Measurement of Newton's Gravitational Constant G

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The original inspiration for Newton's Universal Law of gravitation came from astronomical observations, and comparing the attractive force between celestial bodies to the gravitational acceleration g experienced by objects on the surface of the earth. Newton's Universal Law implied the existence of a radically new phenomenon that had never been observed at the time he proposed it, that in addition to their attraction to the center of the earth, two laboratory objects are also attracted to each other with a force directed along the line between their centers. The small size of Newton's constant G implies that this force is very small for two objects on the scale of two experimental apparatus, and it was not until much later that this prediction was experimentally verified by Henry Cavendish. Using an apparatus similar to the one used by Cavendish, students will measure the attraction between nearby lead spheres, as a function of their separation, and extract a measured value for G.

I. INTRODUCTION

The most obvious effect of the gravitational force on earth is the attraction that every object has toward the center of the earth. Newton explained this familiar phenomenon as a special case of a more general law now known as Newton's Universal Law of Gravitation. According to Newton's Universal Law, all objects are attracted to each other along the line joining their centers, with a force whose magnitude is governed by a single universal constant G known as Newton's constant. The early tests of Newton's Universal Law came from astronomical observations on the scale of the radius of planetary orbits in the solar system, and comparing these observations with the known value of the gravitational acceleration g experienced by all objects on the surface of the earth. The successful description of these two very different phenomena in terms of a single universal law, spanning distances from thousands of km to billions of km, still stands today as one of the landmark successes of classical physics.

In spite of these successes, there is widespread interest today in possible deviations from Newton's Universal Law that may appear when one probes separation distances that are very small or very large. At very large scales, these investigations are motivated by deviations observed in the rotational velocities of stars in the arms of spiral galaxies, whose orbital periods seem to be shorter than what would be expected based on Newton's Universal Law. The standard interpretation of these data is that Newton's Law is correct and that there is additional "dark matter" in the region surrounding these galaxies, which is responsible for increasing the gravitational force felt by stars in the distant reaches of the galaxy. Alternative theories that can also explain these data are based on proposed departures from Newton's Universal Law which may occur at very large length scales.

Equally interesting, and of greater relevance to the experiment described below, is the possibility that Newton's Universal Law may break down at distances much smaller than the radius of the earth. The first precise test of Newtonian gravity at laboratory length scales was initiated by English physicist John Mitchell, and carried to completion by Henry Cavendish in 1798. Mitchell's apparatus consisted of two large weights on the ends of a balance beam suspended in the center by a fine wire called a *torsion fiber*. Fixed masses were placed near the two suspended weights, and the beam was allowed to freely swing in the horizontal plane, subject only to the weak restoring force of the fiber's resistance to being twisted. By measuring small shifts in the equilibrium angle of the balance beam as the fixed masses were moved closer to the suspended ones, and inferring the torsion coefficient of the fiber from the frequency of its small-angle oscillations around the equilibrium, a value for G was obtained that turned out to be consistent with that established on the basis of astronomical data.

Developments in Modern Physics have stimulated renewed interest in looking for deviations from Newtonian gravity at small length scales. Such deviations are predicted by novel theories that invoke large extra dimensions, such as some variants of String Theory. Modern revisions of the Cavendish experiment carried out at progressively smaller scales have pushed the validity limits of the Universal Law down to the scale of 10 μm.

This experiment described below is similar to the one conducted by Cavendish, albeit with bodies that are considerably less massive than those used in the original experiment. Although the precision that you will achieve will not be as good as that achieved by Cavendish, you will probe distances that are smaller by a factor 3-5 than those that were probed in the 1798 experiment.

