Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

Semantics is easy.

Command: order to do something

Study the book until you have understood it.

Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{rcl}
\mathit{com} & ::= & \texttt{SKIP} \\
& \mid & \mathit{string}\ \texttt{::=}\ \mathit{aexp} \\
& \mid & \mathit{com}\ \texttt{;;}\ \mathit{com} \\
& \mid & \texttt{IF}\ \mathit{bexp}\ \texttt{THEN}\ \mathit{com}\ \texttt{ELSE}\ \mathit{com} \\
& \mid & \texttt{WHILE}\ \mathit{bexp}\ \texttt{DO}\ \mathit{com}
\end{array}
$$

Commands

Abstract syntax:

$$
\begin{array}{rcl}
\textbf{datatype}\ com & = & \mathit{SKIP} \\
& | & \mathit{Assign}\ \mathit{string}\ \mathit{aexp} \\
& | & \mathit{Seq}\ \mathit{com}\ \mathit{com} \\
& | & \mathit{If}\ \mathit{bexp}\ \mathit{com}\ \mathit{com} \\
& | & \mathit{While}\ \mathit{bexp}\ \mathit{com}
\end{array}
$$

Com.thy

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Big-step semantics

Concrete syntax:

$$(\mathit{com},\ \mathit{initial\text{-}state}) \Rightarrow \mathit{final\text{-}state}$$

Intended meaning of $(c,\ s) \Rightarrow t$:

Command c started in state s terminates in state t

"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

$$\frac{\neg\ bval\ b\ s \qquad (c_2,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

Big-step rules

$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

$$\frac{(c,\ s_1) \Rightarrow s_2 \qquad \begin{array}{c} bval\ b\ s_1 \\ (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3 \end{array}}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{aligned} w &= WHILE\ b\ DO\ c \\ b &= NotEq\ (V\ ''x'')\ (N\ 2) \\ c &= ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_i &= s(''x'' := i) \end{aligned}$$

$NotEq\ a_1\ a_2 =$
$Not(And\ (Not(Less\ a_1\ a_2))\ (Not(Less\ a_2\ a_1)))$

Logically speaking

$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP,\ s) \Rightarrow t$?
- $(x ::= a,\ s) \Rightarrow t$?
- $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$?

- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?

- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$ (with a new fixed s_2).
No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

(possibly with $\bigwedge\overline{x}$ in front of the $asm_i \Longrightarrow P$)

Reading:

To prove a goal P with assumption asm,
prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

elim attribute

- Theorems with *elim* attribute are used automatically by *blast*, *fastforce* and *auto*
- Can also be added locally, eg *(blast elim: . . .)*
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

$$c \sim c' \;\equiv\; (\forall\, s\; t.\; (c,s) \Rightarrow t \longleftrightarrow (c',s) \Rightarrow t)$$

Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$
$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
$$\vee$$
$$\neg\ bval\ b\ s \wedge t = s$$

$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

`Big_Step.thy`

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

Proof by rule induction, for arbitrary t'.

`Big_Step.thy`

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

- nonterminating computations,
- parallel computations.

We need a finer grained semantics!

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s leaves a "remainder" command c' to be executed in state s'.*

Execution as finite or infinite reduction:

$$(c_1, s_1) \to (c_2, s_2) \to (c_3, s_3) \to \dots$$

Terminology

- A pair (c, s) is called a *configuration*.
- If $cs \rightarrow cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \rightarrow cs'$

The intention:

$$(SKIP,\ s) \ \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;;c_2, s) \ \rightarrow\ (c_1';;c_2, s')}$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s)\ \rightarrow\ (c_1, s)}$$

$$\frac{\neg\ bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s)\ \rightarrow\ (c_2, s)}$$

$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow$$
$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{array}{rcl} w & = & \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b & = & \mathit{Less}\ (V\ ''x'')\ (N\ 1) \\ c & = & ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\ 1) \\ s_n & = & <''x'' := n> \end{array}$$

Small_Step.thy

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1{}',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1{}';;\ c_2,\ s')$

Proof by rule induction.

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \; (SKIP,\; t)$

Small_Step.thy

Equivalence of big and small

Can execution stop prematurely?

That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

$$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$$

by induction on c.

