Large-Angle Motion of a Pendulum

Physics 2502, University of Connecticut
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The pendulum is one of the most familiar examples of simple harmonic motion known to students of physics, and yet it is not a truly harmonic system, even in the case when the effects of friction are negligible. In this experiment students measure the period of a pendulum as a function of its amplitude, and discover that significant deviations from simple harmonic behavior take place even for amplitudes of only a few degrees. A rigid-rod pendulum and a photogate are used to investigate the period of the pendulum as a function of its angular amplitude, measured by analyzing a video taken of the pendulum motion over many periods. These results are compared with the solution obtained by integrating the true equation of motion for the ideal pendulum. Deviations from ideal behavior are investigated in terms of a model that includes both the effects of air drag and imperfections in the pivot.

I. INTRODUCTION

The pendulum is perhaps the most studied mechanical system in introductory physics laboratories. In most of these experiments (Foucalt's pendulum, Kater's pendulum, physical pendulum and coupled pendula) the amplitude is restricted to small angles so that the period approximately obeys the familiar relation,

$$\tau_0 = 2\pi \frac{I}{mgL} \tag{1}$$

where I is the pendulum moment of interia about the pivot, m is its mass, L is the distance from the pivot to its center of mass, and g is the local acceleration of gravity. This is actually the limiting value of the period for small-amplitude oscillations; for oscillations of finite amplitude there are corrections which must be applied in order to accurately predict the period. Although these are often called *large-amplitude* corrections, their effects are present at any non-zero amplitude, and are readily observable for oscillations as small as a degree. The purpose of this experiment is to measure the period of a physical pendulum as a function of its amplitude, and compare these with the predictions based on a model of the ideal pendulum.

II. MODEL OF THE PHYSICAL PENDULUM

Consider the physical pendulum depicted in Fig. 1. Located somewhere along the length of the pendulum shaft, at a distance R from the pivot, is a brightly-colored marker that facilitates automatic tracking of the pendulum position using video analysis software. Ignoring friction, the equation of motion for this system is

$$I \frac{d^2\theta}{dt^2} = -mgL \sin\theta \tag{2}$$

The small-amplitude solution to this equation of motion is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \delta_0) \ , \quad \omega_0 = \sqrt{\frac{mgL}{I}} \tag{3}$$

This is not the exact solution to Eq. 2, but to a related equation of motion that is obtained by replacing the factor $\sin\theta$ on the right-hand side of Eq. 2 with its small-angle approximation θ. Finding exact solutions to Eq. 2 is more difficult because the differential equation is non-linear, so the solution is no longer a perfect sine wave. However, the motion is still periodic, repeating itself every τ seconds. It turns out that a very good description of the motion of a pendulum at amplitudes less than 30° is obtained using the solution Eq. 3, but with the true frequency $\omega = 2\pi/\tau$ in the place of ω_0.

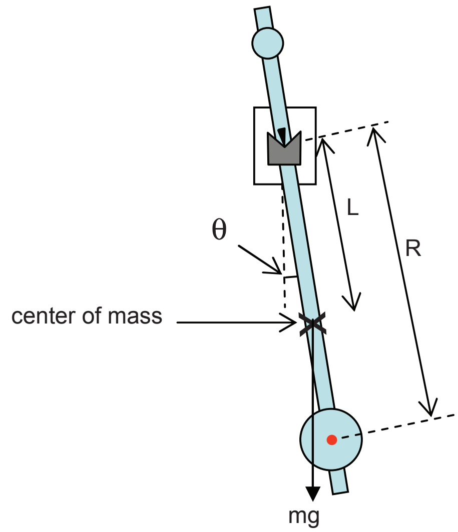

FIG. 1: Schematic diagram of the pendulum used in this experiment. The dashed line is the vertical which is the equilibrium position of the rod. The length L is the distance from the pivot point to the center of mass of the pendulum, while R is the distance from the pivot to the marker used to track the pendulum position in the video.

A. period of the ideal pendulum

Unlike the value of ω_0 in Eq. 3, the value of ω depends on the oscillation amplitude. To work this out, consider the energy carried by the pendulum during its motion. The potential energy of the pendulum is $U = mgL(1 - \cos\theta)$ where the zero of potential energy has been chosen to be at the equilibrium position $\theta = 0$. The total mechanical energy of the pendulum is given by,

$$E = K + U = \frac{1}{2}I\dot{\theta}^2 + mgL(1 - \cos\theta). \tag{4}$$

Consider now the initial condition at $t = 0$ where the bob is released from rest at an angle α. The principle of mechanical energy conservation requires that

$$mgL(1 - \cos\alpha) = \frac{1}{2}I\dot{\theta}^2 + mgL(1 - \cos\theta) \tag{5}$$

