Part III of the Mathematical Tripos

University of Cambridge

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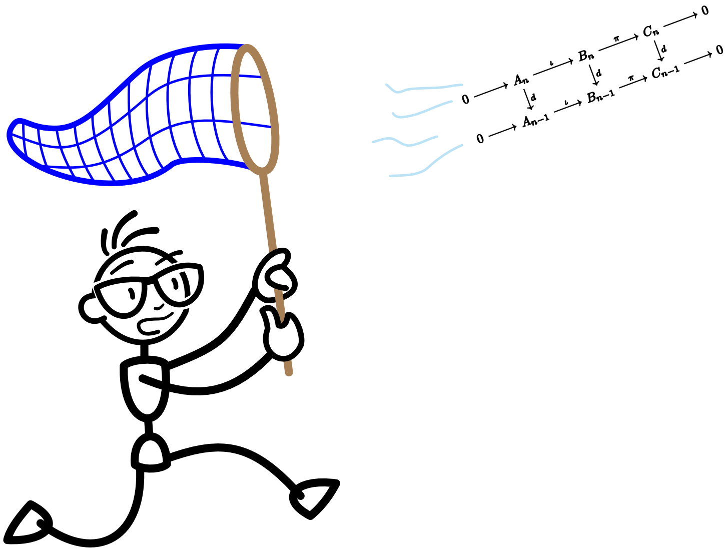
Algebraic Topology



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1 Homotopy

Definition 1.1 Let X, Y be topological spaces, $f_0, f_1 : X \to Y$ continuous maps and $I = [0,1]$. f_0 is *homotopic* to f_1 if there is a continuous map $F : X \times I \to Y$ with

$$\forall x \in X : F(x,0) = f_0(x), \quad F(x,1) = f_1(x). \tag{1.1}$$

Denote by $f_t(x) = F(x,t)$ the path from f_0 to f_1 in $\mathrm{Map}(X,Y) = \{f : X \to Y | f \text{ continuous}\}$.

Convention: all spaces are assumed to be topological and all maps to be continuous for this course.

Example 1.2 The following are all examples of homotopic maps:

(i) $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^m$ with $f_1(0) = x$, i.e. $f_0 \sim f_1$ via $f_t(x) = tx$

(ii) $S^1 = \{z \in \mathbb{C} | |z| = 1\}$, $f_0, f_1 : S^1 \to S^1$; $f_0(z) = z, f_1(z) = -z$, then $f_0 \sim f_1$ via $f_t(z) = \mathrm{e}^{\mathrm{i}\pi t} z$.

(iii) $S^n = \left\{v \in \mathbb{R}^{n+1} | \|v\| = 1\right\}$, $f_0, f_1 : S^n \to S^n$, $f_0(v) = v$, $f_1(v) = -v$ (antipodal map)
$n = 1 :\ f_0 \sim f_1$,
$n = 2 :\ f_0 \nsim f_1$.
In general, $f_0 \sim f_1$ iff n is odd (see ES1 Q9)

(iv) $f_0, f_1 : S^1 \to S^2$ where $f_0(x,y) = (0,0,1)$, $f_1(x,y) = (x,y,0)$, then $f_0 \sim f_1$ via $f_t(x,y) = (tx, ty, \sqrt{1-t^2})$

(v) $D^n = \{v \in R^n | \|v\| \leq 1\}$ with $S^{n-1} \subset D^n$. Say $f : S^{n-1} \to Y$ extends to D^n if there exists $F : D^n \to Y$ with $F|_{S^{n-1}} = f$. f extends to D^n if and only if f is homotopic to a constant map $f_t(v) = F(tv)$.

Lemma 1.3 *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$

Definition 1.4 Define

$$\begin{aligned}[X,Y] = \mathrm{Map}(X,Y)/\sim &= \{\text{homotopy classes of maps } X \to Y\} \\ &= \{\text{path components of } \mathrm{Map}(X,Y)\}\end{aligned}$$

Lemma 1.5 *Suppose $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. If $f_0 \sim f_1$ and $g_0 \sim g_1$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Notation: If $c \in Y$, $c_X : X \to Y$ is given by $c_X(x) = c$ for all $x \in X$.

Corollary 1.6 *Any $f : X \to \mathbb{R}^n$ is homotopic to 0_X.*

Proof. It is $\mathrm{id}_{\mathbb{R}^n} \sim 0_X$, so $f \sim \mathrm{id}_{\mathbb{R}^n} \circ f \sim 0_X \circ f = 0_X$. □

Definition 1.7 X is *contractible* if $\mathrm{id}_X \sim c_X$ for some $c \in X$.

Proposition 1.8 *Y is contractible if and only if $[X, Y]$ has one element for all spaces X.*

Proof. '$\Longrightarrow$': as in Corollary 1.6

'$\Longleftarrow$': $[X, Y]$ has one element, thus $\mathrm{id}_Y \sim c_Y$ for all $c \in Y$. □

Definition 1.9 Spaces X and Y are said to be *homotopic* ($X \sim Y$) if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

Remark 1.10 If spaces X, Y are homeomorphic, then they are also homotopic. However, the converse is not true in general.

Example 1.11 $X \sim \{p\}$ if and only if X is contractible.

Proof. Consider $f : X \to \{p\}, x \mapsto p$ and $g : \{p\} \to X, p \mapsto c \in X$. Then $f \circ g = \mathrm{id}_{\{p\}}$ and $g \circ f = c_X$. But if X is contractible, then $g \circ f \sim \mathrm{id}_X$. Hence,

$$c_X \sim \mathrm{id}_X \iff X \text{ is contractible} \tag{1.2}$$

according to Definition 1.7. □

Lemma 1.12 *If $X_1 \sim X_2$ and $Y_1 \sim Y_2$ then there is a bijection $[X_1, Y_1] \simeq [X_2, Y_2]$.*

Now having developed the notion of homotopy between maps and spaces and what it means for a space to be contractible, one can ask the basic question: given spaces X, Y, are they homotopic? And if so, what is $[X, Y]$? The answer to these questions are to be found in the homotopy groups introduced in the following.

1.1 Homotopy Groups

Definition 1.13 (Maps of Pairs) The map $f : (X, A) \to (Y, B)$ means

(i) $A \subset X, B \subset Y$

(ii) $f : X \to Y$

(iii) $f(A) \subset B$

If $f_0, f_1 : (X, A) \to (Y, B)$, we say $f_0 \sim f_1$ if

$$\exists F : (X \times I, A \times I) \to (Y, B) : F(x, 0) = f_0(x), F(x, 1) = f_1(x). \tag{1.3}$$

Notation: $* = (-1, 0, \ldots, 0) \in S^n$.

Definition 1.14 If $p \in X$, we define the *nth homotopy group* as

$$\pi_n(X, p) = [(S^n, *), (X, p)] = [(D^n, S^{n-1}), (X, p)] = [(I^n, \partial I^n), (X, p)]. \tag{1.4}$$

We denote with π the map

$$\pi : D^n \to D^n/S^{n-1} \simeq S^n, \quad v \mapsto (1 - 2\|v\|, v \cdot \alpha(v)), \tag{1.5}$$

where $\alpha(v) = \sqrt{1-(1-2\|v\|)^2}$.

From the above definition of homotopy groups follow the

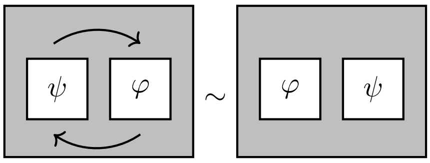
Properties 1.15

- $n > 0$: $\pi_n(X,p)$ is a group: e.g. $n = 2$

  group operation: $\boxed{\psi} + \boxed{\varphi} = \boxed{\psi}\boxed{\varphi}$

  identity element: $\boxed{} + \boxed{\varphi} = \boxed{}\boxed{\varphi} \sim \boxed{\varphi}$, with $\boxed{}$ the constant map $I^n \to p$ (this also applies to the next picture).

- $n > 1$: $\pi_n(X,p)$ is abelian:

  $\psi + \varphi = \boxed{\psi}\boxed{\varphi} \sim$  $\sim \varphi + \psi$

- *induced maps*: $f : (X,p) \to (Y,q)$ induces $f_* : \pi_n(X,p) \to \pi_n(Y,q)$ with $f_*([\gamma]) = [f \circ \gamma]$ well defined by Lemma 1.5. They satisfy the following properties:

  (i) $(\mathrm{id}_{(X,p)})_* = \mathrm{id}_{\pi_n(X,p)}$,

  (ii) $(f \circ g)_* = f_* \circ g_*$.

  Note also that f_* is homotopy invariant: if $f \sim g$, then $f_* = g_*$ since $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma])$.

- this defines a functor

$$\begin{array}{rcl} \left\{\begin{array}{c}\text{pointed spaces}\\ \text{pointed maps}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{groups}\\ \text{homomorphisms}\end{array}\right\} \\ X & \longmapsto & \pi_n(X,p) \\ f : (X,p) \to (Y,q) & \longmapsto & f_* : \pi_n(X,p) \to \pi_n(Y,q) \end{array}$$

2 Homology

The goal here is to define functors

$$\begin{array}{rcl} \left\{\begin{array}{c}\text{topological spaces}\\ \text{continuous groups}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{abelian groups}\\ \text{homomorphisms}\end{array}\right\} \\ X & \longmapsto & H_n(X) \\ f : X \to Y & \longmapsto & f_* : H_n(X) \to H_n(Y) \end{array}$$

such that the following are satisfied:

(i) $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$

(ii) $(f \circ g)_* = f_* \circ g_*$

(iii) if $f \sim g$, then $f_* = g_*$

(iv) $H_n(X) = 0$ if $n > \dim X$ (dimension axiom)

2.1 Chain Complex

Let R be a commutative ring (e.g. $\mathbb{Z}, \mathbb{Q}, Z/p$).

Definition 2.1 A chain complex (C_*, d) over R is

(i) R-modules C_i for $i \in \mathbb{Z}$ and

(ii) homomorphisms $d_i : C_i \to C_{i+1}$ such that

(iii) $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

$$\dots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \dots$$

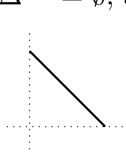
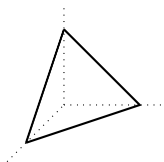
Notation: C_* can either mean $*$ is $i \in \mathbb{Z}$ or $C_* = \bigoplus_{i \in \mathbb{Z}} C_i$ with $d = \sum d_i$ where $d : C_* \to C_{*-1}$. In this notation (iii) in the definition above becomes $d^2 = 0$.

Definition 2.2 (Simplex) Define

$$\Delta^n = \{v = (v_0, \dots, v_n) \in \mathbb{R}^{n+1} | v_i \geq 0, \sum_i v_i = 1\} \tag{2.1}$$

to be the *n-dimensional simplex*.

Example 2.3 It is $\Delta^{-1} = \emptyset$, and

$n = 0$: $\bullet$, $n = 1$:  , $n = 2$:  etc.

Definition 2.4 (Faces) If $I = \{i_0 < i_1 < \dots < i_k\} \subset \{0, 1, \dots, n\}$, then

$$f_I = \{v \in \Delta^n | v_i = 0 \text{ if } i \notin I\} \tag{2.2}$$

is a *k-dimensional face* of Δ^n.

Definition 2.5 (Face maps) Maps

$$F_I : \Delta^k \to f_I, \quad w \mapsto v, \quad v_i = \begin{cases} 0, & i \notin I \\ w_j, & i = \varphi(j) \end{cases}, \tag{2.3}$$

where $\varphi : \{0, \dots, k\} \to I$, $\varphi(j) = i_j$.

Definition 2.6 (reduced chain complex of a simplex) The *reduced chain complex* over $\mathbb{R}$ is defined by $\tilde{S}_k(\Delta^n) = \langle f_I | |I| = k+1 \rangle$ free abelian group with basis the k-dimensional faces f_I. We also have

$$d_k : \tilde{S}_k(\Delta^n) \to \tilde{S}_{k-1}(\Delta^n), \quad d_k(f_I) = \sum_{j=0}^{k} (-1)^j f_{I \setminus \{i_j\}}. \tag{2.4}$$

Example 2.7 $(n = 2)$

$$\begin{array}{ccccccc} C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & C_{-1} \\ \parallel & & \parallel & & \parallel & & \parallel \\ \langle f_{012} \rangle & & \langle f_{01}, f_{12}, f_{02} \rangle & & \langle f_0, f_1, f_2 \rangle & & \langle f_\emptyset \rangle \end{array}$$

We have

$$df_{012} = f_{12} - f_{02} + f_{01}, \quad df_{12} = f_2 - f_1, \quad f_{02} = f_0 - f_2, \quad df_{01} = f_0 - f_1$$

and hence $d^2 f_{012} = 0$.

Proposition 2.8 $d^2 = 0$ *for the chain complex.*

Proof. It is enough to show the above for an arbitrary face, $d^2 f_I = 0$. In order to show this, we use Definition 2.6:

$$\begin{aligned} d^2 f_I &= d \sum_i (-1)^i f_{I \setminus \{i\}} && (2.5) \\ &= \sum_{\substack{i,j \\ j<i}} (-1)^i (-1)^j f_{I \setminus \{j,i\}} + \sum_{\substack{i,j \\ j>i}} (-1)^i (-1)^{j-1} f_{I \setminus \{i,j\}} && (2.6) \\ &= \sum_{\substack{i,j \\ j<i}} (-1)^i (-1)^j f_{I \setminus \{j,i\}} - \sum_{\substack{i,j \\ j<i}} (-1)^i (-1)^j f_{I \setminus \{i,j\}} && (2.7) \\ &= 0 && (2.8) \end{aligned}$$

$\square$

This property of the homomorphisms d is crucial to the definition of homology on a space. Note that $d^2 = 0$ immediately implies $\operatorname{im} d_{i+1} \subset \ker d_i$. We make use of this in the following definition.

