

Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:

A Simple Imperative Language

- ① IMP Commands
- ② Big-Step Semantics
- ③ Small-Step Semantics

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② Big-Step Semantics

③ Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$\begin{array}{l} com ::= \text{SKIP} \\ \quad | \text{ string} ::= aexp \\ \quad | com ; ; com \\ \quad | \text{ IF } bexp \text{ THEN } com \text{ ELSE } com \\ \quad | \text{ WHILE } bexp \text{ DO } com \end{array}$$

Commands

Abstract syntax:

datatype *com* = *SKIP*
| *Assign string aexp*
| *Seq com com*
| *If bexp com com*
| *While bexp com*

Com.thy

① IMP Commands

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Big-step semantics

Concrete syntax:

$$(com, initial-state) \Rightarrow final-state$$

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Command c started in state s terminates in state t

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Command c started in state s terminates in state t

“ \Rightarrow ” here not type!

Big-step rules

$$(SKIP, s) \Rightarrow s$$

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$$(SKIP, s) \Rightarrow s$$

$$(x ::= a, s) \Rightarrow s(x := \text{aval } a \ s)$$

$$\frac{(c_1, s_1) \Rightarrow s_2 \quad (c_2, s_2) \Rightarrow s_3}{(c_1;; c_2, s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \quad (c_1, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \quad (c_1, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

$$\frac{\neg\ bval\ b\ s \quad (c_2, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

Big-step rules

$$\frac{\neg \text{bval } b \ s}{(\text{WHILE } b \text{ DO } c, s) \Rightarrow s}$$

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$$\frac{\begin{array}{c} \textit{bval } b \ s_1 \\ (c, s_1) \Rightarrow s_2 \end{array} \quad (\textit{WHILE } b \textit{ DO } c, s_2) \Rightarrow s_3}{(\textit{WHILE } b \textit{ DO } c, s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{("x" ::= N\ 5;;\ "y" ::= V\ "x",\ s) \Rightarrow ?}$$

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$$\frac{\vdots}{("x'' ::= N\ 5;; "y'' ::= V\ "x'',\ s) \Rightarrow ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow ?}$$

where

- $w = \textit{WHILE } b \textit{ DO } c$
- $b = \textit{NotEq } (V\ "x'')\ (N\ 2)$
- $c = "x'' ::= \textit{Plus } (V\ "x'')\ (N\ 1)$
- $s_i = s("x'' := i)$

Examples: derivation trees

$$\frac{\vdots}{("x'' ::= N\ 5;;\ "y'' ::= V\ "x'',\ s) \Rightarrow ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow ?}$$

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$$\begin{aligned} \textit{NotEq}\ a_1\ a_2 &= \\ \textit{Not}(\textit{And}\ (&\textit{Not}(\textit{Less}\ a_1\ a_2))\ (\textit{Not}(\textit{Less}\ a_2\ a_1)))) \end{aligned}$$

Logically speaking

$$(c, s) \Rightarrow t$$

is just infix syntax for

$$\textit{big_step} (c,s) t$$

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is just infix syntax for

$$big_step\ (c,s)\ t$$

where

$$big_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP, s) \Rightarrow t$?

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 $\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$

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 $\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$
- $(IF \ b \ THEN \ c_1 \ ELSE \ c_2, s) \Rightarrow t \text{ ?}$

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- $(w, s) \Rightarrow t \text{ where } w = WHILE\ b\ DO\ c \quad ?$

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- $(w, s) \Rightarrow t \text{ where } w = WHILE\ b\ DO\ c \quad ?$
 $\neg bval\ b\ s \wedge t = s \vee$
 $bval\ b\ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;; c_2, s_1) \Rightarrow s_3}{\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{\bigwedge s_2. \llbracket (c_1, s_1) \Rightarrow s_2; (c_2, s_2) \Rightarrow s_3 \rrbracket \Longrightarrow P}{P}$$

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Replaces assem $(c_1;; c_2, s_1) \Rightarrow s_3$ by two assems
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No \exists and \wedge !

