Physics 259

Chaos in a Nonlinear RLC Circuit

Based on a laboratory from the Rutgers University Physics 326 Course.
Last modified 3/1/00 by Edward E. Eyler

Purpose:

1.   To study the onset of chaos in a nonlinear RLC circuit and to explore with a MathCad spreadsheet the approach to chaos demonstrated by the Logistic Equation.

References:

"Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator," Paul Lindsay, Physical Review Letters 47, 1349 (1981).

Chaos and Nonlinear Dynamics, by Robert C. Hilborn (Oxford University Press, New York), 1994.

Equipment:

1.   RLC circuit
2.   Oscilloscope
3.   Sine wave generator
4.   Spectrum analyzer (computer with FFT or dedicated instrument)
5.   Accurate AC voltmeter usable at 20 KHz

Background:

A few weeks ago you did an experiment on the forced harmonic motion of a resonant spring-mass system, which demonstrated several important features: (1) the mass undergoes a transient initial motion determined by the initial conditions (such as initial displacement and velocity), (2) the transient motion
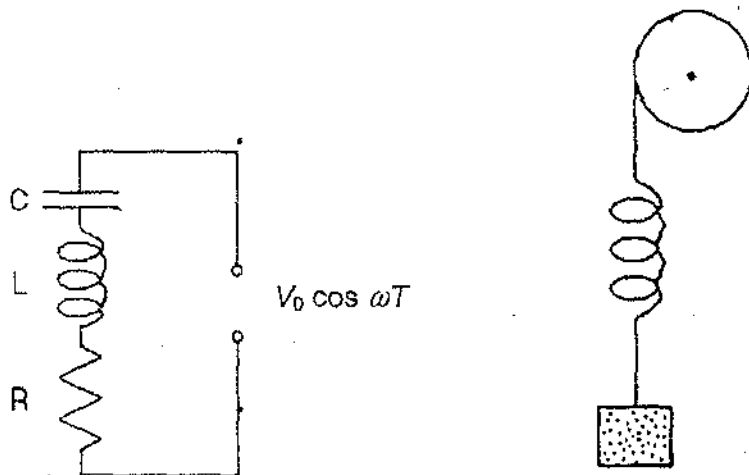
damps out leaving a steady-state motion independent of the initial conditions, (3) the steady state motion is a linear function of the amplitude of the driving force (i.e. if the force is doubled, the amplitude of oscillation doubles). This last feature is a characteristic of a *linear* system. The analogous series RLC circuit shows exactly the same behavior, where L plays the role of the mass, L/C that of the spring constant, and the charge Q stored on the capacitor the role of the amplitude of oscillation.

In this experiment you will study a series resonant RLC circuit, but with one crucial change. In place of using a capacitor whose capacitance is constant, you will use one whose capacitance depends on the magnitude of the voltage across it. This capacitor voltage will then no longer be a linear function of the driving voltage and we have a nonlinear or anharmonic system.

You will view the voltage V_R across the resistance in the circuit shown below as you increase V_0. At first you will observe a voltage oscillating at the frequency ω, with each successive peak of the oscillation having the same magnitude. But when V_0 becomes large enough, the peaks will alternate in amplitude, that is, V_R will no longer have period $T = 2\pi / \omega$, but rather twice the period, $T = 4\pi / \omega$. There has been a *period doubling* of the response of the circuit. As V_0 is further increased, the pattern of peaks again changes so that the same amplitude only occurs every fourth cycle; there has been another period doubling and the period is now $T = 8\pi / \omega$. Each successive period doubling requires a smaller increase in V_0 than the previous one. You will rapidly find, that no matter how good your apparatus, you cannot control V_0 accurately enough to discern successive period doublings and the voltage across the resistor appears to vary randomly with time. But we know from what we've seen for smaller V_0 that the response of the circuit varies in a perfectly well determined manner. This deterministic yet apparently random behavior is called *chaos* (or sometimes, *deterministic chaos*) to distinguish it from truly random behavior.

