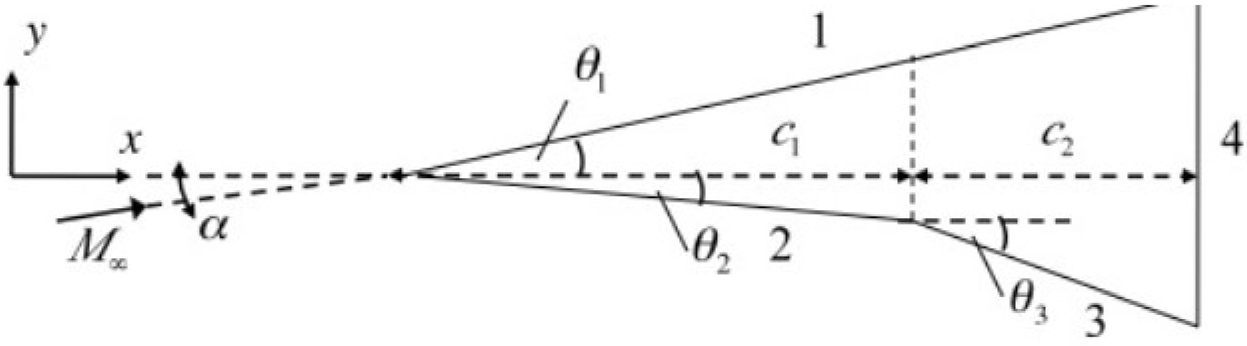
Question 1.



$\theta_1 = 15^o$	c_1 = 1.25 m	M_∞ = 25	h_{alt} = 35 km
$\theta_2 = 5^o$	c_2 = 0.75 m	$\alpha = 10^o$	T_∞ = 237 K
$\theta_3 = 25^o$			p_∞ = 559 pa

Calculate (β-δ), the static pressure (P_1), static temperature (T_1), and static density (ρ_1) on the upper surface (1) using the calorically perfect gas (CPG) assumption and the thermo-chemical equilibrium assumption to solve.

The values for flow deflection will be constants dependent on the geometry.

Face 1:

$$\delta_1 = \theta_1 - \alpha$$

Face 2:

$$\delta_2 = \theta_2 + \alpha$$

Face 3:

$$\delta_3 = \theta_3 - \theta_2$$

a) CPG assumption

$$\tan\delta = 2\cot\beta\left[\frac{M^2\sin^2\beta - 1}{M^2(\gamma+\cos 2\beta)+2}\right] \qquad (1)$$

This can be solved numerically using the Newton or Secant root finding method or by using a on-line calculator or table. For this solution an implementation of the secant method was used.

Secant Method:
For function (where n is the face number and the free stream is relative to the face n)

$$f(\beta^n) = 2\cot\beta^n\left[\frac{M_\infty^2\sin^2\beta^n - 1}{M_\infty^2(\gamma+\cos 2\beta^n)+2}\right] - \tan\delta^n \qquad (2)$$

where δ is known.

Apply the algorithm:

select initial guesses of β_1 and β_0 while tolerance not met loop $\beta_{i+1} = \beta_i - \frac{f(\beta_i)(\beta_i - \beta_{i-1})}{f(\beta_i) - f(\beta_{i-1})}$ check tolerance against $\lvert\beta_{i+1} - \beta_i\rvert$ update shock angle values for next iteration as β_i to equal β_{i+1} and β_{i-1} to equal β_i end loop and return β_{i+1}

From the resulting shock angle β, CPG relations can be used for pressure, density, and temperature ratios to obtain the static flow properties over the surface. **Note these subscripts do not correspond to the surfaces in the figure for this problem.**

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta - 1\right) \qquad (3)$$

$$\frac{T_2}{T_1} = \frac{p_2/p_1}{\rho_2/\rho_1} = \frac{\left[1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta - 1\right)\right]\left[(\gamma-1)M_1^2\sin^2\beta + 2\right]}{(\gamma+1)M_1^2\sin^2\beta} \qquad (4)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta + 2} \qquad (5)$$

