Concrete Semantics
with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1 IMP Commands

2 Big-Step Semantics

3 Small-Step Semantics

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2 Big-Step Semantics

3 Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Study the book until you have understood it.

Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{rcl}
com & ::= & \texttt{SKIP} \\
& \mid & string\ \texttt{::=}\ aexp \\
& \mid & com\ \texttt{;;}\ com \\
& \mid & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\
& \mid & \texttt{WHILE}\ bexp\ \texttt{DO}\ com
\end{array}
$$

Commands

Abstract syntax:

$$\begin{array}{lcl} \textbf{datatype}\ com & = & SKIP \\ & | & Assign\ string\ aexp \\ & | & Seq\ com\ com \\ & | & If\ bexp\ com\ com \\ & | & While\ bexp\ com \end{array}$$

Com.thy

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Big-step semantics

Concrete syntax:

$$(com,\ initial\text{-}state) \Rightarrow final\text{-}state$$

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Command c started in state s terminates in state t

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Intended meaning of $(c,\ s) \Rightarrow t$:

Command c started in state s terminates in state t

"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

Big-step rules

$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

$$\frac{\begin{array}{c} bval\ b\ s_1 \\ (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3 \end{array}}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?}$$

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$$\frac{\vdots}{(''x'' ::= N\,5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{array}{rcl} w & = & \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b & = & \mathit{NotEq}\ (V\ ''x'')\ (N\ 2) \\ c & = & ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\ 1) \\ s_i & = & s(''x'' := i) \end{array}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{aligned} w &= \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b &= \mathit{NotEq}\ (V\ ''x'')\ (N\ 2) \\ c &= ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\ 1) \\ s_i &= s(''x'' := i) \end{aligned}$$

$$\mathit{NotEq}\ a_1\ a_2 =$$
$$\mathit{Not}(\mathit{And}\ (\mathit{Not}(\mathit{Less}\ a_1\ a_2))\ (\mathit{Not}(\mathit{Less}\ a_2\ a_1)))$$

Logically speaking

$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

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$$(c,\ s) \Rightarrow t$$

is just infix syntax for

$$big\_step\ (c,s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

`Big_Step.thy`

Semantics

Rule inversion

What can we deduce from

- $(SKIP,\ s) \Rightarrow t$?

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  $\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3$

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  $\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3$
- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?

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- $(SKIP, s) \Rightarrow t$? $t = s$
- $(x ::= a, s) \Rightarrow t$? $t = s(x := aval\ a\ s)$
- $(c_1;; c_2, s_1) \Rightarrow s_3$?
  $\exists s_2.\ (c_1, s_1) \Rightarrow s_2 \wedge (c_2, s_2) \Rightarrow s_3$
- $(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \Rightarrow t$?
  $bval\ b\ s \wedge (c_1, s) \Rightarrow t\ \vee$
  $\neg\ bval\ b\ s \wedge (c_2, s) \Rightarrow t$

Rule inversion

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- $(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t$?
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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?

Rule inversion

What can we deduce from

- $(\mathit{SKIP},\ s) \Rightarrow t$? $\quad t = s$
- $(x ::= a,\ s) \Rightarrow t$? $\quad t = s(x := \mathit{aval}\ a\ s)$
- $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$?
  $\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3$
- $(\mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2,\ s) \Rightarrow t$?
  $\mathit{bval}\ b\ s \wedge (c_1,\ s) \Rightarrow t\ \vee$
  $\neg\ \mathit{bval}\ b\ s \wedge (c_2,\ s) \Rightarrow t$
- $(w,\ s) \Rightarrow t$ where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$?
  $\neg\ \mathit{bval}\ b\ s \wedge t = s\ \vee$
  $\mathit{bval}\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow\ s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow\ s_2 \wedge\ (c_2,\ s_2) \Rightarrow\ s_3}$$

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$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \land (c_2,\ s_2) \Rightarrow s_3}$$

is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$ (with a new fixed s_2).
No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \dots \quad asm_n \Longrightarrow P}{P}$$

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Reading:

> To prove a goal P with assumption asm,
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Reading:

> To prove a goal P with assumption asm,
> prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

elim attribute

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- Can also be added locally, eg *(blast elim: . . .)*
- Variant: *elim!* applies elim-rules eagerly.

`Big_Step.thy`

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$

$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow\ s' \wedge (w,\ s') \Rightarrow\ t)$$

$$\vee$$

$$\neg\ bval\ b\ s \wedge t = s$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
$$\vee$$
$$\neg\ bval\ b\ s \wedge t = s$$

$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$\begin{gathered} bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t) \\ \vee \\ \neg\ bval\ b\ s \wedge t = s \end{gathered}$$

$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

`Big_Step.thy`

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

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Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

- nonterminating computations,

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- nonterminating computations,
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We need a finer grained semantics!

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2 Big-Step Semantics

3 Small-Step Semantics

Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Small-step semantics

Concrete syntax:

$$(com,state) \to (com,state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
> *to be executed in state s'.*

Small-step semantics

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Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s leaves a "remainder" command c' to be executed in state s'.*

Execution as finite or infinite reduction:

$$(c_1,s_1) \to (c_2,s_2) \to (c_3,s_3) \to \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \rightarrow cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \rightarrow cs'$

The intention:

$$(SKIP,\ s) \text{ is final}$$

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Why?

$SKIP$ is the empty program.

The intention:

$$(SKIP,\ s) \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$(x ::= a,\ s) \ \rightarrow$

Small-step rules

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$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;; c_2, s) \ \rightarrow}$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

$$(SKIP;;\ c,\ s) \ \rightarrow\ (c,\ s)$$

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Small-step rules

$$\frac{bval\ b\ s}{(\mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2, s)\ \rightarrow}$$

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$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s)\ \rightarrow\ (c_1, s)}$$

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \ \to\ (c_1, s)}$$

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$$(WHILE\ b\ DO\ c,\ s) \ \to$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2, s) \ \rightarrow\ (c_1, s)}$$

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow$$
$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \to$$
$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

$$(w,\ s_0) \to \ldots$$

where
$$\begin{array}{rcl} w & = & \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b & = & \mathit{Less}\ (V\ ''x'')\ (N\ 1) \\ c & = & ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\ 1) \\ s_n & = & <''x'' := n> \end{array}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow*$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)

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In two cases a lemma is needed:

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Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1, s) \rightarrow* (c_1', s') \implies (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* ({c_1}',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* ({c_1}';;\ c_2,\ s')$

Proof by rule induction.

From $\rightarrow *$ to $\Rightarrow$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.

From $\rightarrow*$ to $\Rightarrow$

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In the induction step a lemma is needed:

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Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \; (SKIP,\; t)$

Small_Step.thy

Equivalence of big and small

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That is, are there any final configs except $(SKIP, s)$?

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Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

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That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

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    $\Longrightarrow \neg\ \mathit{final}\ (c_1;;\ c_2,\ s)$
- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \to\ ct \Longrightarrow False$

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Together:

$$\textbf{Corollary}\ \mathit{final}\ (c,\ s) = (c = SKIP)$$

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$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

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With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

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4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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We have modeled semantics of imperative languages

But how do we prove imperative programs correct?