Definition 2.9 (Homology groups) If (C_*, d) is a chain complex, it's *ith homology group* is defined as

$$H_i(C_*) = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$$

and again we abuse notation to denote with $*$ either of the two expressions

$$H_*(C_*) = \bigoplus_{i \in \mathbb{Z}} H_i, \qquad H_*(C_*) = \frac{\ker d}{\operatorname{im} d}.$$

Example 2.10 (Unreduced complex of a simplex)

$$\text{Define } S_k(\Delta^n) = \begin{cases} \tilde{S}_k(\Delta^n), & k \geq 0 \\ 0, & k < 0 \end{cases}, \text{ can check that } H_k(S(\Delta^n)) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

Definition 2.11 (Chain maps) If (C, d) and (C', d') are chain complexes over R, a chain map $f : (C, d) \to (C', d')$ are homomorphisms $f_i : C_i \to C'_i$ such that all squares of the following diagram commute:

$$\begin{tikzcd}
\dots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \dots \\
\dots \arrow[r] & C'_{i+1} \arrow[r] & C'_i \arrow[r] & C'_{i-1} \arrow[r] & \dots
\end{tikzcd}$$

i.e. $d'f = fd$ where $f = \sum f_i : C_* \to C'_*$.

Example 2.12 If f_I is a face of Δ^n, there is a chain map $\phi_I : \tilde{S}_*(\Delta^n) \to \tilde{S}_*(\Delta^n)$ with $\phi_I(f_J) = f_{\varphi(J)}$.

If $f : (C, d) \to (C', d')$ is a chain map, then

$$\begin{aligned} dz = 0 \quad &\Longrightarrow \quad d'fz = fdz = 0 \quad \Longrightarrow \quad f(\ker d) \subset \ker d', \\ z = dy \quad &\Longrightarrow \quad fz = fdy = d'fy \quad \Longrightarrow \quad f(\operatorname{im} d) \subset \operatorname{im} d'. \end{aligned}$$

So there is a well-defined map $f_* : H_*(C) \to H_*(C')$, $[z] \mapsto [fz]$.

Notation: If $dx = 0$, write $[x]$ for the image of x in $H_*(C)$.

Lemma 2.13

(i) id_C *is a chain map and* $(\mathrm{id}_C)_* = \mathrm{id}_{H_*(C)}$

(ii) if $f : C \to C'$ *and* $g : C' \to C''$ *are chain maps, then so is* $g \circ f$ *and* $(g \circ f)_* = g_* \circ g_*$

i.e. there is a functor

$$\begin{aligned} H_* : \left\{ \begin{matrix} \text{chain complexes over } R \\ \text{chain maps} \end{matrix} \right\} \quad &\longrightarrow \quad \left\{ \begin{matrix} R \text{ modules} \\ \text{homomorphisms} \end{matrix} \right\} \\ (C, d) \quad &\longmapsto \quad H_*(C) \\ f : C \to C' \quad &\longmapsto \quad f_* : H_*(C) \to H_*(C') \end{aligned}$$

2.2 Singular Chain Complex

The (simplicial) simplices Δ^n introduced above provide (to some extent) a good geometrical intuition about how a chain complex looks like. On the other hand, however, contructing a given space as a chain complex requires, e.g., a triangulation of the space and proof that the homology is independent of our choice of triangulation. This can be quite laborious already for relatively simple space.

Luckily, there are other types of chain complexes and below we introduce the singular chain complex. The most notable advantage of this definition is that there is no triangulation business involved. As such, singular homology is manifestly a property of just the space itself. On the downside, a singular chain complex does not, i.g., consist of a finite number of simplices but can entail an infinite number (not necessarily countable) of singular simplices. Hence, they are monstrous and it is hopeless to try imagine how these chain complexes look like, let alone use it to compute the homology. However, it turns out that the definition is quite powerful in that, knowing the homology of some simple spaces, we can use it to compute homology groups of more complicated ones indirectly. This, however, requires us to do some groundwork first.

Definition 2.14 (Singular chain complex) Let X be a topological space. A *singular k-simplex* in X is a map $\sigma : \Delta^k \to X$.

Definition 2.15 The singular chain complex $C_*(X)$ is given by

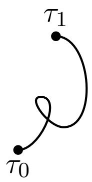
$$\begin{aligned} C_k(X) &= \langle \sigma | \sigma : \Delta^k \to X \text{ continuous} \rangle \\ &= \text{ free abelian group generated by } \sigma' s. \end{aligned}$$

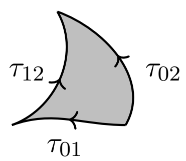
Remark 2.16 Note that we can view an abelian group like $C_k(X)$ above as a $\mathbb{Z}$-module (for $\sigma \in C_k(X)$ write $\sigma + \cdots + \sigma = n\sigma$).

The elements of $C_k(X)$ are finite sums $\sum_{i=1}^N a_i \sigma_i$, $a_i \in \mathbb{Z}$ and the boundary homomorphisms act on these as

$$d(\sigma) = \sum_{i=0}^{k} (-1)^i \sigma \circ F_{\{0,\ldots,k\}\setminus\{i\}} \tag{2.9}$$

with face maps $F_{\{0,\ldots,k\}\setminus\{i\}} : \Delta^{k-1} \to \Delta^k$,

e.g. $k = 1$: 

$d\sigma = \tau_1 - \tau_0$, $k = 2$: 

$d\sigma = \tau_{01} - \tau_{02} + \tau_{12}$

Lemma 2.17 *It is $d^2 = 0$.*

Proof. If $\sigma : \Delta^k \to X$, consider homomorphism $\phi_\sigma : S_*(\Delta^k) \to C_*(X)$ with $f_I \mapsto \sigma \circ F_I$ where $F_I : \Delta^{|I|-1} \to \Delta^k$. d was chosen so $d\phi_\sigma = \phi_\sigma d$ (where on the left hand d act on C_* and on the right on S_*). Then

$$d^2(\sigma) = d^2(\sigma \circ \mathrm{id}_{\Delta^k}) = d^2(\phi_\sigma(f_{\{0,\ldots,k\}})) = \phi_\sigma(d^2(f_{\{0,\ldots,k\}})) = \phi_\sigma(0) = 0,$$

since $d^2 = 0$ on $S_*(\Delta^k)$. $\square$

Definition 2.18 (Reduced singular chain complex) The reduced singular chain complex of X is defined by $\tilde{C}_k(X) = \langle \sigma | \sigma : \Delta^k \to X \rangle$ for $k \geq -1$ and $\tilde{C}_k(X) = 0$ for $k < -1$. For $k \geq 0$, $\tilde{C}_k(X) = C_k(X)$ and $\tilde{C}_{-1}(X) = \langle \sigma_\emptyset \rangle \simeq \mathbb{Z}$ if $\sigma : \Delta^0 \to X$, $d\sigma = \sigma_\emptyset$ the empty simplex ($[\hat{v}_0]$ in simplicial case).

Definition 2.19 $H_n(X) = H_n(C_*(X))$ is the *nth singular homology group* of X. $\tilde{H}_n(X) = H_n(\tilde{C}_*(X))$ is the *nth reduced singular homology group* of X.

Remark 2.20 It is often convenient to have a homology theory where the homology groups of a point vanish in all dimensions, including dimension zero. This is precisely achieved by reduced homology and the main motivation for its introduction. The reduced homology groups can be seen as the ones obtain from the augmented chain complex

$$\ldots \longrightarrow C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$ and with $d\sigma_0 = \sigma_\emptyset$ from Definition 2.18 this ε is the usual boundary map.

Proposition 2.21 *(Homology of a point) If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$. The reduced homology groups are $\tilde{H}_n(X) = 0$ for all n.*

Proof. Since X consists of a single point, there is a unique singular simplex $\sigma_k : \Delta \to X$ (i.e. $C_n(X) \simeq \mathbb{Z}$) such that the boundary is given by

$$d\sigma_k = \sum_{i=0}^{k} (-1)^i \sigma_k \circ F_{\{0,\ldots,k\}\setminus\{i\}} = \sum_{i=0}^{k} (-1)^i \sigma_{k-1} = \begin{cases} \sigma_{k-1}, & k \in 2\mathbb{N} \\ 0, & k \in 2\mathbb{N}+1 \end{cases} \tag{2.10}$$

i.e. the singular chain complex is given by

$$\ldots \longrightarrow C_4 \xrightarrow{\simeq} C_3 \xrightarrow{0} C_2 \xrightarrow{\simeq} C_1 \xrightarrow{0} C_0 \longrightarrow 0 \ .$$

It is immediate that $H_n(X) = 0$ for $n > 0$. Since X is path-connected, we have $H_0(X) = \mathbb{Z}$. For reduced homology we have the augmented chain complex

$$\ldots \longrightarrow C_4 \xrightarrow{\simeq} C_3 \xrightarrow{0} C_2 \xrightarrow{\simeq} C_1 \xrightarrow{0} C_0 \xrightarrow{\simeq} \mathbb{Z} \longrightarrow 0 \ .$$

□

Definition 2.22 (Induced maps) If $f : X \to Y$, define $f_\# : C_*(X) \to C_*(Y)$ by $f_\#(\sigma) = f \circ \sigma$ and extending $f_\#$ linearly via $f_\#(\sum_i n_i \sigma_i) = \sum_i n_i f_\#(\sigma_i) = \sum_i n_i (f \circ \sigma_i)$. The boundary homomorphism acts as

$$\begin{aligned} d(f_\#(\sigma)) &= \sum_{i=0}^{k} (-1)^i (f \circ \sigma) \circ F_{\{0,\dots,k\}\setminus\{i\}} \\ &= \sum_{i=0}^{k} (-1)^i f \circ (\sigma \circ F_{\{0,\dots,k\}\setminus\{i\}}) \\ &= f_\#(d\sigma), \end{aligned}$$

i.e. $f_\#$ is a chain map. Note that f can be any continuous map.

Lemma 2.23 *Let f, g be maps maps such as above. Then*

(i) $(\mathrm{id}_X)_\# = \mathrm{id}_{C_*(X)}$

(ii) $(f \circ g)_\# = f_\# \circ g_\#$

i.e. there is a functor

$$\begin{aligned} C_\# : \left\{\begin{matrix} \text{topological spaces} \\ \text{continuous maps} \end{matrix}\right\} &\longrightarrow \left\{\begin{matrix} \text{chain complexes over } \mathbb{Z} \\ \text{chain maps} \end{matrix}\right\} \\ X &\longmapsto C_*(X) \\ f : X \to Y &\longmapsto f_\# : C_*(X) \to C_\#(Y). \end{aligned}$$

Notation: If $f : X \to Y$, write $f_* : H_*(X) \to H_*(Y)$.

Corollary 2.24 *There is a functor*

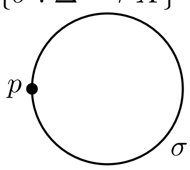
$$\begin{aligned} C_* : \left\{\begin{matrix} \text{topological spaces} \\ \text{continuous maps} \end{matrix}\right\} &\longrightarrow \left\{\begin{matrix} \text{abelian groups} \\ \text{homomorphisms} \end{matrix}\right\} \\ X &\longmapsto H_*(X) \\ f : X \to Y &\longmapsto f_* : H_*(X) \to H_*(Y), \end{aligned}$$

i.e. properties (i) and (ii) from Lemma 2.23 hold for f_*, g_* too.

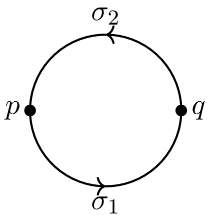
Proof. The composition of functors is a functor again. □

Example 2.25

- $\{\sigma : \Delta \to X\} \longleftrightarrow X$, $\sigma_p(f_0) = p \leftarrow p$
- $\{\sigma : \Delta^1 \to X\} \longleftrightarrow \{\text{paths } \sigma : [0,1] \to X\}$, e.g. $X = S^1$:



$\sigma \in C_1(S^1)$, $\mathrm{d}\sigma = \sigma_p - \sigma_p = 0$



$\mathrm{d}(\sigma_1 + \sigma_2) = \sigma_q - \sigma_p + \sigma_p - \sigma_q = 0$

- find $\tau \in C_2(S^1)$ with $\mathrm{d}\tau = \sigma - (\sigma_1 + \sigma_2)$, i.e. $[\sigma] = [\sigma_1 + \sigma_2]$

Proposition 2.26 *Let X_α denote the path-components of X. Then there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_\alpha(X_\alpha)$.*

Proof. Note that singular simplices always have path-connected image (as σ is continuous and Δ^* is path-connected). Hence $C_n(X)$ splits into direct sum of its subgroups $C_n(X_\alpha)$. The boundary maps d_i preserve this direct sum decomposition (by their very definition) and thus, so do $\ker \mathrm{d}_n$ and $\operatorname{im} \mathrm{d}_{n+1}$. □

Proposition 2.27 *If X is path-connected then $H_0(X) \simeq \mathbb{Z}$. Thus, for generic X, $H_*(X)$ is given by the direct sum of a number of $\mathbb{Z}$'s corresponding to the number of path-components.*

Proof. Let X be path-connected. By definition, $\ker \mathrm{d}_0 = C_0(X)$ since $\mathrm{d}_0 = 0$. Now we define an epimorphism

$$\phi : C_0 \to \mathbb{Z}, \sum_i n_i \sigma_i \mapsto \phi(\sum_i n_i \sigma_i) = \sum_i n_i.$$

We claim that $\ker \phi = \operatorname{im} \mathrm{d}_1$.

