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Around probabilistic cellular automata

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ABSTRACT

We survey probabilistic cellular automata with approaches coming from combinatorics, statistical physics, and theoretical computer science, each bringing a different viewpoint. Some of the questions studied are specific to a domain, and some others are shared, most notably the ergodicity problem.

1. Introduction

Consider a set of cells indexed by $\mathbb{Z}^d, d \geq 1$, each cell containing a letter from a finite alphabet $\mathcal{A}$. The updating of cells is local (each cell updates according to a finite neighborhood), time-synchronous, and space-homogeneous. When the updating is deterministic, we obtain a Cellular Automaton (CA), and when it is random, we obtain a Probabilistic Cellular Automaton (PCA). Alternatively, PCA may be viewed as discrete-time Markov chains on the state space $\mathcal{A}^{\mathbb{Z}^d}$ which are the synchronous counterparts of (finite range) interacting particle systems.

CA are important and widely studied for at least three reasons. First, they are natural from a dynamical point of view. In particular, by Hedlund's theorem [37], they correspond precisely to the functions on $\mathcal{A}^{\mathbb{Z}^d}$ which are continuous (with respect to the product topology) and commuting with translations. Second, they constitute a powerful model of computation, in particular they can "simulate" any Turing machine. Third, due to the amazing gap between the simplicity of their definition and the intricacy of their trajectories, CA are good candidates for modeling "complex systems" appearing in physical and biological processes.

This multiplicity of viewpoints carries over to PCA. First, in theoretical computer science, PCA obtained as a random perturbation of CA have been considered, with at least two different motivations: – to investigate the fault-tolerant computation capability of CA; – to classify elementary CA (Wolfram's program) by using the robustness to errors as a discriminating criterion. Second, in statistical mechanics, general PCA are studied for their connections with Gibbs potentials and Gibbs measures. Third, some specific PCA are linked to important combinatorial models, in particular, directed animals, queues, and directed percolation. Last, PCA inherit from CA the ability to be relevant models for complex systems appearing in physics and biology.

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There exist several books and surveys dedicated to CA, see [43] and the references therein. For PCA, the standard reference is the textbook [75] from 1990. The more recent survey articles [72,74] mostly deal with fault-tolerant computations. In the present survey, our primary goal was to keep the balance between the different viewpoints and to present them in a unified way. We focus on some aspects for each domain, rather than seeking to be exhaustive.

2. Preliminaries on probabilistic cellular automata

2.1. Informal definition of PCA and some examples

Consider an infinite lattice, e.g. $\mathbb{Z}$ or $\mathbb{Z}^2$, divided in regular cells, each cell containing a letter of a finite alphabet. At each time step, the content of a cell changes randomly according to a probability law which depends on the content of a finite neighborhood of the cell. The updates of the different cells are done independently. This is the rough definition of a *Probabilistic Cellular Automaton (PCA)*. In a nutshell, the dynamics is local, random, time-synchronous, and space-homogeneous.

A *Cellular Automaton (CA)* is a degenerated PCA in which, for each neighborhood, the update probability law is concentrated on a single letter, that is, the updates are deterministic.

Consider the specific case of the set of cells $\mathbb{Z}$, the alphabet $\{0, 1\}$, and the neighborhood consisting of the cell itself and its right neighbor (or, the left neighbor and the cell itself). Then, a PCA is entirely determined by the four parameters $(\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11})$, where $\theta_{ij} \in [0, 1]$ is the probability that a cell is updated to 1 if its neighborhood is ij. Here are some examples.

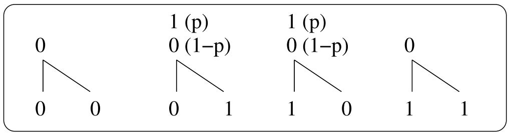

Fig. 1. The noisy additive PCA.

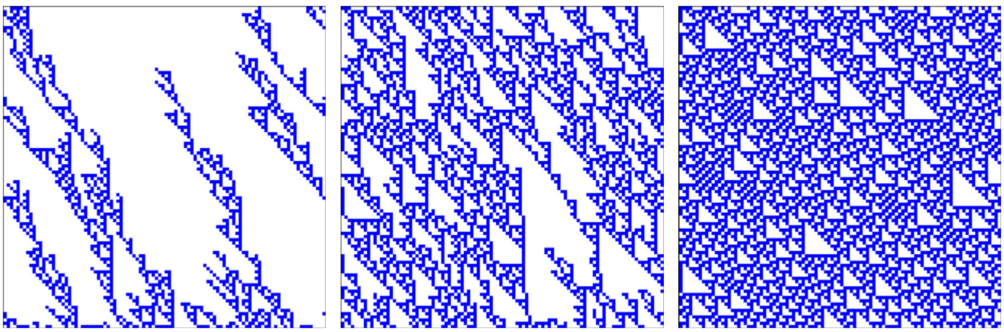

Fig. 2. Space–time diagrams of the noisy additive PCA, for $p = 0.75$, $p = 0.85$, and $p = 1$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Noisy additive PCA Consider the PCA defined by the parameters $(0, p, p, 0)$ for some $p \in (0, 1)$, see Fig. 1.

This PCA can be described as follows. Let the neighborhood of a cell be $(i, j) \in \{0, 1\}^2$, then, the cell is updated in two steps: first, its value is set to $i + j \bmod 2$, and second, with probability $1 - p$, a value 1 is turned into 0. In the limit case $1 - p = 0$, we recover the *additive CA* also known as rule 102 in Wolfram's notation and defined by:

$$F : \{0, 1\}^{\mathbb{Z}} \longrightarrow \{0, 1\}^{\mathbb{Z}}, \qquad x \longmapsto F(x), \qquad F(x)_i = x_i + x_{i+1} \bmod 2. \tag{1}$$

When $p \in (0, 1)$, we get a PCA which can be viewed as a "noisy" version of the additive CA.

In Fig. 2, we have represented the evolution of the noisy additive PCA for different values of the parameter p in so-called *space–time diagrams*. The cells containing a 0, resp. a 1, are painted in white, resp. dark blue. The bottom line is the initial condition, here chosen at random, and the next lines, from bottom to top, are the successive updates of the cells.

One may consider a symmetric variant of the noisy additive PCA: the model in Fig. 3 of parameters $(1 - p, p, p, 1 - p)$ for some $p \in (0, 1)$. Here the cell is updated in two steps: first, its value is set to $i + j \bmod 2$, and second, with probability $1 - p$, a value 1 is turned into 0 and a value 0 is turned into 1.

Some space–time diagrams are shown in Fig. 4. They are qualitatively very different from the ones in Fig. 2.

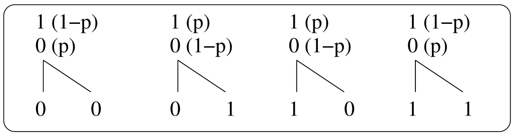

Fig. 3. Symmetric noisy additive PCA.

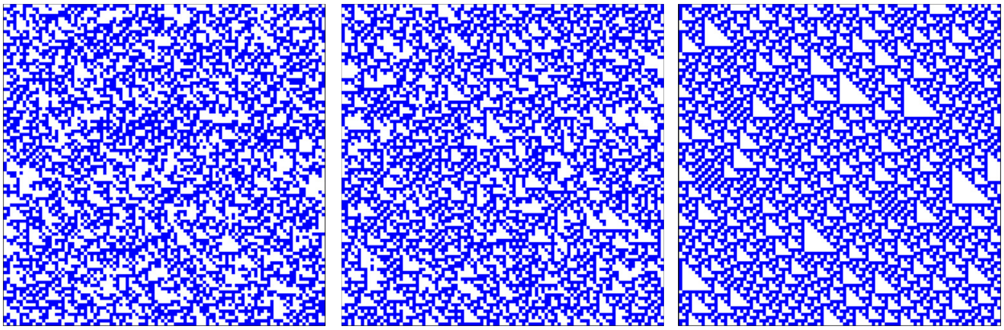

Fig. 4. Space–time diagrams of the symmetric noisy additive PCA, for $p = 0.75, 0.85, 1$.

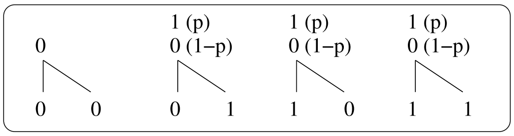

Fig. 5. The Stavskaya PCA.

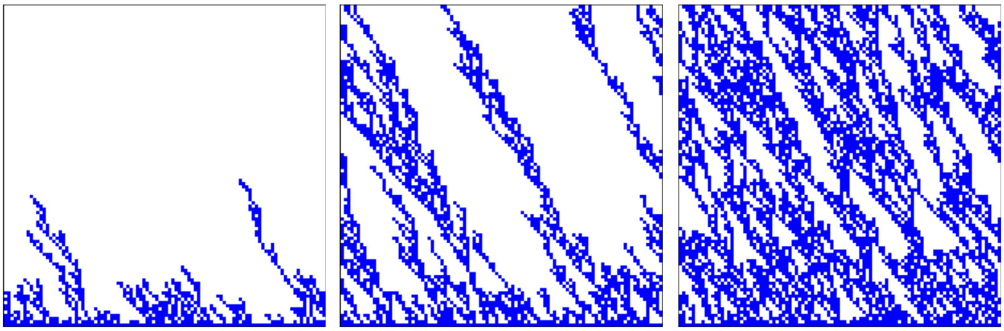

Fig. 6. Space–time diagrams of the Stavskaya PCA, for $p = 0.6$, $p = 0.65$ and $p = 0.71$.

Stavskaya PCA The *Stavskaya PCA* is defined by the parameters $(0, p, p, p)$ for some $p \in (0, 1)$, see Fig. 5.

In Fig. 6, we have represented space–time diagrams of the Stavskaya PCA for different values of the parameter p. Here the initial condition consists only of 1's. The qualitative change of behavior observable between the left and right diagrams can be formalized in terms of ergodicity versus non-ergodicity, see Section 3.

Stavskaya PCA and *percolation PCA*, which are the natural generalizations to larger neighborhoods, play an important role in the study of PCA. This is illustrated in Proposition 3.5 below.

Directed animals PCA The *directed animals PCA* is defined by the parameters $(p, 0, 0, 0)$ for some $p \in (0, 1)$, see Fig. 7.

This PCA gets its name from the role it plays in combinatorics for the enumeration of directed animals, see Section 4.2.

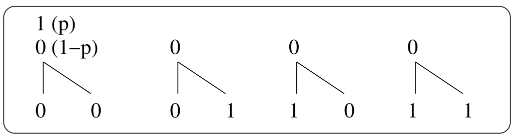

Fig. 7. The directed animals PCA.

2.2. Formal definition of PCA

There are two complementary viewpoints on a PCA. First, it defines a mapping from the set of probability measures on the configuration space into itself. Second, it defines a discrete-time Markov chain on the configuration space. Definition 2.1 corresponds to the first viewpoint and Definition 2.3 corresponds to the second viewpoint.

Let $\mathcal{A}$ be a finite set called the *alphabet* (or the set of *states*), and let $E=\mathbb{Z}^d, d\geq 1$, be the set of *cells* (or *sites*). The set $\mathcal{A}^E$ is called the *configuration space*.

(Other sets of cells may be considered, that is, finite or countable sets with a group structure, e.g. $\mathbb{Z}/n\mathbb{Z}$ or finitely generated free groups. We do not consider them in the present survey, with the exception of $\mathbb{Z}/n\mathbb{Z}$ in Section 7.)

For some finite subset K of E, consider $y=(y_k)_{k\in K}\in\mathcal{A}^K$. The *cylinder* of *base* K defined by y is the set

$$[y]=\{x\in\mathcal{A}^E \mid \forall k\in K, x_k=y_k\}.$$

We denote by $\mathcal{C}(K)$ the set of all cylinders of base K. Let us equip $\mathcal{A}^E$ with the product topology which is the topology generated by cylinders. We denote by $\mathcal{M}(\mathcal{A})$ the set of probability measures on $\mathcal{A}$ and by $\mathcal{M}(\mathcal{A}^E)$ the set of probability measures on $\mathcal{A}^E$ for the σ-algebra generated by all cylinder sets, which corresponds to the Borelian σ-algebra. For $x\in\mathcal{A}^E$, denote by δ_x the Dirac measure concentrated on the configuration x.

Given $K,L\subset E$, we define $K+L=\{u+v \mid u\in K, v\in L\}$.

Definition 2.1. Given a finite set $\mathcal{N}\subset E$, a *transition function* (or *local function*) of *neighborhood* $\mathcal{N}$ is a function $f:\mathcal{A}^{\mathcal{N}}\to\mathcal{M}(\mathcal{A})$. The *probabilistic cellular automaton* (PCA) of transition function f is the mapping $F:\mathcal{M}(\mathcal{A}^E)\longrightarrow\mathcal{M}(\mathcal{A}^E), \mu\longmapsto\mu F$, defined on cylinders by: $\forall K, \forall y=(y_k)_{k\in K}$,

$$\mu F[y]=\sum_{[x]\in\mathcal{C}(K+\mathcal{N})}\mu[x]\prod_{k\in K} f\big((x_{k+v})_{v\in\mathcal{N}}\big)(y_k). \tag{2}$$

To get a feeling for (2), assume that the initial measure is $\delta_x, x\in\mathcal{A}^E$. Then, by application of F, the contents of the cells are updated independently, and the content of the k-th cell is updated to $a\in\mathcal{A}$ with probability $f((x_{k+v})_{v\in\mathcal{N}})(a)$.

It is convenient to work with realizations of PCA. To pave the way for this, let us introduce the notion of update function.

Definition 2.2. Let F be a PCA of local function $f:\mathcal{A}^{\mathcal{N}}\to\mathcal{M}(\mathcal{A})$. An *update function* of F is a measurable function $\phi:\mathcal{A}^{\mathcal{N}}\times[0,1]\to\mathcal{A}$, satisfying, for each $x\in\mathcal{A}^{\mathcal{N}}$ and each $a\in\mathcal{A}$,

$$u\big(\{r\in[0,1] \mid \phi(x,r)=a\}\big)=f(x)(a),$$

where u is the uniform measure on $[0,1]$.

It is possible to define an update function ϕ in a canonical way by setting, if the alphabet is $\mathcal{A}=\{a_1,\dots,a_n\}$,

$$\phi(x,r)=a_i, \quad \text{if } f(x)\big(\{a_1,\dots,a_{i-1}\}\big)\leq r< f(x)\big(\{a_1,\dots,a_i\}\big).$$

Consider a law $\mu\in\mathcal{M}(\mathcal{A}^E)$, a random element $x=(x_i)_{i\in E}\sim\mu$, and a random element $r=(r_i)_{i\in E}\sim u^{\otimes E}$, and assume that x and r are independent. Define

$$y=(y_i)_{i\in E}, \quad y_i=\phi\big((x_{i+v})_{v\in\mathcal{N}}, r_i\big).$$

By construction, we have $y\sim\mu F$.

Definition 2.3. Let F and ϕ be as in Definition 2.2. Consider $\mu\in\mathcal{M}(\mathcal{A}^E)$ and $x=(x_i)_{i\in E}\sim\mu$. Consider the random sequences $(r^t)_{t\in\mathbb{N}}, r^t=(r_i^t)_{i\in E}$, distributed according to $(u^{\otimes E})^{\otimes\mathbb{N}}$ (that is, the r.v.'s r_i^t are i.i.d. and uniform on $[0,1]$), and independent of x. Let $(x^t)_{t\in\mathbb{N}}\in(\mathcal{A}^E)^{\mathbb{N}}$ be defined recursively by

$$x^0=x, \qquad x^{t+1}=\phi\big(x^t,r^t\big).$$

Then $(x^t)_{t\in\mathbb{N}}$ is a realization of the Markov chain associated with the PCA F. We call this sequence a *space–time diagram* of F.

Examples of space–time diagrams appear in Figs. 2, 4, and 6.

Two specializations of PCA One may argue that PCA are a natural mathematical object since they englobe two central objects: cellular automata and finite state Markov chains.

Indeed, recall that a probabilistic cellular automaton is entirely determined by the transition function:

$$f : \mathcal{A}^{\mathcal{N}} \to \mathcal{M}(\mathcal{A}).$$

There are two natural non-trivial specializations of the above: (i) on the left-hand side, set $\mathcal{N} = \{0\}$; (ii) on the right-hand side, replace $\mathcal{M}(\mathcal{A})$ by $\mathcal{A}$. Both correspond to relevant mathematical objects: finite state space Markov chains for (i), cellular automata for (ii).

Indeed assume that $\mathcal{N} = \{0\}$. Then, in a space–time diagram, the different columns evolve independently of one another. Each column is a realization of a Markov chain on the finite state space $\mathcal{A}$. Conversely, it is easily seen that any Markov chain on a finite state space S can be embedded in a PCA over the alphabet S with neighborhood $\{0\}$.

Next definition corresponds to the specialization (ii).

Definition 2.4. A *cellular automaton* (CA) is a PCA, see Definition 2.1, in which the transition function f is such that:

$$\forall x \in \mathcal{A}^{\mathcal{N}}, \exists a \in \mathcal{A}, \qquad f(x) = \delta_a.$$

Thus, the transition function of a CA can be described by a mapping $f : \mathcal{A}^{\mathcal{N}} \longrightarrow \mathcal{A}$, and the CA can be viewed as a mapping $F : \mathcal{A}^E \longrightarrow \mathcal{A}^E$.

Cellular automata are precisely the mappings from $\mathcal{A}^E$ to $\mathcal{A}^E$ which are continuous (with respect to the product topology) and commute with the translations, see [37].

PCA as "combination" of CA Let F be a PCA with alphabet $\mathcal{A}$, neighborhood $\mathcal{N}$, and local function f. It is not difficult to see that there exists a finite family $(G_i)_{i\in I}$ of CA of respective local function $g_i : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$, such that:

$$f = \sum_{i\in I} p_i \delta_{g_i},$$

with $p_i \in (0,1)$, $\sum_i p_i = 1$. So the PCA F can be interpreted as a "combination" of the CA G_i: each cell is updated by applying one of the CA, the CA G_i being chosen with probability p_i, and the different cells are updated independently.

