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Spectra of some self-exciting and mutually exciting point processes

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SUMMARY

In recent years methods of data analysis for point processes have received some attention, for example, by Cox & Lewis (1966) and Lewis (1964). In particular Bartlett (1963*a*, *b*) has introduced methods of analysis based on the point spectrum. Theoretical models are relatively sparse. In this paper the theoretical properties of a class of processes with particular reference to the point spectrum or corresponding covariance density functions are discussed. A particular result is a self-exciting process with the same second-order properties as a certain doubly stochastic process. These are not distinguishable by methods of data analysis based on these properties.

1. INTRODUCTION

Bartlett (1963*a*, *b*) considers a stationary point process $N(t)$, representing the cumulative number of events up to time t, for which

$$\lambda = E\{dN(t)\}/dt$$

is constant and the covariance density

$$\mu(\tau) = E\{dN(t+\tau)\,dN(t)\}/(dt)^2 - \lambda^2 \tag{1}$$

does not depend on t. For $\tau < 0$, $\mu(-\tau) = \mu(\tau)$, but, for $\tau = 0$, $E[\{dN(t)\}^2] = E\{dN(t)\}$ if events cannot occur multiply, so that the complete covariance density becomes

$$\mu^{(c)}(\tau) = \lambda\delta(\tau) + \mu(\tau), \tag{2}$$

where $\delta(\tau)$ is the Dirac delta function and $\mu(\tau)$ is continuous at the origin. The complete spectral density function for $N(t)$ is defined by

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau\omega}\mu^{(c)}(\tau)\,d\tau = \frac{1}{2\pi}\left\{\lambda + \int_{-\infty}^{\infty} e^{-i\tau\omega}\mu(\tau)\,d\tau\right\}. \tag{3}$$

In particular for the Poisson process $\mu(\tau) = 0$ and $f(\omega) = \lambda/(2\pi)$.

This may easily be extended to the simultaneous study of a number of point processes $N_r(t)$ $(r = 1, \ldots, k)$ with covariance densities

$$\mu_{rs}(\tau) = E\{dN_r(t+\tau)\,dN_s(t)\}/(dt)^2 - \lambda_r\lambda_s, \tag{4}$$

where $\lambda_r = E\{dN_r(t)\}/dt$. Now $\mu_{rs}(-\tau) = \mu_{sr}(\tau)$ and delta functions occur at the origin when $r = s$. Then the complete covariance density matrix is

$$\boldsymbol{\mu}^{(c)}(\tau) = \delta(\tau)\,\mathrm{diag}\,(\boldsymbol{\Lambda}) + \boldsymbol{\mu}(\tau), \tag{5}$$

and the spectral density matrix

$$F(\omega) = \frac{1}{2\pi}\left\{\text{diag}\,(\boldsymbol{\Lambda}) + \int_{-\infty}^{\infty} e^{-i\tau\omega}\boldsymbol{\mu}(\tau)\,d\tau\right\} \tag{6}$$

is Hermitian.

2. A SELF-EXCITING POINT PROCESS

We consider a point process $N(t)$ such that

$$\begin{aligned} &\text{pr}\{\Delta N(t) = 1 | N(s)\,(s \leqslant t)\} = \Lambda(t)\,\Delta t + o(\Delta t), \\ &\text{pr}\{\Delta N(t) > 1 | N(s)\,(s \leqslant t)\} = o(\Delta t). \end{aligned} \tag{7}$$

In the doubly stochastic process (Bartlett, 1963*b*), $\Lambda(t)$ is assumed to be a stationary random process and the conditioning event is written as $\{N(s)\,(s \leqslant t): \Lambda(s)\,(-\infty < s < \infty)\}$, so that we could think of $\Lambda(t)$ being determined for all t before the $N(t)$ process is considered. Here in contrast we shall assume that the process is self-exciting in the sense that $\Lambda(t)$ may be written

$$\Lambda(t) = \nu + \int_{-\infty}^{t} g(t-u)\,dN(u), \tag{8}$$

which together with (7) defines the process. In effect one may think of this as a self-exciting shot process in which the current intensity of events is determined by events in the past. This will be a largely local effect if $g(v)$ decays rapidly but may contain longer term effects if $g(v)$ has a hump, remote from the origin. In principle $\Lambda(t)$ should always remain positive but models for which the probability of negative values is small may be useful approximations. We, therefore, assume $g(v) \geqslant 0$ and $g(v) = 0$ $(v < 0)$.

