Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

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3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{lcl}
com & ::= & \texttt{SKIP} \\
 & | & string\ \texttt{::=}\ aexp \\
 & | & com\ \texttt{;;}\ com \\
 & | & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\
 & | & \texttt{WHILE}\ bexp\ \texttt{DO}\ com
\end{array}
$$

Commands

Abstract syntax:

$$\begin{array}{rcl} \textbf{datatype}\ com & = & SKIP \\ & \mid & Assign\ string\ aexp \\ & \mid & Seq\ com\ com \\ & \mid & If\ bexp\ com\ com \\ & \mid & While\ bexp\ com \end{array}$$

Com.thy

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Big-step semantics

Concrete syntax:

$$(\mathit{com},\ \mathit{initial\text{-}state}) \Rightarrow \mathit{final\text{-}state}$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x ::=\ a,\ s) \Rightarrow\ s(x :=\ aval\ a\ s)$$

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$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s_1 \qquad (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

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$$\frac{\vdots}{(''x'' ::= N\,5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{array}{rcl} w & = & \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b & = & \mathit{NotEq}\ (V\ ''x'')\ (N\,2) \\ c & = & ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\,1) \\ s_i & = & s(''x'' := i) \end{array}$$

Examples: derivation trees

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$$\mathit{NotEq}\ a_1\ a_2 =$$
$$\mathit{Not}(\mathit{And}\ (\mathit{Not}(\mathit{Less}\ a_1\ a_2))\ (\mathit{Not}(\mathit{Less}\ a_2\ a_1)))$$

Logically speaking

$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c{,}s)\ t$$

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is just infix syntax for

$$big\_step\ (c{,}s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP, s) \Rightarrow t$?

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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

is logically equivalent to

$$\frac{\begin{array}{c}(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \\ \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P\end{array}}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$ (with a new fixed s_2).
No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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To prove a goal P with assumption asm,
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> To prove a goal P with assumption asm,
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Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

elim attribute

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- Can also be added locally, eg *(blast elim: ...)*
- Variant: *elim!* applies elim-rules eagerly.

`Big_Step.thy`

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$
$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow\ s' \wedge (w,\ s') \Rightarrow\ t)$$

$$\vee$$

$$\neg\ bval\ b\ s \wedge t = s$$

Equivalence proof

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$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$(w',\ s) \Rightarrow t$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

Big_Step.thy

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

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Example problem:

(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

- nonterminating computations,

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- nonterminating computations,
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We need a finer grained semantics!

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Small-step semantics

Concrete syntax:

$$(com,state) \rightarrow (com,state)$$

Small-step semantics

Concrete syntax:

$$(com,state) \to (com,state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Intended meaning of $(c,\ s) \rightarrow (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
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Execution as finite or infinite reduction:

$$(c_1, s_1) \rightarrow (c_2, s_2) \rightarrow (c_3, s_3) \rightarrow \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \rightarrow cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \rightarrow cs'$

The intention:

$$(SKIP,\ s) \ \text{ is final}$$

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Why?

$SKIP$ is the empty program.

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Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a,\ s)\ \rightarrow$$

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$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;;c_2, s) \ \rightarrow}$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

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Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \to$$

$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \to$$
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where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

$$(w,\ s_0) \to \ldots$$

where
$$\begin{aligned} w &= WHILE\ b\ DO\ c \\ b &= Less\ (V\ ''x'')\ (N\ 1) \\ c &= ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_n &= <''x'' := n> \end{aligned}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow*$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction

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Proof by rule induction (of course on $cs \Rightarrow t$)

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In two cases a lemma is needed:

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Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1{}',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1{}';;\ c_2,\ s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1, s) \rightarrow* (c_1', s') \Longrightarrow (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

Proof by rule induction.

From $\rightarrow*$ to $\Rightarrow$

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Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.

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In the induction step a lemma is needed:

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Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \ \longleftrightarrow \ cs \rightarrow\!* \ (SKIP,\ t)$

Small_Step.thy

Equivalence of big and small

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Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

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That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$

by induction on c.

Can execution stop prematurely?

That is, are there any final configs except $(SKIP,s)$?

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \to ct \Longrightarrow False$

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Together:

Corollary $\mathit{final}\ (c, s) = (c = SKIP)$

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$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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(by final $=$ SKIP)

Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

May versus Must

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
program\ exp\ \{ \\
\quad a := 1 \\
\quad while\ (0{<}n)\ do\ \{ \\
\quad\quad a := a + a; \\
\quad\quad n := n - 1 \\
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$$

At the end of the execution, variable a should contain 2^n, where n is the original value of variable n!
and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

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wp of equivalent programs is equal

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Reasoning along syntax of program!

That was easy!

That was easy! But what about $While$?

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Unfolding will continue forever!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$\llbracket wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s \rrbracket \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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Unfold rules still hold:

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$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c_1\ Q\ s\ \textsf{\textit{else}}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate I, such that

$$\bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s$$

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s \rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

Wp_Demo.thy

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If $c = \ WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if\!-\!then\!-\!else$

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$ which we have to split.

wlp_if_eq and wlp_whileI' produce $if-then-else$ which we have to split.

Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![\mathit{wf}\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ **then** $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

lemma $ASSUME\_\Theta\_alt$:
$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta = (\forall\,(f,(P,c,Q))\in\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0)\in R \wedge\ P\ s)\ c\ Q)$

unfolding $ASSUME\_\Theta\_def\ HT'set\_r\_def$ **..**

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Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Simplify specification of pre/postcondition, and invariants

Standard operators

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

IMP2/Introduction.thy

Adding more Operators

C-like syntax

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Arith: $+$,$-$,$*$,$/$ with usual binding

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$while\ (bexp)\ c$

IMP2/Introduction.thy

Program Syntax

More Readable VCs

Idea: Replace $s\ ''x''$ by (Isabelle) variable x.

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Similar: $s_0\ ''x''$ by x_0.

If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$$

`IMP2/Introduction.thy`

More Readable VCs

More Readable Annotations

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All variables that occur in the program!

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Precondition: x interpreted as $s\ ''x''$

Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

IMP2/Introduction.thy

More Readable Annotations

6 Example Verifications

Loop Patterns

Euclid's Algorithm

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We've seen a few loop's already:

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

`IMP2/Examples.thy`

Count-up, Count-Down

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Applications: sqrt, log, ...

`IMP2/Examples.thy`

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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```
l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
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Invariant? $l^2 \leq n < h^2 \wedge \ldots$ (range contains solution)

This program is actually tricky to get right!

`IMP2/Examples.thy`

Bisection

6 Example Verifications

Loop Patterns

Euclid's Algorithm

Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers a, b

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

IMP2/Examples.thy

Euclid

Modified Variables

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Only $a{,}i$ changed: $\forall\, x.\ x \notin \{''a'',\ ''i''\} \longrightarrow s\ x = s'\ x$

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Program modifies at most variables it assigns to

$\pi\text{: } (c,\ s) \Rightarrow t \Longrightarrow \mathit{modifies}\ (\mathit{lhsv}\ \pi\ \ c)\ t\ s$

Modified Variables

We can strengthen correctness statement (automatically)

$wp\ \pi\ \ c\ Q\ s \Longrightarrow wp\ \pi\ \ c\ (\lambda s'.\ Q\ s' \wedge \mathit{modifies}\ (\mathit{lhsv}\ \pi\ \ c)\ s'\ s)\ s$

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For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow wp\ \ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}){\in}R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

Modified Variables

The VCG will automatically rewrite with rule

$$\llbracket modifies\ vs\ s\ s';\ x \notin vs \rrbracket \Longrightarrow s\ x = s'\ x$$

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program_spec computes lhs-variables:

$HT\_mods\ \pi\ mods\ P\ c\ Q \equiv HT\ \pi\ P\ c\ (\lambda s_0\ s.\ modifies\ mods\ s\ s_0 \wedge Q\ s_0\ s)$

IMP2/Examples.thy

Euclid – show modified sets

Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1; \\
\mathit{while}\ (m{>}0)\ \{ \\
\quad n{=}a;\ a = 1; \\
\quad \mathit{while}\ (n{>}0)\ \{ \\
\qquad a{=}2*a;\ n{=}n{-}1 \\
\quad \}; \\
\quad m{=}m{-}1 \\
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\end{array}
$$

What does this compute

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What does this compute?

Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

Modular Proofs

Inner loop invariant: Would like to refer to n right before loop!

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Still, we already have verified inner loop!

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Inner loop invariant: Would like to refer to n right before loop!

In our simple VCG, we can't!

Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

a=1;
while (m>0) {
  n=a;
  inline exp_count_down;
  m=m−1
}

Modular Proofs

a=1;
$while$ (m>0) {
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Reuse existing proof of exp-count-down program!

Modular Proofs

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$$\llbracket HT\ \pi\ P\ c\ Q;\ \bigwedge s.\ P'\ s \Longrightarrow P\ s;\ \bigwedge s_0\ s.\ \llbracket P\ s_0;\ P'\ s_0;\ Q\ s_0\ s\rrbracket \Longrightarrow Q'\ s_0\ s\rrbracket$$
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Modular Proofs

Re-using proofs:

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with modified sets:

$$\llbracket HT\_mods\ \pi\ \ mods\ \ P\ \ c\ \ Q;\ \ P\ \ s;\ \bigwedge s'.\ \llbracket modifies\ \ mods\ \ s'\ \ s;\ \ Q\ \ s\ \ s'\rrbracket \Longrightarrow Q'\ \ s'\rrbracket \Longrightarrow wp\ \pi\ \ c\ \ Q'\ \ s$$

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VCG will automatically use this rule.

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

`IMP2/Examples.thy`

Power-Tower

7 Advanced Verification

Arrays

Data Refinement

Local Variables

Recursion

Arrays

Every variable is of type $int \Rightarrow int$.

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Arithmetic Expressions:

$Vidx{::}char\ list \Rightarrow aexp \Rightarrow aexp$

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$AssignIdx :: char\ list \Rightarrow aexp \Rightarrow aexp \Rightarrow com$

π: $(x[i] ::= a,\ s) \Rightarrow s(x := (s\ x)(aval\ i\ s := aval\ a\ s))$

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$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

Arrays

Every variable is of type $int \Rightarrow int$.

Arithmetic Expressions:

$Vidx :: char\ list \Rightarrow aexp \Rightarrow aexp$

$aval\ (Vidx\ x\ i)\ s = s\ x\ (aval\ i\ s)$

Commands:

$AssignIdx :: char\ list \Rightarrow aexp \Rightarrow aexp \Rightarrow com$

π: $(x[i] ::= a,\ s) \Rightarrow s(x := (s\ x)(aval\ i\ s := aval\ a\ s))$

$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

$ArrayClear :: char\ list \Rightarrow com$

π: $(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

Arrays

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Only with index 0: Bind $VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \ldots)$

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`IMP2/Examples.thy`

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Idea: Halve interval in each step.

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Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

— Donald Knuth

Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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$\uparrow$

l, h

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while (l < h) {
    m = (l + h) / 2;
    if (a[m] < x) l = m + 1
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}
```

Returns smallest i with $x \leq a[i]$

Notes on Binary Search

```
while (l < h) {
    m = (l + h) / 2;
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Otherwise, $m = (l + h)/2$ may overflow!

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Otherwise, $m = (l + h)/2$ may overflow!

Bug in Java Standard Library for > 9 years!

Proving Binary Search

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	
					$\uparrow$	$\uparrow$		
					l	h		

$x = 13$

Invariant:

Proving Binary Search

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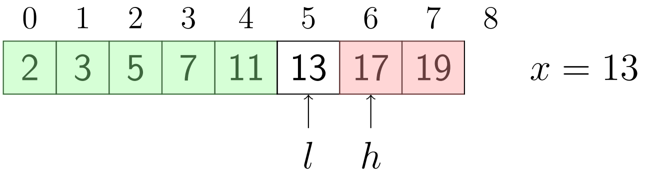
$x = 13$

Invariant:

- $i{<}l \implies a[i] < x$ (strictly smaller than x)

Proving Binary Search



Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)
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- $i{<}l \implies a[i] < x$ (strictly smaller than x)
- $i{\geq}h \implies x \leq a[i]$ (greater or equal to x)
- and the usual bounds

`IMP2/Examples.thy`

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.

In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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Postcondition:

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Precondition: $l \leq h$

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where

$$ran\_sorted\ a\ l\ h \equiv \forall\, i{\in}\{l..{<}h\}.\ \forall\, j{\in}\{l..{<}h\}.\ i \le j \longrightarrow a\ i \le a\ j$$

$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multisets in Isabelle

imports $HOL{-}Library.Multiset$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multiset of elements at indexes in finite set r

Proving Insertion Sort

Separate proof for inner loop!

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j = l + 1;
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  inline inner_loop;
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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$

