Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

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3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Semantics is easy.

Command: order to do something

Study the book until you have understood it.

Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{rcl}
com & ::= & \texttt{SKIP} \\
 & | & string\ \texttt{::=}\ aexp \\
 & | & com\ \texttt{;;}\ com \\
 & | & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\
 & | & \texttt{WHILE}\ bexp\ \texttt{DO}\ com
\end{array}
$$

Commands

Abstract syntax:

$$
\begin{array}{lcl}
\textbf{datatype}\ com & = & SKIP \\
& | & Assign\ string\ aexp \\
& | & Seq\ com\ com \\
& | & If\ \ bexp\ com\ com \\
& | & While\ bexp\ com
\end{array}
$$

Com.thy

1. IMP Commands
2. Big-Step Semantics
3. Small-Step Semantics

Big-step semantics

Concrete syntax:

$$(\mathit{com},\ \mathit{initial\text{-}state}) \Rightarrow \mathit{final\text{-}state}$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x ::=\ a,\ s) \Rightarrow\ s(x :=\ aval\ a\ s)$$

Big-step rules

$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

$$\frac{(c,\ s_1) \Rightarrow s_2 \qquad \begin{array}{c} bval\ b\ s_1 \\ (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3 \end{array}}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ \ ?}$$

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$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{aligned} w &= WHILE\ b\ DO\ c \\ b &= NotEq\ (V\ ''x'')\ (N\ 2) \\ c &= ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_i &= s(''x'' := i) \end{aligned}$$

Examples: derivation trees

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$\mathit{NotEq}\ a_1\ a_2 =$
$\mathit{Not}(\mathit{And}\ (\mathit{Not}(\mathit{Less}\ a_1\ a_2))\ (\mathit{Not}(\mathit{Less}\ a_2\ a_1)))$

Logically speaking

$$(c,\ s) \Rightarrow\ t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

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$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP,\ s) \Rightarrow t$?

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- $(x ::= a, s) \Rightarrow t$? $\quad t = s(x := aval\ a\ s)$
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  $\exists\, s_2.\ (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$

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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow\ s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow\ s_2;\ (c_2,\ s_2) \Rightarrow\ s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$ (with a new fixed s_2).
No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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Reading:

To prove a goal P with assumption asm,
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Reading:

> To prove a goal P with assumption asm,
> prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

$elim$ attribute

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- Can also be added locally, eg *(blast elim: ...)*
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $\;w = \mathit{WHILE}\ b\ \mathit{DO}\ c$

$\quad\quad\; w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
$$\vee$$
$$\neg\ bval\ b\ s \wedge t = s$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

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$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

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$$\vee$$
$$\neg\ bval\ b\ s \wedge t = s$$

$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

`Big_Step.thy`

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

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Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

- nonterminating computations,

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We need a finer grained semantics!

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3. Small-Step Semantics

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Intended meaning of $(c,\ s) \rightarrow (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
> *to be executed in state s'.*

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> *leaves a "remainder" command* c'
> *to be executed in state* s'*.*

Execution as finite or infinite reduction:

$$(c_1, s_1) \rightarrow (c_2, s_2) \rightarrow (c_3, s_3) \rightarrow \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \to cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \to cs'$

The intention:

$$(SKIP,\ s) \ \ \text{is final}$$

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Why?

$SKIP$ is the empty program.

The intention:

$$(SKIP,\ s) \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a,\ s) \ \rightarrow$$

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$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;; c_2, s) \ \rightarrow}$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \ \rightarrow\ (c_1, s)}$$

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow$$

$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow \\ \ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{aligned} w &= \ WHILE\ b\ DO\ c \\ b &= \ Less\ (V\ ''x'')\ (N\ 1) \\ c &= \ ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_n &= \ <''x'' := n> \end{aligned}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow*$

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Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

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Proof by rule induction

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Proof by rule induction (of course on $cs \Rightarrow t$)

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In two cases a lemma is needed:

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1{}',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1{}';;\ c_2,\ s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1, s) \rightarrow* (c_1', s') \implies (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

Proof by rule induction.

From $\rightarrow*$ to $\Rightarrow$

From $\to*$ to $\Rightarrow$

Theorem $cs \to* (SKIP,\ t) \implies cs \Rightarrow t$

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Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.

From $\rightarrow*$ to $\Rightarrow$

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Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow* (SKIP,\ t)$

Small_Step.thy

Equivalence of big and small

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That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$

by induction on c.

Can execution stop prematurely?

That is, are there any final configs except $(SKIP,s)$?

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

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Together:

Corollary $\mathit{final}\ (c, s) = (c = SKIP)$

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$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

4. Weakest Preconditions

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6. Example Verifications

4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\textit{program exp} \ \{ \\
\quad a := 1 \\
\quad \textit{while} \ (0 < n) \ \textit{do} \ \{ \\
\quad\quad a := a + a; \\
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At the end of the execution, variable a should contain 2^n, where n is the original value of variable n!
and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

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wp of equivalent programs is equal

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Reasoning along syntax of program!