"$\supseteq$": Let $\sigma : \Delta^1 \to X$ be a singular 1-simplex. Then $\phi\, \mathrm{d}\tau = \phi(\sigma \circ F_{\{1\}} - \sigma \circ F_{\{0\}}) = 1 - 1 = 0$, i.e. $\mathrm{d}\tau \in \ker \phi$.

"$\subseteq$": Let $\tau \in \ker \phi$, i.e. $\tau = \sum_i n_i \sigma_i$ with $\sum_i n_i = 0$. Each σ_i is just the map to a point in X. We can thus choose an arbitrary but fixed point $x_0 \in X$ and define a (continuous) path to σ_i as $\tau_i : [v_0, v_1] \to X$ s.t. $\tau_i(v_0) = x_0$ and $\tau_i(v_1) = \sigma_i$ (as we assume X to be path-connected). Then, $\mathrm{d}(\sum_i n_i \tau_i) = \sum_i n_i \sigma_i - \sum_i n_i x_0 = \sum_i n_i \sigma_i$ since $\sum_i n_i = 0$. Hence, $\sum_i n_i \sigma_i \in \operatorname{im} \mathrm{d}_1$.

Now we apply the first isomorphism theorem, $C_0(X)/\ker \phi = \operatorname{im} \phi = \mathbb{Z}$ which proves $H_0(X) \simeq \mathbb{Z}$. The second part of the proposition follows immediately from the above and Proposition 2.26. □

2.3 Homotopy Invariance

Goal: if $g_0, g_1 : X \to Y$ and $g_0 \sim g_1$, show $g_{0*} = g_{1*} : H_*(X) \to H_*(Y)$.

Definition 2.28 (Chain homotopy) Chain maps $g_0, g_1 : (C, \mathrm{d}) \to (C', \mathrm{d}')$ are chain homotopic, denoted $g_0 \sim g_1$, if there is a homomorphism $h : C_* \to C'_{*+1}$ with $h(C_i) \subset C'_{i+1}$, such that

$$\mathrm{d}'h + h\mathrm{d} = g_1 - g_0.$$

h is called a chain homotopy.

Lemma 2.29 *Chain homotopy is an equivalence relation.*

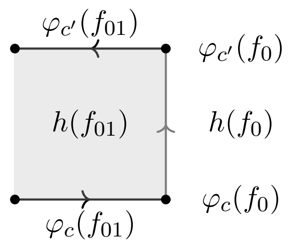
Proof. Suppose $[x] \in H_*(C)$ (i.e. $\mathrm{d}x = 0$ in particular), then $g_{1*}[x] - g_{0*}[x] = [g_1(x) - g_0(x)] = [\mathrm{d}'h(x) + h\mathrm{d}(x)] = [\mathrm{d}'h(0)] = 0$ since $\mathrm{d}'h(x) \in \operatorname{im} \mathrm{d}'$. $\square$

Definition 2.30 (Chain homotopy equivalence) Chain complexes $(C, \mathrm{d}), (C', \mathrm{d}')$ are chain homotopy equivalent, denoted $C \sim C'$, if there exist chain maps $f : C \to C'$ and $g : C' \to C$ such that $g \circ f \sim \mathrm{id}_C$ and $f \circ g \sim \mathrm{id}_{C'}$.

Remark 2.31 Lemma 2.29 shows that $g_{0*} = g_{1*}$ is easy once we know that g_0, g_1 are chain homotopic. Showing that this holds already for $g_0, g_1 : X \to Y$ with $g_0 \sim g_1$ homotopic maps from X to Y is significantly harder as we shall see below.

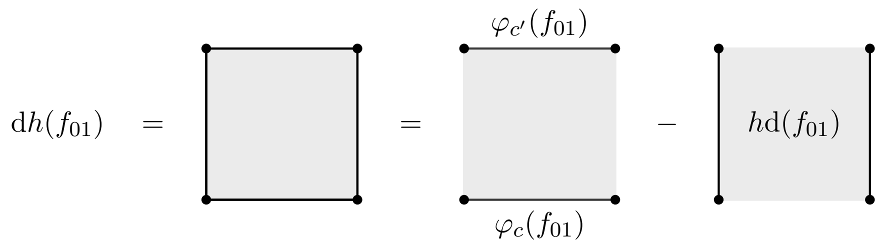
Now consider the folowing setup: Let $c_n, c_n' : \Delta^n \to \Delta^n \times [0,1]$ with $c_n(v) = (v, 0)$, $c_n'(v) = (v, 1)$ and corresponding chain maps $\varphi_{c_n}, \varphi_{c_n'} : S_*(\Delta^n) \to C_*(\Delta^n \times [0,1])$ with $\varphi_{c_n}(f_I) = c_n \circ F_I$.

Example 2.32 Consider



$$\mathrm{d}h(f_0) = \varphi_{c'}(f_0) - \varphi_c(f_0),$$
$$h\mathrm{d}(f_0) = h(0) = 0$$

and we have $\mathrm{d}h(f_{01}) = (\text{top} + \text{bottom}) + \text{sides}$ and $h\mathrm{d}(f_{01}) = -\text{sides}$, $\varphi_{c'}(f_{01}) - \varphi_c(f_{01}) = (\text{top} + \text{bottom})$, i.e.



Proposition 2.33 $\varphi_{c_n} \sim \varphi_{c_n'}$.

Notation: $\Delta^n \times [0,1]$ is a convex subset of $\mathbb{R}^{n+1} \times [0,1]$. If $p_0, \ldots, p_k \in \Delta^n \times [0,1]$, define a map $[p_0 \ldots p_k] : \Delta^k \to \Delta^n \times [0,1], v \mapsto \sum_{i=0}^k v_i p_i$. Then $\mathrm{d}[p_0 \ldots p_k] = \sum_{j=0}^k (-1)^j [p_0 \ldots \hat{p_j} \ldots p_k]$. Call $f_i \times 0 = i, f_i \times 1 = i'$

Proof. (Proposition 2.33) Define a map

$$U_n : S_*(\Delta^n) \to C_{*+1}(\Delta^n \times [0,1]), \quad U_n(f_I) = \sum_{j=0}^k (-1)^j [i_0 \ldots i_j i_j' \ldots i_k].$$

Then,

$$
\begin{aligned}
U_n\mathrm{d}(f_I) &= \sum_{a<b}(-1)^{a+b-1}[i_0\dots \overset{\curlywedge}{i_a}\dots i_b i'_b\dots i'_k] + \sum_{a>b}(-1)^{a+b}[i_0\dots i_b i'_b\dots \overset{\curlywedge}{i'_a}],\\
\mathrm{d}U_n(f_I) &= \sum_{a<b}(-1)^{b+a}[i_0\dots \overset{\curlywedge}{i_a}\dots i_b i'_b\dots i'_k] + \sum_{a>b}(-1)^{b+a+1}[i_0\dots i_b i'_b\dots \overset{\curlywedge}{i'_a}]\\
&\quad + \sum_{b=0}^{k}(-1)^{b+b}[i_0\dots i_{b-1} i'_b\dots i'_k] + \sum_{b=1}^{k+1}(-1)^{b+b-1}[i_0\dots i_{b-1} i'_b\dots i'_k],
\end{aligned}
$$

so $\mathrm{d}U_n(f_I)+U_n\mathrm{d}(f_I) = [i'_0\dots i'_k]-[i_0\dots i_k] = \varphi_{c'_n}(f_I)-\varphi_{c_n}(f_I)$ $\square$

Notation: $\bar{F}_I = F_I\times \mathrm{id}_{[0,1]} : \Delta^k\times[0,1]\to\Delta^n\times[0,1]$.

Lemma 2.34 *The following diagram commutes,*

$$
\begin{tikzcd}
S_*(\Delta^k) \arrow[r, "\varphi_I"] \arrow[d, "U_k"] \arrow[dr, phantom, "\circlearrowleft"] & S_*(\Delta^n) \arrow[d, "U_n"] \\
C_{*+1}(\Delta^k\times[0,1]) \arrow[r, "\bar{F}_{I\#}"] & C_{*+1}(\Delta^n\times[0,1])
\end{tikzcd}.
$$

Theorem 2.35 *Suppose $g_0, g_1 : X\to Y$. If $g_0\sim g_1$, then $g_{0\#}\sim g_{1\#} : C_*(X)\to C_*(Y)$.*

Proof. Let $G : X\times[0,1]\to Y$ be the homotopy. Define $G_\sigma : \Delta^n\times[0,1]\to Y$, $G_\sigma(v,t) = G(\sigma(v),t)$. Then $G_{\sigma\circ F_I} = G_\sigma\circ\bar{F}_I$ $(*)$. Define

$$
h : C_*(X)\to C_{*+1}(Y),\quad h(\sigma) = G_{\sigma\#}(U_n(f_{0\dots n})).
$$

Then we compute

$$
\begin{aligned}
\mathrm{d}h(\sigma) &= \mathrm{d}G_{\sigma\#}(U_n(f_{0\dots n})) = G_{\sigma\#}(\mathrm{d}U_n(f_{0\dots n}))\\
h\mathrm{d}(\sigma) &= h\left(\sum(-1)^j\sigma\circ F_{\overset{\curlywedge}{j}}\right)\\
&= \sum(-1)^j G_{\sigma\#}\circ\bar{F}_{\overset{\curlywedge}{j}\#}(U_{n-1}(f_{0\dots n-1})) \qquad \text{by } (*)\\
&= \sum(-1)^j G_{\sigma\#}(U_n(\varphi_{\overset{\curlywedge}{j}}(f_{0\dots n-1}))) \qquad \text{by Lemma 2.34}\\
&= G_{\sigma\#}(U_n(\sum(-1)^j\varphi_{\overset{\curlywedge}{j}}(f_{0\dots n-1})))\\
&= G_{\sigma\#}(U_n\mathrm{d}(f_{0\dots n-1})),
\end{aligned}
$$

so

$$\begin{aligned}
\mathrm{d}h(\sigma) + h\mathrm{d}(\sigma) =& G_{\sigma\#}(U_n \mathrm{d}(f_{\{0\dots n\}}) + \mathrm{d}U_n(f_{\{0\dots n\}})) \\
=& G_{\sigma\#}((\varphi_{c'_n} - \varphi_{c_n})(f_{\{0\dots n\}})) \qquad \text{by Proposition 2.33} \\
=& G_{\sigma\#}((c'_n - c_n)(F_{\{0\dots n\}})) \\
=& G_\sigma \circ c'_n(F_{\{0\dots n\}}) - G_\sigma \circ c_n(F_{\{0\dots n\}}) \\
=& g_1 \circ \sigma - g_0 \circ \sigma \qquad \text{by } G \text{ homotopy} \\
=& g_{1\#}(\sigma) - g_{0\#}(\sigma)
\end{aligned}$$

□

Corollary 2.36 *If $g_0, g_1 : X \to Y$ with $g_0 \sim g_1$, then $g_{0*} = g_{1*} : H_*(X) \to H_*(Y)$.*

Corollary 2.37 *Suppose $f : X \to Y$ is a homotopy equivalence (i.e. $X \sim Y$). Then the maps $f_* : H_n(X) \to H_*(Y)$ are isomorphisms for all n.*

Proof. Let $g : Y \to X$ be s.t. $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then the induced homomorphism $g_* : H_*(Y) \to H_*(X)$ is the inverse to f_*:

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_*(X)} \,.$$

In the first and third equality we apply properties (i) and (ii) of Corollary 2.24 and in the second one we use Corollary 2.36. □

Corollary 2.38 *Let X be contractible. Then $\tilde{H}_n(X) = 0$ for all n.*

Proof. From Example 1.11 we have X contractible $\iff X \sim \{p\}$. Corollary 2.37 demands $H_*(X) \simeq H_*(\{p\})$ and the statement follows from Proposition 2.21. □

2.4 Homology of a Pair

As is often the case in mathematics, it can be rewarding sometimes to ignore some amount of the data available to a problem, thereby simplifying the objects dealt with and even leading to new results not readily obtainable otherwise. The notion of homology of a pair is one such case. The basic idea is to link the homology of a space X to a subspace A and its quotient, X/A. (Un)fortunately, however, their relation is not simply given by $H_*(X)/H_*(A) = H_*(X/A)$, for if it would be, homology of every space would be trivial. The reason for this is that X can always be embedded in a contractible space, its cone $CX = (X \times [0,1])/(X \times \{0\})$.

There must hence be a more delicate relationship between the two which we are to uncover in this subsection. One important tool that will facilitate the construction of such a relation are exact sequences which we introduce first.

Exact Sequences

Suppose we have a sequence

$$\ldots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \ldots \qquad (2.11)$$

where A_i are $R-$modules and f_i homomorphisms.

Definition 2.39 (Exact sequence) We say (2.11) is exact at A_i, if $\ker f_i = \operatorname{im} f_{i+1}$. We say (2.11) is exact, if it's exact at all A_i (i.e. (A_*, f) is a chain complex with $H_*(A) = 0$).

Example 2.40 Exact sequences reveal information about the homomorphisms:

(i) $0 \longrightarrow A \xrightarrow{\iota} B$ exact at $A \iff \iota$ is injective

(ii) $B \xrightarrow{\pi} C \longrightarrow 0$ exact at $C \iff \pi$ is surjective

(iii) $0 \longrightarrow A \longrightarrow 0$ exact at $A \iff A = 0$

(iv) $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ exact $\iff f$ isomorphism

(v) $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} 0$ exact $\iff \iota : A \hookrightarrow B$ and $\pi : B \twoheadrightarrow C$

(vi) (2.11) exact $\implies$ $0 \longrightarrow \operatorname{coker} f_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} \ker f_{i-1} \longrightarrow 0$

These are said to be short exact sequences (SES).