$\wedge\ mset\_ran\ a\ \{l..{<}h\} = mset\_ran\ a_0\ \{l..{<}h\}$

Insort: Inner Loop

```
key = a[j];
i = j-1;
while (i>=l && a[i]>key) {
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Insort: Inner Loop

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$a[j]$ is moved backwards

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Intuition: ?

$a[j]$ is moved backwards until

Insort: Inner Loop

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Intuition: ?

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Short: Move $a[j]$ backwards over greater elements.

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Invariants easier to find!

Insort: Inner Loop

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assumes $l < j$, $let\ key = a_0\ j$

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Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$

ensures $i \in \{l - 1..{<}j\}$

ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and
$a\ (i + 1) = key$ and
$\forall\, k{\in}\{i + 2..j\}.\ a\ k = a_0\ (k - 1)$

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$

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ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and
$a\ (i + 1) = key$ and
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ensures $l \leq i \longrightarrow a\ i \leq key$ and

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

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ensures $l \leq i \longrightarrow a\ i \leq key$ and
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Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
$\uparrow$						$\uparrow$	$\uparrow$
l						i	j

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Consider intermediate situation

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Consider intermediate situation

- indexes $\leq i$ unchanged: $\forall\, k \in \{l..i\}.\ a\ k = a_0\ k$
- indexes $\geq i+2$ correctly shifted
  $\forall\, k \in \{i+2..j\}.\ a\ k = a_0\ (k-1)$

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- and elements greater than key
  $\forall\, k \in \{i + 2..j\}.\ key < a\ k$
- $+$ the usual bounds: $l - 1 \leq i \wedge i < j$

`IMP2/Examples.thy`

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Prove that this implies expectations of users

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Understand what program does!

Split program into handy parts

Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

7. Advanced Verification

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a\ [l..{<}h]$ as $int\ list$

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Instead of one complex proof, get two simple proofs !

`IMP2/Examples.thy`

Filter, Merge, dedup

7 Advanced Verification

Local Variables

Introduce local variables

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Why?

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Why? Better modularity.

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Don't worry about name-clashes with subroutine's auxiliary variables

Local Variables

Partition variable names into local and global names

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is_global — Variable name starts with "G"

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\textbf{fun } is\_global :: vname \Rightarrow bool \textbf{ where} \\
\quad is\_global\ [] \longleftrightarrow True \\
|\ is\_global\ (CHR\ ''G''\#\_) \longleftrightarrow True \\
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$is\_local\ a = \neg is\_global\ a$

State Combination

$<s_1|s_2>$ – State with locals from s_1, globals from s_2

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$SCOPE\ c$ — Execute c with fresh set of local variables. Restore original local variables afterwards

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$=$

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$\quad = ?$

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$$= wp\ \pi\ c\ (\lambda s'.\ Q\ <s|s'>)\ <<>|s>$$

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Pass information over scope boundaries by globals

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Wrapping Specification

Given specification of body: $HT\ P\ body\ Q$
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How to derive specification for procedure?
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Recall:

$HT\ \pi\ \ P\ \ c\ \ Q \equiv \forall\, s_0.\ \ P\ \ s_0 \longrightarrow wp\ \pi\ \ c\ (Q\ \ s_0)\ \ s_0$

Prologue

$$HT\ \pi\ P\ body\ Q \Longrightarrow$$
$$HT\ \pi\ (wp\ \pi\ prologue\ P)\ (prologue;;\ body)$$
$$(\lambda s_0\ s.\ wp\ \pi\ prologue\ (\lambda s_0.\ Q\ s_0\ s)\ s_0)$$

Intuition: Weakest precondition to enforce P after prologue

Epilogue

$[\![HT\ \pi\ P\ body\ Q;\ \forall\, s.\ \exists\, t.\ \pi{:}\ (epilogue,\ s) \Rightarrow t]\!]$
$\Longrightarrow HT\ \pi\ P\ (body;;\ epilogue)\ (\lambda s_0.\ sp\ \pi\ (Q\ s_0)$
$epilogue)$

Intuition: Strongest postcondition we get from Q after epilogue

Strongest Postconditions

$$sp\ \pi\ \ P\ \ c\ \ t \equiv \exists\, s.\ P\ \ s \wedge \pi\colon (c,\ s) \Rightarrow t$$

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$sp\ \pi\ P\ (c_1;;\ c_2)\ t$

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Parameter assignments: $HT\,\pi\ P\ c\ Q \Longrightarrow$
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Scope: $HT\ \pi\ P\ c\ Q \Longrightarrow$
$HT\ \pi\ (\lambda s.\ P <\!\!<\!\!>|s\!>)\ (SCOPE\ c)\ (\lambda s_0\ s.\ \exists\, l.\ Q <\!\!<\!\!>|s_0\!>$
$<\!l|s\!>)$

`IMP2/Examples.thy`

Merge as Procedure

7 Advanced Verification

- Arrays
- Data Refinement
- Local Variables
- Recursion

Recursive Procedures

Program is map $pname \rightharpoonup com$

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Note: Gets stuck if procedure does not exist!
No problem when proving total correctness

Proof Rules for Recursion

Unfolding: $\pi\ p = Some\ c \Longrightarrow wp\ \pi\ (PCall\ p)\ Q\ s = wp\ \pi\ c\ Q\ s$

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assumes $wf\ R$

$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s){\in}R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$

$\Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$

shows $HT\ \pi\ P\ (PCall\ p)\ Q$

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Idea: Well-Founded induction on state

assumes $wf\ R$
$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s){\in}R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$
$\Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$
shows $HT\ \pi\ P\ (PCall\ p)\ Q$

Show specification for state s, assuming it holds for smaller states s'.

Mutual Recursion

Same idea, but for sets of specifications.

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$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta =$
$(\forall\,(f,\ P,\ c,\ Q){\in}\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge P\ s)\ c\ Q)$
Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

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$PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta \equiv$

$\forall\, P\ c\ Q.\ (f_0,\ P,\ c,\ Q) \in \Theta \wedge P\ s_0 \longrightarrow wp\ \pi\ (c\ s_0)\ (Q\ s_0)\ s_0$

Hoare-triples valid for fixed variant f_0 and state s_0.

Mutual Recursion

lemma $vcg\_HT'setI$:
assumes $wf\ R$
assumes RL: $\bigwedge f_0\ s_0.\ [\![\ ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta\ ]\!] \Longrightarrow PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta$
shows $HT'set\ \pi\ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

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prove that Hoare-triples hold for this state.

$[\![\pi\ p = Some\ c;\ HT\_mods\ \pi\ mods\ P\ c\ Q]\!] \Longrightarrow HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q$

Maps Hoare-Triples to procedure calls

Local Procedures

Idea: Recursive procedure names only valid locally!

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Call procedure with local procedure environment

$HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q \Longrightarrow HT\_mods\ \pi'\ mods\ P$
$(PScope\ \pi\ (PCall\ p))\ Q$

Wrap current procedure environment

Specification of Mutual Recursive Procedures

The IMP2 tools take care of

- wf-relation. Default $less\_than$

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- variants: expression over parameters.
