Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP:
A Simple Imperative Language

1. IMP Commands
2. Big-Step Semantics
3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Semantics is easy.

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Command: order to do something

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Study the book until you have understood it.

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Study the book until you have understood it.

Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{rcl}
com & ::= & \texttt{SKIP} \\
 & \mid & string\ \texttt{::=}\ aexp \\
 & \mid & com\ \texttt{;;}\ com \\
 & \mid & \texttt{IF}\ bexp\ \texttt{THEN}\ com\ \texttt{ELSE}\ com \\
 & \mid & \texttt{WHILE}\ bexp\ \texttt{DO}\ com
\end{array}
$$

Commands

Abstract syntax:

$$
\begin{array}{lcl}
\textbf{datatype}\ com & = & SKIP \\
 & | & Assign\ string\ aexp \\
 & | & Seq\ com\ com \\
 & | & If\ bexp\ com\ com \\
 & | & While\ bexp\ com
\end{array}
$$

Com.thy

Big-step semantics

Concrete syntax:

$$(com,\ initial\text{-}state) \Rightarrow final\text{-}state$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x ::=\ a,\ s) \Rightarrow\ s(x :=\ aval\ a\ s)$$

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$$(SKIP,\ s) \Rightarrow s$$

$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s_1 \qquad (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ \ ?}$$

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$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ \ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ \ ?}$$

where
$$\begin{array}{rcl} w & = & WHILE\ b\ DO\ c \\ b & = & NotEq\ (V\ ''x'')\ (N\ 2) \\ c & = & ''x'' ::= Plus\ (V\ ''x'')\ (N\ 1) \\ s_i & = & s(''x'' := i) \end{array}$$

Examples: derivation trees

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$NotEq\ a_1\ a_2 =$
$Not (And\ (Not (Less\ a_1\ a_2))\ (Not (Less\ a_2\ a_1)))$

Logically speaking

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is just infix syntax for

$$big\_step\ (c,s)\ \ t$$

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$$big\_step\ (c,s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

`Big_Step.thy`

Semantics

Rule inversion

What can we deduce from

- $(SKIP, s) \Rightarrow t$?

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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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To prove a goal P with assumption asm,
prove all $asm_i \Longrightarrow P$

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To prove a goal P with assumption asm,
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Example:

$$\frac{F \vee G \quad F \Longrightarrow P \quad G \Longrightarrow P}{P}$$

$elim$ attribute

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- Can also be added locally, eg *(blast elim: ...)*
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$
$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$\begin{gathered} bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow\ s' \wedge (w,\ s') \Rightarrow\ t) \\ \vee \\ \neg\ bval\ b\ s \wedge t = s \end{gathered}$$

Equivalence proof

$$(w,\ s) \Rightarrow t$$

$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$(w',\ s) \Rightarrow t$$

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$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

`Big_Step.thy`

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

`Big_Step.thy`

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

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$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

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Example problem:

$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

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We need a finer grained semantics!

Small-step semantics

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$$(com, state) \to (com, state)$$

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Intended meaning of $(c,\ s) \to (c',\ s')$:

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Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s leaves a "remainder" command c' to be executed in state s'.*

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Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of* c *in state* s
> *leaves a "remainder" command* c'
> *to be executed in state* s'.

Execution as finite or infinite reduction:

$$(c_1,s_1) \to (c_2,s_2) \to (c_3,s_3) \to \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \to cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \to cs'$

The intention:

$$(SKIP,\ s) \text{ is final}$$

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Why?

$SKIP$ is the empty program.

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$$(SKIP, \, s) \ \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

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Small-step rules

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s)\ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \to$$

$$\ldots$$

where $s = <''x'' := 3,\ ''y'' := 7,\ ''z'' := 5>$.

Small-step examples

$$(''z'' ::= V\ ''x'';;\ ''x'' ::= V\ ''y'';;\ ''y'' ::= V\ ''z'',\ s) \rightarrow$$
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$$(w,\ s_0) \rightarrow \ldots$$

where
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Small_Step.thy

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow *$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

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Proof by rule induction

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In two cases a lemma is needed:

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In two cases a lemma is needed:

Lemma
$(c_1,\ s) \rightarrow* (c_1',\ s') \implies (c_1;;\ c_2,\ s) \rightarrow* (c_1';;\ c_2,\ s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1, s) \rightarrow* (c_1', s') \implies (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

Proof by rule induction.

From $\to*$ to $\Rightarrow$

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Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP,\ t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP,\ t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow\!* \;(SKIP,\ t)$

`Small_Step.thy`

Equivalence of big and small

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We prove the contrapositive

$c \neq SKIP \Longrightarrow \neg\ \mathit{final}(c, s)$

by induction on c.

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- Case $c_1;;\ c_2$: by case distinction:

Can execution stop prematurely?

That is, are there any final configs except $(SKIP, s)$?

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

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Together:

Corollary $\mathit{final}\ (c,\ s) = (c = SKIP)$

Infinite executions

$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\textit{program exp} \; \{ \\
\quad a := 1 \\
\quad \textit{while} \; (0 < n) \; \textit{do} \; \{ \\
\quad\quad a := a + a; \\
\quad\quad n := n - 1 \\
\quad \} \\
\}
\end{array}
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At the end of the execution, variable a should contain 2^n, where n is the original value of variable n!
and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
final state satisfies *postcondition*

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Weakest condition on state, such that program c will satisfy postcondition Q.

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wp of equivalent programs is equal

$$c \sim c' \Longrightarrow wp\ c = wp\ c'$$

Correctness of exp

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$$wp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s$$

$$=$$

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Reasoning along syntax of program!

That was easy!

That was easy! But what about While?

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$$wp\ (WHILE\ b\ DO\ c)\ Q\ s$$
$$=$$

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$=$*if* $bval\ b\ s$ *then* $wp\ c\ (wp\ (WHILE\ b\ DO\ c)\ Q)\ s$ *else* $Q\ s$

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Unfolding will continue forever!

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Obviously, need some inductive argument!

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Unfolding will continue forever!

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But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

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Unfold rules still hold:

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$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{if}\ bval\ b\ s\ \textsf{then}\ wlp\ c_1\ Q\ s\ \textsf{else}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate I, such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s \rrbracket \\ \Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

`Wp_Demo.thy`

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Otherwise, use unfold rules.

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If $c = \ WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$

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Combine rule with splitting!

Wp_Demo.thy

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ **then** $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) \in R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

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shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

lemma $ASSUME\_\Theta\_alt$:
$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta = (\forall (f,(P,c,Q)) \in \Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge P\ s)\ c\ Q)$

unfolding $ASSUME\_\Theta\_def$ $HT'set\_r\_def$ **..**

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

Let's make our VCG more usable

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Add standard arithmetic operators to IMP

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Simplify specification of pre/postcondition, and invariants

Standard operators

We add generic syntax for any unary/binary operator

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

`IMP2/Introduction.thy`

Adding more Operators

C-like syntax

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Arith: $+$,$-$,$*$,$/$ with usual binding

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$while\ (bexp)\ c$

`IMP2/Introduction.thy`

Program Syntax

More Readable VCs

Idea: Replace $s\ ''x''$ by (Isabelle) variable x.

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Similar: $s_0\ ''x''$ by x_0.

If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$$

IMP2/Introduction.thy

More Readable VCs

More Readable Annotations

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All variables that occur in the program!

