Concrete Semantics
with Isabelle/HOL

Peter Lammich

(slides from Concrete Semantics by Nipkow)

2018-10-16

Chapter 1

Introduction

Organization Issues

Course Homepage: `http://www21.in.tum.de/teaching/semantik/WS1819/`

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Tutorials and Homework are the heart and soul of this course!

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Like the state of mathematics in the 19th century
— before set theory and logic entered the scene.

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- This course is about "beyond intuition".

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What does the correctness of a type checker even mean?

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Example:
What does the correctness of a type checker even mean? How is it proved?

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- They provide the worst possible semantics.
- Moreover: compilers may differ!

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- Few languages have a (separate, abstract) semantics.
- If they do, it will be informal (English).

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  GI Dissertationspreis 2003:
  Gerwin Klein: *Verified Java Bytecode Verification*

Standard ML (SML)

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Main achievements: LCF (theorem proving)
SML (functional programming)
CCS, pi (concurrency)

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SML semantics hardly used:

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- too much detail to allow reliable informal proof
- not processable beyond LaTeX, not even executable

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- Real programming languages *are* complex.
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- Complex designs are error-prone.
- Informal mathematical proofs of complex designs are also error-prone.

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Machine-checked language semantics and proofs

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The tool:

Proof Assistant (PA)

or

Interactive Theorem Prover (ITP)

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Potentially addictive</p>

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Time consuming
Potentially addictive
Undermines your naive trust in informal proofs

Terminology

This lecture course:

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Verification = formal correctness proof

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Traditionally:

Formal = mathematical

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Operating system
microkernel (L4)

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Operating system
microkernel (L4)

Gerwin Klein (& Co)
NICTA Sydney
using Isabelle

A happy fact of life

Programming language researchers
are increasingly using PAs

Why verification pays off

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Software Never Dies

1 Background

2 This Course

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- Hot or trendy PLs
- Comparison of PLs or PL paradigms
- Compilers (although they will be one application)

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   Both informally and formally (PA!)

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All exercises require the use of Isabelle/HOL

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- It is the future

- It is the only way to deal with complex languages *reliably*

- I want students to learn how to write correct proofs

- I have seen too many proofs that look more like LSD trips than coherent mathematical arguments

Overview of course

- Introduction to Isabelle/HOL

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- IMP (assignment and while loops) and its semantics

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- Introduction to Isabelle/HOL
- IMP (assignment and while loops) and its semantics
- A compiler for IMP
- Hoare logic for IMP
- Type systems for IMP
- Program analysis for IMP

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A growing number of universities offer related course

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It has applications in compilers, security,
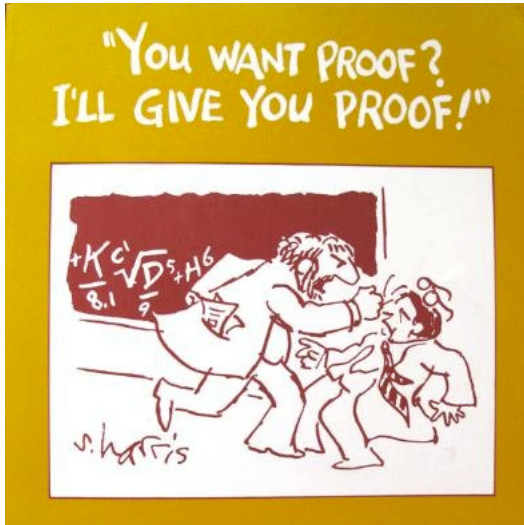software engineering etc.

What you learn in this course goes far beyond PLs

It has applications in compilers, security, software engineering etc.

It is a new approach to informatics

At the end of the course ...

At the end of the course . . .

Part I

Isabelle

Chapter 2

Programming and Proving

3 Overview of Isabelle/HOL

4 Type and function definitions

5 Induction Heuristics

6 Simplification

Which of the following formulas have the same meaning?

1. $A \Longrightarrow (B \Longrightarrow C)$
2. $(A \Longrightarrow B) \Longrightarrow C$
3. $(A \wedge B) \Longrightarrow C$

Notation

Implication associates to the right:

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$$\frac{A_1 \quad \ldots \quad A_n}{B} \quad \text{means} \quad A_1 \Longrightarrow \cdots \Longrightarrow A_n \Longrightarrow B$$

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HOL Formulas:

- For the moment: only $term = term$,
  e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, ...

3 Overview of Isabelle/HOL
  Types and terms
  Interface
  By example: types *bool*, *nat* and *list*
  Summary

Types

Basic syntax:

$$\tau \quad ::=$$

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This language of terms is known as the *λ-calculus*.

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- The step from $(\lambda x.\ t)\ u$ to $t[u/x]$ is called β-*reduction*.

The computation rule of the λ-calculus is the replacement of formal by actual parameters:

$$(\lambda x.\ t)\ u\ =\ t[u/x]$$

where $t[u/x]$ is "t with u substituted for x".

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- The step from $(\lambda x.\ t)\ u$ to $t[u/x]$ is called β-*reduction*.
- Isabelle performs β-reduction automatically.

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$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

Type inference

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In the presence of *overloaded* functions (functions with multiple types) this is not always possible.

User can help with *type annotations* inside the term.
Example: $f\,(x{::}nat)$

Currying

Thou shalt Curry your functions

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

Advantage:

Currying allows *partial application*
$f\ a_1$ where $a_1 :: \tau_1$

Predefined syntactic sugar

- *Infix:* $+$, $-$, $*$, $\#$, $@$, ...

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Prefix binds more strongly than infix:

$$\textbf{!} \quad f\, x + y \;\equiv\; (f\, x) + y \;\not\equiv\; f\, (x + y) \quad \textbf{!}$$

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- *Infix:* $+$, $-$, $*$, $\#$, $@$, ...
- *Mixfix:* *if _ then _ else _*, *case _ of*, ...

