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A non-ergodic probabilistic cellular automaton with a unique invariant measure

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Abstract

We exhibit a Probabilistic Cellular Automaton (PCA) on $\{0, 1\}^{\mathbb{Z}}$ with a neighborhood of size 2 which is non-ergodic although it has a unique invariant measure. This answers by the negative an old open question on whether uniqueness of the invariant measure implies ergodicity for a PCA.

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1. Introduction

Consider a random process on $\Sigma^{\mathbb{Z}^d}$, where Σ is a finite set, with local interactions and a translation invariant dynamic. There are two natural instantiations, one with asynchronous updates of the sites of $\mathbb{Z}^d$, and one with synchronous updates. In the first case, the model is a continuous time Markov process known as a finite range *Interacting Particle System (IPS)*. In the second case, the model is a discrete time Markov chain known as a *Probabilistic Cellular Automaton (PCA)*.

The relevance of IPS in statistical mechanics, as well as in many other contexts, is well established. Let us mention a couple of motivations for studying PCA. First, the investigation of fault-tolerant computational models was the motivation for the Russian school [13,5]. Second, PCA appear in combinatorial problems related to the enumeration of directed animals [7]. Third,

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in the context of the classification of (deterministic) cellular automata (Wolfram's program), robustness to random errors can be used as a discriminating criterion [4].

For IPS and PCA, the first question is to study the equilibrium behavior. An equilibrium is characterized by an *invariant measure*, that is a probability measure on the state space which is left invariant by the dynamic. An invariant measure μ is *attractive* if, for any initial condition, the state of the system converges (weakly) to μ as time goes on.

By a compactness argument, there always exists at least one invariant measure. Therefore, there are, a priori, three possible situations:

(1) several invariant measures;
(2) a unique invariant measure which is not attractive;
(3) a unique invariant measure which is attractive.

In the last case, which corresponds to the nicest possible situation, the model is said to be *ergodic*. Roughly, an ergodic system completely forgets about its initial condition, while a non-ergodic one remembers something forever.

A classical foundational question is whether the intermediate case exists. In other words, does uniqueness of the invariant measure imply convergence to it? For monotone systems, the intermediate case does not exist. But in general, the question is open.

For IPS, this question is *Open Problem* 4 in Chapter 1 of the classical textbook by Liggett [8]. In [10], Mountford proves that the intermediate case does not exist for 1-dimensional IPS (that is $d = 1$). Quoting [10], "it seems more than plausible that the conclusion $(\dots)$ is true in higher dimensions". However, the question remains unsettled. For PCA, the same question is *Unsolved problem* 3.4.3 in [11], or *Unsolved problem 5.7* in [12].

In the present paper, we answer the question for PCA by exhibiting a 1-dimensional PCA, model A, corresponding to the intermediate case (Theorem 3.1). There is a unique invariant measure of the form $(\mu_0 + \mu_1)/2$ and the PCA maps μ_0 to μ_1 and μ_1 to μ_0. Starting from an initial measure μ_0, the probability measure of the state of the system is μ_0 at even times and μ_1 at odd times. Therefore there is no convergence.

Observe that the situations for IPS and PCA are different: in 1-d, the intermediate case exists for PCA, and not for IPS. This is consistent with the situation for Markov processes on a finite state space: in discrete time, periodic phenomena may occur which result in the existence of the intermediate case; in continuous time, the intermediate case does not exist.

To prove the result for model A, we introduce two auxiliary PCA. The first one, model B, corresponds to independently moving particles annihilating when they meet ($p + p \to \emptyset$). The second one, model C, corresponds to independently moving particles merging when they meet ($p + p \to p$). We compute exactly the evolution of the one-dimensional marginals for model C (Theorem 5.2) and models A and B (Proposition 6.1) starting from a "full" configuration. In particular, it proves that the speed of convergence to the invariant measure is of order $1/\sqrt{n}$ for the three models.

Continuous time versions of models B and C have been studied in the IPS literature under the names of *annihilating random walks* and *coalescing random walks*, respectively, see [1,3,6]. The PCA and IPS versions of B and C share the same features: ergodicity with the invariant measure being the "all empty" Dirac measure, and with similar and subexponential speed of convergence. In the IPS setting, the asymptotic speed of convergence was given by Bramson and Griffeath [3] for model C, and by Arratia [1] for model B. Also, the coupling between the models B and C, that we use in Section 6, already appears in [6, Ch. 3, Sec. 5] and in [1] in the continuous time setting. The novelty is that we get exact computations for the PCA models, as opposed to asymptotic results for the IPS ones. At last, let us mention that IPS versions of models B and C on a *finite* set of sites have also been studied, see for instance [2] for B, [9] for C, and the references therein.

2. Probabilistic cellular automaton

Let Σ be a finite set. Denote by $\mathcal{M}(\Sigma)$ the set of probability measures on Σ. Let us equip $X = \Sigma^{\mathbb{Z}}$ with the product topology. Denote by $\mathcal{M}(X)$ the set of probability measures on X for the Borelian σ-algebra. Weak convergence of $(\mu_n)_n$ to μ is denoted by $\mu_n \xrightarrow{w} \mu$. Let K be a finite subset of $\mathbb{Z}$ and consider $x \in \Sigma^K$. The *cylinder* defined by x is the set

$$*x* = \left\{u \in \Sigma^{\mathbb{Z}},\ \forall k \in K, u_k = x_k\right\}.$$

Given $k \in \mathbb{Z}$ and $V = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, we use the notation $k + V$ for $(k + v_1, \ldots, k + v_n)$, and the notation $V(K)$ for $\{i \mid \exists k \in K, \exists v \in V, i = k + v\}$.

Let us introduce probabilistic cellular automata, restricting ourselves to 1-dimensional models.

