Concrete Semantics
with Isabelle/HOL

Peter Lammich

(slides from Concrete Semantics by Nipkow)

2018-10-16

Chapter 1

Introduction

1. Background

2. This Course

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2. This Course

Organization Issues

Course Homepage: `http://www21.in.tum.de/teaching/semantik/WS1819/`

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Homework: IMPORTANT! 40% of final grade

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Tutorials and Homework are the heart and soul of this course!

Why Semantics?

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Like the state of mathematics in the 19th century
— before set theory and logic entered the scene.

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- This course is about “beyond intuition”.

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What does the correctness of a type checker even mean?

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Example:
What does the correctness of a type checker even mean? How is it proved?

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- A compiler gives each individual program a semantics.
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- Because compilers are far too complicated.
- They provide the worst possible semantics.
- Moreover: compilers may differ!

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- Few languages have a (separate, abstract) semantics.
- If they do, it will be informal (English).

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  GI Dissertationspreis 2003:
  Gerwin Klein: *Verified Java Bytecode Verification*

Standard ML (SML)

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Main achievements: LCF (theorem proving)
SML (functional programming)
CCS, pi (concurrency)

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- too much detail to allow reliable informal proof
- not processable beyond LaTeX, not even executable

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- Real programming languages *are* complex.
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- Complex designs are error-prone.
- Informal mathematical proofs of complex designs are also error-prone.

The solution

Machine-checked language semantics and proofs

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The tool:

Proof Assistant (PA)
or
Interactive Theorem Prover (ITP)

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Time consuming
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Undermines your naive trust in informal proofs

Terminology

This lecture course:

Formal = machine-checked
Verification = formal correctness proof

Terminology

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$$\text{Formal} = \text{machine-checked}$$
$$\text{Verification} = \text{formal correctness proof}$$

Traditionally:

$$\text{Formal} = \text{mathematical}$$

Two landmark verifications

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microkernel (L4)

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Gerwin Klein (& Co)
NICTA Sydney
using Isabelle

A happy fact of life

Programming language researchers
are increasingly using PAs

Why verification pays off

Short term: *The software works!*

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Software Never Dies

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2. This Course

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- Comparison of PLs or PL paradigms

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- Hot or trendy PLs
- Comparison of PLs or PL paradigms
- Compilers (although they will be one application)

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  - PL tools
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Both informally and formally (PA!)

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All exercises require the use of Isabelle/HOL

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- It is the future
- It is the only way to deal with complex languages *reliably*
- I want students to learn how to write correct proofs
- I have seen too many proofs that look more like LSD trips than coherent mathematical arguments

Overview of course

- Introduction to Isabelle/HOL

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- IMP (assignment and while loops) and its semantics

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- Introduction to Isabelle/HOL
- IMP (assignment and while loops) and its semantics
- A compiler for IMP
- Hoare logic for IMP
- Type systems for IMP
- Program analysis for IMP

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A growing number of universities offer related course

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It is a new approach to informatics

At the end of the course ...

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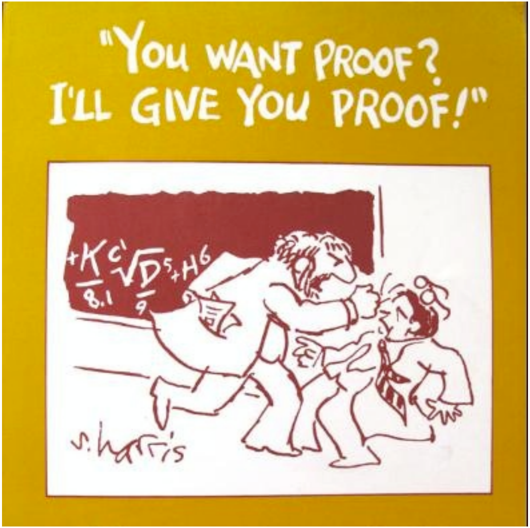

Part I

Isabelle

Chapter 2

Programming and Proving

3. Overview of Isabelle/HOL

4. Type and function definitions

5. Induction Heuristics

6. Simplification

Quiz

Which of the following formulas have the same meaning?