The case $|I| = 2$ is already rich and interesting. All the examples in Section 2.1 correspond to this case.

2.3. Alternative definition of PCA

A slightly different and generalized definition of probabilistic cellular automata is proposed in [2] (for simplicity, we keep the name PCA for this extension).

Definition 2.5. Consider a finite set $\mathcal{N} \subset E$ (the neighborhood) and a *local update* function $\phi : \mathcal{A}^{\mathcal{N}} \times [0,1]^{\mathcal{N}} \to \mathcal{A}$. Define the global update function $\Phi : \mathcal{A}^E \times [0,1]^E \to \mathcal{A}^E$ by:

$$\forall x = (x_i)_{i\in E} \in \mathcal{A}^E, \forall r = (r_i)_{i\in E} \in [0,1]^E, \forall k \in E, \qquad \Phi(x,r)_k = \phi\big((x_{k+v})_{v\in\mathcal{N}}, (r_{k+v})_{v\in\mathcal{N}}\big).$$

The *probabilistic cellular automaton* of local update function ϕ is the mapping $F : \mathcal{M}(\mathcal{A}^E) \longrightarrow \mathcal{M}(\mathcal{A}^E), \mu \longmapsto \mu F$, defined as follows: let $(x,r) \sim \mu \otimes u^{\otimes E}$, then μF is the law of $\Phi(x,r)$.

Let us compare classical PCA (Definition 2.1) and generalized PCA (Definition 2.5). In a classical PCA, states are updated using different sources of randomness, whereas, in a generalized PCA, neighboring states share some sources of randomness. Generalized PCA are defined directly at the level of update functions, as opposed to classical PCA. An advantage of generalized PCA is that the composition of two PCA defined on the same configuration space is a PCA, which is not the case for a classical PCA.

Despite these differences, let us now argue that both models are roughly the same. First, a classical PCA is a generalized PCA in which the local update function $\phi((x_v)_{v\in\mathcal{N}}, (r_v)_{v\in\mathcal{N}})$ does not depend on r_v for $v \neq 0$. Conversely, a generalized PCA can be recoded as the "projection" of a classical PCA. Let us detail this point.

Since a neighborhood can only take a finite number of different values, the $[0,1]$-valued sources of randomness in Definition 2.5 can be replaced by finitely valued sources of randomness. More precisely, a generalized PCA F is associated with a finite neighborhood $\mathcal{N} \subset E$, a finite set $\mathcal{R}$, a local update function $\phi : \mathcal{A}^{\mathcal{N}} \times \mathcal{R}^{\mathcal{N}} \to \mathcal{A}$, a global update function $\Phi : \mathcal{A}^E \times \mathcal{R}^E \to \mathcal{A}^E$ defined accordingly, and a probability distribution $\mathcal{B}_p$, $p = (p_r)_{r\in\mathcal{R}}$ on $\mathcal{R}$. (See Definition 4.1 for the definition of $\mathcal{B}_p$.) We have: $(x,r) \sim \mu \otimes \mathcal{B}_p^{\otimes E} \Longrightarrow \Phi(x,r) \sim \mu F$.

Now consider the classical PCA $\widetilde{F}$ with finite alphabet $\mathcal{A} \times \mathcal{R}$, neighborhood $\mathcal{N}$, and local function $\tilde{f} : (\mathcal{A} \times \mathcal{R})^{\mathcal{N}} \to \mathcal{M}(\mathcal{A} \times \mathcal{R})$ defined by (using the obvious identification between $(\mathcal{A} \times \mathcal{R})^{\mathcal{N}}$ and $A^{\mathcal{N}} \times \mathcal{R}^{\mathcal{N}}$):

$$\circ\circ\circ \quad 0 \;\; 1 \;\; \overset{\text{(p)}}{1 \;\; 0} \;\; 1 \;\; 1 \;\; \overset{\text{(p)}}{1 \;\; 0} \;\; 0 \;\; 0 \;\; 1 \quad \circ\circ\circ$$

Fig. 8. The TASEP PCA.

$$\forall (x,r) \in (\mathcal{A} \times \mathcal{R})^{\mathcal{N}}, \forall (a,b) \in \mathcal{A} \times \mathcal{R}, \qquad \tilde{f}(x,r)(a,b) = \begin{cases} p_b & \text{if } \phi(x,r) = a \\ 0 & \text{if } \phi(x,r) \neq a. \end{cases}$$

The PCA $\widetilde{F}$ can be viewed as operating on two tapes, one with the $\mathcal{A}$-symbols and one with the $\mathcal{R}$-symbols. At each time step, the $\mathcal{A}$-tape is updated by applying ϕ, the update function of F, and the $\mathcal{R}$-tape is updated by choosing brand new random symbols according to $\mathcal{B}_p^{\otimes E}$, independently of everything. Concentrating on the $\mathcal{A}$-tape of the classical PCA $\widetilde{F}$, we recover exactly the behavior of the generalized PCA F.

Summarizing, both classical and generalized PCA correspond basically to the same mathematical object. Depending on the modeling context, it might be more convenient to work with one definition or the other. For instance, the TASEP model, see below, is naturally defined as a generalized PCA. In the present survey, we mainly focus on classical PCA defined according to Definition 2.1. But we also study the TASEP PCA in details in Section 4.3.

TASEP PCA TASEP stands for *Totally Asymmetric Simple Exclusion Process*. It models a flow of non-overtaking particles moving in the same direction. Here, we consider a discrete version of TASEP. The continuous-time version is a standard and widely studied model of interacting particle systems (see Section 2.4).

The set of cells is $\mathbb{Z}$, the alphabet is $\mathcal{A} = \{0,1\}$, a 1 standing for a particle and a 0 for an empty space. A parameter $p \in (0,1)$ is fixed. At each time step, each particle jumps to the right neighboring cell with probability p if this cell is empty, and stands still otherwise.

Observe that the updatings of two adjacent cells are correlated: if a cell is in state 1 and the neighboring right cell is in state 0, then either the states of both cells will change (probability p) or the states of none of them will change (probability $1-p$). So the TASEP should be modeled as a generalized PCA.

Let the neighborhood be $\mathcal{N} = \{-1,0,1\}$. Set $\mathcal{R} = \{i,a\}$ (with i standing for *inactive* and a for *active*) and let the probability distribution on $\mathcal{R}$ be $(1-p,p)$. Define the local update function $\phi : \mathcal{A}^{\mathcal{N}} \times \mathcal{R}^{\mathcal{N}} \to \mathcal{A}$ by, for $x = (x_{-1}, x_0, x_1), r = (r_{-1}, r_0, r_1)$,

$$\phi(x,r) = \begin{cases} 0 & \text{if } [x_{-1} = x_0 = 0] \vee [x_{-1} = 1, x_0 = 0, r_{-1} = i] \vee [x_0 = 1, x_1 = 0, r_0 = a] \\ 1 & \text{if } [x_0 = x_1 = 1] \vee [x_{-1} = 1, x_0 = 0, r_{-1} = a] \vee [x_0 = 1, x_1 = 0, r_0 = i]. \end{cases} \tag{3}$$

The associated PCA is called the *TASEP PCA* (see Fig. 8). This PCA plays a role in combinatorics, see Section 4.3. If $p = 1$, we recover the classical *Traffic* cellular automaton, rule 184 in Wolfram's notation, which is defined by:

$$\mathcal{A} = \{0,1\}, \qquad \mathcal{N} = \{-1,0,1\}, \qquad f(00*) = 0, \qquad f(10*) = 1, \qquad f(*10) = 0, \qquad f(*11) = 1. \tag{4}$$

2.4. Interacting particle systems

The analogs of PCA in continuous time are *(finite-range) interacting particle systems (IPS)*, see Liggett [50]. IPS are characterized by a local function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ (or $\varphi : \mathcal{A}^{\mathcal{N}} \to \mathcal{M}(\mathcal{A})$) for some finite *neighborhood* $\mathcal{N} \subset E$. We attach random and independent clocks to the cells of E. For a given cell, the instants of $\mathbb{R}_+$ at which the clock rings form a Poisson process of parameter 1. Let x^t be the configuration of the process at time $t \in \mathbb{R}_+$. If the clock at cell i rings at instant t, the state of the cell i is updated into $f((x^t_{i+v})_{v \in \mathcal{N}})$ (or according to the probability measure $\varphi((x^t_{i+v})_{v \in \mathcal{N}})$). This defines a transition semigroup $F = (F^t)_{t \in \mathbb{R}_+}$, with $F^t : \mathcal{M}(\mathcal{A}^E) \to \mathcal{M}(\mathcal{A}^E)$. If the initial measure is μ, the distribution of the process at time t is given by μF^t.

In a PCA, all cells are updated at each time step, in a "synchronous" way. On the other hand, for an IPS, the updating is "fully asynchronous". Indeed, the probability of having two clocks ringing at the same instant is 0.

PCA are discrete-time Markov chains, while IPS are continuous-time Markov processes.

3. Ergodicity of PCA

For PCA, the central question is to study equilibrium behaviors. An equilibrium is characterized by an invariant measure, that is a probability measure on the configuration space which is left invariant by the dynamics.

Consider $(\mu_n)_n$ and μ in $\mathcal{M}(\mathcal{A}^E)$. We say that $(\mu_n)_n$ *converges weakly* to μ, denoted by $\mu_n \xrightarrow{w} \mu$, if, for all cylinders $[y]$, we have $\mu_n[y] \longrightarrow \mu[y]$.

Definition 3.1. An *invariant (probability) measure* of F is a probability measure $\mu \in \mathcal{M}(\mathcal{A}^E)$ such that $\mu F = \mu$. The PCA F is *ergodic* if it has a unique invariant measure which is attractive, i.e. if

$$\text{(i)} \quad \left[\exists! \mu \in \mathcal{M}\left(A^E\right), \mu F = \mu\right], \qquad \text{(ii)} \quad \left[\forall \nu \in \mathcal{M}\left(A^E\right), \nu F^n \xrightarrow{w} \mu\right]. \tag{5}$$

The set $\mathcal{M}(\mathcal{A}^E)$ of probability measures on $\mathcal{A}^E$ is compact for the weak topology. Based on this observation, one obtains the following standard result, see for instance [75].

Lemma 3.2. *The set of invariant measures of a PCA is non-empty, convex, and compact.*

Therefore, there are three possible situations for a PCA:

(i) several invariant measures;
(ii) a unique invariant measure which is not attractive;
(iii) a unique invariant measure which is attractive (ergodic case).

To which cases correspond the different examples in Section 2.1?

Proposition 3.3. *The symmetric noisy additive PCA (Fig. 3) is ergodic for all $p \in (0,1)$, the unique invariant measure being the uniform measure. The directed animals PCA (Fig. 7) is ergodic for all $p \in (0,1)$, the unique invariant measure being Markovian.*

For the symmetric noisy additive PCA, the result is a consequence of Theorem 4.3 and Proposition 4.5, see below in Section 4.1.2. For the directed animals PCA, the existence of a Markovian invariant measure is shown in Section 4.2. The ergodicity, a recorded open problem, has just been settled by I. Marcovici and J. Martin (article in preparation).

Proposition 3.4. *Consider the Stavskaya PCA F_p of parameter p (Fig. 5). The Dirac measure $\delta_{0^{\mathbb{Z}}}$ is an invariant measure. There exists $p_* \in (0,1)$ such that:*

$$p < p_* \implies F_p \text{ ergodic}$$
$$p > p_* \implies F_p \text{ has several invariant measures.}$$

This is proved by using a connection with a site percolation model, see for instance [74, §2]. The exact value of p_* is unknown (see Open problem 5) and is experimentally close to $0.7055\cdots$, see [56] (and also Fig. 6!). Proposition 3.4 also holds for all other percolation PCA, with the value of p_* depending on the PCA.

Proposition 3.5. *Let p_* be the critical probability of the Stavskaya PCA as defined in Proposition 3.4. For $p \in (0,p_*)$, the noisy additive PCA (Fig. 1) of parameter p is ergodic, the unique invariant measure being $\delta_{0^{\mathbb{Z}}}$.*

We give the proof which is simple and characteristic: it is based on "coupling", a common argument in the study of PCA.

Proof. Let us fix the parameter p. Let $(x^t)_{t\in\mathbb{N}}$ and $(y^t)_{t\in\mathbb{N}}$ be two realizations of the Markov chain associated with the noisy additive PCA, see Definition 2.3, starting from two different initial conditions, but defined using the same sources of randomness: $x^{t+1} = \phi(x^t, r^t)$, $y^{t+1} = \phi(y^t, r^t)$. For all $t \in \mathbb{N}$, define z^t as follows:

$$z_n^t = i \quad \text{if } x_n^t = y_n^t, \qquad z_n^t = d \quad \text{if } x_n^t \neq y_n^t$$

(i stands for *i*dentical and d for *d*ifferent). Observe the following:

$$\left(z_n^t, z_{n+1}^t\right) = (i,i) \implies z_n^{t+1} = i, \qquad \left(z_n^t, z_{n+1}^t\right) \neq (i,i) \implies \mathbb{P}\left\{z_n^{t+1} = i\right\} \geq 1-p.$$

Therefore the occurrences of d's in $(z^t)_{t\in\mathbb{N}}$ are upper-bounded by the occurrences of d's in the space–time diagram of the PCA on the alphabet $\{i,d\}$ with neighborhood $\mathcal{N} = \{0,1\}$ and local function: $(i,i) \to i$, $(u,v) \in \{(i,d),(d,i),(d,d)\} \to d$ with probability p and i with probability $1-p$. We recognize the Stavskaya PCA of parameter p with i playing the role of 0 and d the role of 1. According to Proposition 3.4, for $p < p_*$, the Stavskaya PCA is ergodic with $\delta_{i^{\mathbb{Z}}}$ as the unique invariant measure. It implies that the occurrences of d's in z^t vanish as t grows, which means that the trajectories $(x^t)_{t\in\mathbb{N}}$ and $(y^t)_{t\in\mathbb{N}}$ coincide asymptotically. □

For p large, the ergodicity of the noisy additive PCA is unsettled. Clearly, $\delta_{0^{\mathbb{Z}}}$ is always an invariant measure, but computer simulations suggest that there might be other ones when p is large (see the middle space–time diagram in Fig. 2).

Open problem 1. What are the values of p for which the noisy additive PCA is ergodic?

We have seen examples of situations (i) and (iii). What about situation (ii)? Let us start with a couple of observations. First, a PCA F with a unique invariant measure μ is attractive for *Cesàro means*, that is,

$$\forall \nu \in \mathcal{M}(\mathcal{A}^E), \quad \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \nu F^i = \mu, \tag{6}$$

for weak convergence. Indeed, set $\nu_n = 1/n \sum_{i=0}^{n-1} \nu F^i$. We have: $\nu_n F = \nu_n + \nu F^n/n - \nu/n$. Therefore, any subsequential limit of $(\nu_n)_n$, say $\bar{\nu}$, must satisfy: $\bar{\nu}F = \bar{\nu}$. It implies that $\bar{\nu} = \mu$ by uniqueness of the invariant measure.

Second, the convergence in (6) does not necessarily imply ergodicity. Assume for instance that $(\nu F^n)_n$ converges to a periodic orbit $(\nu^0, \dots, \nu^{k-1})$, $k \geq 2$, then, we have convergence in Cesàro means of $(\nu F^n)_n$ to the measure $(\nu^0 + \dots + \nu^{k-1})/k$, but no convergence, hence no ergodicity.

The existence of a PCA corresponding to situation (ii) was mentioned as an open problem by Toom [72,74]. It is proved in [15] that situation (ii) occurs for the PCA on $\{0,1\}^{\mathbb{Z}}$ of neighborhood $\mathcal{N} = \{-1, 0\}$, and parameters

$$(\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}) = (1/2, 0, 1, 1/2).$$

The invariant measure is $(\delta_{(01)^{\mathbb{Z}}} + \delta_{(10)^{\mathbb{Z}}})/2$, where $x = (01)^{\mathbb{Z}}$ is the configuration defined by: $\forall n \in \mathbb{Z}, x_{2n} = 0, x_{2n+1} = 1$, and where $(10)^{\mathbb{Z}}$ is defined similarly.

Assessing the ergodicity of a PCA might be difficult, as illustrated by Open problem 1. We give a precise algorithmic statement for this perception in next section.

3.1. Ergodicity is undecidable

Ergodicity is algorithmically undecidable, even for the simplest class of PCA, that is, CA on $\mathbb{Z}$!

Let us stress that we keep the same definition of ergodicity for CA as for PCA (corresponding to the Markov chain terminology). Precisely, a CA F is *ergodic* if: $\exists! \mu, \mu F = \mu$ and $\forall \nu, \nu F^n \xrightarrow{w}_{n\to\infty} \mu$, where $\xrightarrow{w}$ stands for the weak convergence. This definition has restrictive implications for a CA.

First, observe that, if $x_0, \dots, x_{k-1}$ is a periodic orbit of the CA F (i.e., $\forall i, F(x_i) = x_{i+1 \pmod k}$), then, the measure $(\delta_{x_0} + \dots + \delta_{x_{k-1}})/k$ is an invariant measure of F. In particular if x is a fixed point (i.e., $F(x) = x$) then the Dirac measure δ_x is an invariant measure.

A *monochromatic* configuration is a configuration of the type q^E $(E = \mathbb{Z}^d)$ for some $q \in \mathcal{A}$. The image by a CA F of a monochromatic configuration is monochromatic. In particular there exists a monochromatic periodic orbit for F, say $(q_0^E, \dots, q_{k-1}^E)$, and this defines an invariant measure μ as above. Assume that $k \geq 2$. Then $(\delta_{q_0^E} F^n)_n$ does not converge weakly to μ, and F is not ergodic. Therefore, an ergodic F must have a unique monochromatic periodic orbit and this orbit must be of length one (i.e. $\exists q \in \mathcal{A}, F(q^E) = q^E$ and $\forall \ell \in \mathcal{A}, \lim_n F^n(\ell^E) = q^E$).

