Measurements of the Electromagnetic Force Constants

Physics 2502, University of Connecticut
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There are two fundamental constants in the classical theory of electromagnetism, called the permittivity and the permeability of free space. These constants determine the magnitude of the electric and magnetic forces that exist between charges and between currents in a vacuum. Students measure the values of these two constants in this experiment, using a balance to compare electromagnetic forces between charged plates and current-carrying wires to the gravitational force on small masses. The product of these two constants is related to the speed of light, so this experiment can also be considered as a measurement of the speed of light. The measurements are carried out in air, so the speed of light being measured is that in the medium of air.

I. INTRODUCTION

In the theory of electromagnetism, there are two aspects of the interactions between charges and fields: the action of charges in producing fields in the space around them, and the forces caused by the fields in the space around them that act on the charges. Most discussions of the theory are made in terms of the fields $\vec{E}$ and $\vec{B}$, called the *electric field* and the *magnetic induction*, which are measured in volts/cm and gauss, respectively. The force on a body of charge q moving at velocity v in a region of space with fields $\vec{E}$ and $\vec{B}$ is given by the Lorentz force equation.

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) \tag{1}$$

Note that there are no physical constants in Eq. 1. This is because these fields are defined in terms of the forces they produce. In this sense, Eq. 1 can be considered to be a definition of the fields $\vec{E}$ and $\vec{B}$.

The other half of the story is how the fields arise in the first place. They are produced by charges and currents, in a way that is described by the first two of Maxwell's equations, Gauss's law for electric fields from charges, and Ampere's law for magnetic induction from currents. In vacuum, these laws take on the forms

$$\oint_S \vec{E} \cdot \hat{n}\, dA = \frac{Q_S}{\epsilon_0} \tag{2}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C \tag{3}$$

where Q_S is the total charge contained within the Gaussian surface S, and I_C is the total current passing though the Amperean loop C, including the *displacement current.* Notice that Maxwell's equations involve physical constants ϵ_0 and μ_0.

There is an alternate way to present classical electromagnetic theory, that moves the constants into the force equation, and leaves Maxwell's equations without constants. This involves the replacement of electric field with *electric displacement* $\vec{D} = \epsilon_0\vec{E}$ measured in Coulombs per square meter, and magnetic induction with the *magnetic field* $\vec{H} = (1/\mu_0)\vec{B}$ measured in oersteds. It is interesting to note that the original developers of the theory of electric fields and forces chose the first convention, while those working on magnetic forces and fields chose the second, with the result that the *electric field* belongs to one set, while the *magnetic field* belongs to the other. Computing the forces between two charge or current systems involves both the field generating and the field sensing equations, so the result is the same regardless of which convention is used. In what follows, we will use the fields $\vec{E}$ and $\vec{B}$.

II. THE COULOMB BALANCE

The Coulomb balance is a mechanical device that is used to measure the magnitude of electrostatic attraction between two oppositely charged bodies. A side view of the apparatus is shown in Fig. 1, showing the principal parts. When a voltage source is connected between the upper and lower arms of the balance, an electric field is set up between the two plates, as illustrated in Fig. 2. Note that the plates are only parallel when they are in contact, which cannot be the case when they are charged. When they are separated by an angle α, the two plates are no longer parallel, and so the field between them is not uniform, but varies with radial distance α. For small values of α, the electric field lines follow a circular pattern with the pivot point as its center, as shown in the figure, and distortions

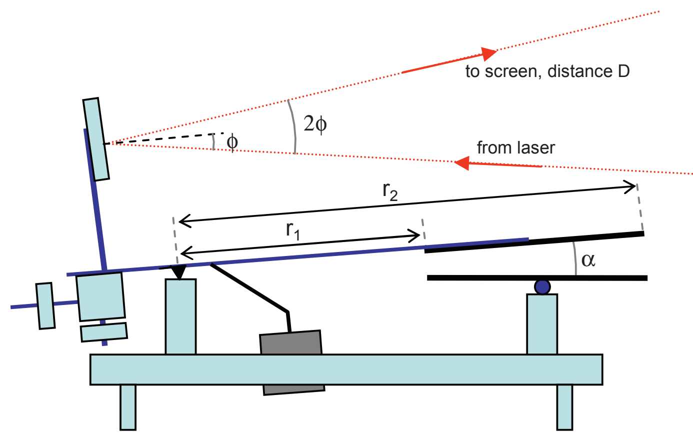

FIG. 1: Sketch of the Coulomb balance used to measure electrostatic forces between two plates. A charge difference is placed on the two plates by connecting a high-voltage source across them, not shown in the figure. Masses are placed in the center of the upper plate when no voltage is applied in order to determine the gravitational force that is equivalent to the electrostatic force at a given potential difference. The displacement of the balance is measured by monitoring displacements of the light spot on a vertical screen located a distance D from the mirror. The screen is to the right, outside the view shown in the figure.