1. $A \Longrightarrow (B \Longrightarrow C)$
2. $(A \Longrightarrow B) \Longrightarrow C$
3. $(A \wedge B) \Longrightarrow C$

Notation

Implication associates to the right:

$$A \Longrightarrow B \Longrightarrow C \quad \text{means} \quad A \Longrightarrow (B \Longrightarrow C)$$

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$$\frac{A_1 \quad \dots \quad A_n}{B} \quad \text{means} \quad A_1 \Longrightarrow \cdots \Longrightarrow A_n \Longrightarrow B$$

3 Overview of Isabelle/HOL

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HOL Formulas:

- For the moment: only $term = term$,
  e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, ...

3 Overview of Isabelle/HOL

- Types and terms
- Interface
- By example: types $bool$, nat and $list$
- Summary

Types

Basic syntax:

$$\tau \quad ::=$$

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Convention: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \equiv \tau_1 \Rightarrow (\tau_2 \Rightarrow \tau_3)$

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	$\mid$	$t\ t$	function application
	$\mid$	$\lambda x.\ t$	function abstraction
	$\mid$	$\ldots$	lots of syntactic sugar

Examples: $f\ (g\ x)\ y$
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Convention: $f\ t_1\ t_2\ t_3\ \equiv\ ((f\ t_1)\ t_2)\ t_3$

This language of terms is known as the λ-*calculus*.

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where $t[u/x]$ is "t with u substituted for x".

Example: $(\lambda x.\ x + 5)\ 3\ =\ 3 + 5$

- The step from $(\lambda x.\ t)\ u$ to $t[u/x]$ is called *β-reduction*.

The computation rule of the λ-calculus is the replacement of formal by actual parameters:

$$(\lambda x.\ t)\ u \ = \ t[u/x]$$

where $t[u/x]$ is "t with u substituted for x".

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- Isabelle performs β-reduction automatically.

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$t\ ::\ \tau$ means "t is a well-typed term of type τ".

$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

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User can help with *type annotations* inside the term.
Example: $f\ (x{::}nat)$

Currying

Thou shalt Curry your functions

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

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- Curried: $f :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: $f' :: \tau_1 \times \tau_2 \Rightarrow \tau$

Advantage:

Currying allows *partial application*
$f\ a_1$ where $a_1 :: \tau_1$

Predefined syntactic sugar

- *Infix:* $+$, $-$, $*$, $\#$, $@$, ...

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Prefix binds more strongly than infix:

! $f\ x + y \ \equiv \ (f\ x) + y \ \not\equiv \ f\ (x + y)$ **!**

Enclose *if* and *case* in parentheses:

! $($*if* _ *then* _ *else* _$)$ **!**

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Syntax:

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imports T_1 ... T_n
begin
(definitions, theorems, proofs, ...)*
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T_i: names of *imported* theories. Import transitive.

Usually: **imports** `Main`

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In `.thy` files:
Types, terms and formulas need to be inclosed in "

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3 Overview of Isabelle/HOL

- Types and terms
- Interface
- By example: types $bool$, nat and $list$
- Summary

isabelle jedit

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- Based on *jEdit* editor

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  when editing `.thy` files (like modern Java IDEs)

Overview_Demo.thy

3 Overview of Isabelle/HOL

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Interface

By example: types $bool$, nat and $list$

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datatype $bool \ = \ True \ | \ False$

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if-and-only-if: $=$

Type nat

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! Numbers and arithmetic operations are overloaded:
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You need type annotations: $1 :: nat$, $x + (y{::}nat)$

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! Numbers and arithmetic operations are overloaded:
0,1,2,... :: $'a$, $\quad + ::\ 'a \Rightarrow 'a \Rightarrow 'a$

You need type annotations: $1 :: nat$, $x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

An informal proof

Lemma $add\ m\ 0 = m$

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Proof by induction on m.

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  The proof is as follows:
  $$\begin{array}{lcll} add\ (Suc\ m)\ 0 & = & Suc\ (add\ m\ 0) & \text{by def. of } add \\ & = & Suc\ m & \text{by IH} \end{array}$$

Type $'a\ list$

Lists of elements of type $'a$

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Syntactic sugar:

- $[] = Nil$: empty list

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Syntactic sugar:

- $[] = Nil$: empty list
- $x \# xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1, \ldots, x_n] = x_1 \# \ldots\ x_n \# []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs, prove

- $P([])$ and
- for arbitrary but fixed x and xs,
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- for arbitrary but fixed x and xs,
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$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \Longrightarrow P(x\#xs)}{P(xs)}$$

`List_Demo.thy`

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$
Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.
- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs =$
  $app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.
  The proof is as follows:
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs$
  $= Cons\ x\ (app\ (app\ xs\ ys)\ zs)$ by definition of app
  $= Cons\ x\ (app\ xs\ (app\ ys\ zs))$ by IH
  $= app\ (Cons\ x\ xs)\ (app\ ys\ zs)$ by definition of app

Large library: `HOL/List.thy`

Included in `Main`.