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Precondition: x interpreted as $s\ ''x''$

Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

`IMP2/Introduction.thy`

More Readable Annotations

4 Weakest Preconditions

5 Towards Simpler Verification of Programs

6 Example Verifications

7 Advanced Verification

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Euclid's Algorithm

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

`IMP2/Examples.thy`

Count-up, Count-Down

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Applications: sqrt, log, ...

IMP2/Examples.thy

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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l=0; h=n+1;
while (l+1 < h)
  m = (l + h) / 2;
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Invariant? $l^2 \leq n < h^2 \wedge \ldots$ (range contains solution)

This program is actually tricky to get right!

`IMP2/Examples.thy`

Bisection

6 Example Verifications

Loop Patterns

Euclid's Algorithm

Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers a, b

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd::int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

IMP2/Examples.thy

Euclid

Modified Variables

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$\mathit{modifies}\ \mathit{vars}\ s_1\ \ s_2 = (\forall\, x.\ x \notin\ \mathit{vars} \longrightarrow s_1\ \ x = s_2\ \ x)$

Program modifies at most variables it assigns to

$\pi\text{:}\ (c,\ s) \Rightarrow t \Longrightarrow \mathit{modifies}\ (\mathit{lhsv}\ \pi\ \ c)\ \ t\ \ s$

Modified Variables

We can strengthen correctness statement (automatically)

$wp\ \pi\ \ c\ Q\ s \Longrightarrow wp\ \pi\ \ c\ (\lambda s'.\ Q\ s' \wedge \mathit{modifies}\ (\mathit{lhsv}\ \pi\ \ c)\ s'\ s)\ s$

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For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow wp\ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}){\in}R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ [\![\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ ]\!] \Longrightarrow Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

Modified Variables

The VCG will automatically rewrite with rule

$$\llbracket modifies\ vs\ s\ s';\ x \notin vs \rrbracket \Longrightarrow s\ x = s'\ x$$

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program_spec computes lhs-variables:

$HT\_mods\ \pi\ mods\ P\ c\ Q \equiv HT\ \pi\ P\ c\ (\lambda s_0\ s.\ modifies\ mods\ s\ s_0 \wedge Q\ s_0\ s)$

`IMP2/Examples.thy`

Euclid – show modified sets

Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1;\\
\mathit{while}\ (m{>}0)\ \{\\
\quad n{=}a;\ a = 1;\\
\quad \mathit{while}\ (n{>}0)\ \{\\
\quad\quad a{=}2*a;\ n{=}n{-}1\\
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What does this compute

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Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

Modular Proofs

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Still, we already have verified inner loop!

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Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

a=1;
$while$ (m>0) {
  n=a;
  $inline$ exp_count_down;
  m=m−1
}

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a=1;
while (m>0) {
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Reuse existing proof of exp-count-down program!

Modular Proofs

Re-using proofs:

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with modified sets:

$$\llbracket HT\_mods\ \pi\ \ mods\ P\ c\ Q;\ P\ s;\ \bigwedge s'.\ \llbracket modifies\ mods\ s'\ s;\ Q\ s\ s'\rrbracket \Longrightarrow Q'\ s'\rrbracket \Longrightarrow wp\ \pi\ \ c\ Q'\ s$$

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

`IMP2/Examples.thy`

Power-Tower

7 Advanced Verification

- Arrays
- Data Refinement
- Local Variables
- Recursion

Arrays

Every variable is of type $int \Rightarrow int$.

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$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

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$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

$ArrayClear :: char\ list \Rightarrow com$

π: $(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

Arrays

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Only with index 0: Bind $VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \dots)$

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`IMP2/Examples.thy`

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

IMP2/Examples.thy

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Idea: Halve interval in each step.

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Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

— Donald Knuth

Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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}
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$\uparrow$ l								$\uparrow$ h

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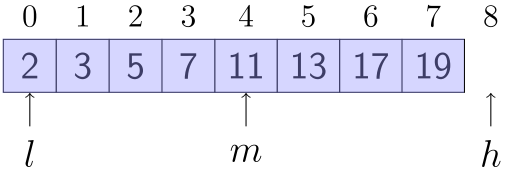

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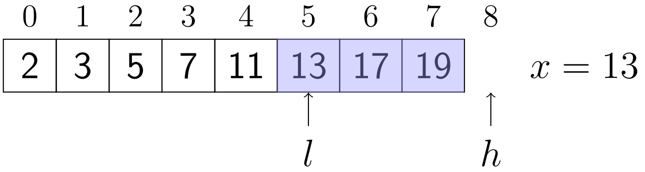


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```
while (l < h) {
    m = (l + h) / 2;
    if (a[m] < x) l = m + 1
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```

Returns smallest i with $x \leq a[i]$

Notes on Binary Search

```
while (l < h) {
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Otherwise, $m = (l + h)/2$ may overflow!

Bug in Java Standard Library for > 9 years!

Proving Binary Search

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Invariant:

Proving Binary Search

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Invariant:

- $i<l \implies a[i] < x$ (strictly smaller than x)

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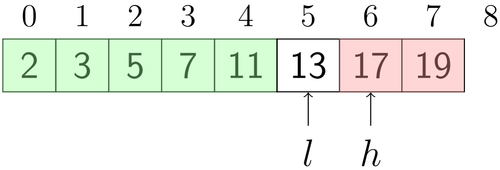

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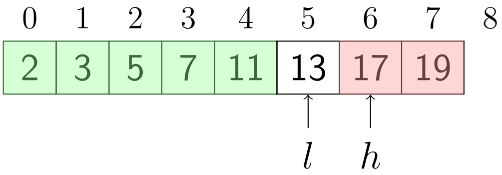
Proving Binary Search



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Invariant:

- $i < l \implies a[i] < x$ (strictly smaller than x)
- $i \geq h \implies x \leq a[i]$ (greater or equal to x)
- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.
In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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where

$$ran\_sorted\ a\ l\ h \equiv \forall\, i \in \{l..<h\}.\ \forall\, j \in \{l..<h\}.\ i \le j \longrightarrow a\ i \le a\ j$$

$$mset\_ran\ a\ r = (\sum i \in r.\ \{\#a\ i\#\})$$

Multisets in Isabelle

imports $HOL{-}Library.Multiset$

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Multiset of elements at indexes in finite set r

Proving Insertion Sort

Separate proof for inner loop!

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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$

$\wedge\ mset\_ran\ a\ \{l..{<}h\} = mset\_ran\ a_0\ \{l..{<}h\}$

Insort: Inner Loop

```
key = a[j];
i = j-1;
while (i>=l && a[i]>key) {
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a[i+1] = key
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Intuition: ?

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Short: Move $a[j]$ backwards over greater elements.

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Invariants easier to find!

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assumes $l < j$, $let\ key = a_0\ j$

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Insort: Inner Loop

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ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and
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assumes $l < j$, $let\ key = a_0\ j$

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ensures $\forall\, k \in \{l..i\}.\ a\ k = a_0\ k$ and
$a\ (i + 1) = key$ and
$\forall\, k \in \{i + 2..j\}.\ a\ k = a_0\ (k - 1)$

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

assumes $l < j$, $let\ key = a_0\ j$
ensures $i \in \{l - 1..{<}j\}$
ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and
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Insort: Finding Invariant

0	1	2	3	4	5	6	7
2	3	5	7	13	17	19	11
$\uparrow$						$\uparrow$	$\uparrow$
l						i	j

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  $\forall\, k \in \{i+2..j\}.\ key < a\ k$
- $+$ the usual bounds: $l - 1 \leq i \wedge i < j$

`IMP2/Examples.thy`

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

7 Advanced Verification

Abstract View

Model $int \Rightarrow int$ not always appropriate

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E.g., list: Understand $a\ [l..{<}h]$ as $int\ list$

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Instead of one complex proof, get two simple proofs !