Definition 2.1. The *alphabet* is a finite set Σ; the set of *sites* is $\mathbb{Z}$. The set of *configurations* is $X = \Sigma^{\mathbb{Z}}$. Given $V \in \mathbb{Z}^n$, a *transition function* of *neighborhood* V is a function $f : \Sigma^V \to \mathcal{M}(\Sigma)$. The *probabilistic cellular automaton* (PCA) F of transition function f is the application $\mathcal{M}(X) \to \mathcal{M}(X)$, $\mu \mapsto \mu F$ defined on cylinders by: $\forall K, \forall y \in \Sigma^K$,

$$\mu F(*y*) = \sum_{x \in \Sigma^{V(K)}} \mu(*x*) \prod_{k \in K} f((x_i)_{i \in k+V})(y_k).$$

Let us look at how F acts on a Dirac measure δ_x. The value of all the sites are updated. The value x_k of the k-th site is changed into the letter $a \in \Sigma$ with probability $f((x_i)_{i\in k+V})(a)$, *independently* of the evolution of the other sites.

By specializing Definition 2.1, we recover two famous models:

- Assume that $V = \{0\}$, then all the sites behave independently. The restriction of the PCA to one site is a Markov chain evolving on Σ. Conversely, any Markov chain on a finite state space E can be realized as (a restriction of) a PCA on the alphabet E with neighborhood $V = \{0\}$.
- Assume that the transition function f is such that: $\forall u \in \Sigma^V$, $f(u)$ is a Dirac probability measure. Then we may view f as a function $\Sigma^V \to \Sigma$. We obtain a (deterministic) *cellular automaton*.

A PCA F may be viewed as a Markov chain on the state space $\Sigma^{\mathbb{Z}}$. Thus we borrow the classical terminology of Markov chains.

Definition 2.2. An *invariant (probability) measure* of F is a probability measure $\mu \in \mathcal{M}(X)$ such that $\mu F = \mu$. The PCA F is *ergodic* if it has a unique invariant measure which is attractive, i.e. if

$$\text{(i) } \left[\exists! \mu \in \mathcal{M}(X), \mu F = \mu\right], \qquad \text{(ii) } \left[\forall \nu \in \mathcal{M}(X), \nu F^n \xrightarrow{w} \mu\right]. \tag{1}$$

Consider for a moment a Markov chain on a finite state space with transition matrix P. Let $\mathcal{G}(P)$ be the graph of the matrix P. Classically, we have

$$\begin{aligned} &\text{(i)} \iff \mathcal{G}(P) \text{ has a unique terminal component} \\ &\text{(i)} + \text{(ii)} \iff \mathcal{G}(P) \text{ has a unique terminal component which is aperiodic.} \end{aligned} \tag{2}$$

Table 1
Finite Markov chain versus “neighborhood 0 PCA”.

Markov chain P	PCA F
$\neg$(i)	$\neg$(i)
(i), $\neg$(ii)	$\neg$(i)
(i), (ii)	(i), (ii)

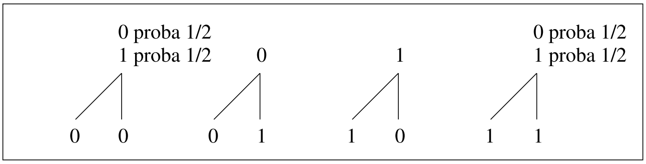

Fig. 1. The transition function of the PCA F_A.

In particular, uniqueness of the invariant measure does not imply ergodicity. The simplest example of a non-ergodic Markov chain with a unique invariant measure is the following: the state space is $X = \{0, 1\}$ and the transition matrix is

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

The unique invariant measure is $\mu = (\delta_0 + \delta_1)/2$ and for $\nu = \delta_0$, we do not have $\nu P^n \xrightarrow{w} \mu$.

For PCA, it was an open question to know if (i) implies (ii) in (1). The purpose of the present paper is to settle the question by proposing a non-ergodic PCA with a unique invariant measure.

To get a hint of the difficulty, consider for instance a PCA F with neighborhood $V = \{0\}$. Recall that each site behaves independently and as a finite Markov chain P. As recalled in (2), P may satisfy either [$\neg$(i)], [(i), $\neg$(ii)], or [(i), (ii)]. We show in Table 1 how this gets reflected on the PCA F.

Let us justify the Table. If μ is an invariant measure of P, then the product measure $\mu^{\otimes\mathbb{Z}}$ is an invariant measure of F. Therefore, if P has several invariant measures, the same holds for F. Assume now that P is ergodic with unique invariant measure μ. One proves easily that $\mu^{\otimes\mathbb{Z}}$ is attractive, so F is ergodic. Let us concentrate now on the intermediate case for P. If P satisfies [(i), $\neg$(ii)] then $\mathcal{G}(P)$ has a unique terminal component which is periodic, say of period 2. Let $(\mu_0 + \mu_1)/2$ be the unique invariant measure of P. Then F has an infinite number of invariant measures. Indeed, consider any $(u_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, and let $(v_i)_{i\in\mathbb{Z}}$ be defined by $v_i = 1 - u_i$ for all i. Then the probability measure $(\otimes_{i\in\mathbb{Z}}\,\mu_{u_i} + \otimes_{i\in\mathbb{Z}}\,\mu_{v_i})/2$ is clearly an invariant measure of F.

3. Statement of the main result

3.1. Model A

Consider the PCA F_A (Fig. 1) on the alphabet $\Sigma = \{0, 1\}$, with neighborhood $V = \{-1, 0\}$, and transition function a defined by:

$$a(00)(1) = 1/2, \qquad a(01)(1) = 0, \qquad a(10)(1) = 1, \qquad a(11)(1) = 1/2.$$

A realization of the Markov chain is obtained as follows. Consider the function

$$\begin{aligned} \mathcal{A} : \{0,1\}^{\mathbb{Z}} \times \mathcal{U}^{\mathbb{Z}} &\to \{0,1\}^{\mathbb{Z}} \\ (x_i)_{i\in\mathbb{Z}}, (u_i)_{i\in\mathbb{Z}} &\mapsto (\tilde{x}_i)_{i\in\mathbb{Z}}, \end{aligned} \tag{4}$$

with $\mathcal{U} = \{\uparrow, \rightarrow\}$, and

$$\tilde{x}_i = \begin{cases} 0 & \text{if } x_{i-1}x_i = 01 \text{ or } (x_{i-1}x_i, u_i) \in \{(00, \rightarrow), (11, \uparrow)\} \\ 1 & \text{if } x_{i-1}x_i = 10 \text{ or } (x_{i-1}x_i, u_i) \in \{(00, \uparrow), (11, \rightarrow)\}. \end{cases}$$