Definition 2.41 (SES of chain complex) $0 \longrightarrow A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \longrightarrow 0$ is a SES of chain complexes if

(i) A_*, B_*, C_* are chain complexes and ι and π are chain maps

(ii) $0 \longrightarrow A_i \xrightarrow{\iota} B_i \xrightarrow{\pi} C_i \longrightarrow 0$ is exact for all i

Lemma 2.42 *(Snake) If* $0 \longrightarrow A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \longrightarrow 0$ *is a SES of chain complexes then there is a homomorphism ∂ and a long exact sequence (LES) as homology,*

$$\begin{tikzcd}
\ldots \arrow[r] & H_{*+1}(A) \arrow[r, "\iota_*"] & H_{*+1}(B) \arrow[r, "\pi_*"] & H_{*+1}(C) \arrow[dll, "\partial"] \\
& H_*(A) \arrow[r, "\iota_*"] & H_*(B) \arrow[r, "\pi_*"] & H_*(C) \arrow[dll, "\partial"] \\
& H_{*-1}(A) \arrow[r, "\iota_*"] & H_{*-1}(B) \arrow[r, "\pi_*"] & H_{*-1}(C) \arrow[r] & \ldots
\end{tikzcd}$$

Definition 2.43 (∂) Given $[c] \in H_n(C), \mathrm{d}c = 0$ and the following, commutative (ι, π chain maps) diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\iota"] \arrow[d, "d"] & B_n \arrow[r, "\pi"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\iota"] & B_{n-1} \arrow[r, "\pi"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Then,

1.) π is surjective, so there exists $b \in B_n$ with $\pi(b) = c$

2.) $\pi(\mathrm{d}b) = \mathrm{d}\pi(b) = \mathrm{d}c = 0$

3.) sequence exact at B_n, thus exists $a \in A_{n-1}$ with $\iota(a) = \mathrm{d}b$

4.) $\iota(\mathrm{d}a) = \mathrm{d}\iota(a) = \mathrm{dd}b = 0$ and as ι injective, $\mathrm{d}a = 0$

$$\begin{tikzcd}
& & a \arrow[dd, mapsto] & \\
& & & A_{n-1} \arrow[dd, "\iota"] \\
b \arrow[rr, mapsto] \arrow[dd, mapsto] & & \mathrm{d}b & \\
& B_n \arrow[rr, "\mathrm{d}"] \arrow[dd, "\pi"'] & & B_{n-1} \\
c & & & \\
& C_n & &
\end{tikzcd}$$

Define $\boxed{\partial[c] = [a]}$.

Now that we have defined ∂, we are ready to prove Lemma 2.42:

Proof. (Lemma 2.42) We have to prove the following six statements:

(i) $\ker \iota_* \subset \operatorname{im} \partial$. Suppose cycle $a \in A_{n-1}$ such that $\iota(a) = \mathrm{d}b$ for some $b \in B_n$ and $\mathrm{d}\pi(b) = \pi(\mathrm{d}b) = \pi(\iota(a)) = \pi(0) = 0$. Thus, $\partial[\pi(b)] = [a]$.

(ii) $\operatorname{im} \partial \subset \ker \iota_*$. It is $\iota_*\partial = 0$ as for $c \in C_n$, $\iota_*\partial[c] = [\mathrm{d}b] = 0$.

(iii) $\ker \partial \subset \operatorname{im} \pi_*$. Suppose $[c] \in \ker \partial$, then there is $a \in A_{n-1}$ s.t. $a = \mathrm{d}a'$ for some $a' \in A_n$. Now note that $[b - \iota(a')] = 0$ since $\mathrm{d}(b - \iota(a')) = \mathrm{d}b - \iota(\mathrm{d}a') = \mathrm{d}b - \iota(a) = 0$ and $\pi(b - \iota(a')) = \pi(b) - \pi(\iota(a')) = \pi(b) = c$, i.e. can always find $[b - \iota(a')]$ that gets mapped to $[c]$ by π_*.

(iv) $\operatorname{im} \pi_* \subset \ker \partial$. It is $\partial\pi_* = 0$ since in this case $\partial[b] = 0$.

(v) $\ker \pi_* \subset \operatorname{im} \iota_*$. Suppose $[b] \in \ker \pi_*$ and $\pi(b)$ a boundary, i.e. there is $c' \in C_{n+1}$ s.t. $\pi(b) = \mathrm{d}c'$. Now π is surjective, hence there is $b' \in B_{n+1}$ with $\pi(b') = c'$. Now note that $\pi(b - \mathrm{d}b') = \pi(b) - \mathrm{d}\pi(b') = \mathrm{d}c' - \mathrm{d}c' = 0$ and hence there exists $a \in A_n$ s.t. $\iota(a) = b - \pi(b')$. But $\iota(\mathrm{d}a) = \mathrm{d}\iota(a) = \mathrm{d}(b - \mathrm{d}b') = \mathrm{d}b = 0$ (last equality holds as b assumed closed in the beginning). Since ι injective, $\mathrm{d}a = 0$ and hence there is $[a]$ s.t. $\iota_*[a] = [b]$.

(vi) $\operatorname{im} \iota_* \subset \ker \pi_*$. This follows from $\pi\iota = 0$ (from exact sequence) and hence $\pi_*\iota_* = (\pi\iota)_* = 0_*$.

□

Remark 2.44 The method of the proof above is sometimes called *diagram chasing*, as we chase elements we need within commutative diagrams.

Example 2.45 $H_*(X) = \begin{cases} \tilde{H}_*(X), & *>0 \\ \tilde{H}_*(X) \oplus \mathbb{Z}, & *=0 \end{cases}$, $X \neq \emptyset$. Let $K_* = \begin{cases} \langle \sigma_\emptyset \rangle, & *=-1 \\ 0, & * \neq -1 \end{cases}$, i.e. $\mathrm{d} \equiv 0$, $H_*(K) = \begin{cases} \langle \sigma_\emptyset \rangle, & *=-1 \\ 0, & * \neq -1 \end{cases}$.

Have SES $0 \longrightarrow K_* \longrightarrow \tilde{C}_*(X) \longrightarrow C_*(X) \longrightarrow 0$ from which we get a LES

$$\begin{gathered} \ldots \longrightarrow \overset{=0}{H_*(K)} \longrightarrow \tilde{H}_*(X) \longrightarrow H_*(X) \longrightarrow \underset{=0}{H_{*-1}(K)} \\ \vdots \\ \overset{=0}{H_0(K)} \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \underset{=\mathbb{Z}}{H_{-1}(K)} \longrightarrow \tilde{H}_{-1}(X) \longrightarrow 0. \end{gathered} \tag{2.12}$$

Note: for $* > 0$, $\tilde{H}_*(X) \simeq H_*(X)$ and for $p \in X, \sigma_p : \Delta^0 \to X, f_0 \mapsto p$ in $\tilde{H}_0(X)$, $\mathrm{d}\sigma_p = \sigma_\emptyset \implies \partial[\sigma_p] = [\sigma_\emptyset]$ and thus ∂ is surjective. From (2.12) we then find $\tilde{H}_{-1}(X) = 0$ and $0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(x) \longrightarrow \mathbb{Z} \longrightarrow 0$. Hence, $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$ by the splitting lemma (cf. ES1 Ex3).

Subcomplexes and Quotients

Definition 2.46 (Subcomplex and quotient of a chain complex) Suppose (C_*, d) is a chain complex. We say A_* is a subcomplex of C_* if

(i) $A_* = \bigoplus_{i \in \mathbb{Z}} A_i$, $A_i \subset C_i$ is a submodule

(ii) $\mathrm{d}(A_i) \subset A_{i-1}$.

If so, (A_*, d) is a chain complex. Let $Q_i = C_i/A_i$, then $\mathrm{d}(A_i) \subset A_{i-1}$, so $\mathrm{d} : C_i \to C_{i-1}$ induces $\mathrm{d}_Q : Q_i \to Q_{i-1}$ with $\mathrm{d}_Q^2 = \mathrm{d}^2 = 0$. We call (Q, d_Q) the quotient complex.

It is easy to see that there is a SES of chain complexes

$$0 \longrightarrow A_* \xrightarrow{\iota} C_* \xrightarrow{\pi} Q_* \longrightarrow 0$$

Example 2.47 Suppose $A \subset X$. If $\sigma : \Delta^k \to X$ has $\operatorname{im} \sigma \subset A$, then $\sigma \circ F_{\{0 \ldots k\} \setminus \{j\}} : \Delta^{k-1} \to A$ as well, so $\mathrm{d}\sigma \in C_*(A)$, i.e. $C_*(A)$ is a subcomplex of $C_*(X)$.

Definition 2.48 (Homology of a pair) If $A \subset X$, we define $C_*(X, A) := C_*(X)/C_*(A)$ and $H_*(X, A) = H_*(C_*(X, A))$ is the homology of the pair (X, A).

We have a SES

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

and the corresponding LES is the LES of (X, A),

$$\ldots \longrightarrow H_*(A) \longrightarrow H_*(X) \longrightarrow H_*(X, A) \xrightarrow{\partial} H_{*-1}(A) \longrightarrow H_{*-1}(X) \longrightarrow H_{*-1}(X, A) \longrightarrow \ldots.$$

Example 2.49 Consider $(X, A) = (D^1, S^0)$. The homologies of D^1 and S^0 are

$$H_*(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}, \qquad H_*(D^1) = \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}.$$

This gives the LES of the pair,

$$\begin{array}{ccccccccccc} H_1(D^1) & \longrightarrow & H_1(D^1, S^0) & \longrightarrow & H_0(S^0) & \longrightarrow & H_0(D^1) & \longrightarrow & H_0(D^1, S^0) & \longrightarrow & 0 \\ \shortparallel & & & & \shortparallel & & \shortparallel & & \shortparallel & & \\ 0 & & & & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow\!\!\!\!\rightarrow & \mathbb{Z} & & 0 & & \end{array}$$

We see that the above LES splits into a SES and we find $H_1(D^1, S^0) = \mathbb{Z}$. This often happens when we deal with homology of pairs. The idea is to make a clever choice of X and A such that the LES splits.

Properties:

(i) For (X, A) a pair and $\iota : A \hookrightarrow X$ an inclusion there is a LES of the pair,

$$\ldots \longrightarrow H_*(A) \xrightarrow{\iota_*} H_*(X) \longrightarrow H_*(X, A) \xrightarrow{\partial} H_{*-1}(A) \longrightarrow \ldots$$

Proof. There is a SES $0 \longrightarrow C_*(A) \xrightarrow{\iota_\#} C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$. □

(ii) Induced maps: Suppose $f : (X, A) \to (Y, B)$ with $f : X \to Y$ and $f(A) \subset B$. If $\sigma : \Delta^k \to A$, then $f_\# : C_*(X) \to C_*(Y)$, $f_\#(\sigma) = f \circ \sigma : \Delta^k \to B$, so $f_\#(C_*(A)) \subset C_*(B)$. Hence, $f_\#$ descends to a chain map

$$f_\#^q : C_*(X, A) = \frac{C_*(X)}{C_*(A)} \longrightarrow \frac{C_*(Y)}{C_*(B)} = C_*(Y, B).$$

Define $f_* : H_*(X, A) \to H_*(Y, B)$ to be the induced map $f_* = (f_\#^q)_*$.

Lemma 2.50 *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "\iota"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & B_* \arrow[r, "\pi"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & C_* \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "\iota'"] & B'_* \arrow[r, "\pi'"] & C'_* \arrow[r] & 0 \end{tikzcd} \tag{2.13}$$

is a commutative diagram of chain complexes and chain maps and the rows are exact. Then we have a commutative diagram of LESs,

$$\begin{tikzcd} \dots \arrow[r] & H_*(A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(B) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(C) \arrow[r, "\partial"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r] & H_*(A') \arrow[r] & H_*(B') \arrow[r] & H_*(C') \arrow[r, "\partial'"] & H_{*-1}(A') \arrow[r] & \dots \end{tikzcd}. \tag{2.14}$$

Proof. (only last commutative square) If $[c] \in H_n(C)$, pick $b \in B_n$, $a \in A_{n-1}$ with $\pi(b) = c$ and $\iota(a) = \mathrm{d}b$, then $\partial[c] = [a]$. Let $a' = f(a), b' = f(b), c' = f(c)$. Then $\pi'(b') = c'$ and $\iota'(a') = \mathrm{d}b'$, so $\partial'[c'] = [a']$ and $\partial' f_*[c] = f_*[a] = f_*\partial[c]$. □

We can reformulate the result above in categorical language as follows: there is a functor

$$\left\{\begin{matrix}\text{SESs of chain complexes} \\ \text{maps like (2.13)}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{LESs of } R\text{-modules} \\ \text{maps like (2.14)}\end{matrix}\right\}.$$

Corollary 2.51 *If $f : (X, A) \to (Y, B)$, there is a commutative diagram*

$$\begin{tikzcd} \dots \arrow[r] & H_*(A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X, A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r] & H_*(B) \arrow[r] & H_*(Y) \arrow[r] & H_*(Y, B) \arrow[r] & H_{*-1}(B) \arrow[r] & \dots \end{tikzcd}.$$

Proof. From existence of f get SES

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \arrow[r, "\iota"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowleft"] & C_*(X) \arrow[r, "\pi"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowleft"] & C_*(X, A) \arrow[r] \arrow[d, "f_\#"] & 0 \\ 0 \arrow[r] & C_*(B) \arrow[r, "\iota'"] & C_*(Y) \arrow[r, "\pi'"] & C_*(Y, B) \arrow[r] & 0 \end{tikzcd},$$

so this follows from Lemma 2.50. □

(iii) Homotopy invariance: If $g_0, g_1 : (X, A) \to (Y, B)$ are homotopic as maps of pairs, then $g_{0*} = g_{1*} : H_*(X, A) \to H_*(Y, B)$.