A CA F is *nilpotent* if the following two equivalent properties hold:

$$\begin{aligned} &\exists q \in \mathcal{A}, \forall x \in \mathcal{A}^E, \exists N \in \mathbb{N}, \forall n \geq N, \quad F^n(x) = q^E \\ \Longleftrightarrow \quad &\exists q \in \mathcal{A}, \exists N \in \mathbb{N}, \forall x \in \mathcal{A}^E, \forall n \geq N, \quad F^n(x) = q^E. \end{aligned}$$

The above equivalence is not difficult to prove using a compactness argument.

Theorem 3.6. *Consider a CA F. We have:*

$$[F \text{ nilpotent}] \quad \Longleftrightarrow \quad [F \text{ ergodic}].$$

In [11, Theorem 3.3], the equivalence between "weakly trace nilpotency" and ergodicity is proved (the result is stated for the set of cells $\mathbb{Z}$ but the proof remains the same for $\mathbb{Z}^d$). The equivalence between "weakly trace nilpotency" and nilpotency is proved in [36] for the set of cells $\mathbb{Z}$, and in [65] for $\mathbb{Z}^d$. By merging the results, we get Theorem 3.6.

Theorem 3.7. *The ergodicity of CA on $\mathbb{Z}$ is undecidable.*

Kari proved in [42] the undecidability of nilpotency for CA on $\mathbb{Z}$, by using a reduction of the tiling problem. Theorem 3.7 appears in [11] as a corollary of Kari's result together with Theorem 3.6.

In [75,73], a result weaker than Theorem 3.7 is proved: the undecidability of ergodicity for PCA on $\mathbb{Z}$ with a transition function $f : \mathcal{A}^{\mathcal{N}} \to \{0, 1/2, 1\}$. The proof in [75,73] is different and uses a reduction of the halting problem of a Turing machine.

Using the above typology, it is unknown if there exists a CA corresponding to situation (ii). Breaking down the question, we get the following.

Open problem 2. Does there exist a CA having a unique invariant measure of the type δ_{q^E} without being nilpotent? Does there exist a CA having a unique invariant measure of the type $(\delta_{q_0^E} + \dots + \delta_{q_{k-1}^E})/k$ for $k \geq 2$? Does there exist a CA such that all periodic configurations converge to the same monochromatic periodic orbit of length greater than or equal to 2?

In the preprint [76], the author claims to answer the first of the three questions by building a non-nilpotent CA with a unique invariant measure δ_{q^E}. The construction is based on Gács CA (see next section).

3.2. The positive-rate problem

Roughly, an ergodic system completely forgets about its initial condition, while a non-ergodic one remembers something forever. Non-ergodicity is the standard situation for a CA (indeed ergodicity is equivalent to nilpotency, see Theorem 3.6, which is a "rare" property). In contrast, one may expect ergodicity to be the standard for a "really random" PCA. This is the basis for an important question, the so-called "positive-rate problem" that we now describe.

Definition 3.8. A PCA of local function $f : \mathcal{A}^{\mathcal{N}} \longrightarrow \mathcal{M}(\mathcal{A})$ is a *positive-rate* PCA if:

$$\forall u \in \mathcal{N}, \forall a \in \mathcal{A}, \quad f(u)(a) > 0.$$

A PCA has a *phase transition* if it has positive-rate and several invariant measures.

The *positive-rate problem* asks whether it is possible to find a PCA with a phase transition. The answer is now known to be positive in all dimension.

There exist simple examples on the set of cells $\mathbb{Z}^d$, $d \geq 2$. A first example is given in Proposition 5.12 and a second one in Theorem 6.5. On the set of cells $\mathbb{Z}$, the only known example is due to Gács [27], and is very complicated, see the discussion in Section 6.1.2.

As a complementary approach, the case of positive-rate PCA of alphabet $\mathcal{A} = \{0, 1\}$ and neighborhood $\mathcal{N} = \{0, 1\}$, has been investigated. Particular attention has been given to the case $\theta_{01} = \theta_{10}$, for which only three parameters remain free. By using various methods (in particular the ones of Section 3.3 below), it is proved in [75, Chap. 7] that for more than 90% of the parameter space, the PCA is ergodic. However for the neighborhood of the two vertices $(\theta_{00}, \theta_{01} = \theta_{10}, \theta_{11}) = (1, 0, 0)$ and $(1, 1, 0)$, ergodicity is not settled. This leaves next question open.

Open problem 3. Does there exist a PCA with phase transition on the set of cells $\mathbb{Z}$, an alphabet of cardinality 2 and the neighborhood $\mathcal{N} = \{0, 1\}$?

More generally, finding simple examples of non-ergodic positive-rate PCA on $\mathbb{Z}$ is a wide open and challenging question. In view of Section 3.1, the next question is also natural.

Open problem 4. Is the ergodicity problem decidable for positive-rate PCA with rational transition probabilities?

We come back to the positive-rate problem in Sections 5.5 and 6.1.

3.3. Ergodicity criteria

Another way to approach the intuition that a "really random" PCA should be ergodic in general, is to derive sufficient conditions ensuring ergodicity. We present briefly some of them, and refer to [75] for details and generalizations.

The coupling method Let F be a PCA on the set of cells $E = \mathbb{Z}^d$, with alphabet $\mathcal{A} = \{0, 1\}$, neighborhood $\mathcal{N}$, and transition function f.

Theorem 3.9. *Assume that we have: $\sum_{a \in \mathcal{A}} \min_{u \in \mathcal{A}^{\mathcal{N}}} f(u)(a) > 1 - 1/|\mathcal{N}|$. Then the PCA is ergodic.*

The proof consists in running the PCA from two arbitrary initial configurations, using the same appropriate update function and the same sources of randomness. The cells where the two space–time diagrams coincide are marked and a percolation argument is invoked [75, Chap. 3]. This is very similar to the argument used in the proof of Proposition 3.5.

Characteristic polynomials Let F be as above. The *characteristic polynomial* of F is the polynomial Q in n variables $X_1, \ldots, X_n$, defined by

$$Q(X_1, \ldots, X_n) = \sum_{(a_i)_i \in \mathcal{A}^{\mathcal{N}}} f\big((a_i)_i\big)(1) \prod_{i \in \mathcal{N}} \big(1 - a_i + (2a_i - 1)X_i\big).$$

Let α_I, $I \subset \{1, \ldots, n\}$, be the coefficient of $\prod_{i \in I} X_i$ in the developed form of Q.

Theorem 3.10. *Assume that we have: $\sum_{I \subset \{1,\ldots,n\}} |\alpha_I| < 1$. Then the PCA is ergodic.*

The proof consists in using the polynomial Q to prove that the PCA is contractive for an appropriate norm [75, Chap. 4].

Example. Assume that $E=\mathbb{Z}$ and $\mathcal{N}=\{0,1\}$. Recall that $\theta_{ij}=f(i,j)(1)$ for $i,j\in\{0,1\}$. The characteristic polynomial is:

$$Q(X_0,X_1)=\theta_{00}+(\theta_{10}-\theta_{00})X_0+(\theta_{01}-\theta_{00})X_1+(\theta_{00}+\theta_{11}-\theta_{01}-\theta_{10})X_0X_1.$$

By Theorem 3.10, the PCA is ergodic if: $|\theta_{00}|+|\theta_{10}-\theta_{00}|+|\theta_{01}-\theta_{00}|+|\theta_{00}+\theta_{11}-\theta_{01}-\theta_{10}|<1$.

4. Combinatorics

In some cases, it is possible to obtain an explicit description of the invariant measure of a PCA. This is interesting on its own right. But it also has unexpected consequences since some of these solvable PCA are connected to other combinatorial models of interest (directed animals, directed percolation ...).

4.1. Bernoulli and Markovian invariant measures

4.1.1. Computation of the image of a product measure by a PCA

For a finite word x over $\mathcal{A}$ and for $a\in\mathcal{A}$, let $|x|_a$ be the number of occurrences of the letter a in x.

Definition 4.1 *(Bernoulli measure).* Consider $p=(p_i)_{i\in\mathcal{A}}$ such that: $\forall i,\ p_i\in[0,1]$, $\sum_{i\in A}p_i=1$. Denote by $\mathcal{B}_p$ the corresponding probability measure on $\mathcal{A}$, called the *Bernoulli measure* of parameter p. The *Bernoulli product measure* induced on $\mathcal{A}^E$ is the probability measure $\mu_p=\mathcal{B}_p^{\otimes E}$. Thus, for any cylinder $[y]$, we have $\mu_p[y]=\prod_{i\in\mathcal{A}}p_i^{|y|_i}$.

When $\mathcal{A}=\{0,1\}$, we simplify the notations by setting $\mathcal{B}_p$, μ_p for $\mathcal{B}_{(1-p,p)}$, $\mu_{(1-p,p)}$.

In the remainder of the section, we restrict our attention to PCA on the alphabet $\mathcal{A}=\{0,1\}$, the set of sites $E=\mathbb{Z}$, and with a neighborhood $\mathcal{N}=\{0,1\}$.

The goal is to get an explicit description of the measure μ_pF, where μ_p is the Bernoulli product measure of parameter $p\in(0,1)$, as a function of the parameters $(\theta_{00},\theta_{01},\theta_{10},\theta_{11})$ defining the PCA.

Let us start with an observation. Consider $(Y_n)_{n\in\mathbb{Z}}\sim\mu_pF$. We have $Y_0\sim\mathcal{B}_q$ where $q=(1-p)^2\theta_{00}+(1-p)p(\theta_{01}+\theta_{10})+p^2\theta_{11}$. Also, $(Y_n)_{n\in\mathbb{Z}}$ is the shuffle of $(Y_{2n})_{n\in\mathbb{Z}}$ and $(Y_{2n+1})_{n\in\mathbb{Z}}$, and these last two sequences are i.i.d. of law μ_q. Therefore, to get an explicit description of μ_pF, all what is needed is the correlation structure of two identical product measures. But this turns out to be non-elementary.

Let us fix $p\in(0,1)$. For $\alpha\in\{0,1\}$, define the function

$$\begin{array}{rccl} g_\alpha: & [0,1] & \longrightarrow & [0,1] \\ & q & \longmapsto & (1-q)(1-p)\theta_{00}^\alpha+(1-q)p\theta_{01}^\alpha+q(1-p)\theta_{10}^\alpha+qp\theta_{11}^\alpha. \end{array} \tag{7}$$

In words, $g_\alpha(q)$ is the probability to have $Y_0=\alpha$, where $Y_0\sim(\mathcal{B}_q\otimes\mathcal{B}_p)f$. Observe that: $g_0(q)+g_1(q)=1$ for all q.

For $\alpha\in\{0,1\}$, we also define the function

$$\begin{array}{rccl} h_\alpha: & [0,1] & \longrightarrow & [0,1] \\ & q & \longmapsto & \begin{cases} [(1-q)p\theta_{01}^\alpha+qp\theta_{11}^\alpha]g_\alpha(q)^{-1} & \text{if } g_\alpha(q)\neq 0 \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{8}$$

Consider X_0,X_1,Y_0 with $(X_0,X_1)\sim\mathcal{B}_q\otimes\mathcal{B}_p$ and Y_0 obtained from (X_0,X_1) by application of the local function f of the PCA (in particular, $Y_0\sim(\mathcal{B}_q\otimes\mathcal{B}_p)f$). In words, $h_\alpha(q)$ is the probability to have $X_1=1$ conditionally to $Y_0=\alpha$.

Proposition 4.2. *Consider the PCA F defined by $(\theta_{00},\theta_{01},\theta_{10},\theta_{11})$. Consider $p\in(0,1)$. For $\alpha_0\cdots\alpha_{n-1}\in\mathcal{A}^n$, the probability of the cylinder $[\alpha_0\cdots\alpha_{n-1}]$ under μ_pF is given by:*

$$\mu_pF[\alpha_0\cdots\alpha_{n-1}]=g_{\alpha_0}(p)\prod_{i=1}^{n-1}g_{\alpha_i}\big(h_{\alpha_{i-1}}\big(h_{\alpha_{i-2}}\big(\ldots h_{\alpha_0}(p)\ldots\big)\big)\big).$$

Proposition 4.2 appears in [53, Prop. 3.1]. We give a sketch of the proof.

Proof. The proof is by induction. Set $X=X^0$ and $Y=X^1$. Assuming that $X\sim\mu_p$. By definition, $\mu_pF[\alpha_0]=\mathbb{P}(Y_0=\alpha_0)=g_{\alpha_0}(p)$. We can decompose the probability $\mu_pF[\alpha_0\alpha_1]$ into

$$\mu_pF[\alpha_0\alpha_1]=\mathbb{P}(Y_0=\alpha_0,Y_1=\alpha_1)=\mathbb{P}(Y_1=\alpha_1\mid Y_0=\alpha_0)\mathbb{P}(Y_0=\alpha_0).$$

By definition, the conditional law of X_1 assuming that $Y_0=\alpha_0$ is given by $\mathcal{B}_{h_{\alpha_0}(p)}$. So the law of (X_1,X_2) is $\mathcal{B}_{h_{\alpha_0}(p)}\otimes\mathcal{B}_p$ and we obtain

$$\mu_p F[\alpha_0\alpha_1] = g_{\alpha_1}\big(h_{\alpha_0}(p)\big) g_{\alpha_0}(p).$$

The general case is proved in a similar way. □

4.1.2. PCA having Bernoulli invariant measures

The goal is to get necessary and sufficient conditions for the product measure μ_p, $p \in [0,1]$, to be an invariant measure of F.

Observe that the measure $\mu_1 = \delta_{1^\mathbb{Z}}$ is invariant for the PCA F if and only if $\theta_{11} = 1$. Similarly, $\mu_0 = \delta_{0^\mathbb{Z}}$ is invariant for F if and only if $\theta_{00} = 0$. The case μ_p, $p \in (0,1)$, is treated in Theorem 4.3.

Theorem 4.3. *The measure μ_p, $p \in (0,1)$, is an invariant measure of the PCA F of parameters $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ if and only if one of the two following conditions is satisfied:*

$$(1-p)\theta_{00} + p\theta_{01} = (1-p)\theta_{10} + p\theta_{11} = p \tag{9}$$

$$(1-p)\theta_{00} + p\theta_{10} = (1-p)\theta_{01} + p\theta_{11} = p. \tag{10}$$

In particular, a PCA has a (non-trivial) Bernoulli product invariant measure if and only if its parameters satisfy:

$$\theta_{00}(1-\theta_{11}) = \theta_{10}(1-\theta_{01}) \quad \textit{or} \quad \theta_{00}(1-\theta_{11}) = \theta_{01}(1-\theta_{10}). \tag{11}$$

The result appears in [75]. Here we give a sketch of the proof from [53] which is different and based on Proposition 4.2.

Proof. Let us assume that F satisfies condition (9) for some $p \in (0,1)$. Then, the function g_1 is given by $g_1(q) = (1-q)p + qp = p$, and $g_0(q) = 1 - g_1(q) = 1 - p$. By Proposition 4.2, we have,

$$\forall \alpha = \alpha_0 \cdots \alpha_{n-1} \in \mathcal{A}^n, \quad \mu_p F[\alpha] = (1-p)^{|\alpha|_0} p^{|\alpha|_1} = \mu_p[\alpha].$$

So μ_p is an invariant measure. Now assume that the PCA F satisfies condition (10). By reversing the space–time direction, we get back to the previous case.

Conversely, assume that $\mu_p F = \mu_p$. It follows from Proposition 4.2 that for any value of the α_i, we must have $g_1(h_{\alpha_{n-1}}(h_{\alpha_{n-2}}(\ldots h_{\alpha_0}(p)\ldots))) = p$. Since g_1 is an affine function, there are only two possibilities: either g_1 is the constant function equal to p; or $h_{\alpha_{n-1}}(h_{\alpha_{n-2}}(\ldots h_{\alpha_0}(p)\ldots)) = p$ for all values of $\alpha_0, \ldots, \alpha_{n-1} \in \mathcal{A}$. After some simple computations, the first case leads to condition (9), and the second case to condition (10). □

Using the same approach as in Theorem 4.3 for larger alphabets and neighborhoods, one obtains sufficient conditions for having a Bernoulli invariant measure. In particular, next result appears in [53, §5.2], see also [77].

Proposition 4.4. *Consider a PCA on the alphabet $\mathcal{A} = \{0, \ldots, n\}$, the neighborhood $\mathcal{N} = \{0, \ldots, \ell\}$, with transition function f. Set $\theta^k_{x_0\cdots x_\ell} = f(x_0, \ldots, x_\ell)(k)$. Consider $p = (p_i)_{i\in\mathcal{A}}$, $\forall i$, $p_i > 0$, $\sum_{i\in\mathcal{A}} p_i = 1$. The product measure μ_p is an invariant measure of the PCA F if one of the two following conditions is satisfied:*

$$\forall x_0, \ldots, x_{\ell-1} \in \mathcal{A}, \forall k \in \mathcal{A}, \quad \sum_{i\in\mathcal{A}} p_i \theta^k_{x_0\cdots x_{\ell-1} i} = p_k \tag{12}$$

$$\forall x_0, \ldots, x_{\ell-1} \in \mathcal{A}, \forall k \in \mathcal{A}, \quad \sum_{i\in\mathcal{A}} p_i \theta^k_{i x_0\cdots x_{\ell-1}} = p_k. \tag{13}$$

When iterating a PCA satisfying (12) or (13) from its Bernoulli invariant measure μ_p, the resulting space–time diagram defines a non-trivial random field with very weak dependences and nice combinatorial properties, see [53] for details.

