

Concrete Semantics

with Isabelle/HOL

Peter Lammich

(slides from Concrete Semantics by Nipkow)

2018-10-16

Chapter 1

Introduction

① Background

② This Course

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Organization Issues

Course Homepage: `http:`

`//www21.in.tum.de/teaching/semantik/WS1819/`

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Homework: IMPORTANT! 40% of final grade

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Tutorials and Homework are the heart and soul of this course!

Why Semantics?

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Like the state of mathematics in the 19th century
— before set theory and logic entered the scene.

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- This course is about “beyond intuition”.

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What does the correctness of a type checker even mean?

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Example:

What does the correctness of a type checker even mean?
How is it proved?

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- A compiler gives each individual program a semantics.
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- Because compilers are far too complicated.
- They provide the worst possible semantics.
- Moreover: compilers may differ!

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- If they do, it will be informal (English).

Bugs

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GI Dissertationspreis 2003:
Gerwin Klein: *Verified Java Bytecode Verification*

Standard ML (SML)

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The Definition of Standard ML. 1990.

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Main achievements: LCF (theorem proving)
SML (functional programming)
CCS, π (concurrency)

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SML semantics hardly used:

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- too much detail to allow reliable informal proof
- not processable beyond \LaTeX , not even executable

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- Real programming languages *are* complex.
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- Complex designs are error-prone.
- Informal mathematical proofs of complex designs are also error-prone.

The solution

Machine-checked language semantics and proofs

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The tool:

Proof Assistant (PA)

or

Interactive Theorem Prover (ITP)

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Undermines your naive trust in informal proofs

Terminology

This lecture course:

Formal = machine-checked

Verification = formal correctness proof

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Verification = formal correctness proof

Traditionally:

Formal = mathematical

Two landmark verifications

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using Coq

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Operating system
microkernel (L4)

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Gerwin Klein (& Co)
NICTA Sydney
using Isabelle

A happy fact of life

Programming language researchers
are increasingly using PAs

Why verification pays off

Short term: *The software works!*

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Software Never Dies

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- Comparison of PLs or PL paradigms

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- Hot or trendy PLs
- Comparison of PLs or PL paradigms
- Compilers (although they will be one application)

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- Techniques for the description and analysis of
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 - PL tools
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Both informally and formally (PA!)

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All exercises require the use of Isabelle/HOL

Why I am so passionate
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- It is the future
- It is the only way to deal with complex languages
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- I want students to learn how to write correct proofs
- I have seen too many proofs that look more like
LSD trips than coherent mathematical arguments

Overview of course

- Introduction to Isabelle/HOL

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- IMP (assignment and while loops) and its semantics

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- Introduction to Isabelle/HOL
- IMP (assignment and while loops) and its semantics
- A compiler for IMP
- Hoare logic for IMP
- Type systems for IMP
- Program analysis for IMP

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A growing number of universities offer related course

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It has applications in compilers, security,
software engineering etc.

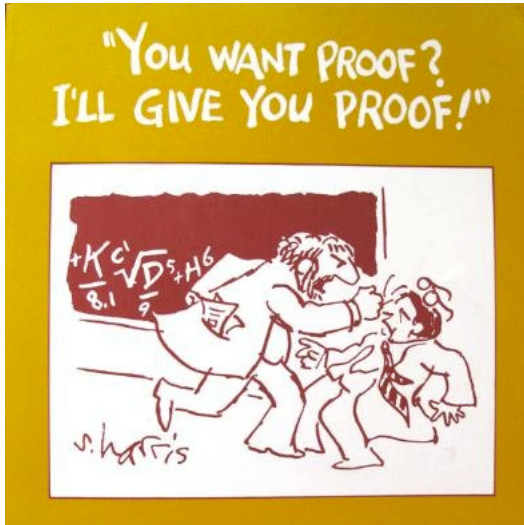
What you learn in this course goes far beyond PLs

It has applications in compilers, security,
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It is a new approach to informatics

At the end of the course . . .

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Part I

Isabelle

Chapter 2

Programming and Proving

- ③ Overview of Isabelle/HOL
- ④ Type and function definitions
- ⑤ Induction Heuristics
- ⑥ Simplification

Quiz

Which of the following formulas have the same meaning?

① $A \implies (B \implies C)$

② $(A \implies B) \implies C$

③ $(A \wedge B) \implies C$

Notation

Implication associates to the right:

$$A \implies B \implies C \quad \text{means} \quad A \implies (B \implies C)$$

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$$\frac{A_1 \quad \dots \quad A_n}{B} \quad \text{means} \quad A_1 \implies \dots \implies A_n \implies B$$

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HOL Formulas:

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- Later: $\wedge, \vee, \longrightarrow, \forall, \dots$

③ Overview of Isabelle/HOL

Types and terms

Interface

By example: types *bool*, *nat* and *list*

Summary

Types

Basic syntax:

$$\tau ::=$$

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$$\tau ::= (\tau)$$

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	$\tau \Rightarrow \tau$	functions

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Convention: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \equiv \tau_1 \Rightarrow (\tau_2 \Rightarrow \tau_3)$

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This language of terms is known as the *λ -calculus*.

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- Isabelle performs β -reduction automatically.

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User can help with *type annotations* inside the term.

Example: $f(x::nat)$

Currying

Thou shalt Curry your functions

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

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- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

Advantage:

Currying allows *partial application*
 $f\ a_1$ where $a_1 :: \tau_1$

Predefined syntactic sugar

- *Infix*: $+$, $-$, $*$, $\#$, $@$, \dots

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Enclose *if* and *case* in parentheses:

$$! \quad (if \, _ \, then \, _ \, else \, _) \quad !$$

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Usually: `imports` Main

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Types, terms and formulas need to be inclosed in "

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③ Overview of Isabelle/HOL

Types and terms

Interface

By example: types *bool*, *nat* and *list*

Summary

isabelle jedit

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- Based on *jEdit* editor

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Overview_Demo.thy

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if-and-only-if: =

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You need type annotations: 1 :: *nat*, *x* + (*y*::*nat*)

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datatype *nat* = 0 | *Suc nat*

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! Numbers and arithmetic operations are overloaded:

$0, 1, 2, \dots :: 'a, \quad + :: 'a \Rightarrow 'a \Rightarrow 'a$

You need type annotations: $1 :: \textit{nat}, x + (y :: \textit{nat})$
unless the context is unambiguous: *Suc* *z*

Nat_Demo.thy

An informal proof

Lemma $\text{add } m \ 0 = m$

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Proof by induction on m .