Prefix binds more strongly than infix:
$$f\,x + y \;\equiv\; (f\,x) + y \;\neq\; f\,(x + y)$$
!   **!**

Enclose *if* and *case* in parentheses:
!   *(if _ then _ else _)*   **!**

Theory = Isabelle Module

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Syntax: `theory` $MyTh$
            `imports` $T_1 \ldots T_n$
            `begin`
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Usually: `imports` `Main`

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In `.thy` files:

Types, terms and formulas need to be inclosed in "

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" normally not shown on slides

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- Processes Isabelle text automatically
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  when editing `.thy` files (like modern Java IDEs)

`Overview_Demo.thy`

3 Overview of Isabelle/HOL

Type *bool*

datatype *bool* = *True* | *False*

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Predefined functions:
∧, ∨, ⟶, ... :: *bool* ⇒ *bool* ⇒ *bool*

A *formula* is a term of type *bool*

if-and-only-if: =

Type nat

datatype $nat = 0 \mid Suc\ nat$

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! Numbers and arithmetic operations are overloaded:
$0,1,2,\ldots :: {'}a,\quad + :: {'}a \Rightarrow {'}a \Rightarrow {'}a$

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You need type annotations: $1 :: nat,\ x + (y{::}nat)$

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! Numbers and arithmetic operations are overloaded:
$0,1,2,\ldots :: {'a},\ \ \ \ +\ ::\ \ {'a} \Rightarrow {'a} \Rightarrow {'a}$

You need type annotations: $1 :: nat,\ x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

An informal proof

Lemma $add\ m\ 0 = m$

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Proof by induction on m.

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  We need to show $add\ (Suc\ m)\ 0 = Suc\ m$.
  The proof is as follows:
  $$
  \begin{aligned}
  add\ (Suc\ m)\ 0 &= Suc\ (add\ m\ 0) \quad \text{by def. of } add \\
  &= Suc\ m \qquad\qquad\ \text{by IH}
  \end{aligned}
  $$

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Syntactic sugar:

- $[]\ =\ Nil$: empty list

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- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)

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Syntactic sugar:

- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1,\ \ldots,\ x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs, prove

- $P([])$ and
- for arbitrary but fixed x and xs,
  $P(xs)$ implies $P(x \# xs)$.

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- for arbitrary but fixed x and xs,
  $P(xs)$ implies $P(x\#xs)$.

$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \implies P(x\#xs)}{P(xs)}$$

List_Demo.thy

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$
Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.

- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs = app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.
  The proof is as follows:
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs$
  $= Cons\ x\ (app\ (app\ xs\ ys)\ zs)$ by definition of app
  $= Cons\ x\ (app\ xs\ (app\ ys\ zs))$ by IH
  $= app\ (Cons\ x\ xs)\ (app\ ys\ zs)$ by definition of app

Large library: `HOL/List.thy`

Included in `Main`.

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Don't reinvent, reuse!

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Predefined: $xs @ ys$ (append), $length$,

Large library: `HOL/List.thy`

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Don't reinvent, reuse!

Predefined: $xs @ ys$ (append), $length$, and map

- **datatype** defines (possibly) recursive data types.

- **fun** defines (possibly) recursive functions by pattern-matching over datatype constructors.

Proof methods

- *induction* performs structural induction on some variable (if the type of the variable is a datatype).

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- *auto* solves as many subgoals as it can, mainly by simplification (symbolic evaluation):

  "=" is used only from left to right!

Proofs

General schema:

lemma $name$: "..."
apply (...)
apply (...)
⋮
done

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lemma $name$: "..."
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If the lemma is suitable as a simplification rule:

lemma $name$[simp]: "..."

Top down proofs

Command

sorry

"completes" any proof.

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Allows top down development:

Assume lemma first, prove it later.

The proof state

1. $\bigwedge x_1 \ldots x_p. \quad A \implies B$

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A local assumption(s)
B actual (sub)goal

Multiple assumptions

$$[\![\, A_1; \, \ldots \, ; \, A_n \,]\!] \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$$\llbracket\ A_1;\ \dots\ ;\ A_n\ \rrbracket \Longrightarrow B$$

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$$; \quad \approx \quad \text{``and''}$$

4 Type and function definitions
  Type definitions
  Function definitions

Type synonyms

type_synonym $name = \tau$

Introduces a *synonym* $name$ for type τ

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Examples

type_synonym $string = char\ list$

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Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$
\textbf{datatype} \ (\alpha_1, \ldots, \alpha_n)t \ \begin{array}{ll} = & C_1 \ \tau_{1,1} \ldots \tau_{1,n_1} \\ | & \ldots \\ | & C_k \ \tau_{k,1} \ldots \tau_{k,n_k} \end{array}
$$

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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)t$

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- *Distinctness*: $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$

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- *Injectivity:* $(C_i \; x_1 \ldots x_{n_i} = C_i \; y_1 \ldots y_{n_i}) =$
  $(x_1 = y_1 \wedge \cdots \wedge x_{n_i} = y_{n_i})$

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Distinctness and injectivity are applied automatically
Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with *case*:

$$(\textbf{\textit{case}}\ xs\ \textbf{\textit{of}}\ \ []\Rightarrow \ldots\ \ |\ \ y\#ys \Rightarrow \ldots\ y\ \ldots\ ys\ \ldots)$$

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Wildcards: _

$$(\textit{\textbf{case}}\ m\ \textit{\textbf{of}}\ \ 0 \Rightarrow Suc\ 0\ \ |\ \ Suc\ \_ \Rightarrow 0)$$

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Need ( ) in context

Tree_Demo.thy

The *option* type

datatype $'a\ option = None\ |\ Some\ 'a$

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If $'a$ has values a_1, a_2, ...
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Typical application:

fun $lookup :: ('a \times 'b)\ list \Rightarrow 'a \Rightarrow 'b\ option$ **where**

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Typical application:

fun $lookup :: ('a \times 'b)\ list \Rightarrow 'a \Rightarrow 'b\ option$ **where**
$lookup\ []\ x = None \mid$
$lookup\ ((a,\ b)\ \#\ ps)\ x =$
  (**if** $a = x$ **then** $Some\ b$ **else** $lookup\ ps\ x$)

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n*n$

Non-recursive definitions

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n*n$

No pattern matching, just $f\ x_1\ \ldots\ x_n\ =\ \ldots$

The danger of nontermination

How about $f\,x = f\,x + 1$?

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Subtract $f\,x$ on both sides.
$$\implies 0 = 1$$

The danger of nontermination

How about $f\,x = f\,x + 1$?

Subtract $f\,x$ on both sides.

$$\Longrightarrow 0 = 1$$

! All functions in HOL must be total !