Let $U = (U_{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{N}}$ be a doubly indexed sequence of i.i.d. r.v.'s with common law

$$\mathbb{P}\left(U_{i,j} = \uparrow\right) = \mathbb{P}\left(U_{i,j} = \rightarrow\right) = 1/2,$$

called the *update process*. Set $U_n = (U_{i,n})_{i\in\mathbb{Z}}$. Given a $\{0,1\}^{\mathbb{Z}}$-valued r.v. $X_0 = (X_{i,0})_{i\in\mathbb{Z}}$, such that $U \perp X_0$, define the sequence of $\{0,1\}^{\mathbb{Z}}$-valued r.v.'s $(X_n)_{n\in\mathbb{N}}$ as follows:

$$X_{n+1} = \mathcal{A}(X_n, U_n). \tag{5}$$

Then $(X_n)_{n\in\mathbb{N}}$ is a realization of model A. The process U is used to randomly update the value of a site, when needed, with $\rightarrow$ being interpreted as "keep" and $\uparrow$ as "switch", and $X_{i,n}$ is the state of site i at time n, so that $X_n = (X_{i,n})_{i\in\mathbb{Z}}$ denotes the state of the system at time n.

3.2. Invariant measure

Let $x = (01)^{\mathbb{Z}}$ be the configuration defined by: $\forall n \in \mathbb{Z}, x_{2n} = 0, x_{2n+1} = 1$. The configuration $(10)^{\mathbb{Z}}$ is defined similarly.

Theorem 3.1. *The PCA F_A has a unique invariant measure which is $\mu = (\delta_{(01)^{\mathbb{Z}}} + \delta_{(10)^{\mathbb{Z}}})/2$. The PCA is non-ergodic.*

On configurations without 00 and 11, the PCA acts as the translation shift. Therefore $\mu = (\delta_{(01)^{\mathbb{Z}}} + \delta_{(10)^{\mathbb{Z}}})/2$ is an invariant measure. Assume that it is the unique one. Then μ is non-attractive, the situation being the same as for (3): consider $\nu = \delta_{(01)^{\mathbb{Z}}}$, then $\nu F_A^n = \delta_{(01)^{\mathbb{Z}}}$ if n is even, and $\nu F_A^n = \delta_{(10)^{\mathbb{Z}}}$ if n is odd.

The purpose of Sections 4 and 5 is to prove Theorem 3.1.

4. Two auxiliary models

We now define two new PCA, that we call respectively *model B* and *model C*. For both models, the alphabet is $\Sigma = \{\circ, \bullet\}$ and the set of sites is $\mathbb{Z}$. Given a configuration $u \in \{\circ, \bullet\}^{\mathbb{Z}}$, the following interpretation holds: if $u_i = \circ$, the site i is "empty"; if $u_i = \bullet$, the site i contains a "particle". At a given time step, a particle decides (independently of the others and independently of the past) to remain at its site with probability $1/2$, or to jump to the site on the right with probability $1/2$. In model B, if two particles collide, then they annihilate. In model C, if two particles collide, they are merged into one particle. Let us define the models more formally.

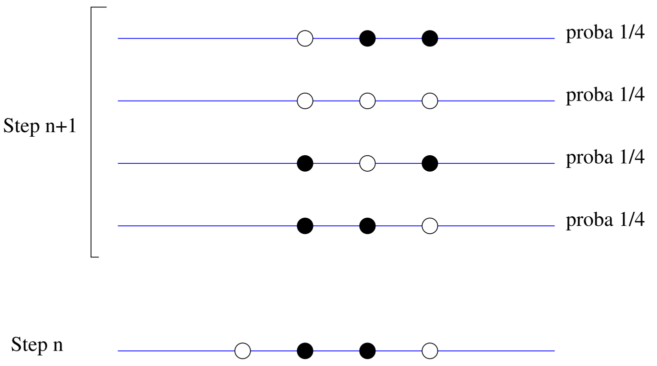

Fig. 2. The transition function of model B.

4.1. Model B

It is the Markov chain on $\{\circ, \bullet\}^{\mathbb{Z}}$ defined as follows. Consider the function

$$\begin{aligned} \mathcal{B} : \{\circ, \bullet\}^{\mathbb{Z}} \times \mathcal{U}^{\mathbb{Z}} &\to \{\circ, \bullet\}^{\mathbb{Z}} \\ (y_i)_{i \in \mathbb{Z}}, (u_i)_{i \in \mathbb{Z}} &\mapsto (\tilde{y}_i)_{i \in \mathbb{Z}}, \end{aligned} \tag{6}$$

with $\mathcal{U} = \{\uparrow, \rightarrow\}$, and

$$\tilde{y}_i = \begin{cases} \bullet & \text{if } (y_{i-1} y_i, u_{i-1} u_i) \in \{(\bullet\circ, \rightarrow \mathcal{U}), (\circ\bullet, \mathcal{U} \uparrow), (\bullet\bullet, \uparrow\uparrow), (\bullet\bullet, \rightarrow\rightarrow)\} \\ \circ & \text{otherwise.} \end{cases}$$

Let U be an update process, defined as in Section 3.1. Given a $\{\circ, \bullet\}^{\mathbb{Z}}$-valued r.v. Y_0, such that $U \perp Y_0$, define the sequence of $\{\circ, \bullet\}^{\mathbb{Z}}$-valued r.v.'s $(Y_n)_{n \in \mathbb{N}}$ as follows:

$$Y_{n+1} = \mathcal{B}(Y_n, U_n). \tag{7}$$

Then $(Y_n)_{n \in \mathbb{N}}$ is a realization of model B (Fig. 2).