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Rightarrow P \quad \dots \quad asm_n \Rightarrow P}{P}$$

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Reading:

To prove a goal P with assumption asm ,
prove all $asm_i \Longrightarrow P$

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Reading:

To prove a goal P with assumption asm ,
prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \vee G \quad F \Longrightarrow P \quad G \Longrightarrow P}{P}$$

elim attribute

- Theorems with *elim* attribute are used automatically by *blast*, *fastforce* and *auto*

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- Can also be added locally, eg (*blast elim: ...*)
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \text{WHILE } b \text{ DO } c$

$w' = \text{IF } b \text{ THEN } c;; w \text{ ELSE SKIP}$

Equivalence proof

$$(w, s) \Rightarrow t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t)$$

$$\vee$$

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Equivalence proof

$$\begin{aligned} & (w, s) \Rightarrow t \\ & \longleftrightarrow \\ & bval\ b\ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t) \\ & \quad \vee \\ & \neg bval\ b\ s \wedge t = s \\ & \longleftrightarrow \\ & (w', s) \Rightarrow t \end{aligned}$$

Equivalence proof

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Using the rules and rule inversions for \Rightarrow .

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c, s) \Rightarrow t \implies (c, s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t' .

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t ?$

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(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t$?

Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a \Rightarrow rule.

Big-step semantics cannot directly describe

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We need a finer grained semantics!

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Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

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Intended meaning of $(c, s) \rightarrow (c', s')$:

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Intended meaning of $(c, s) \rightarrow (c', s')$:

The first step in the execution of c in state s leaves a “remainder” command c' to be executed in state s' .

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Execution as finite or infinite reduction:

$$(c_1, s_1) \rightarrow (c_2, s_2) \rightarrow (c_3, s_3) \rightarrow \dots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \rightarrow cs'$ we say that cs *reduces* to cs' .
- A configuration cs is *final* iff $\nexists cs'. cs \rightarrow cs'$

The intention:

$(SKIP, s)$ is final

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Why?

SKIP is the empty program.

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Why?

SKIP is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a, s) \rightarrow$$

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$$(WHILE\ b\ DO\ c, s) \rightarrow (IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP, s)$$

Small-step rules

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$$(WHILE\ b\ DO\ c, s) \rightarrow (IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP, s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$("z'' ::= V "x'';; "x'' ::= V "y'';; "y'' ::= V "z'', s) \rightarrow$$

...

where $s = \langle "x'' := 3, "y'' := 7, "z'' := 5 \rangle$.

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$$(w, s_0) \rightarrow \dots$$

where

$$\begin{aligned} w &= \text{WHILE } b \text{ DO } c \\ b &= \text{Less } (V "x'') (N 1) \\ c &= "x'' ::= \text{Plus } (V "x'') (N 1) \\ s_n &= \langle "x'' := n \rangle \end{aligned}$$

Small_Step.thy

Semantics

Are big and small-step semantics equivalent?

From \Rightarrow to \rightarrow^*

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Theorem $cs \Rightarrow t \implies cs \rightarrow^* (SKIP, t)$

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In two cases a lemma is needed:

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Lemma

$$(c_1, s) \rightarrow^* (c_1', s') \implies (c_1;; c_2, s) \rightarrow^* (c_1';; c_2, s')$$

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Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From \rightarrow^* to \Rightarrow

Theorem $cs \rightarrow^* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow^* (SKIP, t)$.

In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \iff cs \rightarrow^* (SKIP, t)$

Small_Step.thy

Equivalence of big and small

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We prove the contrapositive

$$c \neq SKIP \implies \neg final(c, s)$$

by induction on c .

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 - $c_1 = SKIP \implies \neg final(c_1;; c_2, s)$

Can execution stop prematurely?

That is, are there any final configs except $(SKIP, s)$?

Lemma $final(c, s) \implies c = SKIP$

We prove the contrapositive

$$c \neq SKIP \implies \neg final(c, s)$$

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP, s) \rightarrow ct \implies False$

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Together:

Corollary $final(c, s) = (c = SKIP)$

Infinite executions

\Rightarrow yields final state iff \rightarrow terminates

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Equivalent:

\Rightarrow does not yield final state iff \rightarrow does not terminate

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Therefore: \Rightarrow correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \dots$

Chapter 8

Hoare Logic

- ④ Weakest Preconditions
- ⑤ Towards Simpler Verification of Programs
- ⑥ Example Verifications
- ⑦ Advanced Verification