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- Pattern-matching over datatype constructors

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- Order of equations matters

- Termination must be provable automatically by size measures

- Proves customized induction schema

Example: separation

fun *sep* :: $'a \Rightarrow{} 'a\ list \Rightarrow{} 'a\ list$ **where**
sep a (x#y#zs) = x # a # sep a (y#zs) |
sep a xs = xs

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**
$ack\ 0 \qquad\quad n \qquad\quad = Suc\ n\ \mid$
$ack\ (Suc\ m)\ 0 \qquad\ = ack\ m\ (Suc\ 0)\ \mid$
$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**
$ack\ 0 \qquad\quad n \qquad\quad = Suc\ n \quad |$
$ack\ (Suc\ m)\ 0 \qquad = ack\ m\ (Suc\ 0) \quad |$
$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$

Terminates because the arguments decrease
lexicographically with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

primrec

- A restrictive version of **fun**

primrec

- A restrictive version of **fun**
- Means *primitive recursive*

primrec

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The essence of primitive recursion:

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\end{array}
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The essence of primitive recursion:

$$f(0) \quad = \ldots \qquad \text{no recursion}$$
$$f(Suc\ n) \quad = \ldots f(n) \ldots$$

$$g([]) \quad = \ldots \qquad \text{no recursion}$$
$$g(x\#xs) \quad = \ldots g(xs) \ldots$$

Basic induction heuristics

Theorems about recursive functions
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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {}'a\ list \Rightarrow {}'a\ list$ **where**
$rev\ [] \quad\quad = []\ |$
$rev\ (x\#xs) \quad = rev\ xs\ @\ [x]$

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lemma $itrev\ xs\ [] = rev\ xs$

Induction_Demo.thy

Generalisation

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- Replace constants by variables

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- Generalize free variables
  - by $arbitrary$ in induction proof
  - (or by universal quantifier in formula)

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In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction

Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0\ \ |$
$div2\ (Suc\ 0) = 0\ \ |$
$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

Computation Induction

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$\rightsquigarrow$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \qquad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

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$\leadsto$ induction rule `div2.induct`:

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prove $P(e)$ assuming $P(r_1)$, ..., $P(r_k)$.

Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal

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Heuristic:
- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal
- ideally the a_i should be variables.

Induction_Demo.thy

Computation Induction

Simplification means . . .

Using equations $l = r$ from left to right

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Terminology: equation $\rightsquigarrow$ *simplification rule*

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Terminology: equation $\rightsquigarrow$ *simplification rule*

Simplification $=$ (Term) Rewriting

An example

$$
\begin{array}{rcll}
0 + n & = & n & (1) \\
(Suc\ m) + n & = & Suc\ (m + n) & (2) \\
(Suc\ m \le Suc\ n) & = & (m \le n) & (3) \\
(0 \le m) & = & True & (4)
\end{array}
$$

Equations:

An example

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$$0 + Suc\ 0 \le Suc\ 0 + x$$

Rewriting:

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$$
\begin{aligned}
0 + Suc\ 0 &\le Suc\ 0 + x & \overset{(1)}{=} \\
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$$0 \leq 0 + x \qquad \overset{(4)}{=}$$
$$True$$

Conditional rewriting

Simplification rules can be conditional:

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Example

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\begin{aligned}
p(0) &= True \\
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\end{aligned}
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We can simplify $f(0)$ to $g(0)$ but
we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

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Principle:

$$[\![\; P_1; \; \ldots; \; P_k \;]\!] \Longrightarrow l = r$$

is suitable as a $simp$-rule only
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$$n < m \implies (n < Suc \ m) = True$$
$$Suc \ n < m \implies (n < m) = True$$

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$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method $simp$

Goal: 1. $\llbracket P_1; \ldots; P_m \rrbracket \Longrightarrow C$

apply$(simp\ add:\ eq_1\ \ldots\ eq_n)$

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- lemmas with attribute $simp$
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- rules from **fun** and **datatype**
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Variations:

- $(simp\ \ldots\ del{:}\ \ldots)$ removes $simp$-lemmas
- add and del are optional

auto versus *simp*

- *auto* acts on all subgoals
- *simp* acts only on subgoal 1

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- *auto* can also be modified:
  (*auto simp add*: ... *simp del*: ...) 📢

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add{:}\ f\_def \dots)$$

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$$(simp\ add:\ f\_def\ \ldots\ )$$

f is the function whose definition is to be unfolded.

Case splitting with $simp/auto$

Automatic:

$$P \ (\textit{if } A \textit{ then } s \textit{ else } t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

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By hand:

$$P \ (\textit{case } e \textit{ of } 0 \Rightarrow a \mid Suc \ n \Rightarrow b)$$
$$=$$
$$(e = 0 \longrightarrow P(a)) \wedge (\forall \, n. \ e = Suc \ n \longrightarrow P(b))$$

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Or $auto$. Similar for any datatype t: $t.split$

Simp_Demo.thy

Chapter 3

Case Study: IMP Expressions

7 Case Study: IMP Expressions

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This section introduces

  arithmetic and boolean expressions

of our imperative language IMP.

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IMP *commands* are introduced later.

Concrete and abstract syntax

Concrete syntax: strings, eg "a+5*b"

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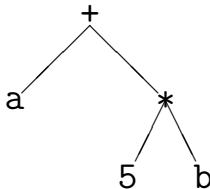
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Concrete and abstract syntax

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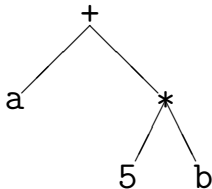
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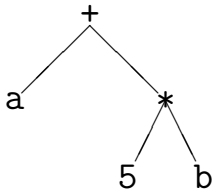
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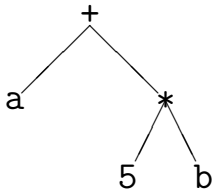
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Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

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Parser: function from strings to trees

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Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

where n can be any natural number and x any variable.

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$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

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x	$V\ ''x''$
x+y	$Plus\ (V\ ''x'')\ (V\ ''y'')$
2+(z+3)	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

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$$N\, 0 \;\neq\; Plus\,(N\,0)\,(N\,0)$$

The (program) state

What is the value of x+1?

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  depends on the value of its variables.
- The value of all variables is recorded in the *state*.
- The state is a function from variable names to
  values:

  type_synonym $val = int$
  type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

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If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

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$$f(a := b)$$

is the function that behaves like f
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$$f(a := b) = (\lambda x.\ if\ x = a\ then\ b\ else\ f\ x)$$

How to write down a state

Some states:

- $\lambda x.\ 0$

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How to write down a state

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- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
- $((\lambda x.\ 0)(''a'' := 5))(''x'' := 3)$

How to write down a state

Some states:

- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
- $((\lambda x.\ 0)(''a'' := 5))(''x'' := 3)$

Nicer notation:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

How to write down a state

Some states:

- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
- $((\lambda x.\ 0)(''a'' := 5))(''x'' := 3)$

Nicer notation:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

BExp.thy

`ASM.thy`

This was easy.

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Because evaluation of expressions always terminates.