If we assume stationarity, then from (8) we have

$$\lambda = E\{\Lambda(t)\} = \nu + \lambda \int_{-\infty}^{t} g(t-u)\,du, \tag{9}$$

or

$$\lambda = \nu \Big/ \left\{1 - \int_0^{\infty} g(v)\,dv\right\}.$$

Thus we must have $\nu > 0$ and

$$\int_0^{\infty} g(v)\,dv < 1.$$

To obtain an equation for the covariance density we observe that, for $\tau > 0$,

$$\begin{aligned} \mu(\tau) &= E\{dN(t+\tau)\,dN(t)\}/(dt)^2 - \lambda^2 \\ &= E\left[\frac{dN(t)}{dt}\left\{\nu + \int_{-\infty}^{t+\tau} g(t+\tau-u)\,dN(u)\right\}\right] - \boldsymbol{\lambda}^2 \\ &= \int_{-\infty}^{\tau} g(\tau-v)\,\mu^{(c)}(v)\,dv. \end{aligned}$$

Applying (2), we find, for $\tau > 0$,

$$\mu(\tau) = \lambda g(\tau) + \int_{-\infty}^{\tau} g(\tau-v)\,\mu(v)\,dv. \tag{10}$$

This integral equation in general is difficult to solve analytically. Since $\mu(\tau)$ is symmetric the equation may be written as

$$\mu(\tau) = \lambda g(\tau) + \int_0^\infty g(\tau+v)\mu(v)dv + \int_0^\tau g(\tau-v)\mu(v)dv \quad (\tau > 0). \tag{11}$$

Standard techniques exist for the numerical solution of integral equations (Mayers, 1962). In (11) the single integral of (10) has been decomposed into two separate integrals. This is convenient as different correction terms are appropriate for the numerical integration of these two types. A finite difference method which decomposes the interval $(0, \infty)$ into disjoint intervals with more pivot points near the origin would be appropriate.

An analytic solution may be obtained when $g(v)$ decays exponentially.

3. The case of exponential decay

If we consider the special case

$$g(v) = \sum_{j=1}^{k} \alpha_j e^{-\beta_j v} \quad (v > 0), \tag{12}$$

with

$$\sum_{j=1}^{k} \alpha_j/\beta_j < 1,$$

then, taking the Laplace transform of (11), we find

$$\mu^*(s) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j + s}\{\mu^*(s) + \lambda + \mu^*(\beta_j)\}$$

and so

$$\mu^*(s) = \frac{\sum_{j=1}^{k} \alpha_j\{\lambda + \mu^*(\beta_j)\}/(\beta_j + s)}{1 - \sum_{j=1}^{k} \alpha_j/(\beta_j + s)}. \tag{13}$$

If we put $s = \beta_r (r = 1, \ldots, k)$ in this equation, we obtain k linear equations in the k unknowns $\mu^*(\beta_r)$ which can easily be solved. Equation (13) will then give a well determined function.

The spectrum of this process is then easily obtained since, from (3), we see

$$f(\omega) = \frac{1}{2\pi}\{\lambda + \mu^*(i\omega) + \mu^*(-i\omega)\}. \tag{14}$$

If we multiply the numerator and denominator in (13) by $\Pi(\beta_j + s)$ we see that

$$\mu^*(s) = p_1(s)/p_2(s),$$

where $p_1(s)$ and $p_2(s)$ are polynomials of degree $(k-1)$ and k, respectively. The general form of $\mu(\tau)$ on inversion will thus be

$$\mu(\tau) = \sum_{j=1}^{k} \gamma_j e^{-\eta_j \tau} \quad (\tau > 0),$$

where $-\eta_j$ are the roots of $p_2(s)$.

In the case $k = 1$ when $g(v) = \alpha e^{-\beta v} (v > 0)$, we find that $\mu^*(\beta) = \frac{1}{2}\alpha\lambda/(\beta - \alpha)$. Therefore,

$$\mu^*(s) = \frac{\alpha\lambda(2\beta - \alpha)}{2(\beta - \alpha)(s + \beta - \alpha)}. \tag{15}$$

Then

$$\mu(\tau) = \frac{\alpha\lambda(2\beta-\alpha)}{2(\beta-\alpha)} e^{-(\beta-\alpha)\tau} \quad (\tau > 0), \tag{16}$$

$$f(\omega) = \frac{\lambda}{2\pi}\left\{1 + \frac{\alpha(2\beta-\alpha)}{(\beta-\alpha)^2+\omega^2}\right\}, \tag{17}$$

where, from (9), $\lambda = \nu\beta/(\beta-\alpha)$.

