

Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:

A Simple Imperative Language

- ① IMP Commands
- ② Big-Step Semantics
- ③ Small-Step Semantics

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② Big-Step Semantics

③ Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$\begin{aligned} com &::= \text{SKIP} \\ &| \text{string} ::= aexp \\ &| com \ ; \ ; \ com \\ &| \text{IF } bexp \text{ THEN } com \text{ ELSE } com \\ &| \text{WHILE } bexp \text{ DO } com \end{aligned}$$

Commands

Abstract syntax:

datatype *com* = *SKIP*
| *Assign string aexp*
| *Seq com com*
| *If bexp com com*
| *While bexp com*

Com.thy

① IMP Commands

② Big-Step Semantics

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Big-step semantics

Concrete syntax:

$$(com, initial-state) \Rightarrow final-state$$

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Command c started in state s terminates in state t

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Command c started in state s terminates in state t

“ \Rightarrow ” here not type!

Big-step rules

$$(SKIP, s) \Rightarrow s$$

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$$(x ::= a, s) \Rightarrow s(x := \textit{aval } a \ s)$$

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$$(SKIP, s) \Rightarrow s$$

$$(x ::= a, s) \Rightarrow s(x := \text{aval } a \ s)$$

$$\frac{(c_1, s_1) \Rightarrow s_2 \quad (c_2, s_2) \Rightarrow s_3}{(c_1;; c_2, s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \quad (c_1, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \quad (c_1, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

$$\frac{\neg\ bval\ b\ s \quad (c_2, s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t}$$

Big-step rules

$$\frac{\neg \text{bval } b \ s}{(\text{WHILE } b \text{ DO } c, s) \Rightarrow s}$$

Big-step rules

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$$\frac{\begin{array}{c} \textit{bval } b \ s_1 \\ (c, s_1) \Rightarrow s_2 \end{array} \quad (\textit{WHILE } b \textit{ DO } c, s_2) \Rightarrow s_3}{(\textit{WHILE } b \textit{ DO } c, s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{("x" ::= N\ 5;;\ "y" ::= V\ "x",\ s) \Rightarrow ?}$$

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$$\frac{\vdots}{("x'' ::= N\ 5;; "y'' ::= V\ "x'',\ s) \Rightarrow ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow ?}$$

where

- $w = \textit{WHILE } b \textit{ DO } c$
- $b = \textit{NotEq } (V\ "x'')\ (N\ 2)$
- $c = "x'' ::= \textit{Plus } (V\ "x'')\ (N\ 1)$
- $s_i = s("x'' := i)$

Examples: derivation trees

$$\frac{\vdots}{("x'' ::= N\ 5;; "y'' ::= V\ "x'',\ s) \Rightarrow ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow ?}$$

where

$$\begin{aligned} w &= \textit{WHILE}\ b\ \textit{DO}\ c \\ b &= \textit{NotEq}\ (V\ "x'')\ (N\ 2) \\ c &= "x'' ::= \textit{Plus}\ (V\ "x'')\ (N\ 1) \\ s_i &= s("x'' := i) \end{aligned}$$

$$\begin{aligned} \textit{NotEq}\ a_1\ a_2 &= \\ \textit{Not}(\textit{And}\ (&\textit{Not}(\textit{Less}\ a_1\ a_2))\ (\textit{Not}(\textit{Less}\ a_2\ a_1)))) \end{aligned}$$

Logically speaking

$$(c, s) \Rightarrow t$$

is just infix syntax for

$$\textit{big_step} \ (c,s) \ t$$

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is just infix syntax for

$$big_step\ (c,s)\ t$$

where

$$big_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP, s) \Rightarrow t$?