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 $add\ 0\ 0 = 0$ holds by definition of add .

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We assume $add\ m\ 0 = m$,

the induction hypothesis (IH).

We need to show $add\ (Suc\ m)\ 0 = Suc\ m$.

The proof is as follows:

$$\begin{aligned} add\ (Suc\ m)\ 0 &= Suc\ (add\ m\ 0) && \text{by def. of } add \\ &= Suc\ m && \text{by IH} \end{aligned}$$

Type *'a list*

Lists of elements of type *'a*

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Syntactic sugar:

- `[]` = *Nil*: empty list

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Syntactic sugar:

- $[] = Nil$: empty list
- $x \# xs = Cons\ x\ xs$:
list with first element x (“head”) and rest xs (“tail”)
- $[x_1, \dots, x_n] = x_1 \# \dots \# x_n \# []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs , prove

- $P([])$ and
- for arbitrary but fixed x and xs ,
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- for arbitrary but fixed x and xs ,
 $P(xs)$ implies $P(x\#xs)$.

$$\frac{P([]) \quad \bigwedge x \, xs. P(xs) \implies P(x\#xs)}{P(xs)}$$

List_Demo.thy

An informal proof

Lemma $app (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$

Proof by induction on xs .

- Case *Nil*: $app (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of *app*.
- Case *Cons* $x\ xs$: We assume $app (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show $app (app (Cons\ x\ xs)\ ys)\ zs = app (Cons\ x\ xs)\ (app\ ys\ zs)$.

The proof is as follows:

$$\begin{aligned} & app (app (Cons\ x\ xs)\ ys)\ zs \\ &= Cons\ x\ (app (app\ xs\ ys)\ zs) && \text{by definition of } app \\ &= Cons\ x\ (app\ xs\ (app\ ys\ zs)) && \text{by IH} \\ &= app (Cons\ x\ xs)\ (app\ ys\ zs) && \text{by definition of } app \end{aligned}$$

Large library: HOL/List.thy

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Predefined: *xs* @ *ys* (append), *length*, and *map*

③ Overview of Isabelle/HOL

Types and terms

Interface

By example: types *bool*, *nat* and *list*

Summary

- **datatype** defines (possibly) recursive data types.
- **fun** defines (possibly) recursive functions by pattern-matching over datatype constructors.

Proof methods

- *induction* performs structural induction on some variable (if the type of the variable is a datatype).

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- *auto* solves as many subgoals as it can, mainly by simplification (symbolic evaluation):

“=” is used only from left to right!

Proofs

General schema:

```
lemma name: "..."  
apply (...)  
apply (...)  
:  
done
```

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```
lemma name: "..."  
apply (...)  
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done
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If the lemma is suitable as a simplification rule:

```
lemma name[simp]:  "..."
```

Top down proofs

Command

sorry

“completes” any proof.

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Allows top down development:

Assume lemma first, prove it later.

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$$1. \bigwedge x_1 \dots x_p. A \implies B$$

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A local assumption(s)

B actual (sub)goal

Multiple assumptions

$$\llbracket A_1; \dots ; A_n \rrbracket \Longrightarrow B$$

abbreviates

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$;$ \approx “and”

- ③ Overview of Isabelle/HOL
- ④ Type and function definitions
- ⑤ Induction Heuristics
- ⑥ Simplification

④ Type and function definitions

Type definitions

Function definitions

Type synonyms

type_synonym *name* = τ

Introduces a *synonym name* for type τ

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Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$\begin{array}{lcl} \mathbf{datatype} \ (\alpha_1, \dots, \alpha_n)t & = & C_1 \ \tau_{1,1} \dots \tau_{1,n_1} \\ & & | \quad \dots \\ & & | \quad C_k \ \tau_{k,1} \dots \tau_{k,n_k} \end{array}$$

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datatype — the general case

$$\text{datatype } (\alpha_1, \dots, \alpha_n)t = \begin{array}{l} C_1 \tau_{1,1} \dots \tau_{1,n_1} \\ \vdots \\ C_k \tau_{k,1} \dots \tau_{k,n_k} \end{array}$$

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Distinctness and injectivity are applied automatically
Induction must be applied explicitly

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Datatype values can be taken apart with *case*:

(case xs of [] \Rightarrow ... | y#ys \Rightarrow ... y ... ys ...)

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Complicated patterns mean complicated proofs!

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Need () in context



Tree_Demo.thy

The *option* type

datatype 'a *option* = *None* | *Some* 'a

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If 'a has values a_1, a_2, \dots

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Typical application:

fun *lookup* :: (*'a* \times *'b*) *list* \Rightarrow *'a* \Rightarrow *'b option* **where**

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lookup [] *x* = *None* |
lookup ((*a*, *b*) # *ps*) *x* =
 (*if* *a* = *x* *then Some b* *else lookup ps x*)

④ Type and function definitions

Type definitions

Function definitions

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n = n*n$

Non-recursive definitions

Example



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No pattern matching, just $f\ x_1 \dots x_n = \dots$

The danger of nontermination

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Subtract $f\ x$ on both sides.

$$\implies 0 = 1$$

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! All functions in HOL must be total !