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But execution of programs may *not* terminate.
Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 4

Logic and Proof Beyond Equality

Syntax (in decreasing precedence):

$$
\begin{array}{rlll}
form & ::= & (form) & | & term = term & | & \neg form \\
& | & form \wedge form & | & form \vee form & | & form \longrightarrow form \\
& | & \forall x.\ form & | & \exists x.\ form
\end{array}
$$

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Examples:

$$\neg\ A \wedge B \vee C \ \equiv\ ((\neg\ A) \wedge B) \vee C$$

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\begin{array}{rcl}
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A \wedge B = B \wedge A & \equiv & {\color{red} A \wedge (B = B) \wedge A} \\
\forall\, x.\ P\ x \wedge Q\ x & \equiv & \forall\, x.\ (P\ x \wedge Q\ x)
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\end{array}
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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y\ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

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$$\forall\, x\, y.\ P\, x\, y\ \equiv\ \forall\, x.\ \forall\, y.\ P\, x\, y$$

Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x.\ Q\ x \ \rightsquigarrow \ P \wedge (\forall x.\ Q\ x) \quad !$$

Mathematical symbols

... and their ascii representations:

$\forall$	\<forall>	ALL
$\exists$	\<exists>	EX
λ	\<lambda>	%
$\longrightarrow$	-->	
$\longleftrightarrow$	<->	
$\wedge$	/\	&
$\vee$	\/	|
$\neg$	\<not>	~
$\neq$	\<noteq>	~=

Sets over type $'a$

$'a\ set$

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$'a \ set$

- $\{\}, \quad \{e_1, \ldots, e_n\}$

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- $e \in A,\quad A \subseteq B$

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- $A \cup B, \quad A \cap B, \quad A - B, \quad -A$

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- $\ldots$

$\in$	\<in>	:
$\subseteq$	\<subseteq>	<=
$\cup$	\<union>	Un
$\cap$	\<inter>	Int

Set comprehension

- $\{x.\ P\}$ where x is a variable

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- $\{x.\ P\}$ where x is a variable
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- $\{x.\ P\}$ where x is a variable
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- Instead: $\{t\ |x\ y\ z.\ P\}$

Set comprehension

- $\{x.\ P\}$ where x is a variable
- But not $\{t.\ P\}$ where t is a proper term
- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists x\ y\ z.\ v = t \land P\}$
  where x, y, z are the free variables in t

$simp$ and $auto$

$simp$: rewriting and a bit of arithmetic

$auto$: rewriting and a bit of arithmetic, logic and sets

$simp$ and $auto$

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- Show you where they got stuck

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- Show you where they got stuck
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Exception: $auto$ acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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- rewriting, logic, sets, relations and a bit of arithmetic.
- incomplete but better than $auto$.

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- Succeeds or fails

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blast

- A complete proof search procedure for FOL . . .

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- A complete proof search procedure for FOL ...
- ... but (almost) without "="

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Automating arithmetic

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- proves linear formulas (no "$*$")

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Automating arithmetic

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- proves linear formulas (no "$*$")
- complete for quantifier-free *real* arithmetic
- complete for first-order theory of *nat* and *int* (Presburger arithmetic)

Sledgehammer

Architecture:

Isabelle

external
ATPs[1]

---
[1]Automatic Theorem Provers
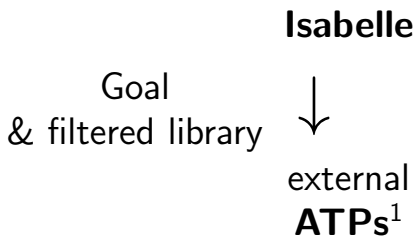
Architecture:

Isabelle

Goal
& filtered library $\downarrow$

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---
[1]Automatic Theorem Provers

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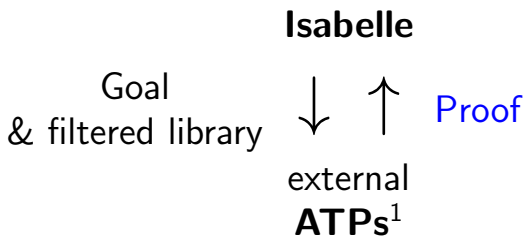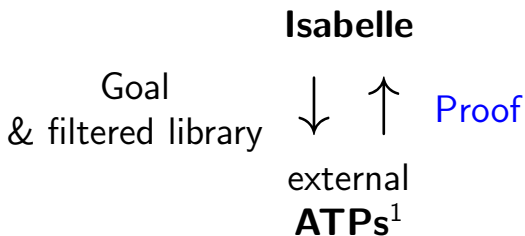
Characteristics:

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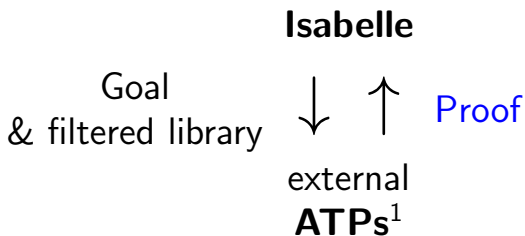
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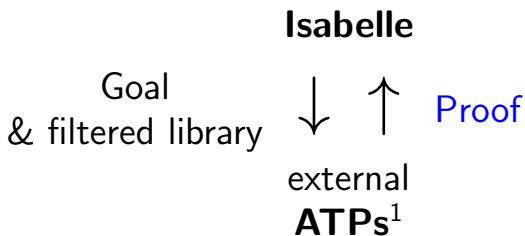
Characteristics:

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- sometimes it doesn't.

Do you feel lucky?

---
[1]Automatic Theorem Provers

by(*proof-method*)

$$\approx$$

apply(*proof-method*)
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

What are these *?-variables* ?

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After you have finished a proof, Isabelle turns all free variables V in the theorem into $?V$.

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  conjI[of "a=b" "False"] $\rightsquigarrow$

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- By unification:
  unifying $?P \wedge ?Q$ with $a{=}b \wedge False$

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- By hand:
  conjI[of "a=b" "False"] $\rightsquigarrow$
  $⟦a = b; \mathit{False}⟧ \Longrightarrow a = b \land \mathit{False}$

- By unification:
  unifying $?P \land ?Q$ with $a{=}b \land \mathit{False}$
  sets $?P$ to $a{=}b$ and $?Q$ to False.