`IMP2/Examples.thy`

Filter, Merge, dedup

7 Advanced Verification

- Arrays
- Data Refinement
- Local Variables
- Recursion

Local Variables

Introduce local variables

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Why?

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Why? Better modularity.

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Don't worry about name-clashes with subroutine's auxiliary variables

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Partition variable names into local and global names

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is_global — Variable name starts with "G"

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\textbf{fun } is\_global :: vname \Rightarrow bool \textbf{ where} \\
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|\ is\_global\ (CHR\ ''G''\#\_) \longleftrightarrow True \\
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$is\_local\ a = \neg is\_global\ a$

State Combination

$<s_1|s_2>$ – State with locals from s_1, globals from s_2

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$SCOPE\ c$ — Execute c with fresh set of local variables. Restore original local variables afterwards

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$=$

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$$= wp\ \pi\ c\ (\lambda s'.\ Q\ <s|s'>)\ <<>|s>$$

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Pass information over scope boundaries by globals

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Wrapping Specification

Given specification of body: $HT\ P\ body\ Q$
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Recall:

$HT\ \pi\ \ P\ \ c\ \ Q \equiv \forall\, s_0.\ P\ s_0 \longrightarrow wp\ \pi\ \ c\ (Q\ s_0)\ s_0$

Prologue

$$HT\ \pi\ P\ body\ Q \Longrightarrow$$
$$HT\ \pi\ (wp\ \pi\ prologue\ P)\ (prologue;;\ body)$$
$$(\lambda s_0\ s.\ wp\ \pi\ prologue\ (\lambda s_0.\ Q\ s_0\ s)\ s_0)$$

Intuition: Weakest precondition to enforce P after prologue

Epilogue

$\llbracket HT\ \pi\ P\ body\ Q;\ \forall\, s.\ \exists\, t.\ \pi\colon (epilogue,\ s) \Rightarrow\ t\rrbracket$
$\Longrightarrow HT\ \pi\ P\ (body;;\ epilogue)\ (\lambda s_0.\ sp\ \pi\ (Q\ s_0)$
$epilogue)$

Intuition: Strongest postcondition we get from Q after epilogue

Strongest Postconditions

$sp\ \pi\ \ P\ \ c\ \ t \equiv \exists\, s.\ P\ \ s \wedge \pi\colon (c,\ s) \Rightarrow\ t$

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Scope: $HT\ \pi\ P\ c\ Q \Longrightarrow$
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IMP2/Examples.thy

Merge as Procedure

7 Advanced Verification

- Arrays
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Recursive Procedures

Program is map $pname \rightharpoonup com$

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Note: Gets stuck if procedure does not exist!
No problem when proving total correctness

Proof Rules for Recursion

Unfolding: $\pi\ p = Some\ c \Longrightarrow wp\ \pi\ (PCall\ p)\ Q\ s = wp\ \pi\ c\ Q\ s$

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assumes $wf\ R$

$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s){\in}R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$
$\quad \Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$

shows $HT\ \pi\ P\ (PCall\ p)\ Q$

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Idea: Well-Founded induction on state

assumes $wf\ R$
$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s){\in}R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$
$\Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$
shows $HT\ \pi\ P\ (PCall\ p)\ Q$

Show specification for state s, assuming it holds for smaller states s'.

Mutual Recursion

Same idea, but for sets of specifications.

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$(\forall\,(f,\ P,\ c,\ Q){\in}\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge P\ s)\ c\ Q)$
Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

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Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

$PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta \equiv$
$\forall\,P\ c\ Q.\ (f_0,\ P,\ c,\ Q) \in \Theta \wedge P\ s_0 \longrightarrow wp\ \pi\ (c\ s_0)\ (Q\ s_0)\ s_0$
Hoare-triples valid for fixed variant f_0 and state s_0.

Mutual Recursion

lemma $vcg\_HT'setI$:
assumes $wf\ R$
assumes RL: $\bigwedge f_0\ s_0.\ \llbracket\ ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta\ \rrbracket \Longrightarrow PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta$
shows $HT'set\ \pi\ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

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Fix variant and state,
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$\llbracket \pi\ p = Some\ c;\ HT\_mods\ \pi\ mods\ P\ c\ Q \rrbracket \Longrightarrow HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q$

Maps Hoare-Triples to procedure calls

Local Procedures

Idea: Recursive procedure names only valid locally!

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Call procedure with local procedure environment

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Call procedure with local procedure environment

$HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q \Longrightarrow HT\_mods\ \pi'\ mods\ P$
$(PScope\ \pi\ (PCall\ p))\ Q$

Wrap current procedure environment

Specification of Mutual Recursive Procedures

The IMP2 tools take care of

- wf-relation. Default $less\_than$

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- variants: expression over parameters.
- localization of procedure environment.

IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

Completeness

Consider program with $HT\,\pi\ P\ c\ Q$

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Can we always find annotations to get provable VCs?

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Only consider while-rule here

Partial Correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ \pi\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s\rrbracket$$
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What invariant shall we use?

$wlp\ \pi\ c\ Q$!

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Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

`IMP2/Examples.thy`

Completeness of While-Rule

Conclusions

IMP2: Verification of simple programs in Isabelle/HOL
  while-language, arrays, local-vars, recursive procedures
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Caveats:
Procedures+Recursion tools not well-tested
VCG is slow for many procedures/inlines

Chapter 9

Compiler

8 Stack Machine

9 Compiler

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Instructions:

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Semantics

Type synonyms:

$$\begin{array}{rcl} stack & = & int\ list \\ config & = & int \times state \times stack \end{array}$$

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$$\begin{array}{rcl} stack & = & int\ list \\ config & = & int \times state \times stack \end{array}$$

Execution of 1 instruction:

$$iexec :: instr \Rightarrow config \Rightarrow config$$

Instruction execution

$$
\begin{array}{l}
\mathit{iexec}\ \mathit{instr}\ (i,\ s,\ \mathit{stk}) = \\
(\textsf{case}\ \mathit{instr}\ \textsf{of}\ \mathit{LOADI}\ n \Rightarrow (i + 1,\ s,\ n\ \#\ \mathit{stk}) \\
\ \mid \mathit{LOAD}\ x \Rightarrow (i + 1,\ s,\ s\ x\ \#\ \mathit{stk}) \\
\ \mid \mathit{ADD} \Rightarrow (i + 1,\ s,\ (\mathit{hd2}\ \mathit{stk} + \mathit{hd}\ \mathit{stk})\ \#\ \mathit{tl2}\ \mathit{stk}) \\
\ \mid \mathit{STORE}\ x \Rightarrow (i + 1,\ s(x := \mathit{hd}\ \mathit{stk}),\ \mathit{tl}\ \mathit{stk}) \\
\ \mid \mathit{JMP}\ n \Rightarrow (i + 1 + n,\ s,\ \mathit{stk}) \\
\ \mid \mathit{JMPLESS}\ n \Rightarrow \\
\qquad (\textsf{if}\ \mathit{hd2}\ \mathit{stk} < \mathit{hd}\ \mathit{stk}\ \textsf{then}\ i + 1 + n\ \textsf{else}\ i + 1, \\
\qquad\ s,\ \mathit{tl2}\ \mathit{stk}) \\
\ \mid \mathit{JMPGE}\ n \Rightarrow \\
\qquad (\textsf{if}\ \mathit{hd}\ \mathit{stk} \le \mathit{hd2}\ \mathit{stk}\ \textsf{then}\ i + 1 + n\ \textsf{else}\ i + 1, \\
\qquad\ s,\ \mathit{tl2}\ \mathit{stk}))
\end{array}
$$

Program execution (1 step)

Programs are instruction lists.