- localization of procedure environment.

IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

Completeness

Consider program with $HT\,\pi\;\;P\;\;c\;\;Q$

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Can we always find annotations to get provable VCs?

Only consider while-rule here

Partial Correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textit{if}\ bval\ b\ s\ \textit{then}\ wlp\ \pi\ c\ I\ s\ \textit{else}\ Q\ s\rrbracket$$
$$\Longrightarrow wlp\ \pi\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

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What invariant shall we use?

$wlp\ \pi\ c\ Q$!

Total Correctness

$[\![wf\ R;\ I\ s_0;$
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Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

IMP2/Examples.thy

Completeness of While-Rule

Conclusions

IMP2: Verification of simple programs in Isabelle/HOL

while-language, arrays, local-vars, recursive procedures

Tools: concrete syntax for programs and specs, VCG

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Conclusions

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while-language, arrays, local-vars, recursive procedures
Tools: concrete syntax for programs and specs, VCG

Not supported:
types (char, float, records), pointers, concurrency, ...
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Caveats:
Procedures+Recursion tools not well-tested
VCG is slow for many procedures/inlines

Chapter 9

Compiler

8 Stack Machine

9 Compiler

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Instructions:

datatype $instr =$

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$\mid JMP\ int$	jump
$\mid JMPLESS\ int$	jump if $<$

Stack Machine

Instructions:

$$
\begin{array}{ll}
\textbf{datatype } \mathit{instr} = & \\
\quad \mathit{LOADI\ int} & \text{load value} \\
\mid \mathit{LOAD\ vname} & \text{load var} \\
\mid \mathit{ADD} & \text{add top of stack} \\
\mid \mathit{STORE\ vname} & \text{store var} \\
\mid \mathit{JMP\ int} & \text{jump} \\
\mid \mathit{JMPLESS\ int} & \text{jump if} < \\
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\end{array}
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Semantics

Type synonyms:

$$\begin{array}{rcl} stack & = & int\ list \\ config & = & int \times state \times stack \end{array}$$

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Execution of 1 instruction:

$$iexec :: instr \Rightarrow config \Rightarrow config$$

Instruction execution

$$
\begin{array}{l}
iexec\ instr\ (i,\ s,\ stk) = \\
(\textsf{case}\ instr\ \textsf{of}\ LOADI\ n \Rightarrow (i + 1,\ s,\ n\ \#\ stk) \\
\ \mid LOAD\ x \Rightarrow (i + 1,\ s,\ s\ x\ \#\ stk) \\
\ \mid ADD \Rightarrow (i + 1,\ s,\ (hd2\ stk + hd\ stk)\ \#\ tl2\ stk) \\
\ \mid STORE\ x \Rightarrow (i + 1,\ s(x := hd\ stk),\ tl\ stk) \\
\ \mid JMP\ n \Rightarrow (i + 1 + n,\ s,\ stk) \\
\ \mid JMPLESS\ n \Rightarrow \\
\qquad (\textsf{if}\ hd2\ stk < hd\ stk\ \textsf{then}\ i + 1 + n\ \textsf{else}\ i + 1, \\
\qquad\ s,\ tl2\ stk) \\
\ \mid JMPGE\ n \Rightarrow \\
\qquad (\textsf{if}\ hd\ stk \le hd2\ stk\ \textsf{then}\ i + 1 + n\ \textsf{else}\ i + 1, \\
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\end{array}
$$

Program execution (1 step)

Programs are instruction lists.

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$$\begin{aligned}
&P \vdash c \rightarrow c' = \\
&(\exists\, i\ s\ stk. \\
&\quad c = (i,\ s,\ stk) \wedge \\
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where	$'a\ \mathit{list}\ !!\ \mathit{int}$	= nth instruction of list
	$\mathit{size} :: {'a}\ \mathit{list} \Rightarrow \mathit{int}$	= list size as integer

Program execution ($*$ steps)

Defined in the usual manner:

$$P \vdash (pc,\ s,\ stk) \rightarrow\!* \ (pc',\ s',\ stk')$$

Compiler.thy

Stack Machine

8 Stack Machine

9 Compiler

Compiling $aexp$

Same as before:

$$acomp\ (N\ n) = [LOADI\ n]$$
$$acomp\ (V\ x) = [LOAD\ x]$$
$$acomp\ (Plus\ a_1\ a_2) = acomp\ a_1\ @\ acomp\ a_2\ @\ [ADD]$$

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$$acomp\ a$$
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Needs lemmas!

$$P \vdash c \rightarrow * \ c' \Longrightarrow P \ @ \ P' \vdash c \rightarrow * \ c'$$

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$$P \vdash (i,\ s,\ stk) \rightarrow\!* \ (i',\ s',\ stk') \Longrightarrow$$
$$P' \mathbin{@} P \vdash (size\ P' + i,\ s,\ stk) \rightarrow\!* \ (size\ P' + i',\ s',\ stk')$$

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Proofs by cases.

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$$bcomp :: bexp \Rightarrow bool \Rightarrow int \Rightarrow instr\ list$$

Example

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$\qquad (Not\ (Less\ (V\ ''z'')\ (V\ ''a'')))$.

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$$]$$

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$]$

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$$bcomp\ (Bc\ v)\ f\ n = (\textsf{if}\ v = f\ \textsf{then}\ [JMP\ n]\ \textsf{else}\ [])$$

$$bcomp\ (Not\ b)\ f\ n = bcomp\ b\ (\neg f)\ n$$

$$bcomp\ (Less\ a_1\ a_2)\ f\ n =$$

$$acomp\ a_1\ @$$
$$acomp\ a_2\ @\ (\textsf{if}\ f\ \textsf{then}\ [JMPLESS\ n]\ \textsf{else}\ [JMPGE\ n])$$

$$bcomp\ (And\ b_1\ b_2)\ f\ n =$$

$$\textsf{let}\ cb_2 = bcomp\ b_2\ f\ n;$$
$$\quad m = \textsf{if}\ f\ \textsf{then}\ size\ cb_2\ \textsf{else}\ size\ cb_2 + n;$$
$$\quad cb_1 = bcomp\ b_1\ False\ m$$
$$\textsf{in}\ cb_1\ @\ cb_2$$

Correctness of $bcomp$

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$$0 \le n \Longrightarrow$$
$$bcomp\ b\ f\ n$$
$$\vdash (0,\ s,\ stk) \rightarrow\!*$$
$$(size\ (bcomp\ b\ f\ n) + (\textit{if}\ f = bval\ b\ s\ \textit{then}\ n\ \textit{else}\ 0),$$
$$s,\ stk)$$

Compiling com

$ccomp :: com \Rightarrow instr\ list$

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$ccomp\ SKIP = []$

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$ccomp\ (c_1;;\ c_2) = ccomp\ c_1\ @\ ccomp\ c_2$

$$ccomp\ (IF\ b\ \ THEN\ c_1\ \ ELSE\ c_2) =$$

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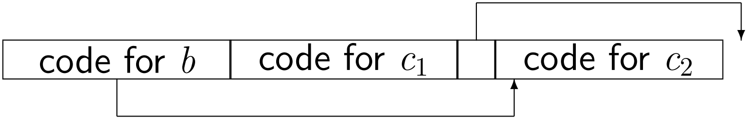
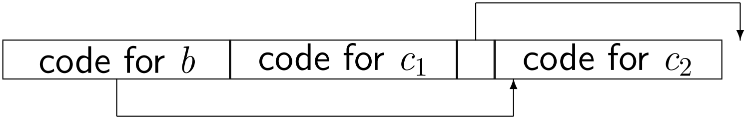


$$ccomp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) =$$

$$\textsf{let}\ cc_1 = ccomp\ c_1;\ cc_2 = ccomp\ c_2;$$
$$\quad cb = bcomp\ b\ False\ (size\ cc_1 + 1)$$
$$\textsf{in}\ cb\ @\ cc_1\ @\ JMP\ (size\ cc_2)\ \#\ cc_2$$



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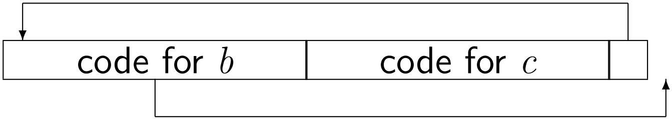
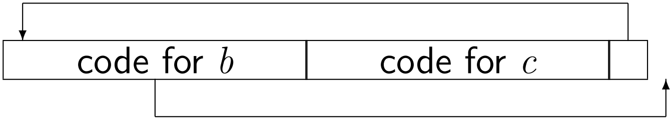


$$ccomp\ (WHILE\ b\ DO\ c) =$$

$$\textsf{\textit{let}}\ cc = ccomp\ c;\ cb = bcomp\ b\ False\ (size\ cc + 1)$$
$$\textsf{\textit{in}}\ cb\ @\ cc\ @\ [JMP\ (-\ (size\ cb + size\ cc + 1))]$$



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Proof by rule induction.

The other direction

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compiled code simulates source code.

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If $ccomp\ c$ with start state s produces result t, and if(!) $(c, s) \Rightarrow t'$, then "$\Longrightarrow$" implies that $ccomp\ c$ with start state s must also produce t' and thus $t' = t$ (why?).

But we have *not* ruled out this potential error:

c does not terminate but $ccomp\ c$ does.

The other direction

Two approaches:

- In the absence of nondeterminism:
  Prove that $ccomp$ preserves nontermination.

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Two approaches:

- In the absence of nondeterminism:
  Prove that $ccomp$ preserves nontermination.
  A nice proof of this fact requires *coinduction*.
  Isabelle supports coinduction, this course avoids it.
- A direct proof: theory $Compiler2$

$ccomp\ c \vdash (0,\ s,\ stk) \rightarrow\!*\ (size\ (ccomp\ c),\ t,\ stk') \Longrightarrow (c,\ s) \Rightarrow t$

Chapter 10

Types

10 A Typed Version of IMP

10 A Typed Version of IMP

10 A Typed Version of IMP

Why Types?

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To prevent mistakes, dummy!

There are 3 kinds of types

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The Good: Static types that *guarantee* absence of certain runtime faults.

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our focus

The Good	Static types that *guarantee* absence of certain runtime faults. Example: no memory access errors in Java.
