PRODUTO ESCALAR, VETORIAL E MISTO

GABARITO EXERCÍCIOS DE FAMILIARIZAÇÃO – PRODUTO ESCALAR

1) Calcule

a) $(1, 0, 1) \bullet (-2, 10, 2) = -2 + 2 = 0$

b) $(\vec{i} + \vec{j} + \vec{k}) \bullet (\vec{i} - \vec{j} - 5\vec{k}) = (1,1,1) \bullet (1,-1,-5) = 1 - 1 - 5 = -5$

c) $(\frac{1}{2}, 0, 0) \bullet (1, 2, 7) = \frac{1}{2}$

d) $(0, 0, 0) \bullet (-1, -9, 10) = 0$

e) $(\sqrt{3}, 1, 0) \bullet (1, 3, -7) = \sqrt{3} + 3$

f) $(2, 1, 1) \bullet (1, -4, 2) = 0$

2) Calcular o ângulo entre os vetores:

a) $\vec{u} = (1, 2, 2)$ e $\vec{v} = (1, -4, 8)$

$\vec{u} = (1, 2, 2)$ e $\vec{v} = (1, -4, 8)$

$|\vec{u}| = \sqrt{1+4+4} = 3$ $\qquad |\vec{v}| = \sqrt{1+16+64} = 9$

$\vec{u} \cdot \vec{v} = 1 - 8 + 16 = 9$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{9}{3 \times 9} = \frac{1}{3}$$

$\therefore \theta = \arccos \frac{1}{3} = 71^\circ$.

b) $\vec{r} = (4, -1, 3)$ e $\vec{s} = (1, 1, -1)$

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$|\vec{r}| = \sqrt{16+1+9} = \sqrt{26}$ $\qquad |\vec{s}| = \sqrt{3}$

$\vec{r} \cdot \vec{s} = 4 - 1 - 3 = 0$

$\cos\theta = 0$

$\theta = \arccos 0 \Rightarrow \theta = 90^\circ \therefore \vec{r} \perp \vec{s}$

3) Qual o valor de m para que os vetores sejam ortogonais?

a) $\vec{u} = (m, 2, 3)$ e $\vec{v} = (2, -1, 2)$

$\vec{u} = (m, 2, 3)$ e $\vec{v} = (2, -1, 2)$

$\vec{u} \cdot \vec{v} = 2m - 2 + 6 = 2m + 4 = 0 \Leftrightarrow 2(m-2) = 0$

$\Leftrightarrow m = -2.$

b) $\vec{r} = (m, 3, 4)$ e $\vec{s} = (m, -2, 3)$

$\vec{r} = (m, 3, 4)$ e $\vec{s}\ (m, -2, 3)$

$\vec{r} \cdot \vec{s} = m^2 - 6 + 12 = m^2 + 6 = 0 \Leftrightarrow m^2 = -6$

$\therefore \nexists\, m \in \mathbb{R} \ / \ \vec{r} \perp \vec{s}$

4) Achar os ângulos diretores do vetor $\vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$

$\vec{v} = \vec{i} + 2\vec{j} + 2\vec{k} \qquad |\vec{v}| = \sqrt{1+4+4} = 3$

$\cos \alpha = \dfrac{x}{|\vec{v}|} = \dfrac{1}{3} \therefore \alpha = \arccos \dfrac{1}{3} \cong 71^\circ$

$\cos \beta = \dfrac{y}{|\vec{v}|} = \dfrac{2}{3} \therefore \beta = \arccos \dfrac{2}{3} \cong 48^\circ$

$\cos \gamma = \dfrac{z}{|\vec{v}|} = \dfrac{2}{3} \therefore \gamma = \arccos \dfrac{2}{3} \cong 48^\circ$

5) Determine a medida, em radianos do arco determinado pelo ângulo formado pelos vetores $\vec{u}$ e $\vec{v}$ nos seguintes casos:

a) $\vec{u} = (1, 0, 1)$ e $\vec{v} = (-2, 10, 2)$

$\vec{u} = (1, 0, 1)$ e $\vec{v} = (-2, 10, 2)$

$\cos \theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{-2 + 0 + 2}{\sqrt{2} \cdot \sqrt{104}} = 0 \therefore \theta = \dfrac{\pi}{2}$

b) $\vec{u} = (3, 3, 0)$ e $\vec{v} = (2, 1, -2)$

$\vec{u} = (3, 3, 0)$ e $\vec{v} = (2, 1, -2)$

$\cos \theta = \dfrac{6 + 3 + 0}{\sqrt{18} \cdot \sqrt{9}} = \dfrac{9}{9\sqrt{2}} = \dfrac{\sqrt{2}}{2} \therefore \theta = \dfrac{\pi}{4}$

c) $\vec{u} = (-1, 1, 1)$ e $\vec{v} = (1, 1, 1)$

$\vec{u} \cdot \vec{v} = -1 + 1 + 1 = 1$ $|\vec{u}| = \sqrt{3}$ $|\vec{v}| = \sqrt{3}$