Remarks. In the above presentation, model B is a Markov chain with synchronous updates and local interactions, but not *stricto sensu* a PCA. Indeed, if Y_0 is deterministic, then the r.v.'s $Y_{i,1}$ and $Y_{i+1,1}$ are not independent, since they are updated using the non-disjoint r.v.'s $\{U_{i-1,0}, U_{i,0}\}$ and $\{U_{i,0}, U_{i+1,0}\}$. However, it is possible to give a PCA presentation of model B on a larger alphabet. Define the sequence of $\left(\{\circ, \bullet\} \times \mathcal{U}\right)^{\mathbb{Z}}$-valued r.v's $(\widetilde{Y}_n)_{n \in \mathbb{N}}$ by $\widetilde{Y}_n = (Y_n, U_n)$. We have:

$$(\widetilde{Y}_{n+1})_i = \left(\mathcal{B}(\widetilde{Y}_n)_i, U_{i,n+1}\right).$$

Thus $(\widetilde{Y}_n)_n$ is a realization of a PCA on the alphabet $\{\circ, \bullet\} \times \mathcal{U}$, with neighborhood $V = \{-1, 0\}$. The same remark holds for model C below.

The continuous time version of model B, with exponential holding times, is called an *annihilating random walk* (cf. [6, Ch. 3, Sec. 5]).

4.2. Model C

It is the Markov chain on $\{\circ, \bullet\}^{\mathbb{Z}}$ defined as follows. Consider the function

$$\begin{aligned} \mathcal{C} : \{\circ, \bullet\}^{\mathbb{Z}} \times \mathcal{U}^{\mathbb{Z}} &\to \{\circ, \bullet\}^{\mathbb{Z}} \\ (z_i)_{i \in \mathbb{Z}}, (u_i)_{i \in \mathbb{Z}} &\mapsto (\tilde{z}_i)_{i \in \mathbb{Z}}, \end{aligned}$$

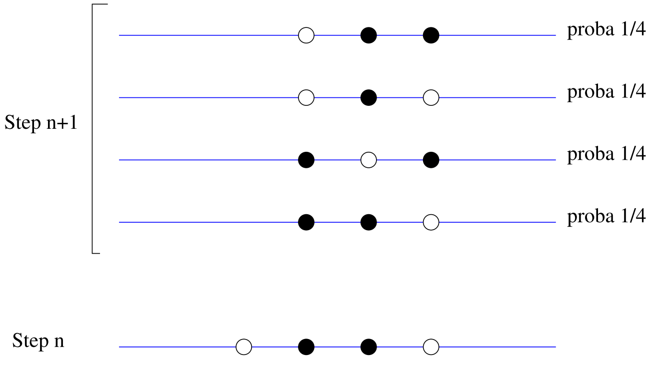

Fig. 3. The transition function of model C.

with

$$\tilde{z}_i = \begin{cases} \bullet & \text{if } (z_{i-1}z_i, u_{i-1}u_i) \in \{(\bullet\circ, \rightarrow \mathcal{U}), (\circ\bullet, \mathcal{U}\uparrow), (\bullet\bullet, \uparrow\uparrow), (\bullet\bullet, \rightarrow \mathcal{U})\} \\ \circ & \text{otherwise.} \end{cases}$$

Let U be an update process. Given a $\{\circ, \bullet\}^{\mathbb{Z}}$-valued r.v. Z_0, such that $U \perp Z_0$, define the sequence of $\{\circ, \bullet\}^{\mathbb{Z}}$-valued r.v's $(Z_n)_{n\in\mathbb{N}}$ as follows:

$$Z_{n+1} = \mathcal{C}(Z_n, U_n).$$

Then $(Z_n)_{n\in\mathbb{N}}$ is a realization of model C (Fig. 3).

The continuous time version of model C with exponential holding times, is called a *coalescing random walk* (cf. [6, Ch. 2, Sec. 9]).

4.3. Links between models A, B, and C

One-step transition of the model B, resp. C, defines the mapping

$$\begin{aligned} F_B : \mathcal{M}(\{\circ, \bullet\}^{\mathbb{Z}}) &\longrightarrow \mathcal{M}(\{\circ, \bullet\}^{\mathbb{Z}}) \\ \mu &\longmapsto \mu F_B, \end{aligned}$$

respectively,

$$\begin{aligned} F_C : \mathcal{M}(\{\circ, \bullet\}^{\mathbb{Z}}) &\longrightarrow \mathcal{M}(\{\circ, \bullet\}^{\mathbb{Z}}) \\ \mu &\longmapsto \mu F_C. \end{aligned}$$

Define

$$\begin{aligned} \varphi : \{0, 1\}^{\mathbb{Z}} &\longrightarrow \{\circ, \bullet\}^{\mathbb{Z}} \\ (x_i)_{i\in\mathbb{Z}} &\longmapsto (y_i)_{i\in\mathbb{Z}}, \end{aligned}$$

with

$$y_i = \begin{cases} \bullet & \text{if } x_i x_{i+1} \in \{00, 11\} \\ \circ & \text{if } x_i x_{i+1} \in \{01, 10\}. \end{cases}$$

By extension, define $\varphi : \mathcal{M}(\{0, 1\}^{\mathbb{Z}}) \rightarrow \mathcal{M}(\{\circ, \bullet\}^{\mathbb{Z}})$.

Lemma 4.1. *The diagram below is commutative:*

$$\begin{array}{ccc} \mathcal{M}(\{0,1\}^{\mathbb{Z}}) & \xrightarrow{F_A} & \mathcal{M}(\{0,1\}^{\mathbb{Z}}) \\ \downarrow \varphi & & \downarrow \varphi \\ \mathcal{M}(\{\circ,\bullet\}^{\mathbb{Z}}) & \xrightarrow{F_B} & \mathcal{M}(\{\circ,\bullet\}^{\mathbb{Z}}) \end{array}$$

If $(X_n)_{n\in\mathbb{N}}$ is a realization of the Markov chain A, then $(\varphi(X_n))_{n\in\mathbb{N}}$ is a realization of the Markov chain B.

Proof. Recall that $\mathcal{A}$ and $\mathcal{B}$ are defined in (4) and (6) respectively. We are going to prove that:

$$\varphi \circ \mathcal{A} = \mathcal{B} \circ (\varphi, \mathrm{Id}). \tag{8}$$

The statement of the lemma follows. Set

$$(x_i)_i, (u_i)_i \xmapsto{\mathcal{A}} (\tilde{x}_i)_i \xmapsto{\varphi} (\tilde{y}_i)_i, \qquad (x_i)_i, (u_i)_i \xmapsto{\varphi,\mathrm{Id}} (y_i)_i, (u_i)_i \xmapsto{\mathcal{B}} (\hat{y}_i)_i.$$

To obtain (8), it is enough to check that $\tilde{y}_0 = \hat{y}_0$. This is done by systematic inspection in Table 2. Each one of the 32 cases mimics the commutative diagram: in the first line, from left to right, (x_{-2}, x_{-1}, x_0), (u_{-1}, u_0), and $(\tilde{x}_{-1}, \tilde{x}_0)$; in the second line, from left to right, (y_{-1}, y_0), (u_{-1}, u_0), and $\tilde{y}_0 = \hat{y}_0$.