The Bad	Static types that have mostly decorative value but do not guarantee anything at runtime. Example: C, C++
The Ugly	Dynamic types that detect errors when it can be too late. Example: "TypeError: ..." in Python.

The ideal

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Robin Milner, *A Theory of Type Polymorphism in Programming*, 1978.

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The most influential slogan and one of the most influential papers in programming language theory.

What could go wrong?

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2. Null pointer exception
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There are type systems for *everything* (and more) but in practice (Java, C#) only 1 is covered.

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A programming language is *type safe* if the execution of a well-typed program cannot lead to certain errors.

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Java and the JVM have been *proved* to be type safe. (Note: Java exceptions are not errors!)

Correctness and completeness

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> *Nontrivial semantic properties of programs*
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Hence there is no (decidable) type system that accepts *all* programs that have a certain semantic property.

Automatic analysis of semantic program properties
is necessarily incomplete.

10 A Typed Version of IMP

Arithmetic

Values:

datatype $val = Iv\ int \mid Rv\ real$

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Arithmetic expresssions:

datatype $aexp =$
$\quad Ic\ int \mid Rc\ real \mid V\ vname \mid Plus\ aexp\ aexp$

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Tags are only used to detect certain errors
and to prove that the type system avoids those errors.

Evaluation of $aexp$

Not recursive function but inductive predicate:

$$taval :: aexp \Rightarrow state \Rightarrow val \Rightarrow bool$$

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Example: evaluation of $Plus\ (V\ ''x'')\ (Ic\ 1)$

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If $s\ ''x'' = Rv\ r$: then there is *no* value v such that $taval\ (Plus\ (V\ ''x'')\ (Ic\ 1))\ s\ v$.

The functional alternative

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Exercise!

Boolean expressions

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We need to detect if things “go wrong”.

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  Would confuse type error and nontermination.
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  Complicates formalization.
- Small step semantics:
  error = semantics gets stuck

Small step semantics

$$\frac{taval\ a\ s\ v}{(x ::= a,\ s) \to (SKIP,\ s(x := v))}$$

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The other rules remain unchanged.

typical exam question: give a derivation with these rules

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- If $s\ ''x'' = \mathit{Rv}\ r$:
  $(c,\ s) \not\to$

10 A Typed Version of IMP

- Remarks on Type Systems
- Typed IMP: Semantics
- Typed IMP: Type System
- Type Safety of Typed IMP

Type system

There are two types:

datatype $ty = Ity \mid Rty$

Type system

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What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

Type system

There are two types:

datatype $ty = Ity \mid Rty$

What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

Depends on the type of $''x''$ and $''y''$!

Type system

There are two types:

datatype $ty = Ity \mid Rty$

What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

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The type of an expression is always relative to a type enviroment Γ. Standard notation:

$$\Gamma \vdash e : \tau$$

Read: *In the context of Γ, e has type τ*

The type of an $aexp$

$\Gamma \vdash a : \tau$

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$$\frac{\Gamma \vdash a_1 : \tau \qquad \Gamma \vdash a_2 : \tau}{\Gamma \vdash Plus\ a_1\ a_2 : \tau}$$

Example

$$\frac{\vdots}{\Gamma \vdash \mathit{Plus}\ (V\ ''x'')\ (\mathit{Plus}\ (V\ ''x'')\ (\mathit{Ic}\ 0))\ :\ \mathit{?}}$$

where $\Gamma\ ''x'' = \mathit{Ity}$.

Well-typed $bexp$

Notation:

$$\Gamma \vdash b$$

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Read: *In context* Γ, b *is well-typed*.

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Example: $\Gamma \vdash Less\ (Ic\ i)\ (Rc\ r)$ does not hold.

Well-typed commands

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$$\frac{\Gamma \vdash b \qquad \Gamma \vdash c}{\Gamma \vdash \mathit{WHILE}\ b\ \mathit{DO}\ c}$$

Syntax-directedness

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- backchaining terminates and requires at most as many steps as the size of the term.

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⑩ A Typed Version of IMP

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Remember:
If $s\ ''x'' = Rv\ r$
then $(''x'' ::= Plus\ (V\ ''x'')\ (Ic\ 1),\ s) \not\rightarrow$

The state must be well-typed w.r.t. Γ.

The type of a value:

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Well-typed state:

$$\Gamma \vdash s \longleftrightarrow (\forall x.\ type\ (s\ x) = \Gamma\ x)$$

Type soundness

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Follows from *progress*:

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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
> *then you can take one more step.*

and *preservation*:

Type soundness

Reduction cannot get stuck:

If everything is ok ($\Gamma \vdash s$, $\Gamma \vdash c$ *),*
and you take a finite number of steps,
and you have not reached SKIP,
then you can take one more step.

Follows from *progress*:

If everything is ok and you have not reached SKIP,
then you can take one more step.

and *preservation*:

If everything is ok and you take a step,
then everything is ok again.

The slogan

$$\text{Progress} \wedge \text{Preservation} \implies \text{Type safety}$$

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Progress	Well-typed programs do not get stuck.
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The slogan

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Progress	Well-typed programs do not get stuck.
Preservation	Well-typedness is preserved by reduction.

Preservation: Well-typedness is an *invariant*.

com

Progress:

$$\llbracket \Gamma \vdash c;\ \Gamma \vdash s;\ c \neq SKIP \rrbracket \Longrightarrow \exists\, cs'.\ (c,\ s) \rightarrow cs'$$

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$bexp$

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$aexp$

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$$[\![\Gamma \vdash a : \tau;\ \Gamma \vdash s]\!] \Longrightarrow \exists\, v.\ \mathit{taval}\ a\ s\ v$$

Preservation:

$$[\![\Gamma \vdash a : \tau;\ \mathit{taval}\ a\ s\ v;\ \Gamma \vdash s]\!] \Longrightarrow \mathit{type}\ v = \tau$$

All proofs by rule induction.

Types.thy

The mantra

Type systems have a purpose:

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The correctness of the prediction must be provable.

Chapter 11

Data-Flow Analyses and Optimization

11. Definite Initialization Analysis

12. Live Variable Analysis

11 Definite Initialization Analysis

12 Live Variable Analysis

Each local variable must have a definitely assigned value when any access of its value occurs. A compiler must carry out a specific conservative flow analysis to make sure that, for every access of a local variable x, x *is definitely assigned before the access; otherwise a compile-time error must occur.*

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Java Language Specification

Each local variable must have a definitely assigned value when any access of its value occurs. A compiler must carry out a specific conservative flow analysis to make sure that, for every access of a local variable x, x is definitely assigned before the access; otherwise a compile-time error must occur.

Java Language Specification

Java was the first language to force programmers to initialize their variables.

Examples: ok or not?

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Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
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Examples: ok or not?

Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
```

```
IF x < x THEN y := y + 1 ELSE y := x
```