④ Weakest Preconditions

⑤ Towards Simpler Verification of Programs

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④ Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

```
program exp {  
  a := 1  
  while (0 < n) do {  
    a := a + a;  
    n := n - 1  
  }  
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and $0 \leq n!$

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Weakest condition on state, such that program c will satisfy postcondition Q .

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wp of equivalent programs is equal

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Correctness of *exp*

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Reasoning along syntax of program!

That was easy!

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Unfolding will continue forever!

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But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall t. (c, s) \Rightarrow t \longrightarrow Q\ t$$

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Cannot reason about termination. This is called ***partial correctness***.

Some obvious facts:

$$c \sim c' \implies wlp\ c = wlp\ c'$$

$$\llbracket wlp\ c\ P\ s; \bigwedge s. P\ s \implies Q\ s \rrbracket \implies wlp\ c\ Q\ s$$

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Relation between wp and wlp

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Unfold rules still hold:

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Let's try to find predicate I , such that

$$\bigwedge s. I \ s \implies \text{if } \text{bval } b \ s \text{ then } \text{wp } c \ I \ s \text{ else } Q \ s$$

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\begin{aligned} & \llbracket I \ s_0; \bigwedge s. I \ s \implies \textit{if } b \textit{val } b \ s \textit{ then } wlp \ c \ I \ s \textit{ else } Q \ s \rrbracket \\ & \implies wlp \ (\textit{WHILE } b \ \textit{DO } c) \ Q \ s_0 \end{aligned}$$

Wp_Demo.thy

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Otherwise, use unfold rules.

Iterate, until all *wlps* gone!

wlp_if_eq and *wlp_whileI'* produce *if-then-else*

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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$$\frac{wf\ r \quad \bigwedge x. \frac{\forall y. (y, x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$$\begin{aligned} & \llbracket wf\ R; I\ s_0; \\ & \bigwedge s. I\ s \implies \text{if } bval\ b\ s \text{ then } wp\ c\ (\lambda s'. I\ s' \wedge (s', s) \in \\ & R)\ s \text{ else } Q\ s \rrbracket \\ & \implies wp\ (WHILE\ b\ DO\ c)\ Q\ s_0 \end{aligned}$$

Or, equivalently

assumes $WF: wf\ R$

assumes $INIT: I\ s_0$

assumes $STEP: \bigwedge s. \llbracket I\ s; bval\ b\ s \rrbracket$
 $\implies wp\ c\ (\lambda s'. I\ s' \wedge (s', s) \in R)\ s$

assumes $FINAL: \bigwedge s. \llbracket I\ s; \neg bval\ b\ s \rrbracket \implies Q\ s$

shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

lemma *ASSUME_Θ_{alt}*:

$$ASSUME_Θ \pi f_0 s_0 R \Theta = (\forall (f, (P, c, Q)) \in \Theta. HT' \pi (\lambda s. (f s, f_0 s_0) \in R \wedge P s) c Q)$$

unfolding *ASSUME_* Θ *_def HT'set_r_def ..*

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Simplify specification of pre/postcondition, and invariants

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$Cmpop::(int \Rightarrow int \Rightarrow bool) \Rightarrow aexp \Rightarrow aexp \Rightarrow bexp$

$BBinop::(bool \Rightarrow bool \Rightarrow bool) \Rightarrow bexp \Rightarrow bexp \Rightarrow bexp$

For example:

$Cmpop (\leq) (Binop (+) (Unop uminus (V "x"))) (N 42)) (N 50)$

IMP2/Introduction.thy

Adding more Operators

C-like syntax

Operators

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Arith: $+$, $-$, $*$, $/$ with usual binding