$\cos\theta = \dfrac{1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3}$ ∴ $\theta = \arccos \dfrac{1}{3}$

d) $\vec{u} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$ e $\vec{v} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3})$

$\vec{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ $\vec{v} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right)$

$\vec{u} \cdot \vec{v} = \frac{3}{4} + \frac{1}{4} = 1$ $|\vec{u}| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ $|\vec{v}| = \sqrt{4} = 2$

$\cos\theta = \dfrac{1}{2}$ ∴ $\theta = \dfrac{\pi}{3}$

e) $\vec{u} = (300, 300, 0)$ e $\vec{v} = (-2000, -1000, 2000)$

$\vec{u} = (300, 300, 0)$ e $\vec{v} = (-2000, -1000, 2000)$
$\vec{u} = 300\,(1, 1, 0)$ $\vec{v} = 1000\,(-2, -1, 2)$
$\vec{u} \cdot \vec{v} = -2 - 1 = -3$
$|\vec{u}| = \sqrt{2}$ e $|\vec{v}| = \sqrt{9} = 3$

$\cos\theta = \dfrac{-3}{3\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$ ∴ $\theta = \dfrac{3\pi}{4}$

6) Determine x de modo que $\vec{u} \perp \vec{v}$ nos casos.

a) $\vec{u} = (x, 0, 3)$ e $\vec{v} = (1, x, 3)$

$\vec{u} = (x, 0, 3)$ e $\vec{v} = (1, x, 3)$

$\vec{u} \cdot \vec{v} = x + 9 = 0 \iff x = -9$

b) $\vec{u} = (x+1, 1, 2)$ e $\vec{v} = (x-1, -1, -2)$

$\vec{u} = (x+1, 1, 2)$ e $\vec{v} = (x-1, -1, -2)$
$\vec{u} \cdot \vec{v} = (x+1)(x-1) - 1 - 4 = 0$
$x^2 - 1 - 1 - 4 = 0 \Rightarrow x^2 = 6 \iff x = \pm\sqrt{6}.$

7) Determine a projeção do vetor $\vec{w}$ na direção do $\vec{v}$ nos casos:

a) $\vec{w}=(-1,1,1)$ e $\vec{v}=(-2,1,2)$

$\vec{w}\cdot\vec{v} = 2+1+2 = 5$

$|\vec{v}|^2 = 4+1+4 = 9$

$\text{proj}_{\vec{v}}\,\vec{w} = \frac{5}{9}(-2,1,2) = \left(-\frac{10}{9}, \frac{5}{9}, \frac{10}{9}\right)$

b) $\vec{w}=(1,3,5)$ e $\vec{v}=(-3,1,0)$

$\vec{w}\cdot\vec{v} = -3+3 = 0$

$\therefore \text{proj}_{\vec{v}}\,\vec{w} = \vec{0}$

8) Decomponha $\vec{w}=(-1,-3,2)$ como soma de dois vetores $\vec{w}_1$ e $\vec{w}_2$ de tal forma que $\vec{w}_1$ seja paralelo e $\vec{w}_2$ seja ortogonal ao vetor $(0,1,3)$.

$\vec{w} = \vec{w}_1 + \vec{w}_2$

$|\vec{u}|^2 = 1+9 = 10 \qquad \vec{w}\cdot\vec{u} = -3+6 = 3$

$\vec{w}_1 = \text{proj}_{\vec{u}}\,\vec{w} = \frac{3}{10}(0,1,3) = \left(0, \frac{3}{10}, \frac{9}{10}\right)$

$\vec{w}_2 = \vec{w} - \vec{w}_1 = (-1,-3,2) - \left(0, \frac{3}{10}, \frac{9}{10}\right) = \left(-1, -\frac{33}{10}, -\frac{29}{10}\right)$

9) Os vetores $\vec{a}$ e $\vec{b}$ são ortogonais, o vetor $\vec{c}$ forma com $\vec{a}$ e $\vec{b}$ ângulos iguais a $\frac{\pi}{3}$.