To complete Theorem 4.3, let us quote a result from [77]. Recall that positive-rate PCA were defined in Definition 3.8.

Proposition 4.5. *Consider a positive-rate PCA F satisfying condition (9) or (10) or (12) or (13), for some $p \in (0,1)$. Then F is ergodic.*

Observe that Proposition 4.5 is not true without the positive-rate assumption. Consider for instance the PCA defined by: $\theta_{00} = p/(1-p), \theta_{01} = 0, \theta_{10} = 0, \theta_{11} = 1$ for some $p \in (0, 1/2]$. It satisfies (9) and (10), but it is not ergodic since $\delta_{1^\mathbb{Z}}$ and μ_p are both invariant.

4.1.3. PCA having Markovian invariant measures

Markovian measures are a natural extension of Benoulli product measures. In a nutshell, Proposition 4.2 can be extended to find conditions for having a Markovian invariant measure.

Definition 4.6 *(Markovian measure).* Consider $a, b \in (0,1)$, the stochastic matrix

$$Q = \begin{pmatrix} 1-a & a \\ 1-b & b \end{pmatrix}, \tag{14}$$

and the vector $\pi = (\pi_0, \pi_1)$ such that $\pi Q = \pi$, $\pi_0 + \pi_1 = 1$, that is, $\pi_0 = (1-b)/(1-b+a)$ and $\pi_1 = a/(1-b+a)$. The *Markovian measure* on $\{0,1\}^{\mathbb{Z}}$ of transition matrix Q is the measure ν_Q defined on cylinders by: $\forall x = x_m \cdots x_n$, $\nu_Q[x] = \pi_{x_m} \prod_{i=m}^{n-1} Q_{x_i, x_{i+1}}$.

For convenience, Definition 4.6 excludes the case $a = 1$ and $1 - b = 1$, for which the associated measure is simply $\nu = 1/2(\delta_{(01)^{\mathbb{Z}}} + \delta_{(10)^{\mathbb{Z}}})$. Proposition 4.7 remains valid in this case. And it is trivial to see that ν is invariant for the PCA F iff $(\theta_{01}, \theta_{10}) \in \{(0,1), (1,0)\}$, which completes Theorem 4.8 below.

The Markovian measure ν_Q is shift-invariant (see Section 5.1). If $a = b$, then $\nu_Q = \mu_a$, the Bernoulli product measure of parameter a.

Let us fix the PCA, that is, the parameters $(\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11})$. Let us fix the parameters a and b in $(0,1)$ (defining Q and π as above). We introduce the analogues of the functions defined in (7) and (8).

For $\alpha \in \{0,1\}$, define the function:

$$\begin{array}{rccl} g_\alpha: & [0,1] & \longrightarrow & (0,1) \\ & r & \longmapsto & (1-r)(1-a)\theta_{00}^\alpha + (1-r)a\theta_{01}^\alpha + r(1-b)\theta_{10}^\alpha + rb\theta_{11}^\alpha. \end{array} \tag{15}$$

In words, $g_\alpha(r)$ is the probability that $Y_0 = \alpha$, where Y_0 is the image of (X_0, X_1) by the PCA and where the law of (X_0, X_1) is given by $\mathbb{P}(X_0 = x_0, X_1 = x_1) = r_{x_0} Q_{x_0, x_1}$ with $r_0 = 1 - r$ and $r_1 = r$. Observe that: $g_0(r) + g_1(r) = 1$ for all r.

For $\alpha \in \{0,1\}$, we also define the function:

$$\begin{array}{rccl} h_\alpha: & [0,1] & \longrightarrow & [0,1] \\ & r & \longmapsto & \begin{cases} [(1-r)a\theta_{01}^\alpha + rb\theta_{11}^\alpha] g_\alpha(r)^{-1} & \text{if } g_\alpha(r) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{16}$$

In words, $h_\alpha(r)$ is the probability to have $X_1 = 1$ conditionally to $Y_0 = \alpha$ if (X_0, X_1) is distributed as above.

Proposition 4.7. *Consider the Markovian measure ν_Q and the PCA F as above. For $\alpha_0 \cdots \alpha_{n-1} \in \mathcal{A}^n$, the probability of the cylinder $[\alpha_1 \cdots \alpha_n]$ under $\nu_Q F$ is given by:*

$$\nu_Q F[\alpha_0 \cdots \alpha_{n-1}] = g_{\alpha_0}(\pi_1) \prod_{i=1}^{n-1} g_{\alpha_i}\big(h_{\alpha_{i-1}}\big(h_{\alpha_{i-2}}\big(\ldots h_{\alpha_0}(\pi_1)\ldots\big)\big)\big).$$

Using this proposition, we obtain necessary and sufficient conditions for having a Markovian invariant measure.

Theorem 4.8. *Consider a PCA F such that: $\exists i, j, \theta_{ij} \in (0,1)$, that is, a PCA which is not a CA. The PCA F has a Markovian invariant measure associated to $a, b \in (0,1)$ if we are in one of the three following cases:*

1. *The parameters satisfy:*

$$(\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}) \in (0,1)^4, \quad \theta_{00}\theta_{11}(1-\theta_{01})(1-\theta_{10}) = \theta_{01}\theta_{10}(1-\theta_{00})(1-\theta_{11}). \tag{17}$$

   In which case, a and b are the unique solutions in $(0,1)$ of the equations:

$$b(1-\theta_{11}) = (1-a)\theta_{00}, \qquad a(1-b)\theta_{01}\theta_{10} = b(1-a)\theta_{00}\theta_{11}.$$

2. *The parameters satisfy:*

$$\begin{array}{lllll} & \theta_{00} = 1, & \theta_{01} \in (0,1], & \theta_{10} = 1, & \theta_{11} \in (0,1) \\ \textit{or} & \theta_{00} = 1, & \theta_{01} = 1, & \theta_{10} \in (0,1], & \theta_{11} \in (0,1). \end{array}$$

   In which case, a and b are the unique solutions in $(0,1)$ of the equations:

$$b(1-\theta_{11}) = (1-a), \qquad a(1-b)\theta_{01}\theta_{10} = b(1-a)\theta_{11}.$$

3. *The parameters satisfy:*

$$\begin{array}{lllll} & \theta_{00}\in(0,1), & \theta_{01}=0, & \theta_{10}\in[0,1), & \theta_{11}=0 \\ \textit{or} & \theta_{00}\in(0,1), & \theta_{01}\in[0,1), & \theta_{10}=0, & \theta_{11}=0. \end{array}$$

In which case, a and b are the unique solutions in $(0,1)$ *of the equations:*

$$b=(1-a)\theta_{00}, \qquad a(1-b)(1-\theta_{01})(1-\theta_{10})=b(1-a)(1-\theta_{00}). \tag{18}$$

For a Markovian measure which is not Bernoulli (i.e. $a\neq b$), the conditions in Theorem 4.8 are necessary and sufficient. The proof of Theorem 4.8 is more technical but similar to the one of Theorem 4.3. The complete proof appears in [53]. Case 1 was first proved in [7], see also [75]. Related statements also appear in [9,78].

Next result, which is the counterpart of Proposition 4.5, is proved in [77].

Proposition 4.9. *Consider a positive-rate PCA F satisfying condition (17). Then F is ergodic.*

When iterating a PCA satisfying condition (17) from its invariant measure, the resulting space–time diagram is "time-reversible", see [77]. This is connected to the results presented in Section 5.

The case of larger alphabets (still with neighborhood $\mathcal{N}=\{0,1\}$) is treated in details in [13] and a result in the spirit of Theorem 4.8 is obtained.

4.2. Directed animals

The results of this section originate from Dhar [21], a paper that has initiated a consequent line of research [80,9,6,48,1].

Back to the directed animals PCA Fix some parameter $p\in(0,1)$. Denote by F_p the directed animals PCA defined in Section 2.1, that is the PCA on the alphabet $\mathcal{A}=\{0,1\}$, with the neighborhood $\mathcal{N}=\{0,1\}$, and defined by $(\theta_{00},\theta_{01},\theta_{10},\theta_{11})=(p,0,0,0)$.

Observe that F_p satisfies the conditions of Theorem 4.8 (case 3). Using Theorem 4.8, we get that F_p has an invariant measure which is Markovian, of the form (with the notations of Definition 4.6) ν_Q with parameters

$$a=\frac{2p^2-p-1+\sqrt{1+2p-3p^2}}{2p^2}, \qquad b=\frac{1+p-\sqrt{1+2p-3p^2}}{2p}. \tag{19}$$

This is obtained by solving Eqs. (18). According to Proposition 3.3, F_p is ergodic for all p, so this Markovian measure is the unique invariant measure.

Directed animals So called "animals" are classical objects in combinatorics. They are related to (site) percolation models. The ultimate goal is to count the number of animals of a given size. There exist two variants: classical and directed animals. Here we consider only directed animals which are simpler to study (see Fig. 9).

Consider the directed infinite graph:

$$(\mathbb{Z}\times\mathbb{N},A), \quad A=\big\{(i,j)\to(i+v,j+1)\ \big|\ (i,j)\in\mathbb{Z}\times\mathbb{N}, v\in\{0,1\}\big\}. \tag{20}$$

Let C be a non-empty finite subset of $\mathbb{Z}$. A *directed animal of base* C is a finite subset E of $\mathbb{Z}\times\mathbb{N}$ such that:

- $E\cap(\mathbb{Z}\times\{0\})=C\times\{0\}$;
- $\forall x\in E, \exists x_0\in C\times\{0\}, x_1,\dots,x_{n-1}\in E, x_n=x, \forall i, x_i\to x_{i+1}$.

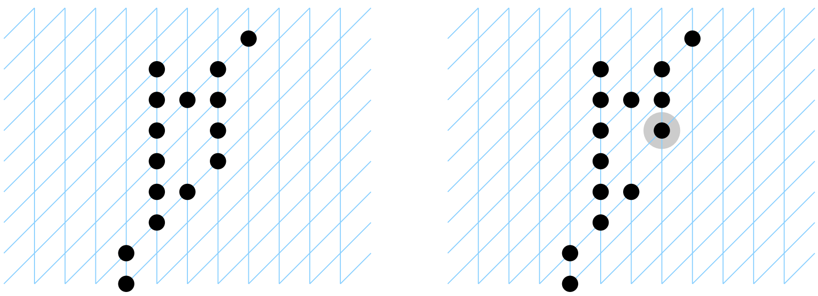

Fig. 9. A directed animal (left), *not* a directed animal (right).

A *directed animal* is a directed animal of base $\{0\}$.

It is customary in combinatorics to count objects according to their size, and to encapsulate all the information in a formal series. The *counting series* of directed animals of base C, respectively of directed animals, is the formal series defined by:

$$S_C(x) = \sum_{E:\text{ DA base } C} x^{|E|}, \qquad S(x) = S_{\{0\}}(x). \tag{21}$$

The coefficient of x^n in $S(x)$ is the number of directed animals of size n.

The goal of the section is to give the proof of next theorem, due to Dhar [21], which is based on the directed animals PCA.

Theorem 4.10. *The counting series of directed animals is given by:*

$$S(x) = \frac{1}{2}\left(\frac{\sqrt{1-2x-3x^2}}{1-3x} - 1\right). \tag{22}$$

This is a perfect result, since $S(x)$ is algebraic and defined in an explicit way. By performing a Taylor expansion around 0 of $S(x)$, we get the successive terms of the counting series: $S(x) = x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + 96x^6 + \cdots$.

Proof. Removing the bottom line of a directed animal provides either the empty set or a new animal on the lines $\{1, 2, \ldots\}$. This simple observation provides a recurrence relation on counting series. Set $V = \{0, 1\}$. We have:

$$S_C(x) = x^{|C|}\left(\sum_{D \subset C+V} S_D(x)\right), \tag{23}$$

with the convention $S_\emptyset(x) = 1$.

Recall that F_p is the directed animals PCA of parameter p. Set $\nu_p = \nu_Q$ for the Markovian invariant measure of F_p. For C a finite subset of $\mathbb{Z}$, denote by $[1 \cdots 1]_C$ the cylinder of the configurations equal to 1 on C, and set

$$s_C = \nu_p\big([1 \cdots 1]_C\big).$$

Consider a sequence of $\{0, 1\}$-valued r.v.'s $(X_i)_{i\in\mathbb{Z}}$, and let $(Y_i)_{i\in\mathbb{Z}}$ be a realization of the image of $(X_i)_i$ by the PCA. Assume that $(X_i)_i \sim \nu_p$ which implies that $(Y_i)_i \sim \nu_p$. Recall that the neighborhood of F_p is $\mathcal{N} = \{0, 1\} = V$. Clearly, we have:

$$s_C = \mathbb{P}\{\forall i \in C, Y_i = 1\} = \mathbb{P}\{\forall i \in C + V, X_i = 0\} p^{|C|}.$$

According to the inclusion–exclusion principle, we get:

$$\mathbb{P}\{\forall i \in C + V, X_i = 0\} = \sum_{D \subset C+V} (-1)^{|D|} \mathbb{P}\{\forall i \in D, X_i = 1\} = \sum_{D \subset C+V} (-1)^{|D|} s_D,$$

with the convention $s_\emptyset = 1$. So we have:

$$s_C = p^{|C|} \sum_{D \subset C+V} (-1)^{|D|} s_D. \tag{24}$$

By comparing (23) and (24), we get that

$$S_C(-p) = (-1)^{|C|} s_C, \qquad S(-p) = -\nu_p\big([1]\big), \tag{25}$$

are possible solutions for the recurrence equations (23). This provides an unexpected relation between two a priori unrelated models.

Now we use the fact that we have an exact expression for the invariant measure ν_p, see (19). We obtain immediately $\nu_p([1]) = a/(1-b+a) = (\sqrt{1+2p-3p^2} - 1 - 3p)/(2+6p)$. By evaluating S formally according to (25), we obtain (22).

The last step consists in arguing that S is indeed the counting series. This requires an argument since the recurrence relations (23) may admit several families of solutions, with only one of them defining the counting series.

One way to proceed is to make a detour via finite sets of cells. Set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Consider directed animals defined in the same way as above but on the graph

$$(\mathbb{Z}_n \times \mathbb{N}, A), \quad A = \big\{(i, j) \to (i + v, j + 1) \,\big|\, (i, j) \in \mathbb{Z}_n \times \mathbb{N}, v \in \{0, 1\}\big\}$$

(to be compared with (20)). The corresponding counting series of directed animals of base $C \subset \mathbb{Z}_n$ is denoted by $S_C^{(n)}$ and we set $S_0^{(n)} = S^{(n)}$. Observe that the number of directed animals of size k is exactly the same on $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Z}_n \times \mathbb{N}$ for n large enough. It implies that $\lim_n S^{(n)} = S$, as formal power series with product topology. Also, the recurrence relations

(23) still hold for the series $S_C^{(n)}$. But now, they form a finite number of linear recurrence relations that uniquely define the series $S_C^{(n)}$ and $S^{(n)}$.

Consider the exact analog of the directed animal PCA of parameter p on the set of cells $\mathbb{Z}_n$. Denote it by $F_p^{(n)}$. The PCA $F_p^{(n)}$ defines a Markov chain on the finite state space $\{0,1\}^{\mathbb{Z}_n}$ which is ergodic (see Section 7). Let $\nu_p^{(n)}$ be the unique invariant measure. The measure $\nu_p^{(n)}$ still has a Markovian structure, see Theorem 7.1. In particular, the quantity $\nu_p^{(n)}([1])$ can be explicited, and we check that $\lim_n \nu_p^{(n)}([1]) = \nu_p([1])$. The analog of (24) still holds, implying the relation: $S^{(n)}(-p) = -\nu_p^{(n)}([1])$. By letting n go to infinity, we conclude that the counting series indeed satisfies $S(-p) = -\nu_p([1])$. $\square$

There exist several alternative proofs of Theorem 4.10 in the literature. Most of them, including the first one by Dhar [21], are based on the PCA connection, see [9,48,1] and the references therein. The above proof is in the same spirit as the ones in [21,9] although slightly different. In [48], the authors propose a construction in which the directed animals "appear" directly in the space–time diagram of the PCA, thus providing an explanation of the magical relation (25). There also exist "combinatorial" proofs of Theorem 4.10 using bijections with other combinatorial objects (heaps of pieces, paths, trees, ...), see [80,6].

The PCA approach is very flexible, that is, the connection with a PCA holds for different extensions and variations of directed animals, for instance directed animals defined on other infinite regular graphs, see [9]. It is arguable that the PCA approach is truly the right one to count directed animals. The difficulty is that the invariant measure of the associated PCA cannot always be made explicit. In this vein, let us mention a well-known open question.

Given a directed animal E, a node x belongs to the *perimeter* of E if $x \in (\mathbb{Z}\times\mathbb{N} - E)$ and $E \cup \{x\}$ is a directed animal. Denote by $\mathcal{P}(E)$ the set of perimeter nodes of E. The *area–perimeter counting series* of directed animals is defined by:

$$S^{\mathrm{AP}}(x,y) = \sum_{E:DA} x^{|E|} y^{|\mathcal{P}(E)|}. \tag{26}$$

The above series is linked to the family of Stavskaya PCA F_p, $p \in (0,1)$, defined as in (5) but with neighborhood $\{-1,0\}$. Consider $u = (u_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ defined by: $u_0 = 1, \forall i \neq 0, u_i = 0$. Let C_p be the (random) set of 1's in a space–time diagram of F_p with initial condition δ_u. Observe that C_p is a (random) directed animal, and observe that we have $\mathbb{P}\{C_p = E\} = p^{|E|}(1-p)^{|\mathcal{P}(E)|}$ if E is some fixed directed animal. We deduce that $\mathbb{P}\{|C_p| < \infty\} = S^{\mathrm{AP}}(p, 1-p)$. In particular, the critical probability p_* of $(F_p)_p$ defined in Proposition 3.4 satisfies:

$$p_* = \sup\bigl\{p \bigm| \mathbb{P}\bigl\{|C_p| < \infty\bigr\} = 1\bigr\} = \sup\bigl\{p \bigm| S^{\mathrm{AP}}(p, 1-p) = 1\bigr\}.$$

Consider the PCA on the alphabet $\mathcal{A} = \{0,1\}$, the neighborhood $\mathcal{N} = \{0,1\}$, defined by

$$(\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}) = (p_1, p_2, p_2, p_2),$$

for $p_1, p_2 \in (0,1)$. For p_1, p_2 large enough, this PCA has a unique invariant measure by application of Theorem 3.9, and we denote it by ν_{p_1,p_2}. It is proved in [9] that (for x, y small enough):

$$S^{\mathrm{AP}}(x,y) = 1 - y - \nu_{1-x-y,1-y}\bigl([1]\bigr).$$

Observe that by setting $x = -p$ and $y = 1$ in the above, we recover the right part of (25).