An example program:

```
$program\ exp\ \{$
  $a := 1$
  $while\ (0{<}n)\ do\ \{$
    $a := a + a;$
    $n := n - 1$
  $\}$
$\}$
```

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and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

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wp of equivalent programs is equal

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$wp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s$

$=$

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Reasoning along syntax of program!

That was easy!

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Unfolding will continue forever!

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But, let's get less ambitious (for first)

Weakest liberal precondition

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

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Unfold rules still hold:

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Let's try to find predicate I, such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

Wp_Demo.thy

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If $c = \mathit{WHILE}\ \_\ \mathit{DO}\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$

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Combine rule with splitting!

`Wp_Demo.thy`

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $\mathit{wf}\ R$
assumes INIT: $I\ s_0$
assumes STEP: $\bigwedge s.\ [\![\ I\ s;\ \mathit{bval}\ b\ s\ ]\!]$
$\Longrightarrow \mathit{wp}\ c\ (\lambda s'.\ I\ s' \wedge (s',s){\in}R)\ s$
assumes FINAL: $\bigwedge s.\ [\![\ I\ s;\ \neg \mathit{bval}\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $\mathit{wp}\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s_0$

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Now we can prove total correctness ...

`Wp_Demo.thy`

Total Correctness

lemma $ASSUME\_\Theta\_alt$:
$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta = (\forall (f,(P,c,Q)) \in \Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge\ P\ s)\ c\ Q)$

unfolding $ASSUME\_\Theta\_def\ HT'set\_r\_def$ **..**

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Let’s make our VCG more usable

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Add standard arithmetic operators to IMP

Let's make our VCG more usable

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Simplify specification of pre/postcondition, and invariants

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

`IMP2/Introduction.thy`

Adding more Operators

C-like syntax

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Arith: $+, -, *, /$ with usual binding

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$while\ (bexp)\ c$

IMP2/Introduction.thy

Program Syntax

More Readable VCs

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Similar: $s_0\ ''x''$ by x_0.

If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$$

IMP2/Introduction.thy

More Readable VCs

More Readable Annotations

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Precondition: x interpreted as $s\ ''x''$

Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

IMP2/Introduction.thy

More Readable Annotations

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

6 Example Verifications
Loop Patterns
Euclid's Algorithm

Common Loop Patterns

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

`IMP2/Examples.thy`

Count-up, Count-Down

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Applications: sqrt, log, ...

`IMP2/Examples.thy`

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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```
l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
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Invariant? $l^2 \leq n < h^2 \wedge \ldots$ (range contains solution)

This program is actually tricky to get right!

IMP2/Examples.thy

Bisection

6 Example Verifications
Loop Patterns
Euclid's Algorithm

Euclid Intro

Compute gcd of positive numbers a, b

Euclid Intro

Compute gcd of positive numbers a, b

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - \ n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

IMP2/Examples.thy

Euclid

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

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Modified Variables

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Only $a{,}i$ changed: $\forall\, x.\ x \notin \{''a'', ''i''\} \longrightarrow s\ x = s'\ x$

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Program modifies at most variables it assigns to

$\pi\!:\ (c,\ s) \Rightarrow\ t \Longrightarrow\ modifies\ (lhsv\ \pi\ \ c)\ t\ s$

Modified Variables

We can strengthen correctness statement (automatically)

$wp\ \pi\ \ c\ Q\ s \Longrightarrow wp\ \pi\ \ c\ (\lambda s'.\ Q\ s' \wedge modifies\ (lhsv\ \pi\ \ c)\ s'\ s)\ s$

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For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow wp\ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}){\in}R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

Modified Variables

The VCG will automatically rewrite with rule

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program_spec computes lhs-variables:

$HT\_mods\ \pi\ mods\ P\ c\ Q \equiv HT\ \pi\ P\ c\ (\lambda s_0\ s.\ modifies\ mods\ s\ s_0 \wedge Q\ s_0\ s)$

IMP2/Examples.thy

Euclid – show modified sets

Modular Proofs

Consider program

```
$a$=1;
$while$ ($m$>0) {
  $n$=$a$; $a$ = 1;
  $while$ ($n$>0) {
    $a$=2*$a$; $n$=$n$−1
  };
  $m$=$m$−1
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What does this compute

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Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

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Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

```
$a$=1;
$while$ ($m$>0) {
  $n$=$a$;
  $inline$ $exp\_count\_down$;
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Reuse existing proof of exp-count-down program!

Modular Proofs

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with modified sets:

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

`IMP2/Examples.thy`

Power-Tower

7 Advanced Verification
Arrays
Data Refinement
Local Variables
Recursion

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Every variable is of type $int \Rightarrow int$.

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Commands:

$AssignIdx :: char\ list \Rightarrow aexp \Rightarrow aexp \Rightarrow com$

π: $(x[i] ::= a,\ s) \Rightarrow s(x := (s\ x)(aval\ i\ s := aval\ a\ s))$

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$aval\ (Vidx\ x\ i)\ s = s\ x\ (aval\ i\ s)$

Commands:

$AssignIdx :: char\ list \Rightarrow aexp \Rightarrow aexp \Rightarrow com$

π: $(x[i] ::= a,\ s) \Rightarrow s(x := (s\ x)(aval\ i\ s := aval\ a\ s))$

$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

$ArrayClear :: char\ list \Rightarrow com$

π: $(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

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IMP2/Examples.thy

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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This algorithm is tricky to implement correctly!

Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

— Donald Knuth

Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
    if (a[m] < x) l = m + 1
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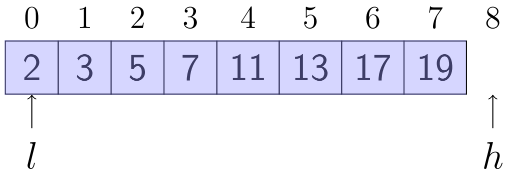
}
```

Binary Search Algorithm



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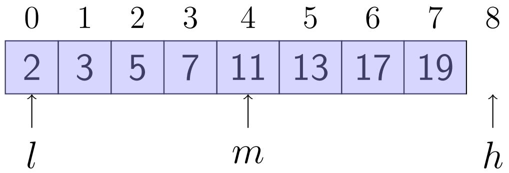

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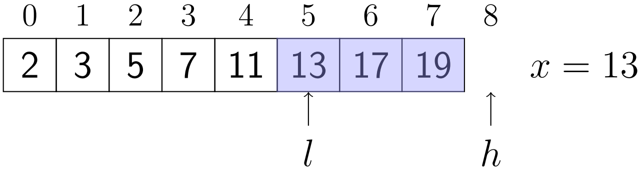

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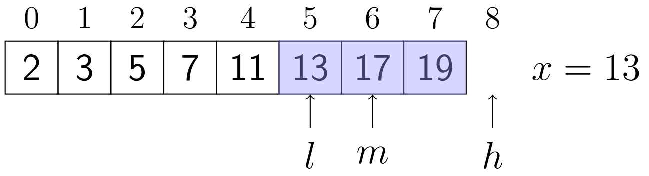

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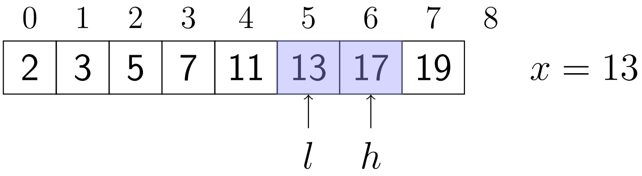

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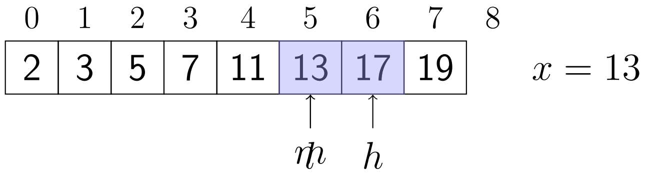

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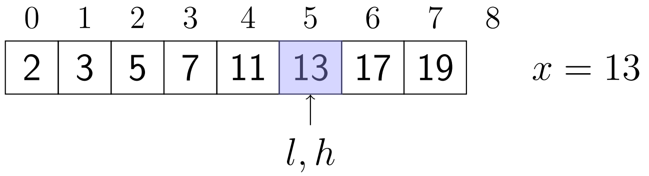

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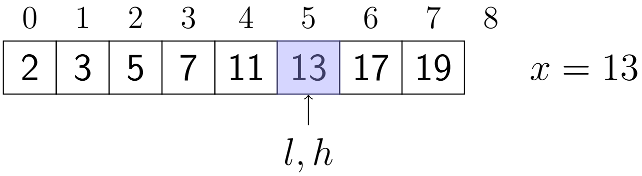

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while (l < h) {
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    else h = m
}
```