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- $(SKIP, s) \Rightarrow t \text{ ?}$ $t = s$
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- $(c_1;; c_2, s_1) \Rightarrow s_3 \ ?$

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 $\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$

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 $\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$
- $(IF \ b \ THEN \ c_1 \ ELSE \ c_2, s) \Rightarrow t \text{ ?}$

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- $(x ::= a, s) \Rightarrow t \quad ? \quad t = s(x := \text{aval } a \ s)$
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 $\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$
- $(IF \ b \ THEN \ c_1 \ ELSE \ c_2, s) \Rightarrow t \quad ?$
 $\text{bval } b \ s \wedge (c_1, s) \Rightarrow t \ \vee$
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- $(w, s) \Rightarrow t \text{ where } w = WHILE \ b \ DO \ c \quad ?$

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- $(w, s) \Rightarrow t \text{ where } w = WHILE \ b \ DO \ c \quad ?$
 $\neg \ bval \ b \ s \wedge t = s \ \vee$
 $bval \ b \ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;; c_2, s_1) \Rightarrow s_3}{\exists s_2. (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{\bigwedge s_2. \llbracket (c_1, s_1) \Rightarrow s_2; (c_2, s_2) \Rightarrow s_3 \rrbracket \implies P}{P}$$

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$$\frac{\bigwedge s_2. [(c_1, s_1) \Rightarrow s_2; (c_2, s_2) \Rightarrow s_3] \Longrightarrow P}{P}$$

Replaces assem $(c_1;; c_2, s_1) \Rightarrow s_3$ by two assems
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No \exists and \wedge !

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Rightarrow P \quad \dots \quad asm_n \Rightarrow P}{P}$$

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Reading:

To prove a goal P with assumption asm ,
prove all $asm_i \Longrightarrow P$

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Reading:

To prove a goal P with assumption asm ,
prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \vee G \quad F \Longrightarrow P \quad G \Longrightarrow P}{P}$$

elim attribute

- Theorems with *elim* attribute are used automatically by *blast*, *fastforce* and *auto*

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- Can also be added locally, eg (*blast elim: ...*)
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \text{WHILE } b \text{ DO } c$

$w' = \text{IF } b \text{ THEN } c;; w \text{ ELSE SKIP}$

Equivalence proof

$$(w, s) \Rightarrow t$$

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$$(w, s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t)$$

$$\vee$$

$$\neg bval\ b\ s \wedge t = s$$

Equivalence proof

$$\begin{aligned} & (w, s) \Rightarrow t \\ & \longleftrightarrow \\ & bval\ b\ s \wedge (\exists s'. (c, s) \Rightarrow s' \wedge (w, s') \Rightarrow t) \\ & \quad \vee \\ & \neg bval\ b\ s \wedge t = s \\ & \longleftrightarrow \\ & (w', s) \Rightarrow t \end{aligned}$$

Equivalence proof

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Using the rules and rule inversions for \Rightarrow .

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c, s) \Rightarrow t \implies (c, s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t' .

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t ?$

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(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t$?

Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c, s) does not terminate iff $\nexists t. (c, s) \Rightarrow t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a \Rightarrow rule.

Big-step semantics cannot directly describe

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We need a finer grained semantics!

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Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Small-step semantics

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Intended meaning of $(c, s) \rightarrow (c', s')$:

Small-step semantics

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Intended meaning of $(c, s) \rightarrow (c', s')$:

The first step in the execution of c in state s leaves a “remainder” command c' to be executed in state s' .

Small-step semantics

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Intended meaning of $(c, s) \rightarrow (c', s')$:

The first step in the execution of c in state s leaves a “remainder” command c' to be executed in state s' .

Execution as finite or infinite reduction:

$$(c_1, s_1) \rightarrow (c_2, s_2) \rightarrow (c_3, s_3) \rightarrow \dots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \rightarrow cs'$ we say that cs *reduces* to cs' .
- A configuration cs is *final* iff $\nexists cs'. cs \rightarrow cs'$

The intention:

$(SKIP, s)$ is final

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Why?

SKIP is the empty program.

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Why?

SKIP is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a, s) \rightarrow$$

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Small-step rules

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$$(WHILE\ b\ DO\ c, s) \rightarrow (IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP, s)$$

Small-step rules

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$$(WHILE\ b\ DO\ c, s) \rightarrow (IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP, s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$("z'' ::= V "x'';; "x'' ::= V "y'';; "y'' ::= V "z'', s) \rightarrow$$

...

where $s = \langle "x'' := 3, "y'' := 7, "z'' := 5 \rangle$.