It has been only since the advent of high speed computers that scientists have realized the extent to which nonlinear systems may exhibit chaotic behavior and that the approach to chaos through period doubling is a universal behavior that does not depend on the details of the system studied. In this experiment we will first observe the chaotic behavior of the nonlinear RLC circuit, and then use a spreadsheet to study the approach to chaos for the Logistic Equation, an extremely simple nonlinear difference equation that exhibits chaos. The Logistic Equation,

$$x(n+1) = a \cdot x(n)\left[1 - x(n)\right]$$

(1)

has been used to model natural populations such as the yearly population swings of lemmings. If the population $x(n)$ during the nth interval is known, the equation gives the population $x(n+1)$ in the next interval (i.e. the next year, for lemmings). The parameter a is called the *growth parameter* and is

essentially the difference between the birth and death rates. We will allow $x(n)$ to vary between 0 and 1, which limits a to a range from 0 to 4.

Just as for the oscillating mass or the RLC circuit, the Logistic Equation shows an initial behavior which depends on the starting condition [$x(0)$], and damps out at large times, leaving a steady state response. It is important to notice that there is no driving term so that the "normal" steady state behavior will be a constant term rather that the steady state oscillation seen for the RLC circuit. Despite this difference the approach to chaos will be the same as that of the RLC circuit.

The behavior of the steady state response of the Logistic Equation depends strongly on a as follows:

$0 \le a < 1$: The growth parameter is too small to sustain any population. Regardless of $x(0)$, the population crashes to zero after a few intervals. $x(n) = 0$ is called a fixed point of the system since $x(n) = x(n+1)$. It is a stable fixed point because after any small disruption the population returns to zero. Stable fixed points are also called attractive fixed points or attractors.

$1 \le a < 3$: For $a = 1$ the fixed point at zero becomes unstable and is called a repellent fixed point or a repeller. At the same time a new attractor appears, $x(n) = 1 - 1/a$. The growth parameter is now large enough to sustain a nonzero steady state population. The transient solution montonically approaches $x(n) = 1 - 1/a$ for a close to 1, but then for larger a it oscillates about the attractor before eventually being damped out.

$3 \le a < 3.4473$: At $a = 3$ the attractor $x(n) = 1 - 1/a$ becomes unstable and the steady state solution oscillates about $x(n) = 1 - 1/a$ with alternate $x(n)$'s having the same value. For $a < 3$, the interval between like values for $x(n)$ in the steady state solution is $\Delta n = 1$, while for $a \ge 3$ the interval is $\Delta n = 2$; the period has doubled. Do not confuse the appearance of this oscillation with the steady state behavior of the RLC circuit before frequency doubling has set in.

$3.4473 \le a < 3.5433$: At $a = 3.4473$ the period doubles again so that the interval between ~qual values of $x(n)$ is now $\Delta n = 4$.

$3.5433 \le a < 3.6$: The steady state solution goes through a series of period doublings as shown in the graph below, until at $a = 3.6$ chaos sets in.

Chaos actually
Sets in @ a =3.57

In your calculations you will see that in the chaotic region, the steady state is so ꞏ that it is impossible for us to predict the solution. The slightest error in our completely throw off our predicted behavior. We also noted that in a linear sys region for the Logistic Equation, that the steady state solution is independent c