Returns smallest i with $x \leq a[i]$

Notes on Binary Search

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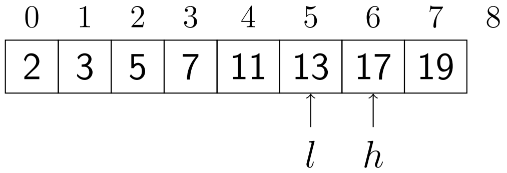
Note: Our language has arbitrary large integers.

Otherwise, $m = (l + h)/2$ may overflow!

Bug in Java Standard Library for > 9 years!

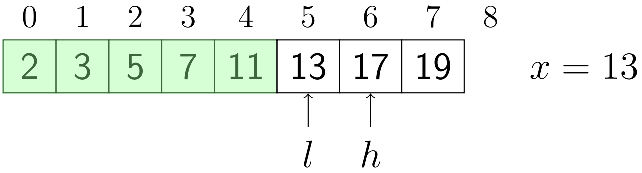
Proving Binary Search



$x = 13$

Invariant:

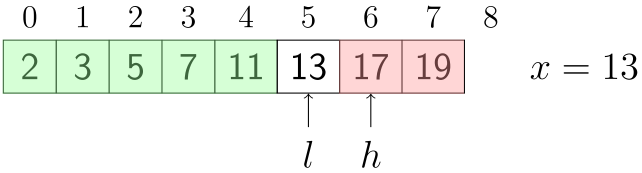
Proving Binary Search



Invariant:

- $i{<}l \Longrightarrow a[i] < x$ (strictly smaller than x)

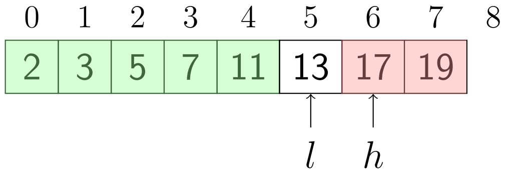
Proving Binary Search



Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)
- $i \geq h \implies x \leq a[i]$ (greater or equal to x)

Proving Binary Search



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Invariant:

- $i{<}l \implies a[i] < x$ (strictly smaller than x)
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- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.
In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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Postcondition:

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where

$$ran\_sorted\ a\ l\ h \equiv \forall\, i{\in}\{l..{<}h\}.\ \forall\, j{\in}\{l..{<}h\}.\ i \le j \longrightarrow a\ i \le a\ j$$

$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

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imports $HOL{-}Library.Multiset$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multiset of elements at indexes in finite set r

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Separate proof for inner loop!

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Invariant of outer loop:

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Invariant of outer loop:

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Insort: Inner Loop

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key = a[j];
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Intuition: ?

$a[j]$ is moved backwards until

Insort: Inner Loop

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Insort: Inner Loop

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Short: Move $a[j]$ backwards over greater elements.

Insort: Inner Loop

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Let's specify this intuition!

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Invariants easier to find!

Insort: Inner Loop

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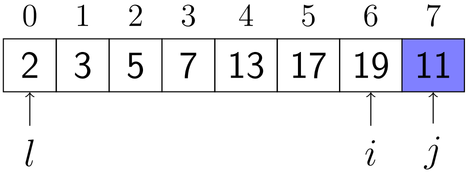
Insort: Finding Invariant



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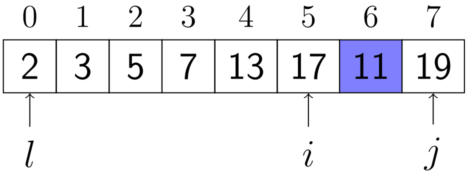

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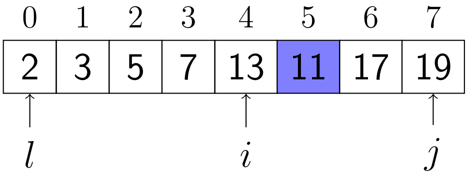

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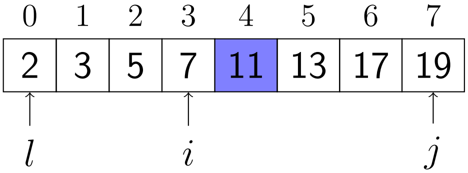

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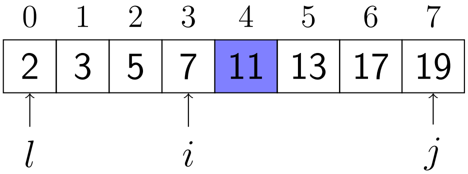
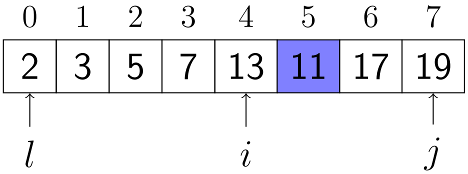


Consider intermediate situation

Insort: Finding Invariant



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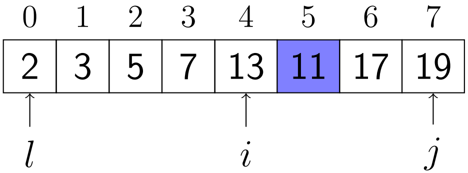

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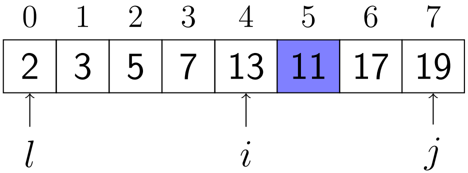

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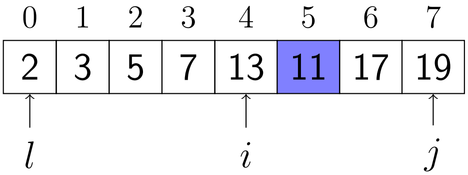

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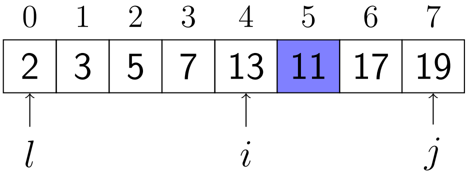

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- $+$ the usual bounds: $l - 1 \leq i \wedge i < j$

IMP2/Examples.thy

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

7 Advanced Verification
Arrays
Data Refinement
Local Variables
Recursion

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a\ [l..{<}h]$ as $int\ list$

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IMP2/Examples.thy

Filter, Merge, dedup

7 Advanced Verification
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Local Variables

Introduce local variables

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Why?