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Unfolding will continue forever!

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Unfolding will continue forever!

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But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

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$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$
$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{if}\ bval\ b\ s\ \textsf{then}\ wlp\ c_1\ Q\ s\ \textsf{else}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate I, such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s \rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

`Wp_Demo.thy`

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If $c =\ WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ c\ (\lambda s'.\ I\ s' \wedge\ s' < s)\ s$
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$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ **then** $wp\ c\ (\lambda s'.\ I\ s' \wedge\ s' <\ s)\ s$ *else* $Q\ s$

Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ **then** $wp\ c\ (\lambda s'.\ I\ s' \wedge\ (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $\mathit{wf}\ R$
assumes INIT: $I\ s_0$
assumes STEP: $\bigwedge s.\ [\![\ I\ s;\ \mathit{bval}\ b\ s\ ]\!]$
$\Longrightarrow \mathit{wp}\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
assumes FINAL: $\bigwedge s.\ [\![\ I\ s;\ \neg \mathit{bval}\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $\mathit{wp}\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow\ wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s){\in}R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Add nice syntax for programs

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Make VCs more readable

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Make VCs more readable

Simplify specification of pre/postcondition, and invariants

Standard operators

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

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Adding more Operators

C-like syntax

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$while\ (bexp)\ c$

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Program Syntax

More Readable VCs

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If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$$

`IMP2/Introduction.thy`

More Readable VCs

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Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

IMP2/Examples.thy

Count-up, Count-Down

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Applications: sqrt, log, ...

`IMP2/Examples.thy`

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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```
l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
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This program is actually tricky to get right!

`IMP2/Examples.thy`

Bisection

Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers $a,\ b$

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - \ n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

IMP2/Examples.thy

Euclid

6 Example Verifications

Modified Variables

Program: $a{=}1;\ i{=}0;\ while\ (i{<}n)\ \{\ a{=}a{*}2;\ i{=}i{+}1\ \}$

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Pre: $n \geq 0$ Post: $a = 2\,\hat{}\,n_0$

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Only $a{,}i$ changed: $\forall\, x.\ x \notin \{''a'',\ ''i''\} \longrightarrow s\ x = s'\ x$

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$modifies\ vars\ s_1\ s_2 = (\forall\, x.\ x \notin vars \longrightarrow s_1\ x = s_2\ x)$

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$\mathit{modifies}\ \mathit{vars}\ s_1\ s_2 = (\forall\, x.\ x \notin \mathit{vars} \longrightarrow s_1\ x = s_2\ x)$

Program modifies at most variables it assigns to

$(c,\ s) \Rightarrow t \Longrightarrow \mathit{modifies}\ (\mathit{lhsv}\ c)\ t\ s$

Modified Variables

We can strengthen correctness statement (automatically)

$$wp\ c\ Q\ s \Longrightarrow wp\ c\ (\lambda s'.\ Q\ s' \wedge modifies\ (lhsv\ c)\ s'\ s)\ s$$

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For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}){\in}R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ ]\!]$
$\Longrightarrow Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

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The VCG will automatically rewrite with rule

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program_spec computes lhs-variables:

$$HT\_mods\ mods\ P\ c\ Q \equiv HT\ P\ c\ Q \wedge mods = lhsv\ c$$

IMP2/Examples.thy

Euclid – show modified sets

Modular Proofs

Consider program

```
a=1;
while (m>0) {
  n=a; a = 1;
  while (n>0) {
    a=2*a; n=n−1
  };
  m=m−1
}
```

What does this compute

Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1; \\
\mathit{while}\ (m{>}0)\ \{ \\
\quad n{=}a;\ a = 1; \\
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What does this compute?

Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

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Still, we already have verified inner loop!

Idea: Split and verify separately!

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```
a=1;
while (m>0) {
  n=a;
  inline exp_count_down;
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}
```

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Reuse existing proof of exp-count-down program!

Modular Proofs

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

IMP2/Examples.thy

Power-Tower

Arrays

Every variable is of type $int \Rightarrow int$.

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$ArrayClear :: char\ list \Rightarrow com$

$(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

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VCG: Guess type from variable usage
Only with index 0: Bind $VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \dots)$

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$V\ x = \ Vidx\ x\ (N\ 0)$

$Assign\ x\ a = \ AssignIdx\ x\ (N\ 0)\ \ a$

VCG: Guess type from variable usage
Only with index 0: Bind $VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \dots)$

Otherwise: Bind $VAR\ (s\ ''x'')\ (\lambda x.\ \dots)$

Arrays

By default, we use index 0.

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`IMP2/Examples.thy`

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Find element in sorted array. In time $O(\log n)$.

Idea: Halve interval in each step.

This algorithm is tricky to implement correctly!

Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

— Donald Knuth

Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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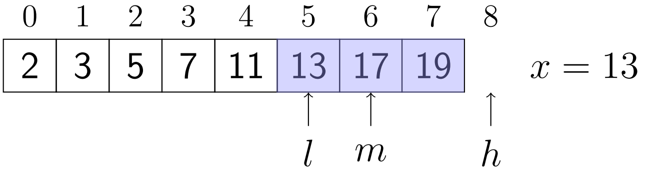


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while (l < h) {
    m = (l + h) / 2;
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```

Returns smallest i with $x \leq a[i]$

Notes on Binary Search

```
while (l < h) {
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Otherwise, $m = (l + h)/2$ may overflow!

