Finite Volume Methods for Hyperbolic Problems

Elasticity and Seismic Waves

- Elasticity equations in 3D
- Plane strain case in 2D
- Heterogeneous medium
- Examples of seismic waves

Elasticity in 3D

Displacement: $\delta(x,y,z,t)$

Strain tensor:

$$\epsilon = \frac{1}{2}(\nabla\delta + (\nabla\delta)^T) = \begin{bmatrix} \epsilon^{11} & \epsilon^{12} & \epsilon^{13} \\ \epsilon^{21} & \epsilon^{22} & \epsilon^{23} \\ \epsilon^{31} & \epsilon^{32} & \epsilon^{33} \end{bmatrix}.$$

Stress tensor:

$$\sigma = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & \sigma^{22} & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & \sigma^{33} \end{bmatrix},$$

Strain and stress are symmetric with 6 distinct elements

Linear elasticity in 3D

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{13} \end{bmatrix} . = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \\ \epsilon^{12} \\ \epsilon^{23} \\ \epsilon^{13} \end{bmatrix} .$$

where λ and μ are the Lamé parameters

Linear elasticity in 3d

Hyperbolic system $q_t + Aq_x + Bq_y + Cq_z = 0$ with

$$q = (\sigma^{11},\ \sigma^{22},\ \sigma^{33},\ \sigma^{12},\ \sigma^{23},\ \sigma^{13},\ u,\ v,\ w)^T$$

and, for example:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \end{bmatrix},$$

where $\rho(x,y) =$ density and $\lambda(x,y), \mu(x,y)$ are
Lamé parameters that characterize the stiffness of material.

Linear elasticity in 3d

Hyperbolic system $q_t + Aq_x + Bq_y + Cq_z = 0$

The eigenvalues of $\breve{A} = n^x A + n^y B + n^z C$ are the same for any unit vector $\vec{n}$, and are given by

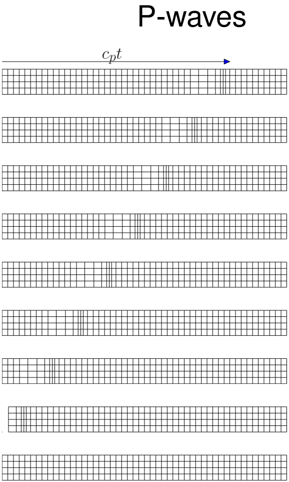
$$\begin{aligned} s^1 &= -c_p, \quad s^2 = c_p, \quad \text{P-waves} \\ s^3 &= -c_s, \quad s^4 = c_s, \quad \text{S-waves} \\ s^5 &= -c_s, \quad s^6 = c_s, \quad \text{S-waves} \\ s^7 &= s^8 = s^9 = 0, \end{aligned}$$

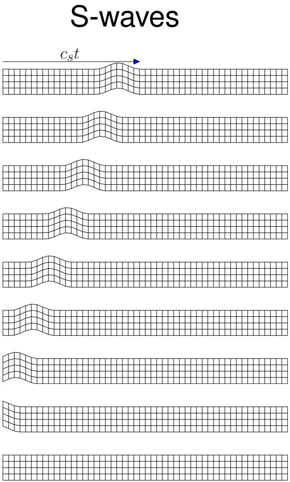
P-waves: compression/expansion in direction $\vec{n}$ of propagation.

S-waves: motion in 2-dimensional plane orthogonal to $\vec{n}$.

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad > \qquad c_s = \sqrt{\frac{\mu}{\rho}}.$$

Elastic waves





Linear elasticity – 2D plane strain

Assume no variation in z-direction.

$$q = [\sigma^{11},\ \sigma^{22},\ \sigma^{12},\ u,\ v]^T,$$

$$\breve{A} = -\begin{bmatrix} 0 & 0 & 0 & n^x(\lambda+2\mu) & n^y\lambda \\ 0 & 0 & 0 & n^x\lambda & n^y(\lambda+2\mu) \\ 0 & 0 & 0 & n^y\mu & n^x\mu \\ n^x/\rho & 0 & n^y/\rho & 0 & 0 \\ 0 & n^y/\rho & n^x/\rho & 0 & 0 \end{bmatrix}.$$