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- Pattern-matching over datatype constructors
- Order of equations matters
- Termination must be provable automatically by size measures
- Proves customized induction schema

Example: separation

fun *sep* :: 'a \Rightarrow 'a list \Rightarrow 'a list **where**
sep a (*x* # *y* # *zs*) = *x* # a # *sep* a (*y* # *zs*) |
sep a *xs* = *xs*



Example: Ackermann

fun *ack* :: *nat* \Rightarrow *nat* \Rightarrow *nat* **where**

ack 0 *n* = *Suc* *n* |

ack (*Suc* *m*) 0 = *ack* *m* (*Suc* 0) |

ack (*Suc* *m*) (*Suc* *n*) = *ack* *m* (*ack* (*Suc* *m*) *n*)

Example: Ackermann

```
fun ack :: nat  $\Rightarrow$  nat  $\Rightarrow$  nat where  
ack 0          n          = Suc n |  
ack (Suc m) 0          = ack m (Suc 0) |  
ack (Suc m) (Suc n) = ack m (ack (Suc m) n)
```

Terminates because the arguments decrease
lexicographically with each recursive call:

- $(\text{Suc } m, 0) > (m, \text{Suc } 0)$
- $(\text{Suc } m, \text{Suc } n) > (\text{Suc } m, n)$
- $(\text{Suc } m, \text{Suc } n) > (m, -)$

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The essence of primitive recursion:

$$\begin{aligned} f(0) &= \dots && \text{no recursion} \\ f(\text{Suc } n) &= \dots f(n) \dots \end{aligned}$$

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The essence of primitive recursion:

$$f(0) = \dots \quad \text{no recursion}$$

$$f(\text{Suc } n) = \dots f(n) \dots$$

$$g([]) = \dots \quad \text{no recursion}$$

$$g(x\#xs) = \dots g(xs) \dots$$

- ③ Overview of Isabelle/HOL
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Basic induction heuristics

Theorems about recursive functions
are proved by induction

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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun *rev* :: 'a list \Rightarrow 'a list **where**

rev [] = [] |

rev (x#xs) = *rev* xs @ [x]

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A tail recursive version:

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 rev (x#xs) = *rev* xs @ [x]

A tail recursive version:

fun *itrev* :: 'a list \Rightarrow 'a list \Rightarrow 'a list **where**
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lemma *itrev* xs [] = *rev* xs

Induction_Demo.thy

Generalisation

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- Replace constants by variables

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- Replace constants by variables
- Generalize free variables
 - by *arbitrary* in induction proof
 - (or by universal quantifier in formula)

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In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction

Example

fun *div2* :: *nat* \Rightarrow *nat* **where**

div2 0 = 0 |

div2 (*Suc* 0) = 0 |

div2 (*Suc*(*Suc* *n*)) = *Suc*(*div2* *n*)

Computation Induction

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\rightsquigarrow induction rule *div2.induct*:

$$\frac{P(0) \quad P(\text{Suc } 0) \quad P(n) \Longrightarrow P(\text{Suc}(\text{Suc } n))}{P(m)}$$

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Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule *f.induct*

How to apply $f.induct$

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(*induction* $a_1 \dots a_n$ *rule: f.induct*)

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Heuristic:

- there should be a call $f\ a_1 \dots a_n$ in your goal
- ideally the a_i should be variables.

Induction_Demo.thy

Computation Induction

- ③ Overview of Isabelle/HOL
- ④ Type and function definitions
- ⑤ Induction Heuristics
- ⑥ Simplification

Simplification means ...

Using equations $l = r$ from left to right

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Terminology: equation \rightsquigarrow *simplification rule*

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Simplification = (Term) Rewriting

An example

Equations:

$$\begin{aligned} 0 + n &= n & (1) \\ (Suc\ m) + n &= Suc\ (m + n) & (2) \\ (Suc\ m \leq Suc\ n) &= (m \leq n) & (3) \\ (0 \leq m) &= True & (4) \end{aligned}$$

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$$0 + Suc\ 0 \leq Suc\ 0 + x$$

Rewriting:

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Simplification rules can be conditional:

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Example

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Example

$$\begin{array}{lcl} p(0) & = & \text{True} \\ p(x) \Longrightarrow f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but
we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.

Isabelle uses *simp*-rules (almost) blindly from left to right.

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$$n < m \Longrightarrow (n < \text{Suc } m) = \text{True} \quad \text{YES}$$

$$\text{Suc } n < m \Longrightarrow (n < m) = \text{True} \quad \text{NO}$$

Proof method *simp*

Goal: 1. $\llbracket P_1; \dots; P_m \rrbracket \Longrightarrow C$

apply(*simp add: eq₁ ... eq_n*)

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Variations:

- (*simp ... del: ...*) removes *simp*-lemmas
- *add* and *del* are optional

auto versus *simp*

- *auto* acts on all subgoals
- *simp* acts only on subgoal 1

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- *auto* applies *simp* and more
- *auto* can also be modified:
(*auto simp add: ... simp del: ...*)

Rewriting with definitions

Definitions (**definition**) must be used **explicitly**:

(simp add: f_def ...)

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f is the function whose definition is to be unfolded.

Case splitting with *simp/auto*

Automatic:

$$\begin{aligned} &P \text{ (if } A \text{ then } s \text{ else } t) \\ &= \\ &(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t)) \end{aligned}$$

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By hand:

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Proof method: (*simp split: nat.split*)

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Proof method: (*simp split: nat.split*)

Or *auto*. Similar for any datatype *t*: *t.split*

Simp_Demo.thy

Chapter 3

Case Study: IMP Expressions

7 Case Study: IMP Expressions

⑦ Case Study: IMP Expressions

This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

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of our imperative language IMP.

IMP *commands* are introduced later.

⑦ Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Stack Machine and Compilation

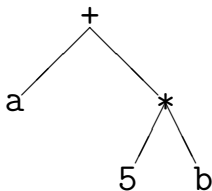
Concrete and abstract syntax

Concrete syntax: strings, eg "a+5*b"

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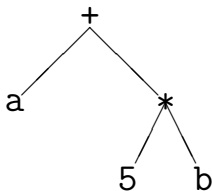
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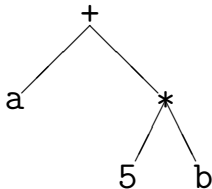


Parser: function from strings to trees

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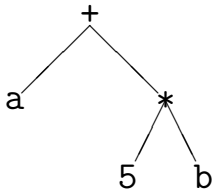
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Linear view of trees: terms, eg *Plus a (Times 5 b)*

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Parser: function from strings to trees

Linear view of trees: terms, eg *Plus a (Times 5 b)*

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \dots$$

where n can be any natural number and x any variable.

