# Concrete Semantics

## with Isabelle/HOL

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# Part II

# Semantics

# Chapter 7

# IMP:
# A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

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# Terminology

Statement: declaration of fact or claim

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Command: order to do something

*Study the book until you have understood it.*

Expressions are *evaluated*, commands are *executed*

# Commands

Concrete syntax:

$$
\begin{array}{rcl}
com & ::= & \texttt{SKIP} \\
& \mid & string\ \texttt{::=}\ aexp \\
& \mid & com\ \texttt{;;}\ com \\
& \mid & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\
& \mid & \texttt{WHILE}\ bexp\ \texttt{DO}\ com
\end{array}
$$

# Commands

Abstract syntax:

$$
\begin{array}{lcl}
\textbf{datatype}\ com & = & \mathit{SKIP} \\
 & \mid & \mathit{Assign}\ \mathit{string}\ \mathit{aexp} \\
 & \mid & \mathit{Seq}\ \mathit{com}\ \mathit{com} \\
 & \mid & \mathit{If}\ \mathit{bexp}\ \mathit{com}\ \mathit{com} \\
 & \mid & \mathit{While}\ \mathit{bexp}\ \mathit{com}
\end{array}
$$

# Com.thy

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# Big-step semantics

Concrete syntax:

$$(\mathit{com},\ \mathit{initial\text{-}state}) \Rightarrow \mathit{final\text{-}state}$$

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Command $c$ started in state $s$ terminates in state $t$

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$$(com,\ initial\text{-}state) \Rightarrow final\text{-}state$$

Intended meaning of $(c,\ s) \Rightarrow t$:

Command $c$ started in state $s$ terminates in state $t$

"$\Rightarrow$" here not type!

# Big-step rules

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$$(x ::=\ a,\ s) \Rightarrow\ s(x :=\ aval\ a\ s)$$

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$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

# Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{\begin{array}{c} bval\ b\ s_1 \\ (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3 \end{array}}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

# Examples: derivation trees

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$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{array}{rcl} w & = & WHILE\ b\ DO\ c \\ b & = & NotEq\ (V\ ''x'')\ (N\ 2) \\ c & = & ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_i & = & s(''x'' := i) \end{array}$$

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where
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$NotEq\ a_1\ a_2 =$
$Not (And\ (Not (Less\ a_1\ a_2))\ (Not (Less\ a_2\ a_1)))$

Logically speaking

$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

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is just infix syntax for

$$big\_step\ (c,s)\ t$$

where

$$big\_step :: \ com \times\ state \Rightarrow\ state \Rightarrow\ bool$$

is an inductively defined predicate.

# Big_Step.thy

Semantics

# Rule inversion

What can we deduce from

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- $(c_1;; c_2, s_1) \Rightarrow s_3$ ?
  $\exists\, s_2.\ (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$

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- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$ ?

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  $bval\ b\ s \wedge (c_1,\ s) \Rightarrow t\ \vee$
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  $bval\ b\ s \wedge (c_1,\ s) \Rightarrow t\ \vee$
  $\neg\ bval\ b\ s \wedge (c_2,\ s) \Rightarrow t$
- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$ ?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

# Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ \llbracket (c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3 \rrbracket \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$ (with a new fixed $s_2$).
No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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Reading:

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Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

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- Theorems with *elim* attribute are used automatically by *blast*, *fastforce* and *auto*
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- Variant: *elim!* applies elim-rules eagerly.

# Big_Step.thy

Rule inversion

# Command equivalence

Two commands have the same input/output behaviour:

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## Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$

$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

# Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

# Big_Step.thy

Command equivalence

# Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary $t'$.

# `Big_Step.thy`

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# The boon and bane of big steps

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Example problem:

$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

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We need a finer grained semantics!

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> *The first step in the execution of $c$ in state $s$ leaves a "remainder" command $c'$ to be executed in state $s'$.*

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> *The first step in the execution of* $c$ *in state* $s$
> *leaves a "remainder" command* $c'$
> *to be executed in state* $s'$.

Execution as finite or infinite reduction:

$$(c_1, s_1) \rightarrow (c_2, s_2) \rightarrow (c_3, s_3) \rightarrow \ldots$$

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- If $cs \rightarrow cs'$ we say that $cs$ *reduces* to $cs'$.
- A configuration $cs$ is *final* iff $\nexists\, cs'.\ cs \rightarrow cs'$

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Why?