- Case $c_1;;\ c_2$: by case distinction:
  - $c_1 = SKIP \Longrightarrow \neg\ \mathit{final}\ (c_1;;\ c_2,\ s)$
  - $c_1 \neq SKIP \Longrightarrow \neg\ \mathit{final}\ (c_1,\ s)$ (by IH)
    $\Longrightarrow \neg\ \mathit{final}\ (c_1;;\ c_2,\ s)$
- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

Together:

Corollary $\mathit{final}\ (c,\ s) = (c = SKIP)$

Infinite executions

$\Rightarrow$ yields final state iff $\rightarrow$ terminates

Lemma $(\exists\, t.\ cs \Rightarrow t) = (\exists\, cs'.\ cs \rightarrow\!*\ cs' \wedge \mathit{final}\ cs')$

Proof: $(\exists\, t.\ cs \Rightarrow t)$

$= (\exists\, t.\ cs \rightarrow\!*\ (\mathit{SKIP},t))$

(by big = small)

$= (\exists\, cs'.\ cs \rightarrow\!*\ cs' \wedge \mathit{final}\ cs')$

(by final = SKIP)

Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

May versus Must

$\rightarrow$ is deterministic:

Lemma $cs \rightarrow cs' \implies cs \rightarrow cs'' \implies cs'' = cs'$
(Proof by rule induction)

Therefore: no difference between

may terminate (there is a terminating $\rightarrow$ path)

must terminate (all $\rightarrow$ paths terminate)

Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4. Partial Correctness

5. Verification Conditions

6. Total Correctness

4 Partial Correctness

5 Verification Conditions

6 Total Correctness

We have proved functional programs correct
(e.g. a compiler).

We have proved properties of imperative languages
(e.g. type safety).

But how do we prove properties of imperative programs?

An example program:

$$''y'' ::= N\ 0;;\ wsum$$

where

$$\begin{array}{l} wsum \equiv \\ WHILE\ Less\ (N\ 0)\ (V\ ''x'') \\ DO\ (''y'' ::= Plus\ (V\ ''y'')\ (V\ ''x'');; \\ \qquad ''x'' ::= Plus\ (V\ ''x'')\ (N\ (-\ 1))) \end{array}$$

At the end of the execution of $''y'' ::= N\ 0;;\ wsum$ variable $''y''$ should contain the sum $1 + \ldots + i$ where i is the initial value of $''x''$.

$$sum\ i = (\textit{if}\ i \leq 0\ \textit{then}\ 0\ \textit{else}\ sum\ (i - 1) + i)$$

A proof via operational semantics

Theorem:

$$(''y'' ::= N\,0;;\ wsum,\ s) \Rightarrow t \Longrightarrow$$
$$t\ ''y'' = sum\ (s\ ''x'')$$

Required Lemma:

$$(wsum,\ s) \Rightarrow t \Longrightarrow$$
$$t\ ''y'' = s\ ''y'' +\ sum\ (s\ ''x'')$$

Proved by rule induction.

Hoare Logic provides a *structured* approach for reasoning about properties of states during program execution:

- Rules of Hoare Logic (almost) syntax directed
- Automates reasoning about program execution
- No explicit induction

But no free lunch:

- Must prove implications between predicates on states
- Needs invariants.

4 Partial Correctness

This is the standard approach.

Formulas are syntactic objects.

Everything is very concrete and simple.

But complex to formalize.

Hence we soon move to a semantic view of formulas.

Reason for introduction of syntactic approach: didactic

For now, we work with a (syntactically) simplified version of IMP.

Hoare Logic reasons about *Hoare triples* $\{P\}\ c\ \{Q\}$ where

- P and Q are *syntactic formulas* involving program variables
- P is the *precondition*, Q is the *postcondition*
- $\{P\}\ c\ \{Q\}$ means that if P is true at the start of the execution, Q is true at the end of the execution — if the execution terminates! (*partial correctness*)

Informal example:

$$\{x = 41\}\ x := x + 1\ \{x = 42\}$$

Terminology: P and Q are called *assertions*.

Examples

$$\{x = 5\} \quad ? \quad \{x = 10\}$$

$$\{True\} \quad ? \quad \{x = 10\}$$

$$\{x = y\} \quad ? \quad \{x \neq y\}$$

Boundary cases:

$$\{True\} \quad ? \quad \{True\}$$

$$\{True\} \quad ? \quad \{False\}$$

$$\{False\} \quad ? \quad \{Q\}$$

The rules of Hoare Logic

$$\{P\}\ SKIP\ \{P\}$$

$$\{Q[a/x]\}\ x := a\ \{Q\}$$

Notation: $Q[a/x]$ means "Q with a substituted for x".

