Concrete Semantics
with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

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2. Big-Step Semantics
3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Study the book until you have understood it.

Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$\begin{array}{rcl} com & ::= & \texttt{SKIP} \\ & | & string \; \texttt{::=} \; aexp \\ & | & com \; \texttt{;;} \; com \\ & | & \texttt{IF} \; bexp \; \texttt{THEN} \; com \; \texttt{ELSE} \; com \\ & | & \texttt{WHILE} \; bexp \; \texttt{DO} \; com \end{array}$$

Commands

Abstract syntax:

$$
\begin{array}{lcl}
\textbf{datatype}\ com & = & \mathit{SKIP} \\
 & | & \mathit{Assign}\ string\ aexp \\
 & | & \mathit{Seq}\ com\ com \\
 & | & \mathit{If}\ bexp\ com\ com \\
 & | & \mathit{While}\ bexp\ com
\end{array}
$$

Com.thy

1. IMP Commands
2. Big-Step Semantics
3. Small-Step Semantics

Big-step semantics

Concrete syntax:

$$(com,\ initial\text{-}state) \Rightarrow final\text{-}state$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x ::=\ a,\ s) \Rightarrow\ s(x :=\ aval\ a\ s)$$

Big-step rules

$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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Big-step rules

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$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

$$\frac{bval\ b\ s_1 \quad (c,\ s_1) \Rightarrow s_2 \quad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ \ ?}$$

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$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{array}{rcl} w & = & WHILE\ b\ DO\ c \\ b & = & NotEq\ (V\ ''x'')\ (N\ 2) \\ c & = & ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_i & = & s(''x'' := i) \end{array}$$

Examples: derivation trees

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$NotEq\ a_1\ a_2 =$
$Not(And\ (Not(Less\ a_1\ a_2))\ (Not(Less\ a_2\ a_1)))$

Logically speaking

$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

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is just infix syntax for

$$big\_step\ (c,s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP,\ s) \Rightarrow t$?

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  $\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3$

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  $\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3$
- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?

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- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?
  $bval\ b\ s \wedge (c_1,\ s) \Rightarrow t\ \vee$
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What can we deduce from

- $(SKIP, s) \Rightarrow t$? $t = s$
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- $(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t$?
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- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?
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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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To prove a goal P with assumption asm,
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Reading:

> To prove a goal P with assumption asm,
> prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \lor G \quad F \Longrightarrow P \quad G \Longrightarrow P}{P}$$

$elim$ attribute

- Theorems with $elim$ attribute are used automatically by $blast$, $fastforce$ and $auto$

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- Can also be added locally, eg *(blast elim: ...)*
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $\;w = \mathit{WHILE}\ b\ \mathit{DO}\ c$
$\qquad\; w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$\begin{gathered} bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t) \\ \vee \\ \neg\ bval\ b\ s \wedge t = s \end{gathered}$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
$$\vee$$
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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Equivalence proof

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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

`Big_Step.thy`

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

- nonterminating computations,

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We need a finer grained semantics!

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3. Small-Step Semantics

Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Small-step semantics

Concrete syntax:

$$(com,state) \to (com,state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

Small-step semantics

Concrete syntax:

$$(\mathit{com},\mathit{state}) \to (\mathit{com},\mathit{state})$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s leaves a "remainder" command c' to be executed in state s'.*

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
> *to be executed in state s'.*

Execution as finite or infinite reduction:

$$(c_1, s_1) \to (c_2, s_2) \to (c_3, s_3) \to \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \to cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \to cs'$

The intention:

$$(SKIP,\ s) \text{ is final}$$

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Why?

$SKIP$ is the empty program.

The intention:

$$(SKIP,\ s)\ \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$$(x{::=}a,\ s) \ \rightarrow$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

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$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;; c_2, s) \ \rightarrow}$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;; c_2, s) \ \rightarrow\ (c_1';; c_2, s')}$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s)\ \rightarrow}$$

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Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \ \rightarrow \ (c_1, s)}$$

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow$$

$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

Small-step examples

$$(''z'' ::= V\ ''x'';; ''x'' ::= V\ ''y'';; ''y'' ::= V\ ''z'',\ s) \rightarrow$$
$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{aligned} w &= \ WHILE\ b\ DO\ c \\ b &= \ Less\ (V\ ''x'')\ (N\ 1) \\ c &= \ ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_n &= \ <''x'' := n> \end{aligned}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow^*$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction

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Proof by rule induction (of course on $cs \Rightarrow t$)

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In two cases a lemma is needed:

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1{}',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1{}';;\ c_2,\ s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1{'},\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1{'};;\ c_2,\ s')$

Proof by rule induction.