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Large library: `HOL/List.thy`

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Don't reinvent, reuse!

Predefined: $xs \ @ \ ys$ (append), $length$, and map

3 Overview of Isabelle/HOL

Types and terms

Interface

By example: types $bool$, nat and $list$

Summary

- **datatype** defines (possibly) recursive data types.
- **fun** defines (possibly) recursive functions by pattern-matching over datatype constructors.

Proof methods

- $induction$ performs structural induction on some variable (if the type of the variable is a datatype).

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- $auto$ solves as many subgoals as it can, mainly by simplification (symbolic evaluation):

  "$=$" is used only from left to right!

Proofs

General schema:

lemma $name$: "..."
apply (...)
apply (...)
⋮
done

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lemma $name$: `"..."`
apply `(...)`
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If the lemma is suitable as a simplification rule:

lemma $name$`[simp]:   "..."`

Top down proofs

Command

sorry

"completes" any proof.

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Allows top down development:

Assume lemma first, prove it later.

The proof state

1. $\bigwedge x_1 \ \dots \ x_p. \ \ A \Longrightarrow B$

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A	local assumption(s)
B	actual (sub)goal

Multiple assumptions

$$[\![\; A_1; \ldots \;; A_n \;]\!] \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$; \quad \approx \quad$ "and"

4 Type and function definitions

Type definitions

Function definitions

Type synonyms

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Introduces a *synonym* $name$ for type τ

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Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$
\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)t \quad = \quad C_1\ \tau_{1,1} \ldots \tau_{1,n_1} \\
\mid \quad \ldots \\
\mid \quad C_k\ \tau_{k,1} \ldots \tau_{k,n_k}
$$

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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1,\ldots,\alpha_n)t$

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Distinctness and injectivity are applied automatically
Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with *case*:

$$(\textit{case}\ xs\ \textit{of}\ \ [] \Rightarrow \ldots \quad | \quad y\#ys \Rightarrow \ldots\ y\ \ldots\ ys\ \ldots)$$

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Need ( ) in context

Tree_Demo.thy

The *option* type

datatype $'a\ option = None \mid Some\ 'a$

The $option$ type

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If $'a$ has values a_1, a_2, ...
then $'a\ option$ has values $None$, $Some\ a_1$, $Some\ a_2$, ...

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fun $lookup :: ('a \times\ 'b)\ list \Rightarrow\ 'a \Rightarrow\ 'b\ option$ **where**

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$\quad (\textit{if}\ a = x\ \textit{then}\ Some\ b\ \textit{else}\ lookup\ ps\ x)$

4. Type and function definitions

    Type definitions

    Function definitions

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

No pattern matching, just $f\,x_1\ \ldots\ x_n\ =\ \ldots$

The danger of nontermination

How about $f\,x = f\,x + 1$?

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Subtract $f\,x$ on both sides.
$\Longrightarrow 0 = 1$

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$\Longrightarrow 0 = 1$

! All functions in HOL must be total **!**

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- Pattern-matching over datatype constructors

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- Pattern-matching over datatype constructors
- Order of equations matters
- Termination must be provable automatically by size measures
- Proves customized induction schema

Example: separation

$$
\begin{array}{l}
\textbf{fun } sep :: {}'a \Rightarrow {}'a\ list \Rightarrow {}'a\ list \textbf{ where} \\
sep\ a\ (x \# y \# zs) = x \ \#\ a\ \#\ sep\ a\ (y \# zs) \quad | \\
sep\ a\ xs = xs
\end{array}
$$