Rearranging Eq. 5 and taking a square root, the angular velocity of the pendulum is

$$\dot{\theta} = \sqrt{2\omega_0^2(\cos\theta - \cos\alpha)} \tag{6}$$

Using the identity, $\cos\alpha = 1 - 2\sin^2\alpha/2$, Eq. 6 can be written as

$$\frac{d\theta}{dt} = 2\omega_0\sqrt{\sin^2\alpha/2 - \sin^2\theta/2} \tag{7}$$

This equation can then be integrated from 0 to α which is the quarter-period time interval t=0 to $\tau/4$. The result is

$$\tau = \frac{2}{\omega_0}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\alpha/2 - \sin^2\theta/2}} \tag{8}$$

There are two features about Eq. 8 which are quite unpleasant. Note first that the integral is improper since the integrand is undefined (infinite) when $\theta = \alpha$ at the upper limit. Extreme care must be taken when numerically evaluating such improper integrals. The second point is that the form of Eq. 8 obscures the fact that, for small α, it approaches the simple result of Eq. 2, independent of amplitude α. To help with the second problem, consider a change of variables from θ to ϕ defined as

$$\sin\phi = \frac{\sin\theta/2}{\sin\alpha/2} \tag{9}$$

such that ϕ advances from 0 to 2π during one full oscillation of the pendulum. Eq. 9 can be used to rewrite Eq. 8 as

$$\frac{\tau}{\tau_0} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{10}$$

where $k = \sin\alpha/2$. Using Eq. 10 makes it is easy to see that $\tau \to \tau_0$ in the limit $k \to 0$, which corresponds to $\alpha \to 0$.

For non-zero k, one must evaluate this integral, which is a member of a special class known as "elliptic integrals". Since this is a definite integral, it can be evaluated numerically and expanded in a power series in the free parameter k. The solution is

$$\frac{\tau}{\tau_0} = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots \tag{11}$$

which is convergent as long as $k < 1$.

B. including effects of friction

Thus far it has been assumed that the mechanical energy E is a constant of the pendulum's motion. This assumption is violated by the presence of dissipative forces, both in the form of the drag produced by the viscosity of the air through which the pendulum moves, and also by the contact friction created as the pivot rotates in the support groove. Both of these effects can be incorporated into the simple harmonic model of the small-oscillation pendulum.

Incorporating air drag into the model leads to the following modifications to Eq. 2,

$$I\frac{d^2\theta}{dt^2} = -mgL\sin\theta - b\frac{d\theta}{dt} \tag{12}$$

The drag coefficient b sets the scale for the torque produced by the drag force. Solving Eq. 12 leads to the familiar damped oscillator solutions, with the exact form depending on whether b is small (under-damped), large (over-damped), or in between (critically damped). In this experiment the value of b is small enough that the scale of the drag torque is much smaller than the scale of the gravitational torque, so the under-damped solution is the appropriate one.

$$\theta(t) = \theta_0 e^{-\Gamma t}\cos(\omega t + \delta_0)\ ,\quad \Gamma = \frac{b}{2I} \tag{13}$$

As explained above, this is only an exact solution to Eq. 12 in the limit of small oscillations, but using the true frequency ω obtained in the previous section in the place of the small-oscillation frequency ω_0 allows this solution to work very well at amplitudes as large as 30°.

The pivot is designed to have a very small a frictional torque, but at very small amplitudes it can be comparable to the torque from air drag. Its small magnitude allows us to treat contact friction as a small perturbation to the motion described by Eq. 13. Kinetic friction produces an approximately constant torque, independent of velocity, that opposes the direction of the swing. Inserting a small constant torque into Eq. 12 causes the equilibrium position of the pendulum to be displaced by a miniscule amount away from $\theta = 0$. Because the friction force always opposes the direction of the motion, this displacement reverses sign each time the pendulum reverses direction. This does not affect the period of the pendulum, but it causes the pendulum to lose a little bit of amplitude upon each half-swing. One way to calculate the damping effect from friction in the mount is to compute the work ΔW_f done by the pendulum against friction during each period

$$\Delta W_f = -\mu_k mg(2r\alpha) \tag{14}$$

where r is the distance from the contact point to the center of rotation in the pivot, and 2α is the total angular distance travelled by the pendulum during between one complete period. The important thing to note here is not the exact value of ΔW but how it is proportional to the oscillation amplitude α. By contrast, the work done by the drag force in Eq. 12 during one period,