Assume `x` and `y` are initialized:

```
WHILE x < y DO z := x; z := z + 1
```

Simplifying principle

We do not analyze boolean expressions to determine program execution.

11 Definite Initialization Analysis

Prelude: Variables in Expressions

Definite Initialization Analysis

Initialization Sensitive Semantics

Theory $Vars$ provides an overloaded function $vars$:

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$$vars :: bexp \Rightarrow vname\ set$$
$$vars\ (Bc\ v) = \{\}$$
$$vars\ (Not\ b) = vars\ b$$
$$vars\ (And\ b_1\ b_2) = vars\ b_1 \cup vars\ b_2$$
$$vars\ (Less\ a_1\ a_2) = vars\ a_1 \cup vars\ a_2$$

Vars.thy

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Modified example from the JLS:

> *Variable x is definitely initialized after $SKIP$ iff x is definitely initialized before $SKIP$.*

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Similar statements for each language construct.

$$D :: vname\ set \Rightarrow com \Rightarrow vname\ set \Rightarrow bool$$

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$D\ A\ c\ A'$ should imply:

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If all variables in A are initialized before c is executed,

$D :: \mathit{vname\ set} \Rightarrow \mathit{com} \Rightarrow \mathit{vname\ set} \Rightarrow \mathit{bool}$

$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution,

$D :: \mathit{vname\ set} \Rightarrow \mathit{com} \Rightarrow \mathit{vname\ set} \Rightarrow \mathit{bool}$

$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution, and all variables in A' are initialized afterwards.

D A $SKIP$ A

$$\frac{D\ A\ SKIP\ A \qquad vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$D \ A \ SKIP \ A$$

$$\frac{vars \ a \subseteq A}{D \ A \ (x ::= a) \ (insert \ x \ A)}$$

$$\frac{D \ A_1 \ c_1 \ A_2 \qquad D \ A_2 \ c_2 \ A_3}{D \ A_1 \ (c_1;; \ c_2) \ A_3}$$

$$D\ A\ SKIP\ A$$

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$$\frac{D\ A_1\ c_1\ A_2 \qquad D\ A_2\ c_2\ A_3}{D\ A_1\ (c_1;;\ c_2)\ A_3}$$

$$\frac{vars\ b \subseteq A \qquad D\ A\ c_1\ A_1 \qquad D\ A\ c_2\ A_2}{D\ A\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ (A_1 \cap A_2)}$$

$$D\ A\ SKIP\ A$$

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$$\frac{vars\ b \subseteq A \qquad D\ A\ c_1\ A_1 \qquad D\ A\ c_2\ A_2}{D\ A\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ (A_1 \cap A_2)}$$

$$\frac{vars\ b \subseteq A \qquad D\ A\ c\ A'}{D\ A\ (WHILE\ b\ DO\ c)\ A}$$

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For variety's sake, we choose a big step semantics.

11 Definite Initialization Analysis

Prelude: Variables in Expressions

Definite Initialization Analysis

Initialization Sensitive Semantics

$$state = vname \Rightarrow val\ option$$

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Notation: $s(x \mapsto y)$ means $s(x := Some\ y)$

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Definition: $dom\ s = \{a.\ s\ a \neq None\}$

Expression evaluation

$aval :: aexp \Rightarrow state \Rightarrow val\ option$

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$\ |\ \_ \Rightarrow None)$

$$bval :: bexp \Rightarrow state \Rightarrow bool\ option$$

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&\quad (Some\ i_1,\ Some\ i_2) \Rightarrow Some(i_1 < i_2) \\
&\ \mid _ \Rightarrow None)
\end{aligned}$$

Big step semantics

$$(com,\ state) \Rightarrow state\ option$$

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A small complication:

$$\frac{(c_1,\ s_1) \Rightarrow Some\ s_2 \qquad (c_2,\ s_2) \Rightarrow s}{(c_1;;\ c_2,\ s_1) \Rightarrow s}$$

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More convenient, because compositional:

$$(com,\ state\ option) \Rightarrow state\ option$$

Error ($None$) propagates:

$$(c,\ None) \Rightarrow None$$

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Execution starting in (mostly) normal states ($Some\ s$):

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$$\frac{aval\ a\ s = None}{(x ::= a,\ Some\ s) \Rightarrow None}$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

$$\frac{bval\ b\ s = Some\ True \qquad (c_1,\ Some\ s) \Rightarrow s'}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ Some\ s) \Rightarrow s'}$$

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$$\frac{\begin{array}{c} bval\ b\ s = Some\ True \\ (c,\ Some\ s) \Rightarrow s' \qquad (WHILE\ b\ DO\ c,\ s') \Rightarrow s'' \end{array}}{(WHILE\ b\ DO\ c,\ Some\ s) \Rightarrow s''}$$

$$\frac{bval\ b\ s = Some\ False}{(WHILE\ b\ DO\ c,\ Some\ s) \Rightarrow Some\ s}$$

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Correctness of D w.r.t. $\Rightarrow$

We want in the end:

Well-initialized programs cannot go wrong.

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$\textit{If } D\ (\textit{dom } s)\ c\ A' \textit{ and } (c, \textit{Some } s) \Rightarrow s'$
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We need to prove a generalized statement:

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We need to prove a generalized statement:

$$\mathit{If}\ \ (c,\ \mathit{Some}\ s) \Rightarrow s'\ \ \mathit{and}\ \ D\ \ A\ \ c\ \ A'\ \ \mathit{and}\ \ A \subseteq \mathit{dom}\ s$$
$$\mathit{then}\ \ \exists\, t.\ s' = \mathit{Some}\ t \wedge A' \subseteq \mathit{dom}\ t.$$

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We need to prove a generalized statement:

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$\textit{then } \exists\, t.\ s' = \textit{Some}\ t \wedge A' \subseteq \textit{dom}\ t.$

By rule induction on $(c,\ \textit{Some}\ s) \Rightarrow s'$.

Proof needs some easy lemmas:

$$vars\ a \subseteq\ dom\ s \Longrightarrow \exists\, i.\ aval\ a\ s = \ Some\ i$$

$$vars\ b \subseteq\ dom\ s \Longrightarrow \exists\, bv.\ bval\ b\ s = \ Some\ bv$$

$$D\ A\ c\ A' \Longrightarrow\ A \subseteq\ A'$$

11 Definite Initialization Analysis

12 Live Variable Analysis

Motivation

Consider the following program:

```
x := y + 1;
y := y + 2;
x := y + 3
```

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Consider the following program:

```
x := y + 1;
y := y + 2;
x := y + 3
```

The first assignment is redundant and can be removed because `x` is dead at that point.

Semantically, a variable x is live before command c if the initial value of x can influence the final state.

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x := 0
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Examples: Is `x` initially dead or live?

```
x := 0
y := x; y := 0; x := 0
```