Now it is known (Bartlett, 1963*b*) that for a doubly stochastic process the covariance density $\mu(\tau)$ is identical with the covariance function of $\Lambda(t)$. Thus the second-order properties of this self-exciting process as exhibited in (16) and (17) could equally well be obtained from a doubly stochastic process where $\Lambda(t)$ has covariance function (16). Such a process can be generated in the form

$$\Lambda(t) = \gamma + \int_{-\infty}^{t} e^{-(\beta-\alpha)(t-u)} dz(u), \tag{18}$$

where $z(u)$ is a process of orthogonal increments with

$$E\{dz(u)\} = m\,du,$$

$$\text{var}\{dz(u)\} = \alpha\lambda(2\beta-\alpha)\,du,$$

where γ and m are any constants satisfying $\lambda = \gamma + m/(\beta-\alpha)$. It is to be expected that the result (13) could also be obtained from a doubly stochastic process where $\Lambda(t)$ has the form

$$\Lambda(t) = \gamma + \sum_{j=1}^{k} \int_{-\infty}^{t} e^{-\psi_j(t-u)} dz_j(u)$$

for a suitable choice of parameters, where $z_j(u)$ are mutually orthogonal processes of orthogonal increments.

Thus it is clear that given a set of data from a point process we cannot expect to discriminate between a self-exciting and a doubly stochastic process on the basis of the estimated point spectrum, covariance density or other equivalent functions such as the variance-time function (Cox & Lewis, 1966, p. 72). Bartlett (1964) found a clustering process with the same second-order properties as a two-dimensional doubly stochastic process; indeed it was completely statistically identical. Vere-Jones (1970) gives a similar one-dimensional example.

4. SOME MUTUALLY EXCITING PROCESSES

Consider k point processes $N_r(t)\,(r = 1, \ldots, k)$ forming a vector process $\mathbf{N}(t)$ such that

$$\begin{aligned} &\text{pr}\{\Delta N_r(t) = 1 | \mathbf{N}(s)\,(s \leqslant t)\} = \Lambda_r(t)\,\Delta t + o(\Delta t), \\ &\text{pr}\{\Delta N_r(t) > 1 | \mathbf{N}(s)\,(s \leqslant t)\} = o(\Delta t) \end{aligned} \tag{19}$$

independently for each r, where

$$\Lambda_r(t) = \nu_r + \sum_{s=1}^{k} \int_{-\infty}^{t} g_{rs}(t-u)\,dN_s(u)$$

or

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\nu} + \int_{-\infty}^{t} \mathbf{G}(t-u)\,d\mathbf{N}(u). \tag{20}$$

The vector of stationary densities is thus

$$\mathbf{\Lambda} = (\mathbf{I} - \mathbf{\Gamma})^{-1}\boldsymbol{\nu}, \tag{21}$$

where

$$\mathbf{\Gamma} = \int_0^\infty \mathbf{G}(v)\,dv,$$

provided $\mathbf{\Lambda} > 0$. For $\tau > 0$

$$\begin{aligned}\boldsymbol{\mu}(\tau) &= E\{d\mathbf{N}(t+\tau)\,d\mathbf{N}'(t)\}/(dt^2) - \mathbf{\Lambda}\mathbf{\Lambda}' \\ &= E\left[\left\{\boldsymbol{\nu} + \int_{-\infty}^{t+\tau} \mathbf{G}(t+\tau-u)\,d\mathbf{N}(u)\right\}\frac{d\mathbf{N}'(t)}{dt}\right] - \mathbf{\Lambda}\mathbf{\Lambda}' \\ &= \int_{-\infty}^{\tau} \mathbf{G}(\tau - v)\,\boldsymbol{\mu}^{(c)}(v)\,dv.\end{aligned}$$

Thus, by (5), we have, for $\tau > 0$,

$$\boldsymbol{\mu}(\tau) = \mathbf{G}(\tau)\,\mathrm{diag}\,(\mathbf{\Lambda}) + \int_{-\infty}^{\tau} \mathbf{G}(\tau - v)\,\boldsymbol{\mu}(v)\,dv. \tag{22}$$

In principle this equation may be solved numerically in the same way as the one-dimensional case, but of course the labour involved increases rapidly and accuracy suffers, except for special cases; see § 5.