Open problem 5. Find an explicit formula for the area–perimeter counting series of directed animals. Find an explicit description of the invariant measure of the PCA of parameters (p_1, p_2, p_2, p_2). Compute the critical probability p_* of the family of Stavskaya PCA.

For a detailed account on this open problem, see [55] and the references therein. A related result of Bacher [4] is the exact computation of the series $(\partial S^{\mathrm{AP}}/\partial y)(x,1)$. The proof is purely combinatorial using heaps of pieces. This suggests the possibility of a reverse interaction between PCA and animals: using combinatorial results on (variants/extensions of) directed animals to prove new results on PCA.

4.3. Queues, percolation, and Young diagram

This section is mostly based on an influential, although unpublished, paper of Jockusch, Propp, and Shor [41].

The distribution of a r.v. X is a *shifted geometric* of parameter $u \in (0,1)$, if:

$$\forall k \in \{1,2,3,\ldots\}, \quad \mathbb{P}\{X = k\} = u(1-u)^{k-1}. \tag{27}$$

Observe that $E[X] = 1/u$.

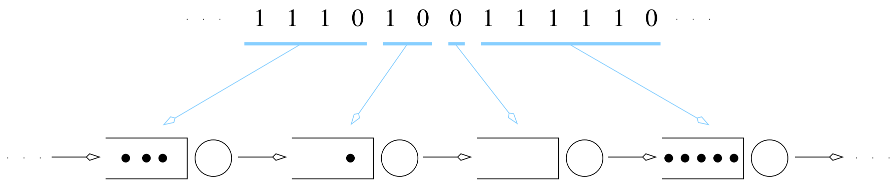

Fig. 10. From the TASEP to tandem queues.

Back to the TASEP PCA Consider the TASEP PCA defined in (3) and denote it by T_p for parameter $p \in (0,1)$. This PCA appears in many different contexts. In particular, there is an important connection between the TASEP PCA and tandem queues, that is described in Fig. 10.

Precisely, a configuration of $\{0,1\}^{\mathbb{Z}}$ is interpreted as follows: each 0 corresponds to a queue with an infinite capacity buffer and the consecutive 1's on its left (if any) correspond to the customers waiting in line at the queue. The dynamics of the TASEP PCA translates as follows for the queuing model: at a given queue, customers are served one by one in their order of arrival, their service time is a shifted geometric of parameter p, and upon being served a customer joins instantaneously the next queue to its right.

This is a standard model in queuing theory, which enjoys remarkable properties. The idea, developed below, is to transfer to the TASEP PCA the results which are known for the queuing model.

For $a \in (0,p)$, define the stochastic matrix

$$Q_a = \begin{pmatrix} (p-a)/p & a/p \\ (p-a)/[p(1-a)] & a(1-p)/[p(1-a)] \end{pmatrix},$$

and let $\pi_a = (\pi_a(0), \pi_a(1))$ be such that: $\pi_a Q_a = \pi_a, \pi_a(0) + \pi_a(1) = 1$, that is:

$$\pi_a(0) = (p-a)/\big(p-a^2\big) \qquad \pi_a(1) = a(1-a)/\big(p-a^2\big).$$

Proposition 4.11. *Consider the TASEP PCA T_p for $p \in (0,1)$. For any $a \in (0,p)$, the Markovian measure ν_{Q_a} associated with Q_a (see Definition 4.6) is an invariant measure of T_p. The invariant measures of T_p which are translation-invariant (see Section 5.1) are precisely the convex combinations of ν_{Q_a}, $a \in (0,p)$, $\delta_{0^{\mathbb{Z}}}$, and $\delta_{1^{\mathbb{Z}}}$.*

Proof. To prove that ν_{Q_a} is an invariant measure, a first way to proceed is simply to check "by hand" that it is left invariant by the dynamic. The verification has to be made for all cylinders. This program is carried out in [41, §4.1].

However, such a proof is not very informative, hiding in particular how the right invariant measures were guessed. Let us sketch instead a queuing theoretic argument. (For the authors of [41], the guess comes from queuing theory in a disguised way. Indeed they adapt an analogous result of Rost [64] on the continuous-time version of the TASEP model, and, in [64], Rost uses explicitly the connection with continuous-time queues.)

Consider a single queue with shifted geometric services of parameter p, that is, at each time step, if the buffer is non-empty, there is a departure with probability p, independently of everything else. Assume that the arrivals to the queue are distributed according to $\mathcal{B}_a^{\otimes\mathbb{Z}}, a \in (0,p)$, that is, at each time step, there is an arrival with probability a, independently of everything else. (The inter-arrival times are independent and distributed according to a shifted geometric of parameter a.) Then, it can be checked immediately that the equilibrium queue-length process (which is a birth-and-death Markov chain) is distributed according to π defined by:

$$\pi_0 = 1 - a/p, \qquad \forall n \geq 1, \quad \pi_n = (1-a/p)\frac{1}{1-p}\left(\frac{(1-p)a}{p(1-a)}\right)^n. \tag{28}$$

Furthermore, the equilibrium departure process from the queue is distributed according to $\mathcal{B}_a^{\otimes\mathbb{Z}}$. This last result is known as a "Burke-type" theorem, since it was proved by Burke in the context of continuous-time queues with exponential services, and by Hsu and Burke [40] in the present context.

Consider now several of the above queues in tandem. The departure process from a queue is the arrival process to the next queue. According to the Burke-type theorem, in equilibrium, the arrival process to any given queue will be distributed according to $\mathcal{B}_a^{\otimes\mathbb{Z}}$. An even stronger result, a so-called "product-form" theorem, holds: in equilibrium, the different queue-lengths are distributed according to π defined in (28) and *independent*, see [40] for a proof.

By letting the number of queues go to infinity, and by using the identification of Fig. 10, one can retrieve the desired result. In particular, the matrix Q_a can be derived from (28). For instance, $(Q_a)_{0,0}$, the probability to have a 0 followed by a 0 in the TASEP, is equal to $\pi_0 = (1-a/p)$, the probability to have an empty queue in the queuing model (see Fig. 10).

To prove that the measures ν_{Q_a} are the only extremal invariant measures, one can use a coupling argument, see [41, §4.2]. This is linked with the uniqueness of invariant arrival–departure processes in the queuing context, see for instance [14]. □

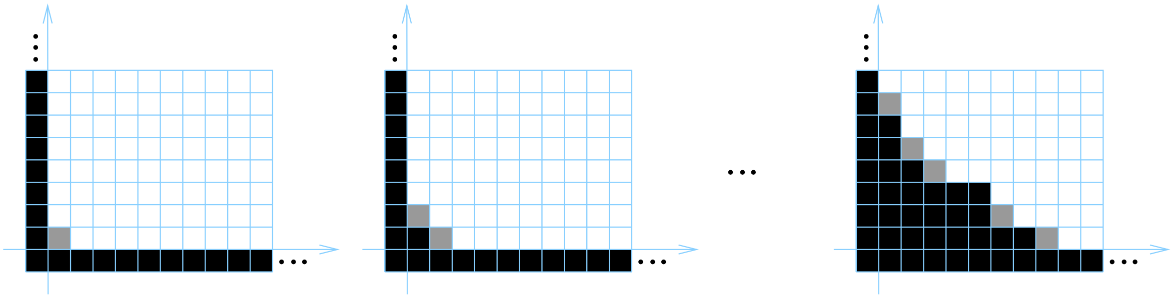

Fig. 11. Random Young diagram.

Random Young diagram and directed last-passage percolation Percolation models are ubiquitous in probability and combinatorics, see for instance [35]. As recalled in Section 3, percolation theory is the key to prove Proposition 3.4 or Theorem 3.9. Here we will see a connection going in reverse direction, that is, using a result on PCA, Proposition 4.11, to prove the percolation result in Theorem 4.12.

Fix a parameter $p \in (0, 1)$. Consider the lattice $\mathbb{Z}^2$ viewed as the union of unit squares with integer corners. Initially, all the squares in $\mathbb{N}^2$ are colored in white, while the other squares are colored in black. White squares turn black according to the following rule. Time is discrete, and at each time step, a white square becomes black with probability p if the two squares below and to the left are black. This is illustrated in Fig. 11. The squares that may turn black at the next time step are colored in gray.

At each time, the black squares within $\mathbb{N}^2$ form a finite sequence of bottom-aligned columns of non-increasing heights. Such shapes are known as *Young diagrams* (or *Ferrers diagrams*), and classically used in combinatorics to represent integer partitions. Therefore our model is often called a *randomly growing Young diagram*, or simply a *random Young diagram*. (Several relevant variants exist, see [79].)

For $(i, j) \in \mathbb{N}^2$, let $L(i, j)$ be the instant at which the square of center $(i + 1/2, j + 1/2)$ turns black. Let $[\cdot]$ denote the integer part. For a fixed $x \geq 0$, the sequence $(L([xn], n))_n$ is super-additive. Therefore, by Kingman's subadditive ergodic theorem [45], for any $x \geq 0$, there exists $\ell(x) \in (0, \infty)$ such that, almost surely,

$$\frac{1}{n} L\big([xn], n\big) \overset{n\to\infty}{\longrightarrow} \ell(x). \tag{29}$$

The function $x \mapsto \ell(x)$ is called the *growth function* of the model. Observe that $\ell(0) = 1/p$.

Here is an alternative way of viewing the model. Let $s(i, j)$, $i, j \in \mathbb{N}$, be the time during which the square of center $(i + 1/2, j + 1/2)$ remains gray. Then $[s(i, j)]_{(i,j)\in\mathbb{N}^2}$ is a collection of i.i.d. random variables with shifted geometric law of parameter p. Observe that we have, for $i, j \geq 1$,

$$L(i, j) = \max\big(L(i - 1, j), L(i, j - 1)\big) + s(i, j).$$

By developing the above recursion, we get, for all $i, j \geq 0$:

$$L(i, j) = \max_{\pi \in \Pi(i,j)} \sum_{(u,v)\in\pi} s(u, v), \tag{30}$$

where $\Pi(i, j)$ is the set of paths defined by:

$$\begin{aligned}\Pi(i, j) = \big\{&(0, 0) = (i_0, j_0), \dots, (i_\ell, j_\ell) = (i, j) \;\big| \\ &\forall k, [i_k = i_{k-1} + 1, j_k = j_{k-1}] \vee [i_k = i_{k-1}, j_k = j_{k-1} + 1]\big\}.\end{aligned} \tag{31}$$

The setting in (30) and (31) defines a *directed* (left–right and bottom–up paths in Π) and *last-passage* (maximum over paths in L) *percolation* model.

Theorem 4.12. *The growth function for the directed last-passage percolation model with shifted geometric distribution of parameter $p \in (0, 1)$ is:*

$$\ell(x) = \frac{1 + x + 2\sqrt{(1 - p)x}}{p}. \tag{32}$$

The original proof of Theorem 4.12 appears in [41,16]. Here we sketch the approach of [67], see also [58].

Proof. Consider the border between the black and the white regions in a Young diagram and encode each vertical segment by a 1, and each horizontal segment by a 0, as in Fig. 12. This encoding can be viewed as a configuration $u \in \{0, 1\}^{\mathbb{Z}}$ with

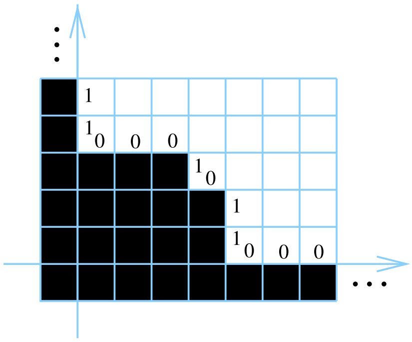

Fig. 12. From random Young diagrams to the TASEP.

the convention that u_0 is the first segment to the right of the main diagonal. For instance, the Young diagram of Fig. 12 has a border configuration: $\cdots u_{-5}u_{-4}\cdots u_6\cdots = \cdots 110001011000\cdots$.

Now, one can easily check that the random Young diagram of parameter p has a border configuration evolving as a TASEP PCA of parameter p and with initial condition:

$$u = (u_i)_{i\in\mathbb{Z}}, \quad \forall i \le -1,\ u_i = 1,\ \forall i \ge 0,\ u_i = 0. \tag{33}$$

View this TASEP PCA as a queuing model using the correspondence of Fig. 10. The initial condition (33) is reinterpreted as an infinite string of customers in the buffer of an initial queue followed by an infinite string of empty queues. Let us label the queues in order by $\mathbb{N}$ (the initial queue being labeled 0) and let us also label the customers in order by $\mathbb{N}$ (customer 0 being the first one to be served at queue 0, followed by customer 1, and so on). Queues and customers keep their label along the evolution.

All in all, we have established a connection between the random Young diagram and a queuing model. It is easily checked that $s(i, j)$, the time during which the square of center $(i + 1/2, j + 1/2)$ remains in gray in the random Young diagram, corresponds to the service time of customer i in queue j in the queuing model. It implies that $L(i, j)$, defined in (30), corresponds to the departure time of customer i from queue j in the queuing model.

Here the queuing model is *not* in equilibrium, since, initially, an infinite number of customers are queued in the buffer of queue 0. The key idea is to reintroduce an (exactly solvable) equilibrium model.

Modify the percolation model as follows. Consider a collection of i.i.d. r.v.'s $s(i, j)$ indexed by $\mathbb{Z}^2$ instead of $\mathbb{N}^2$. For $i_1, i_2, j_1, j_2 \in \mathbb{Z}, i_1 \le i_2$, and $j_1 \le j_2$, define the set of directed paths $\Pi(i_1, j_1)(i_2, j_2)$ going from (i_1, j_1) to (i_2, j_2) using only left–right and bottom–up steps. Define $L(i_1, j_1)(i_2, j_2)$ by replacing $\Pi(i, j)$ by $\Pi(i_1, j_1)(i_2, j_2)$ in (30) (so that $L(i, j) = L(0, 0)(i, j)$).

Modify the queuing model as follows. Consider an infinite tandem of queues labeled by $\mathbb{N}$, and fed by a bi-infinite sequence of customers labeled by $\mathbb{Z}$. Now customer 0 is the first one to arrive at queue 0 *after instant 0*, followed at positive instants by customer 1 and so on, and preceded at negative instants by customer -1 and so on. At any given instant, there exists an integer k such that the customers labeled n, $n \le k$, are within the network, and the customers labeled n, $n > k$, are outside (not yet in) the network. Let $a(k)$, $k \in \mathbb{Z}$, denote the interarrival time between customers k and $k + 1$ at queue 0. Assume that $(a(k))_k$ is a sequence of i.i.d. r.v.'s with shifted geometric law of parameter a for some $a \in (0, p)$. It implies that at each time step a customer arrives in queue 0 with probability a. Assume that the service times are given by the r.v.'s $s(i, j), i \in \mathbb{Z}, j \in \mathbb{N}$, of the extended percolation model.

Let $T(i, j)$, $i \in \mathbb{Z}, j \in \mathbb{N}$, denote the total time spent by customer i in queues 0 up to j. The following pathwise relation, extending (30), links the modified percolation and queuing models:

$$T(0, n) = \sup_{-K<0}\left[L(-K, 0)(0, n) - \sum_{k=-K}^{-1} a(k)\right] = \sup_{x\in\mathbb{R}_+}\left[L\big(-[xn], 0\big)(0, n) - \sum_{k=-[xn]}^{-1} a(k)\right].$$

This relation, first noticed in [69], can be proved recursively. Divide both sides by n, and let n go to infinity. Since $L(-[xn], 0)(0, n) \sim L([xn], n)$, we get:

$$\lim_n \frac{T(0, n)}{n} = \sup_{x>0}\left[\ell(x) - \frac{x}{a}\right]. \tag{34}$$

Now observe that the modified queuing model is exactly the model considered in the proof of Proposition 4.11. In particular, the model is in equilibrium. Then, starting from (28) and after some tedious computations, one obtains that the time spent by a customer in a queue (the *sojourn time*) is distributed as a shifted geometric of parameter $(p - a)/(1 - a)$. Another remarkable result is that the sojourn times of a given customer in successive queues are independent random variables, see for instance [18, Corollary 4.9]. In particular, using the strong law of large numbers, we get that $\lim_n T(0,n)/n = (1-a)/(p-a)$ almost surely. Plug this in (34) to get:

$$\frac{1-a}{p-a} = \sup_{x>0}\left[\ell(x) - \frac{x}{a}\right].$$

Invert the above to get:

$$\ell(x) = \inf_{a:\, a<p}\left[\frac{x}{a} + \frac{1-a}{p-a}\right].$$

Solve this optimization problem to obtain (32). □

Theorem 4.12 is also used in [41] to prove another remarkable combinatorial result: the so-called "arctic circle theorem" for "random domino tilings of the Aztec diamond".

A directed last-passage percolation model can be defined as in (30)–(31) for any probability distribution of the $s(i,j)$ instead of the shifted geometric distribution. The growth function, defined as in (29), exists (and is finite under suitable moment condition for the $s(i,j)$, see [3]). But no explicit formula is known apart from the case in Theorem 4.12 and from the case of the exponential distribution (see [64]). In particular, no explicit formula is known for a Bernoulli distribution.