II. EXPERIMENTAL METHOD

A diagram of the experimental apparatus is shown in Fig. 1. Two small lead spheres are attached to the ends of a light rod which serves as a balance beam. This rigid dumbbell is suspended horizontally by a vertical torsion fiber, all enclosed behind transparent walls to protect the balance from air currents. The two large lead spheres outside the

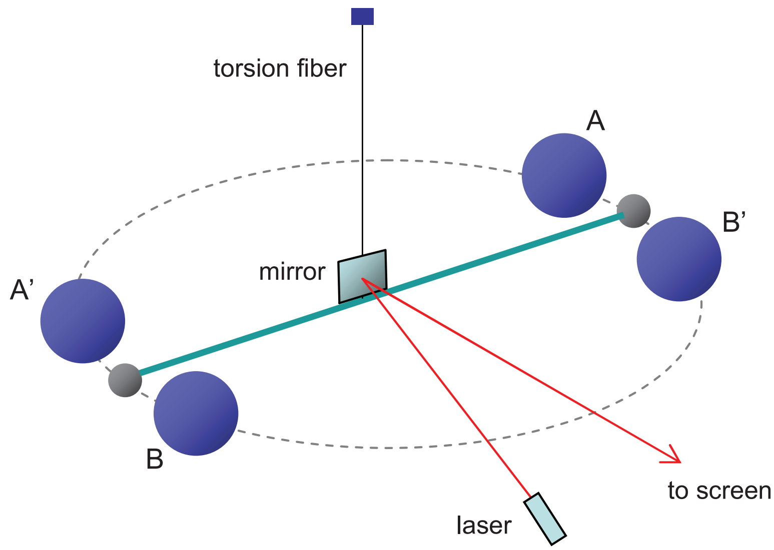

FIG. 1: Diagram of the experimental apparatus used to measure the value of G. The balance beam is free to rotate in the horizontal plane, subject to the gravitational forces between the fixed and suspended masses, and the restoring torque from the fiber.

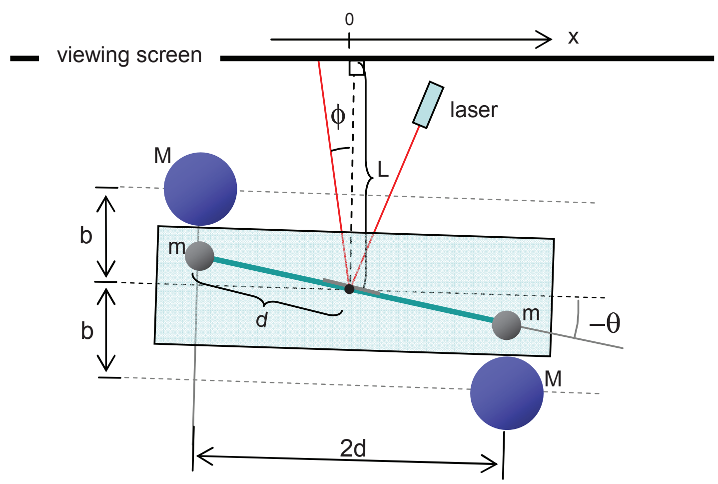

FIG. 2: Diagram of the experimental apparatus, viewed from above with the large masses in positions A, and B. The suspension fiber is indicated by the dot in the center of the figure. The shaded region is the interior of the enclosed volume containing the balance beam. The symmetry axis of the apparatus is deliberately shown be somewhat misaligned from the plane of the viewing screen, as is normal for proper alignment. Some angles are exaggerated for emphasis. Distances are not drawn to scale.

enclosure gravitationally attract the inner spheres. Equilibrium is established when the torque on the dumbbell due to the gravitational forces is balanced by the restoring torque from the torsion fiber. The two large spheres can then be rotated from their initial positions, labeled A and B, to the symmetrically located positions at A' and B'. This displaces the equilibrium and the torsion balance begins to oscillate about a vertical axis. Attached to the center of the rod is a small mirror. By observing the deflection of a beam of light reflected from this mirror, the angular movement of the torsion balance can be measured.