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Why? Better modularity.

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Don't worry about name-clashes with subroutine's auxiliary variables

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Partition variable names into local and global names

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State Combination

$<s_1|s_2>$ – State with locals from s_1, globals from s_2

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Wrapping Specification

Given specification of body: $HT\ P\ body\ Q$
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Recall:

$HT\ \pi\ P\ c\ Q \equiv \forall\, s_0.\ P\ s_0 \longrightarrow wp\ \pi\ c\ (Q\ s_0)\ s_0$

Prologue

$$HT\ \pi\ \ P\ body\ \ Q \Longrightarrow$$
$$HT\ \pi\ \ (wp\ \pi\ \ prologue\ P)\ (prologue;;\ body)$$
$$(\lambda s_0\ \ s.\ wp\ \pi\ \ prologue\ (\lambda s_0.\ Q\ s_0\ \ s)\ s_0)$$

Intuition: Weakest precondition to enforce P after prologue

Epilogue

$$\llbracket HT\ \pi\ \ P\ body\ \ Q;\ \forall\, s.\ \exists\, t.\ \pi\colon\ (epilogue,\ s) \Rightarrow\ t\rrbracket$$
$$\Longrightarrow HT\ \pi\ \ P\ (body;;\ epilogue)\ (\lambda s_0.\ sp\ \pi\ \ (Q\ \ s_0)\ epilogue)$$

Intuition: Strongest postcondition we get from Q after epilogue

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$HT\,\pi\ P\ (c;;\ x[]\ ::=\ y)\ (\lambda s_0\ s.\ \exists\, vx.\ Q\ s_0\ (s(x := vx,\ y := s\ x)))$

Scope: $HT\,\pi\ P\ c\ Q \Longrightarrow$
$HT\,\pi\ (\lambda s.\ P\ {<}{<}{>}|s{>})\ (SCOPE\ c)\ (\lambda s_0\ s.\ \exists\, l.\ Q\ {<}{<}{>}|s_0{>}$
${<}l|s{>})$

`IMP2/Examples.thy`

Merge as Procedure

7 Advanced Verification
Arrays
Data Refinement
Local Variables
Recursion

Recursive Procedures

Program is map $pname \rightharpoonup com$

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Note: Gets stuck if procedure does not exist!

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No problem when proving total correctness

Proof Rules for Recursion

Unfolding: $\pi\ p = Some\ c \Longrightarrow wp\ \pi\ (PCall\ p)\ Q\ s = wp\ \pi\ c\ Q\ s$

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assumes $wf\ R$

$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s) \in R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$
$\Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$

shows $HT\ \pi\ P\ (PCall\ p)\ Q$

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Show specification for state s, assuming it holds for smaller states s'.

Mutual Recursion

Same idea, but for sets of specifications.

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$(\forall\,(f,\ P,\ c,\ Q){\in}\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge P\ s)\ c\ Q)$
Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

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Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

$PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta \equiv$
$\forall\,P\ c\ Q.\ (f_0,\ P,\ c,\ Q) \in \Theta \wedge P\ s_0 \longrightarrow wp\ \pi\ (c\ s_0)\ (Q\ s_0)\ s_0$
Hoare-triples valid for fixed variant f_0 and state s_0.

Mutual Recursion

lemma $vcg\_HT'setI$:
assumes $wf\ R$
assumes RL: $\bigwedge f_0\ s_0.\ [\![\ ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta\ ]\!] \Longrightarrow PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta$
shows $HT'set\ \pi\ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

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Fix variant and state,
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$[\![\pi\ p = Some\ c;\ HT\_mods\ \pi\ mods\ P\ c\ Q]\!] \Longrightarrow HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q$

Maps Hoare-Triples to procedure calls

Local Procedures

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Call procedure with local procedure environment

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Call procedure with local procedure environment

$HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q \Longrightarrow HT\_mods\ \pi'\ mods\ P\ (PScope\ \pi\ (PCall\ p))\ Q$

Wrap current procedure environment

Specification of Mutual Recursive Procedures

The IMP2 tools take care of

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- localization of procedure environment.

IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

Completeness

Consider program with $HT\,\pi\ P\ c\ Q$

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Consider program with $HT\,\pi\ P\ c\ Q$

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Only consider while-rule here

Partial Correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ \pi\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
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What invariant shall we use?

$wlp\ \pi\ c\ Q$!

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$\llbracket wf\ R;\ I\ s_0;$
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Invariant: $wp\ \pi\ c\ Q$

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Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

IMP2/Examples.thy

Completeness of While-Rule

Conclusions

IMP2: Verification of simple programs in Isabelle/HOL
  while-language, arrays, local-vars, recursive procedures
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IMP2: Verification of simple programs in Isabelle/HOL
 while-language, arrays, local-vars, recursive procedures
 Tools: concrete syntax for programs and specs, VCG

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 types (char, float, records), pointers, concurrency, ...
 Tools: ghost variables, compiler, ...

Caveats:
 Procedures+Recursion tools not well-tested
 VCG is slow for many procedures/inlines

Chapter 9

Compiler

8 Stack Machine

9 Compiler

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Instructions:

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Semantics

Type synonyms:

$$\begin{array}{rcl} stack & = & int\ list \\ config & = & int \times state \times stack \end{array}$$

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Execution of 1 instruction:

$$iexec :: instr \Rightarrow config \Rightarrow config$$

Instruction execution

$$
\begin{array}{l}
\textit{iexec instr } (i,\ s,\ stk) = \\
(\textsf{case } \textit{instr } \textsf{of } \textit{LOADI } n \Rightarrow (i + 1,\ s,\ n \ \#\ stk) \\
\ \mid \textit{LOAD } x \Rightarrow (i + 1,\ s,\ s\ x \ \#\ stk) \\
\ \mid \textit{ADD} \Rightarrow (i + 1,\ s,\ (\textit{hd2 stk} + \textit{hd stk}) \ \#\ \textit{tl2 stk}) \\
\ \mid \textit{STORE } x \Rightarrow (i + 1,\ s(x := \textit{hd stk}),\ \textit{tl stk}) \\
\ \mid \textit{JMP } n \Rightarrow (i + 1 + n,\ s,\ stk) \\
\ \mid \textit{JMPLESS } n \Rightarrow \\
\qquad (\textsf{if } \textit{hd2 stk} < \textit{hd stk } \textsf{then } i + 1 + n \textsf{ else } i + 1, \\
\qquad\ s,\ \textit{tl2 stk}) \\
\ \mid \textit{JMPGE } n \Rightarrow \\
\qquad (\textsf{if } \textit{hd stk} \le \textit{hd2 stk } \textsf{then } i + 1 + n \textsf{ else } i + 1, \\
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\end{array}
$$

Program execution (1 step)

Programs are instruction lists.