$SKIP$ is the empty program.

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Why?

$SKIP$ is the empty program. Nothing more to be done.

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

# Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

**Fact** $(SKIP, s)$ is a final configuration.

# Small-step examples

$$("z" ::= V\ "x";;\ "x" ::= V\ "y";;\ "y" ::= V\ "z",\ s) \rightarrow$$

$$\ldots$$

where $s = <"x" := 3,\ "y" := 7,\ "z" := 5>$.

# Small-step examples

$$("z" ::= V\ "x";; "x" ::= V\ "y";; "y" ::= V\ "z",\ s) \rightarrow$$
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where $s = <"x" := 3,\ "y" := 7,\ "z" := 5>$.

$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{array}{rcl} w & = & WHILE\ b\ DO\ c \\ b & = & Less\ (V\ "x")\ (N\ 1) \\ c & = & "x" ::= Plus\ (V\ "x")\ (N\ 1) \\ s_n & = & <"x" := n> \end{array}$$

# Small_Step.thy

Semantics

Are big and small-step semantics equivalent?

# From $\Rightarrow$ to $\rightarrow *$

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**Theorem** $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

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In two cases a lemma is needed:

**Lemma**
$(c_1, s) \rightarrow* (c_1', s') \implies (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

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**Theorem** $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

**Lemma**
$(c_1, s) \rightarrow* (c_1', s') \Longrightarrow (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

Proof by rule induction.

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**Theorem** $cs \rightarrow * \ (SKIP,\ t) \implies cs \Rightarrow t$

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Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.

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Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

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**Theorem** $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

**Lemma** $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

# From $\rightarrow*$ to $\Rightarrow$

**Theorem** $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

**Lemma** $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

# Equivalence

**Corollary** $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \,(SKIP,\, t)$

# Small_Step.thy

Equivalence of big and small

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    $\Longrightarrow \neg\ \mathit{final}\ (c_1;;\ c_2,\ s)$
- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

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Together:

**Corollary** $\mathit{final}\ (c,\ s) = (c = SKIP)$

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

# Chapter 8

# Hoare Logic

## 4 Weakest Preconditions

### Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\textit{program exp} \ \{ \\
\quad a := 1 \\
\quad \textit{while} \ (0 < n) \ \textit{do} \ \{ \\
\quad\quad a := a + a; \\
\quad\quad n := n - 1 \\
\quad \} \\
\}
\end{array}
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and $0 \leq n$!

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Weakest condition on state, such that program $c$ will satisfy postcondition $Q$.

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$wp$ of equivalent programs is equal

$$c \sim c' \Longrightarrow wp\ c = wp\ c'$$

# Correctness of $exp$

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Reasoning along syntax of program!

That was easy!

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Unfolding will continue forever!

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Obviously, need some inductive argument!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

## Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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If $c$ terminates on $s$, then new state satisfies $Q$

Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between $wp$ and $wlp$

$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$
$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c_1\ Q\ s\ \textsf{\textit{else}}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate $I$, such that

$\bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s$

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and $I$ holds for start state.

$wlp\ (WHILE\ b\ DO\ c)\ Q\ s =$
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Intuition: $I$ holds initially, is preserved by iteration, and implies $Q$ at end of loop.

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and $I$ holds for start state.

Intuition: $I$ holds initially, is preserved by iteration, and implies $Q$ at end of loop. $I$ is called *loop invariant*

## While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

# Wp_Demo.thy

Weakest Precondition

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Otherwise, use unfold rules.

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$$P\ s \Longrightarrow wlp\ c\ Q\ s$$

If $c = WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all $wlp$s gone!

$wlp\_if\_eq$ and $wlp\_whileI'$ produce $if{-}then{-}else$

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Combine rule with splitting!

# Wp_Demo.thy

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

# Wellfounded_Demo.thy

For while loop: Find $wf$ relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![\mathit{wf}\ R;\ I\ s_0;$
$\ \bigwedge s.\ I\ s \Longrightarrow$ *if* $\mathit{bval}\ b\ s$ **then** $\mathit{wp}\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow \mathit{wp}\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s_0$

# Or, equivalently

**assumes** $WF$: $wf\ R$
**assumes** $INIT$: $I\ s_0$
**assumes** $STEP$: $\bigwedge s.\ \llbracket\ I\ s;\ bval\ b\ s\ \rrbracket$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
**assumes** $FINAL$: $\bigwedge s.\ \llbracket\ I\ s;\ \neg bval\ b\ s\ \rrbracket \Longrightarrow Q\ s$
**shows** $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

## Or, equivalently

**assumes** $WF$: $wf\ R$
**assumes** $INIT$: $I\ s_0$
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Now we can prove total correctness ...