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype *aexp*

Variable names are strings, values are integers:

type_synonym *vname* = *string*

datatype *aexp* = *N int* | *V vname* | *Plus aexp aexp*

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5	<i>N 5</i>
x	<i>V "x"</i>
x+y	<i>Plus (V "x") (V "y")</i>
2+(z+3)	<i>Plus (N 2) (Plus (V "z") (N 3))</i>

Warning

This is syntax, not (yet) semantics!

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$$N\ 0 \neq Plus\ (N\ 0)\ (N\ 0)$$

The (program) state

What is the value of $x+1$?

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- The value of an expression depends on the value of its variables.
- The value of all variables is recorded in the *state*.
- The state is a function from variable names to values:

type_synonym *val* = *int*

type_synonym *state* = *vname* \Rightarrow *val*

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

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$$f(a := b)$$

is the function that behaves like f
except that it returns b for argument a .

$$f(a := b) = (\lambda x. \text{if } x = a \text{ then } b \text{ else } f\ x)$$

How to write down a state

Some states:

- $\lambda x. 0$

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Nicer notation:

$$<\text{"a"} := 5, \text{"x"} := 3, \text{"y"} := 7>$$

How to write down a state

Some states:

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- $((\lambda x. 0) ("a" := 5)) ("x" := 3)$

Nicer notation:

$$<"a" := 5, "x" := 3, "y" := 7>$$

Maps everything to 0, but $"a"$ to 5, $"x"$ to 3, etc.

AExp.thy

⑦ Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Stack Machine and Compilation

BExp.thy

⑦ Case Study: IMP Expressions

Arithmetic Expressions

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ASM.thy

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Because evaluation of expressions always terminates.

But execution of programs may *not* terminate.

Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 4

Logic and Proof Beyond Equality

⑧ Logical Formulas

⑨ Proof Automation

⑩ Single Step Proofs

⑪ Inductive Definitions

⑧ Logical Formulas

⑨ Proof Automation

⑩ Single Step Proofs

⑪ Inductive Definitions

Syntax (in decreasing precedence):

$$\begin{array}{lcl} \textit{form} & ::= & (\textit{form}) \quad | \quad \textit{term} = \textit{term} \quad | \quad \neg \textit{form} \\ & | & \textit{form} \wedge \textit{form} \quad | \quad \textit{form} \vee \textit{form} \quad | \quad \textit{form} \longrightarrow \textit{form} \\ & | & \forall x. \textit{form} \quad | \quad \exists x. \textit{form} \end{array}$$

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Examples:

$$\neg A \wedge B \vee C \equiv ((\neg A) \wedge B) \vee C$$

Syntax (in decreasing precedence):

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Examples:

$$\begin{aligned} \neg A \wedge B \vee C &\equiv ((\neg A) \wedge B) \vee C \\ s = t \wedge C &\equiv (s = t) \wedge C \end{aligned}$$

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Input syntax: \longleftrightarrow (same precedence as \longrightarrow)

Variable binding convention:

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Similarly for \exists and λ .

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x. Q x \rightsquigarrow P \wedge (\forall x. Q x) \quad !$$

Mathematical symbols

... and their ascii representations:

\forall	<code>\<forall></code>	ALL
\exists	<code>\<exists></code>	EX
λ	<code>\<lambda></code>	%
\longrightarrow	<code>--></code>	
\longleftrightarrow	<code><-></code>	
\wedge	<code>/\</code>	&
\vee	<code>\/</code>	
\neg	<code>\<not></code>	~
\neq	<code>\<noteq></code>	~=

Sets over type $'a$

$'a$ set

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- $\{\}, \quad \{e_1, \dots, e_n\}$

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\in	<code>\<in></code>	:
\subseteq	<code>\<subseteq></code>	<code><=</code>
\cup	<code>\<union></code>	<code>Un</code>
\cap	<code>\<inter></code>	<code>Int</code>

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- $\{x. P\}$ where x is a variable
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- Instead: $\{t \mid x \ y \ z. P\}$
is short for $\{v. \exists x \ y \ z. v = t \wedge P\}$
where x, y, z are the free variables in t

⑧ Logical Formulas

⑨ Proof Automation

⑩ Single Step Proofs

⑪ Inductive Definitions

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

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- Show you where they got stuck

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Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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blast

- A **complete** proof search procedure for FOL ...

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- complete for first-order theory of *nat* and *int* (Presburger arithmetic)

Sledgehammer



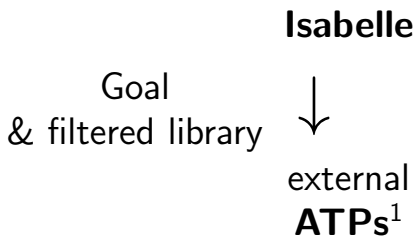
Architecture:

Isabelle

external
ATPs¹

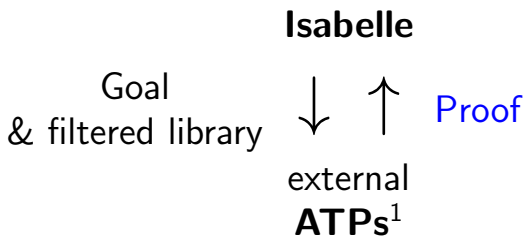
¹Automatic Theorem Provers

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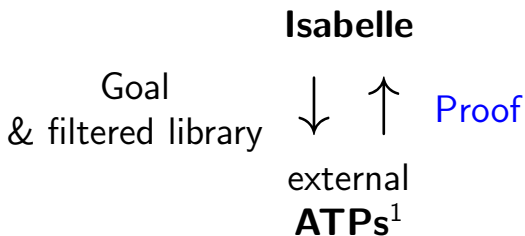
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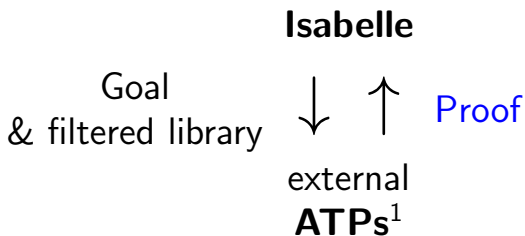


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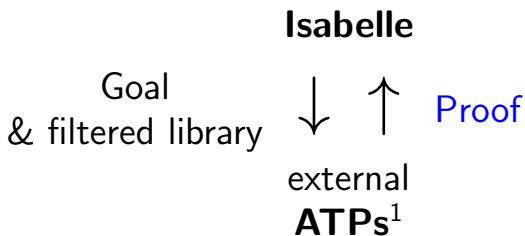


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Do you feel lucky?