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while (*bexp*) c

IMP2/Introduction.thy

Program Syntax

More Readable VCs

Idea: Replace s " x " by (Isabelle) variable x .

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Similar: $s_0 \text{ ''}x\text{''}$ by x_0 .

If subgoal can still be proved for arbitrary (Isabelle) variable x , it can, in particular, be proved for $s \text{ ''}x\text{''}$.

$$(\bigwedge x. P \ x) \Longrightarrow P \ (s \text{ ''}x\text{''})$$

IMP2/Introduction.thy

More Readable VCs

More Readable Annotations

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Postcondition/Invariant: x as $s \text{ ''}x\text{''}$, x_0 as $s_0 \text{ ''}x\text{''}$

IMP2/Introduction.thy

More Readable Annotations

- ④ Weakest Preconditions
- ⑤ Towards Simpler Verification of Programs
- ⑥ Example Verifications
- ⑦ Advanced Verification

⑥ Example Verifications

Loop Patterns

Euclid's Algorithm

Common Loop Patterns

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

IMP2/Examples.thy

Count-up, Count-Down

Approximate Naively

Invert monotonic function, by naively trying all values:

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Applications: sqrt, log, ...

IMP2/Examples.thy

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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```
l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
  if m*m ≤ n then l=m else h=m
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r=l
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Invariant? $l^2 \leq n < h^2 \wedge \dots$ (range contains solution)

This program is actually tricky to get right!

IMP2/Examples.thy

Bisection

⑥ Example Verifications

Loop Patterns

Euclid's Algorithm

Euclid Intro

Compute gcd of positive numbers a , b

Euclid Intro

Compute gcd of positive numbers a, b

Reminder: Divides: $(b \text{ dvd } a) = (\exists k. a = b * k)$

Greatest Common Divisor: $gcd::int \Rightarrow int \Rightarrow int$ such that

$gcd\ a\ b\ \text{dvd}\ a$ and $gcd\ a\ b\ \text{dvd}\ b$ and

$\llbracket a \neq 0; b \neq 0; c\ \text{dvd}\ a; c\ \text{dvd}\ b \rrbracket \implies c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $\gcd(m - n, n) = \gcd(m, n)$

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By modulo. Using: $\gcd \ x \ y = \gcd \ y \ (x \bmod y)$

IMP2/Examples.thy

Euclid

- ④ Weakest Preconditions
- ⑤ Towards Simpler Verification of Programs
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Modified Variables

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Program modifies at most variables it assigns to

$\pi: (c, s) \Rightarrow t \implies \text{modifies} (\text{lhsv } \pi \ c) \ t \ s$

Modified Variables

We can strengthen correctness statement (automatically)

$$wp \ \pi \ c \ Q \ s \Longrightarrow wp \ \pi \ c \ (\lambda s'. Q \ s' \wedge \text{modifies} \ (lhsv \ \pi \ c) \ s' \ s) \ s$$

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For while-rule, we get

lemma *wp_whileI_modset*:

fixes *c*

defines [*simp*]: *modset* \equiv *lhsv c*

assumes *WF*: *wf R*

assumes *INIT*: *I s₀*

assumes *STEP*: $\bigwedge s. \llbracket \text{modifies modset } s\ s_0; I\ s; \text{bval } b\ s \rrbracket$

$\implies wp\ c\ (\lambda s'.\ I\ s' \wedge (s', s) \in R)\ s$

assumes *FINAL*: $\bigwedge s. \llbracket \text{modifies modset } s\ s_0; I\ s; \neg \text{bval } b\ s \rrbracket$

$\implies Q\ s$

shows *wp (WHILE b DO c) Q s₀*

Modified Variables

The VCG will automatically rewrite with rule

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program_spec computes *lhs*-variables:

$$HT_mods \ \pi \ mods \ P \ c \ Q \equiv HT \ \pi \ P \ c \ (\lambda s_0 \ s. \text{modifies} \\ mods \ s \ s_0 \wedge Q \ s_0 \ s)$$

IMP2/Examples.thy

Euclid – show modified sets

Modular Proofs

Consider program