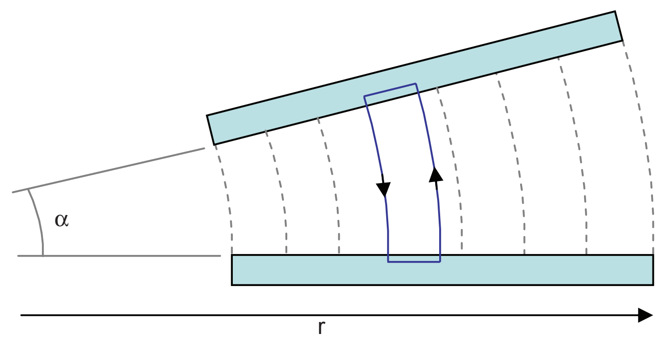

FIG. 2: Diagram used to analyze the electric field pattern produced when two charged conducting plates are supported next to each other in the geometry of the Coulomb balance. The dashed lines represent the electric field between the plates. The closed path in the illustration is used in the text to derive an expression for the electric field between the plates as a function of the radial distance r.

from this pattern are confined to small regions near the edges. These distortions, which are sometimes called *edge effects* are ignored in this treatment.

To derive how the electric field intensity varies with r, consider the closed path shown in the middle of Fig. 2. Electrostatic forces are conservative, so the total work one does in transporting a test charge q around this closed path must be zero. The line segments at the top and bottom of the loop involve no work because they are inside the conductor, where the electric field is zero. The inner arc-shaped path within the gap contributes work $qE(r_i)r_i\alpha$, while the outer path contributes $qE(r_o)r_o\alpha$, if the radii of the two arcs are r_i and r_o, respectively. These two must sum to zero, leading to the result that $r_iE(r_i) = r_oE(r_o)$. Since this must hold for any values of r_i and r_o, it follows that $rE(r) = constant$, or

$$E(r) = \frac{U_0}{r} \tag{4}$$

where U_0 is a constant yet to be determined. To determine it, consider the work done to carry a test charge q along just one of the arc-shaped paths between the two plates, whose absolute value is $q\alpha rE(r) = q\alpha U_0$. This value divided by q is the electric potential between the two plates, which is set by the voltage V coming from the external source. Therefore $U_0 = V/\alpha$.

Gauss's law requires that the field E above a conducting surface with local areal charge density σ is

$$E = \frac{\sigma}{\epsilon_0} \tag{5}$$

which implies that σ also varies like $1/r$ across the surface of the plates. Because the balance is only free to move rotationally, the balance of forces must be computed as a sum of torques that add to zero under conditions of equilibrium.

The torque coming from electrostatic attraction is computed in the following way. Consider the plates as having a width W along the rotation axis, and inner and outer radii r_1 and r_2 as shown in Fig. 1. Break up the area of the upper plate into small stripes of length W and radial width dr. The electrostatic force on this stripe is proportional to the charge on this stripe times the electric field $E(r)$. Actually it is only half that value, because the charges on the surface of the conductor are, roughly speaking, half inside the conductor (where the field is zero) and half outside the conductor (where the field is $E(r)$). The total torque from electrostatic forces is the sum of the forces on all of these stripes, multiplied by their lever arms r. Reducing this to an integral,

$$\begin{aligned}
\tau &= -\frac{1}{2}\int_{r_1}^{r_2} r\frac{U_0}{r}\sigma W\,dr \\
&= -\frac{1}{2}\int_{r_1}^{r_2} r\left(\frac{U_0}{r}\right)^2 \epsilon_0 W\,dr \\
&= -\frac{WV^2\epsilon_0}{2\alpha^2}\ln\frac{r_2}{r_1}
\end{aligned} \tag{6}$$

In the experiment, the electrostatic torque is measured by comparing it to the torque produced by the gravitational force acting on a known mass m, which is treated as a point mass placed on the upper plate at a distance r_0 from the rotation axis. Setting these two torques equal,

$$-mgr_0 = -\frac{WV^2\epsilon_0}{2\alpha^2}\ln\left(\frac{r_2}{r_1}\right) \tag{7}$$

which leads to the final result, in which the electrostatic constant ϵ_0 is computed in terms of known or measured quantities.

$$\epsilon_0 = \frac{2mg\alpha^2 r_0}{WV^2\ln r_2/r_1} \tag{8}$$

III. THE CURRENT BALANCE

The current balance is very similar in its design to the Coulomb balance described above, except that what is measured instead of the electrostatic force is the magnetic force between two current-carrying wires. A side view of the apparatus is shown in Fig. 3. When a current source is connected to the two ends of the conducting wires, shown extending into the page in the figure, a magnetic field is set up in the region around the wires, as illustrated in Fig. 4. The magnetic field lines generated by the current flowing in the upper wire follow a circular pattern around the axis of that wire, which extends into the interior of the lower wire, where a current is present that flows in the opposite direction. This pattern is only exact for a wire of infinite length. Distortions from this pattern are present near the ends of the wires where the current changes direction. These distortions, which are sometimes called *end effects* are ignored in this treatment.