¹Automatic Theorem Provers

by(*proof-method*)

\approx

apply(*proof-method*)
done

Auto_Proof_Demo.thy

8 Logical Formulas

9 Proof Automation

10 Single Step Proofs

11 Inductive Definitions

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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sets $?P$ to $a=b$ and $?Q$ to $False$.

Rule application

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Example: rule: $\llbracket ?P; ?Q \rrbracket \Longrightarrow ?P \wedge ?Q$

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“Backchaining”

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge ?Q} \text{conjI}$$

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They are known as **introduction rules** because they *introduce* a particular connective.

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$(blast\ intro: r)$

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Can greatly increase the search space!

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$$?Q \implies a = a \wedge ?Q$$

The general case:

If r is a theorem $\llbracket A_1; \dots; A_n \rrbracket \implies A$
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\rightsquigarrow

$$a = a \wedge b = b$$

From now on: ? mostly suppressed on slides

Single_Step_Demo.thy

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Phrase theorems like this $\llbracket A_1; \dots; A_n \rrbracket \Longrightarrow A$
not like this $A_1 \wedge \dots \wedge A_n \longrightarrow A$

⑧ Logical Formulas

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Example: even numbers

Informally:

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where

$ev\ 0 \quad |$

$ev\ n \Longrightarrow ev\ (n + 2)$

An easy proof: *ev 4*

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

```
fun evn :: nat  $\Rightarrow$  bool where  
  evn 0 = True |  
  evn (Suc 0) = False |  
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 $\Longrightarrow m = 0 \Longrightarrow evn\ m = True$

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  evn 0 = True |  
  evn (Suc 0) = False |  
  evn (Suc (Suc n)) = evn n
```

A trickier proof: $ev\ m \Longrightarrow evn\ m$

By induction on the *structure* of the derivation of $ev\ m$

Two cases: $ev\ m$ is proved by

- rule $ev\ 0$
 $\Longrightarrow m = 0 \Longrightarrow evn\ m = True$
- rule $ev\ n \Longrightarrow ev\ (n+2)$

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 $\Longrightarrow evn\ m = evn\ (n+2) = evn\ n = True$

Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

Rule induction for ev

To prove

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Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$
- $P\ n \Longrightarrow P(n+2)$

Rule $ev.induct$:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n. \llbracket ev\ n; P\ n \rrbracket \Longrightarrow P(n+2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow \textit{bool}$ **where**

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$\llbracket I\ a_1; \dots ; I\ a_n \rrbracket \Longrightarrow I\ a \mid$

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Note:

- I may have multiple arguments.

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

$\llbracket I\ a_1; \dots ; I\ a_n \rrbracket \Longrightarrow I\ a \mid$

\vdots

Note:

- I may have multiple arguments.
- Each rule may also contain *side conditions* not involving I .

Rule induction in general

To prove

$$I\ x \Longrightarrow P\ x$$

by *rule induction* on $I\ x$

Rule induction in general

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that P is preserved:

Rule induction in general

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that P is preserved:

$$\llbracket I\ a_1; P\ a_1; \dots ; I\ a_n; P\ a_n \rrbracket \Longrightarrow P\ a$$

!

Rule induction is absolutely central
to (operational) semantics
and the rest of this lecture course

!

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau$ *set* **where**

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 \vdots

Difference to **inductive**:

- arguments of I are tupled, not curried

Inductively defined sets

inductive_set $I :: \tau$ *set* **where**

$\llbracket a_1 \in I; \dots ; a_n \in I \rrbracket \implies a \in I \mid$

\vdots

Difference to **inductive**:

- arguments of I are tupled, not curried
- I can later be used with set theoretic operators, eg $I \cup \dots$

Chapter 5

Isar: A Language for Structured Proofs

12 Isar by example

13 Proof patterns

14 Streamlining Proofs

15 Proof by Cases and Induction

Apply scripts

- unreadable

Apply scripts

- unreadable
- hard to maintain

Apply scripts

- unreadable
- hard to maintain
- do not scale

Apply scripts

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- hard to maintain
- do not scale

No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

But: **apply** still useful for proof exploration

A typical Isar proof

```
proof  
  assume  $formula_0$   
  have  $formula_1$  by simp  
   $\vdots$   
  have  $formula_n$  by blast  
  show  $formula_{n+1}$  by ...  
qed
```

A typical Isar proof

proof

assume $formula_0$

have $formula_1$ **by** *simp*

\vdots

have $formula_n$ **by** *blast*

show $formula_{n+1}$ **by** \dots

qed

proves $formula_0 \implies formula_{n+1}$

Isar core syntax

proof = **proof** [method] step* **qed**
| **by** method

Isar core syntax

proof = **proof** [method] step* **qed**
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method = (*simp* ...) | (*blast* ...) | (*induction* ...) | ...

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proof = **proof** [method] step* **qed**
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step = **fix** variables (\wedge)
| **assume** prop (\implies)
| [**from** fact⁺] (**have** | **show**) prop proof

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Isar core syntax

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| [**from** fact⁺] (**have** | **show**) prop proof

prop = [name:] "formula"

fact = name | ...