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Executing one program step:

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$$\begin{array}{l} P \vdash c \rightarrow c' = \\ (\exists\, i\ s\ stk. \\ \quad c = (i,\ s,\ stk) \wedge \\ \quad c' = iexec\ (P\ !!\ i)\ (i,\ s,\ stk) \wedge \\ \quad 0 \leq i \wedge i < size\ P) \end{array}$$

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$$

where $'a\ list\ !!\ int$ = nth instruction of list

$size :: {'a}\ list \Rightarrow int$ = list size as integer

Program execution ($*$ steps)

Defined in the usual manner:

$$P \vdash (pc,\ s,\ stk) \rightarrow\!* \ (pc',\ s',\ stk')$$

Compiler.thy

Stack Machine

8 Stack Machine

9 Compiler

Compiling $aexp$

Same as before:

$$acomp\ (N\ n) = [LOADI\ n]$$
$$acomp\ (V\ x) = [LOAD\ x]$$
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Needs lemmas!

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$$P \vdash (i,\ s,\ stk) \rightarrow\!* \ (i',\ s',\ stk') \Longrightarrow$$
$$P' \ @ \ P \vdash (size\ P' + i,\ s,\ stk) \rightarrow\!* \ (size\ P' + i',\ s',\ stk')$$

$$P \vdash c \rightarrow* c' \Longrightarrow P \mathbin{@} P' \vdash c \rightarrow* c'$$

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Proofs by cases.

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$$bcomp :: bexp \Rightarrow bool \Rightarrow int \Rightarrow instr\ list$$

Example

Let $b = And \quad (Less\ (V\ ''x'')\ (V\ ''y''))$
$\qquad\qquad\qquad (Not\ (Less\ (V\ ''z'')\ (V\ ''a'')))$.

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$$bcomp\ (Not\ b)\ f\ n = bcomp\ b\ (\neg f)\ n$$

$$bcomp\ (Less\ a_1\ a_2)\ f\ n =$$

$$acomp\ a_1\ @$$
$$acomp\ a_2\ @\ (\textsf{if}\ f\ \textsf{then}\ [JMPLESS\ n]\ \textsf{else}\ [JMPGE\ n])$$

$$bcomp\ (And\ b_1\ b_2)\ f\ n =$$

$$\begin{aligned}
&\textsf{let}\ cb_2 = bcomp\ b_2\ f\ n;\\
&\quad m = \textsf{if}\ f\ \textsf{then}\ size\ cb_2\ \textsf{else}\ size\ cb_2 + n;\\
&\quad cb_1 = bcomp\ b_1\ False\ m\\
&\textsf{in}\ cb_1\ @\ cb_2
\end{aligned}$$

Correctness of $bcomp$

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$$
\begin{array}{l}
0 \le n \Longrightarrow \\
bcomp\ b\ f\ n \\
\vdash (0,\ s,\ stk) \rightarrow\!* \\
\quad (size\ (bcomp\ b\ f\ n) + (\textit{if}\ f = bval\ b\ s\ \textit{then}\ n\ \textit{else}\ 0), \\
\quad\ s,\ stk)
\end{array}
$$

Compiling com

$ccomp :: com \Rightarrow instr\ list$

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$ccomp\ SKIP = []$

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$$ccomp\ (c_1;;\ c_2) = ccomp\ c_1\ @\ ccomp\ c_2$$

$$ccomp\ (IF\ b\ \ THEN\ c_1\ \ ELSE\ c_2) =$$

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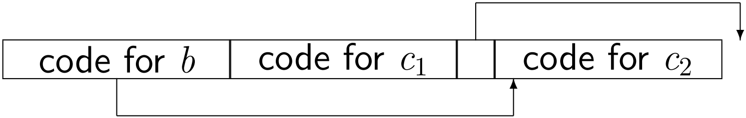
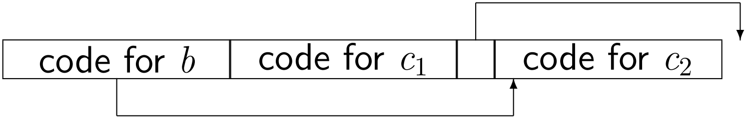


$$ccomp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) =$$

$$\textsf{\textit{let}}\ cc_1 = ccomp\ c_1;\ cc_2 = ccomp\ c_2;$$
$$\quad cb = bcomp\ b\ False\ (size\ cc_1 + 1)$$
$$\textsf{\textit{in}}\ cb\ @\ cc_1\ @\ JMP\ (size\ cc_2)\ \#\ cc_2$$



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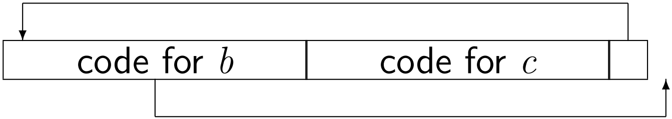
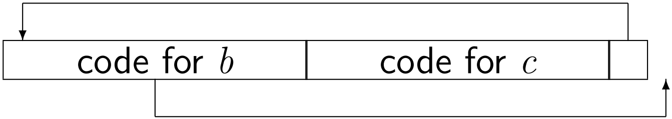


$$ccomp\ (WHILE\ b\ DO\ c) =$$

$$\textsf{\textit{let}}\ cc = ccomp\ c;\ cb = bcomp\ b\ False\ (size\ cc + 1)$$
$$\textsf{\textit{in}}\ cb\ @\ cc\ @\ [JMP\ (-\ (size\ cb + size\ cc + 1))]$$



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Proof by rule induction.

The other direction

We have only shown "$\Longrightarrow$":

compiled code simulates source code.

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and if(!) $(c,\ s) \Rightarrow t'$, then "$\Longrightarrow$" implies
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and thus $t' = t$ (why?).

But we have *not* ruled out this potential error:

c does not terminate but $ccomp\ c$ does.

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Two approaches:

- In the absence of nondeterminism:
  Prove that $ccomp$ preserves nontermination.

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- In the absence of nondeterminism:
  Prove that $ccomp$ preserves nontermination.
  A nice proof of this fact requires *coinduction*.
  Isabelle supports coinduction, this course avoids it.
- A direct proof: theory $Compiler2$

$$ccomp\ c \vdash (0,\ s,\ stk) \rightarrow\!* (size\ (ccomp\ c),\ t,\ stk') \Longrightarrow (c,\ s) \Rightarrow t$$

Chapter 10

Types

10 A Typed Version of IMP

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Why Types?

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To prevent mistakes, dummy!

There are 3 kinds of types

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our focus

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Example: C, C++

The Ugly Dynamic types that detect errors when it can be too late.
Example: "TypeError: ..." in Python.

The ideal

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Robin Milner, *A Theory of Type Polymorphism in Programming*, 1978.

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The most influential slogan and one of the most influential papers in programming language theory.

What could go wrong?

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2. Null pointer exception
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6. and many more ...

There are type systems for *everything* (and more) but in practice (Java, C#) only 1 is covered.