Calculations with the Logistic Equation will also show that the steady state solution is no longer independent of the starting condition and is in fact extremely sensitive to the exact value of $x(0)$. We must conclude then, that when chaos appears we cannot theoretically predict the response of a system even though it is in principle completely deterministic.
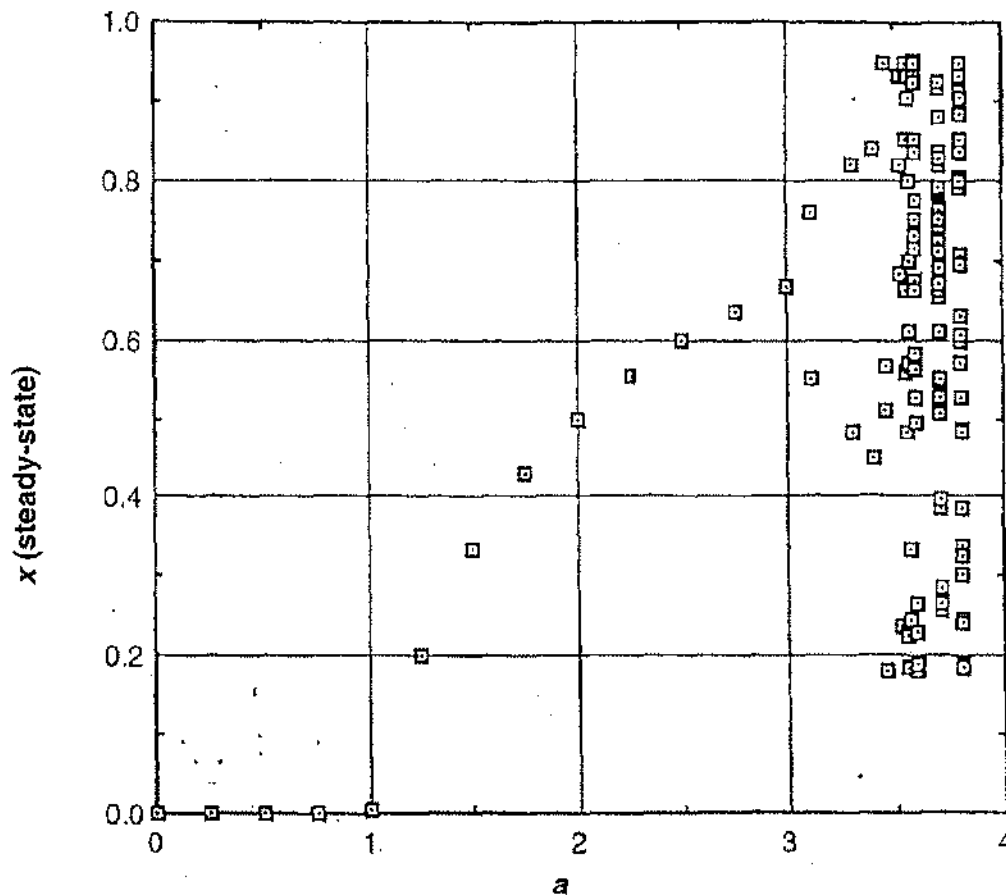

Figure 2. Steady-state solution of the Logistic Equation for x.

The sequence of period doublings that you will observe in this experiment for both the RLC circuit and the Logistic Equation has a universal character that is independent of the particular system. Most of the details of this universal behavior are beyond the scope of this laboratory. However, one aspect is accessible; Feigenbaum has demonstrated that if a_1 is the value of a where the first period doubling sets in, a_2 the value for the second doubling, etc., then for large n,

$$a_{\infty} = a_n + S\delta^{-n},$$

(2)

where a_∞ is the value of a where chaos sets in. S is a constant characteristic of the particular system, but $\delta = 4.6692...$ is a universal constant called the *Feigenbaum number*. Since δ is large compared to 1, this series converges rapidly and Eq. (2) will allow us to estimate δ and predict a_∞ with only a little data. In this experiment you will measure the first few values of a_n and use this data to estimate δ from Eq.(2). To do this use data from three successive period doublings to eliminate S and a_∞ from Eq.(2),

$$\delta = \frac{a_n - a_{n-1}}{a_{n+1} - a_n}, \tag{4}$$

If n is large enough, the value of δ calculated this way should equal Feigenbaum's theoretical value. To get a_∞ from your data, solve Eq.(2) for S,

$$S = \frac{(a_n - a_{n-1})\delta^n}{\delta - 1}, \tag{5}$$

and then use Eq.(2) to get a_∞.