Sabendo-se que $|\vec{a}|=3$, $|\vec{b}|=5$ e $|\vec{c}|=8$, calcule:

a) $(3\vec{a}-2\vec{b})\bullet(\vec{b}+3\vec{c})$

$(3\vec{a}-2\vec{b})\cdot(\vec{b}+3\vec{c}) =$

$= 3\vec{a}\cdot\vec{b} + 9\vec{a}\cdot\vec{c} - 2\vec{b}\cdot\vec{b} - 6\vec{b}\cdot\vec{c}$

Como $\vec{a}$ e $\vec{b}$ são ortogonais então $\vec{a}\cdot\vec{b} = 0$

$\vec{a}\cdot\vec{c} = |\vec{a}||\vec{c}|\cos\frac{\pi}{3} = 3\times 8\times\frac{1}{2} = 12.$

$\vec{b}\cdot\vec{c} = |\vec{b}||\vec{c}|\cos\frac{\pi}{3} = 5\times 8\times\frac{1}{2} = 20.$

$(3\vec{a}-2\vec{b})\cdot(\vec{b}+3\vec{c}) = 3\times 0 + 9\times 12 - 2\times 25 - 6\times 20 =$

$= 108 - 50 - 120 = -62.$

b) $(\vec{a}+\vec{b}+\vec{c})^2$

$(\vec{a}+\vec{b}+\vec{c})^2 =$

$= (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) =$

$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{c} =$

$= |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + 2(\vec{a}\cdot\vec{c}) + 2(\vec{b}\cdot\vec{c}) + |b|^2 + |\vec{c}|^2 =$

$= 9 + 2\times 0 + 2\times 12 + 2\times 20 + 25 + 64 =$

$= 9 + 24 + 40 + 25 + 64 = 162.$

c) $(\vec{a}+2\vec{b}-3\vec{c})^2$

$(\vec{a}+2\vec{b}-3\vec{c})^2 =$

$= (\vec{a}+2\vec{b}-3\vec{c})\cdot(\vec{a}+2\vec{b}-3\vec{c}) =$

$= \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) - 3(\vec{a}\cdot\vec{c}) + 2(\vec{a}\cdot\vec{b}) + 4(\vec{b}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{c}) - 3(\vec{a}\cdot\vec{c}) -$

$-6(\vec{b}\cdot\vec{c}) + 9(\vec{c}\cdot\vec{c}) =$

$= |\vec{a}|^2 + 4|b|^2 + 9|\vec{c}|^2 + 4(\vec{a}\cdot\vec{b}) - 6(\vec{a}\cdot\vec{c}) - 12(\vec{b}\cdot\vec{c}) =$

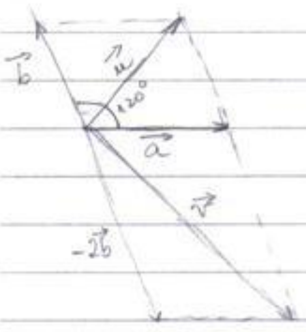
$= 9 + 4\times 25 + 9\times 64 - 6\times 12 - 12\times 20 =$

$= 9 + 100 + 576 - 72 - 240 = 373.$

10) O ângulo entre $\vec{a}$ e $\vec{b}$ mede 120°. Sendo $|\vec{a}|=4$, $|\vec{b}|=3$, $\vec{u}=\vec{a}+\vec{b}$ e $\vec{v}=\vec{a}-2\vec{b}$, o ângulo entre $\vec{u}$ e $\vec{v}$ é agudo, reto ou obtuso?

$\phi = 120^\circ$, $|a| = 4$, $|b| = 3$

$\vec{u} = \vec{a}+\vec{b}$ $\quad \vec{v} = \vec{a}-2\vec{b}$



$\vec{u}\cdot\vec{v} = (\vec{a}+\vec{b})\cdot(\vec{a}-2\vec{b}) =$

$= \vec{a}\cdot(\vec{a}-2\vec{b}) + \vec{b}\cdot(\vec{a}-2\vec{b}) =$

$= \vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - 2\vec{b}\cdot\vec{b} =$

$= |\vec{a}|^2 - 2|b|^2 - \vec{a}\cdot\vec{b}$

Por outro lado, $\vec{a}\cdot b = |\vec{a}||\vec{b}|\cdot\cos 120^\circ =$

$= 4\times 3\left(-\frac{1}{2}\right) = -6.$

$\therefore \vec{u}\cdot\vec{v} = 16 - 2\times 9 + 6 = 16 - 18 + 6 = 4.$

Como o produto escalar entre $\vec{u}$ e $\vec{v}$ é positivo concluímos que o ângulo entre $\vec{u}$ e $\vec{v}$ é agudo.