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A programming language is *type safe* if the execution of a well-typed program cannot lead to certain errors.

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> *Nontrivial semantic properties of programs*
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Hence there is no (decidable) type system that accepts *all* programs that have a certain semantic property.

Automatic analysis of semantic program properties
is necessarily incomplete.

10 A Typed Version of IMP

Arithmetic

Values:

datatype $val = Iv\ int \mid Rv\ real$

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Arithmetic expresssions:

datatype $aexp =$
$\quad Ic\ int \mid Rc\ real \mid V\ vname \mid Plus\ aexp\ aexp$

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Tags are only used to detect certain errors
and to prove that the type system avoids those errors.

Evaluation of $aexp$

Not recursive function but inductive predicate:

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If $s\ ''x'' = Rv\ r$: then there is *no* value v such that $taval\ (Plus\ (V\ ''x'')\ (Ic\ 1))\ s\ v$.

The functional alternative

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Exercise!

Boolean expressions

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  Would confuse type error and nontermination.
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  $big\_step :: com \times state \Rightarrow state\ option \Rightarrow bool$
  Complicates formalization.
- Small step semantics:
  error = semantics gets stuck

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$$\frac{taval\ a\ s\ v}{(x ::= a,\ s) \rightarrow (SKIP,\ s(x := v))}$$

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$$\frac{\mathit{taval}\ a\ s\ v}{(x ::= a,\ s) \rightarrow (\mathit{SKIP},\ s(x := v))}$$

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The other rules remain unchanged.

typical exam question: give a derivation with these rules

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- If $s\ ''x'' = \mathit{Rv}\ r$:
  $(c,\ s) \not\to$

10 A Typed Version of IMP

Type system

There are two types:

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What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

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What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

Depends on the type of $''x''$ and $''y''$!

Type system

There are two types:

datatype $ty = Ity \mid Rty$

What is the type of $Plus\ (V\ ''x'')\ (V\ ''y'')$?

Depends on the type of $''x''$ and $''y''$!

A *type environment* maps variable names to their types:
$tyenv = vname \Rightarrow ty$

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The type of an expression is always relative to a type enviroment Γ. Standard notation:

$$\Gamma \vdash e : \tau$$

Read: *In the context of* Γ*,* e *has type* τ

The type of an $aexp$

$\Gamma \vdash a : \tau$

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$$\Gamma \vdash V\ x : \Gamma\ x$$

$$\frac{\Gamma \vdash a_1 : \tau \qquad \Gamma \vdash a_2 : \tau}{\Gamma \vdash Plus\ a_1\ a_2 : \tau}$$

Example

$$\frac{\vdots}{\Gamma \vdash \ Plus\ (V\ ''x'')\ (Plus\ (V\ ''x'')\ (Ic\ 0))\ :\ ?}$$

where $\Gamma\ ''x'' = Ity$.

Well-typed $bexp$

Notation:

$$\Gamma \vdash b$$

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Example: $\Gamma \vdash Less\ (Ic\ i)\ (Rc\ r)$ does not hold.

Well-typed commands

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$$\frac{\Gamma \vdash b \qquad \Gamma \vdash c_1 \qquad \Gamma \vdash c_2}{\Gamma \vdash IF\ b\ THEN\ c_1\ ELSE\ c_2}$$

$$\frac{\Gamma \vdash b \qquad \Gamma \vdash c}{\Gamma \vdash WHILE\ b\ DO\ c}$$

Syntax-directedness

All three sets of typing rules are *syntax-directed*:

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A syntax-directed set of rules

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- backchaining terminates and requires at most as many steps as the size of the term.

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- the execution of $WHILE$ depends on the execution of $WHILE$.

Well-typed states

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If $s\ ''x'' = Rv\ r$
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Even well-typed programs can get stuck ...
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Remember:
If $s\ ''x'' = Rv\ r$
then $(''x'' ::= Plus\ (V\ ''x'')\ (Ic\ 1),\ s) \not\to$

The state must be well-typed w.r.t. Γ.

The type of a value:

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Well-typed state:

$$\Gamma \vdash s \longleftrightarrow (\forall x.\ type\ (s\ x) = \Gamma\ x)$$

Type soundness

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If everything is ok ($\Gamma \vdash s$, $\Gamma \vdash c$),
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> *If everything is ok (* $\Gamma \vdash s$*,* $\Gamma \vdash c$ *),*
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Follows from *progress*:

Type soundness

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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
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and *preservation*:

Type soundness

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Follows from *progress*:

> *If everything is ok and you have not reached SKIP,*
> *then you can take one more step.*

and *preservation*:

> *If everything is ok and you take a step,*
> *then everything is ok again.*

The slogan

$$\text{Progress} \wedge \text{Preservation} \implies \text{Type safety}$$

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The slogan

$$\text{Progress} \wedge \text{Preservation} \Longrightarrow \text{Type safety}$$

Progress	Well-typed programs do not get stuck.
Preservation	Well-typedness is preserved by reduction.

Preservation: Well-typedness is an *invariant*.

com

Progress:

$$[\![\Gamma \vdash c;\ \Gamma \vdash s;\ c \neq SKIP]\!] \Longrightarrow \exists\, cs'.\ (c,\ s) \rightarrow cs'$$

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$bexp$

Progress:

$$\llbracket \Gamma \vdash\ b;\ \Gamma \vdash\ s \rrbracket \Longrightarrow \exists\, v.\ \ tbval\ b\ \ s\ \ v$$

$aexp$

Progress:

$$[\![\Gamma \vdash a : \tau;\ \Gamma \vdash s]\!] \Longrightarrow \exists\, v.\ \mathit{taval}\ a\ s\ v$$

Preservation:

$$[\![\Gamma \vdash a : \tau;\ \mathit{taval}\ a\ s\ v;\ \Gamma \vdash s]\!] \Longrightarrow \mathit{type}\ v = \tau$$

All proofs by rule induction.

Types.thy

The mantra

Type systems have a purpose:

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> *The static analysis of programs*
> *in order to predict their runtime behaviour.*

The correctness of the prediction must be provable.

Chapter 11

Data-Flow Analyses and Optimization

11 Definite Initialization Analysis

12 Live Variable Analysis

Each local variable must have a definitely assigned value when any access of its value occurs. A compiler must carry out a specific conservative flow analysis to make sure that, for every access of a local variable x*,* x *is definitely assigned before the access; otherwise a compile-time error must occur.*

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Java Language Specification

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Java Language Specification

Java was the first language to force programmers to initialize their variables.

Examples: ok or not?

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Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
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Examples: ok or not?

Assume `x` is initialized:

```
IF x < 1 THEN y := x ELSE y := x + 1;
y := y + 1
```

```
IF x < x THEN y := y + 1 ELSE y := x
```

Assume `x` and `y` are initialized:

```
WHILE x < y DO z := x; z := z + 1
```

Simplifying principle

We do not analyze boolean expressions to determine program execution.

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$vars :: aexp \Rightarrow vname\ set$

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$vars\ (Bc\ v) = \{\}$

$vars\ (Not\ b) = vars\ b$

$vars\ (And\ b_1\ b_2) = vars\ b_1 \cup vars\ b_2$

$vars\ (Less\ a_1\ a_2) = vars\ a_1 \cup vars\ a_2$

Vars.thy

11 Definite Initialization Analysis

Modified example from the JLS:

Variable x is definitely initialized after $SKIP$ iff x is definitely initialized before $SKIP$.