Procedure:

For the nonlinear capacitor we use a silicon rectifier diode, type 1N4003 or similar. It may be advantageous to use two diodes in parallel, to increase the capacitance and lower the resonant frequency into a range more accessible for computer data acquisition. The diode consists of a layer of p-type silicon in contact with a layer of n-type silicon. When the diode is reverse biased, the positive charges in the p-type silicon and the negative charges in the n-type silicon are moved away from the junction by the electric field produced by the applied voltage. When the junction is forward biased they are moved toward the junction. Thus the distance of separation of the charge in the diode and hence its capacitance depends on the voltage. It turns out that this dependence is very nonlinear. The diode also exhibits an additional nonlinearity: when the forward voltage exceeds about 0.6 Volts, the diode begins to turn on and acts more like a resistor than a capacitor. This behavior does not lead to chaos, but it does cause the production of a harmonic spectrum (as opposed to subharmonics), which you will probably be able to observe clearly when driving the circuit well below the resonance frequency.

Nonlinear RLC circuit

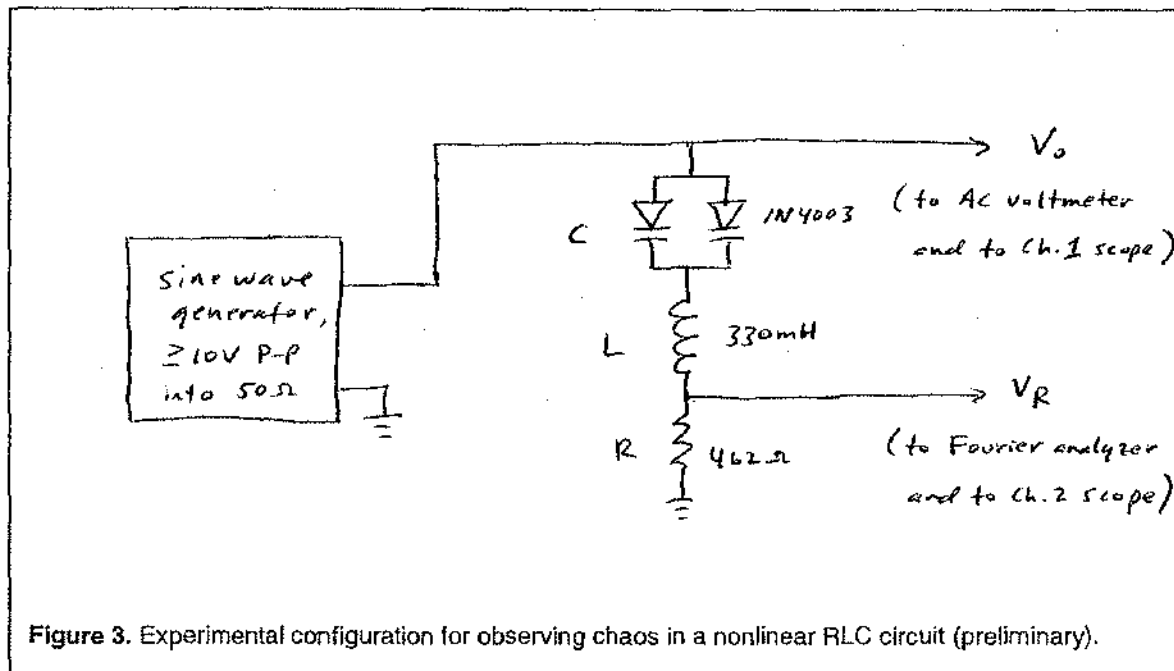1 .Set up the nonlinear RLC circuit as shown in the figure.

Figure 3. Experimental configuration for observing chaos in a nonlinear RLC circuit (preliminary).