Rule application

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Example: rule: $\llbracket ?P;\ ?Q \rrbracket \implies ?P \land ?Q$

Rule application

Example: rule: $\llbracket ?P;\ ?Q \rrbracket \implies ?P \land ?Q$
         subgoal: $1.\ \ldots \implies A \land B$

Rule application

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　　　　　subgoal: 1. $\ldots \implies A \land B$

Result: 1. $\ldots \implies A$
　　　　2. $\ldots \implies B$

Rule application

Example: rule: $\llbracket ?P;\ ?Q \rrbracket \implies ?P \wedge ?Q$
           subgoal: 1. $\ldots \implies A \wedge B$
Result: 1. $\ldots \implies A$
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The general case: applying rule $\llbracket A_1;\ \ldots\ ;\ A_n \rrbracket \implies A$
to subgoal $\ldots \implies C$:

Rule application

Example: rule: $\llbracket \mathit{?P}; \mathit{?Q} \rrbracket \implies \mathit{?P} \land \mathit{?Q}$

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- Unify A and C

Rule application

Example: rule: $[\![?P;\ ?Q]\!] \implies ?P \land ?Q$
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- Replace C with n new subgoals $A_1 \ldots A_n$

Rule application

Example: rule: $\llbracket ?P; \ ?Q \rrbracket \Longrightarrow ?P \land ?Q$
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                apply($rule \ xyz$)

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"Backchaining"

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \land ?Q} \; \texttt{conjI}$$

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$$\frac{?P \quad ?Q}{?P \land ?Q} \; \texttt{conjI}$$

$$\frac{?P \implies ?Q}{?P \longrightarrow ?Q} \; \texttt{impI}$$

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge ?Q} \texttt{ conjI}$$

$$\frac{?P \Longrightarrow ?Q}{?P \longrightarrow ?Q} \texttt{ impI} \qquad \frac{\bigwedge x.\ ?P\ x}{\forall\, x.\ ?P\ x} \texttt{ allI}$$

Typical backwards rules

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They are known as introduction rules
because they *introduce* a particular connective.

Automating intro rules

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If r is a theorem $\llbracket A_1; \ldots; A_n \rrbracket \implies A$ then

$$(blast\ intro:\ r)$$

allows $blast$ to backchain on r during proof search.

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Example:

theorem le_trans: $[\![\ ?x \leq \ ?y; \ ?y \leq \ ?z \ ]\!] \Longrightarrow \ ?x \leq \ ?z$

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    goal 1. $\llbracket\; a \leq b;\; b \leq c;\; c \leq d\; \rrbracket \implies a \leq d$

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    proof **apply**($blast\ intro:\ le\_trans$)

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Also works for $auto$ and $fastforce$

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Also works for $auto$ and $fastforce$

Can greatly increase the search space!

Forward proof: OF

If r is a theorem $A \implies B$

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Example: theorem refl: $\mathit{?t} = \mathit{?t}$

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Example: theorem `refl`: $?t = ?t$

```
conjI[OF refl[of "a"]]
```

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```
conjI[OF refl[of "a"]]
```
$$\rightsquigarrow$$
$$?Q \Longrightarrow a = a \land ?Q$$

The general case:

If r is a theorem $[\![\ A_1;\ \ldots;\ A_n\ ]\!] \Longrightarrow A$
and $r_1,\ \ldots,\ r_m\ (m{\leq}n)$ are theorems then

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conjI[OF refl[of "a"] refl[of "b"]]
```
$$\rightsquigarrow$$
$$a = a \land b = b$$

From now on: *?* mostly suppressed on slides

Single_Step_Demo.thy

$\implies$ versus $\longrightarrow$

$\implies$ is part of the Isabelle framework. It structures
theorems and proof states: $\llbracket A_1; \ldots; A_n \rrbracket \implies A$

$\Longrightarrow$ versus $\longrightarrow$

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$\longrightarrow$ is part of HOL and can occur inside the logical formulas A_i and A.

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$\longrightarrow$ is part of HOL and can occur inside the logical formulas A_i and A.

Phrase theorems like this $\llbracket\, A_1;\, \ldots;\, A_n\, \rrbracket \Longrightarrow A$

not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

Example: even numbers

Informally:

Example: even numbers

Informally:

- 0 is even

Example: even numbers

Informally:

- 0 is even
- If n is even, so is $n + 2$

Example: even numbers

Informally:

- 0 is even
- If n is even, so is $n + 2$
- These are the only even numbers

Example: even numbers

Informally:

- 0 is even
- If n is even, so is $n + 2$
- These are the only even numbers

In Isabelle/HOL:

inductive $ev :: nat \Rightarrow bool$

Example: even numbers

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Example: even numbers

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- 0 is even
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In Isabelle/HOL:

inductive $ev :: nat \Rightarrow bool$
where
$\quad ev\ 0 \quad |$
$\quad ev\ n \implies ev\ (n + 2)$

An easy proof: $ev\ 4$

$$ev\ 0 \implies ev\ 2 \implies ev\ 4$$

Consider

fun *evn* :: *nat* $\Rightarrow$ *bool* **where**
evn 0 = *True* |
evn (*Suc* 0) = *False* |
evn (*Suc* (*Suc* *n*)) = *evn* *n*

Consider

fun $evn :: nat \Rightarrow bool$ **where**
$evn\ 0\ =\ True\ |$
$evn\ (Suc\ 0)\ =\ False\ |$
$evn\ (Suc\ (Suc\ n))\ =\ evn\ n$

A trickier proof: $ev\ m \Longrightarrow evn\ m$

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  $\Longrightarrow m = 0 \Longrightarrow evn\ m = True$

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  $\Longrightarrow m = n{+}2$ and $evn\ n$ (IH)

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  $\Longrightarrow m = 0 \Longrightarrow evn\ m = True$

- rule $ev\ n \Longrightarrow ev\ (n{+}2)$
  $\Longrightarrow m = n{+}2$ and $evn\ n$ (IH)
  $\Longrightarrow evn\ m = evn\ (n{+}2) = evn\ n = True$

Rule induction for ev

To prove

$$ev \ n \Longrightarrow P \ n$$

by *rule induction* on $ev \ n$ we must prove

Rule induction for ev

To prove

$$ev\ n \implies P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$

Rule induction for ev

To prove

$$ev\ n \implies P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$
- $P\ n \implies P(n{+}2)$

Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$
- $P\ n \Longrightarrow P(n+2)$

Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n+2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

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Note:

- I may have multiple arguments.

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**
$\quad [\![\ I\ a_1;\ \dots\ ;\ I\ a_n\ ]\!] \implies I\ a\ \mid$
$\quad \vdots$

Note:

- I may have multiple arguments.
- Each rule may also contain *side conditions* not involving I.