Small-step examples

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$$(w, s_0) \rightarrow \dots$$

where

$$\begin{aligned} w &= \text{WHILE } b \text{ DO } c \\ b &= \text{Less } (V "x'') (N 1) \\ c &= "x'' ::= \text{Plus } (V "x'') (N 1) \\ s_n &= \langle "x'' := n \rangle \end{aligned}$$

Small_Step.thy

Semantics

Are big and small-step semantics equivalent?

From \Rightarrow to \rightarrow^*

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Theorem $cs \Rightarrow t \implies cs \rightarrow^* (SKIP, t)$

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In two cases a lemma is needed:

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In two cases a lemma is needed:

Lemma

$$(c_1, s) \rightarrow^* (c_1', s') \implies (c_1;; c_2, s) \rightarrow^* (c_1';; c_2, s')$$

From \Rightarrow to \rightarrow^*

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Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

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Theorem $cs \rightarrow^* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow^* (SKIP, t)$.

In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \iff cs \rightarrow^* (SKIP, t)$

Small_Step.thy

Equivalence of big and small

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We prove the contrapositive

$$c \neq SKIP \implies \neg final(c, s)$$

by induction on c .

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 - $c_1 = SKIP \implies \neg final(c_1;; c_2, s)$

Can execution stop prematurely?

That is, are there any final configs except $(SKIP, s)$?

Lemma $final(c, s) \implies c = SKIP$

We prove the contrapositive

$$c \neq SKIP \implies \neg final(c, s)$$

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP, s) \rightarrow ct \implies False$

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Together:

Corollary $final(c, s) = (c = SKIP)$

Infinite executions

\Rightarrow yields final state iff \rightarrow terminates

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Equivalent:

\Rightarrow does not yield final state iff \rightarrow does not terminate

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Therefore: \Rightarrow correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \dots$

Chapter 8

Hoare Logic

④ Weakest Preconditions

⑤ Towards Simpler Verification of Programs

⑥ Example Verifications

④ Weakest Preconditions

⑤ Towards Simpler Verification of Programs

⑥ Example Verifications

④ Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

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program exp {  
  a := 1  
  while (0 < n) do {  
    a := a + a;  
    n := n - 1  
  }  
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and $0 \leq n!$

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The RHS of this implication is called *weakest precondition*

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Weakest condition on state, such that program c will satisfy postcondition Q .

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wp of equivalent programs is equal

$$c \sim c' \implies wp\ c = wp\ c'$$

Correctness of *exp*

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Reasoning along syntax of program!

That was easy!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall t. (c, s) \Rightarrow t \longrightarrow Q\ t$$

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Cannot reason about termination. This is called ***partial correctness***.

Some obvious facts:

$$c \sim c' \implies wlp\ c = wlp\ c'$$

$$\llbracket wlp\ c\ P\ s; \bigwedge s. P\ s \implies Q\ s \rrbracket \implies wlp\ c\ Q\ s$$

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Relation between wp and wlp

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Relation between *wp* and *wlp*

$$wp\ c\ Q\ s \implies wlp\ c\ Q\ s$$

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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$

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Let's try to find predicate I , such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\begin{aligned} & \llbracket I \ s_0; \bigwedge s. I \ s \implies \textit{if } b \textit{val } b \ s \textit{ then } wlp \ c \ I \ s \textit{ else } Q \ s \rrbracket \\ & \implies wlp \ (\textit{WHILE } b \ \textit{DO } c) \ Q \ s_0 \end{aligned}$$

Wp_Demo.thy

Weakest Precondition

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Otherwise, use unfold rules.

Iterate, until all *wlps* gone!

wlp_if_eq and *wlp_whileI'* produce *if-then-else*

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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$$\frac{wf\ r \quad \bigwedge x. \frac{\forall y. (y, x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$$\begin{aligned} & \llbracket wf\ R; I\ s_0; \\ & \bigwedge s. I\ s \implies \text{if } bval\ b\ s \text{ then } wp\ c\ (\lambda s'. I\ s' \wedge (s', s) \in \\ & R)\ s \text{ else } Q\ s \rrbracket \\ & \implies wp\ (WHILE\ b\ DO\ c)\ Q\ s_0 \end{aligned}$$