$$\begin{aligned}\Delta W_d &= -b\int_0^\tau \frac{d\theta}{dt}d\theta \\ &= -\frac{\pi b\omega}{2}\alpha^2 \end{aligned} \tag{15}$$

is proportional to the square of the amplitude α. Because of this, drag forces tend to dominate the damping at larger oscillation amplitudes, while pivot friction becomes important at small amplitudes, and causes the damping coefficient Γ in Eq. 13 to increase somewhat at very small amplitudes. Other than this, the effects of kinetic friction on the motion are essentially the same as the drag force, causing the amplitude of the motion to decrease exponentially. It can be incorporated into Eq. 13 by making Γ weakly dependent on the amplitude, increasing somewhat at very small amplitudes. For the purposes of this experiment, treating Γ as a constant is a very good approximation.

C. imperfections in the mount

Empirically one finds that the oscillation period of a pendulum like the one used in this experiment decreases faster at low amplitude than is predicted by Eq. 11. The explanation for this comes from mechanical imperfections in the mount. In the ideal case, viewing the knife edge of the pendulum pivot under a microscope would reveal that the edge is actually rounded in the form of a half-cylinder with a uniform radius r that rolls back and forth on a small flat section at the base of the groove in the mount. Small burrs on this surface, which accumulate with use of the apparatus as the pendulum is unmounted and remounted during experiments, as well as general wear of the surface on which it rocks back and forth, distort the geometry from this ideal shape. These distortions can result in multiple contact points around the tip of the knife edge that the pivot rides up onto as it rocks back and forth during oscillations, all of which are within $\pm r$ of the center of the knife edge. For a crude model of this, consider a knife edge that has been distorted into a square shape, with two corners a distance $2r$ apart sitting on a flat supporting surface. As the pendulum swings right, it rises up onto the left corner a displacement $-r$ from the center of the pivot, and as the pendulum swings back left of center, it switches over and rocks up on the right corner at $+r$. This is just a distortion of the shape of the gravitational potential curve of the pendulum, and does not involve friction, so it cannot result in energy loss of damping. Instead, it causes the oscillation period to decrease by $\Delta\tau$ from the ideal value given in Eq. 11.

$$\Delta\tau = -\left(\frac{2r}{\pi L\alpha}\right)\tau \tag{16}$$

Note that the diameter $2r$ of the knife edge is typically of order 10^{-4} m, while L is of order 1 m, so this frequency shift is only significant for amplitudes α less than 0.1 radians. This effect is seen in Fig. 2, which shows actual measurements of the measured period vs. amplitude taken with an apparatus similar to the one used for this experiment. The red curve shows the prediction based on Eq. 11. The heavy solid lines are measurements, and the dashed line passing through them is an empirical interpolation of those data, including a term proportional to $-1/\alpha$ that causes the period to turn downward at small amplitudes.

At large amplitudes there is a related effect in the pivot that again causes the measured period to decrease relative to that of the ideal pendulum. When the pendulum angle increases above a certain point, the sides of the knife edge in the pivot begin to make contact with the sides of the groove in the support. This induces a new restoring torque that acts in addition to gravity, causing the period to decrease relative to the ideal case assumed in Eq. 11. This effect is seen in Fig. 2 at amplitudes above 0.3 radians, where the measured period falls below the curve for the ideal pendulum. For an apparatus similar to the one used to produce Fig. 2, the range of near-ideal behavior is seen to lie between 0.1 and 0.25 radians.

III. EXPERIMENTAL METHOD

The setup used in this experiment is shown in Fig. 1. At the bottom of its motion, the pendulum passes through a photogate sensor that sends a signal to the data acquisition computer each time the light signal is interrupted. Measure the distance R from the suspension point to the center of the marker that will be used to track the position of the pendulum throughout its swing. Measure this distance several times, and try to obtain an accuracy of ± 1 mm.

