Fractal Geometry
Winter Semester 2018/2019

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Exercise Set 1

1. Let $M \subset \mathbb{R}^n$ and let $\lambda > 0$. Show that
$$\mathcal{H}^s(\lambda M) = \lambda^s \mathcal{H}^s(M),$$
where $\lambda M := \{\lambda x : x \in M\}$.

2. Let $M \subset \mathbb{R}^n$. Show that $\mathcal{H}^0(M)$ is the number of points in M.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Show that
$$\dim_H f(M) \le \dim_H M,$$
for any set $M \subset \mathbb{R}$.

4. (This problem requires some basic measure theory.) Let $M \subset \mathbb{R}^n$ and let μ be a finite measure with $\mu(M) > 0$. Suppose there exist numbers $s \ge 0$, $c > 0$, and $\epsilon_0 > 0$ such that
$$\mu(U) \le c\,|U|^s,$$
for all sets $U \subset \mathbb{R}$ with diameter $|U| \le \epsilon_0$. Show that $\mathcal{H}^s(M) \ge \mu(M)/c$.

5. Show that the function $d_{\mathcal{H}} : \mathcal{H}(\mathsf{X}) \times \mathcal{H}(\mathsf{X}) \to \mathbb{R}$ defined by
$$d_{\mathcal{H}}(A,B) := \max\left\{\max_{a\in A}\min_{b\in B} d(a,b), \max_{b\in B}\min_{a\in A} d(b,a)\right\}$$
is a metric on $\mathcal{H}(\mathsf{X})$.

6. Show that the metric space $(\mathcal{H}(\mathsf{X}), d_{\mathcal{H}})$ is complete.

7. Find an IFS that generates the so-called *Koch curve* $\mathfrak{K}$ shown on the right-hand side of the figure below:

(*The vertices of the generator triangle are* $(\frac{1}{3}, 0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{6})$, *and* $(\frac{2}{3}, 0)$.

8. Consider the IFS $\left(\mathbb{R}; \left\{\frac{1}{2}x, \frac{1}{4}x + \frac{1}{4}, \frac{1}{4}x + \frac{3}{4}\right\}\right)$. Describe the fractal set $\mathfrak{F}$ generated by this IFS. How does $\mathfrak{F}$ differ from the Cantor set?

9. Consider the IFS
$$\left(\mathbb{R}^2; \left\{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right\}\right).$$
Let $A_0 := \{(\frac{1}{2}, y) \,:\, 0 \le y \le 1\}$ and let F be the set-valued map associated with this IFS.
   (a) Compute and plot $A_n := F^n(A_0)$ for $n = 1, 2, 3$.
   (b) Show that the attractor $A = \lim\limits_{n\to\infty} F^n(A_0)$ is given by
$$A = \{(x,y) \,:\, y = x,\, 0 \le x \le 1\}.$$