Because of the finite resistivity of the conductor, the current tends to distribute itself uniformly across the cross sectional area of the wire. This produces a magnetic field pattern that consists of circles centered on the wire axis, as shown in Fig. 4. Ampere's law gives the magnitude of the magnetic field as a function of the distance x from the center of the upper wire.

$$B(x) = \frac{\mu_0 I}{2\pi x} \tag{9}$$

To compute the net magnetic force between the two wires, consider first the case where the wire diameter a is negligible. In that limit, the current in the lower wire samples the magnetic field from the upper wire at a single point

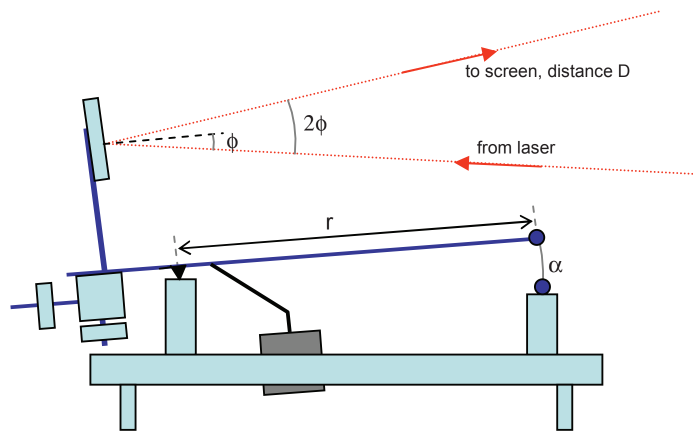

FIG. 3: Sketch of the current balance used to measure magnetic forces between two parallel wires. An alternating current (in and out of the page in the figure) is passed through the two wires whose circular cross sections are visible to the right in the figure. The currents are in opposite directions, creating a repulsive force between the two wires. The force is compensated by adding weights to a small tray on the upper wire, until the magnetic and gravitational forces cancel. The displacement of the balance is measured by monitoring displacements of the light spot on a vertical screen located a distance D from the mirror. The screen is to the right, outside the view shown in the figure.

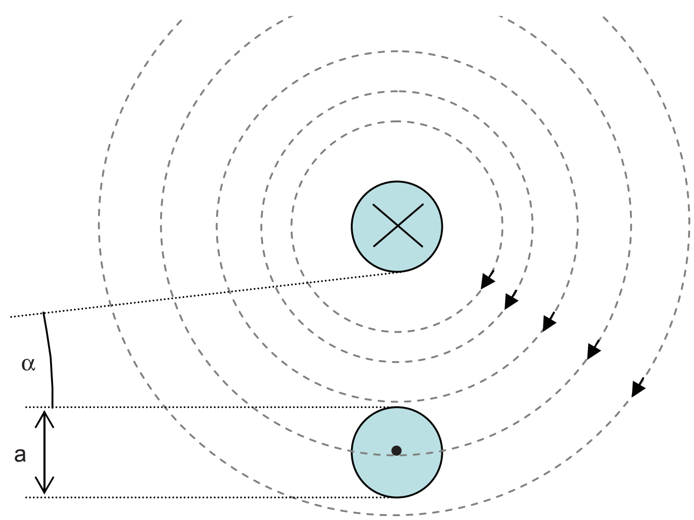

FIG. 4: Diagram used to analyze the magnetic field pattern produced by the current flowing in the upper wire. The force on the lower wire is computed as the action of this field on the current flowing in the lower wire, which is assumed to be uniformly distributed across its cross section.

a distance $x = \alpha r$ from the upper wire, leading to the following expression for the repulsive force in the case of two currents of the same magnitude I and opposite direction flowing in the two wires,

$$\tau = rBIL = \frac{\mu_0 I^2 L}{2\pi\alpha} \tag{10}$$

where L is the length of the parallel section of the two wires.

In this experiment, the approximation of negligible wire diameter a is rather poor because one works with separations distances $r\alpha$ that are only one order of magnitude larger. Because of this, some attention must be given to the corrections to Eq. 10 that occur with a non-zero value of a. There are two independent effects to be considered.

The first is that the distance between the wires can no longer be taken to be αr, but instead should be the distance between the centers of the wires, which is $\alpha r + a$. The second effect comes from the fact that the magnetic field from the upper wire is not constant across the area of the lower wire, but is strongest near the upper surface where it is closest to the upper wire, and weakest near the bottom. To take this into account, consider the current flowing in the lower wire to be composed a circular bundle of many tiny wires, each carrying a portion of the total current that is proportional to its cross sectional area. In the limit of many such wires, the total force becomes an integral over the area of the lower wire,

$$F = L \int_0^{a/2} y\, dy \int_0^{2\pi} d\phi B(x) j(y) \tag{11}$$

where x represents the distance of a point from the center of the upper wire, y is the distance of the same point from the center of the lower wire, ϕ is the azimuthal angle of that point with respect to the center of the lower wire, and $j(y)$ is the current per unit area flowing on the lower wire, here assumed to be constant.

