Physics 258/259 THE CAVENDISH EXPERIMENT

A sensitive torsion balance is used to measure the Newtonian gravitational constant G. The equations of motion of the torsion balance are solved in terms of the experimentally determined parameters, from which G is determined in two different limiting cases.

I. INTRODUCTION

The gravitational torsion balance is shown in Fig.1. Two small lead spheres are attached to the ends of a light rod. This rigid dumbbell is suspended horizontally by a vertical torsion fiber. The two large lead outer spheres gravitationally attract the inner spheres. Equilibrium is established when the torque on the dumbbell due to the gravitational forces is balanced by the restoring torque from the torsion fiber. The two large spheres can then be rotated from their initial positions, labeled A and B, to the symmetrically located

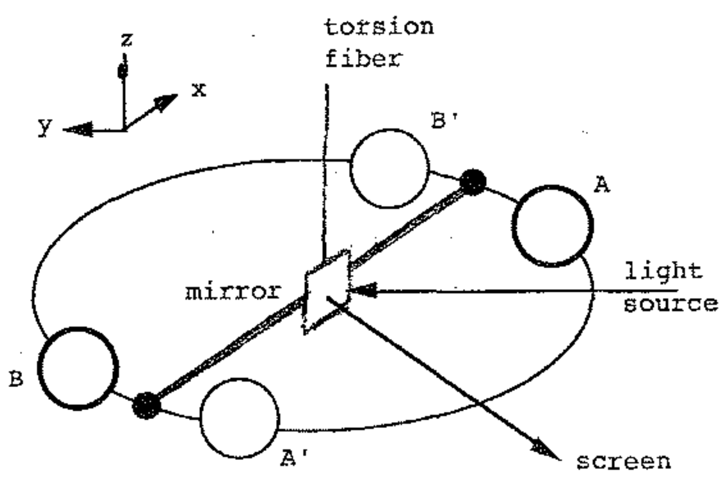

FIG.1. The two large spheres of mass M are initially at the positions A and B. The two small spheres which comprise the torsion balance have a mass m.

positions at A' and B'. This disturbs the equilibrium and the torsion balance begins to oscillate about a vertical axis. Attached to the center of the rod is a small mirror. By observing the deflection of a beam of light reflected from this mirror, the angular movement of the torsion balance can be measured.

In the following section, we will consider the motion of the torsion balance in two limiting cases following the displacement of the large outer spheres. First, in the infinite time limit when mechanical equilibrium is reestablished at a different angular position of the torsion balance, and then in the short time limit, immediately after the displacement.

II. EQUATIONS OF MOTION

A. Forces and Torques

Consider the gravitational force on the small lead sphere on the right hand side in Fig.2.

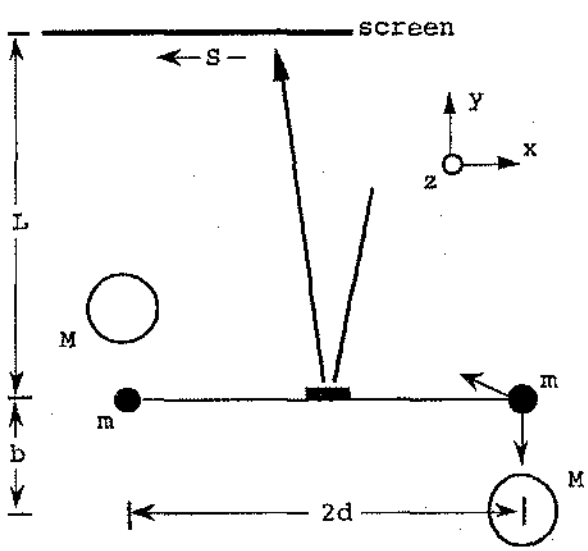

FIG.2. The rotation of the rigid dumbell in the xy plane will cause the position of the reflected light beam to move across the viewing screen.

Show that the net gravitational force on the sphere, perpendicular to the line joining the two small spheres is

$$F_{\perp} = \frac{GMm}{b^2}\left(1 - \frac{b^3}{(b^2+4d^2)^{3/2}}\right) = \frac{GMm\beta}{b^2} \quad . \tag{1}$$

There is an equal in magnitude but opposite in direction force on the small sphere on the left hand side. The gravitational torque on the torsion balance due to these two forces is

$$\Gamma_G = -2dF_{\perp}. \tag{2}$$

There is also a restoring torque on the torsion balance due to the twisting of the torsional fiber of,

$$\Gamma_k = -\kappa\vartheta_i, \tag{3}$$

where the clockwise initial angle ϑ_i is negative. Since the system is initially in mechanical equilibrium, the net torque on the torsion balance must vanish,

$$\sum\Gamma_i = -\kappa\vartheta_i - 2dF_{\perp}=0. \tag{4}$$

The two large spheres are moved to their new symmetric positions at t=0. The force on each of the small spheres suddenly flips in direction, although stays constant in magnitude. The torque on the torsion balance immediately after repositioning the large spheres is

$$\Gamma_G = 2dF_{\perp}. \tag{5}$$

Thus the equation of motion for the torsion balance for t>0 is

$$I\ddot{\vartheta} = -D\dot{\vartheta} - \kappa\vartheta + 2dF_{\perp} \ , \tag{6}$$

where $I=2md^2$ is the moment of inertia of the torsion balance and $D\dot{\vartheta}$ is a frictional damping term. The initial conditions at t=0 are $\vartheta_i=-2dF_\perp/\kappa$ and $\dot{\vartheta}_i=0$.