```
a=1;
while (m>0) {
  n=a; a = 1;
  while (n>0) {
    a=2*a; n=n-1
  };
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```

What does this compute

Modular Proofs

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What does this compute?

Power-tower function: $2^{2^{\cdot^{\cdot^2}}}$ (m times)

Modular Proofs

Inner loop invariant: Would like to refer to n right before loop!

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Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

```
a=1;  
while (m>0) {  
  n=a;  
  inline exp_count_down;  
  m=m−1  
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Reuse existing proof of exp-count-down program!

Modular Proofs

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$$\begin{aligned} & \llbracket HT \ \pi \ P \ c \ Q; \bigwedge s. P' \ s \Longrightarrow P \ s; \bigwedge s_0 \ s. \llbracket P \ s_0; P' \ s_0; Q \\ & \ s_0 \ s \rrbracket \Longrightarrow Q' \ s_0 \ s \rrbracket \\ & \Longrightarrow HT \ \pi \ P' \ c \ Q' \end{aligned}$$

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with modified sets:

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VCG will automatically use this rule.

If inlined program has been proved with **program_spec**

IMP2/Examples.thy

Power-Tower

⑦ Advanced Verification

Arrays

Data Refinement

Local Variables

Recursion

Arrays

Every variable is of type $int \Rightarrow int$.

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$ArrayClear::char\ list \Rightarrow com$
 $\pi: (CLEAR\ x[], s) \Rightarrow s(x := \lambda_. 0)$

Arrays

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Only with index 0: Bind $VAR\ (s\ "x"\ 0)\ (\lambda x. \dots)$

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Only with index 0: $Bind\ VAR\ (s\ "x"\ 0)\ (\lambda x. \dots)$

Otherwise: $Bind\ VAR\ (s\ "x")\ (\lambda x. \dots)$

Arrays

By default, we use index 0.

Abbreviations:

$V\ x = Vid\ x\ (N\ 0)$

$Assign\ x\ a = AssignIdx\ x\ (N\ 0)\ a$

VCG: Guess type from variable usage

Only with index 0: $Bind\ VAR\ (s\ "x"\ 0)\ (\lambda x. \dots)$

Otherwise: $Bind\ VAR\ (s\ "x")\ (\lambda x. \dots)$

IMP2/Examples.thy

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory *IMP2/IMP2_Aux_Lemmas* provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

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Only 5 out of 20 surveyed textbooks had correct implementations

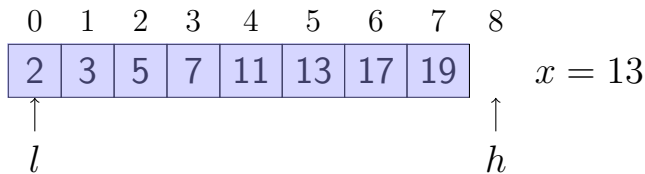
— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8	
2	3	5	7	11	13	17	19		$x = 13$

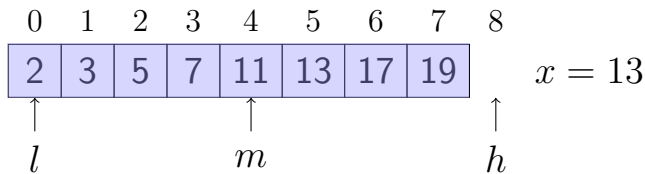
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while (l < h) {  
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while ( $l < h$ ) {  
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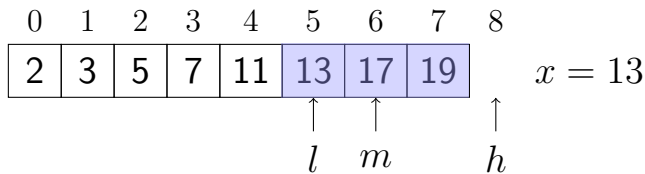
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\uparrow \uparrow
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while (l < h) {  
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Returns **smallest** i with $x \leq a[i]$

Notes on Binary Search

```
while (l < h) {  
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Notes on Binary Search

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Bug in Java Standard Library for > 9 years!

Proving Binary Search

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Invariant:

Proving Binary Search

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Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)