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Similar statements for each language construct.

$$D :: vname\ set \Rightarrow com \Rightarrow vname\ set \Rightarrow bool$$

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$D\ A\ c\ A'$ should imply:

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$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed,

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If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution,

$D :: vname\ set \Rightarrow com \Rightarrow vname\ set \Rightarrow bool$

$D\ A\ c\ A'$ should imply:

If all variables in A are initialized before c is executed, then no uninitialized variable is accessed during execution, and all variables in A' are initialized afterwards.

D A SKIP A

$$\frac{D\ A\ SKIP\ A \qquad vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$\overline{D\ A\ SKIP\ A}$$

$$\frac{vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$\frac{D\ A_1\ c_1\ A_2 \qquad D\ A_2\ c_2\ A_3}{D\ A_1\ (c_1;;\ c_2)\ A_3}$$

$$D\ A\ \mathit{SKIP}\ A$$

$$\frac{\mathit{vars}\ a \subseteq A}{D\ A\ (x ::= a)\ (\mathit{insert}\ x\ A)}$$

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$$\frac{\mathit{vars}\ b \subseteq A \qquad D\ A\ c_1\ A_1 \qquad D\ A\ c_2\ A_2}{D\ A\ (\mathit{IF}\ b\ \mathit{THEN}\ c_1\ \mathit{ELSE}\ c_2)\ (A_1 \cap A_2)}$$

$$D\ A\ SKIP\ A$$

$$\frac{vars\ a \subseteq A}{D\ A\ (x ::= a)\ (insert\ x\ A)}$$

$$\frac{D\ A_1\ c_1\ A_2 \qquad D\ A_2\ c_2\ A_3}{D\ A_1\ (c_1;;\ c_2)\ A_3}$$

$$\frac{vars\ b \subseteq A \qquad D\ A\ c_1\ A_1 \qquad D\ A\ c_2\ A_2}{D\ A\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ (A_1 \cap A_2)}$$

$$\frac{vars\ b \subseteq A \qquad D\ A\ c\ A'}{D\ A\ (WHILE\ b\ DO\ c)\ A}$$

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For variety's sake, we choose a big step semantics.

11 Definite Initialization Analysis

$$state = vname \Rightarrow val\ option$$

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where

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where

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Definition: $dom\ s = \{a.\ s\ a \neq None\}$

Expression evaluation

$aval :: aexp \Rightarrow state \Rightarrow val\ option$

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$(\mathsf{case}\ (aval\ a_1\ s,\ aval\ a_2\ s)\ \mathsf{of}$
$\quad (Some\ i_1,\ Some\ i_2) \Rightarrow Some(i_1{+}i_2)$
$\ |\ \_ \Rightarrow None)$

$bval :: bexp \Rightarrow state \Rightarrow bool\ option$

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$$bval\ (Bc\ v)\ s = Some\ v$$

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$bval\ (Not\ b)\ s =$
$(\textsf{case}\ bval\ b\ s\ \textsf{of}\ None \Rightarrow None$
$\mid Some\ bv \Rightarrow Some\ (\neg\ bv))$

$bval :: bexp \Rightarrow state \Rightarrow bool\ option$

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$(\mathsf{case}\ (aval\ a_1\ s,\ aval\ a_2\ s)\ \mathsf{of}$
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Big step semantics

$$(com,\ state) \Rightarrow state\ option$$

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A small complication:

$$\frac{(c_1,\ s_1) \Rightarrow Some\ s_2 \qquad (c_2,\ s_2) \Rightarrow s}{(c_1;;\ c_2,\ s_1) \Rightarrow s}$$

Big step semantics

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$$\frac{(c_1,\ s_1) \Rightarrow None}{(c_1;;\ c_2,\ s_1) \Rightarrow None}$$

More convenient, because compositional:

$$(com,\ state\ option) \Rightarrow state\ option$$

Error ($None$) propagates:

$$(c,\ None) \Rightarrow None$$

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Execution starting in (mostly) normal states ($Some\ s$):

$$(SKIP,\ s) \Rightarrow s$$

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$$\frac{aval\ a\ s = Some\ i}{(x ::= a,\ Some\ s) \Rightarrow Some\ (s(x \mapsto i))}$$

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Execution starting in (mostly) normal states ($Some\ s$):

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$$\frac{aval\ a\ s = None}{(x ::= a,\ Some\ s) \Rightarrow None}$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

$$\frac{bval\ b\ s = Some\ True \qquad (c_1,\ Some\ s) \Rightarrow s'}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ Some\ s) \Rightarrow s'}$$

$$\frac{bval\ b\ s = Some\ True \qquad (c_1,\ Some\ s) \Rightarrow s'}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ Some\ s) \Rightarrow s'}$$

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$$\frac{\begin{array}{c} bval\ b\ s = Some\ True \\ (c,\ Some\ s) \Rightarrow s' \qquad (WHILE\ b\ DO\ c,\ s') \Rightarrow s'' \end{array}}{(WHILE\ b\ DO\ c,\ Some\ s) \Rightarrow s''}$$

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Correctness of D w.r.t. $\Rightarrow$

We want in the end:

Well-initialized programs cannot go wrong.

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$\mathit{If}\ \ (c,\ \mathit{Some}\ s) \Rightarrow s'\ \ \mathit{and}\ \ D\ A\ c\ A'\ \ \mathit{and}\ \ A \subseteq \mathit{dom}\ s$
$\mathit{then}\ \ \exists\, t.\ s' = \mathit{Some}\ t \wedge A' \subseteq \mathit{dom}\ t.$

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By rule induction on $(c,\ \textit{Some}\ s) \Rightarrow s'$.

Proof needs some easy lemmas:

$$vars\ a \subseteq dom\ s \Longrightarrow \exists\, i.\ aval\ a\ s = Some\ i$$

$$vars\ b \subseteq dom\ s \Longrightarrow \exists\, bv.\ bval\ b\ s = Some\ bv$$

$$D\ A\ c\ A' \Longrightarrow A \subseteq A'$$

Motivation

Consider the following program:

```
x := y + 1;
y := y + 2;
x := y + 3
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Consider the following program:

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x := y + 1;
y := y + 2;
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```

The first assignment is redundant and can be removed because x is dead at that point.

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```
x := 0                                  ☹
y := x; y := 0; x := 0
```