# `Wp_Demo.thy`

Total Correctness

**lemma** $ASSUME\_\Theta\_alt$:
$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta = (\forall (f,(P,c,Q))\in\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0)\in R \wedge\ P\ s)\ c\ Q)$

**unfolding** $ASSUME\_\Theta\_def\ HT'set\_r\_def$ **..**

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

# Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Simplify specification of pre/postcondition, and invariants

# Standard operators

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

# `IMP2/Introduction.thy`

Adding more Operators

# C-like syntax

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$while\ (bexp)\ c$

# IMP2/Introduction.thy

Program Syntax

# More Readable VCs

Idea: Replace $s\ ''x''$ by (Isabelle) variable $x$.

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Similar: $s_0\ ''x''$ by $x_0$.

If subgoal can still be proved for arbitrary (Isabelle) variable $x$, it can, in particular, be proved for $s\ ''x''$.

$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$

# `IMP2/Introduction.thy`

More Readable VCs

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Which variables to interpret? over which states?

All variables that occur in the program!

Precondition: $x$ interpreted as $s\ ''x''$

Postcondition/Invariant: $x$ as $s\ ''x''$, $x_0$ as $s_0\ ''x''$

# IMP2/Introduction.thy

More Readable Annotations

6 Example Verifications

Loop Patterns

Euclid's Algorithm

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

# IMP2/Examples.thy

Count-up, Count-Down

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Applications: sqrt, log, ...

# IMP2/Examples.thy

Approximate from Below

# Bisection

We can compute sqrt more efficiently.

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l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
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Invariant? $l^2 \leq n < h^2 \wedge \ldots$ (range contains solution)

This program is actually tricky to get right!

# IMP2/Examples.thy

Bisection

## 6 Example Verifications

### Loop Patterns

### Euclid's Algorithm

# Euclid Intro

Compute gcd of positive numbers $a$, $b$

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Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

# Euclid Variants

By subtraction. Using $gcd\ (m - n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

# IMP2/Examples.thy

Euclid

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$\mathit{modifies}\ \mathit{vars}\ s_1\ \ s_2 = (\forall\, x.\ x \notin\ \mathit{vars} \longrightarrow s_1\ \ x = s_2\ \ x)$

Program modifies at most variables it assigns to

$\pi\text{: } (c,\ s) \Rightarrow\ t \Longrightarrow\ \mathit{modifies}\ (\mathit{lhsv}\ \pi\ \ c)\ \ t\ \ s$

# Modified Variables

We can strengthen correctness statement (automatically)

$$wp\ \pi\ \ c\ Q\ s \Longrightarrow wp\ \pi\ \ c\ (\lambda s'.\ Q\ s' \wedge modifies\ (lhsv\ \pi\ \ c)\ s'\ s)\ s$$

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For while-rule, we get

**lemma** $wp\_whileI\_modset$:
**fixes** $c$
**defines** $[simp]$: $modset \equiv lhsv\ c$
**assumes** $WF$: $wf\ R$
**assumes** $INIT$: $I\ \mathfrak{s}_0$
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**shows** $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

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**program_spec** computes $lhs$-variables:

$$HT\_mods\ \pi\ mods\ P\ c\ Q \equiv HT\ \pi\ P\ c\ (\lambda s_0\ s.\ \mathit{modifies}\ mods\ s\ s_0 \wedge Q\ s_0\ s)$$

# `IMP2/Examples.thy`

Euclid – show modified sets

# Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1; \\
\textit{while}\ (m{>}0)\ \{ \\
\quad n{=}a;\ a = 1; \\
\quad \textit{while}\ (n{>}0)\ \{ \\
\quad\quad a{=}2{*}a;\ n{=}n{-}1 \\
\quad \}; \\
\quad m{=}m{-}1 \\
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Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ ($m$ times)

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Still, we already have verified inner loop!

Idea: Split and verify separately!