Rule induction in general

To prove

$$I\ x \Longrightarrow P\ x$$

by *rule induction* on $I\ x$

Rule induction in general

To prove

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by *rule induction* on $I \ x$
we must prove for every rule

$$\llbracket \ I \ a_1; \ \ldots \ ; \ I \ a_n \ \rrbracket \Longrightarrow I \ a$$

that P is preserved:

Rule induction in general

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$$I\ x \Longrightarrow P\ x$$

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that P is preserved:

$$[\![\ I\ a_1;\ P\ a_1;\ \ldots\ ;\ I\ a_n;\ P\ a_n\ ]\!] \Longrightarrow P\ a$$

! Rule induction is absolutely central
to (operational) semantics
and the rest of this lecture course !

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau \; set$ **where**

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    $\llbracket \; a_1 \in I; \; \ldots \; ; \; a_n \in I \; \rrbracket \Longrightarrow a \in I \; \mid$

Inductively defined sets

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Difference to **inductive**:

- arguments of I are tupled, not curried

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**
  $\llbracket\ a_1 \in I;\ \dots\ ;\ a_n \in I\ \rrbracket \Longrightarrow a \in I\ \mid$
  $\vdots$

Difference to **inductive**:

- arguments of I are tupled, not curried
- I can later be used with set theoretic operators, eg $I \cup \dots$

Chapter 5

Isar: A Language for Structured Proofs

Apply scripts

- unreadable

Apply scripts

- unreadable
- hard to maintain

Apply scripts

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- do not scale

Apply scripts

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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

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qed

proves $formula_0 \implies formula_{n+1}$

Isar core syntax

$$
\begin{aligned}
\text{proof} \quad = \quad & \textbf{proof} \ [\text{method}] \ \ \text{step}^* \ \ \textbf{qed} \\
\mid \quad & \textbf{by} \ \text{method}
\end{aligned}
$$

Isar core syntax

proof = **proof** [method] step* **qed**
  | **by** method

method = $(simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$

Isar core syntax

proof = **proof** [method] step* **qed**
 | **by** method

method = $(simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$

step = **fix** variables $(\bigwedge)$
 | **assume** prop $(\Longrightarrow)$
 | [**from** fact$^+$] (**have** | **show**) prop proof

Isar core syntax

$$\text{proof} \quad = \quad \textbf{proof} \; [\text{method}] \; \text{step}^* \; \textbf{qed}$$
$$\quad | \quad \textbf{by} \; \text{method}$$

$$\text{method} \quad = \quad (simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$$

$$\text{step} \quad = \quad \textbf{fix} \; \text{variables} \qquad (\bigwedge)$$
$$\quad | \quad \textbf{assume} \; \text{prop} \qquad (\Longrightarrow)$$
$$\quad | \quad [\textbf{from} \; \text{fact}^+] \; (\textbf{have} \mid \textbf{show}) \; \text{prop} \; \text{proof}$$

$$\text{prop} \quad = \quad [\text{name:}] \; \text{"formula"}$$

Isar core syntax

$$\text{proof} \;=\; \textbf{proof}\ [\text{method}]\ \text{step}^{*}\ \textbf{qed}$$
$$\qquad |\quad \textbf{by}\ \text{method}$$

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$$\qquad |\quad [\textbf{from}\ \text{fact}^{+}]\ (\textbf{have}\ |\ \textbf{show})\ \text{prop}\ \text{proof}$$

$$\text{prop} \;=\; [\text{name:}]\ "\text{formula}"$$

$$\text{fact} \;=\; \text{name}\ |\ \ldots$$

Example: Cantor's theorem

lemma $\neg\ surj(f :: {'}a \Rightarrow {'}a\ set)$

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  assume a: $surj\ f$

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  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$

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    by($simp\ add$: $surj\_def$)

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    by $blast$

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  from c **show** $False$

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    by $blast$
  from c **show** $False$
    by $blast$
qed

Isar_Demo.thy

Cantor and abbreviations

Abbreviations

$$
\begin{array}{rcl}
this & = & \text{the previous proposition proved or assumed} \\
\text{then} & = & \textbf{from } this \\
\text{thus} & = & \textbf{then show} \\
\text{hence} & = & \textbf{then have}
\end{array}
$$

using and with

(**have**|**show**) prop **using** facts

using and **with**

(**have**|**show**) prop **using** facts
=
from facts (**have**|**show**) prop

using and **with**

(**have**|**show**) prop **using** facts
=
from facts (**have**|**show**) prop

with facts
=
from facts *this*

Structured lemma statement

lemma
  fixes $f :: {'}a \Rightarrow {'}a \; set$
  assumes s: $surj \; f$
  shows $False$

Structured lemma statement

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   fixes $f :: {'}a \Rightarrow {'}a \; set$
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   shows $False$
proof $-$

Structured lemma statement

lemma
   fixes $f :: {'}a \Rightarrow {'}a \ set$
   assumes s: $surj\ f$
   shows $False$
proof $-$ no automatic proof step

Structured lemma statement

lemma
  fixes $f :: \ 'a \Rightarrow \ 'a \ set$
  assumes s: $surj \ f$
  shows $False$
proof $-$ no automatic proof step
  have $\exists \ a. \ \{x. \ x \notin f \ x\} = f \ a$ **using** s
    by($auto \ simp$: $surj\_def$)

Structured lemma statement

lemma
   fixes $f :: {'}a \Rightarrow {'}a\ set$
   assumes s: $surj\ f$
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   have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
     by$(auto\ simp:\ surj\_def)$
   thus $False$ **by** $blast$
qed

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  thus $False$ **by** $blast$
qed

     Proves $surj\ f \Longrightarrow False$

Structured lemma statement

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proof $-$ no automatic proof step
  have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
    by($auto\ simp$: $surj\_def$)
  thus $False$ **by** $blast$
qed

      Proves *surj f $\Longrightarrow$ False*
      but *surj f* *becomes local fact s in proof.*

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2$ $\ldots$
assumes *a:* P **and** *b:* Q $\ldots$
shows R

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- **fixes** and **assumes** sections optional

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \dots$
assumes *a:* P **and** *b:* $Q \dots$
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

Case distinction

show R
proof *cases*
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg\, P$
  $\vdots$
  show R $\langle proof \rangle$
qed

Case distinction

show R
proof *cases*
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg\, P$
  $\vdots$
  show R $\langle proof \rangle$
qed

have $P \vee Q$ $\langle proof \rangle$
then show R
proof
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume Q
  $\vdots$
  show R $\langle proof \rangle$
qed

