Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP: A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Semantics is easy.

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Study the book until you have understood it.

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Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$\begin{array}{rcl} com & ::= & \texttt{SKIP} \\ & | & string\ \texttt{::=}\ aexp \\ & | & com\ \texttt{;;}\ com \\ & | & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\ & | & \texttt{WHILE}\ bexp\ \texttt{DO}\ com \end{array}$$

Commands

Abstract syntax:

$$\begin{array}{rcl}\textbf{datatype}\ com & = & SKIP \\ & | & Assign\ string\ aexp \\ & | & Seq\ com\ com \\ & | & If\ bexp\ com\ com \\ & | & While\ bexp\ com\end{array}$$

`Com.thy`

Big-step semantics

Concrete syntax:

$$(com,\ initial\text{-}state) \Rightarrow final\text{-}state$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x\ ::=\ a,\ s) \Rightarrow\ s(x\ :=\ aval\ a\ s)$$

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$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s_1 \qquad (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

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where
$$\begin{aligned} w &= WHILE\ b\ DO\ c \\ b &= NotEq\ (V\ ''x'')\ (N\,2) \\ c &= ''x'' ::= Plus\ (V\ ''x'')\ (N\,1) \\ s_i &= s(''x'' := i) \end{aligned}$$

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$NotEq\ a_1\ a_2 =$
$Not(And\ (Not(Less\ a_1\ a_2))\ (Not(Less\ a_2\ a_1)))$

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is just infix syntax for

$$big\_step\ (c{,}s)\ t$$

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$$(c,\ s) \Rightarrow t$$

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$$big\_step\ (c{,}s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP, s) \Rightarrow t$?

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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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To prove a goal P with assumption asm,
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Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

$elim$ attribute

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- Can also be added locally, eg *(blast elim: . . .)*
- Variant: *elim!* applies elim-rules eagerly.

`Big_Step.thy`

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$
$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$(w',\ s) \Rightarrow t$$

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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

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Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

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Example problem:

$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

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We need a finer grained semantics!

Small-step semantics

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$$(com,state) \to (com,state)$$

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Intended meaning of $(c,\ s) \to (c',\ s')$:

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Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
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Execution as finite or infinite reduction:

$$(c_1, s_1) \to (c_2, s_2) \to (c_3, s_3) \to \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \to cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \to cs'$

The intention:

$$(SKIP,\ s) \ \ \text{is final}$$

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Why?

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Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= \ V \ ''x'';; \ ''x'' ::= \ V \ ''y'';; \ ''y'' ::= \ V \ ''z'', \ s) \rightarrow$$

$$\ldots$$

where $s = <''x'' := 3, \ ''y'' := 7, \ ''z'' := 5>$.

Small-step examples

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$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{array}{rcl} w & = & \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b & = & \mathit{Less}\ (V\ "x")\ (N\ 1) \\ c & = & "x" ::= \mathit{Plus}\ (V\ "x")\ (N\ 1) \\ s_n & = & <"x" := n> \end{array}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow*$

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Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

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Lemma
$(c_1,\ s) \rightarrow* (c_1',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1';;\ c_2,\ s')$

From $\Rightarrow$ to $\rightarrow*$

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Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \; (SKIP,\ t)$

`Small_Step.thy`

Equivalence of big and small

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We prove the contrapositive

$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c,s)$

by induction on c.

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- Case $c_1;;\ c_2$: by case distinction:

Can execution stop prematurely?

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We prove the contrapositive

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    $\Longrightarrow \neg\ \mathit{final}\ (c_1;;\ c_2,\ s)$
- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

By rule inversion: $(SKIP,\ s) \to ct \Longrightarrow False$

Together:

Corollary $\mathit{final}\ (c,\ s) = (c = SKIP)$

Infinite executions

$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

May versus Must

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With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

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Introduction

We have proved functional programs correct

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\mathit{program}\ \mathit{exp}\ \{ \\
\quad a := 1 \\
\quad \mathit{while}\ (0 < n)\ \mathit{do}\ \{ \\
\qquad a := a + a; \\
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and $0 \leq n$!

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then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

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wp of equivalent programs is equal

$$c \sim c' \Longrightarrow wp\ c = wp\ c'$$

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$$s\ ''n'' \le 0 \Longrightarrow wp\ exp\ (\lambda s'.\ s'\ ''a'' = 2^{\mathsf{nat}\ (s\ ''n'')})\ s$$

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$$=$$

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$$= \textsf{\textit{if}}\ \mathit{bval}\ b\ s\ \textsf{then}\ wp\ c_1\ Q\ s\ \textsf{else}\ wp\ c_2\ Q\ s$$

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Reasoning along syntax of program!

That was easy!

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Unfolding will continue forever!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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If c terminates on s, then new state satisfies Q

Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$
$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
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Lets try to find predicate I, such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

Wp_Demo.thy

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If $c = \ WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

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Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if\!-\!then\!-\!else$

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness