Hausdorff metric inherits completeness

Theorem 1. *If (X,d) is a complete metric space, then the Hausdorff metric induced by d is also complete.*

Proof. Suppose (A_n) is a Cauchy sequence with respect to the Hausdorff metric. By selecting a subsequence if necessary, we may assume that A_n and A_{n+1} are within 2^{-n} of each other, that is, that $A_n \subset K(A_{n+1}, 2^{-n})$ and $A_{n+1} \subset K(A_n, 2^{-n})$. Now for any natural number N, there is a sequence $(x_n)_{n \geq N}$ in X such that $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Any such sequence is Cauchy with respect to d and thus converges to some $x \in X$. By applying the triangle inequality, we see that for any $n \geq N$, $d(x_n, x) < 2^{-n+1}$.

Define A to be the set of all x such that x is the limit of a sequence $(x_n)_{n \geq 0}$ with $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Then A is nonempty. Furthermore, for any n, if $x \in A$, then there is some $x_n \in A_n$ such that $d(x_n, x) < 2^{-n+1}$, and so $A \subset K(A_n, 2^{-n+1})$. Consequently, the set $\overline{A}$ is nonempty, closed and bounded.

Suppose $\epsilon > 0$. Thus $\epsilon > 2^{-N} > 0$ for some N. Let $n \geq N+1$. Then by applying the claim in the first paragraph, we have that for any $x_n \in A_n$, there is some $x \in X$ with $d(x_n, x) < 2^{-n+1}$. Hence $A_n \subset K(\overline{A}, 2^{-n+1})$. Hence the sequence (A_n) converges to A in the Hausdorff metric. ■

This proof is based on a sketch given in an exercise in [1]. An exercise for the reader: is the set A constructed above closed?

References

1 J. Munkres, *Topology* (2nd edition), Prentice Hall, 1999.

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