Contradiction

show $\neg\, P$
proof
  assume P
  ⋮
  show $False \;\langle proof \rangle$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False\ \langle proof \rangle$
qed

show P
proof ($rule\ ccontr$)
  assume $\neg P$
  $\vdots$
  show $False\ \langle proof \rangle$
qed

$$\longleftrightarrow$$

show $P \longleftrightarrow Q$
proof
  assume P
  ⋮
  show Q $\langle proof \rangle$
next
  assume Q
  ⋮
  show P $\langle proof \rangle$
qed

∀ and ∃ introduction

show $\forall x.\ P(x)$
proof
  fix x local fixed variable
  show $P(x)$ $\langle proof \rangle$
qed

∀ and ∃ introduction

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  show $P(x)$ $\langle proof \rangle$
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show $\exists x.\ P(x)$
proof
  ⋮
  show $P(witness)$ $\langle proof \rangle$
qed

∃ elimination: **obtain**

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have $\exists\, x.\ P(x)$
then obtain x **where** *p:* $P(x)$ **by** *blast*

⋮ x fixed local variable

∃ elimination: **obtain**

have $\exists x.\ P(x)$
then obtain x **where** p: $P(x)$ **by** *blast*

⋮ x fixed local variable

Works for one or more x

obtain example

lemma $\neg\ surj(f :: {}'a \Rightarrow {}'a\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**($auto\ simp\!:\ surj\_def$)

obtain example

lemma $\neg\ surj(f :: {'}a \Rightarrow {'}a\ set)$
proof
  assume $surj\ f$
  hence $\exists\,a.\ \{x.\ x \notin f\,x\} = f\,a$ **by**($auto\ simp$: $surj\_def$)
  then obtain a **where** $\{x.\ x \notin f\,x\} = f\,a$ **by** $blast$

obtain example

lemma $\neg \; surj(f :: {'a} \Rightarrow {'a} \; set)$
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  assume $surj \; f$
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  then obtain a **where** $\{x. \; x \notin f \, x\} = f \, a$ **by** $blast$
  hence $a \notin f \, a \longleftrightarrow a \in f \, a$ **by** $blast$

obtain example

lemma $\neg\; surj(f :: {'a} \Rightarrow {'a}\; set)$
proof
  assume $surj\; f$
  hence $\exists\, a.\; \{x.\; x \notin f\, x\} = f\, a$ **by**($auto\; simp:\; surj\_def$)
  then obtain a **where** $\{x.\; x \notin f\, x\} = f\, a$ **by** $blast$
  hence $a \notin f\, a \longleftrightarrow a \in f\, a$ **by** $blast$
  thus $False$ **by** $blast$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  $\vdots$
  show $x \in B$ $\langle proof \rangle$
qed

Isar_Demo.thy

Exercise

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ **(is** $?L \longleftrightarrow ?R$**)**

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)
proof
  assume $?L$
  ⋮
  show $?R$ $\langle proof \rangle$
next
  assume $?R$
  ⋮
  show $?L$ $\langle proof \rangle$
qed

?thesis

show *formula*
proof -
  ⋮
  show *?thesis* ⟨*proof*⟩
qed

show *formula* *(is ?thesis)*
proof -
  ⋮
  show *?thesis* ⟨*proof*⟩
qed

?thesis

show *formula*   *(is ?thesis)*
proof -
  ⋮
  show *?thesis* ⟨*proof*⟩
qed

Every show implicitly defines *?thesis*

let

Introducing local abbreviations in proofs:

> **let** $?t =$ *"some-big-term"*
> $\vdots$
> **have** *"... $?t$ ... "*

Quoting facts by value

By name:

    have *x0:* $"x > 0"$ ...
    ⋮
    from *x0* ...

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By value:

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    ⋮
    from ‘$x{>}0$‘ ...

Quoting facts by value

By name:

    have *x0:* $"x > 0"$ ...

    ⋮

    from *x0* ...

By value:

    have $"x > 0"$ ...

    ⋮

    from '$x>0$' ...

       ↑   ↑

    back quotes

Isar_Demo.thy

Pattern matching and quotations

Example

lemma
$\exists\, ys\ zs.\ xs = ys\ @\ zs\ \wedge$
$(length\ ys = length\ zs \vee length\ ys = length\ zs + 1)$

Example

lemma
  $\exists\, ys\; zs.\; xs = ys\; @\; zs\; \wedge$
  $(length\; ys = length\; zs \vee length\; ys = length\; zs + 1)$
proof ???

Isar_Demo.thy

Top down proof development

When automation fails

Split proof up into smaller steps.

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

 have ... **using** ...

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have ... **using** ...
apply -   to make incoming facts
                     part of proof state

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                           part of proof state
apply *auto*   or whatever

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

  have . . . **using** . . .
  apply -   to make incoming facts
                      part of proof state

  apply *auto*   or whatever
  apply . . .

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have . . . **using** . . .
apply -   to make incoming facts
                               part of proof state
apply *auto*   or whatever
apply . . .

At the end:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have … **using** …
apply -   to make incoming facts
                            part of proof state

apply *auto*   or whatever
apply …

At the end:

- **done**

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...
apply -   to make incoming facts
                               part of proof state
apply $auto$ or whatever
apply ...

At the end:

- **done**
- Better: convert to structured proof

moreover—ultimately

have P_1 ...
moreover
have P_2 ...
moreover
$\vdots$
moreover
have P_n ...
ultimately
have P ...

moreover—ultimately

have P_1 ...
moreover
have P_2 ...
moreover
$\vdots$
moreover
have P_n ...
ultimately
have P ...

$\approx$

have lab_1: P_1 ...
have lab_2: P_2 ...
$\vdots$
have lab_n: P_n ...
from lab_1 lab_2 ...
have P ...

With names

Local lemmas

have B **if** *name:* $A_1 \ldots A_m$ **for** $x_1 \ldots x_n$
$\langle proof \rangle$

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Local lemmas

    have B **if** *name:* $A_1 \ldots A_m$ **for** $x_1 \ldots x_n$
    $\langle proof \rangle$

proves $\llbracket A_1; \ldots ; A_m \rrbracket \Longrightarrow B$
where all x_i have been replaced by $?x_i$.