Proof. The maps $g_{0\#}, g_{1\#} : C_*(X) \to C_*(Y)$ are chain homotopic via $h(\sigma) = G_{\sigma\#}(U_n(f_{0\dots n}))$, $\sigma : \Delta^n \to X$ and G the homotopy, $G(A \times [0, 1]) \subset B$. If $\sigma : \Delta^n \to A$, then $G_\sigma(\Delta^n \times I) \subset B$, so $h(\sigma) \in C_*(B)$, i.e. $h(C_*(A)) \subset C_*(B)$ and thus descends to

$$h^q : \frac{C_*(X)}{C_*(A)} \to \frac{C_*(Y)}{C_*(B)}, \quad \text{with} \quad \mathrm{d}h + h\mathrm{d} = g^q_{1\#} - g^q_{0\#},$$

i.e. $g^q_{1\#} \sim g^q_{0\#} : C_*(X, A) \to C_*(Y, B)$. □

(iv) Reduced homology: Define $\tilde{C}_*(X, A) = \tilde{C}_*(X)/\tilde{C}_*(A)$, similarly for $\tilde{H}_*(X, A) = H_*(\tilde{C}_*(X, A))$. Again, have LES of pair.

Example 2.52 1.) $H_*(X, A) = \tilde{H}_*(X, A)$ if $A \neq \emptyset$.

Proof. $\tilde{C}_*(X) \simeq C_*(X) \oplus \langle \sigma_\emptyset \rangle$ and $\tilde{C}_*(A) \simeq C_*(A) \oplus \langle \sigma_\emptyset \rangle$, hence for the quotient,

$$\frac{\tilde{C}_*(X)}{\tilde{C}_*(A)} \simeq \frac{C_*(X)}{C_*(A)}.$$

□

2.) If $p \in X$, then $\tilde{H}_*(X) \simeq \tilde{H}_*(X, p) \overset{1.)}{\simeq} H_*(X, p)$

Proof. $\tilde{H}_*(\{p\}) = 0$ by Proposition 2.21. By LES, have

$$\begin{array}{ccccccc} \tilde{H}_*(\{p\}) & \longrightarrow & \tilde{H}_*(X) & \xrightarrow{\pi_*} & \tilde{H}_*(X, \{p\}) & \longrightarrow & \tilde{H}_{*-1}(\{p\}) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

so π_* is an isomorphism. □

3.) $H_*(D^n, S^{n-1}) \simeq \tilde{H}_{*-1}(S^{n-1})$.

Proof. D^n is contractible, so $\tilde{H}_*(D^n) \simeq \tilde{H}_*(\{p\}) = 0$, so LES

$$\begin{array}{ccccccc} \tilde{H}_*(D^n) & \longrightarrow & H_*(D^n, S^{n-1}) & \xrightarrow{\partial} & \tilde{H}_{*-1}(S^{n-1}) & \longrightarrow & \tilde{H}_{*-1}(D^n) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

so ∂ is an isomorphism. □

Collapsing a Pair

Definition 2.53 (Deformation retract) $A \subset U$ is a deformation retract of U if there exists $\pi : (U, A) \to (A, A)$ with $\iota \circ \pi \sim \mathrm{id}_{(U,A)}$ as maps of pairs (with $\iota : (A, A) \to (U, A)$ inclusion as usual).

Example 2.54 S^{n-1} is a deformation retract of $D^n \backslash \{0\}$, with $\pi(v) = v/\|v\|$.

Definition 2.55 (Good pair) The pair (X, A) is good if

(i) $A \subset X$ is closed

(ii) there is some $U \subset X$ open with $A \subset U$ and A is a deformation retract of U

Example 2.56 1.) (D^n, S^{n-1}) is good, $U = D^n \backslash \{0\}$

2.) $(D^n, D^n \backslash \{0\})$ is not good

3.) $A = \{1/n | n \in \mathbb{Z}^*\} \cup \{0\} \subset \mathbb{R}$ is closed but $(\mathbb{R}, A)$ is not good

4.) (smooth mfd., compact mfd.) is good

5.) (simplicial cx., subcx.) is good

If $A \subset X$, have a quotient map $\pi : (X, A) \to (X/A, A/A) = (X/A, \{p_A\})$.

Theorem 2.57 *(Collapsing a pair) If (X, A) is good, then $\pi_* : H_*(X, A) \xrightarrow{\simeq} H_*(X/A, \{p_A\}) \simeq \tilde{H}_*(X/A)$ is an isomorphism.*

In order to prove this theorem we need some more machinery, which we introduce momentarily.

Example 2.58 $D^n/S^{n-1} \simeq S^n$, so $H_*(D^n, S^{n-1}) \simeq \tilde{H}_*(S^n)$.

Proposition 2.59 $\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z}, & * = n \\ 0, & * \neq n \end{cases}$.

Proof. By induction on n:

<u>$n = 0$</u>: $S^0 = \{-1\} \sqcup \{1\}$, so $H_*(S^0) = H_*(\{-1\}) \oplus H_*(\{1\}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}$. But $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$ and thus $\tilde{H}_0(S^0) = \mathbb{Z}$, $\tilde{H}_*(S^0) = 0$ for $* > 0$.

Induction step: $\tilde{H}_*(S^n) \overset{\text{collapsing pair}}{\simeq} H_*(D^n, S^{n-1}) \overset{3.)}{\simeq} \tilde{H}_{*-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & * - 1 = n - 1 \\ 0, & * - 1 \neq n - 1 \end{cases}$. ☐

Corollary 2.60 *If $S^n \sim S^m$, then $n = m$.*

Corollary 2.61 *The map*

$$\begin{tikzcd} \mathrm{id} : S^n \arrow[r] \arrow[d, "\iota"] & S^n \\ D^{n+1} \arrow[ur, "F"] & \end{tikzcd}$$

does not extend to D^{n+1}.

Proof. $F_* \circ \iota_* = \mathrm{id}_{\tilde{H}_*(S^n)}$. But D^{n+1} is contractible, so $\tilde{H}_*(D^{n+1}) = 0$. Hence, $\iota_* : \tilde{H}_*(S^n) \to \tilde{H}_*(D^{n+1})$ is the zero-map. ☐

Corollary 2.62 $\pi_n(S^n, *)$ *is non-trivial.*

Proof. $f : S^n \to X$ is homotopic to a constant $\Leftrightarrow$ f extends to D^{n+1}, so $\mathrm{id}_{S^n} \neq 0$ in $\pi_n(S^n, *)$. $\square$

Example 2.63 What is $H_*(T^2)$?

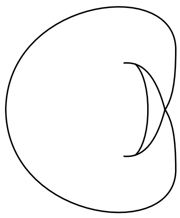
1.) Let $X = S^2$, $A = S^0 = \{p, q\} \subset S^2$. Then $H_*(X, A) = \begin{cases} \mathbb{Z}, & * = 1, 2 \\ 0, & * \neq 1, 2 \end{cases}$

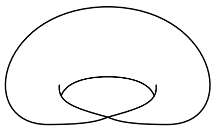
Proof. LES of (X, A),

$$\begin{tikzcd} \tilde{H}_2(A) \arrow[r] \arrow[d, equal] & \tilde{H}_2(S^2) \arrow[r, "\simeq"] \arrow[d, equal] & \tilde{H}_2(X,A) \arrow[r] & \tilde{H}_1(A) \arrow[r] \arrow[d, equal] & \tilde{H}_1(S^2) \arrow[r] \arrow[d, equal] & \tilde{H}_1(X,A) \arrow[r, "\simeq"] & \tilde{H}_0(A) \arrow[r] \arrow[d, equal] & \tilde{H}_0(S^2) \arrow[d, equal] \\ 0 & \mathbb{Z} & & 0 & 0 & & \mathbb{Z} & 0 \end{tikzcd}$$

$\square$

2.) $Y = S^1 \times S^1 = T^2$, $B = S^1 \times 1 \subset T^2$, then $\tilde{H}_*(T^2, B) = \begin{cases} \mathbb{Z}, & * = 1, 2 \\ 0, & * \neq 1, 2 \end{cases}$ using 1.):





Have LES of (T^2, B),

$$\begin{tikzcd} \tilde{H}_2(B) \arrow[r] \arrow[d, equal] & \tilde{H}_2(T^2) \arrow[r] & \tilde{H}_2(T^2,B) \arrow[r] \arrow[d, equal] & \tilde{H}_1(B) \arrow[r, "\iota_*"] \arrow[d, equal] & \tilde{H}_1(T^2) \arrow[r] & \tilde{H}_1(T^2,B) \arrow[r] \arrow[d, equal] & \tilde{H}_0(B) \arrow[d, equal] \\ 0 & & \mathbb{Z} & \mathbb{Z} & & \mathbb{Z} & 0 \end{tikzcd}$$

<u>Claim</u>: $\iota_* : \tilde{H}_1(B) \to \tilde{H}_1(T^2)$ is injective:
$\pi : S^1 \times S^1 \to S^1, (x, y) \mapsto x$ then $\pi \circ \iota = \mathrm{id}_{S^1}$, $\pi_* \circ \iota_* = \mathrm{id}_{\tilde{H}_*(S^1)} \Rightarrow \iota_*$ is injective.
Split LES into SESs

$$\begin{tikzcd} 0 \arrow[r] & H_2(T^2) \arrow[r] \arrow[d, equal] & H_2(T^2,B) \arrow[r] \arrow[d, equal] & \ker \iota_* \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} & \mathbb{Z} & 0 & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & H_*(B) \arrow[r] \arrow[d, equal] & H_*(T^2) \arrow[r] & H_*(T^2,B) \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} & & \mathbb{Z} & \end{tikzcd}$$

We arrive at $H_*(T^2) = \begin{cases} \mathbb{Z}, & * = 2 \\ \mathbb{Z}^2, & * = 1 \\ 0, & \text{else} \end{cases}$.

2.5 Subdivision, Excision and Collapsing

Subdivision

Suppose $\mathcal{U} = \{U_\alpha\}$ is an open cover of X.

Notation: If $\sigma : \Delta^k \to X$, write $\sigma \vartriangle \mathcal{U}$ if $\operatorname{im}\sigma \subset$ some U_α.

Definition 2.64 $C_k^{\mathcal{U}}(X) = \langle \sigma | \sigma : \Delta^k \to X, \sigma \vartriangle \mathcal{U} \rangle$.

If $\operatorname{im}\sigma \subset U_\alpha$, then $\operatorname{im}\sigma \circ F_I \subset U_\alpha$, so $C_*^{\mathcal{U}}(X)$ is a subcomplex of $C_*(X)$. Let $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ be the inclusion map.

Theorem 2.65 *(Subdivision) Let $\mathcal{U}$ be an open cover of X. If $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ is the inclusion, the induced map $i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.*

In particular, this ensures that the homology of X is independent of the cover $\mathcal{U}$ we choose to compute $H_*^{\mathcal{U}}(X)$. This can be useful in cases where there are covers that admit to compute the homology easily compared to others.

In order to prove this theorem we have to do some preparatory work first. We start by setting some notations about face maps. If $I \subset \{0, \ldots, n\}$, we let $F_I : \Delta^{|I|} \to \Delta^n$be the corresponding face map. There is a chain map $\varphi_k : S_*(\Delta^k) \to C_*(\Delta^k)$ given by $\varphi_k(e_I) = F_I$. Let $f_I : S_*(\Delta^k) \to S_*(\Delta^n)$ be the chain map given by $f_I(e_I) = e_{I\circ J} := e_{i_{j_0} \cdots i_{j_l}}$, where $l = |J|$. Then we have $F_{I\circ J} = F_I \circ F_J$, which implies $\varphi_n \circ f_I = F_{I\#} \circ \varphi_k$. Finally, we define $\mathbf{e}_n := e_{01\ldots n} \in S_n(\Delta^n)$ to be the top-dimensional face of Δ^n.

Definition 2.66 If X is a space, the cone on X is $CX = (X \times [0,1])/(X \times \{0\})$.

A map $f : X \to Y$ induces a map $Cf : CX \to CY$ given by $Cf(x,t) = (f(x), t)$. The cone $C\Delta^{n-1}$ can be identified with Δ^n by the map ϕ which sends $(x_0, \ldots, x_{n-1}, t)$ to $(1-t, tx_0, \ldots, tx_{n-1})$. Thus, $\sigma : \Delta^{n-1} \to X$ induces a map $c\sigma : \Delta^n \to CX$ given by $c\sigma = C(\sigma \circ \phi^{-1})$. Hence, we have a map $c : C_*(X) \to C_{*+1}(CX)$ given by $c(e_\sigma) = e_{c\sigma}$. If $i : X \to CX$ is the map given by $i(x) = (x, 1)$, it follows easily from the definition that

$$\mathrm{d}c(e_\sigma) = i_\#(e_\sigma) - c(\mathrm{d}e_\sigma).$$

Let $\pi : C\Delta^n \to \Delta^n, (\mathbf{v}, t) \mapsto t\mathbf{v} + (1-t)\mathbf{b}$ where $\mathbf{b} = \frac{1}{n+1}(1, \ldots, 1)$ is the barycenter of Δ^n, and define $\beta = \pi_\# \circ c : C_*(\Delta^n) \to C_{*+1}(\Delta^n)$. Since $\pi \circ i = 1_{\Delta^n}$, we have

$$\mathrm{d}\beta(e_\sigma) = e_\sigma - \beta(\mathrm{d}e_\sigma).$$

Definition 2.67 (Barycentric subdivision) Define a chain map $B_n : S_*(\Delta^n) \to C_*(\Delta^n)$ inductively. First, $B_0 : S_0(\Delta^0) \to C_0(\Delta^n)$ is uniquely defined by the requirement that $B_0(e_0) = e_p$, where p is the unique point of Δ_0. In general, if I is a proper subset of $\{0, \ldots, n\}$, we define

$$B_n(e_I) = F_{I\#}(B_{|I|}(e_{|I|})).$$

Finally, we define

$$B_n(\mathbf{e}_n) = \beta(B_n(\mathrm{d}\mathbf{e}_n)) \in C_n(\Delta^n).$$

Observe that all the singular simplices appearing in the image of B_n are given by affine linear maps.