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x := 0                              ☹
y := x; y := 0; x := 0              ☺
WHILE b DO y := x; x := 1
```

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```

At the end of a command, we may be interested in the value of *only some of the variables*, At the end of a command, we may be interested in the value of *only some of the variables*, e.g. *only the global variables* at the end of a procedure.

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Then we say that x is live before c *relative to* the set of variables X.

Liveness analysis

$L :: com \Rightarrow vname\ set \Rightarrow vname\ set$

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$$\begin{array}{lcl} L\ SKIP\ X & = & X \\ L\ (x ::= a)\ X & = & \end{array}$$

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$$\begin{aligned}
L\ SKIP\ X &= X\\
L\ (x ::= a)\ X &= vars\ a \cup (X - \{x\})\\
L\ (c_1;;\ c_2)\ X &= L\ c_1\ (L\ c_2\ X)
\end{aligned}$$

$$\begin{aligned}
&L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X =\\
&\quad vars\ b \cup L\ c_1\ X \cup L\ c_2\ X
\end{aligned}$$

Example:

$$\begin{aligned}
&L\ (''y'' ::= V\ ''z'';;\ ''x'' ::= Plus\ (V\ ''y'')\ (V\ ''z''))\\
&\quad \{''x''\} \;=
\end{aligned}$$

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\multicolumn{3}{l}{L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =} \\
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$$L\ (''y'' ::= V\ ''z'';;\ ''x'' ::= Plus\ (V\ ''y'')\ (V\ ''z''))$$
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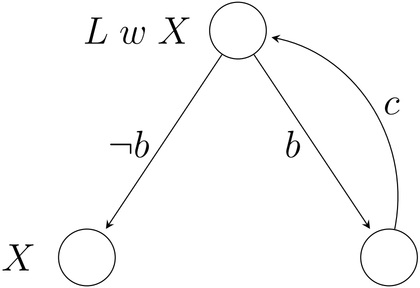
$WHILE\ b\ DO\ c$



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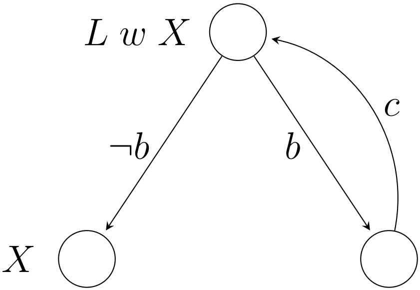
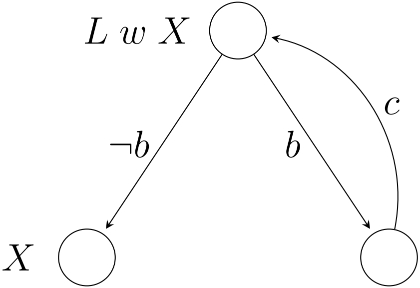


$L\ w\ X$ must satisfy

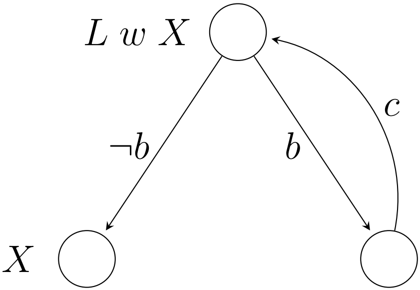
$WHILE\ b\ DO\ c$



$L\ w\ X$ must satisfy

$vars\ b \subseteq L\ w\ X$ (evaluation of b)

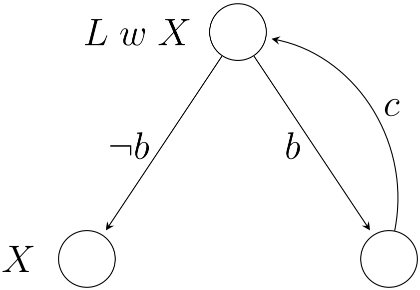
$WHILE\ b\ DO\ c$



$L\ w\ X$ must satisfy

$vars\ b$	$\subseteq$	$L\ w\ X$	(evaluation of b)
X	$\subseteq$	$L\ w\ X$	(exit)

$\mathit{WHILE}\ b\ \mathit{DO}\ c$



$L\ w\ X$ must satisfy

$$\begin{array}{lcll} \mathit{vars}\ b & \subseteq & L\ w\ X & \text{(evaluation of } b\text{)} \\ X & \subseteq & L\ w\ X & \text{(exit)} \\ L\ c\ (L\ w\ X) & \subseteq & L\ w\ X & \text{(execution of } c\text{)} \end{array}$$

We define

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$L\ c\ (L\ w\ X) \subseteq L\ w\ X$?

$$
\begin{array}{lcl}
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\end{array}
$$

$$L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =\ vars\ b \cup L\ c_1\ X \cup L\ c_2\ X$$

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Example:

$$L\ (WHILE\ Less\ (V\ ''x'')\ (V\ ''x'')\ DO\ ''y'' ::= V\ ''z'')\\ \{''x''\}\ =$$

$$
\begin{array}{lcl}
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$$L\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =\ vars\ b \cup L\ c_1\ X \cup L\ c_2\ X$$

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Example:

$$L\ (WHILE\ Less\ (V\ ''x'')\ (V\ ''x'')\ DO\ ''y'' ::= V\ ''z'')$$
$$\{''x''\}\ =\ \{''x'',''z''\}$$

Gen/kill analyses

A data-flow analysis $A :: com \Rightarrow \tau\ set \Rightarrow \tau\ set$
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Many standard data-flow analyses are gen/kill.
In particular liveness analysis.

Liveness via gen/kill

$kill :: com \Rightarrow vname\ set$

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$kill\ SKIP \qquad\qquad\qquad\qquad =$

Liveness via gen/kill

$$
\begin{array}{lcl}
\textcolor{blue}{kill :: com \Rightarrow vname\ set} & & \\
kill\ SKIP & = & \{\}
\end{array}
$$

Liveness via gen/kill

$$
\begin{array}{lcl}
\textit{kill} :: \textit{com} \Rightarrow \textit{vname set} & & \\
\textit{kill SKIP} & = & \{\} \\
\textit{kill}\ (x ::= a) & = &
\end{array}
$$

Liveness via gen/kill

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$$\begin{array}{lcl} kill\ SKIP & = & \{\} \\ kill\ (x ::= a) & = & \{x\} \end{array}$$

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\end{array}
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\textit{kill}\ (\textit{IF}\ b\ \textit{THEN}\ c_1\ \textit{ELSE}\ c_2) & = & \textit{kill}\ c_1 \cap \textit{kill}\ c_2
\end{array}
$$

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\end{array}
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kill\ (WHILE\ b\ DO\ c) & = & \{\}
\end{array}$$

$gen :: com \Rightarrow vname\ set$

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$$gen\ SKIP \qquad =$$

$$gen :: com \Rightarrow vname\ set$$

$$gen\ SKIP \quad = \quad \{\}$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = &
\end{array}
$$

$gen :: com \Rightarrow vname\ set$

$$\begin{array}{lcl} gen\ SKIP & = & \{\} \\ gen\ (x ::= a) & = & vars\ a \end{array}$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = &
\end{array}
$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1)
\end{array}
$$

$gen :: com \Rightarrow vname\ set$

$$
\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1) \\
\multicolumn{3}{l}{gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ =}
\end{array}
$$

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gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1) \\
\multicolumn{3}{l}{gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ =} \\
\multicolumn{3}{l}{\quad vars\ b \cup gen\ c_1 \cup gen\ c_2}
\end{array}$$

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gen\ SKIP & = & \{\} \\
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$$gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2) =$$

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$$gen :: com \Rightarrow vname\ set$$

$$\begin{array}{lcl}
gen\ SKIP & = & \{\} \\
gen\ (x ::= a) & = & vars\ a \\
gen\ (c_1;;\ c_2) & = & gen\ c_1 \cup (gen\ c_2 - kill\ c_1)
\end{array}$$

$$gen\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ =$$
$$\quad vars\ b \cup gen\ c_1 \cup gen\ c_2$$

$$gen\ (WHILE\ b\ DO\ c)\ =\ vars\ b \cup gen\ c$$

$$L\ c\ X = gen\ c \cup (X - kill\ c)$$

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Proof by induction on c.

$$L\ c\ X = gen\ c \cup (X - kill\ c)$$

Proof by induction on c.

$\Longrightarrow$

$$L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Digression: definite initialization via gen/kill

$A\ c\ X$: the set of variables initialized after c
if X was initialized before c

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12 Live Variable Analysis

Correctness of L

Dead Variable Elimination

True Liveness

Comparisons

$(.,.) \Rightarrow .$ and L should roughly be related like this:

The value of the final state on X
only depends on
the value of the initial state on $L\ c\ X$.

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The value of the final state on X
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Put differently:

If two initial states agree on $L \; c \; X$
then the corresponding final states agree on X.

Equality on