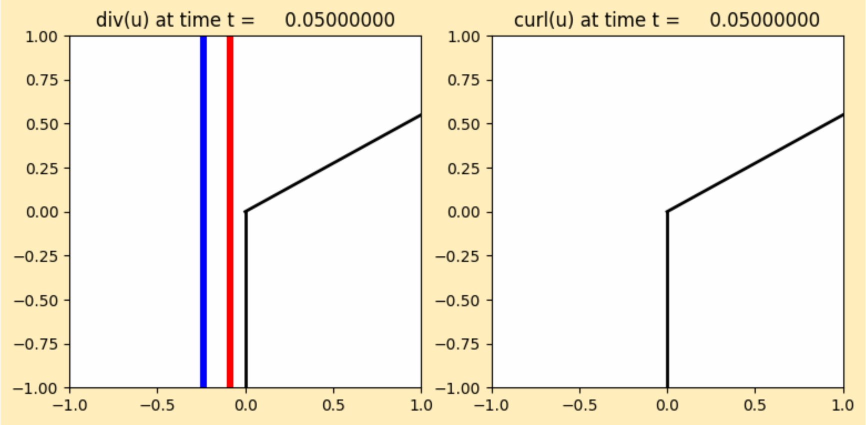
The eigenvalues of $\breve{A}$ are

$$\breve{s}^1 = -c_p, \quad \breve{s}^2 = c_p, \quad \breve{s}^3 = -c_s, \quad \breve{s}^4 = c_s, \quad \breve{s}^5 = 0,$$

where

$$c_p = \sqrt{\frac{\lambda+2\mu}{\rho}} \qquad > \qquad c_s = \sqrt{\frac{\mu}{\rho}}.$$

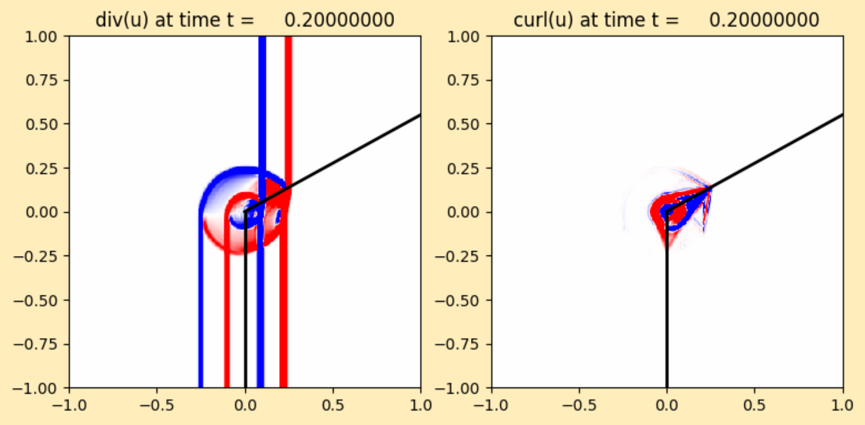
Elastic wave hitting corner



div(u) shows P-waves

curl(u) shows S-waves

Elastic wave hitting corner



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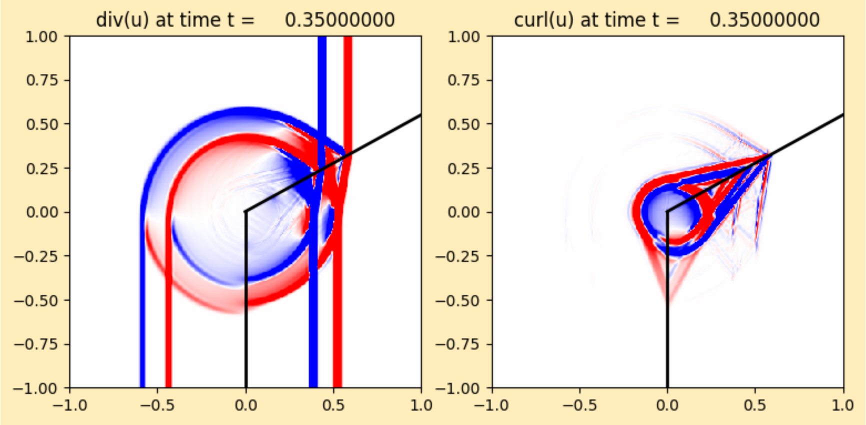

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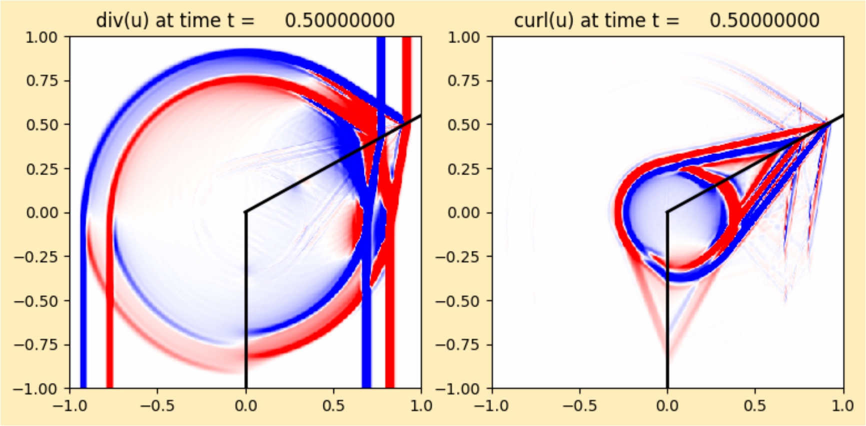
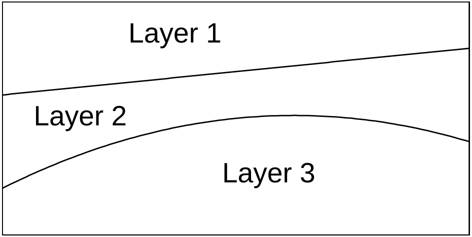