If the process X is defined by (5), relation (8) entails that the process Y, defined by $Y_n = \varphi(X_n)$, satisfies relation (7). □

Lemma 4.2. *Model B is dominated by model C: for $x, u \in \{\circ, \bullet\}^{\mathbb{Z}} \times \mathcal{U}^{\mathbb{Z}}$,*

$$\mathcal{B}(x, u) \leq \mathcal{C}(x, u),$$

where $\leq$ is the coordinate-wise product ordering on $\{\circ, \bullet\}^{\mathbb{Z}}$, with $\circ \leq \bullet$.

Proof. This can be checked directly on the definitions of $\mathcal{B}$ and $\mathcal{C}$. Intuitively, particles are merged in model C, and annihilate in model B. □

Lemma 4.3. *The following implications hold:*

$$\begin{aligned} &[C \text{ is ergodic with invariant measure } \delta_{\circ^{\mathbb{Z}}}] \\ &\quad \Longrightarrow [B \text{ is ergodic with invariant measure } \delta_{\circ^{\mathbb{Z}}}] \\ &\quad \Longleftrightarrow [A \text{ is non-ergodic with invariant measure } (\delta_{(01)^{\mathbb{Z}}} + \delta_{(10)^{\mathbb{Z}}})/2]. \end{aligned}$$

Proof. This is a direct consequence of Lemmas 4.1 and 4.2. □

Therefore, in order to prove Theorem 3.1, it is sufficient to prove that model C is ergodic with invariant measure $\delta_{\circ^{\mathbb{Z}}}$. This is the purpose of next section.

5. Model *C* is ergodic

Lemma 5.1. *Model C is monotone, that is: for $z \in \{\circ, \bullet\}^{\mathbb{Z}}$, $\tilde{z} \in \{\circ, \bullet\}^{\mathbb{Z}}$, $u \in \mathcal{U}^{\mathbb{Z}}$,*

$$z \leq \tilde{z} \Longrightarrow \mathcal{C}(z, u) \leq \mathcal{C}(\tilde{z}, u),$$

where $\leq$ is the coordinate-wise product ordering.

Table 2
The 32 possible cases.

111 or 000	$\rightarrow\rightarrow$	11 or 00	101 or 010	$\rightarrow\rightarrow$	10 or 01
$\bullet\bullet$	$\rightarrow\rightarrow$	$\bullet$	$\circ\circ$	$\rightarrow\rightarrow$	$\circ$
111 or 000	$\rightarrow\uparrow$	10 or 01	101 or 010	$\rightarrow\uparrow$	10 or 01
$\bullet\bullet$	$\rightarrow\uparrow$	$\circ$	$\circ\circ$	$\rightarrow\uparrow$	$\circ$
111 or 000	$\uparrow\rightarrow$	01 or 10	101 or 010	$\uparrow\rightarrow$	10 or 01
$\bullet\bullet$	$\uparrow\rightarrow$	$\circ$	$\circ\circ$	$\uparrow\rightarrow$	$\circ$
111 or 000	$\uparrow\uparrow$	00 or 11	101 or 010	$\uparrow\uparrow$	10 or 01
$\bullet\bullet$	$\uparrow\uparrow$	$\bullet$	$\circ\circ$	$\uparrow\uparrow$	$\circ$
110 or 001	$\rightarrow\rightarrow$	11 or 00	100 or 011	$\rightarrow\rightarrow$	10 or 01
$\bullet\circ$	$\rightarrow\rightarrow$	$\bullet$	$\circ\bullet$	$\rightarrow\rightarrow$	$\circ$
110 or 001	$\rightarrow\uparrow$	11 or 00	100 or 011	$\rightarrow\uparrow$	11 or 00
$\bullet\circ$	$\rightarrow\uparrow$	$\bullet$	$\circ\bullet$	$\rightarrow\uparrow$	$\bullet$
110 or 001	$\uparrow\rightarrow$	01 or 10	100 or 011	$\uparrow\rightarrow$	10 or 01
$\bullet\circ$	$\uparrow\rightarrow$	$\circ$	$\circ\bullet$	$\uparrow\rightarrow$	$\circ$
110 or 001	$\uparrow\uparrow$	01 or 10	100 or 011	$\uparrow\uparrow$	11 or 00
$\bullet\circ$	$\uparrow\uparrow$	$\circ$	$\circ\bullet$	$\uparrow\uparrow$	$\bullet$

Proof. It can be checked directly on the definition of $\mathcal{C}$. $\square$

With this monotonicity, to get the ergodicity, it is enough to prove that $\delta_{\bullet\mathbb{Z}} F_C^n \to \delta_{\circ\mathbb{Z}}$. Indeed, consider two realizations of model C, one, say $Z = (Z_n)_n$, that starts with all sites occupied, the other, say $\tilde{Z} = (\tilde{Z}_n)_n$, that starts with an arbitrary initial condition, their evolution using the same update process U. According to Lemma 5.1, at any time $n \in \mathbb{N}$, $\tilde{Z}_n \leq Z_n$.

From now on, we focus on the process Z. Recall that for each n, $Z_n = (Z_{k,n})_{k\in\mathbb{Z}}$ is the state of the system at time n. The process Z_n is stationary, i.e. invariant by translation, since Z_0, U, and $\mathcal{C}$ are invariant too. Define

$$d_n = \mathbb{P}\left(Z_{k,n} = \bullet\right) = \mathbb{P}\left(Z_{0,n} = \bullet\right). \tag{9}$$

This is the density of particles at time n. The density d_n can also be viewed as an evaluation of the distance between Z_n and $\delta_{\circ\mathbb{Z}}$. Indeed, for any finite subset E of $\mathbb{Z}$, consider the Hamming distance on $\{\circ, \bullet\}^E$, and denote by $\mathcal{W}_H$ the corresponding Wasserstein distance on $\mathcal{M}(\{\circ, \bullet\}^E)$. Setting $Z_{E,n} = (Z_{k,n})_{k\in E}$, we have: $\mathcal{W}_H\left(Z_{E,n}, \delta_{\circ E}\right) = |E| d_n$.