Example: Ackermann

$$
\begin{aligned}
&\textbf{fun } ack :: nat \Rightarrow nat \Rightarrow nat \textbf{ where}\\
&ack\ 0\ n = Suc\ n\ \mid\\
&ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0)\ \mid\\
&ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)
\end{aligned}
$$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$$
\begin{array}{lll}
ack\ 0 & n & = Suc\ n\ \ | \\
ack\ (Suc\ m)\ 0 & & = ack\ m\ (Suc\ 0)\ \ | \\
ack\ (Suc\ m)\ (Suc\ n) & & = ack\ m\ (ack\ (Suc\ m)\ n)
\end{array}
$$

Terminates because the arguments decrease *lexicographically* with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

primrec

- A restrictive version of **fun**

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$$\begin{array}{lll} f(0) & = \dots & \text{no recursion} \\ f(Suc\ n) & = \dots f(n) \dots & \end{array}$$

$$\begin{array}{lll} g([]) & = \dots & \text{no recursion} \\ g(x\#xs) & = \dots g(xs) \dots & \end{array}$$

Basic induction heuristics

Theorems about recursive functions
are proved by induction

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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {'a}\ list \Rightarrow {'a}\ list$ **where**

$$
\begin{aligned}
&rev\ [] &&= [] \mid \\
&rev\ (x\#xs) &&= rev\ xs\ @\ [x]
\end{aligned}
$$

A tail recursive reverse

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lemma $itrev\ xs\ [] = rev\ xs$

`Induction_Demo.thy`

Generalisation

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- Replace constants by variables

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- Replace constants by variables
- Generalize free variables
    - by $arbitrary$ in induction proof
    - (or by universal quantifier in formula)

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In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction

Example

fun $div2 :: nat \Rightarrow nat$ **where**

$div2\ 0 = 0 \ \ |$

$div2\ (Suc\ 0) = 0 \ \ |$

$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

Computation Induction

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$\leadsto$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \qquad\qquad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

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Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

If $f :: \tau_1 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \tau'$:

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal
- ideally the a_i should be variables.

`Induction_Demo.thy`

Computation Induction

3. Overview of Isabelle/HOL

4. Type and function definitions

5. Induction Heuristics

6. Simplification

Simplification means . . .

Using equations $l = r$ from left to right

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As long as possible

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Terminology: equation $\rightsquigarrow$ *simplification rule*

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Simplification $=$ (Term) Rewriting

An example

Equations:

$$
\begin{aligned}
0 + n &= n && (1)\\
(Suc\ m) + n &= Suc\ (m + n) && (2)\\
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(0 \leq m) &= True && (4)
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$$0 + Suc\ 0 \;\leq\; Suc\ 0 + x$$

Rewriting:

An example

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$$

$$
\begin{array}{rcll}
0 + Suc\ 0 & \leq & Suc\ 0 + x & \stackrel{(1)}{=} \\
Suc\ 0 & \leq & Suc\ 0 + x &
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Conditional rewriting

Simplification rules can be conditional:

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Example

$$\begin{array}{rcl} p(0) & = & True \\ p(x) \Longrightarrow\ f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

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$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method *simp*

Goal: 1. $[\![\ P_1; \ldots; P_m\ ]\!] \Longrightarrow C$

apply$(simp\ add{:}\ eq_1\ \ldots\ eq_n)$

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- rules from **fun** and **datatype**

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Variations:

- $(simp\ \ldots\ del{:}\ \ldots)$ removes $simp$-lemmas
- add and del are optional

auto versus *simp*

- *auto* acts on all subgoals
- *simp* acts only on subgoal 1

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- *auto* applies *simp* and more
- *auto* can also be modified:
  $(auto\ simp\ add\!:\ \ldots\ \ simp\ del\!:\ \ldots)$

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add\colon f\_def \ldots)$$

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f is the function whose definition is to be unfolded.

Case splitting with $simp/auto$

Automatic:

$$P\ (\textit{if}\ A\ \textit{then}\ s\ \textit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

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By hand:

$$P\ (\textsf{\textit{case}}\ e\ \textsf{\textit{of}}\ 0 \Rightarrow a \mid Suc\ n \Rightarrow b)$$
$$=$$
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Proof method: $(simp\ split\text{:}\ nat.split)$

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Or $auto$. Similar for any datatype t: $t.split$

Simp_Demo.thy

Chapter 3

Case Study: IMP Expressions

7 Case Study: IMP Expressions

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This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

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of our imperative language IMP.

IMP *commands* are introduced later.

