TÓPICO 6: ÂNGULOS E DISTÂNCIAS

GABARITO DOS EXERCÍCIOS DE FAMILIARIZAÇÃO

Exercício 1: Determine a medida em radianos do ângulo entre as retas:

a) $r:\begin{cases} x = 3 + t \\ y = t \\ z = -1 - 2t \end{cases}$ $(t \in \mathbb{R})$ e s: $\frac{x+2}{-2} = \frac{y-3}{1} = \frac{z}{1}$

$r:\begin{cases} x = 3 + t \\ y = t \\ z = -1 - 2t \end{cases}$ $\quad s: \frac{x+2}{-2} = \frac{y-3}{1} = \frac{z}{1}$

$\vec{r} = (1, 1, -2)$ $\quad \vec{s} = (-2, 1, 1)$

$|\vec{r}| = \sqrt{6}$ $\quad |\vec{s}| = \sqrt{6}.$

$\vec{r} \cdot \vec{s} = (1, 1, -2) \cdot (-2, 1, 1) = -2 + 1 - 2 = -3$

$\cos\theta = \frac{3}{6} = \frac{1}{2}$

$\theta = \arccos\frac{1}{2} = \frac{\pi}{3}$

b) r: $\overrightarrow{OX} = (1, 1, 9) + \lambda(0, 1, -1)$ $(\lambda \in \mathbb{R})$ e s: $\begin{cases} x - 1 = y \\ z = 4 \end{cases}$.

se $y = t \in \mathbb{R}$, $\quad s:\begin{cases} x = 1 + t \\ y = t \\ z = 4 \end{cases}$

$\vec{r} = (0, 1, -1)$ $\quad \vec{s} = (1, 1, 0)$ $\quad \vec{r} \cdot \vec{s} = 1$

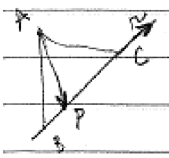
$|\vec{r}| = \sqrt{2}$ $\quad |\vec{s}| = \sqrt{2}$

$\cos\theta = \frac{1}{2}$

$\theta = \arccos\frac{1}{2} = \frac{\pi}{3}$

Exercício 2: Determine os vértices B e C do triângulo equilátero *ABC*, sabendo que $A(1, 1, 0)$ e que o lado *BC* está contido na reta r de equação r: $\overrightarrow{OX} = (0, 0, 0) + \lambda(0, 1, -1)$ $(\lambda \in \mathbb{R})$.

$A(1, 1, 0)$ $\quad r: X = (0,0,0) + \lambda(0, 1, -1)$



$\vec{r} = (0, 1, -1)$ $\quad |\vec{r}| = \sqrt{2}$

$\forall P \in r, \quad P = (0, \lambda, -\lambda)$

$\overrightarrow{AP} = (1, 1 - \lambda, \lambda)$

$|\overrightarrow{AP}| = \sqrt{1 + (1-\lambda)^2 + \lambda^2} = \sqrt{2\lambda^2 - 2\lambda + 2}$

Como ABC é equilátero o ângulo entre $\vec{r}$ e $\overrightarrow{AP}$ é $60°$

Logo $\cos 60° = \dfrac{|(0,1,-1)\cdot(1,1-\lambda,\lambda)|}{\sqrt{2}\cdot\sqrt{2(\lambda^2-\lambda+1)}} = \dfrac{|1-\lambda-\lambda|}{2\sqrt{\lambda^2-\lambda+1}} = \dfrac{1}{2}$

$\Rightarrow \dfrac{|1-2\lambda|}{\sqrt{\lambda^2-\lambda+1}} = 1 \Rightarrow |1-2\lambda| = \sqrt{\lambda^2-\lambda+1}$

$\cancel{1}-4\lambda+4\lambda^2 = \lambda^2-\lambda+\cancel{1}$

$3\lambda^2-3\lambda = 0 \Rightarrow 3(\lambda^2-\lambda) = 0$

$\lambda^2-\lambda = 0 \iff \lambda(\lambda-1) = 0 \iff \lambda = 0$ ou $\lambda = 1$

Se $\lambda = 0$, $B(0,0,0)$

Se $\lambda = 1$, $C(0,1,-1)$

Exercício 3: Determine a medida θ entre os planos $\pi_1: x - y + z = 0$ e $\pi_2: x + y + z = 0$.

$\pi_1: x - y + z = 0 \qquad \vec{n}_1 = (1,-1,1) \qquad |\vec{n}_1| = \sqrt{3}$

$\pi_2: x + y + z = 0 \qquad \vec{n}_2 = (1,1,1) \qquad |\vec{n}_2| = \sqrt{3}$

$(1,-1,1)\cdot(1,1,1) = 1-1+1 = 1$

$\cos\theta = \dfrac{1}{3} \quad \therefore \quad \theta = \arccos\dfrac{1}{3}$

Exercício 4: Calcule a distância entre os pontos $A(6,5,2)$ e $B(7,3,4)$.