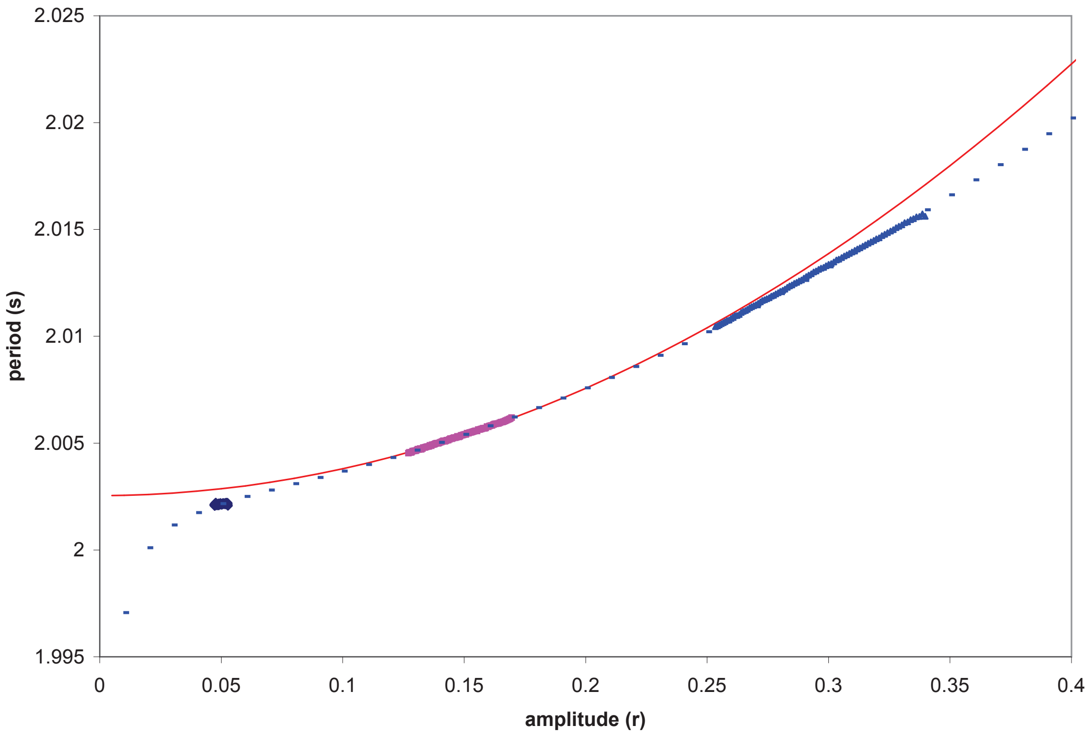

FIG. 2: Plot of the measured period of an apparatus similar to the one used in this experiment, as a function of oscillation amplitude. The red curve is the theoretical prediction based on Eq. 11. The heavy solid lines are measurements, and the dashed line is an interpolation of the measurements over the full dynamic range of the pendulum.

Mount the pendulum on the pivot and start it swinging with a few-degree amplitude. On the data acquisiton computer, start the photogate program. In the program graphical window, change the minimum and maximum period values to reasonable values like 0.5 and 5.0 seconds and set the total number of periods to 90, then press GO. The program displays the time interval between each photogate pulse, which is a half-period when the photogate is aligned with the equilibrium position of the pendulum. Carefully slide the photogate back and forth (taking care not to move the arms of the photogate into the path of the pendulum) until the two half-periods are approximately equal. A 1% difference is ok.

Set up the video camera and align it so that it is viewing the pendulum along a line of sight that is perpendicular to the plane of the pendulum motion and that passes through the pendulum marker when it is in the equilibrium position. Check that the field of view in the camera is large enough that the marker never goes out of view, even when the amplitude is the largest allowed, typically around 30°. Set the camera to record video at medium resolution (640 x 320 is good). The frame rate should be between at least 30 fps. Higher frame rates are permitted, but will result in larger video files.

Place an object with distance markings (a meter stick is ideal) in the view of the camera just behind the pendulum. This will be useful later on for calibrating the distance scale in the video. Level the calibration object so that at edge is clearly visible to define the horizontal axis. Try to arrange the stand for the photogate so that the pendulum marker is visible to the camera at all points during its swing. When everything is aligned, start the data acquisition program and the video recording when the pendulum is at rest.

Based on the period vs. amplitude curve shown in Fig. 2, chose three starting amplitudes within the region between 0.1 radians and 0.25 radians. Manually raise the pendulum to its maximum amplitude near 15°. After a pause, release the pendulum and allow it to swing freely, recording video and period data for a total of 90 periods. When the data acquisition finishes, enter a descriptive name for the data file and save it. Stop the video recording and save the video file. This gives about 3 minutes of video, which produces a reasonably small file that can be easily transfered from the camera to the data acquisition computer. Save the file together with the period data file, using names that clearly identify them as going together.