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$$\begin{aligned}
&P \vdash c \rightarrow c' = \\
&(\exists\, i\ s\ stk. \\
&\quad c = (i,\ s,\ stk) \wedge \\
&\quad c' = iexec\ (P\ !!\ i)\ (i,\ s,\ stk) \wedge \\
&\quad 0 \leq i \wedge i < size\ P)
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\end{array}
$$

where	$'a\ list\ !!\ int$	= nth instruction of list
	$size :: 'a\ list \Rightarrow int$	= list size as integer

Program execution ($*$ steps)

Defined in the usual manner:

$$P \vdash (pc,\ s,\ stk) \rightarrow* (pc',\ s',\ stk')$$

`Compiler.thy`

Stack Machine

8 Stack Machine

9 Compiler

Compiling $aexp$

Same as before:

$$acomp\ (N\ n) = [LOADI\ n]$$
$$acomp\ (V\ x) = [LOAD\ x]$$
$$acomp\ (Plus\ a_1\ a_2) = acomp\ a_1\ @\ acomp\ a_2\ @\ [ADD]$$

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Correctness theorem:

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Needs lemmas!

$$P \vdash c \rightarrow\!* \ c' \Longrightarrow P \ @ \ P' \vdash c \rightarrow\!* \ c'$$

$$P \vdash c \rightarrow* c' \Longrightarrow P \mathbin{@} P' \vdash c \rightarrow* c'$$

$$P \vdash (i,\ s,\ stk) \rightarrow* (i',\ s',\ stk') \Longrightarrow$$
$$P' \mathbin{@} P \vdash (size\ P' + i,\ s,\ stk) \rightarrow* (size\ P' + i',\ s',\ stk')$$

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Proofs by cases.

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$$bcomp :: bexp \Rightarrow bool \Rightarrow int \Rightarrow instr\ list$$

Example

Let $b = And \quad (Less\ (V\ ''x'')\ (V\ ''y''))$
$\qquad\qquad\qquad (Not\ (Less\ (V\ ''z'')\ (V\ ''a'')))$.

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$\qquad\qquad\qquad (Not\ (Less\ (V\ ''z'')\ (V\ ''a'')))$.

$bcomp\ b\ False\ 3 =$

$[LOAD\ ''x'',$
$\ LOAD\ ''y'',$

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$bcomp\ (Less\ a_1\ a_2)\ f\ n =$