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Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

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proof default proof: assume *surj*, show *False*

Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof default proof: assume *surj*, show *False*

assume *a*: *surj f*

Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof default proof: assume *surj*, show *False*

assume $a: \text{surj } f$

from a **have** $b: \forall A. \exists a. A = f a$

Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof default proof: assume *surj*, show *False*

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by(*simp add: surj_def*)

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from b **have** $c: \exists a. \{x. x \notin f x\} = f a$

Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

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from *a* **have** *b*: $\forall A. \exists a. A = f\ a$

by(*simp add: surj_def*)

from *b* **have** *c*: $\exists a. \{x. x \notin f\ x\} = f\ a$

by *blast*

Example: Cantor's theorem

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof default proof: assume *surj*, show *False*

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from *a* **have** *b*: $\forall A. \exists a. A = f\ a$

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by(*simp add: surj_def*)

from *b* **have** *c*: $\exists a. \{x. x \notin f\ x\} = f\ a$

by *blast*

from *c* **show** *False*

by *blast*

Example: Cantor's theorem

```
lemma  $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$   
proof   default proof: assume surj, show False  
  assume a: surj f  
  from a have b:  $\forall A. \exists a. A = f\ a$   
    by(simp add: surj_def)  
  from b have c:  $\exists a. \{x. x \notin f\ x\} = f\ a$   
    by blast  
  from c show False  
    by blast  
qed
```

Isar_Demo.thy

Cantor and abbreviations

Abbreviations

<i>this</i>	=	the previous proposition proved or assumed
then	=	from <i>this</i>
thus	=	then show
hence	=	then have

using and with

(have|show) prop **using** facts

using and with

(have|show) prop **using** facts
=
from facts **(have|show)** prop

using and with

(**have|show**) prop **using** facts
=
from facts (**have|show**) prop

with facts
=
from facts *this*

Structured lemma statement

lemma

fixes $f :: 'a \Rightarrow 'a \text{ set}$

assumes $s: \text{surj } f$

shows False

Structured lemma statement

lemma

fixes $f :: 'a \Rightarrow 'a \text{ set}$

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proof —

Structured lemma statement

lemma

fixes $f :: 'a \Rightarrow 'a \text{ set}$

assumes $s: \text{surj } f$

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proof — no automatic proof step

Structured lemma statement

lemma

fixes $f :: 'a \Rightarrow 'a \text{ set}$

assumes $s: \text{surj } f$

shows False

proof — **no automatic proof step**

have $\exists a. \{x. x \notin f x\} = f a$ **using** s

by $(\text{auto simp: surj_def})$

Structured lemma statement

lemma

fixes $f :: 'a \Rightarrow 'a \text{ set}$

assumes $s: \text{surj } f$

shows False

proof — **no automatic proof step**

have $\exists a. \{x. x \notin f\ x\} = f\ a$ **using** s

by $(\text{auto simp: surj_def})$

thus False **by** blast

qed

Structured lemma statement

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fixes $f :: 'a \Rightarrow 'a \text{ set}$

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Proves $\text{surj } f \Longrightarrow \text{False}$

Structured lemma statement

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fixes $f :: 'a \Rightarrow 'a \text{ set}$

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shows False

proof — **no automatic proof step**

have $\exists a. \{x. x \notin f\ x\} = f\ a$ **using** s

by $(\text{auto simp: surj_def})$

thus False **by** blast

qed

Proves $\text{surj } f \Longrightarrow \text{False}$

but $\text{surj } f$ becomes local fact s in proof.

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \dots$
assumes $a: P$ **and** $b: Q \dots$
shows R

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fixes $x :: \tau_1$ **and** $y :: \tau_2 \dots$
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- **fixes** and **assumes** sections optional

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \dots$
assumes $a: P$ **and** $b: Q \dots$
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

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Case distinction

```
show  $R$   
proof cases  
  assume  $P$   
   $\vdots$   
  show  $R$   $\langle proof \rangle$   
next  
  assume  $\neg P$   
   $\vdots$   
  show  $R$   $\langle proof \rangle$   
qed
```

Case distinction

show R
proof *cases*
 assume P
 :
 show R $\langle proof \rangle$
next
 assume $\neg P$
 :
 show R $\langle proof \rangle$
qed

have $P \vee Q$ $\langle proof \rangle$
then show R
proof
 assume P
 :
 show R $\langle proof \rangle$
next
 assume Q
 :
 show R $\langle proof \rangle$
qed

Contradiction

```
show  $\neg P$   
proof  
  assume  $P$   
   $\vdots$   
  show  $False$   $\langle proof \rangle$   
qed
```


Contradiction

```
show  $\neg P$   
proof  
  assume  $P$   
   $\vdots$   
  show False  $\langle proof \rangle$   
qed
```

```
show  $P$   
proof (rule ccontr)  
  assume  $\neg P$   
   $\vdots$   
  show False  $\langle proof \rangle$   
qed
```



```
show  $P \longleftrightarrow Q$ 
proof
  assume  $P$ 
  :
  show  $Q$   $\langle proof \rangle$ 
next
  assume  $Q$ 
  :
  show  $P$   $\langle proof \rangle$ 
qed
```

\forall and \exists introduction

show $\forall x. P(x)$

proof

fix x local fixed variable

show $P(x)$ $\langle proof \rangle$

qed

\forall and \exists introduction

show $\forall x. P(x)$

proof

fix x local fixed variable

show $P(x)$ $\langle proof \rangle$

qed

show $\exists x. P(x)$

proof

\vdots

show $P(witness)$ $\langle proof \rangle$

qed

\exists elimination: **obtain**

\exists elimination: **obtain**

have $\exists x. P(x)$

then obtain x **where** $p: P(x)$ **by** *blast*

\vdots x fixed local variable

\exists elimination: **obtain**

have $\exists x. P(x)$

then obtain x **where** $p: P(x)$ **by** *blast*

\vdots x fixed local variable

Works for one or more x

obtain example

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof

assume $\text{surj } f$

hence $\exists a. \{x. x \notin f\ x\} = f\ a$ **by** $(\text{auto simp: surj_def})$

obtain example

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof

assume $\text{surj } f$

hence $\exists a. \{x. x \notin f x\} = f a$ **by** $(\text{auto simp: surj_def})$

then obtain a **where** $\{x. x \notin f x\} = f a$ **by** blast

obtain example

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof

assume $\text{surj } f$

hence $\exists a. \{x. x \notin f x\} = f a$ **by** $(\text{auto simp: surj_def})$

then obtain a **where** $\{x. x \notin f x\} = f a$ **by** blast

hence $a \notin f a \longleftrightarrow a \in f a$ **by** blast

obtain example

lemma $\neg \text{surj}(f :: 'a \Rightarrow 'a \text{ set})$

proof

assume $\text{surj } f$

hence $\exists a. \{x. x \notin f x\} = f a$ **by** $(\text{auto simp: surj_def})$

then obtain a **where** $\{x. x \notin f x\} = f a$ **by** blast

hence $a \notin f a \longleftrightarrow a \in f a$ **by** blast

thus False **by** blast

qed

Set equality and subset

show $A = B$

proof

show $A \subseteq B$ $\langle proof \rangle$

next

show $B \subseteq A$ $\langle proof \rangle$

qed

Set equality and subset

show $A = B$

proof

show $A \subseteq B$ $\langle proof \rangle$

next

show $B \subseteq A$ $\langle proof \rangle$

qed

show $A \subseteq B$

proof

fix x

assume $x \in A$

\vdots

show $x \in B$ $\langle proof \rangle$

qed

Isar_Demo.thy

Exercise

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14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)

