Fractal Geometry
Winter Semester 2018/2019

PD Dr. Peter Massopust
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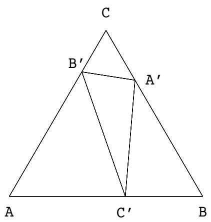
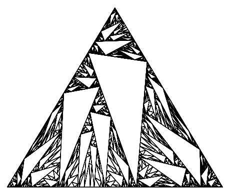
Exercise Set 2

1. Consider the figure below on the left where $\{A, B, C\} \cap \{A', B', C'\} = \emptyset$. Define ratios $r := \frac{\overline{AB'}}{\overline{AC}}$, $s := \frac{\overline{BC'}}{\overline{BA}}$, and $t := \frac{\overline{CA'}}{\overline{CB}}$. Define an IFS $(\triangle(A, B, C); \{f_1, f_2, f_3\})$ by requiring





that $(A, B, C) \overset{f_1}{\longmapsto} (C', A, B')$, $(A, B, C) \overset{f_2}{\longmapsto} (C', A', B)$, and $(A, B, C) \overset{f_3}{\longmapsto} (C, A', B')$. Choosing $A := (0, 0)$, $B := (1, 0)$, and $C := (0.5, 1)$, derive formulae for the maps f_1, f_2, and f_3. The attractor $\mathfrak{S}(r, s, t)$ is a three-parameter family of Sierpiński triangles. A representative of this family is depicted on the right.

2. Let (X, d) be a complete metric space and let $f : [0, 1] \times \mathsf{X} \to \mathsf{X}$ be a parameterized contraction on X, i.e., $f(t, \cdot)$, $t \in [0, 1]$, is a contraction on X. Assume that for fixed $x \in \mathsf{X}$, f is continuous on $[0, 1]$. Prove that the fixed point x_f^* of f depends continuously on t, that is, $x_f^* : [0, 1] \to \mathsf{X}$ is continuous.

3. Let (X, d) be a complete metric space, let $1 < N \in \mathbb{N}$, and let $f_i : [0, 1] \times \mathsf{X} \to \mathsf{X}$, $i = 1, \ldots N$, be a family of contractions on X depending continuously on a parameter $t \in [0, 1]$. Show that the mapping $F : [0, 1] \times \mathcal{H}(\mathsf{X}) \to \mathcal{H}(\mathsf{X})$ defined by

$$F(t, A) := \bigcup_{i=1}^{N} f_i(t, A)$$

is continuous in t (in the metric space $(\mathcal{H}(\mathsf{X}), d_{\mathcal{H}})$).

4. Combining the previous two problems, prove the following: Let (X, d) be a complete metric space and let $(\mathsf{X}, \{f_1, \ldots, f_N\})$ be an IFS whose mappings $f_i : [0, 1] \times \mathsf{X} \to \mathsf{X}$, $i = 1, \ldots N$ depend continuously on the parameter $t \in [0, 1]$. Then the attractor $\mathfrak{A}(t)$ of this IFS also depends continuously on $t \in [0, 1]$, with respect to the Hausdorff metric $d_{\mathcal{H}}$.

5. Consider the IFS $(\mathbb{C}, \{f_1 = \lambda(\cdot) - 1, f_2 = \lambda(\cdot) + 1\})$, $\lambda \in \mathbb{C}$. Show that for $\lambda = \frac{1}{2}(1 + i)$, the fractal set generated by this IFS is the famous *twin dragon*. What happens with the attractor when $\frac{1}{2}$ is replaced by α with $0 < \alpha < \frac{1}{2}$?