div(u) shows P-waves

curl(u) shows S-waves

Seismic waves in layered earth



Layers 1 and 3: $\rho = 2, \;\; \lambda = 1, \;\; \mu = 1, \;\; c_p \approx 1.2, \;\; c_s \approx 0.7$

Layer 2: $\rho = 5, \;\; \lambda = 10, \;\; \mu = 5, \;\; c_p = 2.0, \;\; c_s = 1$

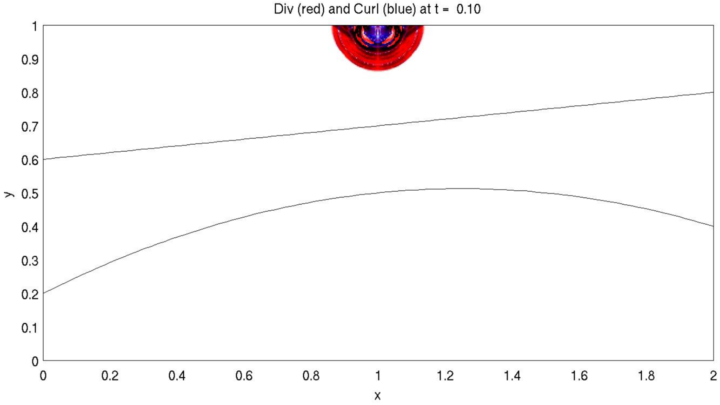
Impulse at top surface at $t = 0$ (seismic exploration/testing).

Solved on uniform Cartesian grid (600×300).

Cell average of material parameters used in each finite volume cell.

Seismic wave in layered medium

Red = div(u) [P-waves], Blue = curl(u) [S-waves]



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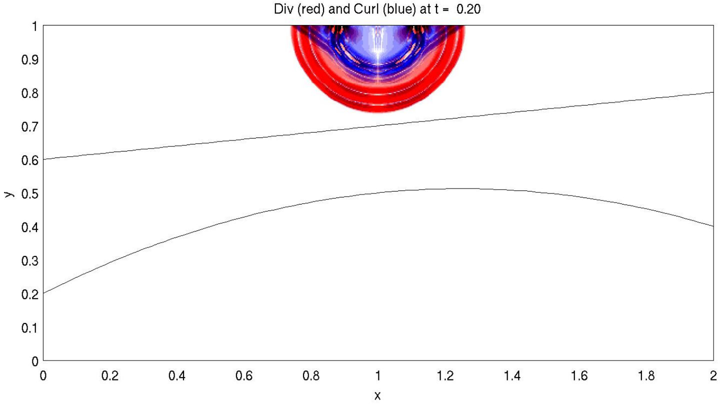

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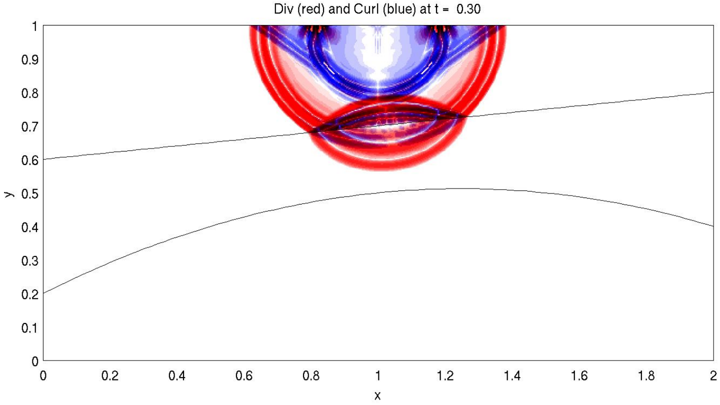

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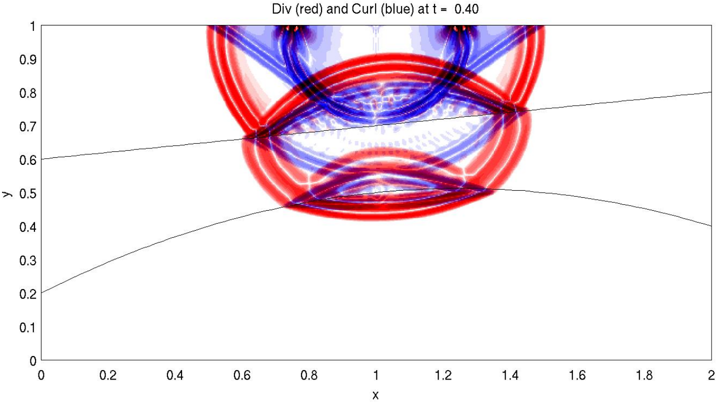