Repeat the above measurements for a series of successively smaller initial amplitude values, until the final set has

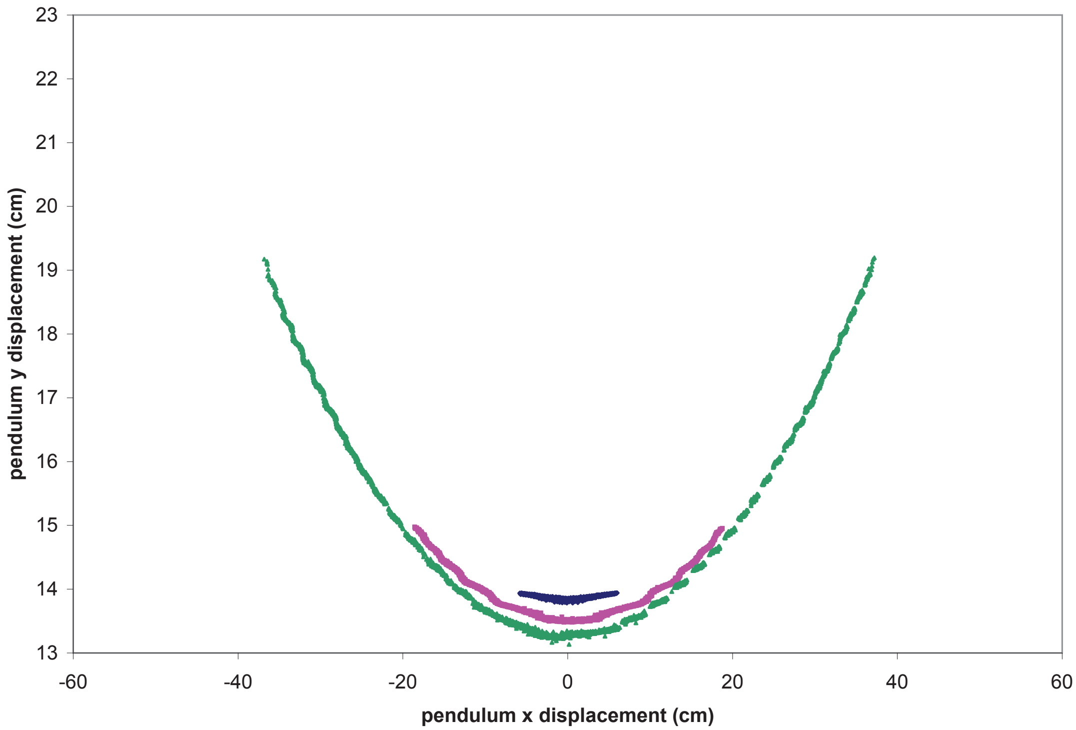

FIG. 3: Plot of the position measurements of the pendulum marker, for the data shown in Fig. 2. Note that the apex of the curves are all centered at $x = 0$ and the maximum limits in y for each curve are left-right symmetric, showing that the axis alignment is correct.

an amplitude of about 6-7 cm. A total of 3 or 4 runs is sufficient.

IV. VIDEO ANALYSIS

The video analysis should be carried out in the laboratory period, so that the video data does not need to be carried away by each student. Stop the photogate program on the data acquisition computer, and start the *Tracker*[1] video analysis program. Open the first video file. The first frame of the video should appear in the window. Use the slider at the bottom of the video screen to advance the video to where you can see the hand releasing the pendulum, then step forward frame-by-frame until the person is out of the view and the pendulum has reached the maximum height. Record the frame number as the start frame. Now press the button at the lower right corner of the screen called "clip settings", and enter the start frame number. Leave the end frame unchanged, and press OK. Move the frame slider back to the start frame. Select menu item *Tracks* → *New* → *Point Mass*. Right-click on the new button just created in the *Mass Control* menu, and select *Autotracker*. This brings up the Autotracker wizard tool. First the program wants you to select the pendulum marker in the video. Center the loop cursor on the marker in the video and left-click. You can now resize and reposition the target image to optimize image recognition. After this you select the size and position of the search box in which the autotracker looks for the marker. The search box should be large enough that the box centered on the marker in frame n will always contain the marker image in frame $n + 1$. Smaller search boxes make analysis faster, so try several settings here until the optimum is found. The autotracker supports a feature called "look-ahead", which causes it to try and extrapolate forward from previous points where it thinks the marker will be, and center the search box there. Running with this enabled can allow you to use a smaller search box, but generally the tracking is more stable with it turned off.

Allow the autotracker to analyze the entire clip. This can take a while. You can monitor its progress by watching the frame number change in the autotracker control window. If it fails to find the marker in any frame, it will stop and give you a change to reposition the search window, or skip that frame. Skipping a few frames will have negligible effect on the quality of your results.