☹

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x := 0                        ☹
y := x; y := 0; x := 0        ☺
WHILE b DO y := x; x := 1
```

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Examples: Is `x` initially dead or live?

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x := 0                      ☹
y := x; y := 0; x := 0      ☺
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```

At the end of a command, we may be interested in the value of *only some of the variables*, At the end of a command, we may be interested in the value of *only some of the variables*, e.g. *only the global variables* at the end of a procedure.

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Then we say that x is live before c *relative to* the set of variables X.

Liveness analysis

$L :: com \Rightarrow vname\ set \Rightarrow vname\ set$

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$$L\ c\ X \ = \ \text{live before } c \text{ relative to } X$$

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$L\ SKIP\ X \qquad =$

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$L :: com \Rightarrow vname\ set \Rightarrow vname\ set$

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Liveness analysis

$L :: com \Rightarrow vname\ set \Rightarrow vname\ set$

$$L\ c\ X\ =\ \text{live before } c \text{ relative to } X$$

$$\begin{aligned} L\ SKIP\ X &= X \\ L\ (x ::= a)\ X &= \end{aligned}$$

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&\quad vars\ b \cup L\ c_1\ X \cup L\ c_2\ X
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$$

Example:

$$L\ (''y'' ::= V\ ''z'';;\ ''x'' ::= Plus\ (V\ ''y'')\ (V\ ''z''))$$
$$\{''x''\}\ =$$

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$$L\ c\ X\ =\ \text{live before } c \text{ relative to } X$$

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$$L\ (''y'' ::= V\ ''z'';;\ ''x'' ::= Plus\ (V\ ''y'')\ (V\ ''z''))$$
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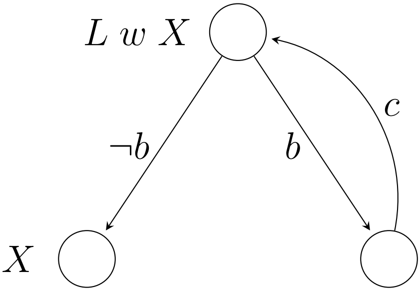
$WHILE\ b\ DO\ c$



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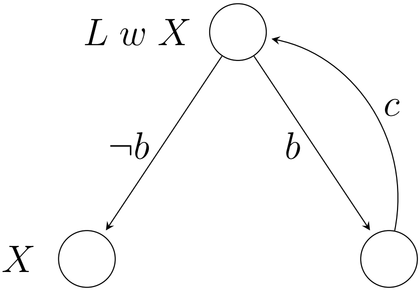
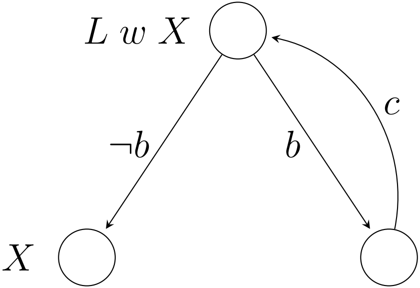


$L\ w\ X$ must satisfy

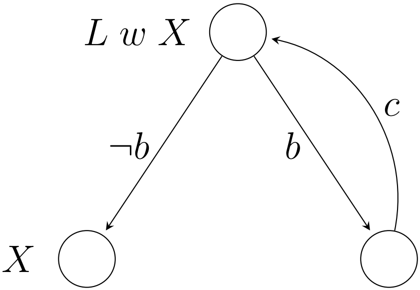
$WHILE\ b\ DO\ c$



$L\ w\ X$ must satisfy

$vars\ b \subseteq L\ w\ X$ (evaluation of b)

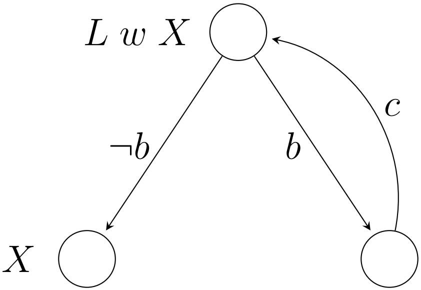
$WHILE\ b\ DO\ c$



$L\ w\ X$ must satisfy

$vars\ b$	$\subseteq$	$L\ w\ X$	(evaluation of b)
X	$\subseteq$	$L\ w\ X$	(exit)

$\mathit{WHILE}\ b\ \mathit{DO}\ c$



$L\ w\ X$ must satisfy

$$\begin{array}{lcll}
\mathit{vars}\ b & \subseteq & L\ w\ X & \text{(evaluation of } b\text{)} \\
X & \subseteq & L\ w\ X & \text{(exit)} \\
L\ c\ (L\ w\ X) & \subseteq & L\ w\ X & \text{(execution of } c\text{)}
\end{array}$$

We define

$$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$$

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$\Longrightarrow$

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$\Longrightarrow$

$vars\ b \subseteq L\ w\ X$ ✓

$X \subseteq L\ w\ X$ ✓

$L\ c\ (L\ w\ X) \subseteq L\ w\ X$?

$$
\begin{array}{lcl}
L\ SKIP\ X & = & X \\
L\ (x ::= a)\ X & = & vars\ a \cup (X - \{x\}) \\
L\ (c_1;;\ c_2)\ X & = & L\ c_1\ (L\ c_2\ X)
\end{array}
$$

$$L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =\ vars\ b \cup L\ c_1\ X \cup L\ c_2\ X$$

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Example:

$$L\ (WHILE\ Less\ (V\ ''x'')\ (V\ ''x'')\ DO\ ''y'' ::= V\ ''z'')$$
$$\{''x''\}\ =$$

$$
\begin{array}{lcl}
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L\ (c_1;;\ c_2)\ X & = & L\ c_1\ (L\ c_2\ X)
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$$

$$L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =\ vars\ b \cup L\ c_1\ X \cup L\ c_2\ X$$

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Example:

$$L\ (WHILE\ Less\ (V\ ''x'')\ (V\ ''x'')\ DO\ ''y'' ::= V\ ''z'')$$
$$\{''x''\}\ =\ \{''x'',''z''\}$$

Gen/kill analyses

A data-flow analysis $A :: com \Rightarrow \tau\ set \Rightarrow \tau\ set$
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Gen/kill analyses are extremely well-behaved, e.g.

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$$X_1 \subseteq X_2 \Longrightarrow A\ c\ X_1 \subseteq A\ c\ X_2$$
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Many standard data-flow analyses are gen/kill.

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Many standard data-flow analyses are gen/kill.
In particular liveness analysis.

Liveness via gen/kill

$kill :: com \Rightarrow vname\ set$

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$kill\ SKIP \qquad\qquad\qquad\qquad =$

Liveness via gen/kill

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$$kill\ SKIP \quad = \quad \{\}$$

Liveness via gen/kill

$$kill :: com \Rightarrow vname\ set$$

$$\begin{aligned} kill\ SKIP &= \{\} \\ kill\ (x ::= a) &= \end{aligned}$$

Liveness via gen/kill

$$\mathit{kill} :: \mathit{com} \Rightarrow \mathit{vname\ set}$$

$$\begin{array}{lcl}
\mathit{kill}\ \mathit{SKIP} & = & \{\} \\
\mathit{kill}\ (x ::= a) & = & \{x\}
\end{array}$$

Liveness via gen/kill

$kill :: com \Rightarrow vname\ set$

$$
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kill\ (x ::= a) & = & \{x\} \\
kill\ (c_1;;\ c_2) & = &
\end{array}
$$

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kill\ SKIP & = & \{\} \\
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kill\ (WHILE\ b\ DO\ c) & = &
\end{array}
$$

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$$\begin{array}{lcl}
kill\ SKIP & = & \{\} \\
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kill\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) & = & kill\ c_1 \cap kill\ c_2 \\
kill\ (WHILE\ b\ DO\ c) & = & \{\}
\end{array}$$

$gen :: com \Rightarrow vname\ set$

$$gen :: com \Rightarrow vname\ set$$

$$gen\ SKIP \qquad =$$

$$gen :: com \Rightarrow vname\ set$$

$$gen\ SKIP \quad = \quad \{\}$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = &
\end{array}
$$

$$gen :: com \Rightarrow vname\ set$$

$$\begin{array}{lcl} gen\ SKIP & = & \{\} \\ gen\ (x ::= a) & = & vars\ a \end{array}$$

$gen :: com \Rightarrow vname\ set$

$$\begin{aligned}
gen\ SKIP &= \{\} \\
gen\ (x ::= a) &= vars\ a \\
gen\ (c_1;;\ c_2) &=
\end{aligned}$$

$$gen :: com \Rightarrow vname\ set$$

$$\begin{array}{lcl} gen\ SKIP & = & \{\} \\ gen\ (x ::= a) & = & vars\ a \\ gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1) \end{array}$$

$$gen :: com \Rightarrow vname\ set$$

$$\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
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\multicolumn{3}{l}{gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ =}
\end{array}$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1) \\
\multicolumn{3}{l}{gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ =} \\
\multicolumn{3}{l}{\quad vars\ b \cup gen\ c_1 \cup gen\ c_2}
\end{array}
$$

$$gen :: com \Rightarrow vname\ set$$

$$\begin{array}{lll}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1)
\end{array}$$

$$gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) =$$

$$\quad vars\ b \cup gen\ c_1 \cup gen\ c_2$$

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$$\color{blue}{gen :: com \Rightarrow vname\ set}$$

$$
\begin{array}{lcl}
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Proof by induction on c.