Theorem 5.2. *Let T be the time that a simple symmetric random walk on $\mathbb{Z}$ needs to reach 2, starting from 0. We have*

$$d_n = \mathbb{P}(T > 2n) \tag{10}$$

$$= 4^{-n}\binom{2n+1}{n}. \tag{11}$$

In particular, $d_n \sim 2/\sqrt{\pi n}$, hence converges to 0 as n grows.

In continuous time, when the particles perform a simple symmetric random walk, Bramson and Griffeath [3] obtain the same asymptotic behavior for d_n, up to a scaling factor, as expected.

Corollary 5.3. *Model C is ergodic with unique invariant measure $\delta_{\circ\mathbb{Z}}$.*

Proof. We first prove (11), assuming (10). Let $S = (S_k)_{k\in\mathbb{N}}$ be a realization of the simple symmetric random walk on $\mathbb{Z}$, starting from 0. Define $M_k = \max\{S_i, 0 \leq i \leq k\}$, the maximum of the random walk at time k. Recall that $T = \inf\{i \geq 0 \mid S_i = 2\}$. We have

$$\begin{aligned}\mathbb{P}(T > 2n) &= \mathbb{P}(M_{2n} \leq 1) = 1 - \mathbb{P}(M_{2n} \geq 2)\\ &= 1 - \sum_{\ell \in \mathbb{Z}} \mathbb{P}(M_{2n} \geq 2,\ S_{2n} = \ell)\\ &= 1 - \mathbb{P}(S_{2n} \geq 2) - \sum_{\ell \leq 1} \mathbb{P}(M_{2n} \geq 2, S_{2n} = \ell).\end{aligned}$$

According to the reflection principle, for $\ell \leq 1, \mathbb{P}(M_{2n} \geq 2, S_{2n} = \ell) = \mathbb{P}(S_{2n} = 4 - \ell)$. Therefore,

$$\begin{aligned}\mathbb{P}(T > 2n) &= 1 - \mathbb{P}(S_{2n} \geq 2) - \mathbb{P}(S_{2n} \geq 3)\\ &= 1 - \mathbb{P}(S_{2n} \leq -2) - \mathbb{P}(S_{2n} \geq 3)\\ &= \mathbb{P}(S_{2n} \in \{0, 2\})\\ &= 4^{-n}\left(\binom{2n}{n} + \binom{2n}{n+1}\right) = 4^{-n}\binom{2n+1}{n}.\end{aligned}$$

Using Stirling's formula, we get

$$4^{-n}\binom{2n+1}{n} \sim \frac{2}{\sqrt{\pi n}}.$$

Now let us prove (10). Recall that $Z = (Z_n)_{n \in \mathbb{N}}$ is a realization of model C with $Z_0 = \bullet^{\mathbb{Z}}$. One can extend the definition of Z via coupling from the past. Consider the i.i.d. r.v.'s $(U_{k,n})_{(k,n) \in \mathbb{Z} \times \mathbb{Z}}$ with $\mathbb{P}(U_{i,j} = \uparrow) = \mathbb{P}(U_{i,j} = \rightarrow) = 1/2$. For each $s \in \mathbb{Z}$, define $Z^{(s)} = (Z_n^{(s)})_{n \geq s}$ by

$$Z_s^{(s)} = \bullet^{\mathbb{Z}}, \quad \forall n \geq s, \qquad Z_{n+1}^{(s)} = \mathcal{C}(Z_n^{(s)}, (U_{k,n})_{k \in \mathbb{Z}}).$$

The starting time of $Z^{(s)}$ is s, but, besides that, the dynamic is the same as that of process Z. Observe that $Z^{(0)} = Z$. More generally, $Z^{(s)}$ has the same distribution as $(Z_{-s+n})_{n \geq s}$. Thus, we have

$$d_n = \mathbb{P}\left(Z_{0,0}^{(-n)} = \bullet\right).$$

In Fig. 4, we have represented space–time diagrams for the model. The point of coordinate (k, n) corresponds to site k at step n. We have also represented the updating variables with the following convention: at the point (k, n), there is an arrow pointing north if $U_{k,n} = \uparrow$ and an arrow pointing north-east if $U_{k,n} = \rightarrow$. This allows to visualize the evolution of particles in the processes $Z^{(s)}$. In Fig. 4(b) and (c), the processes $Z^{(-3)}$ and $Z^{(-5)}$ are represented; the gray nodes are the ones whose color depend on updating variables outside of the represented window. In Fig. 4(d), the particles painted in orange (gray) are those that merged into the particle present at time 0 and site 0.

Let $I_n, n \geq 0$, be the set of indices of particles present at time $-n$ in $Z^{(-n)}$ that merge into particle 0 at time 0. Either $I_n = [\![a_n, b_n]\!], a_n \leq b_n$, in which case $Z_{0,0}^{(-n)} = \bullet$, or $I_n = \varnothing$, in which case $Z_{0,0}^{(-n)} = \circ$. We focus on $|I_n|$. For instance, in Fig. 4(d), we have $(|I_n|)_{n \in [\![0,6]\!]} = (1, 2, 1, 2, 3, 4, 3)$. Observe that

$$d_n = \mathbb{P}\left(Z_{0,0}^{(-n)} = \bullet\right) = \mathbb{P}(|I_n| \geq 1). \tag{12}$$

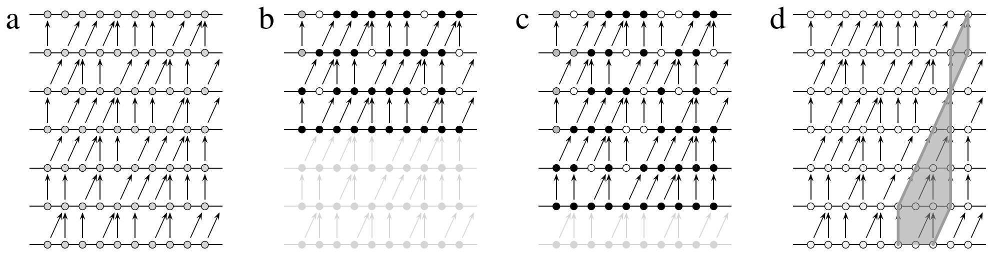

Fig. 4. (a) The updating $(U_{k,n})_{(k,n)\in[\![-9,0]\!]\times[\![-6,0]\!]}$. (b) The process $Z^{(-3)}$ during the span $[\![-3,0]\!]$. (c) The process $Z^{(-5)}$ during the span $[\![-5,0]\!]$. (d) The merging of particles.