Fig. 2 shows the distances and angles that must be measured in the experiment. The angle θ of the balance beam is measured relative to the symmetry axis of the apparatus, as shown. Place a vertical mark on the viewing screen from which a line perpendicular to the screen passes through the wire of the balance. This defines the origin of the x measurements to be carried out later. Find the bubble level, a small round clear plastic disk filled with liquid with a small bubble of air that appears in the center of the top window when it is level. Place the bubble level on the top of the rectangular enclosure of the balance and observe the displacement of the bubble from the center of the inner circle. The top surface of the box is not perfectly smooth, so you should make observations at several positions and take the average. Level the balance by adjusting the screws in its feet until the bubble in the level is approximately centered. Look through the side window of the box to make sure that the pin marking the mid-point of the balance hangs freely above the center point. If you ever need to move the balance while you are aligning it, move it very slowly and try to minimize vibrations.

Place the laser on a stable support at the same height as the window of the apparatus and align it so that the beam spot is centered on the mirror that sits on the balance beam. The laser should be stable, and once measurements have begun, it should not be disturbed for the duration of the experiment. Make sure that the reflected spot on the viewing screen is bright and symmetrical.

Make sure that the large lead masses are removed from the holding rings. Observe the motion of the balance by watching the reflected laser spot move on the screen. The natural oscillation period of the balance is about 10 minutes, so motion of the spot may seem at first to be very slow. To see if it is moving, hold a finger at the spot position on the viewing screen; if the spot moves off the finger within 10-15 seconds then the beam is moving at a reasonable speed. If somehow it gets moving too fast or exhibits rapid vibrations, leave the apparatus untouched for 15-20 minutes and it will settle down on its own. Watch the motion of the spot for 10 minutes to see one complete period. The goal in observing these initial oscillations is to find the limits of motion of the balance beam, and mark them with vertical lines on the viewing screen. These outer limits are caused by the balance beam striking the inside surface of the glass windows of the enclosure. You know when this happens because the beam spot on the viewing screen appears to “bounce”, suddenly reversing its direction at a certain point.

If the balance has been sitting quietly for more than an hour, the oscillations may have decayed away to the point that it never reaches the window on one side or the other, or both. If you suspect that this is the case, gently tap the glass sides of the balance enclosure. After the vibrations settle down, you should see that the slow oscillation amplitude has increased. Repeat this until you clearly see the spot bouncing from both ends of its oscillation range. If it gets moving too fast, the end-point positions will appear to be different on each swing, but once it settles down, the end-points will be reproducible to better than the size of the laser spot. Mark the limits with vertical lines on the screen. Make sure that the oscillation range between the two limits is reasonably close to centered on the $x = 0$ mark you made earlier, within 2-3%. If this is not the case, move the laser (not the balance) until the full range of the spot motion is centered on $x = 0$ at the 2-3% level. If you moved the laser, redraw the new limits on the viewing screen. The $x = 0$ mark should not be moved.

At this point, the alignment of the apparatus is nearly complete. Note that the central plane of the apparatus is not exactly parallel to the viewing screen. These two planes are normally misaligned by a small angle, as suggested by the slight tilt from horizontal of the dashed lines in Fig. 2. This is properly taken into account in the data analysis.

Set up the camera that will be used to measure the motion of the beam spot using time-lapse photography. Set the time-lapse interval to 5 seconds. Place a meter stick next to the viewing screen within the view window of the camera, to be used to set the scale later when analyzing the images. Set the zoom on the camera so that both limits of the laser spot motion marked earlier are just within the viewport. Disable the flash and set the resolution to 640×480 pixels. Begin a measurement sequence with the camera, with no exterior weights loaded on the balance. Watch the first couple of oscillations of the laser spot. Review the sequence of images taken with the camera, and visually estimate where the mid-point of the oscillations is located on the screen. This needs to be close to the $x = 0$ point on the screen, within $\pm 10\%$ of the distance between the limit points.

If the apparatus has not been aligned for several days, this usually requires adjustment of the zero-point of the torsion wire. This is adjusted by turning the large knurled nut at the top of the central tube on the balance. Without shaking the balance, carefully loosen the plastic lock nut, then turn the knurled nut in the direction you want the zero-point to shift. Usually one over-estimates how much rotation is needed, so start off with a very small adjustment, and increase it as needed. After each adjustment, tighten the lock nut again to hold the zero-point fixed. Finding the zero point and getting it properly centered on the screen is the most time-consuming aspect of this entire experiment. Exercising patience at this stage and getting the alignment right will pay off later. Once the zero-point has been found and centered, the alignment is complete and should not be touched. At some later point during the experiment you may notice that the zero-point seems to have shifted somewhat. This is caused by long-term “memory” in the torsion fiber, and can be taken into account in the data analysis. If you ever redo the alignment, you must start the data-taking again from the beginning.