Since the integral is over $y\, dy\, d\phi$, we need to find how $B(x)$ depends on y and ϕ. Using the law of cosines, this relationship is

$$x = \sqrt{x_0^2 + y^2 + 2x_0 y \cos\phi} \tag{12}$$

leading to the result for the total force

$$F = \frac{\mu_0 I L j}{2\pi} \int_0^{a/2} y\, dy \int_0^{2\pi} d\phi \frac{1}{\sqrt{x_0^2 + y^2 + 2x_0 y \cos\phi}} \tag{13}$$

where $x_0 = \alpha r + a$ is the distance between the centers of the two wires. This integral is very difficult to compute exactly, but a good approximation can be obtained by noticing that the constant x_0 is about an order of magnitude larger than y, so one can expand the integrand in powers of y/x_0. The first few terms in the Taylor series are

$$(x_0^2 + y^2 + 2x_0 y \cos\phi)^{-\frac{1}{2}} = \frac{1}{x_0}\left[1 - \frac{y}{x_0}\cos\phi - \left(\frac{1}{2} - \frac{3\cos^2\phi}{2}\right)\frac{y^2}{x_0^2} + \left(\frac{3\cos\phi}{2} - \frac{5\cos^3\phi}{2}\right)\frac{y^3}{x_0^3} + \mathcal{O}\left(\frac{y}{x_0}\right)^4\right] \tag{14}$$

Keeping only the first non-zero correction in y/x_0 leads to

$$F = \frac{\mu_0 I L j}{2\pi x_0} \int_0^{a/2} y\, dy \int_0^{2\pi} d\phi \left[1 - \left(\frac{1}{2} - \frac{3\cos^2\phi}{2}\right)\frac{y^2}{x_0^2} + \mathcal{O}\left(\frac{y^4}{x_0^4}\right)\right] \tag{15}$$

This integral is easy to evaluate, leading to

$$F = \frac{\mu_0 L I^2}{2\pi x_0}\left[1 + \frac{a^2}{32x_0^2} + \mathcal{O}\left(\frac{a}{x_0}\right)^4\right] \tag{16}$$

Equating the torque from this magnetic force to the weight of a known mass m leads to an expression for the magnetic permeability constant in terms of known or measured quantities.

$$\mu_0 = \frac{64\pi x_0^3 m g}{L I^2 (32x_0^2 + a^2)} \tag{17}$$

IV. EXPERIMENTAL PROCEDURE

The following initial set-up steps apply to either the Coulomb balance or the current balance. The upper arm of the balance sits on two knife edges which rest on the flat surfaces of two posts near the rear corners of the base plate. The balance is a delicate instrument and should be handled carefully. Particularly sensitive to damage are the two knife edge suspension points and the surfaces upon which they rest. The upper balance assembly can be raised off of the knife edges by a lift mechanism. The balance should be lifted off the knife edges any time it is not in use.

Attached to the upper arm and hanging under it is a vertical metal plate. When the upper arm is properly aligned, this plate should pass through a slot in the base plate. Mounted in the base plate on either side of the slot are magnets which induce a damping force on the plate whenever it moves up or down inside the slot. When everything is

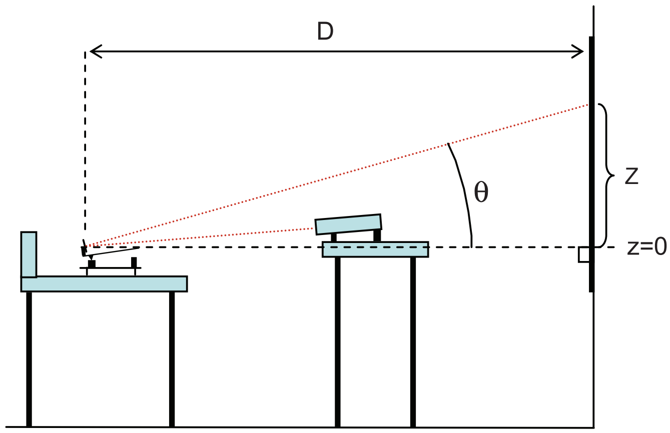

FIG. 5: Diagram of the scheme used to precisely measure the angular displacement of the balances used for this experiment. Changes in the angle θ of the reflected laser beam with respect to the horizontal are determined by measuring the height z of the beam spot on the viewing screen shown by the heavy vertical bar at the right side of the figure.

properly aligned, the upper arm should be free to oscillate with an amplitude of several degrees without the damping plate hitting anything or scraping on the sides of the slot. The oscillations should damp away with a time constant between 5 and 10 seconds. Make sure that the base plate is resting on a firm level surface, and that it does not rock on its supporting feet.

There are two adjustable knobs on the back side of the upper arm of the balance. The one that extends horizontally out the back is used to set the equilibrium position of the balance. Adjust it by rotating it in one direction or the other, until the equilibrium position of the balance is found with the electrodes a few mm apart. You may need to adjust this later on during the measurements to find an optimal setting. The knob that is hanging under the upper arm is used to adjust the sensitivity of the balance. Raising it up makes the balance more sensitive, so that its equilibrium position moves a larger amount in response to a given torque. Higher sensitivity also means that the balance is more susceptible to vibrations and air currents, such that finding the equilibrium position can be more difficult. Finding an optimal compromise between stability and sensitivity is one of the most important challenges in conducting this experiment.

Throughout the measurement, it is important to minimize any air currents in the vicinity of the balance. This is true for both measurements, but the Coulomb balance is particularly sensitive because of the large area of the plates. People in the room should minimize their movements while measurements are being taken, so that the balance can reach equilibrium for a long enough period for measurements to be taken.