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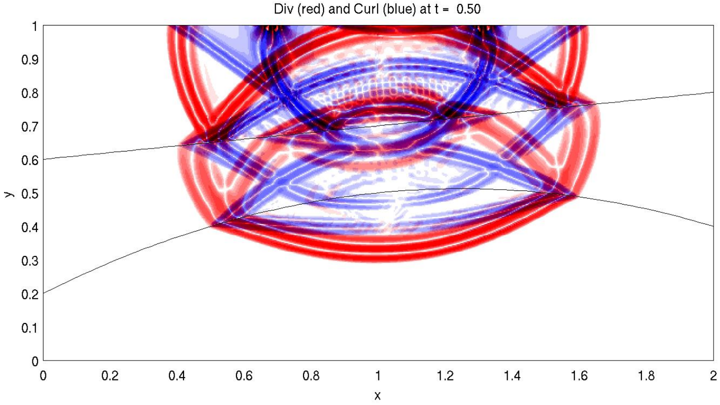

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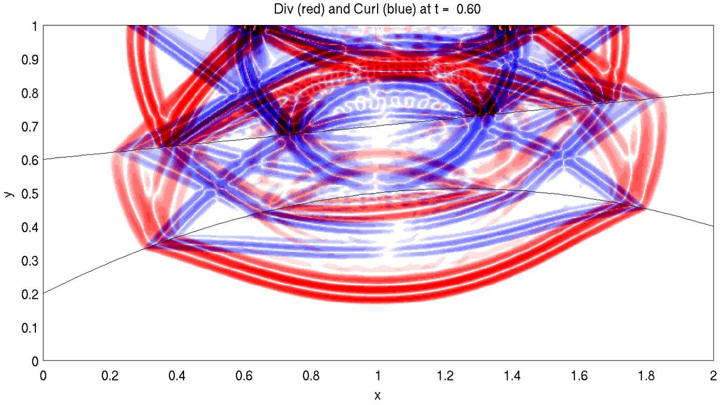

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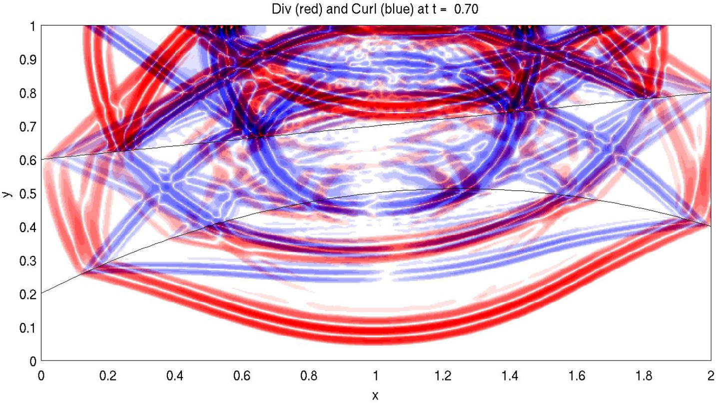

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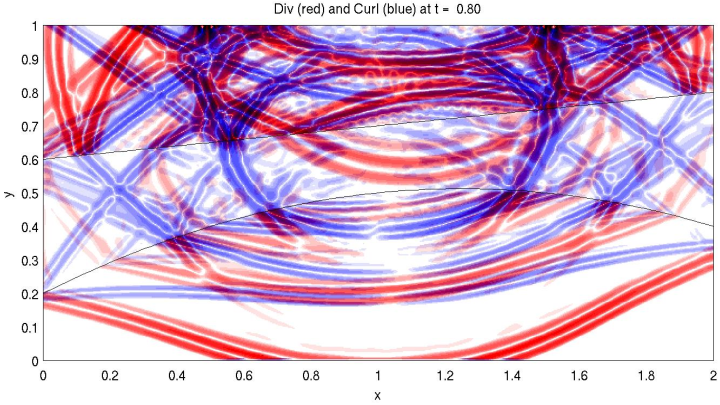

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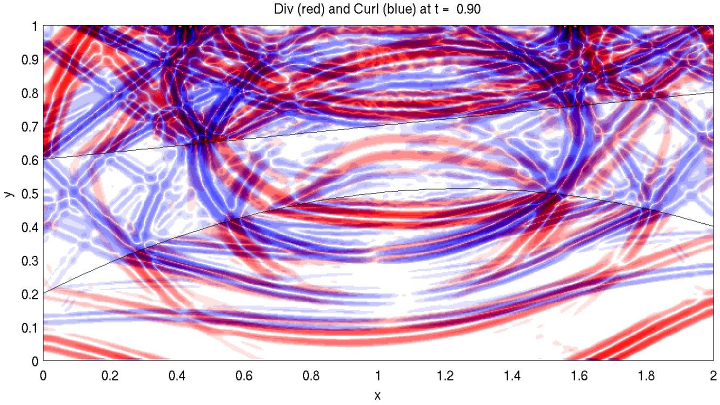