Before you save the results, you should first include some calibration data. Slide the frame slider back to frame 1, when the pendulum was at rest. Select the tape measure tool from the toolbar (looks like a double-arrow line with a number next to it) and click on the image to add it. Click on the two ends of the arrow, one at a time, and drag them to the two ends of the meter stick (or other distance reference) in the image. Click on the number above the arrow and enter the true distance in cm. Record the angle of the horizontal line in the image (displayed just below the toolbar). Next select the axis cross-hairs tool from the toolbar, and place axes on the picture. Click on the origin of the cross-hairs and move them to the rest position of the marker. Enter the angle recorded above into the box for the axis angle. Check that the vertical axis aligns with the pendulum rod, confirming that the horizontal and vertical axes are perpendicular in the video. If this is not the case, it indicates problems with your camera alignment. Save the results to disk, using the same file name as the video. A new file with the extension *.trk* is created.

Open the *.trk* file in a browser. With a little work, you should be able to find the coordinate origin and axis rotation angles recorded in video pixel units. The time interval Δt between frames is recorded in the file as *delta_t*. Open a new spreadsheet and create columns t, u, and v. Extract the tracker coordinates recorded in the file as x and y values, and enter them into the spreadsheet as u and v. There will be several thousand of these, so do not attempt to do this manually. See the TA if you need help finding tools to do this extraction. Create new rows at the top of the spreadsheet to hold calibration data for the coordinates of the origin u_0 and v_0, the scale factors S_u and S_v, and the rotation angle β. Create new columns next to those for u and v and label them x and y, filling them according to the linear transformation

$$\begin{aligned} x &= (u-u_0)S_u\cos\beta + (v-v_0)S_v\sin\beta \\ y &= -(u-u_0)S_u\sin\beta + (v-v_0)S_v\cos\beta \end{aligned}$$

Apply this tranformation to the entire columns of data, then convert the x and y values to θ through the equation

$$\theta = \tan^{-1}\left(\frac{x}{R-y}\right) \tag{17}$$

Finally, fill in the values in the t column, starting at 0 in the first row of the table and increasing by δt going down the column. Save the spreadsheet to disk frequency to prevent loss of your work.

Within the spreadsheet, create duplicates of the first worksheet, copying all of the data and formulas. Rename the worksheets, one for each of your video runs. Repeat the above video analysis for each of your videos. Extract the data from each of the *.trk* files and overwrite the calibration and u, v columns on worksheets 2-n with the corresponding data from the files. The formulas on these worksheets will automatically carry out the transformations and produce the θ values. Plots of y vs. x should look similar to Fig. 3, except that the small and large oscillation amplitudes should be less extreme than the ones shown in the example.

Save this spreadsheet together with the period data files collected earlier. Each student should carry away copies of these files for further data analysis.

V. DATA ANALYSIS

The challenge of data analysis for this experiment is to match up the amplitude information extracted from the video with the period information collected by the photogate program, so that the period can be plotted as a function of amplitude. This is not a simple task because the period measurements are taken only once per period, while the amplitude information is in the local maxima and minima that appear the oscillating function $\theta(t)$ that was sampled once per camera frame, typically 30 or 60 times per second. One way to proceed would be to manually scroll down through the θ column, find the maximum and minimum amplitude for each period, subtract them and divide by 2, and record the result as the amplitude for that period. Actually carrying out the analysis this way would be exhausting however, because there are 90 periods in each data sets, and several data sets to analyze. A much more convenient and powerful technique called *a linear filter* can be used to accomplish the same thing in an automated fashion. Not only is the linear filter a much more efficient method than manual inspection, but it actually produces amplitude estimates that are more precise than inspection could produce.

The theoretical basis for the linear filter to be used in this analysis is presented in Appendix A. The emphasis here is not why it works, but how to use it. Think of the filter as an operator that takes in one column S of a spreadsheet as input and produces a new column A of the same length as output, where A contains the amplitude of the oscillating function S. The basic formula for the filter is

$$A_m = \sum_{n=0}^{N-1} S_{m+n} f_n \tag{18}$$

where the numbers $f_1 \cdots f_{N-1}$ are the N values of the filter column vector f. In the current analysis task, the column of $\theta(t)$ values serves as the input vector S. The filter vector f needed for this purpose is actually column of N complex numbers. According to Eq. 18, if the numbers f_n are complex then this implies that the outputs A are also complex. For the purposes of a spreadsheet application that deals only with real numbers, Eq. 18 is can be rewritten as,

$$\begin{aligned} \Re A_m &= \sum_{n=0}^{N-1} S_{m+n} \Re f_n \\ \Im A_m &= \sum_{n=0}^{N-1} S_{m+n} \Im f_n \\ Amag_m &= |A_m| = \sqrt{\Re A_m^2 + \Im A_m^2} \end{aligned} \tag{19}$$

where $Amag_m$ stands for the absolute magnitude of the complex number A_m.