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Proof by induction on c.

$\Longrightarrow$

$$L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Digression: definite initialization via gen/kill

$A\ c\ X$: the set of variables initialized after c
if X was initialized before c

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$$
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\mathit{gen}\ (\mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2) & = & \mathit{gen}\ c_1 \cap \mathit{gen}\ c_2 \\
\mathit{gen}\ (\mathit{WHILE}\ b\ \mathit{DO}\ c) & = & \{\}
\end{array}
$$

$$\mathit{kill}\ c \ = \ \{\}$$

12 Live Variable Analysis

- Correctness of L
- Dead Variable Elimination
- True Liveness
- Comparisons

$(.,.) \Rightarrow .$ and L should roughly be related like this:

The value of the final state on X
only depends on
the value of the initial state on $L\ c\ X$.

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The value of the final state on X
only depends on
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Put differently:

If two initial states agree on $L \; c \; X$
then the corresponding final states agree on X.

Equality on

An abbreviation:

$$f = g \ on\ X \quad \equiv \quad \forall\, x \in X.\ f\, x = g\ x$$

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Two easy theorems (in theory $Vars$):

$$s_1 = s_2\ on\ vars\ a \Longrightarrow aval\ a\ s_1 = aval\ a\ s_2$$

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Two easy theorems (in theory $Vars$):

$$s_1 = s_2 \ on \ vars \ a \Longrightarrow aval \ a \ s_1 = aval \ a \ s_2$$
$$s_1 = s_2 \ on \ vars \ b \Longrightarrow bval \ b \ s_1 = bval \ b \ s_2$$

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If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction.
For the two $WHILE$ cases we do not need the definition of $L\ w$ but only the characteristic property

$$vars\ b \cup X \cup L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Optimality of $L\ w$

The result of L should be as small as possible: the more dead variables, the better

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> $L\ w\ X$ *should be the least set such that*
> $vars\ b \cup X \cup L\ c\ (L\ w\ X) \subseteq L\ w\ X.$

Follows easily from $L\ c\ X = gen\ c \cup (X - kill\ c)$:

> $vars\ b \cup X \cup L\ c\ P \subseteq P \Longrightarrow$
> $L\ (WHILE\ b\ DO\ c)\ X \subseteq P$

12 Live Variable Analysis

Bury all assignments to dead variables:

$bury :: com \Rightarrow vname\ set \Rightarrow com$

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\multicolumn{3}{l}{bury\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =}
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$$

$$
\begin{array}{l}
bury\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ = \\
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$$

Correctness of $bury$

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$$bury\ c\ UNIV \sim c$$

where $UNIV$ is the set of all variables.

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The two directions need to be proved separately.

$$(c,\ s) \Rightarrow s' \Longrightarrow (bury\ c\ UNIV,\ s) \Rightarrow s'$$

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Follows from generalized statement:

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction, like for correctness of L.

$$(bury\ c\ UNIV,\ s) \Rightarrow s' \Longrightarrow (c,\ s) \Rightarrow s'$$

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Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command,

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Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command, e.g.

$$\begin{aligned}
&(bc_1;;\ bc_2 = bury\ c\ X) = \\
&(\exists\, c_1\ c_2. \\
&\quad c = c_1;;\ c_2 \wedge \\
&\quad bc_2 = bury\ c_2\ X \wedge bc_1 = bury\ c_1\ (L\ c_2\ X))
\end{aligned}$$

12 Live Variable Analysis

Correctness of L

Dead Variable Elimination

True Liveness

Comparisons

Terminology

Let $f :: \tau \Rightarrow \tau$ and $x :: \tau$.

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If $f\ x \le x$ then x is a *pre-fixpoint* (*pfp*) of f.

If $x \le y \Longrightarrow f\ x \le f\ y$ for all x,y, then f is *monotone*.

Application to $L\ w$

Remember the specification of $L\ w$:

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Application to $L\ w$

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This is the same as saying that $L\ w\ X$ should be a pfp of

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and in particular of $L\ c$.

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$L\ (''x'' ::= V\ ''y'')\ \{\} = \{''y''\}$

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$L\ (x ::= a)\ X =$
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$L\ (x ::= a)\ X =$
$(\textit{if}\ x \in X\ \textit{then}\ vars\ a \cup (X - \{x\})\ \textit{else}\ X)$

But then

$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

is not correct anymore.

$$L\ (x ::= a)\ X =$$
$$(\textsf{\textit{if}}\ x \in X\ \textsf{\textit{then}}\ vars\ a \cup (X - \{x\})\ \textsf{\textit{else}}\ X)$$

$$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$$

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$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

Let $w = WHILE\ b\ DO\ c$
where $b\ =\ Less\ (N\ 0)\ (V\ y)$
and $c\ =\ y ::= V\ x;;\ x ::= V\ z$
and $distinct\ [x,\ y,\ z]$

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Then $L\ w\ \{y\} = \{x,\ y\}$, but z is live before w !

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$$y ::= V\ x \qquad\qquad x ::= V\ z\ \ \{y\}$$

$L\ (x ::= a)\ X =$
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$$y ::= V\ x\ \ \{y\}\ \ x ::= V\ z\ \ \{y\}$$

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Then $L\ w\ \{y\} = \{x,\ y\}$, but z is live before w !

$\{x\}\ \ y ::= V\ x\ \ \{y\}\ \ x ::= V\ z\ \ \{y\}$

$\Longrightarrow\ L\ w\ \{y\} = \{y\} \cup \{y\} \cup \{x\}$

$$\begin{aligned} b &= Less\ (N\ 0)\ (V\ y) \\ c &= y ::= V\ x;;\ x ::= V\ z \end{aligned}$$

$L\ w\ \{y\} = \{x,\ y\}$ is not a pfp of $L\ c$:

$$
\begin{aligned}
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$L\ w$ for true liveness

Define $L\ w\ X$ *as the least pfp of*
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Existence of least fixpoints

Theorem (Knaster-Tarski) Let $f :: \tau\ set \Rightarrow \tau\ set$.

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is the least pre-fixpoint and least fixpoint of f.

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$L\ (x ::= a)\ X =$
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Corollary f_w is monotone.

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Proof Show $f^i(\{\}) \subseteq p$ for any pfp p of f (by induction on i).

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From the library theory `While_Combinator`:

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Limiting the number of iterations

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where $g_w\ P = vars\ b \cup X \cup Lb\ c\ P$

Theorem $L\ c\ X \subseteq Lb\ c\ X$

Proof by induction on c. In the $WHILE$ case:

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If $Lb\ w\ X = V_w$: $L\ w\ X \subseteq V_w$ (by Lemma)

12 Live Variable Analysis

- Correctness of L
- Dead Variable Elimination
- True Liveness
- Comparisons

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