Check the image sequence recorded by the camera to make sure that the zoom, focus, brightness, and snapshot interval settings are optimal, then start a measurement run. A recording run consists of a continuous sequence of images taken without moving the camera, the laser, or the balance. To be useful, a measurement run should subtend at least 5 complete oscillations, requiring no less than 1 hour of continuous observation with the camera, and 500-1000 images.

Software tools for converting the still images to a speed-up video of the run are provided on the laboratory computers.

Create and view the video of the initial run taken without weights present. A slight drift in the zero-position of the oscillations is inevitable and will be taken into account in the data analysis. If the drift is so large that there is a risk that the beam will run into the limit points before the oscillations die away, the apparatus will need to be allowed to settle for a day or two, after which you should repeat the zero-point alignment, and start the measurements again. Once you are satisfied with the results from the initial run, save the movie for later analysis under the name `unloaded_1`. The software script that generates the movie from the time-lapse sequence also reports the average interval between frames. Be sure to record this number in your log book for each run, so that you can convert later from frame number to seconds.

Use a tape measure to record the distance L from the torsion fiber to the viewing screen, together with its uncertainty. Measure the masses M of the two external lead spheres. You should find that the two masses are equal within the uncertainty ΔM of your measurement. Record M and ΔM. Measure the distance $2d$ by holding a ruler up to the glass window of the balance and sighting the centers of the two spheres along the rules on the ruler. Have more than one experimenter repeat the measurement, and share the results only after all measurements have been made, recording the average value as the value of $2d$ and the RMS of your measurements divided by $\sqrt{N}$ as its uncertainty.

The most difficult distance to measure in this experiment – and the most critical in terms of its effect on the uncertainty of the final result – is the determination of the distance b. To measure this, place the two lead spheres in their holder on the balance, and swivel the holder until the lead spheres barely touch the glass on both sides. Over years of use, these lead spheres have many bumps and bruises on their surface so their radii are not uniform. You may need to turn them several times until you find smooth spots where both spheres come into contact with the glass at the same time. Check that this is the case in both the front-left and front-right orientations. Once they have been placed, do not move the external spheres in the holder again until after the gravitational measurements are complete. Once they are placed, swivel the holder out away from the glass and use a vernier caliper to measure the diameter of the two spheres along the axis containing the points where they touch the glass. Record average of the two measurements together with its uncertainty. Use the vernier caliper to measure the total thickness of the housing, including the glass windows, and record this together with its uncertainty. Compute the sum of the housing thickness plus the sphere diameter and record this value as $2b$, together with its uncertainty.

Move the external mass holder to the front-left position, with the mass just barely touching the glass, and start a new run with the camera. When this run is complete, without stopping the camera, rotate the mass holder to the front-right position and take a second run. Save the images, and convert them to video format, recording the average interval between frames as reported by the video conversion script. Remove the two external masses and set them aside on a bench, then take a new run with the camera, called `unloaded_2`. When these images are converted to video format and the frame interval recorded, you are finished with the data collection phase of the experiment.

III. THEORETICAL MODEL

Consider the apparatus represented in Fig. 2. The positions of the small spheres mounted on the ends of the balance beam are denoted as $\vec{x}_1$ and $\vec{x}_2$, where $\vec{x}_1$ refers to the mass on the right in the figure. The positions of the large spheres are denoted as $\vec{y}_1$ and $\vec{y}_2$, where $\vec{y}_1$ refers to the sphere on the lower side of the enclosure in the figure. All of these vectors lie in the same plane, taken to be $z = 0$. The origin of the coordinate system is the intersection of the torsion wire axis with the $z = 0$ plane. In the orientation shown in the figure,

$$\begin{aligned} \vec{x}_1 &= (d\cos\theta, d\sin\theta) \\ \vec{x}_2 &= (-d\cos\theta, -d\sin\theta) \\ \vec{y}_1 &= (d, -b) \\ \vec{y}_2 &= (-d, b) \end{aligned} \tag{1}$$