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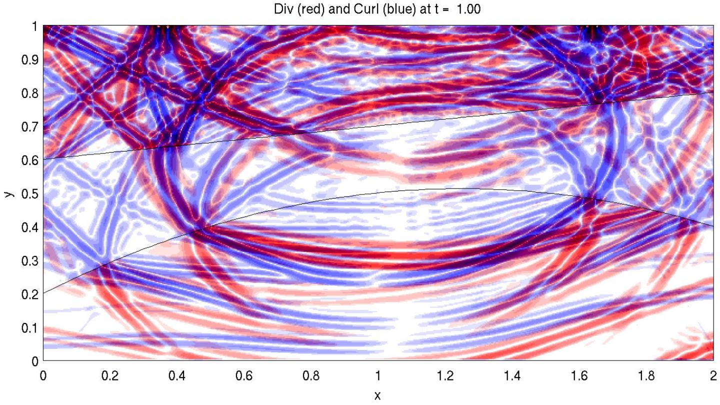
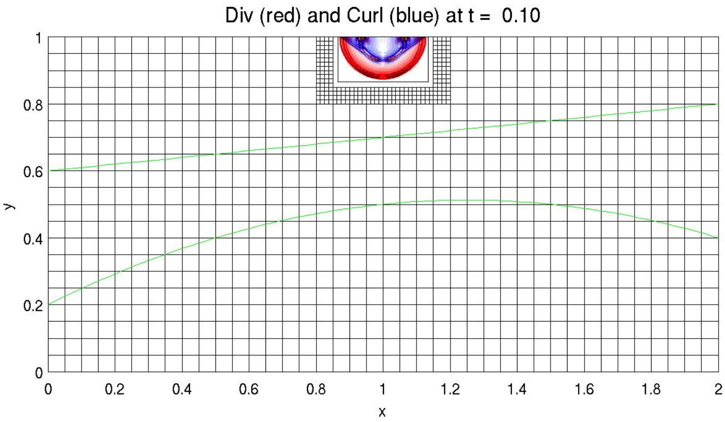
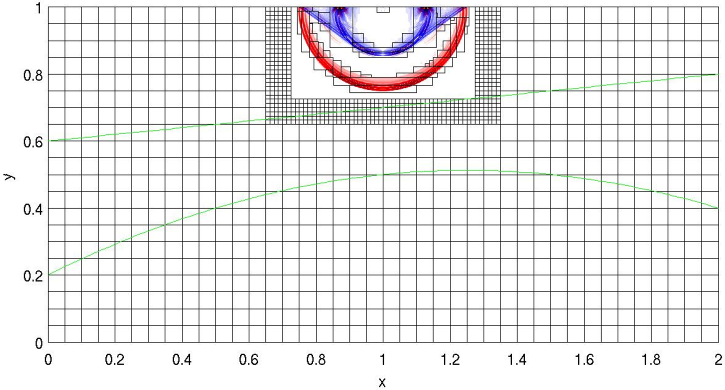
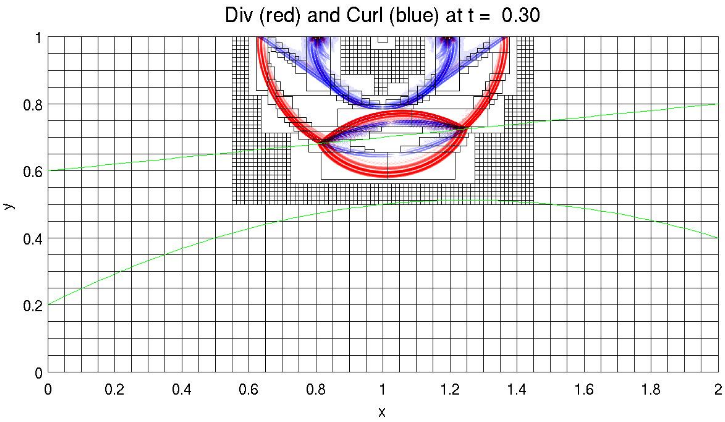
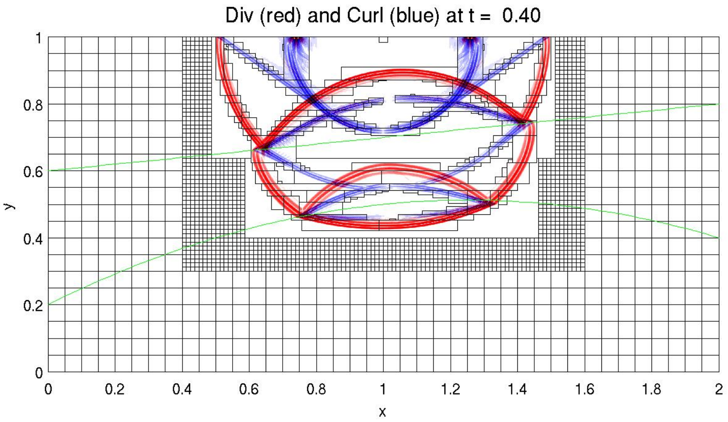