$\vec{AB} = (1,-2,2)$

$d(A,B) = \sqrt{1+4+4} = 3$

Exercício 5: Calcule a distância do ponto $P(2,0,7)$ à reta r: $\frac{x}{2} = \frac{y-2}{2} = \frac{z+3}{1}$.

$P(2,0,7) \qquad r: \dfrac{x}{2} = \dfrac{y-2}{2} = \dfrac{z+3}{1}$

$\vec{r} = (2,2,1) \quad |\vec{r}| = \sqrt{9} = 3$

$Q(0,2,-3) \in r. \quad \vec{u} = \vec{QP} = (2,-2,10)$

$$\vec{u} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 10 \\ 2 & 2 & 1 \end{vmatrix} = -22\vec{i} + 18\vec{j} + 8\vec{k} = (-22, 18, 8)$$

$d(P,r) = \dfrac{\sqrt{484+324+64}}{3} = \dfrac{\sqrt{872}}{3}$

Exercício 6: Calcule a distância do ponto ao plano nos seguintes casos:

a) $P(-4,2,5)$ e $\pi: 2x+y+2z+8=0$

$P(-4,2,5) \quad \pi: 2x+y+2z+8=0$

$\vec{n}=(2,1,2) \quad |\vec{n}|=\sqrt{9}=3$

$$d(P,\pi)=\frac{|2(-4)+1\times 2+2\times 5+8|}{3}=\frac{12}{3}=4$$

b) $P(1,2,-1)$ e $\pi: 3x-4y-5z+1=0$

$P(1,2,-1) \quad \pi: 3x-4y-5z+1 \quad \vec{n}=(3,-4,-5), |\vec{n}|=5\sqrt{2}$

$$d(P,\pi)=\frac{|3\times 1-4\times 2-5(-1)+1|}{5\sqrt{2}}=\frac{1}{5\sqrt{2}}=\frac{\sqrt{2}}{10}.$$

Exercício 7: Calcule a distância entre as retas nos seguintes casos:

a) $r: \begin{cases} y=1 \\ x+2=\frac{z-4}{-2} \end{cases}$ e $s: \begin{cases} x=3 \\ y=2t-1 \quad (t\in\mathbb{R}) \\ z=-t+3 \end{cases}$

$r: \begin{cases} y=1 \\ x+2=\frac{z-4}{-2} \end{cases} \quad s: \begin{cases} x=3 \\ y=2t-1 \\ z=-t+3 \end{cases}$

$\vec{r}=(1,0,-2) \quad \vec{s}=(0,2,-1)$

$P(-2,1,4)\in r \quad Q(3,-1,3)\in s.$

$\overrightarrow{PQ}=(5,-2,-1)$

$$[\vec{r},\vec{s},\overrightarrow{PQ}]=\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & -1 \\ 5 & -2 & -1 \end{vmatrix}=-2+20-2=16$$

r e s são reversas.

$$\vec{r}\times\vec{s}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 2 & -1 \end{vmatrix}=4\vec{i}+\vec{j}+2\vec{k}=(4,1,2)$$

$|\vec{r}\times\vec{s}|=\sqrt{16+1+4}=\sqrt{21}$

$$d(r,s)=\frac{16}{\sqrt{21}} \text{ u.c.}$$

b) $r: \overrightarrow{OX} = (-1,2,0) + t(1,3,1)\ (t \in \mathbb{R})$ e $s: \begin{cases} 3x - 2z - 3 = 0 \\ y - z - 2 = 0 \end{cases}$

$r: X = (-1,2,0) + \lambda(1,3,1)$ $\quad s: \begin{cases} 3x - 2z - 3 = 0 & \vec{n}_1 = (3,0,-2) \\ y - z - 2 = 0 & \vec{n}_2 = (0,1,-1) \end{cases}$

$P(-1,2,0) \in r \quad \vec{r} = (1,3,1)$

$$\vec{s} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 2\vec{i} + 3\vec{j} + 3\vec{k} = (2,3,3)$$

$Q \in s$: $z = 0$ e $y = 2 \Rightarrow 3x = 3 \Rightarrow x = 1 \quad Q(1,2,0) \in s$

$\overrightarrow{QP} = (-2,0,0)$

$$[\vec{r}, \vec{s}, \overrightarrow{QP}] = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ -2 & 0 & 0 \end{vmatrix} = -18 + 6 = -12$$

$$\vec{r} \times \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 6\vec{i} - \vec{j} - 3\vec{k} = (6,-1,-3).$$