According to Newton's universal law of gravitation, the presence of the external spheres produces a net force on the balance mass at x_1 that is given by

$$\vec{F}_1 = GMm\left[\left(\frac{\vec{y}_1 - \vec{x}_1}{|\vec{y}_1 - \vec{x}_1|^3}\right) + \left(\frac{\vec{y}_2 - \vec{x}_1}{|\vec{y}_2 - \vec{x}_1|^3}\right)\right] \tag{2}$$

where M is the mass of the large spheres and m is the mass of the small spheres on the ends of the balance beam. The gravitational force on the rod connecting the two small spheres is neglected in this analysis. A similar expression gives the force $\vec{F}_2$ that the two external masses produce on the small mass on the other end of the balance beam at $\vec{x}_2$, obtained from Eq. 2 by replacing $\vec{x}_1$ everywhere with $\vec{x}_2$. The fact that $\vec{x}_1 = -\vec{x}_2$ and $\vec{y}_1 = -\vec{y}_2$ implies that

$\vec{F}_1 = -\vec{F}_2$, so the net force on the balance beam is zero. The net torque, however, does not sum to zero. Consider just the torque on the balance beam about the origin that arises from force $\vec{F}_1$.

$$\begin{aligned}\vec{\tau}_1 &= GMm\left[\vec{x}_1 \times \left(\frac{\vec{y}_1 - \vec{x}_1}{|\vec{y}_1 - \vec{x}_1|^3}\right) + \vec{x}_1 \times \left(\frac{\vec{y}_2 - \vec{x}_1}{|\vec{y}_2 - \vec{x}_1|^3}\right)\right] \\ &= GMmDd\left[\frac{-\sin(\theta+\theta_M)}{|\vec{y}_1 - \vec{x}_1|^3} + \frac{\sin(\theta+\theta_M)}{|\vec{y}_2 - \vec{x}_1|^3}\right]\hat{z}\end{aligned} \tag{3}$$

where $D^2 = d^2 + b^2$ and $\theta_M = \sin^{-1}(b/D)$. The torque $\vec{\tau}_2$ on small mass 2 is the same, leading to the following expression for the net gravitational torque on the balance beam.

$$\vec{\tau}_G = -2GMmDd\ \sin(\theta - \theta_M)\left[\frac{1}{|\vec{y}_1 - \vec{x}_1|^3} - \frac{1}{|\vec{y}_2 - \vec{x}_1|^3} -\right]\hat{z} \tag{4}$$

In this analysis, gravitational forces from all of the other objects surrounding the apparatus, including the camera, the support for the apparatus, the building in which the experiment is housed, and significant geological features in the local region, the moon, the earth itself, the sun, etc., all of which may be expected to produce gravitational forces at least as large as those from the lead sphere, or even larger. To understand why these background forces do not affect the experiment, note that the alignment procedure guarantees that the net gravitational force from all of the neglected external objects has a zero projection onto the rotation plane of the balance. This is because a freely hanging wire naturally aligns with the direction of the net force, and the balance is constrained to move such that the height of its center of mass remains fixed.

This leaves open the question of the net background torque. This torque does not sum to zero, and may be larger than the torque from the lead spheres, but it does not matter because it is canceled exactly by an opposite torque $\vec{\tau}_w$. The torsion wire produces a harmonic restoring torque that attracts the beam back toward its equilibrium orientation θ_0.

$$\vec{\tau}_w = -K(\theta - \theta_0) \tag{5}$$

The presence of a constant external torque simply shifts the equilibrium position θ_0 such that it is completely canceled out by the wire. From this point of view, it is impossible to measure the θ_0 of the wire itself because it is impossible to get rid of all of the background gravitational torques. Thus, when we measure the point θ_0 in the presence of this background but without the lead spheres present, we are effectively measuring the net background torque and getting rid of its effect on the experiment. This justifies ignoring all other masses except the ones that are moved during the experiment. Keep in mind that other things in the environment that move during the experiment, including human bodies, can throw off the results. Thus it is suggested that everyone keep away from the vicinity of the balance during periods of measurement with the camera.