Displacements of the balance will be measured using a laser beam reflecting off the mirror on the upper arm of the balance onto a viewing screen some distance away, as shown in Fig. 5. The viewing screen should be mounted on a fixed vertical surface. A white board mounted on the wall is a good choice, allowing displacements to be recorded using a marker. Mount the mirror in the space between the screen and the balance and align it so that the beam is centered on the mirror. The incident beam direction should be pointing slightly downward, as shown in the figure, so that the reflected spot remains within the range of the white board. Check that the reflected beam spot on the viewing screen is visible without obstruction through the full range of motion of the upper balance arm. Check that the plane swept out by the reflected laser beam, as the upper arm of the balance moves through its range of motion, intersects the screen at right angles to within a few degrees.

Carefully mark the point labeled $z = 0$ on the screen. This will serve as the origin for all measurements of z throughout the experiment. Measure and record the distance D, together with its error. Once D has been measured and the point $z = 0$ marked on the screen, make sure the laser and balance are not moved. If either one is disturbed after this, stop the measurements and go back and start from the beginning again.

A. Coulomb balance procedure

Carry out the above steps to set up the balance for measurements. Make sure that the orientation of the lower plate is such that the two plates are parallel and touching everywhere when they come together. Align the upper arm on its mounts so that the edges and corners of the upper and lower plates line up as exactly as possible. Measure the width W of the upper and lower plates using a vernier caliper. Take the average of your two measurements as the value for W, and record your measurement error. Measure the length of the two plates, and record their average values as H, together with its measurement error. On the upper plate, measure the distance r_1, which is the radial distance between the rocking axis defined by the knife edges and the inner edge of the plate, as shown in Fig. 1. The value of r_2 is obtained as $r_1 + H$. On the top of the upper plate is a cross-hair marking the spot where you will place weights during the measurement. Measure the distance H_2 from the cross hair to the center of the inner edge of the plate, and record its value and error. Confirm that $H_2 = H/2$ within measurement error.

Prepare several small weights with masses between 20 and 200 mg to be used during the measurement. One convenient way to do this is to cut out strips of office paper with different areas. Before cutting the paper, measure the mass per unit area of the paper by weighing an entire sheet and measuring its area. Be aware that the density of office paper can vary from one package to the next, and is highly sensitive to the ambient humidity in the area where it is stored. If you chose to weigh more than one sheet at a time for increased accuracy, make sure that all of the sheets came from the same package and were stored together during the time since the package was opened. Determine the masses of your paper weights by measuring their dimensions as you cut them out, together with your measurement errors. Make sure that the equilibrium position of the unweighted balance is set high enough that the plates do not come into contact when the heaviest of your masses is placed in the center of the upper plate, but that they do come together when a penny is used.

Because the electric force measured in the Coulomb balance is attractive and increases with decreasing separation between the plates, special restrictions must hold in order to achieve stable equilibrium. A detailed analysis of these conditions is presented in App. A. In summary, stable equilibrium with high voltage present on the plates can only be achieved if the separation angle α is greater than $(2/3)\alpha_0$, where α_0 is the equilibrium separation angle of the unweighted plates when the voltage is zero. To minimize edge effects, you should make sure that the equilibrium settings with the weights present are in the range where the plates are 5-15 mm apart. Putting all of this together, set the unweighted equilibrium point such that the plates are about 10-20 mm apart. Adjust the sensitivity so that adding the lightest weights produces a significant displacement from α_0, but not sinking past $(2/3)\alpha_0$. As you increase the weight added to the upper plate, watch that the displacement does not push the balance below $(2/3)\alpha_0$. When it does, stop and switch over to measuring in high-voltage mode (see below), then come back later and decrease the sensitivity of the balance in order to do the heavier weights. Remember that it is a waste of your time to measure displacements past $(2/3)\alpha_0$ because, if you do, you will not be able to achieve stability at an equivalent displacement using electric forces later. Stability under electric forces is only possible under the condition that $(2/3)\alpha_0 < \alpha < \alpha_0$.

Place the penny on the upper plate and record the displacement of the beam spot on the viewing screen as $m = \infty$. Take off the penny and measure the unweighted equilibrium displacement of the spot on the screen, marking it as $m = 0$ on the screen. Record the z positions and errors Δz for these two limiting mass values. The z positions for all intermediate masses between 0 and ∞ will lie between these two limits. Using a meter stick, mark the point that is 2/3 the way from $m = \infty$ to the $m = 0$ mark. In all of the subsequent measurements in this run, you should make sure that you do not go below this line. One by one, starting with the smallest mass, place each one on the cross-hairs in the middle of the upper plate and record the values of z and errors Δz for each. Be careful in placing the weights on the plate that you do not knock or disturb its alignment in any way. If you are using weights cut from a sheet of paper, fold the paper so that it can be easily centered on the cross hairs and picked up without touching the apparatus, or use tweezers to place and remove them.