CPG assumption results:

Face↓	(β-δ)	P [Pa]	T [K]	ρ [kg/m^3]
1	0.031187	5,599.3	622.31	0.031362
2	0.059923	40,660	3,103.0	0.045673
3	0.133536	433,750	8487.4	0.17813

b) Thermo-chemical equilibrium assumption

This approach requires an iterative procedure as follows:

Select initial guess for ε_1 and ε_0, usually 0.1 and 0.001 respectively

Loop while $|f(\epsilon)| > tolerance$

(1) $\epsilon = \rho_1 / \rho_2$

(2) $$\tan\beta = \frac{(1-\epsilon) - \sqrt{(1-\epsilon)^2 - 4\epsilon\tan^2\delta}}{2\epsilon\tan\delta}$$

(3) $u_{1n} = u_1 \sin\beta$

(4) $u_{2n} = \epsilon u_{1n}$

(5) $p_2 = p_1 + \rho_1 u_{1n}[1-\epsilon]$

(6) $$h_2 = h_1 + \frac{u_{1n}^2}{2}[1-\epsilon^2]$$

(7) $\tilde{h}_2 = \tilde{h}_2(\rho_{2,} p_2)$

(8) $f(\epsilon) = h_2(\epsilon) - \tilde{h}_2(\epsilon)$

(9) Apply secant method to step (8) to obtain next value for ε_{i+1}

$$\epsilon_{i+1} = \epsilon_i - \frac{f(\epsilon_i)(\epsilon_i - \epsilon_{i-1})}{f(\epsilon_i) - f(\epsilon_{i-1})}$$

The values for static pressure, and density fall out of this method directly. To obtain the static temperature the relation T = T(ρ,p) is used.

Thermo-chemical equilibrium assumption results:

Face↓	(β-δ)	P [Pa]	T [K]	ρ [kg/m^3]
1	0.030751	5,580.8	605.27	0.031689
2	0.047308	39,113	2,437.9	0.055454
3	0.070615	458,340	4,067.1	0.34981

c) Comparison of results obtained in parts (a) and (b).

Percent Difference Results:

Face↓	(β-δ) [%]	P [%]	T [%]	ρ [%]
1	1.41813	0.33217	2.81376	1.03142
2	26.66596	3.95533	27.28175	17.63855
3	89.10380	5.36565	108.68590	49.07832

From these results it can be seen that the pressure is the least effect by the CPG assumption vs the equilibrium assumption. The temperature was most effected by the CPG assumption. Both of these results make sense based on the assumptions made for each method. For equilibrium the vibrational mode is being considered. Energy that would have been seen in the translational mode (measured by temperature) is now being used in the vibration of the molecules. Pressure however is only dependent of the number of collisions on the surface, which is not effected by energy but rather momentum. Finally a correlation can be drawn between the errors seen by the shock layer thickness (β-δ) and the density. In hypersonic flow the shock layer will be smaller using the Equilibrium approach as opposed to using the CPG approach. With a smaller shock layer there smaller volume between the surface and the shock boundary. As a result, there will be more compression of the mass behind the shock, causing a higher density.

d) Calculate the compressibility factor (z) and the molecular weight (M) of the mixtures on each face.

For air R_0 = 286.9 [J/kg K] and M_0 = 28.97 [kg/kg-mol]

In class the following relations relation was derived.

$$z=\frac{M_0}{M}=\frac{R}{R_0} \qquad (6)$$

From this it state equation can be written in terms of the reference gas constant at STP, and the static state variables on a face, and the compressibility. From this the compressibility can be solved for in terms of known variables.

$$p=\rho R_0\left(\frac{R}{R_0}\right)T=\rho R_0 z T \quad => \quad z=\frac{p}{\rho R_0 T} \qquad (7)$$

Finally once compressibility of the mixture is known, the molar mass (M) of the mixture can be found from the reference molar mass of air at STP and the compressibility.

$$M=\frac{M_0}{z} \qquad (8)$$

Results: **using the equilibrium results from part (b)

Face↓	z	M [kg/kg-mol]
1	1.0142	29.551
2	1.0084	29.720
3	1.1229	26.689

e) Calculate the pressure coefficient on each face using the results from CPG, thermo-chemical equilibrium, and the hypersonic similarity parameter (HSP).