2. Set the sine wave generator to a frequency of about ~~20~~ *35* kHz and minimum amplitude. We will choose to look at the voltage V_R across the ~~voltage-variable capacitor, because it is easiest to observe the appearance of new frequencies by looking at the sharp "pulses" that occur here each time that the diode is in its nonconducting state.~~ *resistor* First search for the LC resonance, then fix the signal generator frequency to resonance or slightly below it (a few percent below it seems to work particularly well.) Now you can gradually increase the signal amplitude V_0, looking for the appearance of subharmonics using the oscilloscope. You can use two different modes to display V_R in a useful manner; you can either look at V_R V_R versus time using the time sweep for the x-axis, or you can look at V_R versus V_0 using an *x-y* display mode, which is known as a Lissajous figure. The Lissajous figure is probably the easiest way to look for period doubling using a time-domain analysis, although you may be confused by higher harmonics, which are produced as an irrelevant side-effect by the circuit. Practice using both modes of operation.

3. Experiment with the amplitude and frequency until you can clearly see at least three successive period doublings (the fourth will probably be unobservable, since very small adjustments will cause chaotic behavior rather than an orderly period doubling).✗

4. Look at the FFT of the spectrum (your instructor will help you set this up, using either a spectrum analyzer or a computed FFT on the lab computer). Set up the oscilloscope at the same time. Starting with

V_0 small, slowly increase its value and look for evidence for period doubling using the Lissajous figure. What happens to the FFT spectrum when this happens? Record the amplitude and frequencies of peaks in the FFT spectrum as you increase V_0 still further to observe additional doubling and the onset of chaos. If using the computer, record data for further offline analysis (your instructor will provide details). You may notice that once you have reached the chaotic region, there are still values of V_0 for which you get simple deterministic behavior (period tripling and quintupling).

5. Using the AC voltmeter to measure V_0 and the oscilloscope to measure the amplitude(s) of the peak value(s) of $\frac{V_R}{V_C}$, prepare a graph of $\frac{V_R}{V_C}$ versus V_0, analogous to the attached graph for period doubling for the Logistic Equation. Carefully determine the values of V_0 where period doubling occurs and use Eqs. (2)-(4) to estimate a_∞, S, and δ. (Note that V_0 plays the role of the growth parameter a.) When the frequency is close to a doubling point, you may find it useful to average the spectrum over 50 samples (if this feature is available) to better determine the actual point. For the 3rd or higher points you should try reducing the frequency span to better isolate the point.

The Logistic Equation

1. Set up a MathCad spreadsheet to calculate $x(n)$ for various values of $x(0)$ and a. In order to examine the transient response of the equation, graph $x(n)$ for the first 20 values of n, and examine the steady state response graph of $x(n)$ for about 40 cycles when n is large. (To get the steady state solution near $a = 3$, you may have to go up to $n \sim 1000$ to avoid the transient.) Set a graph on the screen so that you can watch the effect of changing a or $x(0)$.

2. Examine the behavior of the solutions for $a < 3$ for various values of $x(0)$ in order to verify that the equation behaves as described in the BACKGROUND section. In particular, verify that the steady state solution is independent of the initial condition.

3. Examine the behavior of the solutions for the four cases $a = 2.99$ and 3.01 with $x=0.667$ and 0.5, where period doubling first sets in. Does the steady state value depend on the initial value for any of these cases?

4. Determine to four decimal places (3.xxxx) the values of a where period doubling sets for at least five period doublings. To do this examine the steady state solution values of $x(n)$ for n, $n+1$, $n+2$, ... to see how many cycles you have to go through before the pattern repeats. Then vary a until you find the value where the period doubles. Use a value of n near 1000 and increase the accuracy of $x(n)$ as much as necessary to see a change in pattern. Use Eqs. (2)-(4) to estimate a_∞, S, and δ.

5. For a in the chaotic regime, $a > 3.6$, examine the steady state solution to see the dependence on extremely small changes in a and in $x(0)$. The effect is easiest to see on a small graph if you use values near 3.9. For example, use a starting point of 0.60 and record the 200th iteration. Then change the starting point to 0.6001. How much does the 200th iteration change?