Turn on the high-voltage supply and test it using a volt meter and a high-voltage probe that has been approved by your TA for this purpose. After confirming that the supply is working, set the output voltage to zero and turn off the supply. Remove all of the masses from the upper plate of the balance and connect the high-voltage supply to its two electrical terminals. Unless you know that the power supply is internally protected against shorts, a large resistor (10 MΩ) should be used between the supply and the terminal on the high-voltage side to prevent damage to the supply, should an arc accidentally occur between the two plates. *This resistor is does not make the apparatus safe to touch when the high voltage is present!* Establish and respect a safety zone of 20-30 cm in all directions around the apparatus while the high-voltage supply is connected. Besides body parts, this also includes all clothing and any items such as notebooks, pens, or any electronics that is not required for operating the balance. A good rule is to announce "voltage on" each time that high voltage is present on the balance, and "voltage off" each time it is removed, so that everyone present in the room is aware of conditions on the bench.

Turn on the high-voltage supply and slowly ramp up the voltage, watching the spot on the screen for the first sign of a shift in the equilibrium position. This should occur within the first 200 volts. Now increase the voltage more slowly, waiting between steps for the balance to find its new equilibrium, until you reach the displacement that you recorded for the lightest weight. Using the high-voltage probe, record the voltage reading that gives the best match to that displacement, then vary it above and below this point to find your measurement error on V. Continue increasing the voltage until you find values for V and ΔV for each of the heavier masses.

If it does happen that you increase V high enough that the plates dip below the $(2/3)\alpha_0$ point, they will eventually come together and touch, creating a spark. This is not a disaster, but requires prompt action. Turn off the high-voltage supply using the on/off switch, then turn the control knob down to zero. The balance usually swings strongly after sparking, so wait for it to settle down again, then turn the supply on again and ramp the voltage back up again. If air currents are responsible for the oscillations that caused the spark, try to reduce them before continuing your measurements. There is a practical minimum distance for the separation between the plates close to 1 mm, below which it is practically very difficult to make a measurement without sparking. If you find that your heaviest mass results in a plate spacing that is too small, you may need to adjust the equilibrium balance position to a higher setting and repeat the measurement.

Turn off the high-voltage supply and disconnect it from the balance. Use a lead to short the two arms of the balance together, then announce "voltage off", before touching the apparatus with your hands. You will need to remember to remove the shorting cable before turning on the voltage again later. Increase the equilibrium position by adjusting the knob on the back of the upper plate, then repeat all of the above measurements a second time, starting with a re-measurement of the $m = \infty$ point with the penny, and the $m = 0$ point without any weights present. You will find that with a higher equilibrium position, larger high voltage values will be required to achieve a match with the displacement for a given weight. If this requires raising the high voltage higher than the supply can safely provide, you may chose to create and substitute some new lighter weights instead. If you do this, keep in mind that the percent errors on the paper masses will increase substantially for masses less than 10 mg.

B. current balance procedure

Carry out the steps outlined at the head of this section to set up the balance for measurements. Make sure that the upper and lower wires of the current balance are straight and parallel so that they are touching all along their length when the two arms come together. Align the upper arm on its mounts so that the two wires are lined up as exactly as possible, when viewed from above. Measure the length L of the section of the two wires where they are running parallel to each other. Take the average of your two measurements as the value for L, and record your measurement error. On the upper arm, measure the distance r, which is the radial distance between the rocking axis defined by the knife edges and the center of the wire, as shown in Fig. 3. Use a vernier caliper to measure the diameters of the upper and lower wires. Record the average value as a, together with your measurement error. On the top of the upper wire is a small tray where you will place weights during the measurement. Make sure that the center of the tray is aligned with the center of the wire, so that a second measurement for r is not needed.

Prepare several small weights with masses between 20 and 200 mg to be used during the measurement. One convenient way to do this is to cut out strips of office paper with different areas. Before cutting the paper, measure the mass per unit area of the paper by weighing an entire sheet and measuring its area. Be aware that the density of office paper can vary from one package to the next, and is highly sensitive to the ambient humidity in the area where it is stored. If you chose to weigh more than one sheet at a time for increased accuracy, make sure that all of the sheets came from the same package and were stored together during the time since the package was opened. Determine the masses of your paper weights by measuring their dimensions as you cut them out, together with your measurement errors.

Carefully place the penny in the tray on the upper wire, making sure that the wires come into full contact when at rest. Record the displacement of the beam spot on the viewing screen as $z = 0$. All other displacements will be measured using this position as the origin. Adjust the equilibrium position of the balance so that it comes to rest with the wires several mm (less than 10) apart. Adjust the sensitivity of the balance so that there is little oscillation of the bars after a minute of settling time, or at least that the center of the oscillations can be accurately determined.

The earth's magnetic field is sufficiently strong to produce forces that can distort the results of this measurement, depending on how the balance is oriented. To avoid this complication, an alternating-current supply is used. An AC current supply capable of producing 20 A into a few Ohms resistance can be built using a pair of Variacs and a step-down transformer. Turn the power switches to off on both of the Variacs. Connect the Variacs in series, and the output of the second one to the primary winding of the transformer. Connect the secondary winding of the transformer to the two terminals of a current meter, making sure that the meter is capable of at least 10 A. Turn the first Variac to zero and the second one to its maximum setting. Plug the first Variac into a power outlet and turn it on. Slowly ramp up the current by turning up the output of the first Variac, that was set to zero in a preceding step. Watch the current meter as you ramp it up, and confirm that you can reach 10 A before going above half the maximum output on the first Variac. Ramp the current back down to zero by lowering the output of the first Variac. The control on the first Variac is the current control knob for the remainder of this experiment.