Or, equivalently

assumes $WF: wf\ R$

assumes $INIT: I\ s_0$

assumes $STEP: \bigwedge s. \llbracket I\ s; bval\ b\ s \rrbracket$
 $\implies wp\ c\ (\lambda s'. I\ s' \wedge (s', s) \in R)\ s$

assumes $FINAL: \bigwedge s. \llbracket I\ s; \neg bval\ b\ s \rrbracket \implies Q\ s$

shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

④ Weakest Preconditions

⑤ Towards Simpler Verification of Programs

⑥ Example Verifications

Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Add nice syntax for programs

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Simplify specification of pre/postcondition, and invariants

Standard operators

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For example:

$Cmpop (\leq) (Binop (+) (Unop uminus (V "x"))) (N 42)) (N 50)$

IMP2/Introduction.thy

Adding more Operators

C-like syntax

Operators

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Operators

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while ($bexp$) c

IMP2/Introduction.thy

Program Syntax

More Readable VCs

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If subgoal can still be proved for arbitrary (Isabelle) variable x , it can, in particular, be proved for $s \text{ ''}x\text{''}$.

$$(\bigwedge x. P \ x) \Longrightarrow P \ (s \text{ ''}x\text{''})$$

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More Readable VCs

More Readable Annotations

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Postcondition/Invariant: x as $s \text{ ''}x\text{''}$, x_0 as $s_0 \text{ ''}x\text{''}$

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More Readable Annotations

④ Weakest Preconditions

⑤ Towards Simpler Verification of Programs

⑥ Example Verifications

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

IMP2/Examples.thy

Count-up, Count-Down

Approximate Naively

Invert monotonic function, by naively trying all values:

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Applications: sqrt, log, ...

IMP2/Examples.thy

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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l=0; h=n+1;
while (l+1 < h)
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This program is actually tricky to get right!

IMP2/Examples.thy

Bisection

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Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers a, b

Reminder: Divides: $(b \text{ dvd } a) = (\exists k. a = b * k)$

Greatest Common Divisor: $gcd::int \Rightarrow int \Rightarrow int$ such that

$gcd\ a\ b\ \text{dvd}\ a$ and $gcd\ a\ b\ \text{dvd}\ b$ and

$\llbracket a \neq 0; b \neq 0; c\ \text{dvd}\ a; c\ \text{dvd}\ b \rrbracket \implies c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $\gcd(m - n, n) = \gcd(m, n)$

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By modulo. Using: $\gcd x \ y = \gcd y \ (x \bmod y)$

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Euclid

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Program modifies at most variables it assigns to

$(c, s) \Rightarrow t \implies \text{modifies } (\text{lhsv } c) \ t \ s$



Modified Variables

We can strengthen correctness statement (automatically)

$$wp\ c\ Q\ s \implies wp\ c\ (\lambda s'.\ Q\ s' \wedge \text{modifies}\ (lhsv\ c)\ s'\ s)\ s$$

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For while-rule, we get

lemma *wp_whileI_modset*:

fixes *c*

defines [*simp*]: *modset* \equiv *lhsv c*

assumes *WF*: *wf R*

assumes *INIT*: $I\ s_0$

assumes *STEP*: $\bigwedge s. \llbracket \text{modifies}\ modset\ s\ s_0; I\ s; bval\ b\ s \rrbracket$

$\implies wp\ c\ (\lambda s'. I\ s' \wedge (s', s) \in R)\ s$

assumes *FINAL*: $\bigwedge s. \llbracket \text{modifies}\ modset\ s\ s_0; I\ s; \neg bval\ b\ s \rrbracket$

$\implies Q\ s$

shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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program_spec computes *lhs*-variables:

$$HT_mods \ mods \ P \ c \ Q \equiv HT \ P \ c \ Q \wedge mods = lhsv \ c$$



IMP2/Examples.thy

Euclid – show modified sets

Modular Proofs

Consider program

```
a=1;  
while (m>0) {  
  n=a; a = 1;  
  while (n>0) {  
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}
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What does this compute

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Power-tower function: $2^{2^{\cdot^{\cdot^2}}}$ (m times)

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Idea: Split and verify separately!

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a=1;  
while (m>0) {  
    n=a;  
    inline exp_count_down;  
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Reuse existing proof of exp-count-down program!

Modular Proofs

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VCG will automatically use this rule.

If inlined program has been proved with  **program_spec**

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Power-Tower

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Every variable is of type $int \Rightarrow int$.

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$ArrayClear::char\ list \Rightarrow com$
 $(CLEAR\ x[], s) \Rightarrow s(x := \lambda_.\ 0)$



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Only with index 0: Bind $VAR\ (s\ "x"\ 0)\ (\lambda x. \dots)$

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IMP2/Examples.thy

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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$\forall i \in \{0..<42\}. a\ i > 0$

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Elements 0 to 41 are positive

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$\forall i \in \{0..<42\}. a\ i > 0$ means?

Elements 0 to 41 are positive

$\forall i \in \{l..<h\}. \forall j \in \{l..<h\}. i \leq j \longrightarrow a\ i \leq a\ j$

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory *IMP2/IMP2_Aux_Lemmas* provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Only 5 out of 20 surveyed textbooks had correct implementations

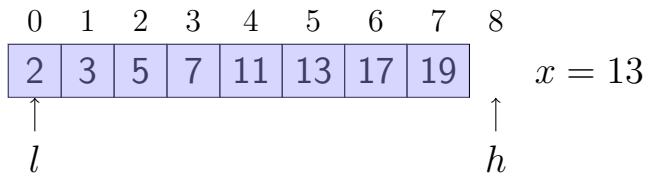
— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8	
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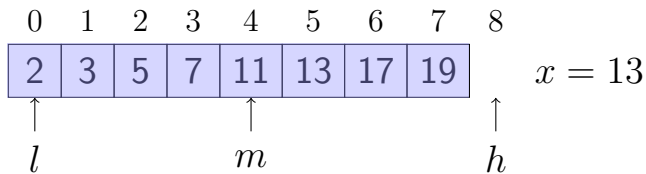
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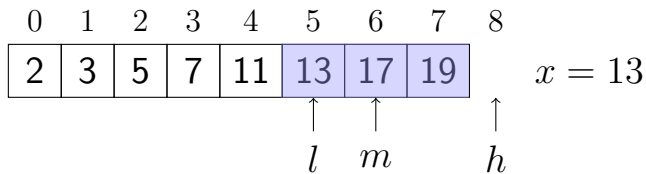
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2	3	5	7	11	13	17	19		$x = 13$
					\uparrow l		\uparrow h		

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while (l < h) {  
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```

Returns **smallest** i with $x \leq a[i]$

Notes on Binary Search

```
while (l < h) {  
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Notes on Binary Search

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Note: Our language has arbitrary large integers.

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Otherwise, $m = (l + h)/2$ may overflow!

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Bug in Java Standard Library for > 9 years!

Proving Binary Search

0	1	2	3	4	5	6	7	8	
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					\uparrow	\uparrow			
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Invariant:

Proving Binary Search

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- $i < l \implies a[i] < x$ (strictly smaller than x)

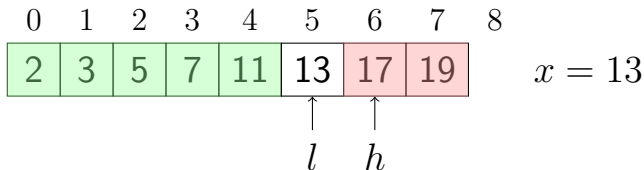
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Proving Binary Search



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- $i < l \implies a[i] < x$ (strictly smaller than x)
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- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