We have $a_0 = b_0 = 0$, and $|I_0| = 1$. Define

$$\rho = \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1.$$

We check the following:

- Assume that $a_n < b_n$. Then,

$$a_{n+1} = \begin{cases} a_n - 1 & \text{if } U_{a_n-1,-n-1} = \rightarrow \\ a_n & \text{if } U_{a_n-1,-n-1} = \uparrow \end{cases}, \qquad b_{n+1} = \begin{cases} b_n - 1 & \text{if } U_{b_n,-n-1} = \rightarrow \\ b_n & \text{if } U_{b_n,-n-1} = \uparrow \end{cases}.$$

  Thus, $|I_{n+1}| - |I_n| \in \{0, \pm 1\}$, and the conditional law of $|I_{n+1}| - |I_n|$ is ρ.
- Assume that $a_n = b_n$. Then

$$I_{n+1} = \begin{cases} \varnothing & \text{if } (U_{a_n-1,-n-1}, U_{a_n,-n-1}) = (\uparrow, \rightarrow) \\ [\![a_n - 1, a_n - 1]\!] & \text{if } (U_{a_n-1,-n-1}, U_{a_n,-n-1}) = (\rightarrow, \rightarrow) \\ [\![a_n - 1, a_n]\!] & \text{if } (U_{a_n-1,-n-1}, U_{a_n,-n-1}) = (\rightarrow, \uparrow) \\ [\![a_n, a_n]\!] & \text{if } (U_{a_n-1,-n-1}, U_{a_n,-n-1}) = (\uparrow, \uparrow). \end{cases}$$

  For instance, the third case appears between lines -3 and -2 in Fig. 4(d). Here again, the conditional distribution of $|I_{n+1}| - |I_n|$ is ρ.
- If $I_n = \varnothing$, then $I_{n+1} = \varnothing$.

Consequently $(|I_n|)_{n\in\mathbb{N}}$ is a random walk with step ρ, starting from 1, and killed when it reaches 0. Using (12), we obtain (10). $\square$

6. Speed of convergence for models *A* and *B*

Let $(A_n)_{n\in\mathbb{N}}$ be a realization of model A, with $A_0 \sim \mu$, $\mu \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})$. The possible limits for weakly converging subsequences of $(A_n)_{n\in\mathbb{N}}$ are of the form $p\delta_{(01)^{\mathbb{Z}}} + (1-p)\delta_{(10)^{\mathbb{Z}}}$ for $p \in [0, 1]$. An evaluation of the distance to the limits is given by

$$\mathbb{P}\left(A_{0,n}A_{1,n} \in \{00, 11\}\right).$$

Since model A is not monotone, we do not know for which initial measure μ this distance will be maximized. Hence we evaluate the "speed of convergence" for model A by the quantity:

$$d_n^A = \max_{\mu\in\mathcal{M}(\{0,1\}^{\mathbb{Z}})} \mathbb{P}\left(A_{0,n}A_{1,n} \in \{00, 11\}\right).$$

The quantity d_n^A is also the speed of convergence to $\delta_{\circ^{\mathbb{Z}}}$ for model B. Indeed we have $\mathbb{P}\left(A_{0,n}A_{1,n} \in \{00, 11\}\right) = \mathbb{P}\left(\varphi(A_n)_0 = \bullet\right)$, which implies that

$$d_n^A = \max_{\nu \in \mathcal{M}(\{\circ,\bullet\}^{\mathbb{Z}})} \mathbb{P}\left(B_{0,n} = \bullet\right),$$

where $(B_n)_n$ denotes a realization of model B and ν denotes its initial distribution ($B_0 \sim \nu$).

Recall that $d_n = 4^{-n}\binom{2n+1}{n}$ is the speed of convergence for model C, see (9) and Theorem 5.2.

Proposition 6.1. *We have*

$$\frac{1}{2}d_{n-1} \leq d_n^A \leq d_n.$$

Let $(A_n)_n$ be a realization of model A, with $A_0 \sim \mu$. If μ is the uniform distribution on $\{0,1\}^{\mathbb{Z}}$, i.e. the r.v.'s $A_{i,0}$ are i.i.d. with $\mathbb{P}\left(A_{0,0} = 0\right) = \mathbb{P}\left(A_{0,0} = 1\right) = 1/2$, then we shall see that $\mathbb{P}\left(A_{0,n}A_{1,n} \in \{00, 11\}\right) = d_n/2$. If $\mu = \delta_{1^{\mathbb{Z}}}$, then we have $\mathbb{P}\left(A_{0,n}A_{1,n} \in \{00, 11\}\right) = d_{n-1}/2$, which is larger than $d_n/2$. The results can be translated to model B: the density of particles at step n is $d_n/2$ if the initial distribution is uniform, and it is $d_{n-1}/2$ if the initial distribution is $\delta_{\bullet^{\mathbb{Z}}}$.

The end of the section is devoted to the proof of Proposition 6.1, through the study of model A with the two initial distribution mentioned previously.