An abbreviation:

$$f = g \ on \ X \quad \equiv \quad \forall\, x \in X.\ f\, x = g\ x$$

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Two easy theorems (in theory $Vars$):

$$s_1 = s_2\ on\ vars\ a \Longrightarrow aval\ a\ s_1 = aval\ a\ s_2$$

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$$s_1 = s_2 \ on \ vars \ a \Longrightarrow aval \ a \ s_1 = aval \ a \ s_2$$
$$s_1 = s_2 \ on \ vars \ b \Longrightarrow bval \ b \ s_1 = bval \ b \ s_2$$

Correctness of L

If $(c, s) \Rightarrow s'$ *and* $s = t \; on \; L \; c \; X$
then $\exists \, t'. \; (c, t) \Rightarrow t' \wedge s' = t' \; on \; X$.

Correctness of L

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction.

Correctness of L

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
then $\exists\, t'.\ (c,\ t) \Rightarrow t' \wedge s' = t'\ on\ X$.

Proof by rule induction.
For the two $WHILE$ cases we do not need the definition of $L\ w$ but only the characteristic property

$$vars\ b \cup X \cup L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

Optimality of L w

The result of L should be as small as possible: the more dead variables, the better

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Follows easily from $L\ c\ X = gen\ c \cup (X - kill\ c)$:

$$vars\ b \cup X \cup L\ c\ P \subseteq P \Longrightarrow$$
$$L\ (WHILE\ b\ DO\ c)\ X \subseteq P$$

12 Live Variable Analysis

- Correctness of L
- Dead Variable Elimination
- True Liveness
- Comparisons

Bury all assignments to dead variables:

$$bury :: com \Rightarrow vname\ set \Rightarrow com$$

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bury\ (c_1;;\ c_2)\ X & = & bury\ c_1\ (L\ c_2\ X);;\ bury\ c_2\ X
\end{array}
$$

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\end{array}
$$

$bury\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ X\ =$

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Correctness of $bury$

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$$bury\ c\ UNIV \sim c$$

where $UNIV$ is the set of all variables.

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The two directions need to be proved separately.

$$(c,\ s) \Rightarrow s' \Longrightarrow (bury\ c\ UNIV,\ s) \Rightarrow s'$$

$$(c,\ s) \Rightarrow s' \Longrightarrow (bury\ c\ UNIV,\ s) \Rightarrow s'$$

Follows from generalized statement:

If $(c,\ s) \Rightarrow s'$ *and* $s = t\ on\ L\ c\ X$
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Proof by rule induction, like for correctness of L.

$$(bury\ c\ UNIV,\ s) \Rightarrow s' \Longrightarrow (c,\ s) \Rightarrow s'$$

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Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command,

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then $\exists\, t'.\ (c,\ t) \Rightarrow t' \wedge s' = t'\ on\ X$.

Proof very similar to other direction, but needs inversion lemmas for $bury$ for every kind of command, e.g.

$$\begin{aligned}
&(bc_1;;\ bc_2 = bury\ c\ X) = \\
&(\exists\, c_1\ c_2. \\
&\quad c = c_1;;\ c_2 \wedge \\
&\quad bc_2 = bury\ c_2\ X \wedge bc_1 = bury\ c_1\ (L\ c_2\ X))
\end{aligned}$$

12 Live Variable Analysis

- Correctness of L
- Dead Variable Elimination
- True Liveness
- Comparisons

Terminology

Let $f :: \tau \Rightarrow \tau$ and $x :: \tau$.

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If $f\ x \leq x$ then x is a *pre-fixpoint* (*pfp*) of f.

If $x \leq y \Longrightarrow f\ x \leq f\ y$ for all x,y, then f is *monotone*.

Application to $L\ w$

Remember the specification of $L\ w$:

$$vars\ b \cup X \cup L\ c\ (L\ w\ X) \subseteq L\ w\ X$$

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Application to $L\ w$

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This is the same as saying that $L\ w\ X$ should be a pfp of

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and in particular of $L\ c$.

True liveness

$$L\ (''x'' ::= V\ ''y'')\ \{\} = \{''y''\}$$

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$L\ (x ::= a)\ X =$
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$L\ (x ::= a)\ X =$
$(\textit{if}\ x \in X\ \textit{then}\ vars\ a \cup (X - \{x\})\ \textit{else}\ X)$

But then

$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

is not correct anymore.

$$L\ (x ::= a)\ X =$$
$$(\textsf{\textit{if}}\ x \in X\ \textsf{\textit{then}}\ vars\ a \cup (X - \{x\})\ \textsf{\textit{else}}\ X)$$

$$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$$

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$L\ (WHILE\ b\ DO\ c)\ X = vars\ b \cup X \cup L\ c\ X$

Let $w = WHILE\ b\ DO\ c$
where $b\ = \ Less\ (N\ 0)\ (V\ y)$
and $c\ = \ y ::= V\ x;;\ x ::= V\ z$
and $distinct\ [x,\ y,\ z]$

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$$y ::= V\ x \qquad x ::= V\ z\ \ \{y\}$$

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$$y ::= V\ x\ \ \{y\}\ \ x ::= V\ z\ \ \{y\}$$

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Then $L\ w\ \{y\} = \{x,\ y\}$, but z is live before w !

$\{x\}\ \ y ::= V\ x\ \ \{y\}\ \ x ::= V\ z\ \ \{y\}$

$\Longrightarrow\ L\ w\ \{y\} = \{y\} \cup \{y\} \cup \{x\}$

$$\begin{aligned} b &= \mathit{Less}\ (N\ 0)\ (V\ y) \\ c &= y ::= V\ x;;\ x ::= V\ z \end{aligned}$$

$L\ w\ \{y\} = \{x,\ y\}$ is not a pfp of $L\ c$:

$$b = Less\ (N\ 0)\ (V\ y)$$
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$L\ w$ for true liveness

Define $L\ w\ X$ *as the least pfp of*
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Existence of least fixpoints

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is the least pre-fixpoint and least fixpoint of f.

Proof of Knaster-Tarski

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$L\ (x ::= a)\ X =$
$(\textit{if}\ x \in X\ \textit{then}\ vars\ a \cup (X - \{x\})\ \textit{else}\ X)$

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Corollary f_w is monotone.

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Proof Show $f^i(\{\}) \subseteq p$ for any pfp p of f (by induction on i).

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More precisely: $k \leq |\mathit{vars}\ c| + 1$

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Example

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From the library theory `While_Combinator`:

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where $g_w\ P = vars\ b \cup X \cup Lb\ c\ P$

Theorem $L\ c\ X \subseteq Lb\ c\ X$

Proof by induction on c. In the $WHILE$ case:

If $Lb\ w\ X = g_w^i\ \{\}$: $\forall P.\ L\ c\ P \subseteq Lb\ c\ P$ (IH) $\Longrightarrow$
$\forall P.\ f_w\ P \subseteq g_w\ P \Longrightarrow f_w(g_w^i\ \{\}) = g_w\ (g_w^i\ \{\}) = g_w^i\ \{\}$
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Limiting the number of iterations

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If $Lb\ w\ X = V_w$: $L\ w\ X \subseteq V_w$ (by Lemma)

12 Live Variable Analysis

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  - live variables *may* be used (on some program path) before they are assigned.

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