Div (red) and Curl (blue) at t = 0.10
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x

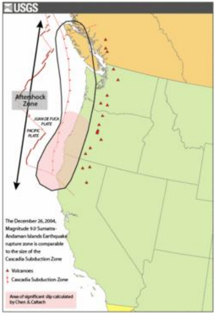
Div (red) and Curl (blue) at t = 0.20

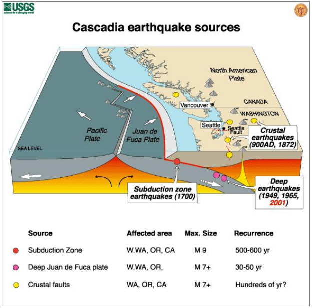



Div (red) and Curl (blue) at t = 0.30
y
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
0.2
0.4
0.6
0.8
1


Div (red) and Curl (blue) at t = 0.40
y
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Cascadia Subduction Zone (CSZ)



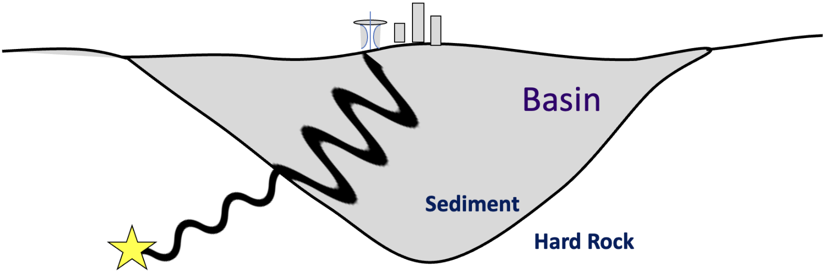


- 1200 km long off-shore fault stretching from northern California to southern Canada.
- Last major rupture: magnitude 9.0 earthquake on January 26, 1700.
- Tsunami recorded in Japan with run-up of up to 5 meters.
- Historically there appear to be magnitude 8 or larger quakes every 500 years on average.

Local geology significantly impacts the strength of ground shaking

Seattle sits within a deep sedimentary basin

***Sedimentary basins** and **shallow soft soils** **amplify** shaking*



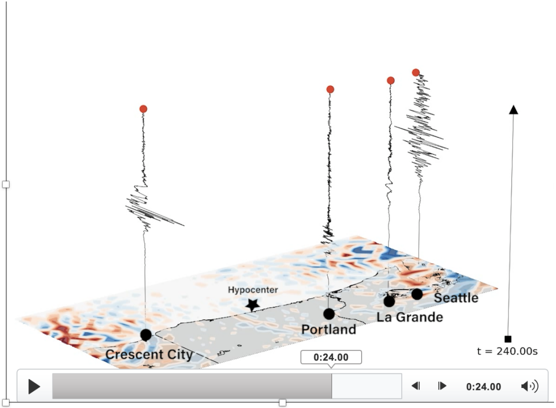
[Figure: Erin Wirth, USGS]

Simulation of a CSZ Mw 9 earthquake

Full 3D simulation using seismic modeling code.

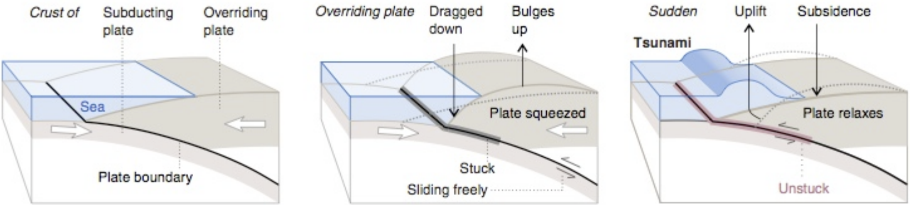
Capture motion at surface for earthquake hazard studies

50 Earthquake Simulations are online – available to engineers, emergency managers, the public, etc.



Arthur Frankel & Erin Wirth (USGS), Nasser Marafi (UW)

Tsunamis caused by subduction zone earthquakes



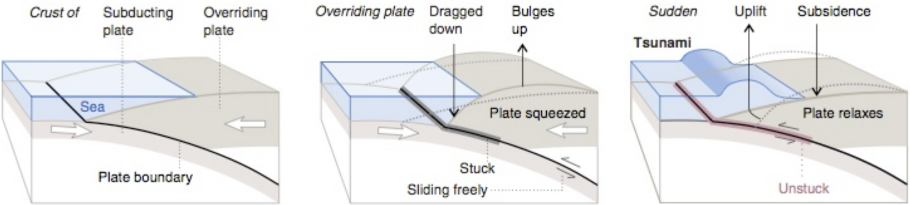
OVERALL, a tectonic plate descends, or "subducts," beneath an adjoining plate. But it does so in a stick-slip fashion.