We define a new PCA, called model D, which is a coupling of models B and C. The alphabet is $\{\circ, b, g\}$ and the set of sites is $\mathbb{Z}$. Given a configuration $u \in \{\circ, b, g\}^{\mathbb{Z}}$, the interpretation is as follows: if $u_i = \circ$ then site i is empty; if $u_i = b$ then site i contains a *blue* particle; if $u_i = g$ then site i contains a *green* particle. Particles move as in models B and C. When two particles collide, they get merged into one particle as in model C. In absence of collision, particles keep their color. In case of a collision, the merged particle is colored according to the rules:

$$b + b \to g, \qquad g + g \to g, \qquad b + g \to b, \qquad g + b \to b. \tag{13}$$

We have represented a realization of model D on Fig. 5. The "question mark" nodes are the ones whose color depend on updating variables outside of the represented window. Define

$$\pi_B : \{\circ, b, g\} \longrightarrow \{\circ, \bullet\}, \qquad \pi_B(\circ) = \circ, \qquad \pi_B(b) = \bullet, \qquad \pi_B(g) = \circ,$$
$$\pi_C : \{\circ, b, g\} \longrightarrow \{\circ, \bullet\}, \qquad \pi_C(\circ) = \circ, \qquad \pi_C(b) = \bullet, \qquad \pi_C(g) = \bullet.$$

We keep the same notations for the product applications: $\pi_B : \{\circ, b, g\}^{\mathbb{Z}} \to \{\circ, \bullet\}^{\mathbb{Z}}$, $(u_i)_i \mapsto (\pi_B(u_i))_i$, and $\pi_C : \{\circ, b, g\}^{\mathbb{Z}} \to \{\circ, \bullet\}^{\mathbb{Z}}$, $(u_i)_i \mapsto (\pi_C(u_i))_i$.

Lemma 6.2. *If $(D_n)_n$ is a realization of model D, then $(\pi_B(D_n))_n$ is a realization of model B, and $(\pi_C(D_n))_n$ is a realization of model C. As consequences, model D is ergodic with unique invariant measure $\delta_{\circ^{\mathbb{Z}}}$, and $d_n^A \leq d_n$.*

Let $(D_n)_n$ be a realization of model D with D_0 being defined as follows: the r.v.'s $D_{i,0}$ are i.i.d. with $\mathbb{P}\left(D_{0,0} = b\right) = \mathbb{P}\left(D_{0,0} = g\right) = 1/2$. At step n, the colors of the remaining particles are still i.i.d. and uniform: whatever the shape of the binary tree of coalescences leading to the presence of a particle at a given position at time n (see Fig. 5 for an example), if the colors of the initial particles are independent and if one of these particle's color is uniform, then, due to (13),

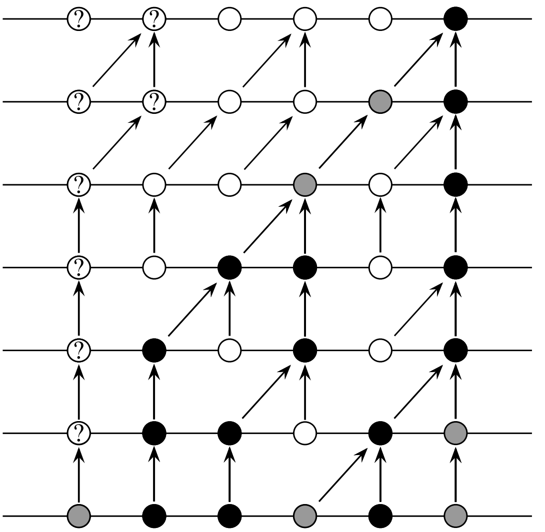

Fig. 5. A realization of process D (gray for g(reen), black for b(lue)).

the resulting color will still be uniformly distributed. Therefore we have

$$\begin{aligned}\mathbb{P}\left(\pi_B(D_{0,n}) = \bullet\right) &= \mathbb{P}\left(D_{0,n} = b\right) = \frac{1}{2}\mathbb{P}\left(D_{0,n} \in \{b, g\}\right)\\ &= \frac{1}{2}\mathbb{P}\left(\pi_C(D_{0,n}) = \bullet\right) = \frac{1}{2}d_n,\end{aligned} \tag{14}$$

where the last equality follows from Theorem 5.2.

Now let $(\widetilde{D}_n)_n$ be a realization of model D with $\widetilde{D}_0 = b^{\mathbb{Z}}$. Define $E = (E_i)_i$ by

$$E_i = \begin{cases} b & \text{if } \widetilde{D}_{i,1} = b \\ g & \text{if } \widetilde{D}_{i,1} = \circ \text{ or } g. \end{cases}$$

The r.v.'s $(E_i)_i$ are i.i.d. with $\mathbb{P}(E_0 = b) = \mathbb{P}(E_0 = g) = 1/2$. Let us justify this point. The state at time 1 of a realization of model A that starts from $0^{\mathbb{Z}}$ is uniformly distributed by definition. Hence, the state at time 1 of a realization of model B that starts from $\bullet^{\mathbb{Z}}$ is uniformly distributed. And E has the same law as the latter up to the transformation $b \leftrightarrow \bullet$, $g \leftrightarrow \circ$.

So we have $E \sim D_0$. Observe also that $\pi_B(\widetilde{D}_1) = \pi_B(E)$. We deduce that, for all $n \geq 1$, we have $\pi_B(\widetilde{D}_n) \sim \pi_B(D_{n-1})$. In particular, using (14),

$$\mathbb{P}\left(\pi_B(\widetilde{D}_{0,n}) = \bullet\right) = \mathbb{P}\left(\widetilde{D}_{0,n} = b\right) = \mathbb{P}\left(D_{0,n-1} = b\right) = d_{n-1}/2.$$

This completes the proof of Proposition 6.1.

Conclusion. The following question remains: does there exist a *positive-rates* PCA which is non-ergodic with a unique invariant measure?

Let us provide some context. By definition, a PCA has *positive-rates* if all its probability transitions are different from 0 and 1 (more formally, if $f : \Sigma^V \to \mathcal{M}(\Sigma)$ is the transition function, then $\forall u \in \Sigma^V$, $\forall v \in \Sigma$, $f(u)(v) \in (0, 1)$). It had been a long standing conjecture that all 1-dimensional positive-rates PCA are ergodic. In [5], Gács disproved the conjecture by exhibiting a complex counter-example with several invariant measures. The existence of the intermediate case (unique but non-attractive invariant measure) remains open. A priori, it is not possible to perturbate model A to get a positive rates example.

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