Example: pattern matching

```
show  $formula_1 \longleftrightarrow formula_2$  (is  $?L \longleftrightarrow ?R$ )  
proof  
  assume  $?L$   
   $\vdots$   
  show  $?R$   $\langle proof \rangle$   
next  
  assume  $?R$   
   $\vdots$   
  show  $?L$   $\langle proof \rangle$   
qed
```

?thesis

show *formula*

proof -

⋮

show *?thesis* $\langle proof \rangle$

qed

?thesis

show *formula* (*is ?thesis*)

proof -

⋮

show *?thesis* $\langle proof \rangle$

qed

?thesis

```
show formula (is ?thesis)  
proof -  
  ⋮  
  show ?thesis  $\langle proof \rangle$   
qed
```

Every **show** implicitly defines *?thesis*

let

Introducing local abbreviations in proofs:

let *?t* = "*some-big-term*"

⋮

have "... *?t* ... "

Quoting facts by value

By name:

have $x0$: " $x > 0$ " ...

:

from $x0$...

Quoting facts by value

By name:

```
have x0: "x > 0" ...  
:  
from x0 ...
```

By value:

```
have "x > 0" ...  
:  
from 'x>0' ...
```



Quoting facts by value

By name:

```
have x0: "x > 0" ...  
:  
from x0 ...
```

By value:

```
have "x > 0" ...  
:  
from 'x>0' ...
```


back quotes

Isar_Demo.thy

Pattern matching and quotations

14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Example

lemma

$$\exists ys\ zs. xs = ys @ zs \wedge \\ (length\ ys = length\ zs \vee length\ ys = length\ zs + 1)$$

Example

lemma

$\exists ys\ zs. xs = ys @ zs \wedge$
 $(length\ ys = length\ zs \vee length\ ys = length\ zs + 1)$

proof ???

Isar_Demo.thy

Top down proof development

When automation fails

Split proof up into smaller steps.

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... using ...

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... using ...

apply -

to make incoming facts
part of proof state

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Or explore by **apply**:

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part of proof state

apply *auto*

or whatever

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When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...

apply -

to make incoming facts
part of proof state

apply *auto*

or whatever

apply ...

At the end:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...

apply -

to make incoming facts
part of proof state

apply *auto*

or whatever

apply ...

At the end:

- **done**

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...

apply -

to make incoming facts
part of proof state

apply *auto*

or whatever

apply ...

At the end:

- **done**
- Better: convert to structured proof

14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

moreover—ultimately

have $P_1 \dots$

moreover

have $P_2 \dots$

moreover

⋮

moreover

have $P_n \dots$

ultimately

have $P \dots$

moreover—ultimately

have $P_1 \dots$

moreover

have $P_2 \dots$

moreover

\vdots

moreover

have $P_n \dots$

ultimately

have $P \dots$

\approx

have $lab_1: P_1 \dots$

have $lab_2: P_2 \dots$

\vdots

have $lab_n: P_n \dots$

from $lab_1 lab_2 \dots$

have $P \dots$

With names

14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Local lemmas

have B **if** *name:* $A_1 \dots A_m$ **for** $x_1 \dots x_n$
 $\langle proof \rangle$

Local lemmas

have B **if** *name*: $A_1 \dots A_m$ **for** $x_1 \dots x_n$
 $\langle proof \rangle$

proves $\llbracket A_1; \dots ; A_m \rrbracket \implies B$

Local lemmas

have B **if** *name*: $A_1 \dots A_m$ **for** $x_1 \dots x_n$
 $\langle proof \rangle$

proves $\llbracket A_1; \dots ; A_m \rrbracket \implies B$

where all x_i have been replaced by $?x_i$.

Proof state and Isar text

Proof state and Isar text

In general: **proof** *method*

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \dots x_n. \llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow B$$

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \dots x_n. \llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow B$$

How to prove each subgoal:

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \dots x_n. \llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow B$$

How to prove each subgoal:

```
fix  $x_1 \dots x_n$   
assume  $A_1 \dots A_m$   
 $\vdots$   
show  $B$ 
```

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \dots x_n. \llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow B$$

How to prove each subgoal:

```
fix  $x_1 \dots x_n$   
assume  $A_1 \dots A_m$   
:  
show  $B$ 
```

Separated by **next**

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Isar_Induction_Demo.thy

Proof by cases

Datatype case analysis

datatype $t = C_1 \vec{\tau} \mid \dots$

Datatype case analysis

datatype $t = C_1 \vec{\tau} \mid \dots$

```
proof (cases "term")  
  case ( $C_1\ x_1\ \dots\ x_k$ )  
     $\dots\ x_j\ \dots$   
next  
   $\vdots$   
qed
```


Datatype case analysis

datatype $t = C_1 \vec{\tau} \mid \dots$

```
proof (cases "term")  
  case ( $C_1\ x_1 \dots x_k$ )  
     $\dots\ x_j \dots$   
next  
 $\vdots$   
qed
```

where **case** ($C_i\ x_1 \dots x_k$) \equiv

```
fix  $x_1 \dots x_k$   
assume  $\underbrace{C_i}_{\text{label}}\ \underbrace{term = (C_i\ x_1 \dots x_k)}_{\text{formula}}$ 
```