$acomp\ a_1$ @
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```
let $cb_2 = bcomp\ b_2\ f\ n;$
    $m = \textsf{if}\ f\ \textsf{then}\ size\ cb_2\ \textsf{else}\ size\ cb_2 + n;$
    $cb_1 = bcomp\ b_1\ False\ m$
in $cb_1\ @\ cb_2$
```

Correctness of $bcomp$

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$$
\begin{array}{l}
0 \le n \Longrightarrow \\
bcomp\ b\ f\ n \\
\vdash (0,\ s,\ stk) \rightarrow\!* \\
\quad (size\ (bcomp\ b\ f\ n) + (\textit{if}\ f = bval\ b\ s\ \textit{then}\ n\ \textit{else}\ 0), \\
\quad\ s,\ stk)
\end{array}
$$

Compiling com

$ccomp :: com \Rightarrow instr\ list$

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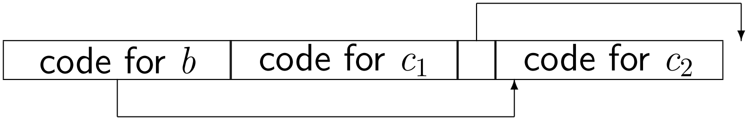
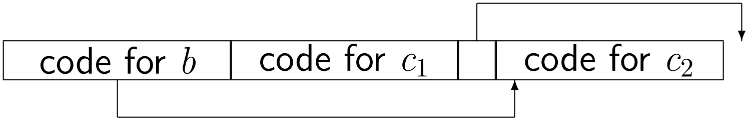


$ccomp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) =$

let $cc_1 = ccomp\ c_1;\ cc_2 = ccomp\ c_2;$
$\quad cb = bcomp\ b\ False\ (size\ cc_1 + 1)$
in $cb\ @\ cc_1\ @\ JMP\ (size\ cc_2)\ \#\ cc_2$



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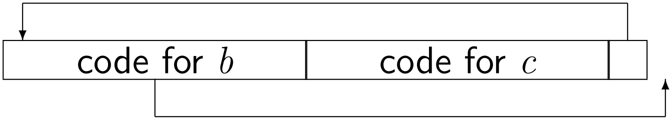
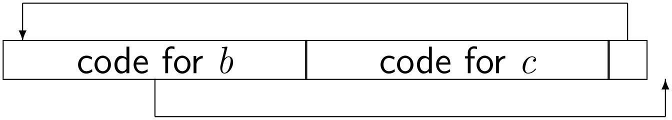


$ccomp\ (WHILE\ b\ DO\ c) =$

let $cc = ccomp\ c;\ cb = bcomp\ b\ False\ (size\ cc + 1)$
in $cb\ @\ cc\ @\ [JMP\ (-\ (size\ cb + size\ cc + 1))]$



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Proof by rule induction.

The other direction

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and thus $t' = t$ (why?).

But we have *not* ruled out this potential error:

c does not terminate but $ccomp\ c$ does.

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- In the absence of nondeterminism:
  Prove that $ccomp$ preserves nontermination.

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- A direct proof: theory *Compiler2*

$$ccomp\ c \vdash (0,\ s,\ stk) \rightarrow\!* (size\ (ccomp\ c),\ t,\ stk') \Longrightarrow (c,\ s) \Rightarrow t$$

Chapter 10

Types

10 A Typed Version of IMP

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Remarks on Type Systems

Typed IMP: Semantics

Typed IMP: Type System

Type Safety of Typed IMP

Why Types?

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To prevent mistakes, dummy!

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The Bad	Static types that have mostly decorative value but do not guarantee anything at runtime. Example: C, C++
The Ugly	Dynamic types that detect errors when it can be too late. Example: "TypeError: ..." in Python.

The ideal

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The most influential slogan and one of the most influential papers in programming language theory.

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There are type systems for *everything* (and more) but in practice (Java, C#) only 1 is covered.

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> *Nontrivial semantic properties of programs*
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How about completeness? Remember Rice:

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Hence there is no (decidable) type system that accepts *all* programs that have a certain semantic property.

Automatic analysis of semantic program properties
is necessarily incomplete.

10 A Typed Version of IMP
Remarks on Type Systems
Typed IMP: Semantics
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Arithmetic

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Arithmetic expresssions:

datatype $aexp =$
$\quad Ic\ int \mid Rc\ real \mid V\ vname \mid Plus\ aexp\ aexp$

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Tags are only used to detect certain errors
and to prove that the type system avoids those errors.

Evaluation of $aexp$

Not recursive function but inductive predicate:

$$taval :: aexp \Rightarrow state \Rightarrow val \Rightarrow bool$$

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If $s\ ''x'' = Rv\ r$: then there is *no* value v such that $taval\ (Plus\ (V\ ''x'')\ (Ic\ 1))\ s\ v$.

The functional alternative

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Exercise!

Boolean expressions

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  Complicates formalization.

com: big or small steps?

We need to detect if things "go wrong".

- Big step semantics:
  Cannot model error by absence of final state.
  Would confuse $\overset{\text{type}}{\text{error}}$ and nontermination.
  Could introduce an extra error-element, e.g.
  $big\_step :: com \times state \Rightarrow state\ option \Rightarrow bool$
  Complicates formalization.
- Small step semantics:
  error = semantics gets stuck

Small step semantics

$$\frac{taval\ a\ s\ v}{(x ::= a,\ s) \rightarrow (SKIP,\ s(x := v))}$$

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The other rules remain unchanged.

typical exam question: give a derivation with these rules

Example

Let $c \;=\; (''x'' ::= Plus\ (V\ ''x'')\ (Ic\ 1))$.

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- If $s\ ''x'' = \mathit{Iv}\ i$:
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- If $s\ ''x'' = \mathit{Rv}\ r$:

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Let $c \;=\; (''x'' ::= Plus\ (V\ ''x'')\ (Ic\ 1))$.

- If $s\ ''x'' = Iv\ i$:
  $(c,\ s) \to (SKIP,\ s(''x'' := Iv\ (i + 1)))$
- If $s\ ''x'' = Rv\ r$:
  $(c,\ s) \not\to$

10 A Typed Version of IMP
Remarks on Type Systems
Typed IMP: Semantics
Typed IMP: Type System
Type Safety of Typed IMP

Type system

There are two types:

datatype $ty = Ity \mid Rty$

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A *type environment* maps variable names to their types:

$tyenv = vname \Rightarrow ty$

The type of an expression is always relative to a type enviroment Γ. Standard notation:

$$\Gamma \vdash e : \tau$$

Read: *In the context of* Γ, e *has type* τ

The type of an $aexp$

$\Gamma \vdash a : \tau$

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$$\Gamma \vdash V\ x : \Gamma\ x$$

$$\frac{\Gamma \vdash a_1 : \tau \qquad \Gamma \vdash a_2 : \tau}{\Gamma \vdash Plus\ a_1\ a_2 : \tau}$$

Example

$$\frac{\vdots}{\Gamma \vdash \ Plus\ (V\ ''x'')\ (Plus\ (V\ ''x'')\ (Ic\ 0))\ :\ ?}$$

where $\Gamma\ ''x'' = Ity$.

Well-typed $bexp$

Notation:

$$\Gamma \vdash b$$

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Read: *In context Γ, b is well-typed*.

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$$\Gamma \vdash \; Bc \; v$$

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$$\frac{\Gamma \vdash a_1 : \tau \qquad \Gamma \vdash a_2 : \tau}{\Gamma \vdash Less\ a_1\ a_2}$$

Example: $\Gamma \vdash Less\ (Ic\ i)\ (Rc\ r)$ does not hold.

Well-typed commands

Notation:

$$\Gamma \vdash c$$

Well-typed commands

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$$\frac{\Gamma \vdash b \qquad \Gamma \vdash c}{\Gamma \vdash \mathit{WHILE}\ b\ \mathit{DO}\ c}$$

Syntax-directedness

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A syntax-directed set of rules

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A syntax-directed set of rules

- is executable by backchaining without backtracking and
- backchaining terminates and requires at most as many steps as the size of the term.

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Remember:
If $s\ ''x'' = Rv\ r$
then $(''x'' ::= Plus\ (V\ ''x'')\ (Ic\ 1),\ s) \not\to$

The state must be well-typed w.r.t. Γ.

The type of a value:

$$type\ (Iv\ i) = Ity$$
$$type\ (Rv\ r) = Rty$$

The type of a value:

$$\begin{aligned}&type\ (Iv\ i) = Ity\\&type\ (Rv\ r) = Rty\end{aligned}$$

Well-typed state:

$$\Gamma \vdash s \longleftrightarrow (\forall x.\ type\ (s\ x) = \Gamma\ x)$$

Type soundness

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Follows from *progress*:

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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
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Reduction cannot get stuck:

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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
> *then you can take one more step.*

and *preservation*:

Type soundness

Reduction cannot get stuck:

> *If everything is ok (* $\Gamma \vdash s$, $\Gamma \vdash c$ *),*
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> *and you have not reached SKIP,*
> *then you can take one more step.*

Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
> *then you can take one more step.*

and *preservation*:

> *If everything is ok and you take a step,*
> *then everything is ok again.*

The slogan

$$\text{Progress} \wedge \text{Preservation} \implies \text{Type safety}$$

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Progress Well-typed programs do not get stuck.

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Progress	Well-typed programs do not get stuck.
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The slogan

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Progress	Well-typed programs do not get stuck.
Preservation	Well-typedness is preserved by reduction.

Preservation: Well-typedness is an *invariant*.

com

Progress:

$$\llbracket \Gamma \vdash c;\ \Gamma \vdash s;\ c \neq SKIP \rrbracket \Longrightarrow \exists\, cs'.\ (c,\ s) \rightarrow cs'$$

com

Progress:

$$[\![\Gamma \vdash c;\ \Gamma \vdash s;\ c \neq SKIP]\!] \Longrightarrow \exists\, cs'.\ (c,\ s) \rightarrow cs'$$

Preservation:

$$[\![(c,\ s) \rightarrow (c',\ s');\ \Gamma \vdash c;\ \Gamma \vdash s]\!] \Longrightarrow \Gamma \vdash s'$$

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$bexp$

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$$\llbracket \Gamma \vdash a : \tau;\ \Gamma \vdash s \rrbracket \Longrightarrow \exists\, v.\ \mathit{taval}\ a\ s\ v$$

Preservation:

$$\llbracket \Gamma \vdash a : \tau;\ \mathit{taval}\ a\ s\ v;\ \Gamma \vdash s \rrbracket \Longrightarrow \mathit{type}\ v = \tau$$

All proofs by rule induction.

Types.thy

The mantra

Type systems have a purpose:

*The static analysis of programs
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The correctness of the prediction must be provable.

Chapter 11

Data-Flow Analyses and Optimization

11 Definite Initialization Analysis

12 Live Variable Analysis

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Each local variable must have a definitely assigned value when any access of its value occurs. A compiler must carry out a specific conservative flow analysis to make sure that, for every access of a local variable x*,* x *is definitely assigned before the access; otherwise a compile-time error must occur.*

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Java Language Specification

Each local variable must have a definitely assigned value when any access of its value occurs. A compiler must carry out a specific conservative flow analysis to make sure that, for every access of a local variable x, x is definitely assigned before the access; otherwise a compile-time error must occur.

Java Language Specification

Java was the first language to force programmers to initialize their variables.

Examples: ok or not?

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Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
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Examples: ok or not?

Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
```

```
IF x < x THEN y := y + 1 ELSE y := x
```

Assume `x` and `y` are initialized:

```
WHILE x < y DO z := x; z := z + 1
```

Simplifying principle

We do not analyze boolean expressions to determine program execution.