Open problem 6. Determine the growth function of the directed last-passage percolation model associated with the variables $s(i,j)$ of law: $\mathbb{P}\{s(0,0)=1\} = \mathbb{P}\{s(0,0)=2\} = 1/2$.

In terms of PCA, it amounts to finding explicitly the invariant measures of a TASEP-like PCA on the alphabet $\{0,1,2\}$ defined by the local rules: $10 \to 01$ with probability $1/2$, $10 \to 20$ with probability $1/2$, and $20 \to 01$ with probability 1. (To be compared with the TASEP PCA simply defined by the local rule: $10 \to 01$ with probability p.)

5. Statistical mechanics

The connection with equilibrium statistical mechanics is essential to understand PCA. Here, the results are presented without proofs.

In Definition 2.3, the space–time diagrams of a PCA were defined from time 0 on. We need to extend this by defining "space–time diagrams starting at time $-\infty$".

Consider a PCA F on the set of cells $\mathbb{Z}^d$. Let μ be an invariant measure of F. Let $X^0 = (x^0(u,n))_{u\in\mathbb{Z}^d, n\in\mathbb{N}}$, with $(x^0(u,0))_{u\in\mathbb{Z}^d} \sim \mu$, be an associated space–time diagram (see Definition 2.3). It is possible to start the evolution of the PCA F at instant $-N$, $N\in\mathbb{N}$, instead of 0. Let us denote by $X^{-N} = (x^{-N}(u,n))_{u\in\mathbb{Z}^d, n\geq -N}$, with $(x^{-N}(u,-N))_{u\in\mathbb{Z}^d} \sim \mu$, an associated space–time diagram. Let $\mu^{-N} \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d\times\{-N,-N+1,\dots\}})$ be the law of X^{-N}. By the invariance of μ, the laws $(\mu^{-N})_N$ are consistent (i.e., for $M>N$, the restriction of μ^{-M} to $\mathcal{A}^{\mathbb{Z}^d\times\{-N,-N+1,\dots\}}$ is μ^{-N}). By Kolmogorov extension theorem, there exists a uniquely defined law $\mu^{-\infty} \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^{d+1}})$ whose restrictions are the laws μ^{-N}.

Roughly, $\mu^{-\infty}$ is the law of a space–time diagram starting at time $-\infty$. The measure $\mu^{-\infty}$ (or a random field distributed according to $\mu^{-\infty}$) is still called a *space–time diagram* of F.

5.1. Translation-invariant space–time diagram of a PCA

Consider $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^k})$ and a random element $(x_u)_{u\in\mathbb{Z}^k} \sim \mu$. We say that μ is *translation-invariant* (or *stationary*) if:

$$\forall v \in \mathbb{Z}^k, \quad (x_{u+v})_{u\in\mathbb{Z}^k} \sim \mu. \tag{35}$$

On $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, a measure μ is translation-invariant iff $\mu\sigma = \mu$ where σ is the *shift* defined by: $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}, (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$.

Consider a PCA F on the set of cells $\mathbb{Z}^d$. If μ is a translation-invariant measure on $\mathbb{Z}^d$ then μF is also translation-invariant. Using this fact, one can prove the following which completes Lemma 3.2.

Lemma 5.1. *The set of invariant measures for a PCA which are translation-invariant is non-empty, convex and compact.*

There may exist invariant measures which are not translation-invariant. Consider for instance either the TASEP PCA or the Traffic CA defined in (3) and (4). Then for $x = \cdots 000111\cdots$, we have that δ_x is an invariant measure which is not translation invariant.

Consider a PCA F on the set of cells $\mathbb{Z}^d$ and let μ be a translation-invariant invariant measure of F. Let $\mu^{-\infty} \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^{d+1}})$ be the corresponding space–time diagram defined as above. Using the invariance by F and the translation-invariance on $\mathbb{Z}^d$ of μ, we obtain that $\mu^{-\infty}$ is translation-invariant on $\mathbb{Z}^{d+1}$. We say that $\mu^{-\infty}$ (or a random field distributed according to $\mu^{-\infty}$) is a *translation-invariant space–time diagram* of F.

5.2. Gibbs measures

We first introduce some background of statistical mechanics. Let $G = (\Gamma, E)$ be an infinite and locally finite undirected graph with vertices Γ and edges E. We are concerned by two types of graphs. First, G may be the Cayley graph of the group $(\mathbb{Z}^d, +)$, that is, the graph with vertices $\mathbb{Z}^d$ and with edges between vertices at distance 1 (for the usual ℓ_1 metric). For simplicity, we will then say that G is the graph $\mathbb{Z}^d$. Second, G may be a graph with set of vertices $\mathbb{Z}^d \times \{0, 1\}$.

Use the notation $C \Subset \Gamma$ to specify that C is a finite subset of Γ.

Definition 5.2 *(Gibbs potential).* Let $\mathcal{A}$ be a finite set. A *Gibbs potential*, or simply *potential*, on the graph $G = (\Gamma, E)$ is a family $\varphi = (\varphi_C)_{C \Subset \Gamma}$ of functions $\varphi_C : \mathcal{A}^C \to \mathbb{R}$. The potential φ has a *finite range* if there exists L in $\mathbb{N}$ such that $\varphi_C \equiv 0$ as soon as the set C contains two elements at distance larger than L in the graph metric.

In the following, we only consider finite range potentials, even if not specified.

For $K \subset \Gamma, x \in \mathcal{A}^\Gamma$, let $x_K \in \mathcal{A}^K$ be the restriction of x to K. By convention, for finite sets C, K with $C \subset K$ and $x \in \mathcal{A}^K$, we set $\varphi_C(x) = \varphi_C(x_C)$. For a set $K \subset \Gamma$, we define $\mathcal{V}(K)$ as the union of the sets $C \Subset \Gamma$ such that $C \cap K \neq \emptyset$ and $\varphi_C \not\equiv 0$. We also define $\partial K = \mathcal{V}(K) \setminus K$.

Definition 5.3 *(Gibbs measure).* Let φ be a Gibbs potential on the graph $G = (\Gamma, E)$. A probability measure μ on $\mathcal{A}^\Gamma$ is a *Gibbs measure with potential* φ if: $\forall x \in \mathcal{A}^\Gamma, \forall J \Subset \Gamma, \forall K \Subset \Gamma, \mathcal{V}(K) \subset J$,

$$\mu\big([x_{J \setminus K}]\big) > 0 \quad \Longrightarrow \quad \mu\big([x_K] \big| [x_{J \setminus K}]\big) = \frac{1}{Z(x_{\partial K})} \exp\Big(- \sum_{C \subset \mathcal{V}(K)} \varphi_C(x)\Big), \tag{36}$$

where $Z(x_{\partial K})$ is a normalizing factor depending only on $x_{\partial K}$. The set of Gibbs measures with potential φ is denoted by $\mathcal{G}(\varphi)$.

Observe in particular that (36) implies that: $\mu([x_K]|[x_{J \setminus K}]) = \mu([x_K]|[x_{\partial K}])$, as soon as $\mu([x_{J \setminus K}]) > 0$. For a proof of next lemma, see for instance [30].

Lemma 5.4. *Let φ be a Gibbs potential. The set of Gibbs measures $\mathcal{G}(\varphi)$ is non-empty, convex, and compact.*

There might be several Gibbs measures associated to the same potential. In which case, we say that there is a *phase transition*.

For comparison, in the classical approach, Kolmogorov extension theorem defines a (unique) probability measure given a family of consistent finite-dimensional distributions. Here, a (non-necessarily unique) probability measure is specified through a family of finite-dimensional *conditional* distributions. This is referred to as the *DLR approach*, in tribute to Dobrushin, Lanford, and Ruelle.

By definition, Gibbs measures satisfy a Markov property: indeed, the conditional measures depend only on a finite number of neighbors. Thus, Gibbs measures are *Markov random fields*. Let us mention that there is in fact an equivalence between Markov random fields and Gibbs measures with finite range potentials (Hammersley–Clifford theorem), see [68,30].

Next result is very classical, see for instance [30] for a proof.

Theorem 5.5. *Let φ be a Gibbs potential on the graph $\mathbb{Z}$. There is a unique Gibbs measure with potential φ.*

The above is often stated as: "there is no phase transition in $\mathbb{Z}$". On the other hand, on the graph $\mathbb{Z}^d, d \geq 2$, there are simple Gibbs potentials with phase transition. We come back to this last point in Section 5.5.

5.3. PCA and equilibrium statistical mechanics

In this section, we present the correspondence between the translation-invariant space–time diagrams of a positive-rate PCA on the set of cells $\mathbb{Z}^d$, and the Gibbs measures of a related potential on the graph $\mathbb{Z}^{d+1}$.

Consider a positive-rate PCA F on the set of cells $\mathbb{Z}^d$ of neighborhood $\mathcal{N}$ and local function f. For $(k, n) \in \mathbb{Z}^d \times \mathbb{Z}$, define the set $N(k, n) = \{(k + v, n); v \in \mathcal{N}\} \cup \{(k, n + 1)\}$. Define the Gibbs potential φ on $\mathbb{Z}^{d+1}$ by: $\forall x = (x_k^n)_{k \in \mathbb{Z}^d, n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}^{d+1}}, \forall (k, n) \in \mathbb{Z}^d \times \mathbb{Z}$,

$$\varphi_{N(k,n)}(x) = -\log f\big(\big(x_{k+v}^n\big)_{v \in \mathcal{N}}\big)\big(x_k^{n+1}\big), \tag{37}$$

and $\varphi_C \equiv 0$ for all other finite sets C. Next result appears in [31].

Proposition 5.6. *Let F be a positive-rate PCA on $\mathbb{Z}^d$ and let φ be the Gibbs potential on $\mathbb{Z}^{d+1}$ defined by (37). The translation-invariant Gibbs measures for φ correspond exactly to the translation-invariant space–time diagrams for F.*

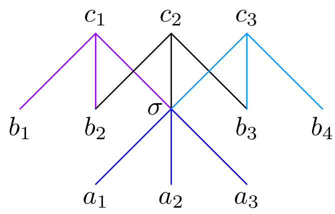

Fig. 13. Illustration of Proposition 5.6.

In Proposition 5.6, the difficulty consists in showing that any translation-invariant Gibbs measure corresponds to a translation-invariant space–time diagram for F. The proof uses conditional entropy and the variational principle. The other direction, stating that a translation-invariant space–time diagram for F is a Gibbs measure for φ, is easier.

It follows from Proposition 5.6 that any translation-invariant invariant measure of a PCA on $\mathbb{Z}^d$ is the projection of a translation-invariant Gibbs measure on $\mathbb{Z}^{d+1}$. Projections of Gibbs measures are not necessarily Gibbs measures. Hence, the translation-invariant invariant measures are not necessarily Gibbs.

Example. Let us consider a PCA on $\mathbb{Z}$, of neighborhood $\mathcal{N} = \{-1, 0, 1\}$. We consider a portion of space–time diagram as in Fig. 13, with time going up. A consequence of Proposition 5.6 is that, if the space–time diagram is translation-invariant, the conditional distribution of the central cell knowing all the other values of the space–time diagram is equal to its conditional distribution knowing the states of the 10 neighboring cells represented on Fig. 13.

Conditionally to the values of these 10 neighboring cells, the central cell takes the value σ with probability:

$$\frac{1}{Z} f(a_1, a_2, a_3)(\sigma) f(b_1, b_2, \sigma)(c_1) f(b_2, \sigma, b_3)(c_2) f(\sigma, b_3, b_4)(c_3),$$

where $Z = \sum_{\alpha \in \mathcal{A}} f(a_1, a_2, a_3)(\alpha) f(b_1, b_2, \alpha)(c_1) f(b_2, \alpha, b_3)(c_2) f(\alpha, b_3, b_4)(c_3)$.

5.4. Reversibility

In the previous section, we have seen the link between PCA on $\mathbb{Z}^d$ and Gibbs measures on $\mathbb{Z}^{d+1}$. Here we investigate a different and natural question: how to relate PCA on $\mathbb{Z}^d$ and Gibbs measures on $\mathbb{Z}^d$.

First, here is a result proved for instance in [19].

Proposition 5.7. *Let F be a positive-rate PCA on the set of cells $\mathbb{Z}^d$. Let φ be a Gibbs potential on $\mathbb{Z}^d$. If F has a translation-invariant invariant measure belonging to $\mathcal{G}(\varphi)$, then all the translation-invariant invariant measures of F belong to $\mathcal{G}(\varphi)$.*

It is not obvious that for a given φ, there exist PCA satisfying the required condition in Proposition 5.7.

To go further, we need to introduce the notion of reversibility. Let F be a PCA on the alphabet $\mathcal{A}$ and the set of cells E, and let μ be an invariant measure of F. Let (X^0, X^1) be a pair of $\mathcal{A}^E$-valued r.v.'s such that $X^0 \sim \mu$ and X^1 is obtained from X^0 by one iteration of the PCA. We say that μ is *reversible* if $(X^0, X^1) \sim (X^1, X^0)$. We say that a PCA is *reversible* if it has at least one reversible invariant measure.

Reversibility is defined similarly for interacting particle systems (recall that IPS were introduced in Section 2.4). A key point is that for any Gibbs potential φ on $\mathbb{Z}^d$, there exists an IPS on $\mathbb{Z}^d$ such that $\mathcal{G}(\varphi)$ is equal to the set of *reversible* invariant measures of the IPS, see [50]. For PCA, it is *not* always true, and it requires some conditions on the potential. This was first observed in [20] and will be apparent in Theorem 5.9 below.

We say that a subset V of $\mathbb{Z}^d$ is *symmetric* if: $i \in V \Longrightarrow -i \in V$. Given a PCA, we may assume that its neighborhood is symmetric without loss of generality. Indeed, starting from a non-symmetric neighborhood, we may artificially increase it to make it symmetric, without changing the global function.

Definition 5.8. Let us consider a PCA F on $\mathbb{Z}^d$ with a symmetric neighborhood $\mathcal{N}$. The *doubling graph* of F is the undirected graph with vertices $\mathbb{Z}^d \times \{0, 1\}$ and edges between (k, t) and $(k + v, 1 - t)$ for any $(k, v, t) \in \mathbb{Z}^d \times \mathcal{N} \times \{0, 1\}$.

An example of a doubling graph is given in Fig. 14.

Let V be a symmetric and finite subset of $\mathbb{Z}^d$. Consider functions $(\phi, (\phi_v)_{v \in V})$ satisfying:

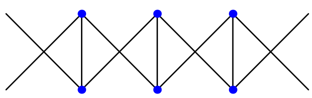

Fig. 14. Doubling graph of a PCA on $\mathbb{Z}$ with neighborhood $\mathcal{N} = \{-1, 0, 1\}$.

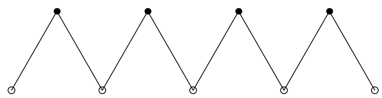

Fig. 15. Symmetric doubling graph associated to the neighborhood $\mathcal{N}_1$.

$$\phi : \mathcal{A} \to \mathbb{R}, \qquad \forall v \in V, \quad \phi_v : \mathcal{A}^2 \to \mathbb{R}, \qquad \phi_v(a,b) = \phi_{-v}(b,a). \tag{38}$$

The *potential associated with* $(\phi, (\phi_v)_v)$ is the potential φ on $\Gamma = \mathbb{Z}^d \times \{0,1\}$ defined by:

$$\forall u \in \Gamma, \quad \varphi_{\{u\}} = \phi, \qquad \forall j \in \mathbb{Z}^d, \forall k \in j + \mathcal{N}, \quad \varphi_{\{(j,0),(k,1)\}}(x_j, y_k) = \phi_{j-k}(x_j, y_k), \tag{39}$$

and $\varphi_C \equiv 0$ otherwise. By definition, we have: $\varphi_{\{(k,0),(j,1)\}}(y_k, x_j) = \varphi_{\{(j,0),(k,1)\}}(x_j, y_k)$.

Next result is proved in [46,75,77].

Theorem 5.9. *Let F be a positive-rate PCA on $\mathbb{Z}^d$ of neighborhood $\mathcal{N}$. A necessary and sufficient condition for F to be reversible is the existence of functions $(\phi, (\phi_v)_{v\in\mathcal{N}})$ as in (38) and such that the local function f of F satisfies:*

$$f\big((x_v)_{v\in\mathcal{N}}\big)(\alpha) = \frac{1}{Z(x_{\mathcal{N}})} \exp\Big(-\phi(\alpha) - \sum_{v\in\mathcal{N}} \phi_v(x_v, \alpha)\Big), \tag{40}$$

where $Z(x_{\mathcal{N}}) = \sum_{\gamma\in\mathcal{A}} \exp(-\phi(\gamma) - \sum_{v\in\mathcal{N}} \phi_v(x_v,\gamma))$. Let φ be the potential associated with $(\phi, (\phi_v)_v)$ as in (39). Under the above condition, the reversible measures of F are exactly the projections on $\mathbb{Z}^d$ of the Gibbs measures on $\mathbb{Z}^d \times \{0,1\}$ of potential φ that are equal on both copies of $\mathbb{Z}^d$. They are themselves Gibbs measures on $\mathbb{Z}^d$, of potential $\hat{\varphi}$ defined by:

$$\forall k \in \mathbb{Z}, \quad \hat{\varphi}_{\{k\}} = \phi, \qquad \hat{\varphi}_{k+\mathcal{N}}\big((x_{k+v})_{v\in\mathcal{N}}\big) = -\log Z(x_{k+\mathcal{N}}),$$

and $\hat{\varphi}_C \equiv 0$ for all other sets C.

The strength of Theorem 5.9 is reinforced by the next result which appears in [19, Prop. 3.3].