Disconnect the current meter from the transformer output, and in its place connect the transformer to the terminals of the current balance. Using a heavy copper lead, connect the far terminal of the upper wire to the far terminal of the lower wire. This way, the currents through the two wires are going in opposite directions in the straight parallel section. The current meter is not connected in series with the balance for this measurement because it may be damaged by the currents as high as we wish to use. Instead, you will measure the current indirectly by measuring the voltage drop along the heavy copper lead that conducts the current between the upper and lower arms of the balance. Disconnect the copper lead from the apparatus and measure its resistance. This resistance will increase as the wire heats up under heavy current load, so you will need to measure it several times throughout the experiment. The resistance is small, so you need to measure the resistance of the voltmeter leads and subtract them away to get an accurate value. Measure this resistance with a second ohmmeter and a different pair of leads to obtain an estimate of your measurement error. Then reconnect the copper leads to the balance, and place a voltmeter across its two ends to record the voltage drop. Make sure the voltmeter is in AC mode.

Because the current balance is arranged to measure repulsive rather than attractive forces, there are no restrictive conditions for stable equilibrium with the current balance like there were for the Coulomb balance. The only restriction on the permissible range of α is that the two wires should be more than 2 diameters apart at equilibrium, otherwise higher-order terms that have been dropped in Eq. 16 might contribute significantly and throw off the analysis.

With no weights in the tray and no current flowing in the wires, measure the equilibrium position z_0 of the balance. Now place the smallest mass into the weight tray. This should cause the upper arm of the balance to move downward. Slowly ramp up the current through the wires until the displacement moves back to its original position z_0. Record the value of the voltage drop across the copper lead as an indicator of the current in the wire. Adjust the current above and below the central value and obtain an estimate of the measurement error on the current needed to balance the weight of the mass. Turn off the current, quickly remove the copper lead, and measure its resistance. At currents of 1-2 A there should not be much change in the resistance, but at higher currents there may be a substantial increase due to heating. This is why the measurement should take place promptly after turning off the current source, before the wire has had a change to cool off.

At this point you should also check for hot spots along the wire. A defective lead may develop hot spots, often near the end terminals, that can lead to melting of the insulation. Any time you smell a scent of burning you should turn off the current source immediately and investigate the cause of the smell. In no case should you increase the current above 20 A.

Turn off the current supply and disconnect it from the balance. Increase the equilibrium position by adjusting the knob on the back of the upper plate, then repeat all of the above measurements a second time, starting with a re-measurement of the $z = 0$ point with the penny. You will find that with a higher equilibrium position, larger current values will be required to achieve equilibrium given weight. If this requires raising the current higher than 20 A, you may chose to create and substitute some new lighter weights instead. If you do this, keep in mind that the percent errors on the paper masses will increase substantially for masses less than 10 mg.

V. DATA ANALYSIS

For the Coulomb balance, Eq. 8 predicts that a plot of $(V/\alpha)^2$ vs. m will yield a straight line with a zero y-intercept. Experimental values for α are obtained using the relations,

$$\begin{aligned} \theta &= \arctan\left(\frac{z}{D}\right) \\ \alpha &= \frac{\theta - \theta_\infty}{2} \end{aligned} \tag{18}$$

where z is the spot displacement on the viewing screen relative to the point $z = 0$ shown in Fig. 5, and θ_∞ is the value of θ at the $m = \infty$ point, where α is defined to be zero. Fit your data for $(V/\alpha)^2$ vs. m to a straight line using a least-squares fit. Be sure to propagate the errors from V and α into total errors for the points on the y axis of the fit plot. Compute the contribution to the errors coming from the uncertainties on the m values as Δm multiplied by the slope of the best-fit line. These should be added in quadrature to the errors coming from V and α in computing the total error Δy for each data point. Verify that the y-intercept of this line is zero within the error. Use the slope of this line and its error, together with the values of the other parameters in Eq. 8 and their errors, to extract a measured value for ϵ_0 and its uncertainty. Compare this result to the accepted value for ϵ_0.

For the current balance, Eq. 17 predicts a linear relationship between I^2 and m for a given equilibrium displacement α. Experimental values for α are obtained using Eq. 18. Fit your data for I^2 vs. m to a straight line using a least-squares fit. Be sure to propagate the errors from I into errors on I^2 to obtain errors for the points on the y axis of the fit plot. Compute the contribution to the errors coming from the uncertainties on the m values as Δm multiplied by the slope of the best-fit line. These should be added in quadrature to the errors coming from I in computing the total error Δy for each data point. Verify that the y-intercept of this line is zero within the error. Use the slope of this line and its error, together with the values of the other parameters in Eq. 17 and their errors, to extract a measured value for μ_0. Compare this result to the accepted value for μ_0.

One of the triumphs of Maxwell's classical theory of electromagnetism is its prediction for the speed of light waves in vacuum. If light is an electromagnetic wave then Maxwell's equations predict that it should travel at a unique speed c in vacuum, regardless of its wavelength. This value is

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \tag{19}$$

Use your measured values for ϵ_0 and μ_0 to estimate the speed of light in vacuum, and your uncertainty on this value. Compare your estimate with the published value of c.