Lemma 2.68 *B_n is a chain map.*

Proof. This is proved by induction on n. The case $n = 0$ is trivial. Given that B_k is a chain map for $k < n$, it follows from the definition that B_n is a chain map when restricted to $S_*(\Delta^n)$ where $* < n$. Thus, the only thing to check is that $B_n(\mathrm{d}\mathbf{e}_n) = \mathrm{d}B_n(\mathbf{e}_n)$. We compute

$$\begin{aligned}
\mathrm{d}B_n(\mathbf{e}_n) =& \mathrm{d}\beta(B_n(\mathrm{d}\mathbf{e}_n)) \\
=& B_n(\mathrm{d}\mathbf{e}_n) - \beta(\mathrm{d}B_n(\mathrm{d}\mathbf{e}_n)) \\
=& B_n(\mathrm{d}\mathbf{e}_n) - \beta(B_n(\mathrm{d}^2\mathbf{e}_n)) \\
=& B_n(\mathrm{d}\mathbf{e}_n).
\end{aligned}$$

We have used the fact thet statement holds in gradings $< n$ in passing from the second to the third line. $\square$

Next, we want to define a chain homotopy $T_n : S_*(\Delta_n) \to C_{*+1}(\Delta^n)$. As with the chain map B_n, we define T_n inductively. First, let T_0 be the trivial map. Next, if I is a proper subset of $0, \ldots, n$, define

$$T_n(e_I) = F_{I\#}(T_{|I|}(\mathbf{e}_{|I|})).$$

Finally, we define

$$T_n(\mathbf{e}_n) = \beta(B_n(\mathbf{e}_n) - \varphi_n(e_n) - T_n(\mathrm{d}\mathbf{e}_n)).$$

Lemma 2.69 $\mathrm{d}T_n + T_n\mathrm{d} = B_n - \varphi_n$.

Proof. This is proved by induction on n. The case $n = 0$ is easily verified, since $T_0 = 0$ and $B_0 = \phi_0$. Suppose the result holds for all $k < n$. As in the case of B_n, we need only verify the identity when both sides are applied to $\mathbf{e}_n$; the other cases follow from the induction hypothesis. For $\mathbf{e}_n$, we compute

$$\begin{aligned}
\mathrm{d}T_n(\mathbf{e}_n) =& \mathrm{d}\beta(B_n(\mathbf{e}_n) - \varphi_n(\mathbf{e}_n) - T_n(\mathrm{d}\mathbf{e}_n)) \\
=& B_n(\mathbf{e}_n) - \varphi_n(\mathbf{e}_n) - T_n(\mathrm{d}\mathbf{e}_n) - \beta(B_n(\mathrm{d}\mathbf{e}_n) - \varphi_n(\mathrm{d}\mathbf{e}_n) - \mathrm{d}T_n(\mathrm{d}\mathbf{e}_n)) \\
=& B_n(\mathbf{e}_n) - \varphi_n(\mathbf{e}_n) - T_n(\mathrm{d}\mathbf{e}_n) - \beta(T_n(\mathrm{d}^2\mathbf{e}_n)) \\
=& B_n(\mathbf{e}_n) - \varphi_n(\mathbf{e}_n) - T_n(\mathrm{d}\mathbf{e}_n)
\end{aligned}$$

where we have used the fact that the identity holds for e_I with $|I| < n$ in going from the second to the third line. So $\mathrm{d}T_n(\mathbf{e}_n) + T_n(\mathrm{d}\mathbf{e}_n) = B_n(\mathbf{e}_n) - \varphi_n(\mathbf{e}_n)$ as desired. $\square$

Lemma 2.70 *If $F_I : \Delta^k \to \Delta^n$ is a face map, then $B_n \circ f_I = F_{I\#} \circ B_k$ and $T_n \circ f_I = F_{I\#} \circ T_k$.*

Proof. We have

$$B_n(f_I(e_J)) = B_n(e_{I\circ J}) = (F_{I\circ J})_\#(B_{|J|}(\mathbf{e})_{|J|}) = F_{I\#}(F_{J\#}(B_{|J|}(\mathbf{e}_{|J|}))) = F_{I\#}(B_k(e_J)).$$

The proof of the second statement is identical, but with B's replaced by T's. $\square$

If X is a space, define $B : C_*(X) \to C_*(X)$ by $B(e_\sigma) = \sigma_\#(B_n(\mathbf{e}_n))$ for $\sigma : \Delta^n \to X$. It is clear from the definition that if $g : X \to Y$, then $B(g_\#(e_\sigma)) = g_\#(B_n(e_\sigma))$.

Lemma 2.71 *B is a chain map.*

Proof. We compute

$$\begin{aligned}
B(\mathrm{d}e_\sigma) =& \sum (-1)^j B(e_{\sigma\circ F_{\hat{j}}}) \\
=& \sum (-1)^j \sigma_\#(F_{\hat{j}\#}(B_{n-1}(\mathbf{e}_{n-1}))) \\
=& \sum (-1)^j \sigma_\#(B_n(f_{\hat{j}}(\mathbf{e}_{n-1}))) \qquad \text{(by Lemma 2.70)} \\
=& \sigma_\#(B_n(\mathrm{d}\mathbf{e}_n)) \\
=& \sigma_\#(\mathrm{d}B_n(\mathbf{e}_n)) \qquad (B_n \text{ is a chain map}) \\
=& \mathrm{d}B(e_\sigma).
\end{aligned}$$

$\square$

Lemma 2.72 *B is chain homotopic to $1_{C_*(X)}$.*

Proof. Let us define $T : C_*(X) \to C_{*+1}(X)$ by $T(e_\sigma) = \sigma_\#(T_n(\mathbf{e}_n))$. As in the previous lemma, we compute

$$\begin{aligned}
T(\mathrm{d}e_\sigma) =& \sum (-1)^j T(e_{\sigma\circ F_{\hat{j}}}) \\
=& \sum (-1)^j \sigma_\#(F_{\hat{j}\#}(T_{n-1}(\mathbf{e}_{n-1}))) \\
=& \sum (-1)^j \sigma_\#(T_n(f_{\hat{j}}(\mathbf{e}_{n-1}))) \qquad \text{(by Lemma 2.70)} \\
=& \sigma_\#(T_n(\mathrm{d}\mathbf{e}_n)).
\end{aligned}$$

Somewhat more easily, we have $\mathrm{d}T(e_\sigma) = \sigma_\#(\mathrm{d}T_n(e_n))$, so

$$\mathrm{d}T(e_\sigma) + T\mathrm{d}(e_\sigma) = \sigma_\#(\mathrm{d}T_n(e_n) + T_n(\mathrm{d}e_n)) = \sigma_\#(B_n(e_n) - \varphi_n(e_n)) = B(e_\sigma) - e_\sigma.$$

$\square$

Now let $\mathbf{F}_n = \phi_n(\mathbf{e}_n) \in C_n(\Delta^n)$ be the singular simplex corresponding to the map 1_{Δ^n}. If $\sigma : \Delta^n \to X$, then $e_\sigma = \sigma_\#(\mathbf{F}_n)$, so $B_r(e_\sigma) = \sigma_\#(B_r(\mathbf{F}_n))$. The simplices appearing in $B_r(\mathbf{F}_n)$ are all affine linear simplices obtained by iteratively applying barycentric subdivision to Δ^n.

Lemma 2.73 *If Δ is an affine linear simplex of dimension n, and Δ' is a simplex obtained by applying barycentric subdivision to Δ, then* $\operatorname{diam}(\Delta') \leq \frac{n}{n+1} \operatorname{diam}(\Delta)$.

Proof. Let $\mathbf{v}_0, \ldots, \mathbf{v}_n$ be the vertices of Δ, so $d = \operatorname{diam}(\Delta) = \max \|\mathbf{v}_i - \mathbf{v}_j\|$. We induct on n. Suppose $\mathbf{v}, \mathbf{v}'$ are two vertices of Δ'. If $\mathbf{v}, \mathbf{v}'$ lie in a k-dimensional proper face Δ_I of Δ, they are vertices of a simplex appearing in the barycentric subdivision of Δ_I. By induction we have $\|\mathbf{v} - \mathbf{v}'\| \leq \frac{k}{k+1} \operatorname{diam}(\Delta_I) \leq \frac{n}{n+1} d$. So it suffices to consider the case where $\mathbf{v} = \frac{n}{n+1}(\mathbf{v}_0 + \cdots + \mathbf{v}_n)$ is the barycenter. Without loss of generality, we may assume the other vertex is of the form $\frac{1}{k+1}(\mathbf{v}_0 + \cdots + \mathbf{v}_n)$ for some k. Then,

$$\mathbf{v}' - \mathbf{v} = \frac{1}{(n+1)(k+1)}[(n-k)\mathbf{v}_0 + \cdots + (n-k)\mathbf{v}_k - (k+1)\mathbf{v}_{k+1} - \cdots - (k+1)\mathbf{v}_n].$$

The sum in the parentheses on the RHS can be rearranged into a sum of $(n-k)(k+1)$ terms of the form $\mathbf{v}_i - \mathbf{v}_j$, so

$$\left\|\mathbf{v}' - \mathbf{v}\right\| \leq \frac{n-k}{n+1} \max \|\mathbf{v}_i - \mathbf{v}_j\| = \frac{n-k}{n+1} d \leq \frac{n}{n+1} d.$$

□

Corollary 2.74 *If we normalise Δ^n to have diameter 1, then every simplez appearing in $B_r(\boldsymbol{F}_n)$ has diameter less or equal to $\left(\frac{n}{n+1}\right)^r$.*

We will use the following standard fact about metric spaces:

Lemma 2.75 *If $\{U_i\}$ is an open cover of a compact metric space X, then there is some $\epsilon > 0$ so that any $A \subset X$ with diameter $< \epsilon$ is contained in some U_i.*

Proposition 2.76 *Let $\mathcal{U}$ be an open cover of X. If $c \in C_*(X)$, there is some $r > 0$ so that $B_r(c) \in C_*^{\mathcal{U}}(X)$.*

Proof. Since any $c \in C_*(X)$ is a finite linear combination of singular simplices, it suffices to prove the claim in the case where $c = e_\sigma$. Let $V_i = \sigma^{-1}(U_i)$. Then the V_i are an open cover of Δ^n, so we can apply Lemma 2.75 to find an $\epsilon > 0$ such that any $A \subset X$ with diameter $< \epsilon$ is contained in some V_i. By Corollary 2.74, we can choose r so that the diameter of any simplex in $B_r(\mathbf{F}_n)$ is $< \epsilon$. Thus, any simplex appearing in $B_r(\mathbf{F}_n)$ is contained in some V_i. Since $B_r(e_\sigma) = \sigma_\#(B_r(\mathbf{F}_n))$, every singular simplex appearing in $B_r(e_\sigma)$ is contained in some U_i. □

Proof. (of Theorem 2.65) We first show the map $i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$ is surjective. Given $[c] \in H_*(X)$, we apply Proposition 2.76 to see that $B_r(c) \in C_*^{\mathcal{U}}(X)$ for some r. Now $B_r \sim 1_{C_*(X)}$, so $[c] = [B_r(c)]$. It follows that i_* is surjective. Next, we show that i_* is injective. If $[c] \in H_*^{\mathcal{U}}$ and $c = dy$ for some $y \in C_*(X)$, then we can find r so that

$B_r(y) \in C_*^{\mathcal{U}}(X)$. Since B is a chain map, $B_r(c) = B_r(\mathrm{d}y) = \mathrm{d}B_r(y)$, so $[c] = [B_r(c)] = 0$ in $H_*^{\mathcal{U}}(X)$. □

There is a very useful application of the subdivision theorem which we write down at once.

Proposition 2.77 *(Mayer-Vietoris sequence) Suppose $U_1, U_2 \subset X$ are open and $U_1 \cup U_2 = X$, i.e. $\mathcal{U} = \{U_1, U_2\}$ is an open cover of X,*

$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & \\ U_1 \cap U_2 \arrow[ur, "l_1"] \arrow[dr, "l_2"'] & & X \\ & U_2 \arrow[ur, "j_2"'] & \end{tikzcd} .$$

There is a LES

$$\ldots \longrightarrow H_*(U_1 \cap U_2) \xrightarrow{l_{1*} \oplus l_{2*}} H_*(U_1) \oplus H_*(U_2) \xrightarrow{j_{1*} - j_{2*}} H_*(X) \xrightarrow{\partial} H_{*-1}(U_1 \cap U_2) \longrightarrow \ldots$$

Proof. There is a SES

$$0 \longrightarrow C_*(U_1 \cap U_2) \xrightarrow{l_{1\#} \oplus l_{2\#}} C_*(U_1) \oplus C_*(U_2) \xrightarrow{j_{1\#} - j_{2\#}} C_*^{\mathcal{U}}(X) \longrightarrow 0.$$

Take LES of homology and use $H_*^{\mathcal{U}}(X) \simeq H_*(X)$. □

We obtain a similar sequenece for the reduced homology $\tilde{H}_*(X)$.