BETWEEN EARTHQUAKES the plates slide freely at great depth, where hot and ductile. But at shallow depth, where cool and brittle, they stick together. Slowly squeezed, the overriding plate thickens.

DURING AN EARTHQUAKE the leading edge of the overriding plate breaks free, springing seaward and upward. Behind, the plate stretches; its surface falls. The vertical displacements set off a tsunami.

Source: Atwater et al., 2005.

Tsunamis caused by subduction zone earthquakes



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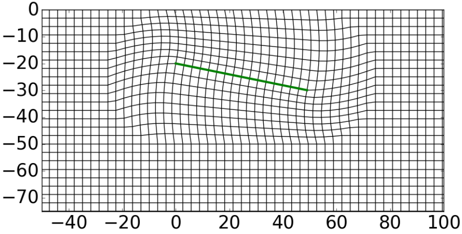
- Uplift/subsidence of seafloor over large area creates long-wavelength disturbance.
- Propagates as waves across ocean at 400–500 mph.

Imposing fault slip

Mapped grid used so fault aligned with cell edges:



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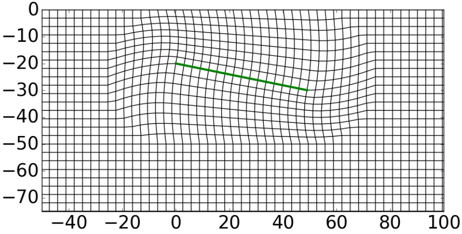
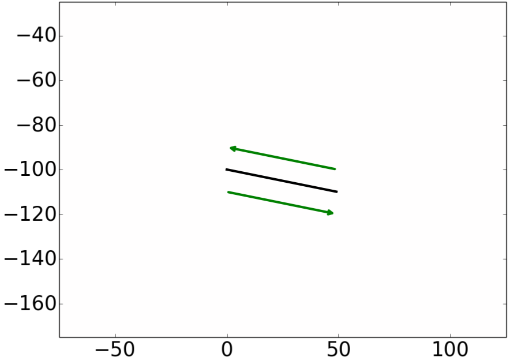


Riemann solver is modified at these grid edges:

During duration of rupture, wave strengths are modified to impose jump in tangential velocity at interface in Riemann solution (with continuity of normal velocity and traction)