# Modular Proofs

$a$=1;
$while$ ($m$>0) {
  $n$=$a$;
  $inline$ $exp\_count\_down$;
  $m$=$m$−1
}

# Modular Proofs

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Reuse existing proof of exp-count-down program!

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$$\llbracket HT\, \pi\;\; P\;\; c\;\; Q;\; \bigwedge s.\;\; P'\; s \Longrightarrow P\;\; s;\; \bigwedge s_0\;\; s.\; \llbracket P\;\; s_0;\;\; P'\;\; s_0;\;\; Q\;\; s_0\;\; s\rrbracket \Longrightarrow Q'\; s_0\;\; s\rrbracket \Longrightarrow HT\, \pi\;\; P'\;\; c\;\; Q'$$

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with modified sets:

$$\llbracket HT\_mods\ \pi\ \ mods\ \ P\ \ c\ \ Q;\ \ P\ \ s;\ \bigwedge s'.\ \llbracket modifies\ \ mods\ \ s'\ \ s;\ \ Q\ \ s\ \ s'\rrbracket \Longrightarrow Q'\ s'\rrbracket \Longrightarrow wp\ \pi\ \ c\ \ Q'\ s$$

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$$\llbracket HT\_mods\ \pi\ \ mods\ \ P\ \ c\ \ Q;\ \ P\ \ s;\ \bigwedge s'.\ \llbracket modifies\ \ mods\ \ s'\ s;\ \ Q\ \ s\ \ s'\rrbracket \Longrightarrow Q'\ s'\rrbracket \Longrightarrow wp\ \pi\ \ c\ \ Q'\ s$$

VCG will automatically use this rule.

# Modular Proofs

Re-using proofs:

$$\llbracket HT\ \pi\ \ P\ c\ Q;\ \bigwedge s.\ P'\ s \Longrightarrow P\ s;\ \bigwedge s_0\ s.\ \llbracket P\ s_0;\ P'\ s_0;\ Q\ s_0\ s\rrbracket \Longrightarrow Q'\ s_0\ s\rrbracket \Longrightarrow HT\ \pi\ \ P'\ c\ Q'$$

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

# `IMP2/Examples.thy`

Power-Tower

7 Advanced Verification

Arrays

Data Refinement

Local Variables

Recursion

# Arrays

Every variable is of type $int \Rightarrow int$.

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# `IMP2/Examples.thy`

Array-Sum

# Reasoning about Arrays

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

# IMP2/Examples.thy

Sortedness Check

# Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Idea: Halve interval in each step.

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*Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...*

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Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

# Binary Search Algorithm

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
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$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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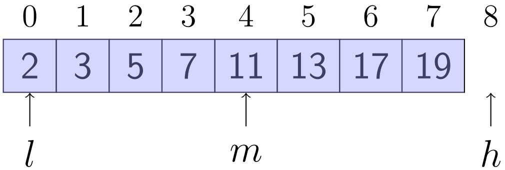


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$\uparrow$

$l, h$

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$\uparrow$ $l, h$ (at index 5)

```
while (l < h) {
    m = (l + h) / 2;
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    else h = m
}
```

Returns smallest $i$ with $x \leq a[i]$

# Notes on Binary Search

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Bug in Java Standard Library for $> 9$ years!

# Proving Binary Search

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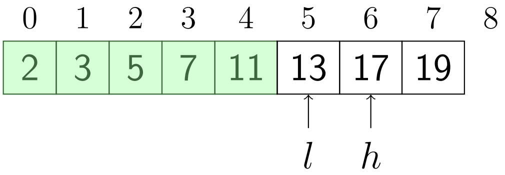
Invariant:

# Proving Binary Search

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Invariant:

- $i{<}l \implies a[i] < x$ (strictly smaller than $x$)

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Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than $x$)
- $i \geq h \implies x \leq a[i]$ (greater or equal to $x$)
- and the usual bounds

# IMP2/Examples.thy

Binary Search

# Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.

In each iteration, move next element to its position

# Specifying Sorting Algorithms

Precondition: $l \leq h$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multiset of elements at indexes in finite set $r$

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Separate proof for inner loop!

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Invariant of outer loop:

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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$

$\wedge\ mset\_ran\ a\ \{l..{<}h\} = mset\_ran\ a_0\ \{l..{<}h\}$

# Insort: Inner Loop

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Intuition:

# Insort: Inner Loop

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Intuition: ?