Proof state and Isar text

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In general: **proof** *method*

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Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \ldots \ x_n. \ [\![\ A_1; \ \ldots \ ; \ A_m \ ]\!] \Longrightarrow B$$

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Proof state and Isar text

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Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ldots x_n. \; \llbracket A_1; \ldots ; A_m \rrbracket \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ldots x_n$
assume $A_1 \ldots A_m$
⋮
show B

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ldots x_n.\ [\![\ A_1;\ \ldots\ ;\ A_m\ ]\!] \Longrightarrow B$$

How to prove each subgoal:

> **fix** $x_1 \ldots x_n$
> **assume** $A_1 \ldots A_m$
> $\vdots$
> **show** B

Separated by **next**

Isar_Induction_Demo.thy

Proof by cases

Datatype case analysis

datatype $t = C_1 \vec{\tau} \mid \dots$

Datatype case analysis

datatype $t \;=\; C_1 \; \vec{\tau} \;\mid\; \ldots$

> **proof** $(cases \; "term")$
>   **case** $(C_1 \; x_1 \; \ldots \; x_k)$
> $\ldots \; x_j \; \ldots$
> **next**
> $\vdots$
> **qed**

Datatype case analysis

datatype $t = C_1 \; \vec{\tau} \; | \; \dots$

> **proof** $(cases \; "term")$
>   **case** $(C_1 \; x_1 \; \dots \; x_k)$
> $\dots \; x_j \; \dots$
> **next**
> $\vdots$
> **qed**

---

where   **case** $(C_i \; x_1 \; \dots \; x_k)$ $\equiv$

         fix $x_1 \; \dots \; x_k$
         assume $\underbrace{C_i}_{\text{label}}: \quad \underbrace{term = (C_i \; x_1 \; \dots \; x_k)}_{\text{formula}}$

Isar_Induction_Demo.thy

Structural induction for nat

Structural induction for nat

show $P(n)$
proof ($induction\ n$)
  case 0
  $\vdots$
  show *?case*
next
  case ($Suc\ n$)
  $\vdots$
  show *?case*
qed

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **let** $?case = P(0)$
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$
  show $?case$
qed

Structural induction for nat

show $P(n)$
proof ($induction\ n$)
  case 0 $\equiv$ **let** $?case = P(0)$
  $\vdots$
  show $?case$
next
  case ($Suc\ n$) $\equiv$ **fix** n **assume** Suc: $P(n)$
  $\vdots$ **let** $?case = P(Suc\ n)$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof ($induction\ n$)
  case 0
  $\vdots$
  show $?case$
next
  case ($Suc\ n$)
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **assume** 0: $A(0)$
  $\vdots$ **let** $?case = P(0)$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$

  case 0 $\equiv$ **assume** 0: $A(0)$

  $\vdots$ **let** $?case = P(0)$

  show $?case$

next

  case $(Suc\ n)$ $\equiv$ **fix** n

  $\vdots$ **assume** Suc: $A(n) \Longrightarrow P(n)$
                                     $A(Suc\ n)$

  $\vdots$ **let** $?case = P(Suc\ n)$

  show $?case$

qed

Named assumptions

In a proof of
$$A_1 \implies \ldots \implies A_n \implies B$$
by structural induction:

Named assumptions

In a proof of
$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of
    case C

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we have

    C.IH   the induction hypotheses

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Named assumptions

In a proof of
$$A_1 \implies \ldots \implies A_n \implies B$$

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In the context of
    case C

we have

     $C.IH$ the induction hypotheses

  $C.prems$ the premises A_i

       C $C.IH + C.prems$

A remark on style

- **case** $(Suc\ n)$ ... **show** *?case*
  is easy to write and maintain

A remark on style

- **case** $(Suc\ n)\ \dots$ **show** *?case*
  is easy to write and maintain
- **fix** n **assume** *formula* $\dots$ **show** *formula$'$*
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. *?case*)

`Isar_Induction_Demo.thy`

Rule induction

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: ...
$\vdots$
$rule_n$: ...

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$ **show** $I\ x\ y \Longrightarrow P\ x\ y$
where
$rule_1$: ...
$\vdots$
$rule_n$: ...

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: ...
$\vdots$
$rule_n$: ...

show $I\ x\ y \Longrightarrow P\ x\ y$
proof *(induction rule: I.induct)*

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: ...
⋮
$rule_n$: ...

show $I\ x\ y \Longrightarrow P\ x\ y$
proof (*induction rule*: *I.induct*)
  case $rule_1$
  ...
  show *?case*
next
⋮
next
  case $rule_n$
  ...
  show *?case*
qed

Fixing your own variable names

case $(rule_i \ x_1 \ \ldots \ x_k)$

Renames the first k variables in $rule_i$ (from left to right) to $x_1 \ \ldots \ x_k$.

Named assumptions

In a proof of

$$I \ldots \implies A_1 \implies \ldots \implies A_n \implies B$$

by rule induction on $I \ldots$:

Named assumptions

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by rule induction on $I \ldots$:
In the context of

    case R

we have

    $R.IH$ the induction hypotheses

Named assumptions

In a proof of

$$I \ldots \implies A_1 \implies \ldots \implies A_n \implies B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$ the induction hypotheses

$R.hyps$ the assumptions of rule R

Named assumptions

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$$I \ldots \implies A_1 \implies \ldots \implies A_n \implies B$$

by rule induction on $I \ldots$:
In the context of

    case R

we have

    $R.IH$ the induction hypotheses

  $R.hyps$ the assumptions of rule R

$R.prems$ the premises A_i

Named assumptions

In a proof of

$$I \ldots \implies A_1 \implies \ldots \implies A_n \implies B$$

by rule induction on $I \ldots$:
In the context of

    case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i
R	$R.IH + R.hyps + R.prems$

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0\ |$
$evSS$: $ev\ n \implies ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?

Rule inversion

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That it was proved by either $ev0$ or $evSS$!

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$$ev\ n \Longrightarrow n = 0 \lor (\exists\, k.\ n = Suc\ (Suc\ k) \land ev\ k)$$

Rule inversion

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$ev\ n \Longrightarrow n = 0 \ \lor \ (\exists\, k.\ n = Suc\ (Suc\ k) \ \land \ ev\ k)$

Rule inversion $=$ case distinction over rules

Isar_Induction_Demo.thy

Rule inversion

Rule inversion template

from *'ev n'* **have** *P*
proof *cases*
  case *ev0* $n = 0$
  ⋮
  show *?thesis* ...
next
  case (*evSS k*) $n = Suc\ (Suc\ k),\ ev\ k$
  ⋮
  show *?thesis* ...
qed

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Rule inversion template

from *'ev n'* **have** P
proof *cases*
  case *ev0* $n = 0$
  ⋮
  show *?thesis* ...
next
  case (*evSS k*) $n = Suc\ (Suc\ k),\ ev\ k$
  ⋮
  show *?thesis* ...
qed

Impossible cases disappear automatically

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