Seismic waves from instantaneous slip

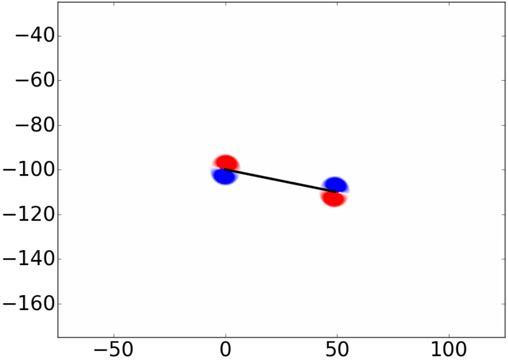
$t = 0$ seconds



Red: compression, Blue: tension

Seismic waves from instantaneous slip

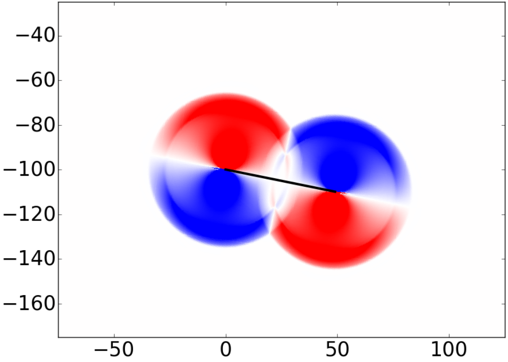
$t = 1.5$ seconds



Red: compression, Blue: tension

Seismic waves from instantaneous slip

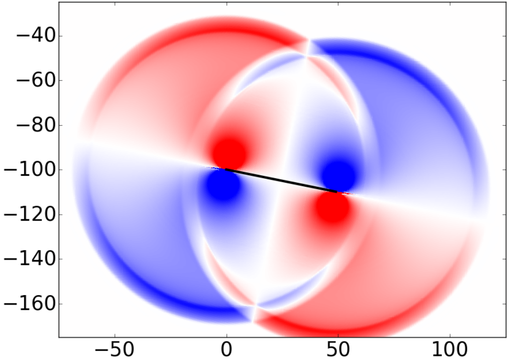
$t = 5$ seconds



Red: compression, Blue: tension

Seismic waves from instantaneous slip

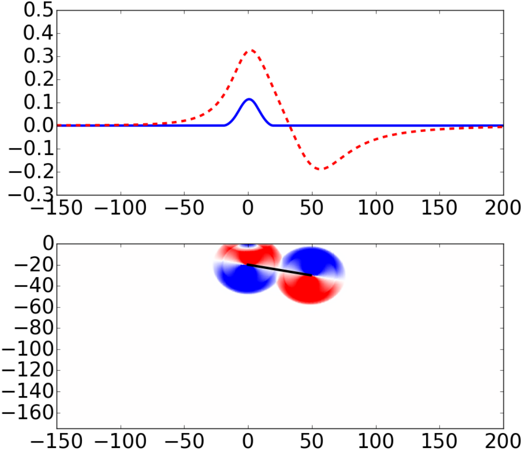
$t = 10$ seconds



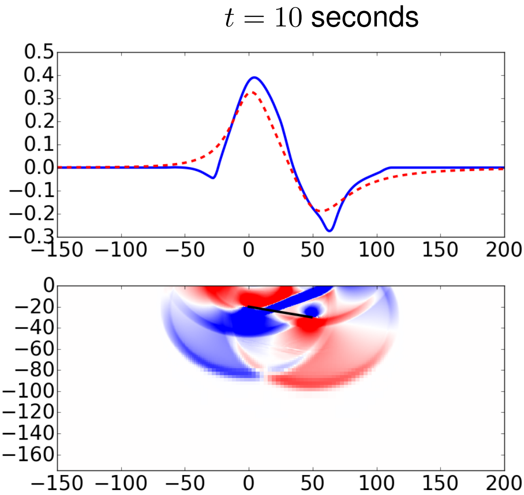
Red: compression, Blue: tension

Seismic waves vs. Okada solution

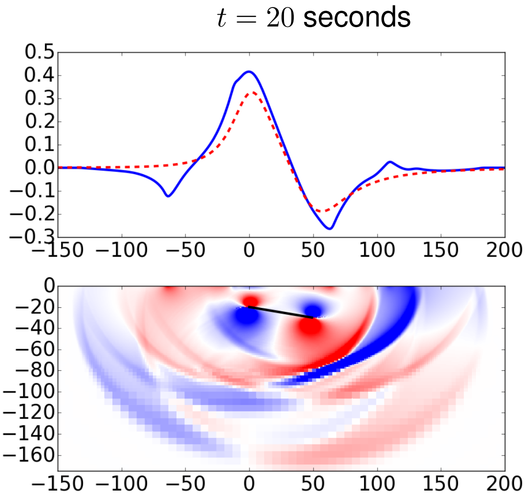
$t = 4$ seconds



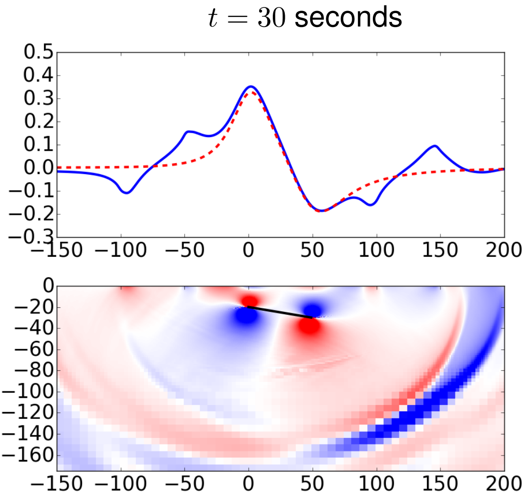
Seismic waves vs. Okada solution



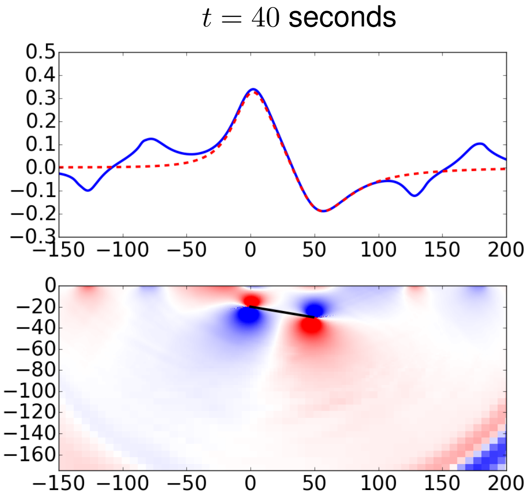
Seismic waves vs. Okada solution



Seismic waves vs. Okada solution



Seismic waves vs. Okada solution



Seismic waves vs. Okada solution