Example 2.78 Let $X = S^n$ and choose $U_1 = S^n \backslash \{p\}, U_2 = S^n \backslash \{q\}$. Then $U_1 \cap U_2 \simeq \operatorname{int} D^n \backslash \{0\} \simeq S^{n-1}$. Mayer-Vietoris yields the sequence

$$\begin{tikzcd} \tilde{H}_*(U_1) \oplus \tilde{H}_*(U_2) \arrow[r] \arrow[d, equal] & \tilde{H}_*(S^n) \arrow[r] & \tilde{H}_{*-1}(U_1 \cap U_2) \arrow[r] \arrow[d, equal] & \tilde{H}_{*-1}(U_1) \oplus \tilde{H}_{*-1}(U_2) \arrow[d, equal] \\ 0 & & \tilde{H}_{*-1}(S^{n-1}) & 0 \end{tikzcd} \tag{2.15}$$

and thus $\tilde{H}_*(S^n) \simeq \tilde{H}_{*-1}(S^{n-1})$.

Excision

Suppose $A \subset X$, $\mathcal{U}$ is an open cover of X. Let $\mathcal{U}_A = \{U_\alpha \cap A\}$ an open cover of A. Then $C_*^{\mathcal{U}_A}(A)$ is a subcomplex of $C_*^{\mathcal{U}}(X)$. Define $C_*^{\mathcal{U}}(X, A) := \frac{C_*^{\mathcal{U}}(X)}{C_*^{\mathcal{U}_A}(A)}$.

Lemma 2.79 *(Five lemma) Suppose we have a commuting diagram*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] \arrow[dr, phantom, "\circlearrowleft"] & A_2 \arrow[r] \arrow[d, "f_2"] \arrow[dr, phantom, "\circlearrowleft"] & A_3 \arrow[r] \arrow[d, "f_3"] \arrow[dr, phantom, "\circlearrowleft"] & A_4 \arrow[r] \arrow[d, "f_4"] \arrow[dr, phantom, "\circlearrowleft"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

with exact rows and f_1, f_2, f_4, f_5 all isomorphisms. Then also f_3 is an isomorphism.

Proof. We show (i) f_2, f_4 monomorphisms, f_1 epimorphism $\implies$ f_3 monomorphism and (ii) f_2, f_4 epimorphism, f_5 monomorphism $\implies$ f_3 epimorphism.

(i):

$$\begin{tikzcd}
\exists x_1 \arrow[r, mapsto] \arrow[d, mapsto] & x_2', \exists x_2 \arrow[r, mapsto] \arrow[d, mapsto] & x_3 \in \ker f_3 \arrow[r] \arrow[d, mapsto] & x_4 = 0 \arrow[d, mapsto] \\
\exists y_1 \arrow[r, mapsto] & y_2 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

But f_2 monomorphism $\Rightarrow x_2' = x_2 \Rightarrow x_3 = 0$ (exact at A_2).

(ii):

$$\begin{tikzcd}
\exists x_2 \arrow[r, mapsto] \arrow[d, mapsto] & x_3', \exists x_3 \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dd, mapsto, bend right] & \exists x_4 \arrow[r, mapsto] \arrow[d, mapsto] & x_5 = 0 \arrow[d, mapsto] \\
\exists y_2 \arrow[dr, mapsto] & y_3', y_3 \arrow[r, mapsto] \arrow[r, mapsto, bend right] & y_4 \arrow[r, mapsto] & y_5 = 0 \\
 & y_3'' = y_3 - y_3' \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

Thus, $x_3' + x_3 \mapsto y_3'' + y_3' = y_3$, so f_3 epimorphism.

□

Corollary 2.80 $H_*^{\mathcal{U}}(X, A) \simeq H_*(X, A)$.

Proof. There is a map of SESs

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & C_*^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C_*^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

so we get a map of LESs of homology

$$\begin{tikzcd}
H_*^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & H_*^{\mathcal{U}}(X) \arrow[r] \arrow[d] & H_*^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & H_{*-1}^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & H_{*-1}^{\mathcal{U}}(X) \arrow[d] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X, A) \arrow[r] & H_{*-1}(A) \arrow[r] & H_{*-1}(X)
\end{tikzcd}$$

Arrows 1,2,4,5 are isomorphisms by subdivision, so 3 is isomorphism by the Five lemma.

□

Theorem 2.81 *(Excision) Suppose $B \subset A \subset X$, $j : (X - B, A - B) \to (X, A)$ inclusion. If $\bar{B} \subset \operatorname{int} A$, then $j_* : H_*(X - B, A - B) \to H_*(X, A)$ is an isomorphism.*

Proof. $\bar{B} \subset \operatorname{int} A$, so $\mathcal{U} = \{X - \bar{B}, \operatorname{int} A\}$ is an open cover of X. Then

$$\begin{aligned} C_*^{\mathcal{U}}(X) =& \langle \sigma | \sigma \vartriangle \mathcal{U}, \operatorname{im}\sigma \cap \bar{B} = \emptyset \rangle \oplus \langle \sigma | \sigma \vartriangle U, \operatorname{im}\sigma \cap B \neq \emptyset \rangle \\ =& C_*^{\mathcal{U}'}(X - B) \oplus \langle \sigma | \operatorname{im}\sigma \subset \operatorname{int} A \rangle, \end{aligned}$$

where $\mathcal{U}' = U_{X-B}$. Similarly, $C_*^{\mathcal{U}_A}(A) = C_*^{\mathcal{U}'_A}(A - B) \oplus \langle \sigma | \operatorname{im}\sigma \subset \operatorname{int} A \rangle$, so

$$\frac{C_*^{\mathcal{U}}(X)}{C_*^{\mathcal{U}_A}(A)} \simeq \frac{C_*^{\mathcal{U}'}(X - B)}{C_*^{\mathcal{U}'_A}(A - B)},$$

i.e.

$$\begin{tikzcd} j_\#^{\mathcal{U}} : C_*^{\mathcal{U}'}(X - B, A - B) \arrow[r] \arrow[d, "\iota'"] & C_*^{\mathcal{U}}(X, A) \arrow[d, "\iota"] \\ C_*(X - B, A - B) \arrow[r, "j_\#"] & C_*(X, A) \end{tikzcd}$$

is an isomorphism. By Corollary 2.80, ι'_* and ι_* are isomorphisms and $j_*^{\mathcal{U}}$ is an isomorphism because $j_\#^{\mathcal{U}}$ is. Hence, j_* is an isomorphism. $\square$

Example 2.82 (i) $H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{p\}) = \begin{cases} \mathbb{Z}, & * = n \\ 0, & * \neq n \end{cases}$.

Proof. $\mathbb{R}^n \backslash \{p\} \simeq \mathbb{R}^n \backslash \{0\} \sim S^{n-1}$, LES

$$\begin{tikzcd} \tilde{H}_*(\mathbb{R}^n) \arrow[r] \arrow[d, equal] & H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{p\}) \arrow[r, "\partial"] & \tilde{H}_{*-1}(\mathbb{R}^n \backslash \{p\}) \arrow[r] & \tilde{H}_{*-1}(\mathbb{R}^n) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

so $\partial : H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{p\}) \xrightarrow{\simeq} H_{*-1}(S^{n-1})$. NB: $\tilde{H}_*(\mathbb{R}^n / (\mathbb{R}^n \backslash \{p\}))$ does not depend in n. $\square$

(ii) If $U \subset \mathbb{R}^n$ is open, then $H_*(U, U \backslash \{p\}) = \begin{cases} \mathbb{Z}, & * = n \\ 0, & * \neq n \end{cases}$.

Proof. $C = \mathbb{R}^n \backslash U$ is closed in $\mathbb{R}^n$, so $\bar{C} \subset \mathbb{R}^n \backslash \{p\}$. By excision,

$$H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{p\}) \simeq H_*(\mathbb{R}^n \backslash C, \mathbb{R}^n \backslash \{p\} \backslash C) = H_*(U, U \backslash \{p\}).$$

$\square$

Corollary 2.83 *If $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open and $U \simeq V$ (homeomorphic), then $n = m$.*

Proof. If $f : U \xrightarrow{\simeq} V$, then $(U, U \backslash \{p\}) \overset{\simeq}{\mapsto} (V, V \backslash \{f(p)\})$, so

$$H_*(U, U \backslash \{p\}) \simeq H_*(V, V \backslash \{f(p)\}).$$

$\square$

Deformation Retractions

Suppose $A \subset U$, let $\iota : A \to U$ be the inclusion. If $\pi : U \to A$, have maps of pairs

$$(U, A) \xrightarrow{\tilde{\pi}} (A, A) \xrightarrow{\tilde{\iota}} (U, A)$$

Definition 2.84 $\pi : U \to A$ is a deformation retraction if $\tilde{\iota} \circ \tilde{\pi} \sim \mathrm{id}_{(U,A)}$ as maps of pairs.

Lemma 2.85 *If $\pi : U \to A$ is a deformation retract so is $\pi' : U/A \to A/A$.*

Lemma 2.86 *(LES of triple) Suppose $B \subset A \subset X$. Then there is a LES*

$$\ldots \longrightarrow H_*(A, B) \xrightarrow{j_*} H_*(X, B) \xrightarrow{i_*} H_*(X, A) \xrightarrow{\partial} H_{*-1}(A, B) \longrightarrow \ldots$$

with j_*, i_* induced by inclusions of pairs.

Proof. There is a SES

$$0 \longrightarrow \frac{C_*(A)}{C_*(B)} \xrightarrow{i_\#} \frac{C_*(X)}{C_*(B)} \xrightarrow{j_\#} \frac{C_*(X)}{C_*(A)} \longrightarrow 0.$$

□

Lemma 2.87 *Suppose $A \subset U \subset X$ and A is a deformation retract of U. Then the map $\iota_* : H_*(X, A) \to H_*(X, U)$ is an isomorphism.*

Proof. $\iota : A \to U$ is a homotopy equivalence (A deformation retract), so $\iota_* : H_*(A) \xrightarrow{\simeq} H_*(U)$. The LES of (U, A) gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\operatorname{coker} \iota_*)_n & \longrightarrow & H_*(U, A) & \longrightarrow & (\ker \iota_*)_{n-1} & \longrightarrow & 0 \\ & & \shortparallel & & & & \shortparallel & & \\ & & 0 & & & & 0 & & \end{array}$$

so $H_*(U, A) = 0$. The LES of the triple (X, U, A) gives

$$\begin{array}{ccccccc} H_*(U, A) & \longrightarrow & H_*(X, A) & \xrightarrow{\iota_*} & H_*(X, U) & \longrightarrow & H_{*-1}(U, A) \\ \shortparallel & & & & & & \shortparallel \\ 0 & & & & & & 0 \end{array}$$

so ι_* is an isomorphism. □

We are now in a position to prove Theorem 2.65, whose proof we had postponed in order to build some more machinery first.

Proof. (of Theorem 2.65)

$$\begin{tikzcd} H_*(X, A) \arrow[r, "\iota_*"] \arrow[d, "\pi_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X, U) \arrow[d, "\pi_{2*}"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X - A, U - A) \arrow[l, "j_*"'] \arrow[d, "\pi_{3*}"] \\ H_*(X/A, A/A) \arrow[r, "\iota'_*"] & H_*(X/A, U/A) & H_*(X/A - A/A, U/A - A/A) \arrow[l, "j'_*"'] \end{tikzcd}$$

Now $\pi_3 : (X - A, U - A) \to (X/A - A/A, U/A - A/A)$ is a homeomorphism, so π_{3*} is an isomorphism. A is closed and U is open, so $\bar{A} \subset \operatorname{int} U$. By excision, j_*, j'_* are isomorphisms, hence, so is π_{2*}. By Lemma 2.85 and 2.87, ι_* and ι'_* are isomorphisms. Therefore, finally, π_* is an isomorphism, as advertised. $\square$

2.6 Maps $S^n \to S^n$

Fix generators for $\tilde{H}_n(S^n) \simeq \mathbb{Z}$ (e.g. $S^0 = \{-1, 1\}, [S^0] = \sigma_1 - \sigma_{-1}$ generates $\tilde{H}_0(S^0)$). We have isomorphisms

$$\begin{tikzcd} \underset{}{\overset{[S^n]}{\tilde{H}_n(S^n)}} & \overset{[D^n,S^{n-1}]}{H_n(D^n, S^{n-1})} \arrow[l, "p_*"'] \arrow[r, "f_*"] \arrow[d, "\partial"] & \overset{[I^n,\partial I^n]}{H_n(I^n, \partial I^n)} \simeq \mathbb{Z} \\ & \underset{[S^{n-1}]}{\tilde{H}_{n-1}(S^{n-1})} & \end{tikzcd}.$$

If $f : S^n \to S^n$, then the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from $\mathbb{Z}$ to itself and hence we must have $f_*[S^n] = k[S^n]$ with $k \in \mathbb{Z}$.

Definition 2.88 (Degree of a map) Let $f : S^n \to S^n$ as above, then we set $\deg f := k$ the degree of f.

Properties:

(i) $\deg(f \circ g) = \deg f \cdot \deg g$, for $(f \circ g)_* = f_* \circ g_*$

(ii) $f \sim g \implies \deg f = \deg g$, for $f_* = g_*$

(iii) $\deg \mathrm{id}_{S^n} = 1$, for $(\mathrm{id}_{S^n})_* = \mathrm{id}$

(iv) If $f : S^n \to S^n$ is not surjctive, then $\deg f = 0$.