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Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)
- $i \geq h \implies x \leq a[i]$ (greater or equal to x)
- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j < h) {
    key = a[j];
    i = j - 1;
    while (i >= l && a[i] > key) {
        a[i + 1] = a[i];
        i = i - 1;
    };
    a[i + 1] = key;
    j = j + 1;
}
```

Idea: Build sorted array from start.

In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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Postcondition:

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where

$$ran_sorted\ a\ l\ h \equiv \forall i \in \{l..<h\}. \forall j \in \{l..<h\}. i \leq j \longrightarrow a\ i \leq a\ j$$

$$mset_ran\ a\ r = (\sum_{i \in r}. \{\#a\ i\# \})$$

Multisets in Isabelle

imports *HOL–Library.Multiset*

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$mset_ran\ a\ r = (\sum_{i \in r}. \{\#a\ i\# \})$

Multiset of elements at indexes in finite **set** r

Proving Insertion Sort

Separate proof for inner loop!

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j = l + 1;  
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Specification of inner loop: ?

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Invariant of outer loop:

$\text{ran_sorted } a \ l \ j$

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Insert: Inner Loop

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key = a[j];  
i = j - 1;  
while (i >= 0 && a[i] > key) {  
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    i = i - 1;  
};  
a[i + 1] = key
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Intuition:

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Intuition: ?

$a[j]$ is moved backwards

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Intuition: ?

$a[j]$ is moved backwards until

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Intuition: ?

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$a[j]$ is moved backwards until
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Short: Move $a[j]$ backwards over greater elements.

Insert: Inner Loop

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Let's specify this intuition!

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Invariants easier to find!

Insert: Inner Loop

Move $a[j]$ backwards over greater elements.

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Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a[j]$

Insert: Inner Loop

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Insert: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a_0 j$

ensures $i \in \{l - 1..<j\}$

ensures $\forall k \in \{l..i\}. a k = a_0 k$ and

Insert: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a[j]$

ensures $i \in \{l - 1..j\}$

ensures $\forall k \in \{l..i\}. a[k] \geq key$ and
 $a[i + 1] = key$ and

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Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a_0 j$

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ensures $\forall k \in \{l..i\}. a\ k = a_0\ k$ and

$a\ (i + 1) = key$ and

$\forall k \in \{i + 2..j\}. a\ k = a_0\ (k - 1)$

Insert: Inner Loop

Move $a[j]$ backwards over greater elements.

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ensures $l \leq i \longrightarrow a i \leq key$ and

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ensures $l \leq i \longrightarrow a i \leq key$ and

$\forall k \in \{i + 2..j\}. key < a k$

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
\uparrow						\uparrow	\uparrow
l						i	j

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	11	19
\uparrow					\uparrow		\uparrow
l					i		j

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	11	17	19
\uparrow				\uparrow			\uparrow
l				i			j

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	11	13	17	19
\uparrow			\uparrow				\uparrow
l			i				j

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	11	13	17	19
\uparrow			\uparrow				\uparrow
l			i				j

Consider intermediate situation

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	11	17	19
\uparrow				\uparrow			\uparrow
l				i			j

Consider intermediate situation

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	11	17	19
\uparrow				\uparrow			\uparrow
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Insert: Finding Invariant

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- + the usual bounds: $l-1 \leq i \wedge i < j$

IMP2/Examples.thy

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

⑦ Advanced Verification

Arrays

Data Refinement

Local Variables

Recursion

Abstract View

Model $int \Rightarrow int$ not always appropriate

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IMP2/Examples.thy

Filter, Merge, dedup

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Local Variables

Introduce local variables

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Why?

Local Variables

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Why? Better modularity.

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Don't worry about name-clashes with subroutine's auxiliary variables

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SCOPE c — Execute c with fresh set of local variables.
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Given specification of body: $HT\ P\ body\ Q$