In the exponential case

$$\mathbf{G}(v) = \sum_{m=1}^{n} \boldsymbol{\alpha}^{(m)} e^{-\beta^{(m)} v},$$

we may take the element by element Laplace transform of (22), and using $\boldsymbol{\mu}(-\tau) = \boldsymbol{\mu}'(\tau)$, we find

$$\boldsymbol{\mu}^*(s) = \{\mathbf{I} - \mathbf{G}^*(s)\}^{-1}\left[\mathbf{G}^*(s)\,\mathrm{diag}\,(\mathbf{\Lambda}) + \sum_{m=1}^{n} \boldsymbol{\alpha}^{(m)}\boldsymbol{\mu}^{*\prime}\{\beta^{(m)}\}/\{\beta^{(m)} + s\}\right]. \tag{23}$$

By inserting $s = \beta^{(m)}$ $(m = 1, \ldots, n)$ in (23), we obtain a set of linear equations for the unknown constants $\mu_{ij}^*\{\beta^{(m)}\}$, the solution of which may be substituted into (23) to give the complete solution for $\boldsymbol{\mu}^*(s)$. The spectral matrix is then given by

$$\mathbf{F}(\omega) = \frac{1}{2\pi}\{\mathrm{diag}\,(\mathbf{\Lambda}) + \boldsymbol{\mu}^*(i\omega) + \boldsymbol{\mu}^*(-i\omega)\}. \tag{24}$$

From (23) we may infer the general form

$$\boldsymbol{\mu}(\tau) = \sum_r \mathbf{\Gamma}_r(\tau)\,e^{-\eta_r \tau} \quad (\tau > 0),$$

where the elements of $\mathbf{\Gamma}_r(\tau)$ are polynomials in τ, usually constant.

Some results may be obtained explicitly in simple cases. For example, in the bivariate case, $k = 2$, with simple exponential decays we will assume that for $i = 1, 2$ and $j = 1, 2$

$$g_{ij}(v) = \alpha_{ij} e^{-\beta_{ij} v} \quad (v > 0).$$

Hence

$$g_{ij}^*(s) = \frac{\alpha_{ij}}{\beta_{ij} + s}.$$

Then (23) yields

$$\boldsymbol{\mu}^*(s) = \mathbf{H}(s)/[\{1 - g_{11}^*(s)\}\{1 - g_{22}^*(s)\} - g_{12}^*(s)\,g_{21}^*(s)], \tag{25}$$

where

$$h_{11}(s) = \{1-g_{22}^*(s)\}\{g_{11}^*(s)\lambda_1 + g_{11}^*(s)\mu_{11}^*(\beta_{11}) + g_{12}^*(s)\mu_{12}^*(\beta_{12})\} + g_{12}^*(s)\{g_{21}^*(s)\lambda_1 + g_{21}^*(s)\mu_{11}^*(\beta_{21}) + g_{22}^*(s)\mu_{12}^*(\beta_{22})\},$$

$$h_{12}(s) = \{1-g_{22}^*(s)\}\{g_{12}^*\lambda_2 + g_{11}^*(s)\mu_{21}^*(\beta_{11}) + g_{12}^*(s)\mu_{22}^*(\beta_{12})\} + g_{12}^*(s)\{g_{22}^*(s)\lambda_2 + g_{21}^*(s)\mu_{21}^*(\beta_{21}) + g_{22}^*(s)\mu_{22}^*(\beta_{22})\}. \quad (26)$$

The values of $h_{22}(s)$ and $h_{21}(s)$ are obtained from these by interchanging suffices.

Case 1: *Non-interacting.* If $\alpha_{12} = \alpha_{21} = 0$, the cross-covariances are zero and each process is self-exciting. Then

$$\mu_{ii}^*(s) = g_{ii}^*(s)\{\lambda_i + \mu_{ii}^*(\beta_{ii})\}/\{1-g_{ii}^*(s)\} = \frac{\alpha_{ii}\lambda_i(2\beta_{ii}-\alpha_{ii})}{2(\beta_{ii}-\alpha_{ii})(\beta_{ii}-\alpha_{ii}+s)}$$

as in (15).