Proof. If f not surjective, have $y_0 \in S^n - f(S^n)$ such that we have

$$\begin{tikzcd} f : S^n \arrow[rr] \arrow[dr] & & S^n \\ & S^n - \{y_0\} \arrow[ur, hook, "\iota"] & \end{tikzcd}.$$

But $S^n - \{y_0\} \sim D^n$, so $H_*(S^n - \{y_0\}) = 0$. Therefore, $\iota_* = 0$ and thus $f_* = 0$. $\square$

(v) If $f : S^n \xrightarrow{\simeq} S^n$ is a homeomorphism, then $\deg f = \pm 1$.

Proof. $1 = \deg \mathrm{id}_{S^n} = \deg(f \circ f^{-1}) = \deg f \cdot \deg f^{-1} \implies \deg f$ is a unit in $\mathbb{Z}$. $\square$

Proposition 2.89 *If $p : S^n \to S^n$ is a reflection in a hyperplane, then $\deg p = -1$.*

Proof. The reflection fixe the points in a subsphere S^{n-1} and exchanges the two hemispheres. Give S^n a Δ-complex structure with the two n-simplices Δ_1^n, Δ_2^n the two hemispheres. Then $\Delta_1^n - \Delta_2^n$ generates $H_n(S^n)$. □

Corollary 2.90 *If $A : S^n \to S^n, v \mapsto -v$ is the antipodal map, then* $\deg A = (-1)^{n+1}$.

Proof. $A = p_1 \circ p_2 \circ \cdots \circ p_{n+1}$, where $p_i(v) = (v_1, \ldots, -v_i, \ldots, v_{n+1})$ is a reflection. □

Corollary 2.91 *If n is even, $A \not\simeq \mathrm{id}_{S^n}$.*

Hurewicz Homomorphism

Definition 2.92 The Hurewicz homomorphism is $\psi : \pi_n(X,p) \to H_n(X)$ with $\psi([\tilde{\alpha}]) = \tilde{\alpha}_*[S^n]$, $\psi([\alpha]) = \alpha_*[I^n, \partial I^n]$, where

$$\begin{tikzcd} \tilde{\alpha} : (S^n, *) \arrow[r] & (X,p) \\ (I^n, \partial I^n) \arrow[u, "\pi"] \arrow[ur, "\alpha"'] & \end{tikzcd}.$$

ψ is well defined: $\alpha \sim \beta$, then $\alpha_* = \beta_*$.

Proposition 2.93 *We have $\psi([\alpha + \beta]) = \psi([\alpha]) + \psi([\beta])$.*

The proof of Proposition 2.93 is postponed at this point and will be provided after some more work. According to the proposition above, $\psi : \pi_n(S^n, *) \to H_n([S^n]) \simeq \mathbb{Z}, f \mapsto \deg f$ is a homomorphism. Moreover, $\psi(\mathrm{id}_{S^n}) = 1$ and therefore ψ is surjective. Define $R : I^n \to I^n, (x, \vec{x}) \to (1-x, \vec{x})$.

Lemma 2.94 *It is $[\alpha] + [\alpha \circ R] = 0$ in $\pi_n(S^n)$, i.e. $[\alpha \circ R] = -[\alpha]$.*

Corollary 2.95 $R_*[I^n, \partial I^n] = -[I^n, \partial I^n]$.

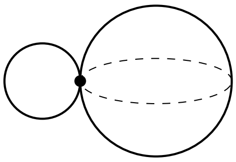
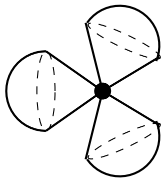
Proof. $0 = [\alpha + \alpha \circ R] = [\alpha] + \deg R\,[\alpha]$, which implies that if $p : S^{n-1} \to S^{n-1}$ is a reflection, $\deg p = \deg R = -1$. □

Definition 2.96 (Wedge Product) If (X_α, p_α), $\alpha \in A$ is a collection of spaces X_α, $p_\alpha \in X_\alpha$, we define their wedge product as

$$\bigvee_{\alpha \in A} (X_\alpha, p_\alpha) = \bigsqcup_{\alpha \in A} X_\alpha / \bigsqcup_{\alpha \in A} p_\alpha.$$

This works best if X_α's are homogeneous, i.e. if for $p, q \in X_\alpha$, there is $f_{pq} : X_\alpha \overset{\simeq}{\to} X_\alpha$ with $f_{pq}(p) = q$.

Example 2.97 S^n is a homogeneous space. Then we can drop the points p_α from the notation, e.g.

$S^1 \vee S^2$:  or $S^2 \vee S^2 \vee S^2$: 

Lemma 2.98 *If (X_α, p_α) is a good pair for all $\alpha \in A$ then there are isomorphisms*

$$\bigoplus_{\alpha \in A} \tilde{H}_*(X_\alpha) \underset{\bar{\pi}}{\overset{\bar{\iota}}{\rightleftarrows}} \tilde{H}_*(\bigvee_{\alpha \in A}(X_\alpha, p_\alpha))$$

with

$$\iota_\alpha : X_\alpha \to \bigvee_{\alpha \in A}(X_\alpha, p_\alpha), \quad X_\alpha \ni x \mapsto x,$$
$$\pi_\alpha : \bigvee_{\alpha \in A}(X_\alpha, p_\alpha) \to X_\alpha, \quad X_\beta \ni x \mapsto p_\alpha$$

and $\bar{\iota} = \sum_{\alpha \in A} \iota_{\alpha *}$ *and* $\bar{\pi} = \bigoplus_{\alpha \in A} \pi_{\alpha *}$.

Proof. □

2.7 Cellular Homology

3 Cohomology and Products

As was mentioned briefly in the last chapter, there is a covariant additive homology functor $F : C(R\text{-Mod}) \to R\text{-Mod}$ from the category of complexes over R-Mod to the category of R-modules R-Mod, for R a ring. One might now wonder if there exists a corresponding contravariant functor too. In this section we give a positive answer to this question. The corresponding contravariant functor is called the cohomology functor.

3.1 Homology with Coefficients

So far we have mainly considered $R = \mathbb{Z}$ in our homology computations. However, sometimes it might be easier to consider a different ring. This is not as easy as rewriting the homology as $H_*(X) \otimes R$ but requires some more work which we present in this section.

Tensor Product

Let R commutative ring. If M and N are R-modules, there is an R-module $M \otimes_R N = M \otimes N = \langle m \otimes n | m \in M, n \in N \rangle / \sim$ with identifications

$$\begin{aligned}(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n,\\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2,\\ r(m \otimes n) &= rm \otimes n = m \otimes rn,\end{aligned}$$

for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $r \in R$.

Remark 3.1 Note that in order for $M \otimes N$ to be an R-module again commutativity of R is crucial (alternatively, M, N could be R-bimidules for generic R). In general, $M \otimes N$ is merely an abelian group.

Example 3.2 $R \otimes N \simeq N$ via $r \otimes n \mapsto rn$.
$R = \mathbb{Z}$, $\mathbb{Q} \otimes \mathbb{Z}/a = 0$ since $x \otimes y = ax \otimes \frac{y}{a} = 0 \otimes \frac{y}{a} = 0$.

Properties:

(i) $M \otimes N \simeq N \otimes M$

(ii) $(M_1 \oplus M_2) \otimes N \simeq M_1 \otimes N \,\oplus\, M_2 \otimes N$ ($\Longrightarrow R^m \otimes R^n = R^{n \cdot m}$)

(iii) $R^m \otimes M \simeq M^m$

(iv) if $f : M_1 \to M_2, g : N_1 \to N_2$ are homomorphisms, so is $f \otimes g : M_1 \otimes N_1 \to M_2 \otimes N_2$, $m \otimes n \mapsto f(m) \otimes g(n)$

Chain Complexes

If (C_*, d) is a chain complex over R and M is an R-module, then $(C_* \otimes M, \mathrm{d} \otimes \mathrm{id}_M)$ is a chain complex, $(\mathrm{d} \otimes \mathrm{id}_M)^2 = (\mathrm{d}^2 \otimes \mathrm{id}_M^2) = 0$.

Example 3.3 $C_*^{\mathrm{cell}}(\mathbb{R}P^2) = \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}$ is a complex over $R = \mathbb{Z}$.
$C_*^{\mathrm{cell}}(\mathbb{R}P^2) \otimes \mathbb{Z}/2 = \mathbb{Z}/2 \xrightarrow{\cdot 2 = 0} \mathbb{Z}/2 \xrightarrow{\cdot 0} \mathbb{Z}/2$.

$$H_*(C_*^{\mathrm{cell}}(\mathbb{R}P^2) \otimes \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = 0, 1, 2 \\ 0, & \text{else} \end{cases} \neq H_*^{\mathrm{cell}}(\mathbb{R}P^2) \otimes \mathbb{Z}/2.$$

NB: $H_*(C \otimes M) \neq H_*(C) \otimes M$.

Lemma 3.4 *If $f : C_* \to C'_*$ is a chain map then $f \otimes \mathrm{id}_M : C_* \otimes M \to C'_* \otimes M$ is a chaiin map. If $f \sim g$, then $f \otimes \mathrm{id}_M \sim g \otimes \mathrm{id}_M$.*

Definition 3.5 (Homology with coefficients) If X is a space and G a $\mathbb{Z}$-module (i.e. abelian group), $C_*(X; G) := C_*(X) \otimes_{\mathbb{Z}} G$ is the singular chain complex of X with coefficients in G, define $H_*(X; G) := H_*(C_*(X; G))$ the homology with coefficients in G.

Best choices for G are $G = \mathbb{R}, \mathbb{Q}, \mathbb{Z}/a$ which are all rings. Note that if R is a ring, $C_*(X;R)$ is a chain complex over R.
Maps: If $g \in G$, there is a chain map $C_*(X) \to C_*(X;G)$, $x \mapsto x \otimes g$ that induces $H_*(X) \to H_*(X;G)$, $[x] \mapsto [x \otimes g]$. Also, if $f : X \to Y$, $f_\# \otimes \mathrm{id}_G : C_*(X;G) \to C_*(Y;G)$ is a chain map and induces $f_* : H_*(X;G) \to H_*(Y;G)$.

Remark 3.6 $C_*(X;\mathbb{Z}) = C_*(X)$.

Lemma 3.7 *There is a commutative square*

$$\begin{tikzcd}
H_*(X) \arrow[r, "f_*"] \arrow[d, "\bullet\otimes g"] \arrow[dr, phantom, "\circlearrowleft"] & H_*(Y) \arrow[d, "\bullet\otimes g"] \\
H_*(X;G) \arrow[r, "f_*"] & H_*(Y;G)
\end{tikzcd}$$

Definition 3.8 If X is a FCC, $C_*^{\mathrm{cell}}(X;G) := C_*^{\mathrm{cell}}(X) \otimes_{\mathbb{Z}} G$ has homology $H_*^{\mathrm{cell}}(X;G)$.

Theorem 3.9 *If X is a FCC, then $H_*(X;G) \simeq H_*^{cell}(X;G)$.*

Proof. (Sketch) $H_*(\bullet;G)$ is a functor

$$\begin{cases} \text{pairs of spaces} \\ \text{maps of pairs} \end{cases} \longrightarrow \begin{cases} \text{abelian groups} \\ \text{homomorphisms} \end{cases}$$

and $C_*(X,A;G) = C_*(X,A) \otimes G \simeq \frac{C_*(X;G)}{C_*(A;G)}$.

(i) If $f \sim g$, then $f_* = g_*$

(ii) LES of a pair: if $f : (X,A) \to (Y,B)$, there is a commuting diagram with exact rows,

$$\begin{tikzcd}
H_*(A;G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X;G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_*(X,A;G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_{*-1}(A;G) \arrow[d] \\
H_*(B;G) \arrow[r] & H_*(Y;G) \arrow[r] & H_*(Y,B;G) \arrow[r] & H_{*-1}(B;G)
\end{tikzcd}$$

(iii) Excision: if $\bar{B} \subset \operatorname{int} A$, then $j_* : H_*(X-B, A-B;G) \xrightarrow{\simeq} H_*(X,A;G)$

(iv) $H_*(\{p\};G) = \begin{cases} G, & * = 0 \\ 0, & \text{else} \end{cases}$

Functoriality together with properties (i) to (iii) mean that $H_*(\bullet;G)$ is a generalised homology theory.
Define $\tilde{H}_*(X;G) = \ker(H_*(X;G) \xrightarrow{f_*} H_*(\{p\}))$, $f : X \to \{p\}$. Then show

(i) $\tilde{H}_*(S^n;G) \simeq \tilde{H}_*(D^n, S^{n-1};G) = \begin{cases} G, & * = n \\ 0, & \text{else} \end{cases}$

(ii) if $f : S^n \to S^n$, have diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[d, "\bullet\otimes g"] & H_n(S^n) \arrow[d, "\bullet\otimes g"] \\ H_n(S^n;G) \arrow[r, "f_*"] & H_n(S^n;G) \end{tikzcd}$$

$\implies f_* : H_n(S^n;G) \to H_n(S^n;G)$ is multiplication by $\deg f$.

(iii) Then run proof of cellular homology as before.

$\square$

Example 3.10 $H_*(\mathbb{R}P^n;\mathbb{Z}/2) \simeq H_*^{\text{cell}}(\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = 0,1,\ldots,n \\ 0, & \text{else} \end{cases}$, e.g.

$C_*^{\text{cell}}(\mathbb{R}P^3;\mathbb{Z}/2) = \mathbb{Z}/2 \xrightarrow{\cdot 0} \mathbb{Z}/2 \xrightarrow{\cdot 2 = 0} \mathbb{Z}/2 \xrightarrow{\cdot 0} \mathbb{Z}/2$

3.2 Cohomology

Definition 3.11 (Hom) If M, N are R-modules,

$$\mathrm{Hom}(M,N) = \{\phi : M \to N | \phi \text{ is a homomorphism}\}$$

is an R-module.

$$
\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \simeq H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "d_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}
$$