Before these formulas can be implemented, one must first create concrete instances of the filter functions $\Re f_n$ and $\Re f I_n$. Let us define the N complex numbers f_n as,

$$f_n = \frac{2}{N} e^{2\pi i \frac{n}{N}} \tag{20}$$

with $n = 0 \cdots N-1$. With this definition, the column $\Re f_0 \cdots \Re f_{N-1}$ and $\Im f_0 \cdots \Im f_{N-1}$ contain cosine and sine functions, respectively sampled at N equal intervals between 0 and 2π. In your spreadsheet, create two new columns to the right of the t column created above, and label them “fR filter” and “fI filter”. Underneath each, create formulas to implement Eq. 20 that extend down just N cells from the top. Adjust your choice of N so that the oscillations in the filter match the oscillations in $\theta(t)$ as close as possible. N has to be an integer, so the match cannot be exact, but the closer the better. To the right of the filter columns, create a new column called “amplitude”, and enter a formula to compute $Amag_m$ from Eq. 19 in it. With Microsoft Excel, $Amag_m$ can be computed in only one step by making use of the special function `SUMPRODUCT`. Plot the resulting column $Amag$ vs. t in a graph together with the input column θ vs. t, and verify that the results in the amplitude column agree with what you would have gotten by taking the peak heights for each period and connecting them by a continuous smooth curve.

Now that you have extracted the amplitude from the oscillating function, you would like to sample the amplitude function once per period, to be able to match it up row-for-row with the periods measured using the photogate. The most straight-forward way to do this is to fit the amplitude function to a simple polynomial function, then enter the polynomial formula into a new “S amplitude” column along-side the photogate period data. The amplitude is a smooth and slowly decreasing function, so a second-order polynomial should give an excellent fit. Create 3 cells in the upper portion of your worksheet to contain the 3 parameters of a second-order polynomial, and create a new column to the right of the $Amag$ column, called “amplitude fit” to contain the fit function values. After the “amplitude fit” column add one for “amplitude residuals” and enter the usual formula for the squared difference between the $Amag$ data and “amplitude fit” functions divided by the square of the error that you assign based on fluctuations that you observe the $Amag$ values in neighboring cells. Up near the parameters list, add a cell for chi-squared and its degrees of freedom. Enter a formula for the sum of residuals in the chi-squared cell, and use the Solver to find the best-fit values for the polynomial coefficients. Do this for each of the amplitude data sets you collected.

Now you are ready to combine the amplitudes extracted from the video with the periods measured using the photogate. Find the photogate period data contained in plain text files saved by the photogate program. Add two new column to the right of the existing table, and name them “t photogate” and “τ photogate”. Import the period values from the file produced by the photogate program into the “τ photogate” column. There should be about 90 rows in this column at this point. Find the extra-long period that coincides with the start of free oscillations, and assign the value of zero to $tphotogate$ for that cell. Erase all of the period data from the spreadsheet before that point. For all of the “t photogate” cells below the first one, insert the formula adding the value of the “photogate period” in the preceeding row to the preceeding “t photogate”. This produces a cummulative sum of the measured periods “τ photogate” in the “t photogate” column. Add a new column to the right of “τ photogate” called ”amplitude from fit” and enter the polynomial formula from the fit performed above, to compute the amplitude for the time “t photogate”, based on the fit to the camera video data for this run.

Create a graph of the measured “τ photogate” values vs. “amplitude from fit”. Include data from all of your runs in the same plot. The resulting plot should look similar to the one shown in Fig. 2. Add vertical error bars to the data points in the plot that represent your best estimate for the errors in the photogate period data. You can estimate these by looking at sequential numbers in a measured sequence, and noting the size of the random fluctuations between consecutive measurements. Add a curve to the plot that represents the function $\tau(\alpha)$ in Eq. 11. To get a good description of your data, you will need to adjust the value of the unknown parameter τ_0 in Eq. 11 to minimize the chi-square between the curve and the data points with their errors.

Use a chi-square test to make a quantitative comparison between your data and Eq. 11. If there are systematic deviations between the data and the curve in some ranges of amplitude, as seen in Fig. 2, comment on what effects contribute to these deviations, and repeat the chi-square test, including only data within a restricted amplitude range where the effects of these distortions are small. Study the effects of each of the higher-order terms in Eq. 11 in influencing the quality of the fit.

Acknowledgments

This write-up was written by Prof. Richard Jones, based on an earlier laboratory described by Prof. Doug Hamilton that studied the large-angle motion of a bifilar pendulum, with periods measured using a stop-watch (1988), and later a photogate (2004).

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[1] Douglas Brown, Open Source Physics project, Tracker video analysis program, v. 3.10, available for download at `http://www.cabrillo.edu/~dbrown/tracker/`, Jan. 20, 2011.