B. Final Deflection Solution

Following the symmetric displacement of the large spheres, the torsion balance undergoes a damped oscillatory motion and then reaches a final equilibrium position at an angle ϑ_f. This angle can be found from the equilibrium condition for the net torque,

$$\Gamma_f = 2dF_\perp - \kappa\vartheta_f = 0. \tag{7}$$

The net angular displacement, $\Delta\vartheta=\vartheta_f-\vartheta_i$, can be found from Eq.(4) and Eq.(7) as

$$\Delta\vartheta = 4dF_\perp/\kappa. \tag{8}$$

We now want to relate the torsion constant κ to the period τ of the oscillatory motion. Neglecting the small damping term, the equation of motion predicts that the oscillation period is

$$\tau = 2\pi\sqrt{\frac{I}{\kappa}}\,, \tag{9}$$

where $I=2md^2$. By solving Eq.(9) for κ and then substituting into Eq.(8), the net angular displacement of the torsion balance is

$$\Delta\vartheta = \frac{F_\perp\tau^2}{2\pi^2 md}. \tag{10}$$

By combining Eq.(1) with Eq.(10), the gravitational constant can be written in terms of experimentally accessible parameters as,

$$G = \frac{2\pi^2 b^2 d}{M\tau^2\beta}\Delta\vartheta\,. \tag{11}$$

During the angular displacement of the torsion balance, the light beam will move through an angle $\Delta\varphi$ and a distance ΔS where

$$\Delta\vartheta = \Delta\varphi/2 = \Delta S/2L, \tag{12}$$

and the factor of 2 arises from the law of reflection at the mirror.

C. Initial Acceleration Solution

Consider the $t\rightarrow 0^+$ limit of Eq.(6), which would be the equation of motion for the torsion balance just after the large spheres have been displaced. In this limit, $\dot{\vartheta}\rightarrow 0$ and $\vartheta\rightarrow\vartheta_i$, and thus Eq.(6) becomes

$$I\ddot{\vartheta}(t\rightarrow 0^+) = -\kappa\vartheta_i + 2dF_\perp \ . \tag{13}$$

By substituting the expression for ϑ_i from Eq.(4) into Eq.(13), the initial angular acceleration of the torsion balance is

$$\ddot{\vartheta}(t\rightarrow 0^+) = \frac{2F_\perp}{md} \ . \tag{14}$$

After substituting Eq.(1) into Eq.(14), the gravitational constant can be related to a second set of measurable quantities as,

$$G = \frac{b^2 d\ddot{\vartheta}}{2M\beta} \ . \tag{15}$$

The angular acceleration is related to the constant initial linear acceleration of the position of the light spot on the screen by $\ddot{\vartheta} \approx \ddot{S}/2L$.

III. PROCEDURE

The gravitational torsion balance will be aligned and in equilibrium when you come into the laboratory. It takes about two hours to return to a full equilibrium condition so be careful not to touch, bump or otherwise disturb the balance. Set up a viewing screen at a distance of $L\sim 5m$. Align the He-Ne laser light source and note its position on the screen. This is the initial ($S_i=0$) position. Observe the position for several minutes to ensure that the balance is in equilibrium.

Carefully move the outer spheres to their diagonally symmetric positions, being sure not to otherwise disturb the balance. Be especially careful not the let the bracket which holds the large spheres slam into the stops. Start the stopwatch/timer as you move the spheres. Mark the position of the light beam on the screen every ten seconds for the first two minutes, and every 30 seconds thereafter for a total elapsed time of about one hour. Wait until the light spot is stationary and carefully record the final equilibrium position S_f. Then measure L, M, d and b. Note that the small lead balls are located along the center-line of the case.

IV. DATA ANALYSIS

Use the data for the position of the reflected light beam during the initial two minutes to find the value of $\ddot{S}$. A plot of S as a function of t^2 should be helpful. Carefully consider the time interval over which $\ddot{S}$ is constant and Eq.(15) remains valid. Use this result to find the initial value of $\ddot{\vartheta}$. Then find G from Eq.(15). Estimate the average oscillation period τ from your data. Use Eq.(11) to find a value for G.

Compare your two measurements and their uncertainties. Which one is more accurate? Are their any systematic errors in the measurements? Compare your measured values to other published measurements of G.