When both the gravitational torque $\vec{\tau}_G$ and $\vec{\tau}_w$ are taken into account, the condition for static equilibrium of the balance becomes

$$K(\theta - \theta_0) = -2GMmDd\ \sin(\theta + \theta_M)\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \tag{6}$$

where

$$\begin{aligned}r_1^2 &= d^2(1-\cos\theta)^2 + (b + d\sin\theta)^2 \\ r_2^2 &= d^2(1+\cos\theta)^2 + (b - d\sin\theta)^2\end{aligned} \tag{7}$$

A system obeying Eq. 5 in the form of Hooke's Law undergoes simple harmonic motion with a natural frequency given by

$$\omega = \sqrt{\frac{K}{I}} \tag{8}$$

where the moment of inertia I of the beam can be very well approximated as $2md^2$. The angular frequency ω of the beam oscillations is easier to measure than K, so the equilibrium condition Eq. 6 can be rewritten as

$$\theta - \theta_0 = -\frac{GMD}{d\omega^2}\sin(\theta + \theta_M)\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \tag{9}$$

It turns out that in the experiment one does not measure θ directly. Instead one measures the position of a laser spot on a viewing screen, which can easily be converted into the angle ϕ shown in Fig. 2.

$$\phi = \phi_c + 2\theta \tag{10}$$

where the constant offset ϕ_c incorporates a number of unmeasured offsets, including the angle between the mirror and the balance beam axis, the angle between the mid-plane of the balance enclosure and the viewing plane, and the incidence angle of the laser in the frame of the apparatus. While all of these offsets are not individually measured in the experiment, their sum ϕ_c is measured as the angle ϕ corresponding to the midpoint between the two limits in ϕ where the ends of the beam strike the glass walls of the enclosure.

Measuring the average value ϕ_0 about which the beam oscillates when the external masses are removed allows the value of θ_0 to be determined as

$$\theta_0 = \frac{1}{2}(\phi_0 - \phi_c) \tag{11}$$

Placing the external masses in their supports and rotating them to the front-left position leads to the beam oscillating with the same period about a new equilibrium θ_1, corresponding to observed angle ϕ_1 of the spot on the viewing screen.

$$\begin{aligned} \theta_1 &= \frac{1}{2}(\phi_1 - \phi_c) \\ &= -\frac{GMD}{d\omega^2}\sin(\theta_1 + \theta_M)\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \end{aligned} \tag{12}$$

Similar relations hold for the front-right configuration of the external masses, for which the shifted equilibrium position is labeled θ_2.

$$\begin{aligned} \theta_2 &= \frac{1}{2}(\phi_2 - \phi_c) \\ &= -\frac{GMD}{d\omega^2}\sin(\theta_2 - \theta_M)\left(\frac{1}{s_1^3} - \frac{1}{s_2^3}\right) \end{aligned} \tag{13}$$

where the distances s_1,s_2 are defined in a similar manner to the r_1,r_2 in Eq. 7 as

$$\begin{aligned} s_1^2 &= d^2(1-\cos\theta)^2 + (b - d\sin\theta)^2 \\ s_2^2 &= d^2(1+\cos\theta)^2 + (b + d\sin\theta)^2 \end{aligned} \tag{14}$$

Note that r_1 and r_2 must be evaluated at angle $\theta = \theta_1$ in Eq. 12, whereas s_1 and s_2 in Eq. 13 must be evaluated at $\theta = \theta_2$. Solving Eqs. 12 and 13 for G yields two independently measured values for this important physical constant.

IV. DATA ANALYSIS

Use the Open Source Physics Tracker [2] application to analyze the three image sequences, `unloaded_1, left_right, and unloaded_2`, and digitize the position of the laser spot as a function of frame number. Be sure to set the scale and orientation of the axes in the image, using the meter stick that is visible in the image, before starting the autotracker. Save the results from each run into output files containing 3 columns of data for t, x and y. Import these data into three separate worksheets in a spreadsheet application for further analysis. Correct the t columns on each sheet to reflect the correct time interval between frames for that run. The y columns can be eliminated.

