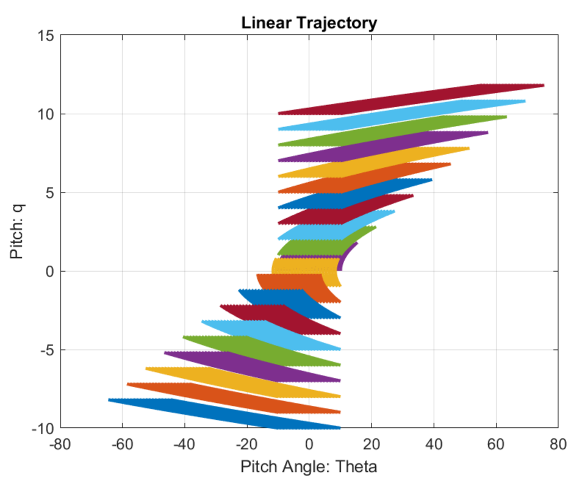
Determining the controllability of the states yields:

$$W = 8\rho\omega_e C_t D^4 \begin{bmatrix} 0 & \frac{b}{Jyy} & 0 & 0 \\ \frac{b}{Jyy} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-bg}{Jyy} \\ 0 & 0 & \frac{-bg}{Jyy} & 0 \end{bmatrix}$$

Which has full rank, thus the states $\delta\theta, \delta q, \delta X_{NED}$ and $\delta u$ are controllable. Determining the observability of the states yields:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -g & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which has full rank, thus the states $\delta\theta, \delta q, \delta X_{NED}$ and $\delta u$ are observable.
The phase portrait for states $\delta\theta$ and $\delta q$:




For Aileron input:

$$\begin{vmatrix} \delta\dot{\phi} \\ \delta\dot{p} \\ \delta\dot{Y}_{NED} \\ \delta\dot{v} \end{vmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \delta\phi \\ \delta p \\ \delta Y_{NED} \\ \delta v \end{vmatrix} + \begin{bmatrix} 0 \\ \frac{8b\rho\omega_e C_t D^4}{Jxx} \\ 0 \\ 0 \end{bmatrix} \delta A$$

$$\begin{vmatrix} \delta\phi \\ \delta p \\ \delta Y_{NED} \\ \delta v \\ \delta accel_y \end{vmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} \delta\phi \\ \delta p \\ \delta Y_{NED} \\ \delta v \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta A$$

The eigenvalues and eigenvectors of the system:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\vec{v_1} = \begin{vmatrix} 0 \\ 0 \\ g \\ 0 \end{vmatrix}, \vec{v_2} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ g \end{vmatrix}, \vec{v_3} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \vec{v_4} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix},$$

The matrix exponential:

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The modes:

$$e^{At}\vec{v_1} = e^{\lambda_1 t}\vec{v_1} = \begin{vmatrix} \frac{gt^2}{2} \\ gt \\ g \\ 0 \end{vmatrix}, e^{At}\vec{v_2} = e^{\lambda_2 t}\vec{v_2} = \begin{vmatrix} \frac{gt^3}{6} \\ \frac{gt^2}{2} \\ gt \\ g \end{vmatrix}$$

$$e^{At}\vec{v_3} = e^{\lambda_3 t}\vec{v_3} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, e^{At}\vec{v_4} = e^{\lambda_4 t}\vec{v_4} = \begin{vmatrix} t \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

Given that the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \nless 0$ the system is not stable.