Isar_Induction_Demo.thy

Structural induction for *nat*

Structural induction for nat

```
show  $P(n)$   
proof (induction  $n$ )  
  case 0  
   $\vdots$   
  show  $?case$   
next  
  case ( $Suc\ n$ )  
   $\vdots$   
  show  $?case$   
qed
```

Structural induction for nat

show $P(n)$

proof (*induction* n)

case 0

\equiv **let** $?case = P(0)$

\vdots

show $?case$

next

case ($Suc\ n$)

\vdots
 \vdots
 \vdots

show $?case$

qed

Structural induction for nat

show $P(n)$

proof (*induction* n)

case 0

\equiv **let** $?case = P(0)$

\vdots

show $?case$

next

case ($Suc\ n$)

\equiv **fix** n **assume** $Suc: P(n)$

\vdots

let $?case = P(Suc\ n)$

show $?case$

qed

Structural induction with \Rightarrow

show $A(n) \Rightarrow P(n)$

proof (*induction n*)

case 0

\vdots

show *?case*

next

case (*Suc n*)

\vdots

\vdots

show *?case*

qed

Structural induction with \Rightarrow

show $A(n) \Rightarrow P(n)$

proof (*induction n*)

case 0

\equiv **assume** 0: $A(0)$

\vdots

let $?case = P(0)$

show $?case$

next

case ($Suc\ n$)

\vdots

\vdots

show $?case$

qed

Structural induction with \implies

show $A(n) \implies P(n)$

proof (*induction n*)

case 0

\equiv **assume** 0: $A(0)$

\vdots

let $?case = P(0)$

show $?case$

next

case ($Suc\ n$)

\equiv **fix** n

\vdots

assume Suc : $A(n) \implies P(n)$
 $A(Suc\ n)$

\vdots

let $?case = P(Suc\ n)$

show $?case$

qed

Named assumptions

In a proof of

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

Named assumptions

In a proof of

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of

case C

Named assumptions

In a proof of

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of

case C

we have

$C.IH$ the induction hypotheses

Named assumptions

In a proof of

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of

case C

we have

C.IH the induction hypotheses

C.premis the premises A_i

Named assumptions

In a proof of

$$A_1 \implies \dots \implies A_n \implies B$$

by structural induction:

In the context of

case C

we have

$C.IH$ the induction hypotheses

$C.prem_s$ the premises A_i

C $C.IH + C.prem_s$

A remark on style

- **case** (*Suc n*) ... **show** *?case*
is easy to write and maintain

A remark on style

- **case** (*Suc n*) ... **show** *?case*
is easy to write and maintain
- **fix** *n* **assume** *formula* ... **show** *formula'*
is easier to read:
 - all information is shown locally
 - no contextual references (e.g. *?case*)

15 Proof by Cases and Induction

Rule Induction

Rule Inversion

Isar_Induction_Demo.thy

Rule induction

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow \text{bool}$

where

$\text{rule}_1: \dots$

\vdots

$\text{rule}_n: \dots$

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow \text{bool}$

where

$\text{rule}_1: \dots$

\vdots

$\text{rule}_n: \dots$

show $I\ x\ y \Longrightarrow P\ x\ y$

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow \text{bool}$

where

$\text{rule}_1: \dots$

\vdots

$\text{rule}_n: \dots$

show $I\ x\ y \Longrightarrow P\ x\ y$

proof (*induction rule: I.induct*)

Rule induction

```
inductive  $I :: \tau \Rightarrow \sigma \Rightarrow \text{bool}$   
where  
   $\text{rule}_1: \dots$   
   $\vdots$   
   $\text{rule}_n: \dots$ 
```

```
show  $I\ x\ y \Longrightarrow P\ x\ y$   
proof (induction rule: I.induct)  
  case  $\text{rule}_1$   
     $\dots$   
    show  $?case$   
next  
   $\vdots$   
next  
  case  $\text{rule}_n$   
     $\dots$   
    show  $?case$   
qed
```

Fixing your own variable names

case ($rule_i \ x_1 \ \dots \ x_k$)

Renames the first k variables in $rule_i$ (from left to right) to $x_1 \ \dots \ x_k$.

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

In the context of

case R

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

In the context of

case R

we have

R.IH the induction hypotheses

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

In the context of

case R

we have

R.IH the induction hypotheses

R.hyps the assumptions of rule R

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

In the context of

case R

we have

R.IH the induction hypotheses

R.hyps the assumptions of rule R

*R.prem*s the premises A_i

Named assumptions

In a proof of

$$I \dots \Longrightarrow A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \dots$:

In the context of

case R

we have

R.IH the induction hypotheses

R.hyps the assumptions of rule R

*R.prem*s the premises A_i

R $R.IH + R.hyps + R.prem$ s

15 Proof by Cases and Induction

Rule Induction

Rule Inversion

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**

$ev0:$ $ev\ 0 \mid$

$evSS:$ $ev\ n \Longrightarrow ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**

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$evSS:$ $ev\ n \Longrightarrow ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?

That it was proved by either $ev0$ or $evSS$!

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**

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What can we deduce from $ev\ n$?

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$ev\ n \Longrightarrow n = 0 \vee (\exists k. n = Suc\ (Suc\ k) \wedge ev\ k)$

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**

$ev0$: $ev\ 0 \mid$

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$$ev\ n \Longrightarrow n = 0 \vee (\exists k. n = Suc\ (Suc\ k) \wedge ev\ k)$$

Rule inversion = case distinction over rules

Isar_Induction_Demo.thy

Rule inversion

Rule inversion template

from $\text{'ev } n\text{'}$ **have** P

proof *cases*

case $ev0$

$n = 0$

\vdots

show $?thesis \dots$

next

case $(evSS\ k)$

$n = Suc\ (Suc\ k),\ ev\ k$

\vdots

show $?thesis \dots$

qed

Rule inversion template

from $\text{'ev } n\text{'}$ **have** P

proof *cases*

case $ev0$

$n = 0$

\vdots

show $?thesis \dots$

next

case $(evSS\ k)$

$n = Suc\ (Suc\ k),\ ev\ k$

\vdots

show $?thesis \dots$

qed

Impossible cases disappear automatically