Question: 1(a)

Bayes' Formula: $f(x|y) = \frac{f(y|x)f(x)}{f(y)}$

Where:

Prior density of x is: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$

Observable y is given by: $y = x^2 + \eta$, where η $\sim N(0,1)$. Thus, given x, y is normally distributed with mean x^2 and variance 1:

$$y|x \sim N(x^2, 1)$$

Density of y given x is: $f(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x^2)^2}{2}}$

Using Bayes' Formula Posterior Density :

$$f(x|y) \propto f(y|x)\, f(x)$$

Substitute the expression for $f(x|y)$ and $f(x)$:

$$f(x|y) \propto \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x^2)^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}\right)$$

$$f(x|y) \propto e^{-\frac{(y-x^2)^2}{2}} e^{-\frac{(x-1)^2}{2}}$$

Combining the exponents:

$$f(x|y) \propto e^{-\frac{1}{2}\left[(y-x^2)^2 + (x-1)^2\right]}$$