Case 2: *One-way interaction.* If $\alpha_{21} = 0$, the behaviour of $N_1(t)$ does not affect the future of the $N_2(t)$ process which will be a simple self-exciting process, and

$$\Lambda_1(t) = \nu_1 + \int_{-\infty}^{t} \alpha_{11} e^{-\beta_{11}(t-u)}\, dN_1(u) + \int_{-\infty}^{t} \alpha_{12} e^{-\beta_{12}(t-u)}\, dN_2(u).$$

The result is particularly simple if $\alpha_{22} = 0$ also, so that $N_2(t)$ is just a Poisson process. Then substituting in (25) and (26) we find

$$\mu_{21}^*(s) = \mu_{22}^*(s) = 0,$$

$$\mu_{11}^*(s) = \frac{1}{\beta_{11}-\alpha_{11}+s}\left[\alpha_{11}\{\lambda_1 + \mu_{11}^*(\beta_{11})\} + \frac{(\beta_{11}+s)}{(\beta_{12}+s)}\mu_{12}^*(\beta_{12})\right], \quad (27)$$

$$\mu_{12}^*(s) = \frac{\alpha_{12}(\beta_{11}+s)\lambda_2}{(\beta_{11}-\alpha_{11}+s)(\beta_{12}+s)}.$$

Solving for $\mu_{11}^*(\beta_{11})$, we find

$$\mu_{11}^*(\beta_{11}) = \frac{\alpha_{11}\lambda_1 + \lambda_2\beta_{11}\alpha_{12}^2\{\beta_{12}(\beta_{11}+\beta_{12}-\alpha_{11})\}^{-1}}{2(\beta_{11}-\alpha_{11})}.$$

Then inversion of (27) yields, for $\tau > 0$,

$$\mu_{21}(\tau) = \mu_{22} = 0,$$

$$\mu_{12}(\tau) = \frac{\lambda_2\alpha_{12}}{\beta_{11}-\beta_{12}-\alpha_{11}}\{(\beta_{11}-\beta_{12})e^{-\beta_{12}\tau} - \alpha_{11}e^{-(\beta_{11}-\alpha_{11})\tau}\}, \quad (28)$$

$$\mu_{11}(\tau) = \left[\alpha_{11}\{\lambda_1 + \mu_{11}^*(\beta_{11})\} - \frac{\alpha_{11}\alpha_{12}\mu_{12}^*(\beta_{12})}{\beta_{11}-\beta_{12}-\alpha_{11}}\right]e^{-(\beta_{11}-\alpha_{11})\tau} + \frac{\alpha_{12}\mu_{12}^*(\beta_{12})(\beta_{11}-\beta_{12})}{\beta_{11}-\beta_{12}-\alpha_{11}}e^{-\beta_{12}\tau},$$

which has the expected general form.

We observe that

(i) if $\alpha_{12} = 0$, (28) reduces to (16);

(ii) if $\alpha_{11} = 0$, then $N_1(t)$ is 'simply excited' by the $N_2(t)$ processes. Then equation (28) gives

$$\mu_{12}(\tau) = \lambda_2\alpha_{12}e^{-\beta_{12}\tau},$$

$$\mu_{11}(\tau) = \tfrac{1}{2}\lambda_2\alpha_{12}^2 e^{-\beta_{12}\tau}/\beta_{12}.$$

This agrees with the result for the doubly stochastic process which in fact it is; see (16). As mentioned in § 3, $\mu_{11}(\tau)$ is also consistent with a self-excited process. In this case the self-excited and simply-excited processes may be discriminated by looking at the cross-covariance $\mu_{12}(\tau)$.

5. Process with known exciter

If the interaction between two processes is one-way in the sense that

$$\mathrm{pr}\,[A\,|\{N_1(s), N_2(s)\}\,(-\infty < s \leqslant t)] = \mathrm{pr}\,\{A\,|N_2(s)\,(-\infty < s \leqslant t)\}$$

for any set A in the Borel field of the process $\{N_2(s), t \leqslant s\}$, i.e. the future behaviour of $N_2(s)$ is independent of the past of $N_1(s)$ when the past of $N_2(s)$ is known, then it is possible to allow $N_2(t)$ to behave in a more general manner.