For both CPG and equilibrium, the results obtained in parts (a) and (b) can be used to calculate the pressure coefficients. The HSP will require an additional calculation as will be shown shortly.

$$c_p = \frac{p - p_\infty}{q_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) \qquad (9)$$

Equation 9 applies to all cases not only to CPG even though a constant specific heat ratio (γ). This can be shown, but is just accepted here.

For the HSP the relations in equations 10 and 11 are used to obtain the pressure ratio.

$$\mathrm{K} = M_{up-stream}\,\delta \qquad (10)$$

$$\frac{p_{down-stream}}{p_{up-stream}} = 1 + \frac{\gamma(\gamma+1)}{4}\mathrm{K}^2 + \gamma\,\mathrm{K}^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{\mathrm{K}^2}} \qquad (11)$$

Notice that the pressure ratio is not with the free-stream pressure but the pressure upstream of the shock. As a result face 3 will need to be corrected to give a pressure coefficient value that is with respect to the free-stream. The procedure to do this is as follows.

1. Calculate the pressure coefficient on face 3 with respect to the conditions seen on face 2. ie.

$$\frac{p_3}{p_2} = 1 + \frac{\gamma(\gamma+1)}{4}\mathrm{K}^2 + \gamma\,\mathrm{K}^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{\mathrm{K}^2}} \qquad (12)$$

were $\mathrm{K} = M_2\theta$ **Note: The Mach number on face two was found using the CPG assumption. This is acceptable because the HSP is derived using this assumption in addition to other assumptions.

2. Using equation 9, the pressure on face 3 and the pressure on face 2 were backed out. **The Mach number is still from the CPG assumption for face 3.

3. Finally the pressure coefficient on face three was re calculated using the pressure found on face 3 using the HSP assumptions and the free-stream conditions.

Results:

Face↓	Equilibrium	CPG	HSP
1	0.020534	0.020610	0.020639
2	0.157646	0.163972	0.167118
3	1.871829	1.771270	1.808017

f) Find L/D using each method from part (e)

Derivation of cl and cd in terms of free-stream conditions and pressure coefficients on each face.

X and Y-direction forces, A and N respectively.

$$N = -p_1 A_1 \cos\theta_1 + p_2 A_2 \cos\theta_2 + p_3 A_3 \cos\theta_3$$

$$A = p_1 A_1 \sin\theta_1 + p_2 A_2 \sin\theta_2 + p_3 A_3 \sin\theta_3 - p_\infty A_4$$

From the geometry, the areas can be expressed as:

$$A_1 = \frac{c_1 + c_2}{\cos\theta_1} \qquad A_2 = \frac{c_1}{\cos\theta_2} \qquad A_3 = \frac{c_2}{\cos\theta_3} \qquad A_4 = (c_1 + c_2)\tan\theta_1 + c_1 \tan\theta_2 + c_2 \tan\theta_3$$

With some algebra, N and A can be re-written as: ($S = c_1 + c_2$)

$$N = -(p_1 - p_\infty) S + (p_2 - p_\infty) c_1 + (p_3 - p_\infty) c_2$$

$$A = (p_1 - p_\infty) S \tan\theta_1 + (p_2 - p_\infty) c_1 \tan\theta_2 + (p_3 - p_\infty) c_2 \tan\theta_3$$