11 Definite Initialization Analysis

Prelude: Variables in Expressions

Definite Initialization Analysis

Initialization Sensitive Semantics

Theory $Vars$ provides an overloaded function $vars$:

$vars :: aexp \Rightarrow vname\ set$

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$vars\ (Bc\ v) = \{\}$
$vars\ (Not\ b) = vars\ b$
$vars\ (And\ b_1\ b_2) = vars\ b_1 \cup vars\ b_2$
$vars\ (Less\ a_1\ a_2) = vars\ a_1 \cup vars\ a_2$

Vars.thy

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Modified example from the JLS:

> *Variable x is definitely initialized after $SKIP$ iff x is definitely initialized before $SKIP$.*

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Similar statements for each language construct.

$$D :: vname\ set \Rightarrow com \Rightarrow vname\ set \Rightarrow bool$$

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If all variables in A are initialized before c is executed,

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$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution,

$D :: \mathit{vname\ set} \Rightarrow \mathit{com} \Rightarrow \mathit{vname\ set} \Rightarrow \mathit{bool}$

$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution, and all variables in A' are initialized afterwards.

D A SKIP A

$$\frac{D\ A\ SKIP\ A \qquad vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$D\ A\ SKIP\ A$$

$$\frac{vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$\frac{D\ A_1\ c_1\ A_2 \qquad D\ A_2\ c_2\ A_3}{D\ A_1\ (c_1;;\ c_2)\ A_3}$$

$$D\ A\ \mathit{SKIP}\ A$$

$$\frac{\mathit{vars}\ a \subseteq A}{D\ A\ (x ::= a)\ (\mathit{insert}\ x\ A)}$$

$$\frac{D\ A_1\ c_1\ A_2 \qquad D\ A_2\ c_2\ A_3}{D\ A_1\ (c_1;;\ c_2)\ A_3}$$

$$\frac{\mathit{vars}\ b \subseteq A \qquad D\ A\ c_1\ A_1 \qquad D\ A\ c_2\ A_2}{D\ A\ (\mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2)\ (A_1 \cap A_2)}$$

$$D\ A\ \mathit{SKIP}\ A$$

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$$\frac{\mathit{vars}\ b \subseteq A \qquad D\ A\ c\ A'}{D\ A\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ A}$$

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- Small step semantics:
  semantics shorter, correctness proof longer

For variety's sake, we choose a big step semantics.

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Prelude: Variables in Expressions
Definite Initialization Analysis
Initialization Sensitive Semantics

$$state = vname \Rightarrow val\ option$$

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where

datatype $'a\ option = None \mid Some\ 'a$

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Notation: $s(x \mapsto y)$ means $s(x := Some\ y)$

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where

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Definition: $dom\ s = \{a.\ s\ a \neq None\}$

Expression evaluation

$aval :: aexp \Rightarrow state \Rightarrow val\ option$

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$aval\ (Plus\ a_1\ a_2)\ s =$
$(\textsf{case}\ (aval\ a_1\ s,\ aval\ a_2\ s)\ \textsf{of}$
$\quad (Some\ i_1,\ Some\ i_2) \Rightarrow Some(i_1{+}i_2)$
$\ |\ \_ \Rightarrow None)$

$bval :: bexp \Rightarrow state \Rightarrow bool\ option$

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$(\textsf{case}\ bval\ b\ s\ \textsf{of}\ None \Rightarrow None$
$\mid Some\ bv \Rightarrow Some\ (\neg\ bv))$

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$bval\ (And\ b_1\ b_2)\ s =$
$(\textsf{case}\ (bval\ b_1\ s,\ bval\ b_2\ s)\ \textsf{of}$
$\quad (Some\ bv_1,\ Some\ bv_2) \Rightarrow Some(bv_1 \wedge bv_2)$
$\ \mid \_ \Rightarrow None)$

$bval :: bexp \Rightarrow state \Rightarrow bool\ option$

$bval\ (Bc\ v)\ s = Some\ v$

$bval\ (Not\ b)\ s =$
$(\textsf{case}\ bval\ b\ s\ \textsf{of}\ None \Rightarrow None$
$\ \mid Some\ bv \Rightarrow Some\ (\neg\ bv))$

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$bval\ (Less\ a_1\ a_2)\ s =$
$(\textsf{case}\ (aval\ a_1\ s,\ aval\ a_2\ s)\ \textsf{of}$
$\quad (Some\ i_1,\ Some\ i_2) \Rightarrow Some(i_1 < i_2)$
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Big step semantics

$$(com,\ state) \Rightarrow state\ option$$

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A small complication:

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More convenient, because compositional:

$$(com,\ state\ option) \Rightarrow state\ option$$

Error ($None$) propagates:

$$(c,\ None) \Rightarrow None$$

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We want in the end:

Well-initialized programs cannot go wrong.

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$\mathit{then}\ \ \exists\, t.\ s' = \mathit{Some}\ t \wedge A' \subseteq \mathit{dom}\ t.$

By rule induction on $(c,\ \mathit{Some}\ s) \Rightarrow s'$.

Proof needs some easy lemmas:

$$vars\ a \subseteq\ dom\ s \Longrightarrow \exists\, i.\ aval\ a\ s =\ Some\ i$$

$$vars\ b \subseteq\ dom\ s \Longrightarrow \exists\, bv.\ bval\ b\ s =\ Some\ bv$$

$$D\ A\ c\ A' \Longrightarrow\ A \subseteq\ A'$$

11 Definite Initialization Analysis

12 Live Variable Analysis

Motivation

Consider the following program:

```
    x := y + 1;
    y := y + 2;
    x := y + 3
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Consider the following program:

```
x := y + 1;
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The first assignment is redundant and can be removed because x is dead at that point.

