

## Exercise Set 1

1. Let  $M \subset \mathbb{R}^n$  and let  $\lambda > 0$ . Show that

$$\mathcal{H}^s(\lambda M) = \lambda^s \mathcal{H}^s(M),$$

where  $\lambda M := \{\lambda x : x \in M\}$ .

2. Let  $M \subset \mathbb{R}^n$ . Show that  $\mathcal{H}^0(M)$  is the number of points in  $M$ .

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function. Show that

$$\dim_H f(M) \leq \dim_H M,$$

for any set  $M \subset \mathbb{R}$ .

4. (This problem requires some basic measure theory.) Let  $M \subset \mathbb{R}^n$  and let  $\mu$  be a finite measure with  $\mu(M) > 0$ . Suppose there exist numbers  $s \geq 0$ ,  $c > 0$ , and  $\epsilon_0 > 0$  such that

$$\mu(U) \leq c|U|^s,$$

for all sets  $U \subset \mathbb{R}$  with diameter  $|U| \leq \epsilon_0$ . Show that  $\mathcal{H}^s(M) \geq \mu(M)/c$ .

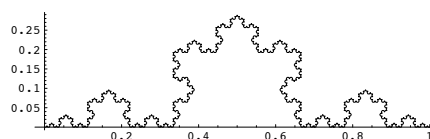
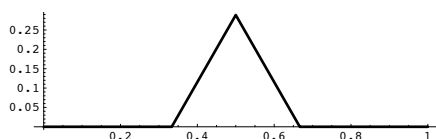
5. Show that the function  $d_{\mathcal{H}} : \mathcal{H}(X) \times \mathcal{H}(X) \rightarrow \mathbb{R}$  defined by

$$d_{\mathcal{H}}(A, B) := \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(b, a) \right\}$$

is a metric on  $\mathcal{H}(X)$ .

6. Show that the metric space  $(\mathcal{H}(X), d_{\mathcal{H}})$  is complete.

7. Find an IFS that generates the so-called *Koch curve*  $\mathfrak{K}$  shown on the right-hand side of the figure below:



(The vertices of the generator triangle are  $(\frac{1}{3}, 0)$ ,  $(\frac{1}{2}, \frac{\sqrt{3}}{6})$ , and  $(\frac{2}{3}, 0)$ ).

8. Consider the IFS  $(\mathbb{R}; \{\frac{1}{2}x, \frac{1}{4}x + \frac{1}{4}, \frac{1}{4}x + \frac{3}{4}\})$ . Describe the fractal set  $\mathfrak{F}$  generated by this IFS. How does  $\mathfrak{F}$  differ from the Cantor set?

9. Consider the IFS

$$\left( \mathbb{R}^2; \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\} \right).$$

Let  $A_0 := \{(\frac{1}{2}, y) : 0 \leq y \leq 1\}$  and let  $F$  be the set-valued map associated with this IFS.

- (a) Compute and plot  $A_n := F^n(A_0)$  for  $n = 1, 2, 3$ .

- (b) Show that the attractor  $A = \lim_{n \rightarrow \infty} F^n(A_0)$  is given by

$$A = \{(x, y) : y = x, 0 \leq x \leq 1\}.$$