These equations can now be used in the lift and drag equations:

$$L = N\cos\alpha - A\sin\alpha \qquad D = N\sin\alpha + A\cos\alpha$$

And then into the lift and drag coefficient equations:

$$c_L = \frac{L}{q_\infty S} \qquad c_D = \frac{D}{q_\infty S}$$

With some more algebra it can be shown then that:

$$c_L = \left(-c_{p1} + c_{p2}\frac{c_1}{S} + c_{p3}\frac{c_2}{S}\right)\cos\alpha - \left(c_{p1}\tan\theta_1 + c_{p2}\frac{c_1 \tan\theta_2}{S} + c_{p1}\frac{c_2 \tan\theta_3}{S}\right)\sin\alpha$$

$$c_D = \left(-c_{p1} + c_{p2}\frac{c_1}{S} + c_{p3}\frac{c_2}{S}\right)\sin\alpha + \left(c_{p1}\tan\theta_1 + c_{p2}\frac{c_1 \tan\theta_2}{S} + c_{p1}\frac{c_2 \tan\theta_3}{S}\right)\cos\alpha$$

From these values the L/D term can be found as:

$$\frac{L}{D} = \frac{c_L}{c_D}$$

Results:

	Equilibrium	CPG	% error	HSP	%error
c_l	0.70879	0.6784	4.2788	0.69280	2.2569
c_d	0.47169	0.44886	4.8403	0.45809	2.8826
L/D	1.5027	1.5115	0.59010	1.5124	0.64432

The best method used here is the equilibrium method.

Question 2.

a) Derive an expression for the Gas constant of the given mixture in terms of moles of each species. Y_1 is the number of moles of N_2 and Y_2 is the number of moles of He.

Staring with:

$$\tilde{R}=\sum_{i=1}^{N} c_i R_i = c_{N_2} R_{N_2} + c_{He} R_{He}$$

using the relation:

$$c_i = \frac{\rho_i}{\tilde{\rho}} = \frac{N_i}{N} \quad \text{where} \quad N = Y_1 + Y_2$$

Rearrange:

$$\tilde{R} = \frac{Y_1}{Y_1+Y_2} R_{N_2} + \frac{Y_2}{Y_1+Y_2} R_{He}$$

Substitute the values and pull out the universal gas constant:

$$\tilde{R} = \frac{8314.32}{Y_1+Y_2}\left[\frac{Y_1}{28.0} + \frac{Y_2}{4.002602}\right]$$

b) Derive expressions for sensible internal energy (**e**) and enthalpy (**h**) per unit mass of mixture in terms of Y_1 and Y_2.

For a diatomic molecule (N_2):

$$e = e_{trans.} + e_{rot.} + e_{vib.} + e_{el.}$$

For an atom (He):

$$e = e_{trans.} + e_{el.}$$

Neglecting the electronic excitation these simplify to:

For a diatomic gas (N_2):

$$e = e_{trans.} + e_{rot.} + e_{vib.}$$

For an monoatomic gas (He):

$$e = e_{trans.}$$

Each term in these equations can then further be defined as follows:

$$e_{trans.} = \frac{3}{2} RT \qquad e_{rot.} = RT \qquad e_{vib.} = \frac{\theta_v / T}{e^{\theta_v / T} - 1} RT$$

Substituting these terms back into the equations above results in:

For a diatomic gas (N_2):

$$e = \frac{5}{2} RT + \frac{\theta_v / T}{e^{\theta_v / T} - 1} RT$$

For an monoatomic gas (He):

$$e = \frac{3}{2} RT$$

For the enthalpy per unit mass:

$$h = e + RT$$

For a diatomic gas (N_2):

$$h = \frac{7}{2} RT + \frac{\theta_v / T}{e^{\theta_v / T} - 1} RT$$

For an monoatomic gas (He):

$$h = \frac{5}{2} RT$$

To find the sensible internal energy and enthalpy per unit mass for a mixture, the following equations are used:

$$e_s = \sum_{i=1}^{N} c_i (e_s)_i \qquad h_s = \sum_{i=1}^{N} c_i (h_s)_i$$

Expanding these and making the appropriate substitutions results in:

$$e_s = c_1(e_s)_1 + c_2(e_s)_2 = \frac{Y_1}{Y_1+Y_2}\left(\frac{5}{2}R_{N_2} + \frac{\theta_{vN_2}}{e^{\theta_{vN_2}/T}-1}R_{N_2}T\right) + \frac{Y_2}{Y_1+Y_2}\left(\frac{3}{2}R_{He}T\right)$$