# Insort: Inner Loop

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Short: Move $a[j]$ backwards over greater elements.

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Invariants easier to find!

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# Insort: Finding Invariant

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|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 13 | 17 | 19 | 11 |
| $\uparrow$ | | | | | | $\uparrow$ | $\uparrow$ |
| $l$ | | | | | | $i$ | $j$ |

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- $+$ the usual bounds: $l - 1 \leq i \wedge i < j$

# `IMP2/Examples.thy`

Insertion Sort

# Summary so Far

Understand what program does!

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Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

## 7 Advanced Verification

- Arrays
- Data Refinement
- Local Variables
- Recursion

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# `IMP2/Examples.thy`

Filter, Merge, dedup

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Don't worry about name-clashes with subroutine's auxiliary variables

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# Wrapping Specification

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Recall:

$$HT\ \pi\ \ P\ \ c\ \ Q \equiv \forall\, s_0.\ P\ s_0 \longrightarrow wp\ \pi\ \ c\ (Q\ s_0)\ s_0$$

# Prologue

$$HT\ \pi\ P\ body\ Q \Longrightarrow$$
$$HT\ \pi\ (wp\ \pi\ prologue\ P)\ (prologue;;\ body)$$
$$(\lambda s_0\ s.\ wp\ \pi\ prologue\ (\lambda s_0.\ Q\ s_0\ s)\ s_0)$$

Intuition: Weakest precondition to enforce $P$ after prologue

# Epilogue

$[\![HT\ \pi\ \ P\ body\ \ Q;\ \forall\, s.\ \exists\, t.\ \pi\colon\ (epilogue,\ s)\ \Rightarrow\ t]\!]$
$\Longrightarrow HT\ \pi\ \ P\ (body;;\ epilogue)\ (\lambda s_0.\ sp\ \pi\ \ (Q\ \ s_0)$
$epilogue)$

Intuition: Strongest postcondition we get from $Q$ after epilogue

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$$sp\ \pi\ P\ c\ t \equiv \exists\, s.\ P\ s \wedge \pi \colon (c,\ s) \Rightarrow\ t$$

Selected rules:

$sp\ \pi\ P\ (x[]\ ::=\ y)\ t \longleftrightarrow \exists\, vx.\ \textit{let}\ s\ =\ t(x := vx)\ \textit{in}\ t\ x = s\ y \wedge P\ s$

$sp\ \pi\ P\ (x[]\ ::=\ y)\ t \longleftrightarrow t\ x = t\ y \wedge (\exists\, vx.\ P\ (t(x := vx,\ y := t\ x)))$

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$$HT\ P'\ (scope\ \{\ p_1\ =\ G_1;\ \ldots;\ p_n\ =\ G_n;\ body;\ G{=}x\ \})\ Q'$$

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# IMP2/Examples.thy

Merge as Procedure

## 7 Advanced Verification

# Recursive Procedures

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Show specification for state $s$, assuming it holds for smaller states $s'$.

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Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

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Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

$PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta \equiv$
$\forall\ P\ c\ Q.\ (f_0,\ P,\ c,\ Q) \in \Theta \wedge P\ s_0 \longrightarrow wp\ \pi\ (c\ s_0)\ (Q\ s_0)\ s_0$
Hoare-triples valid for fixed variant $f_0$ and state $s_0$.

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**lemma** $vcg\_HT'setI$:
**assumes** $wf\ R$
**assumes** $RL$: $\bigwedge f_0\ s_0.\ [\![\ ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta\ ]\!] \Longrightarrow PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta$
**shows** $HT'set\ \pi\ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

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$[\![\pi\ p = Some\ c;\ HT\_mods\ \pi\ mods\ P\ c\ Q]\!] \Longrightarrow HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q$

Maps Hoare-Triples to procedure calls

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$HT\_mods \; \pi \; mods \; P \; (PCall \; p) \; Q \Longrightarrow HT\_mods \; \pi' \; mods \; P$
$(PScope \; \pi \; p) \; Q$

Wrap current procedure environment

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- variants: expression over parameters.
- localization of procedure environment.

# IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

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Can we always find annotations to get provable VCs?

Only consider while-rule here

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$wlp\ \pi\ c\ Q$!

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Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

# IMP2/Examples.thy

Completeness of While-Rule

# Conclusions

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Caveats:
Procedures+Recursion tools not well-tested
VCG is slow for many procedures/inlines