The physical constants ϵ_0 and μ_0 actually apply only to measurements made in vacuum, whereas your measurements are made in air. Use the published value of the index of refraction of light in air to argue that corrections for the presence of air in this experiment are expected to be much smaller than the experimental uncertainties coming from the measurement.

Acknowledgments

This document was written by Prof. Richard Jones, based on an earlier write-up by Prof. Ed Eyler and on information contained in the User's Manuals for the Coulomb balance and the current balance issued by their manufacturer, Sargent-Welch Scientific Co., 7300 N. Linder Ave., Skokie, Ill. 60076.

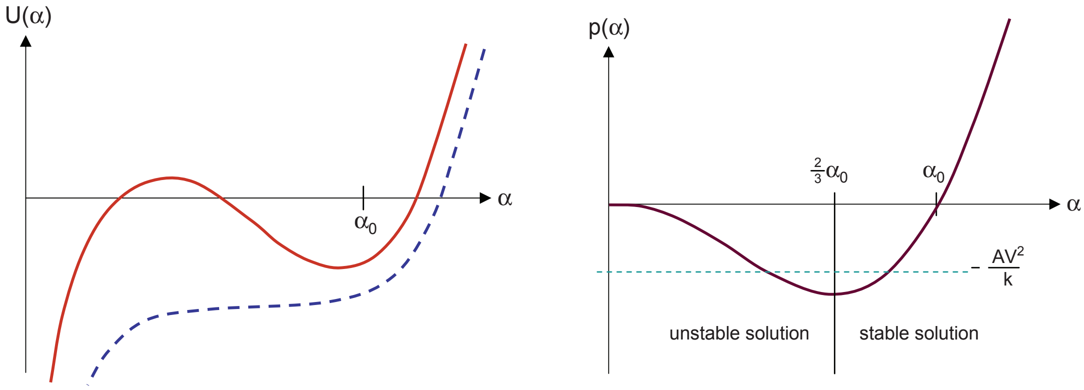

FIG. A.1: Illustration of the conditions for stability in the Coulomb balance. The first panel shows the potential energy of the balance as a function of displacement α for a small value of the high voltage V (solid red curve) and a large value of V (dashed line). The second panel illustrates the behavior of the function $p(\alpha)$ defined in Eq. A3 and the correspondence of its two positive roots to stable and unstable equilibrium solutions.

APPENDIX A: STABILITY OF THE COULOMB BALANCE

The conditions for equilibrium of a balance are that the center of gravity is directly under the pivot point. Angular displacements in either direction from this equilibrium result in a restoring torque that tends to bring it back toward the equilibrium angle, here denoted α_0. The potential $U(\alpha)$ corresponding to this torque is

$$U_0(\alpha) = \frac{1}{2}k(\alpha - \alpha_0)^2 \tag{A1}$$

where k is the Hooke's Law coefficient of the restoring force. The zero subscript on U_0 is there because this potential only holds when $V = 0$. When V differs from zero, there is an additional torque due to electric forces that is given by Eq. 6. Including this force leads to the combined potential

$$U(\alpha) = \frac{1}{2}k(\alpha - \alpha_0)^2 - \frac{AV^2}{\alpha} \tag{A2}$$

where the constant $A = (W\epsilon_0/2)\ln r_2/r_1$. The overall shape of the potential is governed by the size of the voltage V that appears in the second term in Eq. A2. For small values of V, the shape of the potential is like the solid curve shown in the first panel of Fig. A.1. This potential exhibits a stable equilibrium about a position that is close to α_0 but somewhat less. The dashed curve shows what happens to the shape of the potential curve for larger values of V. Here the local minimum in the potential has disappeared and there is no longer a stable equilibrium anywhere except at $\alpha = 0$.

To find out under what conditions the stable equilibrium exists, one solves for the roots of the first derivative of the potential. If there are two positive roots then the lesser of the two corresponds to the unstable equilibrium at the local maximum in the potential, and the greater root corresponds to the stable equilibrium at the local minimum. If there is no positive root then there is no stable equilibrium. The zeroes of the function $dU/d\alpha$ satisfy the equation

$$\alpha^2(\alpha - \alpha_0) = -\frac{AV^2}{k} \tag{A3}$$

Calling the left-hand side of Eq. A3 $p(\alpha)$, the roots of this equation are illustrated in the second panel of Fig. A.1. The minimum in the function $p(\alpha)$ appears at $\alpha = (2/3)\alpha_0$. If V is not too large, there are two positive roots, corresponding to the unstable and stable equilibria of the potential, as shown in the figure. If V is increased further, the two roots eventually meet at $(2/3)\alpha_0$, and then disappear.

This analysis leads to the following simple conclusions. If an equilibrium exists at all with $V > 0$, then it exists in the window between $(2/3)\alpha_0$ and α_0. Conversely, for any displacement α within this window, it is possible to find a value of V for which α represents a stable equilibrium of the system. For any displacement α outside this window, it is impossible to find a value of V for which α represents a stable equilibrium.