$$e_s = \frac{8314.32\,T}{Y_1+Y_2}\left[\frac{Y_1}{28.0}\left(\frac{5}{2} + \frac{3390/T}{e^{3390/T}-1}\right) + \frac{Y_2}{4.002602}\left(\frac{3}{2}\right)\right]$$

$$h_s = c_1(h_s)_1 + c_2(h_s)_2 = \frac{Y_1}{Y_1+Y_2}\left(\frac{7}{2}R_{N_2} + \frac{\theta_{vN_2}}{e^{\theta_{vN_2}/T}-1}R_{N_2}T\right) + \frac{Y_2}{Y_1+Y_2}\left(\frac{5}{2}R_{He}T\right)$$

$$h_s = \frac{8314.32\,T}{Y_1+Y_2}\left[\frac{Y_1}{28.0}\left(\frac{7}{2} + \frac{3390/T}{e^{3390/T}-1}\right) + \frac{Y_2}{4.002602}\left(\frac{5}{2}\right)\right]$$

c) Iterative procedure to solve for conditions behind a normal shock, no reactions, vibrational excitation is present.

Input u_1 and T_1

Select initial guess for ε_1 and ε_0, usually 0.1 and 0.001 respectively

Loop while $|f(\epsilon)| > tolerance$

(1) let $\epsilon = \rho_1/\rho_2$

(2) $u_2 = \epsilon u_1$

(3) $\tilde{R} = \frac{8314.32}{Y_1+Y_2}\left[\frac{Y_1}{28.0}+\frac{Y_2}{4.002602}\right]$

(4) $T_2 = \epsilon\left[T_1 + \frac{u_1^2}{\tilde{R}}(1-\epsilon)\right]$

(5) $h_1 = h_1(T_1) = \frac{8314.32 T_1}{Y_1+Y_2}\left[\frac{Y_1}{28.0}\left(\frac{7}{2}+\frac{3390/T_1}{e^{3390/T_1}-1}\right)+\frac{Y_2}{4.002602}\left(\frac{5}{2}\right)\right]$

(6) $h_2 = h_1 + \frac{u_1^2}{2}[1-\epsilon^2]$

(7) $\tilde{h}_2 = \tilde{h}_2(T_2) = \frac{8314.32 T_2}{Y_1+Y_2}\left[\frac{Y_1}{28.0}\left(\frac{7}{2}+\frac{3390/T_2}{e^{3390/T_2}-1}\right)+\frac{Y_2}{4.002602}\left(\frac{5}{2}\right)\right]$

(8) $f(\epsilon) = h_2(\epsilon) - \tilde{h}_2(\epsilon)$

(9) Apply secant method to step (8) to obtain next value for ε_{i+1}

$$\epsilon_{i+1} = \epsilon_i - \frac{f(\epsilon_i)(\epsilon_i - \epsilon_{i-1})}{f(\epsilon_i) - f(\epsilon_{i-1})}$$

end loop

$$\frac{\rho_2}{\rho_1} = \frac{1}{\epsilon}$$
$$\frac{T_2}{T_1} = \frac{T_2}{T_1}$$
$$p_2 = \rho_2 \tilde{R} T_2$$
$$\frac{p_2}{p_1} = \frac{p_2}{p_1}$$

d) Would there be a simpler approach to solve for a mono-atomic gas through a normal shock?

Yes there would be a simpler solution than using the iterative procedure. Because it is a mono-atomic gas, the CPG assumption (constant specific heat ratio) can be made so long as there is no electronic excitation of the molecules.