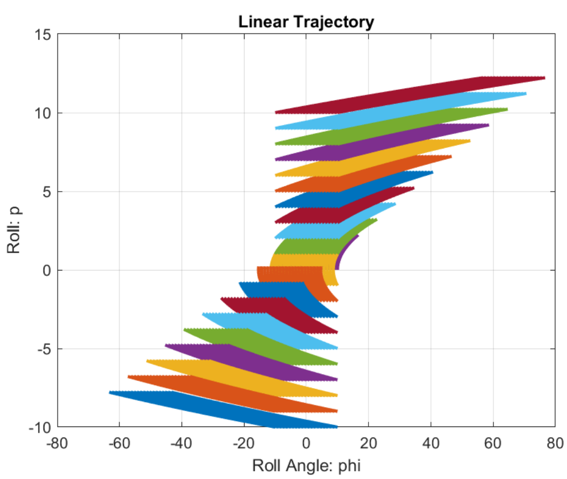
Determining the controllability of the states yields:

$$W = 8\rho\omega_e C_t D^4 \begin{bmatrix} 0 & \frac{b}{Jxx} & 0 & 0 \\ \frac{b}{Jxx} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{bg}{Jxx} \\ 0 & 0 & \frac{bg}{Jxx} & 0 \end{bmatrix}$$

Which has full rank, thus the states $\delta\phi, \delta p, \delta Y_{NED}$ and $\delta v$ are controllable. Determining the observability of the states yields:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which has full rank, thus the states $\delta\phi, \delta p, \delta Y_{NED}$ and $\delta v$ are observable.
The phase portrait for states $\delta\phi$ and $\delta p$:




For Rudder input:

$$\begin{vmatrix}\delta\dot{\psi}\\ \delta\dot{r}\end{vmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{vmatrix}\delta\psi\\ \delta r\end{vmatrix} + \begin{bmatrix}0\\ -\frac{8d\rho\omega_e C_t D^4}{Jzz}\end{bmatrix}\delta R$$

$$\begin{vmatrix}\delta\psi\\ \delta r\end{vmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{vmatrix}\delta\psi\\ \delta r\end{vmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}\delta R$$

The eigenvalues and eigenvectors of the system:

$$\lambda_1 = \lambda_2 = 0$$

$$\vec{v_1} = \begin{vmatrix}1\\ 0\end{vmatrix}, \vec{v_2} = \begin{vmatrix}t\\ 1\end{vmatrix}$$

The matrix exponential:

$$e^{At} = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$$

The modes:

$$e^{At}\vec{v_1} = e^{\lambda_1 t}\vec{v_1} = \begin{vmatrix}1\\ 0\end{vmatrix}$$

$$e^{At}\vec{v_2} = e^{\lambda_2 t}\vec{v_2} = \begin{vmatrix}t\\ 1\end{vmatrix}$$

Given that the eigenvalues $\lambda_1 = \lambda_2 \not< 0$ the system is not stable.

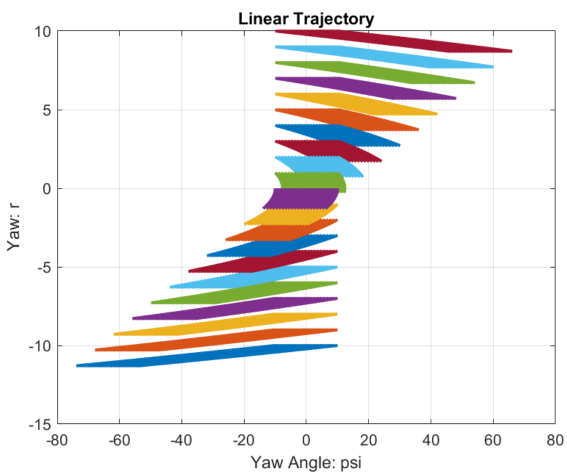
Determining the controllability of the states by verifying the rank of controllability matrix yields:

$$W = 8\rho\omega_e C_t D^4 \begin{bmatrix}0 & -\frac{1}{J_{zz}}\\ -\frac{1}{J_{zz}} & 0\end{bmatrix}$$

Which has full rank, thus the states $\delta\psi$ and $\delta r$ are controllable. Determining the observability of the states yields:

$$Q = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

Which has full rank, thus the states $\delta\psi$ and $\delta r$ are observable
The phase portrait for states $\delta\psi$ and $\delta r$:


Linear Trajectory
Yaw: r
Yaw Angle: psi
10
5
0
-5
-10
-15
-80
-60
-40
-20
0
20
40
60
80