Suppose $N_2(t)$ is a point process with known spectrum $f_{22}(\omega)$ and that

$$\mathrm{pr}\,\{\Delta N_1(t) = 1\,|N_1(s)\,(-\infty < s \leqslant t), \quad N_2(s)\,(-\infty < s < \infty)\} = \Lambda_1(t)\,\Delta t + o(\Delta t), \tag{29}$$

where

$$\Lambda_1(t) = \nu_1 + \int_{-\infty}^{t} g_{11}(t-u)\,dN_1(u) + \int_{-\infty}^{t} g_{12}(t-u)\,dN_2(u).$$

Notice that the essential difference between this and the previous theory lies in the conditioning event (29).

As before we have, for $\tau > 0$,

$$\mu_{11}(\tau) = \lambda_1 g_{11}(\tau) + \int_{-\infty}^{\tau} g_{11}(\tau-v)\,\mu_{11}(v)\,dv + \int_{-\infty}^{\tau} g_{12}(\tau-v)\,\mu_{21}(v)\,dv. \tag{30}$$

However, by virtue of the conditioning event in (29), the similar equation

$$\mu_{12}(\tau) = \int_{-\infty}^{\tau} g_{11}(\tau-u)\,\mu_{12}(u)\,du + \int_{-\infty}^{\tau} g_{12}(\tau-u)\,\mu_{22}^{(c)}(u)\,du \tag{31}$$

holds for all values of τ.

Hence on taking transforms we find the cross-spectrum

$$f_{12}(\omega) = \frac{g_{12}^*(i\omega) f_{22}(\omega)}{1 - g_{11}^*(i\omega)}. \tag{32}$$

The value of $\mu_{12}(\tau)$ may be obtained by inversion, probably numerically, and we note $\mu_{21}(\tau) = \mu_{12}(-\tau)$ and $f_{21}(\omega) = f_{12}(-\omega)$. The third term in (30) is thus a known function of τ and (30) becomes an integral equation which may be solved by the same methods as (10).

The case $g_{11}(u) = 0$ is a doubly stochastic process. Therefore (30) holds for all τ and one may take the transform to obtain

$$\begin{aligned} f_{11}(\omega) &= \tfrac{1}{2}\lambda_1/\pi + g_{12}^*(i\omega) f_{21}(\omega) \\ &= \tfrac{1}{2}\lambda_1/\pi + |g_{12}^*(i\omega)|^2 f_{22}(\omega) \end{aligned}$$

which is the usual result for the doubly stochastic process.

The above results can be applied to the k-variate process of § 4 when the transition matrix $\mathbf{G}(u)$ can be partitioned as

$$\mathbf{G}(u) = \begin{bmatrix} \mathbf{G}_{11}(u) & \mathbf{G}_{12}(u) \\ \mathbf{0} & \mathbf{G}_{22}(u) \end{bmatrix},$$

where the vector process $\mathbf{N}_2(t)$ does not depend on the past of the vector process $\mathbf{N}_1(t)$, given the past of $\mathbf{N}_2(t)$. Equations (30) and (31) are to be interpreted as matrix equations and (32) becomes

$$\mathbf{F}_{12}(\omega) = \{\mathbf{I} - \mathbf{G}_{11}^*(i\omega)\}^{-1}\mathbf{G}_{12}^*(i\omega)\mathbf{F}_{22}(\omega).$$

This is particularly useful when $\mathbf{G}(u)$ is triangular, so $N_i(t)$ only depends on $N_j(t)$ $(j = i, i+1, \ldots, k)$. In this case the solution for $N_{k-1}(t)$ can be found from the properties of $N_k(t)$. The solution for $N_{k-2}(t)$ from the properties of $N_{k-1}(t)$ and $N_k(t)$ and so on recursively solving each one by applying the methods of this section.

6. APPLICATIONS

The object of this study is to produce a class of theoretical models which may be applicable to a variety of problems. We have concentrated on the second-order properties since these are most amenable to data analysis for testing goodness of fit. Some possible applications are suggested.

The self-exciting process is a possible epidemic model in large populations in so far as the occurrence of a number of cases increases the probability of further cases. The mutually exciting processes could provide models for epidemics in which different types of cases are considered (children, adults, animals) and for associated diseases such as shingles and chicken pox. Other applications might be in complex equipment: for example, the computer (Lewis, 1964) and the human body, as a model of neuron firing (Coleman & Gastwirth, 1969). Another application might be to the emission of particles from a radiating body which is excited by impact of other particles.

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