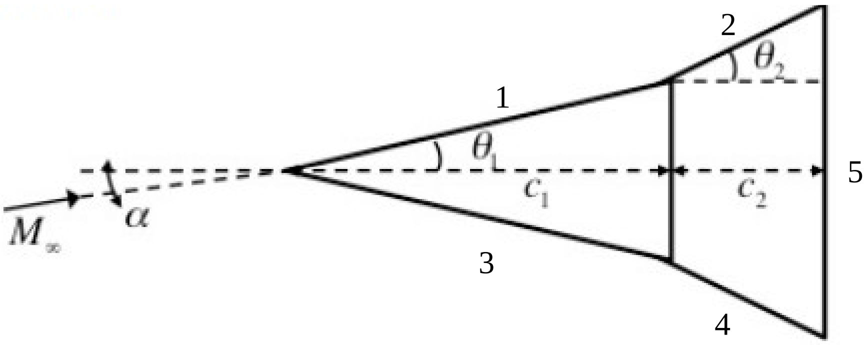
The specific heat ratio for a mono-atomic gas will be 1.66. As a result the following relations can be used for the normal shock relations:

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)$$

$$\frac{T_2}{T_1}=\frac{p_2/p_1}{\rho_2/\rho_1}=\frac{\left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right]\left[\left(\gamma-1\right)M_1^2+2\right]}{\left(\gamma+1\right)M_1^2}$$

$$\frac{\rho_2}{\rho_1}=\frac{\left(\gamma+1\right)M_1^2}{\left(\gamma-1\right)M_1^2+2}$$

Question 3.



θ_1 = 15° C₁ = 2.0 m α = 5°

θ_2 = 25° c_2 = 1.0 m

Calculate (β-δ), the static pressure (P_1), static temperature (T_1), and static density (ρ_1) on the upper surface (1) using the calorically perfect gas (CPG) assumption and the thermo-chemical equilibrium assumption to solve.

The values for flow deflection will be constants dependent on the geometry.

Face 1:

$$\delta_1 = \theta_1 - \alpha$$

Face 2 and 4:

$$\delta_{2,4} = \theta_2 - \theta_1$$

Face 3:

$$\delta_3 = \theta_1 + \alpha$$

Newtonian theory:

$$c_p = 2\sin^2\delta$$

Note that $c_{p2} = c_{p4}$ because the deflection angle is the same on face 2 and 4.

Approaching this problem similarly to problem 1, the following equations were found.

X and Y-direction forces, A and N respectively.

$$N = -p_1 A_1 \cos\theta_1 - p_2 A_2 \cos\theta_2 + p_3 A_3 \cos\theta_1 + p_4 A_4 \cos\theta_2$$

$$A = p_1 A_1 \sin\theta_1 + p_2 A_2 \sin\theta_2 + p_3 A_3 \sin\theta_1 + p_4 A_4 \sin\theta_2 - p_\infty A_5$$

From the geometry, the areas can be expressed as:

$$A_1 = A_3 = \frac{c_1}{\cos\theta_1} \qquad A_2 = A_4 = \frac{c_2}{\cos\theta_2} \qquad A_4 = 2(c_1 \tan\theta_1 + c_2 \tan\theta_2)$$

With some algebra, N and A can be re-written as: ($S = c_1 + c_2$)

$$N = -(p_1 - p_\infty)c_1 - (p_2 - p_\infty)c_2 + (p_3 - p_\infty)c_1 + (p_4 - p_\infty)c_2$$

$$A = (p_1 - p_\infty)c_1 \tan\theta_1 + (p_2 - p_\infty)c_2 \tan\theta_2 + (p_3 - p_\infty)c_1 \tan\theta_1 + (p_4 - p_\infty)c_2 \tan\theta_2$$

These equations can now be used in the lift and drag equations:

$$c_l = c_N \cos\alpha - c_A \sin\alpha \qquad c_d = c_N \sin\alpha + c_A \cos\alpha$$

where $c_L = \frac{L}{q_\infty S}$ $\quad c_N = \frac{N}{q_\infty S}$

With some algebra it can be shown then that:

$$c_N = \frac{2c_1}{S}[\sin^2\delta_1 + \sin^2\delta_3]$$

$$c_A = \frac{2c_1 \tan\theta_1}{S}[\sin^2\delta_1 + \sin^2\delta_3] + \frac{4c_2 \tan\theta_2}{S}\sin^2\delta_{2,4}$$

From these values the L/D term can be found as:

$$\frac{L}{D} = \frac{c_L}{c_D}$$

L/D for this geometry was found to be: 2.3681