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Bug in Java Standard Library for > 9 years!

Proving Binary Search

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2	3	5	7	11	13	17	19	
					$\uparrow$	$\uparrow$		
					l	h		

$x = 13$

Invariant:

Proving Binary Search

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Invariant:

- $i<l \implies a[i] < x$ (strictly smaller than x)

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Invariant:

- $i{<}l \implies a[i] < x$ (strictly smaller than x)
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- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
  key = a[j];
  i = j-1;
  while (i>=l && a[i]>key) {
    a[i+1] = a[i];
    i=i-1
  };
  a[i+1] = key
  j=j+1
}
```

Idea: Build sorted array from start.
In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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Postcondition:

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where

$ran\_sorted\ a\ l\ h \equiv \forall\, i \in \{l..<h\}.\ \forall\, j \in \{l..<h\}.\ i \leq j \longrightarrow a\ i \leq a\ j$

$mset\_ran\ a\ r = (\sum i \in r.\ \{\#a\ i\#\})$

Multisets in Isabelle

imports $HOL{-}Library.Multiset$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multiset of elements at indexes in finite set r

Proving Insertion Sort

Separate proof for inner loop!

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Invariant of outer loop:

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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$

$\wedge\ mset\_ran\ a\ \{l..{<}h\} = mset\_ran\ a_0\ \{l..{<}h\}$

Insort: Inner Loop

```
key = a[j];
i = j-1;
while (i>=l && a[i]>key) {
  a[i+1] = a[i];
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Intuition:

Insort: Inner Loop

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$a[j]$ is moved backwards

Insort: Inner Loop

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Intuition: ?

$a[j]$ is moved backwards until

Insort: Inner Loop

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```

Intuition: ?

$a[j]$ is moved backwards until
previous element is $\leq a[j]$

Insort: Inner Loop

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Short: Move $a[j]$ backwards over greater elements.

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Let's specify this intuition!

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Invariants easier to find!

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

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assumes $l < j$, $let\ key = a_0\ j$

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Move $a[j]$ backwards over greater elements.

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ensures $i \in \{l - (1::'a)..<j\}$

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$

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ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and

Insort: Inner Loop

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$a\ (i + (1::'b)) = key$ and

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$
ensures $i \in \{l - (1::'a)..<j\}$
ensures $\forall\, k \in \{l..i\}.\ a\ k = a_0\ k$ and
$a\ (i + (1::'b)) = key$ and
$\forall\, k \in \{i + (2::'a)..j\}.\ a\ k = a_0\ (k - (1::'a))$

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$

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ensures $l \leq i \longrightarrow a\ i \leq key$ and

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, *let* $key = a_0\ j$

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$a\ (i + (1::'b)) = key$ and
$\forall\, k{\in}\{i + (2::'a)..j\}.\ a\ k = a_0\ (k - (1::'a))$

ensures $l \le i \longrightarrow a\ i \le key$ and
$\forall\, k{\in}\{i + (2::'a)..j\}.\ key < a\ k$

Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
↑						↑	↑
l						i	j

Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	11	19
$\uparrow$					$\uparrow$		$\uparrow$
l					i		j

Insort: Finding Invariant

0	1	2	3	4	5	6	7
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↑				↑			↑
l				i			j

Insort: Finding Invariant

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$\uparrow$			$\uparrow$				$\uparrow$
l			i				j

Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	11	13	17	19
$\uparrow$			$\uparrow$				$\uparrow$
l			i				j

Consider intermediate situation

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Consider intermediate situation

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- indexes $\geq i+2$ correctly shifted
  $\forall k \in \{i + (2::'a)..j\}.\ a\ k = a_0\ (k - (1::'a))$

Insort: Finding Invariant

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- and elements greater than key
  $\forall\, k \in \{i + (2{::}'a)..j\}.\ key < a\ k$

Insort: Finding Invariant

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  $\forall\, k \in \{i + (2::'a)..j\}.\ key < a\ k$
- + the usual bounds: $l - (1::'a) \leq i \wedge i < j$

IMP2/Examples.thy

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Prove that this implies expectations of users

Summary so Far

Understand what program does!

Split program into handy parts

Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

6 Example Verifications

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a\ [l..<h]$ as $int\ list$

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Idea: Do proof at level of understanding first

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then show that implementation is correct!

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Idea: Do proof at level of understanding first

then show that implementation is correct!

Instead of one complex proof, get two simple proofs !

IMP2/Examples.thy

Filter, Merge, dedup