Elastic Energy Problem

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1 Context

Let the displacement field of an elastic solid be $\mathbf{u}(\mathbf{r},t) = \sum_i^3 \mathbf{u}_i(\mathbf{r},t)\hat{\mathbf{e}}_i$, as a function of which it can be shown that the kinetic and potential energies densities of a linear elastic solid are

$$T = \frac{1}{2}\sum_i \rho \dot{u}_i^2 \tag{1.0.1}$$

$$V = \frac{1}{2}\sum_{ijkl} C_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \tag{1.0.2}$$

Making the Linear Elastic Lagrangian

$$L = T - V = \frac{1}{2}\int_V \left(\sum_i \rho \dot{u}_i^2 - \sum_{ijkl} C_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right) dV \tag{1.0.3}$$

The theory of RUS (Resonant Ultrasound Spectroscopy, a subset of the theory of elasticity) proposes a harmonic solution to the displacement field, e.g. one of the form

$$\mathbf{u}(\mathbf{r},t) = \mathbf{u}(\mathbf{r})e^{i\omega t} \tag{1.0.4}$$

Classically, the theory of RUS goes on to argue that this produces the following Linear Elastic Lagrangian

$$L = T - V = \frac{1}{2}\int_V \left(\sum_i \rho \omega^2 u_i^2 - \sum_{ijkl} C_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right) dV \tag{1.0.5}$$

Where in the above, $u_i = u_i(\mathbf{r})$, the spatial component only of 1.0.4 .

2 Problem

Classical RUS theory seems to have derived the result 1.0.5 from the displacement field 1.0.4 (which we note is complex) by supposing that

$$T = \frac{1}{2}\sum_i \rho \left|\dot{u}_i(\mathbf{r},t)\right|^2 \equiv \frac{1}{2}\sum_i \rho \left|\dot{u}_i^2(\mathbf{r},t)\right| \qquad \text{Why?} \tag{2.0.1}$$

where $|z|$ is the complex modulus, defined

$$|z| = \sqrt{zz^*} = \sqrt{\mathrm{Re}\,[z]^2 + \mathrm{Im}\,[z]^2} \qquad \forall z \in \mathbb{C} \tag{2.0.2}$$

such that the kinetic energy becomes

$$T = \frac{1}{2}\sum_i \rho \dot{u}_i(\mathbf{r},t)\dot{u}_i^*(\mathbf{r},t) \tag{2.0.3}$$

$$= \frac{1}{2}\sum_i \rho\omega^2 u_i(\mathbf{r})^2 \tag{2.0.4}$$

Question 1: What allows us to suppose that kinetic energy is the sum of the complex moduli of the velocity of the displacement field? I.e., what makes 2.0.1 a legal expression based on the displacement field 1.0.4 ?

As for the potential energy, RUS theory supposes something even stranger (in my opinion) than it does for the kinetic energy, specifically that for the complex displacement field 1.0.4

$$V = \frac{1}{2}\sum_{ijkl} C_{ijkl}\frac{\partial u_i(\mathbf{r},t)}{\partial x_j}\left(\frac{\partial u_k(\mathbf{r},t)}{\partial x_l}\right)^* \qquad \text{Why?} \tag{2.0.5}$$

such that, trivially, the potential energy becomes

$$V = \frac{1}{2}\sum_{ijkl} C_{ijkl}\frac{\partial u_i(\mathbf{r})}{\partial x_j}\frac{\partial u_k(\mathbf{r})}{\partial x_l} \tag{2.0.6}$$

Question 2: What allows us to suppose that the potential energy is the sum of the product of the complex conjugates of the spatial derivatives of the displacement field?