APPENDIX A: BASICS OF LINEAR FILTERS

Consider a periodic function $S(t)$ with period τ such that $S(t+\tau) = S(t)$ for all t. Fourier's theorem states that S can be written in terms of its Fourier expansion as

$$S(t) = \sum_{p=0}^{\infty} a_p \cos(p\omega t) + \sum_{n=p}^{\infty} b_p \sin(p\omega t) \ , \quad \omega = \frac{2\pi}{\tau} \tag{A1}$$

where a_n and b_n indicate the amplitude and phase of the component in S that oscillates at frequency $n\omega$. Often in experiments the functional form of $S(t)$ is not known, but it is sampled at regular intervals Δt, so that what one knows about S is contained in the list of values $S(t_0), S(t_1), S(t_2), \ldots$ where $t_{n+1} = t_n + \Delta t$. Let us define a linear filter as an operator which acts on the S_n to produce a new list of numbers A_n as follows,

$$A_m = \frac{2}{N} \sum_{n=0}^{N-1} S_{m+n} \cos\left(2\pi q \frac{n}{N}\right) \tag{A2}$$

where the parameters q and N are properties of the filter whose meaning is made clear below. Substituting the Fourier expansion for S into Eq. A2, the sum can be readily computed.

$$\begin{aligned} A_m &= \frac{1}{N} \sum_{p=0}^{\infty} a_p \left\{ \mathcal{S}(p,-q) \cos\left(p\omega t_m + (p\omega N\Delta t + 2\pi q)\frac{N-1}{2N}\right) + \mathcal{S}(p,+q) \cos\left(p\omega t_m + (p\omega N\Delta t - 2\pi q)\frac{N-1}{2N}\right) \right\} \\ &+ \frac{1}{N} \sum_{p=0}^{\infty} b_p \left\{ \mathcal{S}(p,-q) \sin\left(p\omega t_m + (p\omega N\Delta t + 2\pi q)\frac{N-1}{2N}\right) + \mathcal{S}(p,+q) \sin\left(p\omega t_m + (p\omega N\Delta t - 2\pi q)\frac{N-1}{2N}\right) \right\} \end{aligned} \tag{A3}$$

where

$$\mathcal{S}(p,q) = \frac{\sin\left[\frac{p\omega N\Delta t - 2\pi q}{2}\right]}{\sin\left[\frac{p\omega N\Delta t - 2\pi q}{2N}\right]} \tag{A4}$$

These equations simplify greatly in the special case where Δt evenly divides τ, so that it is possible to chose N such that $\tau = N\Delta_t$. Under these conditions, the factor $\mathcal{S}(p,q)$ reduces to $N\delta_{pq}$ and Eq. A3 becomes

$$A_m = a_q \cos(q\omega t_m) + b_q \sin(q\omega t_m) \tag{A5}$$

Defining a similar set of sine filters,

$$\begin{aligned} B_m &= \frac{2}{N}\sum_{n=0}^{N-1} S_{m+n}\sin\left(2\pi q\frac{n}{N}\right) \\ &= b_q\cos(q\omega t_m) - a_q\sin(q\omega t_m) \end{aligned} \tag{A6}$$

one can combine the two filters together to produce

$$A_m^2 + B_m^2 = a_q^2 + b_q^2 \tag{A7}$$

That is, the sum in quadrature of the A_m and B_m filtered versions of S yields the squared amplitude of the q'th harmonic in S.

In practice, it is often not possible to tune the sampling interval Δt so that the period τ is an integer multiple of Δt. In that case, the values of $\mathcal{S}(p,q)$ with $p \neq q$ are not exactly zero, and unwanted terms in Eq. A3 bleed through the filter. In such cases, it is often possible to detune the value of q somewhat from a whole-integer value, and suppress a particular unwanted term. For example, consider a signal with a period of 20.1 s being sampled at a rate of 1 Hz. Chosing $N = 20$ gives a close but imperfect match between the filter period $N\Delta t$ and the signal period τ. Because of this, a $q = 1$ filter designed to extract the amplitude of the fundamental harmonic picks up small contributions from other harmonics as well, due to the off-diagonal terms in the $\mathcal{S}(p,q)$ matrix. If a term involving the off-diagonal term $\mathcal{S}(1,-1)$ is creating a problem, it can be suppressed by shifting $q = 1$ down to $q = 0.995$. This small shift has a negligible effect on the leading term ($\mathcal{S}(1,1) \approx \mathcal{S}(1,0.995) \approx 1$), but it kills the problematic term completely because $\mathcal{S}(1,-0.995) = 0$.