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j = l + 1;
while (j < h) {
    key = a[j];
    i = j - 1;
    while (i >= l && a[i] > key) {
        a[i + 1] = a[i];
        i = i - 1;
    };
    a[i + 1] = key;
    j = j + 1;
}
```



Idea: Build sorted array from start.

In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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$$mset_ran\ a\ \{l..<h\} = mset_ran\ a_0\ \{l..<h\}$$

 where

$$ran_sorted\ a\ l\ h \equiv \forall i \in \{l..<h\}. \forall j \in \{l..<h\}. i \leq j \longrightarrow a\ i \leq a\ j$$

$$mset_ran\ a\ r = (\sum_{i \in r}. \{\#a\ i\# \})$$



Multisets in Isabelle

imports *HOL–Library.Multiset*

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Multiset of elements at indexes in finite **set** r

Proving Insertion Sort

Separate proof for inner loop!

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    inline inner_loop;  
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Insert: Inner Loop

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Intuition: ?

$a[j]$ is moved backwards until

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Short: Move $a[j]$ backwards over greater elements.

Insert: Inner Loop

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Let's specify this intuition!



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Invariants easier to find!

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ensures $i \in \{l - (1::'a)..<j\}$

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 $a[i + (1::'b)] = key$ and



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Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a_0 j$

ensures $i \in \{l - (1::'a)..<j\}$

ensures $\forall k \in \{l..i\}. a k = a_0 k$ and

$a (i + (1::'b)) = key$ and

$\forall k \in \{i + (2::'a)..j\}. a k = a_0 (k - (1::'a))$



Insert: Inner Loop

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ensures $l \leq i \longrightarrow a\ i \leq key$ and



Insert: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, let $key = a_0 j$

ensures $i \in \{l - (1::'a) .. < j\}$

ensures $\forall k \in \{l..i\}. a k = a_0 k$ and
 $a (i + (1::'b)) = key$ and
 $\forall k \in \{i + (2::'a) .. j\}. a k = a_0 (k - (1::'a))$

ensures $l \leq i \longrightarrow a i \leq key$ and
 $\forall k \in \{i + (2::'a) .. j\}. key < a k$



Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
\uparrow						\uparrow	\uparrow
l						i	j



Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	11	19
\uparrow					\uparrow		\uparrow
l					i		j

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	11	17	19
\uparrow				\uparrow			\uparrow
l				i			j

Insert: Finding Invariant

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l			i				j

Insert: Finding Invariant

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Consider intermediate situation

Insert: Finding Invariant

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Consider intermediate situation

Insert: Finding Invariant

0	1	2	3	4	5	6	7
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Consider intermediate situation

- indexes $\leq i$ unchanged: $\forall k \in \{l..i\}. a_k = a_0[k]$

Insert: Finding Invariant

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\uparrow				\uparrow		\uparrow	
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Consider intermediate situation

- indexes $\leq i$ unchanged: $\forall k \in \{l..i\}. a[k] = a_0[k]$
- indexes $\geq i+2$ correctly shifted
 $\forall k \in \{i + (2::'a)..j\}. a[k] = a_0[k - (1::'a)]$

Insert: Finding Invariant

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Consider intermediate situation

- indexes $\leq i$ unchanged: $\forall k \in \{l..i\}. a[k] = a_0[k]$
- indexes $\geq i+2$ correctly shifted
 $\forall k \in \{i + (2::'a)..j\}. a[k] = a_0[k - (1::'a)]$
- and elements greater than key
 $\forall k \in \{i + (2::'a)..j\}. key < a[k]$

Insert: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	11	17	19
\uparrow				\uparrow			\uparrow
l				i			j

Consider intermediate situation

- indexes $\leq i$ unchanged: $\forall k \in \{l..i\}. a[k] = a_0[k]$
- indexes $\geq i+2$ correctly shifted
 $\forall k \in \{i + (2::'a)..j\}. a[k] = a_0[k - (1::'a)]$
- and elements greater than key
 $\forall k \in \{i + (2::'a)..j\}. key < a[k]$
- + the usual bounds: $l - (1::'a) \leq i \wedge i < j$



IMP2/Examples.thy

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Summary so Far

Understand what program does!

Split program into handy parts

Specify what parts do (independently of users)



Prove that this implies expectations of users



Prove parts separately and assemble to bigger parts

⑥ Example Verifications

Loop Patterns

Euclid's Algorithm

Advanced Verification

Arrays

Data Refinement

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a [l..<h]$ as $int\ list$

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Instead of one proof, get two ???

Abstract View

Model $int \Rightarrow int$ not always appropriate 

E.g., list: Understand $a[l..<h]$ as *int list*

Idea: Do proof at level of understanding **first**
then show that implementation is correct!

Instead of one **complex** proof, get two **simple** proofs !



IMP2/Examples.thy

Filter, Merge, dedup