On the `unloaded_1` worksheet, create a new column for the angle ϕ and fill in its values using the formula,

$$\phi = -\tan^{-1}\left(\frac{x}{L}\right) \tag{15}$$

where L is the perpendicular distance from the torsion fiber to the screen. Locate the midpoint x_c between the two limits marked on the viewing screen, and record the value of ϕ_c obtained from it using Eq. 15. Be sure to record the measurement errors on each.

Plot the sequence ϕ *vs* t and verify that the equilibrium position around which the oscillations are centered is reasonably stable throughout the run. Carry out a least-squares fit of these ϕ data to the following model function,

$$\Phi(t) = a_0 e^{-\lambda_0 t}\sin(\omega(t - t_0)) + a_1 e^{(} - \lambda_1 t) + a_\infty \tag{16}$$

where free parameters a_0, λ_0, ω, t_0, a_1, λ_1, and a_∞ are varied to minimize the χ^2. To define the χ^2, you need to estimate the measurement error on ϕ. This should be determined mainly by the error on the x position of the laser spot, propagated through Eq. 15. A fraction of the laser spot diameter would be a good starting point for this estimate. A good way to check your estimate is to compare the minimum χ^2 value returned from the fit with the number of degrees of freedom N in the fit. These two should not match exactly, but the agreement should be within roughly $\pm\sqrt{2N}$. Plot the best-fit curve on top of the data and verify visually that the fit was successful. Deviations between the data points and the fit curve should scarcely be visible in the plot.

If deviations are apparent, try excluding a sequence of points from the fit starting at the beginning of the run. It may be that the balance was still shaking from the initial setup, and needed time to settle down. To exclude a range of data from the fit, simply redefine the χ^2 cell in the spreadsheet to exclude those rows from the sum. Do not remove these points from the graph, as it is interesting to see what happens to the agreement between the curve and the data in regions where the data are excluded from the fit. In the end, your fit should cover at least the last 3 complete oscillations of the system. It is not legitimate to "cut out" data from the middle of the run, just because they do not agree with the model. The only part you may exclude is an initial sequence, and only because it may be affected by transients.

Carry out a jack-knife analysis to obtain the errors on each of the fit parameters. In a case like this, with hundreds of points in the data set, it is sufficient to carry out only 5-10 jack-knife fits. To do 10 fits, free up a block on your spreadsheet with 12 columns and 10 rows. Label the first column χ_s^2 and assign it the formula equal to the overall chi_s^2 cell minus the sum of *every tenth entry in the residuals squared column.* Entering a formula to do this requires a little spreadsheet magic using the `SUMPRODUCT` and `MOD` built-in functions. You should be able to figure this out, or find a recipe for how to do it online. Copy this formula down all 10 rows of the block, then carry out 10 fits to minimize the 10 χ_s^2 functions so defined. After each fit, copy the best-fit values of the 11 free parameters into the 11 columns to the right of the χ_s^2 column. When you are done, the error on the 11 free fit parameters can be computed, based on the standard deviation of the corresponding column in the jack-knife fit results block, in the usual fashion for jack-knife error analysis.

Repeat the above analysis for the data from the `unloaded_2` run. Check whether the oscillation frequency f and the two damping constants λ_0 and λ_1 are consistent between the two unloaded runs. If this is not the case, investigate the source of the inconsistency, and find a way to include it in the errors that you assigned to the measured coordinates. Once these two independent fits are seen to be consistent with one another, you are ready to do a joint fit to the front-left and front-right data sets.