Examples:

$$\begin{array}{lll} \{\qquad\} & x := 5 & \{x = 5\} \\ \{\qquad\} & x := x{+}5 & \{x = 5\} \\ \{\qquad\} & x := 2{*}(x{+}5) & \{x > 20\} \end{array}$$

Alternative explanation of assignment rule:

$$\{Q[a]\}\ x := a\ \{Q[x]\}$$

The assignment axiom allows us
to compute the precondition from the postcondition.

There is a version to compute the postcondition from the precondition, but it is more complicated. (Exercise!)

More rules of Hoare Logic

$$\frac{\{P_1\}\ c_1\ \{P_2\} \qquad \{P_2\}\ c_2\ \{P_3\}}{\{P_1\}\ c_1;c_2\ \{P_3\}}$$

$$\frac{\{P \wedge\ b\}\ c_1\ \{Q\} \qquad \{P \wedge \neg\ b\}\ c_2\ \{Q\}}{\{P\}\ IF\ b\ THEN\ c_1\ ELSE\ c_2\ \{Q\}}$$

$$\frac{\{P \wedge\ b\}\ c\ \{P\}}{\{P\}\ WHILE\ b\ DO\ c\ \{P \wedge \neg\ b\}}$$

In the While-rule, P is called an *invariant* because it is preserved across executions of the loop body.

The *consequence* rule

So far, the rules were syntax-directed. Now we add

$$\frac{P' \longrightarrow P \qquad \{P\}\ c\ \{Q\} \qquad Q \longrightarrow Q'}{\{P'\}\ c\ \{Q'\}}$$

Preconditions can be strengthened,
postconditions can be weakened.

Two derived rules

Problem with assignment and While-rule:
special form of pre and postcondition.
Better: combine with consequence rule.

$$\frac{P \longrightarrow Q[a/x]}{\{P\}\ x := a\ \{Q\}}$$

$$\frac{\{P \wedge b\}\ c\ \{P\} \qquad P \wedge \neg\, b \longrightarrow Q}{\{P\}\ \mathit{WHILE}\ b\ \mathit{DO}\ c\ \{Q\}}$$

Example

$$\{x = i\}$$

$$y := 0;$$
$$\mathit{WHILE}\ 0 < x\ \mathit{DO}\ (y := y{+}x;\ x := x{-}1)$$

$$\{y = \mathit{sum}\ i\}$$

Example proof exhibits key properties of Hoare logic:

- Choice of rules is syntax-directed and hence automatic.
- Proof of ";" proceeds from right to left.
- Proofs require only invariants and arithmetic reasoning.

4 Partial Correctness

- Introduction
- The Syntactic Approach
- The Semantic Approach
- Soundness and Completeness

Assertions are predicates on states

$$assn = state \Rightarrow bool$$

Alternative view: *sets of states*

Semantic approach simplifies meta-theory, our main objective.

Validity

$$\models \{P\}\ c\ \{Q\}$$

$$\longleftrightarrow$$

$$\forall\, s\ t.\ P\ s \wedge (c,\ s) \Rightarrow t \longrightarrow Q\ t$$

"$\{P\}\ c\ \{Q\}$ is valid"

In contrast:

$$\vdash \{P\}\ c\ \{Q\}$$

"$\{P\}\ c\ \{Q\}$ is provable/derivable"

Provability

$$\vdash \{P\}\ SKIP\ \{P\}$$

$$\vdash \{\lambda s.\ Q\ (s[a/x])\}\ x ::= a\ \{Q\}$$

where $s[a/x] \equiv s(x := aval\ a\ s)$

Example: $\{x{+}5 = 5\}\ x := x{+}5\ \{x = 5\}$ in semantic terms:

$$\vdash \{P\}\ x ::= Plus\ (V\ x)\ (N\ 5)\ \{\lambda t.\ t\ x = 5\}$$

where
$$\begin{aligned} P &= (\lambda s.\ (\lambda t.\ t\ x = 5)(s[Plus\ (V\ x)\ (N\ 5)/x])) \\ &= (\lambda s.\ (\lambda t.\ t\ x = 5)(s(x := s\ x + 5))) \\ &= (\lambda s.\ s\ x + 5 = 5) \end{aligned}$$

$$\frac{\vdash \{P\}\ c_1\ \{Q\} \qquad \vdash \{Q\}\ c_2\ \{R\}}{\vdash \{P\}\ c_1;;\ c_2\ \{R\}}$$