From $\rightarrow*$ to $\Rightarrow$

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In the induction step a lemma is needed:

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Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \;(SKIP,\; t)$

Small_Step.thy

Equivalence of big and small

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That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$

by induction on c.

Can execution stop prematurely?

That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

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$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \to ct \Longrightarrow False$

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Together:

Corollary $\mathit{final}\ (c,\ s) = (c = SKIP)$

Infinite executions

$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

May versus Must

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

4. Weakest Preconditions

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6. Example Verifications

4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\textit{program exp} \; \{ \\
\quad a := 1 \\
\quad \textit{while } (0 < n) \textit{ do } \{ \\
\qquad a := a + a; \\
\qquad n := n - 1 \\
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\end{array}
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$$

At the end of the execution, variable a should contain 2^n, where n is the original value of variable n!
and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

Some obvious facts:

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Consequence rule:

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wp of equivalent programs is equal

$$c \sim c' \Longrightarrow wp\ c = wp\ c'$$

Correctness of exp

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$$= \textsf{if}\ bval\ b\ s\ \textsf{then}\ wp\ c_1\ Q\ s\ \textsf{else}\ wp\ c_2\ Q\ s$$

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Reasoning along syntax of program!

That was easy!

That was easy! But what about While?

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$$=$$

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$=$*if* $bval\ b\ s$ *then* $wp\ c\ (wp\ (WHILE\ b\ DO\ c)\ Q)\ s$ *else*
$Q\ s$

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Unfolding will continue forever!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$

$$\llbracket wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s \rrbracket \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$
$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c_1\ Q\ s\ \textsf{\textit{else}}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate I, such that

$\bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s$

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and I holds for start state.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

`Wp_Demo.thy`

Weakest Precondition

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If $c =\ WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ **then** $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ \llbracket\ I\ s;\ bval\ b\ s\ \rrbracket$
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assumes $FINAL$: $\bigwedge s.\ \llbracket\ I\ s;\ \neg bval\ b\ s\ \rrbracket \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Add nice syntax for programs

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Make VCs more readable

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Make VCs more readable

Simplify specification of pre/postcondition, and invariants

Standard operators

We add generic syntax for any unary/binary operator

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N$
$42))\ (N\ 50)$

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Adding more Operators

C-like syntax

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Arith: $+,-,*,/$ with usual binding

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$while\ (bexp)\ c$

`IMP2/Introduction.thy`

Program Syntax

More Readable VCs

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If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$$

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More Readable VCs

More Readable Annotations

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Precondition: x interpreted as $s\ ''x''$

Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

IMP2/Introduction.thy

More Readable Annotations

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

IMP2/Examples.thy

Count-up, Count-Down

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Applications: sqrt, log, ...

`IMP2/Examples.thy`

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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l=0; h=n+1;
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  m = (l + h) / 2;
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This program is actually tricky to get right!

IMP2/Examples.thy

Bisection

6 Example Verifications

Loop Patterns

Euclid's Algorithm

Advanced Verification

Arrays

Data Refinement

Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers a, b

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

`IMP2/Examples.thy`

Euclid

6 Example Verifications

Modified Variables

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Program modifies at most variables it assigns to

$(c,\ s) \Rightarrow t \Longrightarrow \mathit{modifies}\ (\mathit{lhsv}\ c)\ t\ s$

Modified Variables

We can strengthen correctness statement (automatically)

$$wp\ c\ Q\ s \Longrightarrow wp\ c\ (\lambda s'.\ Q\ s' \wedge modifies\ (lhsv\ c)\ s'\ s)\ s$$

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For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}) \in R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ ]\!]$
$\Longrightarrow Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

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The VCG will automatically rewrite with rule

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program_spec computes lhs-variables:

$$HT\_mods\ mods\ P\ c\ Q \equiv HT\ P\ c\ Q \wedge mods = lhsv\ c$$

`IMP2/Examples.thy`

Euclid – show modified sets

Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1; \\
\mathit{while}\ (m{>}0)\ \{ \\
\quad n{=}a;\ a = 1; \\
\quad \mathit{while}\ (n{>}0)\ \{ \\
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What does this compute?

Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

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Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

```
a=1;
while (m>0) {
  n=a;
  inline exp_count_down;
  m=m-1
}
```

Modular Proofs

$a{=}1;$
$while\ (m{>}0)\ \{$
$\quad n{=}a;$
$\quad inline\ exp\_count\_down;$
$\quad m{=}m{-}1$
$\}$

Reuse existing proof of exp-count-down program!

Modular Proofs

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If inlined program has been proved with **program_spec**

`IMP2/Examples.thy`

Power-Tower

6 Example Verifications

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$ArrayClear :: char\ list \Rightarrow com$

$(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

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VCG: Guess type from variable usage

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VCG: Guess type from variable usage
Only with index 0: Bind $VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \ldots)$

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IMP2/Examples.thy

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

`IMP2/Examples.thy`

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Idea: Halve interval in each step.

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Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

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Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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    if (a[m] < x) l = m + 1
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```

Returns smallest i with $x \le a[i]$

Notes on Binary Search

```
while (l < h) {
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Bug in Java Standard Library for > 9 years!

Proving Binary Search

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					l	h		

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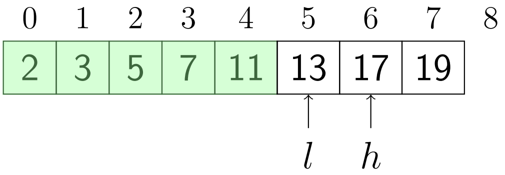
Invariant:

Proving Binary Search



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- $i < l \implies a[i] < x$ (strictly smaller than x)

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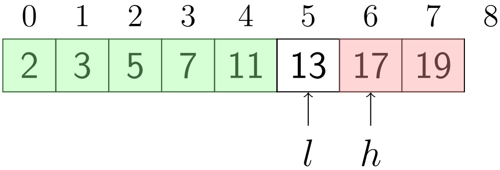

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Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)
- $i \geq h \implies x \leq a[i]$ (greater or equal to x)
- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.

In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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Postcondition:

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where

$ran\_sorted\ a\ l\ h \equiv \forall\, i \in \{l..<h\}.\ \forall\, j \in \{l..<h\}.\ i \leq j \longrightarrow a\ i \leq a\ j$

$mset\_ran\ a\ r = (\sum i \in r.\ \{\#a\ i\#\})$

Multisets in Isabelle

imports $HOL{-}Library.Multiset$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multiset of elements at indexes in finite set r

Proving Insertion Sort

Separate proof for inner loop!

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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$

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Insort: Inner Loop

```
key = a[j];
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while (i>=l && a[i]>key) {
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Insort: Inner Loop

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$a[j]$ is moved backwards

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Intuition: ?

$a[j]$ is moved backwards until

Insort: Inner Loop

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```

Intuition: ?
$a[j]$ is moved backwards until
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Intuition: ?

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Short: Move $a[j]$ backwards over greater elements.

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Invariants easier to find!

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Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

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$\forall\, k{\in}\{i + (2{::}'a)..j\}.\ a\ k = a_0\ (k - (1{::}'a))$

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ensures $l \le i \longrightarrow a\ i \le key$ and
$\forall\, k{\in}\{i + (2{::}'a)..j\}.\ key < a\ k$

Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
l						i	j

Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	11	19
l					i		j

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$\uparrow$				$\uparrow$			$\uparrow$
l				i			j

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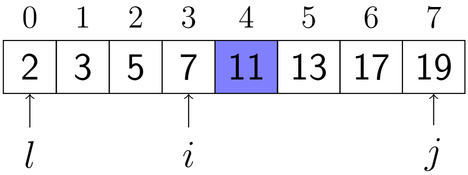
2	3	5	7	11	13	17	19
$\uparrow$ l			$\uparrow$ i				$\uparrow$ j

Insort: Finding Invariant



Consider intermediate situation

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- indexes $\leq i$ unchanged: $\forall\, k \in \{l..i\}.\ a\ k = a_0\ k$

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  $\forall\, k \in \{i + (2{::}'a)..j\}.\ a\ k = a_0\ (k - (1{::}'a))$

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  $\forall\, k \in \{i + (2::'a)..j\}.\ key < a\ k$
- $+$ the usual bounds: $l - (1::'a) \leq i \wedge i < j$

IMP2/Examples.thy

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Prove that this implies expectations of users

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Split program into handy parts

Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a\ [l..<h]$ as $int\ list$

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Instead of one proof, get two

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Instead of one complex proof, get two simple proofs !

IMP2/Examples.thy

Filter, Merge, dedup