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Recall:

$HT\ \pi\ P\ c\ Q \equiv \forall s_0. P\ s_0 \longrightarrow wp\ \pi\ c\ (Q\ s_0)\ s_0$

Prologue

$$\begin{aligned} HT \pi P \text{ body } Q &\implies \\ HT \pi (wp \pi \text{ prologue } P) (\text{prologue};; \text{body}) \\ &(\lambda s_0 s. wp \pi \text{ prologue } (\lambda s_0. Q s_0 s) s_0) \end{aligned}$$

Intuition: Weakest precondition to enforce P after prologue

Epilogue

$$\begin{aligned} & \llbracket HT \pi P \textit{body} Q; \forall s. \exists t. \pi: (\textit{epilogue}, s) \Rightarrow t \rrbracket \\ & \implies HT \pi P (\textit{body};; \textit{epilogue}) (\lambda s_0. sp \pi (Q s_0) \\ & \textit{epilogue}) \end{aligned}$$

Intuition: Strongest postcondition we get from Q after epilogue

Strongest Postconditions

$$sp\ \pi\ P\ c\ t \equiv \exists s. P\ s \wedge \pi: (c, s) \Rightarrow t$$

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IMP2/Examples.thy

Merge as Procedure

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No problem when proving **total** correctness

Proof Rules for Recursion

Unfolding: $\pi \ p = \textit{Some } c \implies \textit{wp } \pi \ (\textit{PCall } p) \ Q \ s = \textit{wp } \pi \ c \ Q \ s$

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assumes $wf \ R$

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Show specification for state s , assuming it holds for smaller states s' .

Mutual Recursion

Same idea, but for sets of specifications.

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All Hoare-Triples in Θ valid. Annotation n ignored!

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$ASSUME_{\Theta} \pi f_0 s_0 R \Theta =$

$(\forall (f, P, c, Q) \in \Theta. HT' \pi (\lambda s. (f s, f_0 s_0) \in R \wedge P s) c Q)$

Hoare-triples valid for states less than $f_0 s_0$. Annotation is **variant**.

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Hoare-triples valid for states less than $f_0 \ s_0$. Annotation is **variant**.

$PROVE_{\Theta} \pi f_0 \ s_0 \ \Theta \equiv$

$\forall P \ c \ Q. (f_0, P, c, Q) \in \Theta \wedge P \ s_0 \longrightarrow wp \ \pi \ (c \ s_0) \ (Q \ s_0) \ s_0$

Hoare-triples valid for fixed variant f_0 and state s_0 .

Mutual Recursion

lemma $vcg_HT'setI$:
assumes $wf\ R$
assumes RL : $\bigwedge f_0\ s_0. \llbracket ASSUME_Θ\ \pi\ f_0\ s_0\ R\ Θ \rrbracket \implies$
 $PROVE_Θ\ \pi\ f_0\ s_0\ Θ$
shows $HT'set\ \pi\ Θ$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

Mutual Recursion

lemma *vcg_HT'setI*:
assumes *wf R*
assumes *RL*: $\bigwedge f_0 \ s_0. \llbracket \text{ASSUME_}\Theta \ \pi \ f_0 \ s_0 \ R \ \Theta \rrbracket \implies$
PROVE_ $\Theta \ \pi \ f_0 \ s_0 \ \Theta$
shows *HT'set* $\pi \ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

$$\llbracket \pi \ p = \text{Some } c; \text{HT_mods } \pi \ \text{mods } P \ c \ Q \rrbracket \implies \text{HT_mods } \pi \ \text{mods } P \ (P\text{Call } p) \ Q$$

Maps Hoare-Triples to procedure calls

Local Procedures

Idea: Recursive procedure names only valid locally!

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Call procedure with local procedure environment

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Call procedure with local procedure environment

$$HT_mods \pi \text{ mods } P (P\text{Call } p) Q \Longrightarrow HT_mods \pi' \text{ mods } P \\ (P\text{Scope } \pi p) Q$$

Wrap current procedure environment

Specification of Mutual Recursive Procedures

The IMP2 tools take care of

- wf-relation. Default *less_than*

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- variants: expression over parameters.
- localization of procedure environment.

IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

Completeness

Consider program with $HT \pi P c Q$

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Can we always find annotations to get provable VCs?

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Can we always find annotations to get provable VCs?

Only consider while-rule here

Partial Correctness

$$\begin{aligned} & \llbracket I \ s_0; \bigwedge s. I \ s \implies \textit{if } b \textit{val } b \ s \textit{ then } wlp \ \pi \ c \ I \ s \textit{ else } Q \ s \rrbracket \\ & \implies wlp \ \pi \ (\textit{WHILE } b \textit{ DO } c) \ Q \ s_0 \end{aligned}$$

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What invariant shall we use?

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What invariant shall we use?

$$wlp \ \pi \ c \ Q!$$

Total Correctness

$$\begin{aligned} & \llbracket wf\ R; \ I\ s_0; \\ & \quad \bigwedge s. I\ s \implies \text{if } bval\ b\ s \text{ then } wp\ \pi\ c\ (\lambda s'. I\ s' \wedge (s', s) \\ & \quad \in R) \ s \text{ else } Q\ s \rrbracket \\ & \implies wp\ \pi\ (WHILE\ b\ DO\ c)\ Q\ s_0 \end{aligned}$$

Invariant: $wp\ \pi\ c\ Q$

Total Correctness

$$\llbracket wf\ R; I\ s_0;$$
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Variant?

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Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

IMP2/Examples.thy

Completeness of While-Rule

Conclusions

IMP2: Verification of simple programs in Isabelle/HOL
while-language, arrays, local-vars, recursive procedures
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Caveats:
Procedures+Recursion tools not well-tested
VCG is slow for many procedures/inlines