$$\frac{\begin{array}{c}\vdash \{\lambda s.\ P\ s \wedge \mathit{bval}\ b\ s\}\ c_1\ \{Q\} \\ \vdash \{\lambda s.\ P\ s \wedge \neg\ \mathit{bval}\ b\ s\}\ c_2\ \{Q\}\end{array}}{\vdash \{P\}\ \mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2\ \{Q\}}$$

$$\frac{\vdash \{\lambda s.\ P\ s \wedge \mathit{bval}\ b\ s\}\ c\ \{P\}}{\vdash \{P\}\ \mathit{WHILE}\ b\ \mathit{DO}\ c\ \{\lambda s.\ P\ s \wedge \neg\ \mathit{bval}\ b\ s\}}$$

$$\frac{\begin{array}{c}\forall s.\ P'\ s \longrightarrow P\ s \\ \vdash \{P\}\ c\ \{Q\} \\ \forall s.\ Q\ s \longrightarrow Q'\ s\end{array}}{\vdash \{P'\}\ c\ \{Q'\}}$$

Hoare_Examples.thy

4 Partial Correctness

Soundness

Everything that is provable is valid:

$$\vdash \{P\}\ c\ \{Q\} \Longrightarrow \models \{P\}\ c\ \{Q\}$$

Proof by induction, with a nested induction in the While-case.

Towards completeness: $\models \implies \vdash$

Weakest preconditions

The weakest precondition
of command c w.r.t. postcondition Q:

$$wp\ c\ Q = (\lambda s.\ \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t)$$

The set of states that lead (via c) into Q.

A foundational semantic notion, not merely for the completeness proof.

Nice and easy properties of wp

$$wp\ SKIP\ Q = Q$$

$$wp\ (x ::= a)\ Q = (\lambda s.\ Q\ (s[a/x]))$$

$$wp\ (c_1;;\ c_2)\ Q = wp\ c_1\ (wp\ c_2\ Q)$$

$$wp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q =$$
$$(\lambda s.\ \textsf{if}\ bval\ b\ s\ \textsf{then}\ wp\ c_1\ Q\ s\ \textsf{else}\ wp\ c_2\ Q\ s)$$

$$\neg\ bval\ b\ s \Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s = Q\ s$$

$$bval\ b\ s \Longrightarrow$$
$$wp\ (WHILE\ b\ DO\ c)\ Q\ s =$$
$$wp\ (c;;\ WHILE\ b\ DO\ c)\ Q\ s$$

Completeness

$$\models \{P\}\ c\ \{Q\} \Longrightarrow \vdash \{P\}\ c\ \{Q\}$$

Proof idea: do not prove $\vdash \{P\}\ c\ \{Q\}$ directly, prove something stronger:

Lemma $\vdash \{wp\ c\ Q\}\ c\ \{Q\}$
Proof by induction on c, for arbitary Q.

Now prove $\vdash \{P\}\ c\ \{Q\}$ from $\vdash \{wp\ c\ Q\}\ c\ \{Q\}$ by the consequence rule because

Fact $\models \{P\}\ c\ \{Q\} \longleftrightarrow (\forall\, s.\ P\ s \longrightarrow wp\ c\ Q\ s)$
Follows directly from defs of $\models$ and wp.

$$\vdash \{P\}\ c\ \{Q\} \quad \longleftrightarrow \quad \models \{P\}\ c\ \{Q\}$$

Proving program properties by Hoare logic ($\vdash$)
is just as powerful as by operational semantics ($\models$).

WARNING

Most texts that discuss completeness of Hoare logic state or prove that Hoare logic is only "relatively complete" but not complete.

Reason: the standard notion of completeness assumes some abstract mathematical notion of $\models$.

Our notion of $\models$ is defined within the same (limited) proof system (for HOL) as $\vdash$.

4 Partial Correctness

5 Verification Conditions

6 Total Correctness

Idea:

Reduce provability in Hoare logic to provability in the assertion language: automate the Hoare logic part of the problem.

More precisely:

From $\{P\}\ c\ \{Q\}$ *generate an assertion* A*, the verification condition, such that* $\vdash \{P\}\ c\ \{Q\}$ *iff* A *is provable.*

Method:

Simulate syntax-directed application of Hoare logic rules. Collect all assertion language side conditions.

A problem: loop invariants

Where do they come from?

A trivial solution:

Let the user provide them!

How?

Each loop must be annotated with its invariant!

How to synthesize loop invariants automatically
is an important research problem.

Which we ignore for the moment.

But come back to later.

Terminology:

VCG = Verification Condition Generator

All successful verification technology for imperative programs relies on

- VCGs (of one kind or another)
- and powerful (semi-)automatic theorem provers.

The (approx.) plan of attack

1. Introduce annotated commands with loop invariants
2. Define functions for *computing*
   - weakest preconditions: $pre :: com \Rightarrow assn \Rightarrow assn$
   - verification conditions: $vc :: com \Rightarrow assn \Rightarrow bool$
3. Soundness: $vc\ c\ Q \Longrightarrow\ \vdash \{\ ?\ \}\ c\ \{Q\}$
4. Completeness: if $\vdash \{P\}\ c\ \{Q\}$ then c can be annotated (becoming C) such that $vc\ C\ Q$.

The details are a bit different ...

Annotated commands

Like commands, except for While:

$$\textbf{datatype}\ \mathit{acom} = \mathit{Askip} \mid \mathit{Aassign}\ \mathit{vname}\ \mathit{aexp} \mid \mathit{Aseq}\ \mathit{acom}\ \mathit{acom} \mid \mathit{Aif}\ \mathit{bexp}\ \mathit{acom}\ \mathit{acom} \mid \mathit{Awhile}\ \mathit{assn}\ \mathit{bexp}\ \mathit{acom}$$

Concrete syntax: like commands, except for WHILE:

$$\{I\}\ \mathit{WHILE}\ b\ \mathit{DO}\ c$$

Weakest precondition

$pre :: acom \Rightarrow assn \Rightarrow assn$

$pre\ SKIP\ Q = Q$

$pre\ (x ::= a)\ Q = (\lambda s.\ Q\ (s[a/x]))$

$pre\ (C_1;;\ C_2)\ Q = pre\ C_1\ (pre\ C_2\ Q)$

$pre\ (IF\ b\ THEN\ C_1\ ELSE\ C_2)\ Q =$
$(\lambda s.\ \textit{if}\ bval\ b\ s\ \textit{then}\ pre\ C_1\ Q\ s\ \textit{else}\ pre\ C_2\ Q\ s)$

$pre\ (\{I\}\ WHILE\ b\ DO\ C)\ Q = I$

Warning

In the presence of loops,
$pre\ C$ may not be the weakest precondition
but may be anything!

Verification condition

$vc :: acom \Rightarrow assn \Rightarrow bool$

$vc\ SKIP\ Q = True$

$vc\ (x ::= a)\ Q = True$

$vc\ (C_1;;\ C_2)\ Q = (vc\ C_1\ (pre\ C_2\ Q) \wedge vc\ C_2\ Q)$

$vc\ (IF\ b\ THEN\ C_1\ ELSE\ C_2)\ Q =$
$(vc\ C_1\ Q \wedge vc\ C_2\ Q)$

$vc\ (\{I\}\ WHILE\ b\ DO\ C)\ Q =$
$((\forall s.\ (I\ s \wedge bval\ b\ s \longrightarrow pre\ C\ I\ s) \wedge$
$\quad (I\ s \wedge \neg\ bval\ b\ s \longrightarrow Q\ s)) \wedge$
$vc\ C\ I)$

Verification conditions only arise from loops:

- the invariant must be invariant
- and it must imply the postcondition.

Everything else in the definition of vc is just bureaucracy: collecting assertions and passing them around.

Hoare triples operate on com,
functions pre and vc operate on $acom$.
Therefore we define

$$strip :: acom \Rightarrow com$$

$$strip\ SKIP = SKIP$$
$$strip\ (x ::= a) = x ::= a$$
$$strip\ (C_1;;\ C_2) = strip\ C_1;;\ strip\ C_2$$
$$strip\ (IF\ b\ THEN\ C_1\ ELSE\ C_2) =$$
$$IF\ b\ THEN\ strip\ C_1\ ELSE\ strip\ C_2$$
$$strip\ (\{I\}\ WHILE\ b\ DO\ C) = WHILE\ b\ DO\ strip\ C$$

Soundness of vc & pre w.r.t. $\vdash$

$$vc\ C\ Q \Longrightarrow \vdash \{pre\ C\ Q\}\ strip\ C\ \{Q\}$$

Proof by induction on C, for arbitrary Q.