Proposition 5.10. *Let F be a reversible positive-rate PCA. Then any translation-invariant invariant measure of F is reversible.*

Example 5.11. Consider the set of cells $\mathbb{Z}$, the alphabet $\mathcal{A} = \{0,1\}$, and the neighborhood $\mathcal{N} = \{-1,0,1\}$. Applying Theorem 5.9, we get that the reversible positive-rate PCA form a three parameters family: the PCA described by a local function f of the type:

$$\forall (x,y,z) \in \mathcal{A}^{\mathcal{N}}, \quad f(x,y,z)(0) = \frac{1}{1 + c_1 c_2^{x+z} c_3^{y}}, \qquad f(x,y,z)(1) = 1 - f(x,y,z)(0),$$

where c_1, c_2, c_3, are positive real.

Asymmetric neighborhood For some asymmetric neighborhoods, instead of expanding the neighborhood, it may be relevant to modify the representation of the space–time diagram in order to recover a symmetric neighborhood. For example, if $E = \mathbb{Z}$, $\mathcal{N} = \{0,1\}$, it is natural to shift by 1/2 the image X^1 of the initial configuration X^0, which amounts to consider that the neighborhood is in fact $\mathcal{N}' = \{-1/2, 1/2\}$. The corresponding doubling graph is then represented as in Fig. 15. The notion of reversibility and all the results of this section can be adapted to this new context.

Back to directed animals Let us go back to the directed animals PCA F_p, $p \in (0,1)$. One can check that its unique invariant measure, the Markovian measure ν_p described in (19), is a reversible invariant measure. Let us interpret this in the light of the above results.

Consider the doubling graph of the PCA built as in Fig. 15. The local rule of F_p can be described as in (40) for some functions $(\phi, (\phi_v)_v)$. If we interpret 1's as particles and 0's as empty sites, the Gibbs potential associated with $(\phi, (\phi_v)_v)$, see (39), corresponds exactly to the hardcore lattice gas model on the doubling graph. Now observe that the doubling graph can be identified with $\mathbb{Z}$. And, in $\mathbb{Z}$, it is well known that there is a unique Gibbs measure for the hardcore potential, which is a Markovian measure. When taking the projection of this measure on odd (or even) sites, we obtain the measure ν_p, the reversible invariant measure of the PCA. (So the result of Theorem 5.9 holds true for this PCA although it does not have positive rates.)

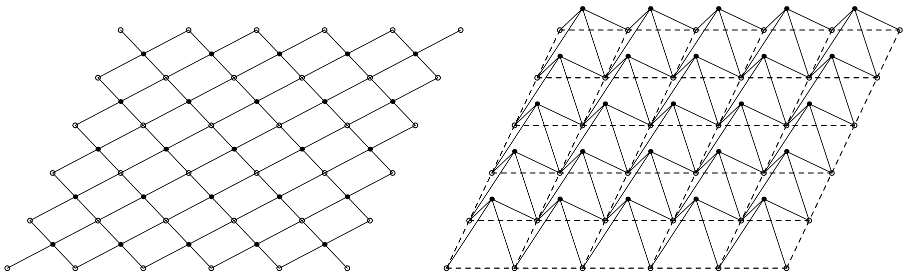

Fig. 16. From $\mathbb{Z}^2$ to $\mathbb{Z}^2 \times \{0,1\}$.

5.5. Back to the positive-rate problem

Here, starting from the Ising model and using the above approach, we build a non-ergodic positive-rate PCA on $\mathbb{Z}^2$. This example comes from [46,75,77].

Set $\mathcal{A} = \{-1, 1\}$. Fix some parameter $\beta \in (0, \infty)$. Consider the graph $\mathbb{Z}^2$ and denote the set of edges by E. Consider the Gibbs potential on $\mathbb{Z}^2$ defined by:

$$\forall (u, v) \in E, \quad \varphi_{\{u,v\}} : \mathcal{A}^2 \to \mathbb{R}, \qquad (a, b) \mapsto -\beta ab,$$

and $\varphi_C \equiv 0$ for all other subsets C. Observe that φ can be seen as the potential associated with $(\phi, (\phi_e)_e)$ where $\phi \equiv 0$ and, for all edge e, $\phi_e(a, b) = -\beta ab$.

The potential φ corresponds to the classical Ising model. It is well-known that, for β large enough, this potential has a phase transition: there exist at least two translation-invariant Gibbs measures, of density of 1's respectively strictly larger and strictly smaller than $1/2$.

Now twist the graph $\mathbb{Z}^2$ by lifting the vertices (i, j) such that $i + j$ is odd, see Fig. 16 (right). This can be viewed as a new graph with vertices indexed by $\mathbb{Z}^2 \times \{0, 1\}$, the copy $\mathbb{Z}^2 \times \{0\}$ corresponding to the even vertices in the original graph. This new graph can be seen as the doubling graph of a PCA on $\mathbb{Z}^2$ with an original neighborhood $\mathcal{N} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ having been symmetrized as $\mathcal{N}' = \{(\pm 1/2, \pm 1/2)\}$ (see the explanation at the end of the previous section for the case $\mathbb{Z}$).

Let F be the positive-rate PCA on the set of cells $\mathbb{Z}^2$ associated with the potential φ, that is, the PCA with local function f defined as in (40). If we set $\varepsilon = \exp(-4\beta)$, we get:

$$f(x, y, z, t)(1) = \frac{\varepsilon^2}{1+\varepsilon^2}, \frac{\varepsilon}{1+\varepsilon}, \frac{1}{2}, \frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon^2}, \tag{41}$$

if there are respectively 0, 1, 2, 3, or 4 times the state 1 among x, y, z, t.

Applying Theorem 5.9, any Gibbs measure of potential φ on the doubling graph provides an invariant measure μ for the PCA when projecting it on the grid on which the PCA is defined (on the right of Fig. 16, this grid is the dashed grid whereas the doubling graph is represented with continuous lines).

As a consequence, if β is large enough (corresponding to small values of ε), the PCA F has at least two different invariant measures of density of 1's respectively strictly larger and strictly smaller than $1/2$. Summarizing the above, we get next result.

Proposition 5.12. *Consider the positive-rate PCA on the set of cells $\mathbb{Z}^2$, the alphabet $\mathcal{A} = \{-1, 1\}$, the neighborhood $\mathcal{N} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, with local function f defined by (41). For ε small enough, this PCA is non-ergodic with at least two invariant measures.*

6. Computation model and dynamical systems

Cellular automata can be viewed as computation models in which both the program and the input are encoded in the initial configuration and where the computation corresponds to the space–time diagram. CA are known to form a Turing-complete model of computation, see for instance [59] and the references therein. More precisely, it is not difficult to simulate any Turing machine, and in particular a universal one, as a CA. Even some simple CA are computationally universal, like rule 110 in Wolfram's notation which is defined by: $\mathcal{A} = \{0, 1\}, \mathcal{N} = \{-1, 0, 1\}$,

$$f(000) = f(100) = f(111) = 0, \qquad f(001) = f(010) = f(011) = f(101) = f(110) = 1.$$

This is proved by Cook in [17].

For any model of computation, a natural question is its tolerance to faults: can it still compute in the presence of errors? We are going to investigate this question for CA. One of the key ideas is to relate it to the positive-rate problem for PCA defined in Section 3.2.

For more detailed surveys on this aspect of PCA, with many open problems, see [72,74].

6.1. Eroding, classifying, computing with errors

Let us define three properties, each one of them corresponding to a different type of error-correcting capacity.

Eroder property Consider a CA F on the set of cells $E=\mathbb{Z}^d$, and the alphabet $\mathcal{A}=\{0,1\}$, satisfying $F(0^E)=0^E$ and $F(1^E)=1^E$. Define

$$X_0=\{x\in\mathcal{A}^E \mid \#\{i, x_i=1\}<\infty\}, \qquad X_1=\{x\in\mathcal{A}^E \mid \#\{i, x_i=0\}<\infty\}.$$

Definition 6.1. The CA F on the set of cells $E=\mathbb{Z}^d$ and the alphabet $\mathcal{A}=\{0,1\}$ is an *eroder* if:

$$\forall x\in X_0, \exists n, \quad F^n(x)=0^E, \qquad \forall x\in X_1, \exists n, \quad F^n(x)=1^E. \tag{42}$$

The eroder property can be interpreted as the capacity of erasing a finite number of "errors" in the initial configuration. The notion of eroder can be extended to a PCA by requiring that (42) holds almost surely. The difficulty in designing a (P)CA with the eroder property is the following: at the frontier between a 0-zone and a 1-zone, how to decide if we are in the presence of a 0-island in a 1-ocean, or vice-versa? We will see below how to cope with the difficulty.

The eroder property is undecidable for general CA and becomes decidable for monotone CA (defined in (44)), see [71,74].

Density classification property The "density classification problem" consists in deciding, in a decentralized way, if a configuration of $\{0,1\}^E$ contains more 0's or more 1's. Recall that μ_p, $p\in(0,1)$, denotes the Bernoulli product measure of parameter p on $\{0,1\}^E$ (see Definition 4.1).

Definition 6.2. The CA or PCA F on $E=\mathbb{Z}^d$ and the alphabet $\mathcal{A}=\{0,1\}$ *classifies the density* if:

$$\forall p<1/2, \quad \lim_{n\to\infty}\mu_p F^n=\delta_{0^E}, \qquad \forall p>1/2, \quad \lim_{n\to\infty}\mu_p F^n=\delta_{0^E},$$

for the weak convergence.

Density classification amounts to erasing an infinite number of "errors" in the initial configuration, the "errors" being the symbols which are in minority. Here, the difficulty is to gather a global information on the density with a local mechanism.

Fault-tolerance property Consider a CA F with local function $f:\mathcal{A}^{\mathcal{N}}\to\mathcal{A}$. Introduce random errors as follows. For $\varepsilon\in(0,1)$, define the *noisy* version of F as the positive-rate PCA F_ε with local function $f_\varepsilon:\mathcal{A}^{\mathcal{N}}\to\mathcal{M}(\mathcal{A})$ defined by:

$$f_\varepsilon=(1-\varepsilon)\delta_f+\varepsilon\,\mathrm{Unif}, \tag{43}$$

where Unif is the uniform distribution on $\mathcal{A}$. For a PCA F of local function φ, we define in the same way the positive-rate PCA F_ε of local function $\varphi_\varepsilon=(1-\varepsilon)\varphi+\varepsilon\,\mathrm{Unif}$.

The PCA F_ε can indeed be interpreted as a "noisy" or "faulty" version of F: the computations are done according to F but at each time and in each cell, an error may occur with probability ε in which case the new cell value is chosen uniformly.

Definition 6.3. The CA or PCA F is *fault-tolerant* if there exists $\varepsilon_0\in(0,1)$ such that, for all $\varepsilon\in(0,\varepsilon_0)$, the noisy PCA F_ε has several invariant measures.

Fault-tolerance is linked with the positive-rate problem defined in Section 3.2. Indeed, F is fault-tolerant iff F_ε has a phase transition (Definition 3.8) for ε small enough, that is, iff F_ε answers positively the positive-rate problem for ε small enough.

Fault-tolerance is related to *reliable computation*, that is, the capacity of computing in presence of errors. Let us illustrate this point.

First, by definition, fault-tolerance is the capacity to remember part of the initial condition when "errors" occur in the whole space–time diagram. Indeed, the ergodicity of F_ε corresponds to a complete asymptotic forgetting of the initial condition. Conversely, the existence of several invariant measures for F_ε implies that "something" can be remembered forever about the initial condition, the "something" being captured by the limiting invariant measure.

Second, we can sometimes use this property to build a truly reliable computation scheme. Assume for instance that T is a CA on the alphabet $\{0,1\}$ and the set of cells $E=\mathbb{Z}^d$ such that $T(0^E)=0^E$, $T(1^E)=1^E$, and such that, for $\varepsilon\in(0,1)$ small enough, for all n, the measure $\delta_{0^E}T_\varepsilon^n$ is "close" to δ_{0^E}, and the measure $\delta_{1^E}T_\varepsilon^n$ is "close" to δ_{1^E}. In words, T remembers the value of an initial monochromatic configuration in the presence of noise.

Let U be any CA on the alphabet $\{0,1\}$ and the set of cells $\mathbb{Z}$, for instance a computationally universal one. Build a CA V on the set of cells $\mathbb{Z}^{d+1}$ as follows. View $\mathbb{Z}^{d+1}$ as consisting of a horizontal line of d-dimensional vertical layers. At

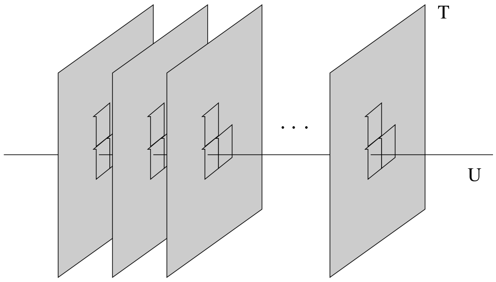

Fig. 17. A CA on $\mathbb{Z}^3$ which can compute in the presence of errors.

each time step, first, apply the CA T on each d-dimensional vertical layer, and, second, apply the CA U on each horizontal line. Consider an initial condition on $\mathbb{Z}^{d+1}$ which is obtained by choosing an initial condition on $\mathbb{Z}$ and by repeating it identically on each horizontal line (thus, the initial configuration is monochromatic on each d-dimensional vertical layer). By construction, the dynamics of the CA V is simply to repeat on each horizontal line the dynamics of U. In particular, the configurations obtained by successive applications of V remain monochromatic on each d-dimensional vertical layer. See Fig. 17 for $\mathbb{Z}^{d+1} = \mathbb{Z}^3$.

Now consider the noisy PCA V_ε for a small error parameter $\varepsilon \in (0,1)$. In good cases, due to the "remembering" ability of the vertical layers, if we start from an initial condition x as above, computation will still be possible on horizontal lines, that is, $\delta_x V_\varepsilon^n$ will be "close" to $\delta_x V^n$ for all n.

A precise situation in which the above approach works is by choosing Toom CA for T (see Definition 6.4) as illustrated in Fig. 17. See [29] for a precise statement and a proof.

6.1.1. The set of cells $\mathbb{Z}^d$, $d \geq 2$

We provide a simple CA $\mathcal{T}$ with the three above properties on the set of cells $\mathbb{Z}^2$. It also provides a solution in higher dimensions by decomposing $\mathbb{Z}^d$ into $\mathbb{Z}^2$-layers and by applying $\mathcal{T}$ on each layer.

Let $\mathcal{A} = \{0,1\}$, and, for $n \in \mathbb{N}, n \geq 1$, let us denote by $\mathtt{maj}_{2n+1} : \mathcal{A}^{2n+1} \to \mathcal{A}$, the *majority* function defined by:

$$\mathtt{maj}_{2n+1}(u_1,\dots,u_{2n+1}) = \begin{cases} 0 & \text{if } |u_1u_2\cdots u_{2n+1}|_0 \geq n+1 \\ 1 & \text{if } |u_1u_2\cdots u_{2n+1}|_1 \geq n+1. \end{cases}$$

To design a CA on $\mathbb{Z}^2$ with the three above error-correcting capacities, the first natural idea is to apply the majority rule on a cell and its four direct neighbors, that is to consider the CA with global function:

$$F(x)_{i,j} = \mathtt{maj}_5(x_{i,j-1}, x_{i-1,j}, x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$

Unfortunately, this does not work. Indeed, under the repeated action of the CA, a 2×2 square of four cells in state 1 (resp. 0) remains in state 1 (resp. 0) forever. Therefore, the CA is not an eroder and does not classify the density. On the other hand, F may have the fault-tolerance property. This is suggested by computer simulations and preliminary results, see [5] and the references therein.

Open problem 7. Prove that the CA on $\mathbb{Z}^2$ corresponding to the majority rule applied to a cell and its four direct neighbors is fault-tolerant.

The situation is the same for any majority CA with a symmetric neighborhood containing the origin: fault-tolerance is open, and the eroder and density-classification properties do not hold since there exist finite patterns which remain unchanged by the repeated action of the CA (see for instance [12, Lemma 3.1]).

To overcome the difficulty, the key idea is to consider the majority CA but on an asymmetric neighborhood.

Definition 6.4. *Toom CA* is the CA $\mathcal{T}$ on the set of cells $\mathbb{Z}^2$, the alphabet $\mathcal{A} = \{0,1\}$, the neighborhood $\mathcal{N} = \{(0,0),(0,1),(1,0)\}$, defined by the local function $\mathtt{maj}_3$, so that $\mathcal{T}$ is defined by:

$$\mathcal{T}(x)_{i,j} = \mathtt{maj}_3(x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$

Toom CA is also known as the *NEC* CA, with NEC standing for North–East–Center, a description of the shape of the neighborhood, see the left part of Fig. 18.

Theorem 6.5. *Let $\mathcal{T}$ be Toom CA. Then,*

- *$\mathcal{T}$ has the eroder property;*

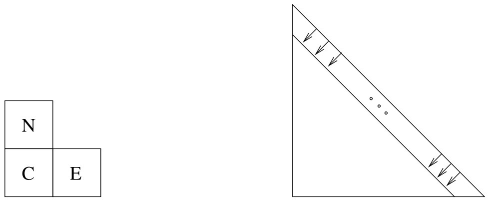

Fig. 18. Around Toom cellular automaton.

- *$\mathcal{T}$ classifies the density;*
- *$\mathcal{T}$ is fault-tolerant, that is, $\mathcal{T}_\varepsilon$ has several invariant measures for ε small enough.*

The eroder property was first observed by Toom. The proof is not complicated, and follows from the "triangle-shrinking" property illustrated in the right part of Fig. 18: if the set of "errors" of x lies within the outer triangle, then the set of "errors" of $\mathcal{T}(x)$ lies within the inner triangle. The density classification result is proved in [12]. The fault-tolerant property is obtained by proving that, for ε small enough, the noisy PCA $\mathcal{T}_\varepsilon$ has at least one invariant measure close to "all 0" and one close to "all 1". The original proof is due to Toom [71], and is complicated. Alternative proofs have been proposed [49,29,8] and a recent and precise investigation of the low-noise regime appears in [63]. As already mentioned, Toom CA can also be used as a building block to get a reliable universal CA on $\mathbb{Z}^3$, following the methodology of Fig. 17, see [29].