Import the `left,right` data for t, x, y into a fresh worksheet. Prepare the data for fitting in the same fashion as before, correcting t to represent real time, eliminating y, and converting x and Δx into ϕ and $\Delta\phi$. Graph the measured values of ϕ *vs* t on a single plot, showing the oscillations around the left equilibrium point, followed by the right. Set up two identical side-by-side parameter blocks representing the free parameters in Eq. 16. The first set will be used to fit the first half of the data in this worksheet, and the second set to fit the second half. As before, form a single column for the fit function $\Phi(t)$ and enter the formula from Eq. 16. Half-way down the column, near where the weights were switched from left to right, edit the fit formula to use the second parameter set instead of the first, then copy this formula down the rest of the column. Enter beginning guesses for the values in the parameter blocks, then plot the approximate fit function on top of the measured points in the graph.

In the cell for the χ^2 value for parameter block 1, enter the sum of the residual squares for the first half of the data. You should only include in this sum the rows that correspond to the region of clean behavior, excluding points early in the run period that might have been contaminated by transients, and also points when you intervened to switch between the two mass configurations. You should only exclude data from the beginning of the run and during the transition period between the two configurations; do not clip data out of the middle of one of the oscillation periods.

Carry out independent fits of the two oscillation periods within run 2, using parameters from the two parameter blocks. Refine the ranges included in the χ^2 sums until the fits show no major systematic deviations from the data. Then verify that the measurement error you used on the x values makes sense by comparing the best-fit χ^2 value with the number of degrees of freedom. Repeat the jack-knife error analysis for both of these periods.

In the next step, you will introduce the constraint that the values of the frequency and damping rates should be the same before and after the lead masses where switched. Henceforth subscript 1 will be used to denote parameters in the first block, and 2 to label those in the second block. In the place of the guessed value you entered for $\omega[2]$ in the second parameter block, create a formula equating it to $\omega[1]$ in the first block. Do the same thing for $\lambda_0[2]$ and $\lambda_1[2]$, constraining them to always equal $\lambda_0[1]$ and $\lambda_1[1]$, respectively. In a new cell, form a global χ_T^2 as the sum of the original $\chi^2[1]$ and $\chi^2[2]$ values, and do a least-squares fit on the value of χ_T^2, allowing only the 11 unconstrained parameters to vary, 7 from parameter block 1 and 4 from block 2.

Use the jack-knife technique to estimate the errors on the 11 parameters returned from the global fit. You should find that introducing the constraints between the two parameter blocks and fitting both sets at once results in somewhat smaller errors on the fit parameters than were obtained fitting the two periods separately. The parameters of particular interest in extracting a value for G are the frequency ω and the two offset values $a_\infty[1]$ and $a_\infty[2]$.

Use a similar jack-knife analysis on the `unloaded_1` and `unloaded_2` data sets, and extract errors on the value of a_∞ for the unloaded case. Verify that the values from the early and later run periods are in agreement within errors. To help distinguish between the different values for a_∞ coming from the different fits, they will be renamed as follows. The average of a_∞ from the `unloaded_1` and `unloaded_2` fits will be called ϕ_0 and its error $\Delta\phi_0$. The value of $a_\infty[1]$ from the first fit parameter set from the weighted run (front-left) will be called ϕ_1, and that from the second parameter set (front-right) will be called ϕ_2. Their associated errors are $\Delta\phi_1$ and $\Delta\phi_2$, respectively.

To obtain G, first convert ϕ_0, ϕ_1, and ϕ_2 into θ_0, θ_1, and θ_2 using Eq. 15, being careful to propagate their errors correctly. Use Eqs. 12 and 13 and the measured values of all other relevant experimental quantities to to extract the value of G from your measurements. You should obtain two independent estimates for G, one from θ_1 and the other from θ_2. Compute the error on each of these estimated values, and verify that the two are in agreement within their errors.

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[1] E.G. Adelberger, J.H. Gundlach, B.R. Heckel, S. Hoedl, and S. Schlamminger, "Torsion Balance Experiments: A Low-energy Frontier of Particle Physics", Prog. Part. Nucl. Phys., Vol. 62, Issue 1 (2009) 102-134.

[2] D. Brown, in Open Source Physics: A User's Guide with Examples (2006), WWW Document, (`http://www.compadre.org/Repository/document/ServeFile.cfm?ID=7379&DocID=530`).