$|\vec{r} \times \vec{s}| = \sqrt{36+1+9} = \sqrt{46} \quad d(r,s) = \dfrac{12}{\sqrt{46}}$

c) $r: \begin{cases} y = -2x + 3 \\ z = 2x \end{cases}$ e $s: \begin{cases} x = -1 - 2t \\ y = 1 + 4t \\ z = -3 - 4t \end{cases} \quad (t \in \mathbb{R})$

$r: \begin{cases} y = -2x + 3 \\ z = 2x \end{cases} \quad s: \begin{cases} x = -1 - 2t \\ y = 1 + 4t \\ z = -3 - 4t \end{cases}$

$x = \lambda$

$\vec{r} = (1,-2,2) \quad \vec{s} = (-2,4,-4)$

$Q(0,3,0) \in r \quad P(-1,1,-3) \in s$

$\overrightarrow{PQ} = (1,2,3)$

Como $\vec{s} = -2\vec{r}$ então $r // s$.

$$d(r,s) = d(Q,s) = \frac{|\overrightarrow{PQ} \times \vec{s}|}{|\vec{s}|}$$

$$\overrightarrow{PQ} \times \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -2 & 4 & -4 \end{vmatrix} = -20\vec{i} - 2\vec{j} + 8\vec{k} = (-20,-2,8)$$

$|\overrightarrow{PQ} \times \vec{s}| = \sqrt{400+4+64} = \sqrt{468} = 6\sqrt{13}$

$|\vec{s}| = \sqrt{4+16+16} = \sqrt{36} = 6.$

$$d(r,s) = \frac{6\sqrt{13}}{6} = \sqrt{13}.$$

Exercício 8: Calcule a distância entre os planos $\pi_1: 3x - 2y + z - 2 = 0$ e $\pi_2: 3x - 2y + z = 0$.

$P \in \pi \Rightarrow x = y = 0 \Rightarrow z = 2 \quad \therefore P(0,0,2)$

$$d(P, \pi_2) = \frac{|3 \times 0 - 2 \times 0 + 2|}{\sqrt{9+4+1}} = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$

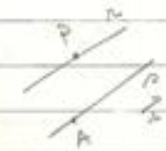
$$\therefore d(\pi_1, \pi_2) = \frac{\sqrt{14}}{7}$$

Exercício 9: Calcule a distância entre as retas paralelas: $r: \frac{x-2}{3} = \frac{y-1}{2} = z$ e $s: \frac{x-3}{3} = \frac{y-1}{2} = z + 1$.

$\therefore \; r: \frac{x-2}{3} = \frac{y-1}{2} = z \qquad s: \frac{x-3}{3} = \frac{y-1}{2} = z+1$

$P = (2, 1, 0) \in r \qquad A = (3, 1, -1) \in s$

$\vec{s} = (3, 2, 1)$



$\vec{u} = \vec{PA} = (1, 0, -1)$

$$\vec{u} \wedge \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 3 & 2 & 1 \end{vmatrix} = +2\vec{i} - 4\vec{j} + 2\vec{k} = (2, -4, 2)$$

$$d(P, s) = \frac{\sqrt{4+16+4}}{\sqrt{9+4+1}} = \frac{\sqrt{24}}{\sqrt{14}} = \frac{\sqrt{12}}{\sqrt{7}} = \frac{2\sqrt{3}}{\sqrt{7}} = \frac{2\sqrt{21}}{7}$$

$$\therefore d(r, s) = \frac{\sqrt{84}}{7} \text{ u.c.}$$

Exercício 10: Determine a distância entre as retas reversas: $r: \frac{x-1}{2} = \frac{y-2}{-1} = z$ e $s: \begin{cases} \frac{x-2}{5} = z \\ y = z - 1 \end{cases}$.

$r: \frac{x-1}{2} = \frac{y-2}{-1} = z$

$P = (1, 2, 0) \in r$

$\vec{r} = (2, -1, 1)$

$s: \begin{cases} \frac{x-2}{5} = z \\ y = z - 1 \end{cases}$ $\quad z = \lambda \Rightarrow \begin{cases} x = 5\lambda + 2 \\ y = \lambda - 1 \\ z = \lambda \end{cases}$

$Q = (2, -1, 0) \in s$

$\vec{s} = (5, 1, 1)$

$\vec{u} = P - Q = (-1, 3, 0)$

$$[\vec{r}, \vec{s}, \vec{u}] = \begin{vmatrix} 2 & -1 & 1 \\ 5 & 1 & 1 \\ -1 & 3 & 0 \end{vmatrix} = 1 + 15 + 1 - 6 = 11$$

$$\vec{r} \wedge \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix} = -2\vec{i} + 3\vec{j} + 7\vec{k} = (-2, 3, 7)$$

$$\| \vec{r} \wedge \vec{s} \| = \sqrt{4+9+49} = \sqrt{62}$$

$$\therefore\ d(r, s) = \frac{11}{\sqrt{62}} \text{ u. c.}$$