Say that a CA on the alphabet $\mathcal{A} = \{0, 1\}$ is *monotone* if its local function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ satisfies:

$$\forall x, y \in \mathcal{A}^{\mathcal{N}}, \quad x \le y \quad \Longrightarrow \quad f(x) \le f(y), \tag{44}$$

where $x \le y$ means $x_v \le y_v$ for all $v \in \mathcal{N}$. Toom CA and, more generally, any majority CA are monotone. Next result which completes Theorem 6.5 is also due to Toom [71].

Theorem 6.6. *Let F be a CA on the alphabet $\mathcal{A} = \{0, 1\}$ and the set of cells $\mathbb{Z}^d$, $d \ge 1$. Assume that F is monotone and has the eroder property. Then F is fault-tolerant.*

6.1.2. The set of cells $\mathbb{Z}$

There exists simple CA with the eroder property. There exist no known examples of CA or PCA that classify the density. The only known example of a fault-tolerant CA is the example of Gács already discussed in Section 3.2. Let us go into some details.

The first important point is that, on the set of cells $\mathbb{Z}$, there is no monotone CA having the eroder property, see [49, Section 6.2]. A consequence is that Theorem 6.6 becomes void. Another consequence is that CA having the eroder property should be searched for outside the class of majority CA.

The *Gács–Kurdyumov–Levin (GKL)* cellular automaton, introduced in [28], is the CA with neighborhood $\mathcal{N} = \{-3, -1, 0, 1, 3\}$ defined by: for $x \in \mathcal{A}^{\mathbb{Z}}$, $i \in \mathbb{Z}$,

$$\mathtt{Gkl}(x)_i = \begin{cases} \mathtt{maj}_3(x_i, x_{i+1}, x_{i+3}) & \text{if } x_i = 1 \\ \mathtt{maj}_3(x_i, x_{i-1}, x_{i-3}) & \text{if } x_i = 0. \end{cases} \tag{45}$$

The GKL CA is not monotone: for $x = 11000$ and $y = 11100$, we have $\mathtt{gkl}(x) = 1$ and $\mathtt{gkl}(y) = 0$ where $\mathtt{gkl}$ is the local function. Next proposition is proved in [28], see also [32] (see Fig. 19).

Proposition 6.7. *The GKL CA has the eroder property.*

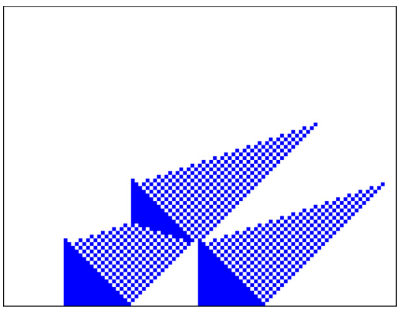

Fig. 19. Space–time diagram of GKL illustrating the eroder property.

Now consider the density classification. There is no known (P)CA with the property. It is conjectured in [12] that GKL classifies the density. Here is another candidate proposed in [12]. The *majority-traffic* PCA of parameter $\alpha \in (0,1)$ is the PCA on the alphabet $\mathcal{A} = \{0,1\}$ of neighborhood $\mathcal{N} = \{-1,0,1\}$ and local function:

$$(x,y,z) \longmapsto \alpha \delta_{\mathtt{maj}_3(x,y,z)} + (1-\alpha)\delta_{\mathtt{traf}(x,y,z)},$$

where $\mathtt{traf}$ is the local function defined in (4). The rough idea is as follows: starting from a configuration with a minority of 1's, the bulks of 1's should be spread out by the traffic rule and the isolated 1's should be eliminated by the majority rule.

Open problem 8. Does GKL or majority-traffic classify the density on $\mathbb{Z}$? Does there exist a CA or PCA classifying the density on $\mathbb{Z}$?

Let us move on to the fault-tolerance property. By analogy with Theorem 5.5, it had been conjectured in the 60's that there might exist no positive-rate PCA with a phase transition on the set of cells $\mathbb{Z}$, and in particular, no fault-tolerant (P)CA.

In [33], it was proved that the majority CA with neighborhood $\mathcal{N} = \{-1,0,1\}$ is not fault-tolerant. The GKL CA was originally introduced as a candidate to be fault-tolerant [28]. It is still unknown if it is the case or not, although the actual belief is that it is not fault-tolerant, see [32,60]. In particular, in [60], Park considers a "biased-noisy" GKL PCA, that is, the PCA obtained as in (43) for $f = \mathtt{gkl}$ but replacing the uniform measure Unif by the Bernoulli measure $\mathcal{B}_p$ for $p \in (0,1)$, $p \neq 1/2$. He proves that for ε and p small enough, the PCA is ergodic.

Open problem 9. Prove that $\mathtt{Gkl}_\varepsilon$ is ergodic for all $\varepsilon \in (0,1)$.

The breakthrough was provided by Gács who proved the conjecture to be wrong in [27] (the result was first suggested in [26]). Gács defines a CA, let us call it G, with remarkable properties: it is fault-tolerant and it is capable of universal computations in the presence of errors without the need for an additional dimension (that is, G is computationally universal and for ε small enough, $\delta_x G_\varepsilon^n$ is "close" to $\delta_x G^n$ for all $x \in \mathcal{A}^{\mathbb{Z}}$ and all $n \in \mathbb{N}$).

It is impossible to describe G here due to its complexity (in [34], the cardinal of $\mathcal{A}$ is evaluated to be at least 2^{24}). In fact, the example is so complicated that few people claim to have a complete understanding of it, see the discussion in [34].

Concerning fault-tolerance, the main challenge is to find simple examples in $\mathbb{Z}$, see the related Open problem 3.

6.2. Cellular automata: rigidity and randomization

In Section 4.1, we have seen that a positive-rate PCA having a Bernoulli invariant measure is ergodic (Proposition 4.5). On the other hand, a CA having a non-degenerate Bernoulli invariant measure is non-ergodic, since it has at least a second invariant measure supported by a monochromatic periodic orbit (see Section 3.1).

Given a CA, a *rigidity* result consists in proving that any invariant measure satisfying some properties (so that measures supported by periodic orbits and other measures that are too "degenerated" are excluded) is the uniform measure. Given a CA, a *randomization* result consists in proving that, starting from a large class of initial measures, the iterates of the CA converge to the uniform measure.

Rigidity and randomization can be viewed as weakened versions of ergodicity adapted to the CA context. We give a flavor of the type of results existing in the literature. For a more detailed account as well as a description of various other ergodic-type properties of CA, see the survey [61].

Bernoulli invariant measures and conservation Let us first state a classical result on CA which is proved in [37].

Proposition 6.8. *Let F be a cellular automaton, and let λ be the uniform Bernoulli product measure. We have:*

$$F \text{ is surjective} \quad \Longleftrightarrow \quad \lambda F = \lambda.$$

Given a finite and non-empty word $u \in \mathcal{A}^+$, let $u^{\mathbb{Z}} = \cdots uuu \cdots \in \mathcal{A}^{\mathbb{Z}}$ be a periodic bi-infinite word of period u (the starting position is indifferent). If $F : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is a CA, then we clearly have: $\forall u \in \mathcal{A}^+, \exists v \in \mathcal{A}^+, |u| = |v|, F(u^{\mathbb{Z}}) = v^{\mathbb{Z}}$. For simplicity, we then write $v = F(u)$.

Next theorem, which refines Proposition 6.8, appears in [44].

Theorem 6.9. *Consider a CA F on the alphabet $\mathcal{A}$. The Bernoulli product measure μ_p, $p = (p_i)_{i \in \mathcal{A}}$, $p_i > 0$ for all i, is invariant for F if and only if:*

$$(i) \quad F \text{ is surjective}, \quad \text{and} \quad (ii) \quad \forall u \in \mathcal{A}^+, \quad \sum_{i \in \mathcal{A}} |u|_i \log(p_i) = \sum_{i \in \mathcal{A}} \big|F(u)\big|_i \log(p_i).$$

In words, the result says that a CA preserves the Bernoulli product measure μ_p iff, when attributing a weight $\log p_i$ to the letter $i \in \mathcal{A}$, the total weight of the period in a periodic configuration is preserved by the CA. The existence of such a quantity that is preserved by the CA is an obstacle to randomization.

Rigidity Consider the additive CA defined in (1). It is convenient to view it as operating on the alphabet $\mathcal{A} = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, so that the global function becomes: $x \mapsto F(x)$, $F(x)_k = x_k + x_{k+1}$. The additive CA is clearly surjective. By Proposition 6.8, the uniform measure $\lambda = \mu_{1/2}$ is an invariant measure of F, but F also has many other invariant measures, such as $\delta_{0^{\mathbb{Z}}}$, or measures coming from various periodic orbits.

Define $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$ (the *shift* on $\mathcal{A}^{\mathbb{Z}}$). A measure μ on $\mathcal{A}^{\mathbb{Z}}$ is *shift-mixing* if for any cylinders $[u]$ and $[v]$,

$$\mu\big([u] \cap \sigma^{-n}[v]\big) \longrightarrow_{n\to\infty} \mu[u]\mu[v].$$

Next proposition, which is a rigidity result for the additive CA, is proved in [57].

Proposition 6.10. *Consider the additive CA F on $\mathbb{Z}_2^{\mathbb{Z}}$. The uniform measure is the only invariant measure of F which is shift-mixing and has full support $\mathbb{Z}_2^{\mathbb{Z}}$.*

A *linear* CA is a cellular automaton on the alphabet $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$, with local function $f : (x_i)_{i\in\mathcal{N}} \longmapsto \sum_{i\in\mathcal{N}} a_i x_i$ for some $a_i, i \in \mathcal{N}$. There exist rigidity results for linear CA on $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, p prime, and neighborhood $\mathcal{N} = \{0, 1\}$, based on a "positive entropy" criterion [39]. The techniques and results have also been extended to larger classes of so-called *bipermutative* CA. See [61] for precise statements and references.

Open problem 10. Consider the CA F on $\mathcal{A} = \mathbb{Z}/6\mathbb{Z}$ and $\mathcal{N} = \{0, 1\}$ defined by the local function $f(x, y) = (3x \bmod 6) + \lceil y/2 \rceil$. Prove that the uniform measure is the only invariant measure of F which is non-atomic and shift-invariant.

The restriction to $\mathcal{A}^{\mathbb{N}}$ of the CA F corresponds to the multiplication by 3 in base-6 notation. Open problem 10 is a restatement of the very famous $(2\times, 3\times)$ Furstenberg conjecture. For an explanation of the connection, see [61, §2F].

Randomization Next result was originally proved in [51] for the CA G on $\mathbb{Z}_2^{\mathbb{Z}}$ defined by $G(x)_i = x_{i-1} + x_{i+1}$. The version below appears in [52].

Proposition 6.11. *Let us consider the additive CA on $\mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}}$ defined by $F(x)_k = x_k + x_{k+1}$. Let μ_p be the Bernoulli product measure of parameter $p \in (0, 1)$. We have*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu_p F^i = \mu_{1/2},$$

for the weak convergence.

The additive CA F is said to *randomize in Cesàro mean* the Bernoulli measures. This result has been generalized to classes of linear CA and to larger classes of initial measures, with tools coming from stochastic processes (e.g. [24]) and harmonic analysis (e.g. [62]).

For the additive CA F, the iterates of a Bernoulli product measure $\mu_p, p \neq 1/2$, do not converge simply (without Cesáro means) to $\mu_{1/2}$. Indeed, the following scaling property holds:

$$\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}, \quad F^{2^n}(x)_k = x_k + x_{k+2^n} \bmod 2.$$

Consequently, we have $\mu_p F^{2^n}[1] = 2p(1-p)$, a quantity that does not converge to $1/2$. Similar arguments show that linear CA do not randomize (without Cesàro means), see [38, Ch. 3].

Open problem 11. Does there exist a CA randomizing all Bernoulli product measures (without Cesàro means)?

Candidates are proposed and studied experimentally in [38, Ch. 3] and [54, Ch. 5].

7. Finite set of cells

Up to now, we have considered PCA on the infinite set of cells $\mathbb{Z}^d$. Let us now consider a finite set of cells. Precisely, consider a PCA defined as in Definition 2.1 but with the set of cells $E = \mathbb{Z}/n\mathbb{Z}$ or $E = (\mathbb{Z}/n\mathbb{Z})^d$. It corresponds to an arrangement of the cells as a ring or as a d-dimensional discrete torus. Following Definition 2.3, this PCA defines a Markov chain on the *finite* state space $\mathcal{A}^E$.

(Another option is to consider a finite set of cells of the type $E = \{1, \dots, n\}^d$, but this requires to twist the PCA definition by specifying boundary conditions. The cases $E = (\mathbb{Z}/n\mathbb{Z})^d$ and $E = \{1, \dots, n\}^d$ are often referred to as, respectively, "periodic" and "open" boundary conditions. Here we discuss only the periodic case.)

Let us reconsider some of the results of Sections 3 to 6 in this new context.

Concerning the ergodicity question, the picture is completely different for a finite set of cells. For any PCA, ergodicity is decidable. Indeed, a PCA is ergodic iff the graph of its transition matrix has a single terminal strongly connected component. This follows from basic Markov chain theory, see for instance [10]. In particular, a positive-rate PCA is always ergodic.

On a finite set of cells, the study of ergodicity is often replaced by the study of the "relaxation time", that is, the convergence time to equilibrium. If the invariant measure is of the form $\delta_x, x \in \mathcal{A}^E$, then the relaxation time may simply be the maximum over all possible initial conditions of the expected convergence time to x. More generally, one may consider the evolution in k of the total variation distance between the invariant measure and the state of the Markov chain after k steps. It is standard to focus on a family of PCA depending on a parameter and to study the influence of the parameter on the relaxation time. In some cases, one observes a drastic change between "slow" and "fast" relaxation times as the parameter varies. See for instance [60,23].

Let us focus on combinatorial properties. Let the set of cells be $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. A *Bernoulli product measure* on $\mathcal{A}^{\mathbb{Z}_n}$ is defined in exactly the same way as on $\mathcal{A}^{\mathbb{Z}}$. To define a Markovian measure on $\mathcal{A}^{\mathbb{Z}_n}$, one needs to be more careful: the measure has to be consistent with the cyclic structure of the cells. Let Q be a stochastic matrix of dimension $\mathcal{A} \times \mathcal{A}$. The *(cyclic) Markovian measure* on $\mathcal{A}^{\mathbb{Z}_n}$ associated with Q is the measure ν_Q defined by, $\forall x = (x_0, \dots, x_{n-1}) \in \mathcal{A}^{\mathbb{Z}_n}$,

$$\nu_Q(x) = Z^{-1} \prod_{i=0}^{n-1} Q_{x_i, x_{i+1[n]}}, \quad Z = \sum_x \prod_{i=0}^{n-1} Q_{x_i, x_{i+1[n]}}.$$

The complexity lies in evaluating the normalizing constant Z. Next result is proved in [9].

Theorem 7.1. *The exact analog of Theorem 4.8 holds on the set of cells $\mathbb{Z}_n$.*

The analog of Theorem 7.1 for a larger alphabet is treated in [13]. On the other hand, there is no analog of Theorem 4.3: if (9) or (10) is satisfied, then the invariant measure is not $\mathcal{B}_p^{\otimes \mathbb{Z}_n}$ in general.

Many specific models of PCA on a finite set of cells have been studied; for instance, the directed animals PCA and the TASEP PCA of Sections 4.2 and 4.3. In both cases, the invariant distribution can be made explicit and has a Markovian structure. This is an application of Theorem 7.1 for the directed animals and this is proved in [66] for the TASEP.

The eroder and the fault-tolerance properties are irrelevant for finite sets of cells. On the other hand, the density classification property is relevant and has been widely studied on $E = \mathbb{Z}_n$. Here the goal is to design a CA or PCA whose trajectories converge to 0^E or to 1^E if the initial configuration contains more 0's or more 1's, respectively (the equality case is not specified). Perfect classification is impossible, that is, there exist no given CA or PCA that solves the density classification problem for all values of n, see [47,12]. The quest for the best – although imperfect – models has mobilized a large amount of research efforts both for CA and for PCA, see [25,22].

Let us focus on a last aspect connecting finite and infinite sets of cells. Consider a given local function, and the corresponding sequence of PCA F_n with set of cells $E = (\mathbb{Z}_n)^d$ for increasing values of n. Let F be the PCA with the same local function on the infinite set of cells $E = \mathbb{Z}^d$. Can we relate the behaviors of $(F_n)_n$ with the one of F? In particular, is the (non-)ergodicity of F somehow reflected by the behaviors of F_n?

First, we have the lemma below which is easy to check.

Lemma 7.2. *If π_n is an invariant measure of F_n, viewed as a periodic measure on $\mathcal{A}^{\mathbb{Z}}$, then, any accumulation point of $(\pi_n)_n$ (for the weak convergence) is an invariant measure of F.*

For instance, this provides an alternative proof of Theorem 4.8 as a corollary of Theorem 7.1. Lemma 7.2 is also exploited in the directed animals context in [1].

Second, several experimental studies support the following idea: ergodicity and fast relaxation time for F_n should imply ergodicity for F. A recent theoretical confirmation appears in [70] where the following is proved. Let p_* be the parameter threshold between ergodicity and non-ergodicity for the Stavskaya PCA on $\mathbb{Z}$, see Proposition 3.4. On $E = \mathbb{Z}_n$, let the "relaxation time" be the expected time to reach δ_{0^E} starting from 1^E. If $p < p_*$, the relaxation time is logarithmic in n, and if $p > p_*$, the relaxation time is exponential in n.

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