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y := x; y := 0; x := 0
```

☹

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WHILE b DO y := x; x := 1
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At the end of a command, we may be interested in the value of *only some of the variables*, At the end of a command, we may be interested in the value of *only some of the variables*, e.g. *only the global variables* at the end of a procedure.

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Then we say that x is live before c *relative to* the set of variables X.

Liveness analysis

$L :: com \Rightarrow vname\ set \Rightarrow vname\ set$

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$$L\ c\ X\ =\ \text{live before } c \text{ relative to } X$$

$$\begin{aligned} L\ SKIP\ X &= X \\ L\ (x ::= a)\ X &= \end{aligned}$$

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$$
\begin{array}{lcl}
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L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X &= \\
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Example:

$$L\ (''y'' ::= V\ ''z'';;\ ''x'' ::= Plus\ (V\ ''y'')\ (V\ ''z''))$$
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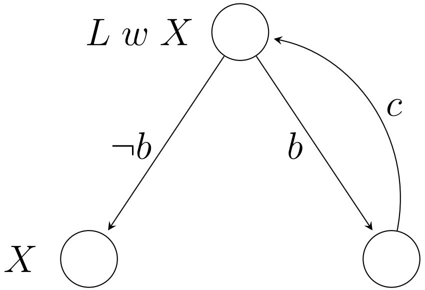
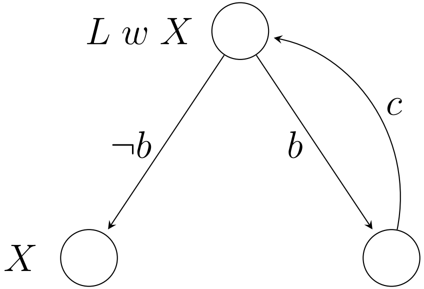
WHILE b DO c



$WHILE\ b\ DO\ c$



$L\ w\ X$ must satisfy

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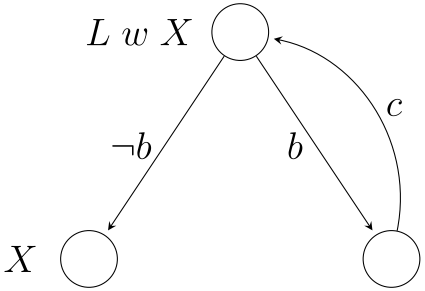
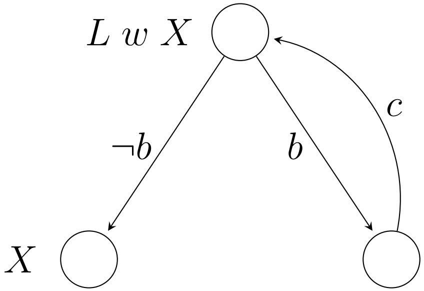


$L\ w\ X$ must satisfy

$vars\ b \subseteq L\ w\ X$ (evaluation of b)

$WHILE\ b\ DO\ c$



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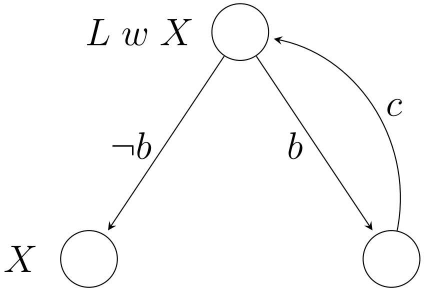

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$L\ c\ (L\ w\ X)$	$\subseteq$	$L\ w\ X$	(execution of c)

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Example:

$$L\ (\mathit{WHILE}\ \mathit{Less}\ (V\ ''x'')\ (V\ ''x'')\ \mathit{DO}\ ''y'' ::= V\ ''z'') \\ \{''x''\}\ =$$

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$$L\ (WHILE\ Less\ (V\ ''x'')\ (V\ ''x'')\ DO\ ''y'' ::= V\ ''z'')$$
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In particular liveness analysis.

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$kill\ SKIP \qquad\qquad\qquad =$

Liveness via gen/kill

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kill\ (WHILE\ b\ DO\ c) & = &
\end{array}$$

Liveness via gen/kill

$kill :: com \Rightarrow vname\ set$

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Proof by induction on c.

$$L\ c\ X = gen\ c \cup (X - kill\ c)$$

Proof by induction on c.

$\Longrightarrow$

$$L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Digression: definite initialization via gen/kill

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\end{array}$$

$$kill\ c = \{\}$$

12 Live Variable Analysis
Correctness of L
Dead Variable Elimination
True Liveness
Comparisons

$(.,.) \Rightarrow .$ and L should roughly be related like this:

The value of the final state on X
only depends on
the value of the initial state on $L\ c\ X$.

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The value of the final state on X
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Put differently:

If two initial states agree on $L \; c \; X$
then the corresponding final states agree on X*.*

Equality on

An abbreviation:

$$f = g \ on \ X \quad \equiv \quad \forall x \in X.\ f\,x = g\ x$$

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Two easy theorems (in theory $Vars$):

$$s_1 = s_2\ on\ vars\ a \Longrightarrow aval\ a\ s_1 = aval\ a\ s_2$$

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Two easy theorems (in theory $Vars$):

$$s_1 = s_2 \ on \ vars \ a \Longrightarrow aval \ a \ s_1 = aval \ a \ s_2$$
$$s_1 = s_2 \ on \ vars \ b \Longrightarrow bval \ b \ s_1 = bval \ b \ s_2$$

Correctness of L

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Correctness of L

If $(c, s) \Rightarrow s'$ *and* $s = t \ on \ L \ c \ X$
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Proof by rule induction.

Correctness of L

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction.
For the two $WHILE$ cases we do not need the definition of $L\ w$ but only the characteristic property

$$vars\ b \cup X \cup L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Optimality of $L\ w$

The result of L should be as small as possible: the more dead variables, the better

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Follows easily from $L\ c\ X = gen\ c \cup (X - kill\ c)$:

$$vars\ b \cup X \cup L\ c\ P \subseteq P \Longrightarrow$$
$$L\ (WHILE\ b\ DO\ c)\ X \subseteq P$$

12 Live Variable Analysis

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Bury all assignments to dead variables:

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Correctness of $bury$

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where $UNIV$ is the set of all variables.

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The two directions need to be proved separately.

$$(c,\ s) \Rightarrow s' \Longrightarrow (bury\ c\ UNIV,\ s) \Rightarrow s'$$

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Follows from generalized statement:

> *If* $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction, like for correctness of L.

$$(bury\ c\ UNIV,\ s) \Rightarrow s' \Longrightarrow (c,\ s) \Rightarrow s'$$

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Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command,

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Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command, e.g.

$$\begin{aligned}
&(bc_1;;\ bc_2 = bury\ c\ X) = \\
&(\exists\, c_1\ c_2. \\
&\quad c = c_1;;\ c_2 \wedge \\
&\quad bc_2 = bury\ c_2\ X \wedge bc_1 = bury\ c_1\ (L\ c_2\ X))
\end{aligned}$$

12 Live Variable Analysis

- Correctness of L
- Dead Variable Elimination
- True Liveness
- Comparisons

Terminology

Let $f :: \tau \Rightarrow \tau$ and $x :: \tau$.

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If $x \leq y \Longrightarrow f\ x \leq f\ y$ for all x,y, then f is *monotone*.

Application to $L\ w$

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Application to $L\ w$

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This is the same as saying that $L\ w\ X$ should be a pfp of

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and in particular of $L\ c$.

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$L\ (''x'' ::= V\ ''y'')\ \{\} = \{''y''\}$

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Better:

$L\ (x ::= a)\ X =$
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But then

$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

is not correct anymore.

$$L\ (x ::= a)\ X =$$
$$(\textsf{\textit{if}}\ x \in X\ \textsf{\textit{then}}\ vars\ a \cup (X - \{x\})\ \textsf{\textit{else}}\ X)$$

$$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$$

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$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

Let $w = WHILE\ b\ DO\ c$
where $b\ =\ Less\ (N\ 0)\ (V\ y)$
and $c\ =\ y ::= V\ x;;\ x ::= V\ z$
and $distinct\ [x,\ y,\ z]$

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and $distinct\ [x,\ y,\ z]$

Then $L\ w\ \{y\} = \{x,\ y\}$, but z is live before w !

$L\ (x ::= a)\ X =$
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Then $L\ w\ \{y\} = \{x,\ y\}$, but z is live before w !

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$L\ w$ for true liveness

Define $L\ w\ X$ *as the least pfp of*
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Existence of least fixpoints

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then

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is the least pre-fixpoint and least fixpoint of f.

Proof of Knaster-Tarski

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Corollary f_w is monotone.

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Proof Show $f^i(\{\}) \subseteq p$ for any pfp p of f (by induction on i).

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Proof by induction on c. In the $WHILE$ case:

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Limiting the number of iterations

Fix some small k (eg 2) and define Lb like L except

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If $Lb\ w\ X = V_w$: $L\ w\ X \subseteq V_w$ (by Lemma)

12 Live Variable Analysis
Correctness of L
Dead Variable Elimination
True Liveness
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