Corollary:

$$[\![vc\ C\ Q;\ \forall\, s.\ P\ s \longrightarrow pre\ C\ Q\ s]\!]$$
$$\Longrightarrow \vdash \{P\}\ strip\ C\ \{Q\}$$

How to prove some $\vdash \{P\}\ c\ \{Q\}$:

- Annotate c yielding C, i.e. $strip\ C = c$.
- Prove Hoare-free premise of corollary.

But is premise provable if $\vdash \{P\}\ c\ \{Q\}$ is?

$$\llbracket vc\ C\ Q;\ \forall\, s.\ P\ s \longrightarrow pre\ C\ Q\ s \rrbracket$$
$$\Longrightarrow\ \vdash \{P\}\ strip\ C\ \{Q\}$$

Why could premise not be provable
although conclusion is?

- Some annotation in C is not invariant.
- vc or pre are wrong
  (e.g. accidentally always produce $False$).

Therefore we prove completeness:
suitable annotations exist such that premise is provable.

Completeness of vc & pre w.r.t. $\vdash$

$$\vdash \{P\}\ c\ \{Q\} \Longrightarrow$$
$$\exists\, C.\ strip\ C = c \wedge vc\ C\ Q \wedge (\forall\, s.\ P\ s \longrightarrow pre\ C\ Q\ s)$$

Proof by rule induction. Needs two monotonicity lemmas:

$$[\![\forall\, s.\ P\ s \longrightarrow P'\ s;\ pre\ C\ P\ s]\!] \Longrightarrow pre\ C\ P'\ s$$

$$[\![\forall\, s.\ P\ s \longrightarrow P'\ s;\ vc\ C\ P]\!] \Longrightarrow vc\ C\ P'$$

4. Partial Correctness

5. Verification Conditions

6. Total Correctness

- Partial Correctness:
  if command terminates, postcondition holds
- Total Correctness:
  command terminates *and* postcondition holds

Total Correctness = Partial Correctness + Termination

Formally:

$$(\models_t \{P\}\ c\ \{Q\}) =$$
$$(\forall s.\ P\ s \longrightarrow (\exists t.\ (c,\ s) \Rightarrow t \wedge Q\ t))$$

Assumes that semantics is deterministic!

Exercise: Reformulate for nondeterministic language

$\vdash_t$: A proof system for total correctness

Only need to change the $WHILE$ rule.

Some measure function $state \Rightarrow nat$
must decrease with every loop iteration

$$\frac{\bigwedge n.\ \vdash_t \{\lambda s.\ P\ s \wedge bval\ b\ s \wedge n = f\ s\}\ c\ \{\lambda s.\ P\ s \wedge f\ s < n\}}{\vdash_t \{P\}\ WHILE\ b\ DO\ c\ \{\lambda s.\ P\ s \wedge \neg\ bval\ b\ s\}}$$

$WHILE$ rule can be generalized from a function to a relation:

$$\frac{\bigwedge n.\ \vdash_t \{\lambda s.\ P\ s \wedge bval\ b\ s \wedge T\ s\ n\}\ c\ \{\lambda s.\ P\ s \wedge (\exists\, n'{<}n.\ T\ s\ n')\}}{\vdash_t \{\lambda s.\ P\ s \wedge (\exists\, n.\ T\ s\ n)\}\ WHILE\ b\ DO\ c\ \{\lambda s.\ P\ s \wedge \neg\ bval\ b\ s\}}$$

Hoare_Total.thy

Example

Soundness

$$\vdash_t \{P\}\ c\ \{Q\} \Longrightarrow \models_t \{P\}\ c\ \{Q\}$$

Proof by induction, with a nested induction on n in the While-case.

Completeness

$$\models_t \{P\}\ c\ \{Q\} \Longrightarrow \vdash_t \{P\}\ c\ \{Q\}$$

Follows easily from

$$\vdash_t \{wp_t\ c\ Q\}\ c\ \{Q\}$$

where

$$wp_t\ c\ Q = (\lambda s.\ \exists\, t.\ (c,\ s) \Rightarrow\ t \wedge\ Q\ t).$$

Proof of $\vdash_t \{wp_t\ c\ Q\}\ c\ \{Q\}$ is by induction on c.
In the $WHILE\ b\ DO\ c$ case, use the $WHILE$ rule with

$$\frac{\neg\ bval\ b\ s}{T\ s\ 0} \qquad \frac{bval\ b\ s \qquad (c,\ s) \Rightarrow s' \qquad T\ s'\ n}{T\ s\ (n+1)}$$

$T\ s\ n$ means that $WHILE\ b\ DO\ c$ started in state s needs n iterations to terminate.