7 Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Stack Machine and Compilation

Concrete and abstract syntax

Concrete syntax: strings, eg `"a+5*b"`

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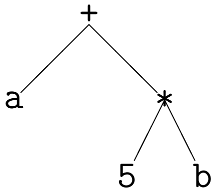
Abstract syntax: trees, eg



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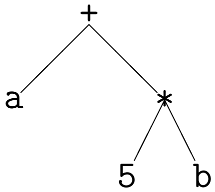

Parser: function from strings to trees

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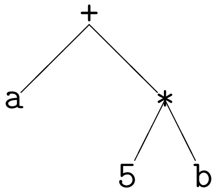

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Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

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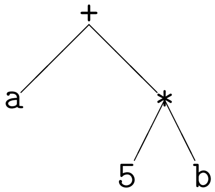

Parser: function from strings to trees

Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

where n can be any natural number and x any variable.

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

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x	$V\ ''x''$

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`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$

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Concrete	Abstract
5	$N\ 5$
`x`	$V\ ''x''$
`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$
`2+(z+3)`	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

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This is syntax, not (yet) semantics!

$$N\,0 \;\neq\; Plus\;(N\,0)\;(N\,0)$$

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type_synonym $val = int$
type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

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$$f(a := b) = (\lambda x.\ \mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ f\ x)$$

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Nicer notation:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

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Nicer notation:

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Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

7 Case Study: IMP Expressions

BExp.thy

7 Case Study: IMP Expressions

ASM.thy

This was easy.

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Because evaluation of expressions always terminates.

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Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 4

Logic and Proof Beyond Equality

Syntax (in decreasing precedence):

$$
\begin{array}{lclclcl}
\textit{form} & ::= & (\textit{form}) & \mid & \textit{term} = \textit{term} & \mid & \neg\textit{form} \\
& \mid & \textit{form} \wedge \textit{form} & \mid & \textit{form} \vee \textit{form} & \mid & \textit{form} \longrightarrow \textit{form} \\
& \mid & \forall x.\ \textit{form} & \mid & \exists x.\ \textit{form} & &
\end{array}
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$$\neg\ A\ \wedge\ B \vee\ C\ \ \equiv\ \ ((\neg\ A)\ \wedge\ B)\ \vee\ C$$

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Examples:

$$\begin{array}{rcl} \neg\ A \wedge\ B \vee\ C & \equiv & ((\neg\ A) \wedge\ B) \vee\ C \\ s =\ t \wedge\ C & \equiv & (s =\ t) \wedge\ C \end{array}$$

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s = t \wedge C & \equiv & (s = t) \wedge C \\
A \wedge B = B \wedge A & \equiv & \color{red}{A \wedge (B = B) \wedge A} \\
\forall\, x.\ P\ x \wedge Q\ x & \equiv & \forall\, x.\ (P\ x \wedge Q\ x)
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$$

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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y \ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

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Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x.\ Q\ x \rightsquigarrow P \wedge (\forall x.\ Q\ x) \quad !$$

Mathematical symbols

. . . and their ascii representations:

<table>
<tr><td>$\forall$</td><td>\<forall></td><td>ALL</td></tr>
<tr><td>$\exists$</td><td>\<exists></td><td>EX</td></tr>
<tr><td>λ</td><td>\<lambda></td><td>%</td></tr>
<tr><td>$\longrightarrow$</td><td>--></td><td></td></tr>
<tr><td>$\longleftrightarrow$</td><td><-></td><td></td></tr>
<tr><td>$\wedge$</td><td>/\</td><td>&</td></tr>
<tr><td>$\vee$</td><td>\/</td><td>|</td></tr>
<tr><td>$\neg$</td><td>\<not></td><td>~</td></tr>
<tr><td>$\neq$</td><td>\<noteq></td><td>~=</td></tr>
</table>

Sets over type $'a$

$'a \ set$

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- $\{\}$, $\{e_1, \ldots, e_n\}$

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- $\ldots$

$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

Set comprehension

- $\{x.\ P\}$ where x is a variable

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Set comprehension

- $\{x.\ P\}$ where x is a variable
- But not $\{t.\ P\}$ where t is a proper term
- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

$simp$ and $auto$

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- Show you where they got stuck

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Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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- rewriting, logic, sets, relations and a bit of arithmetic.
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- A complete proof search procedure for FOL . . .

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Sledgehammer

Architecture:

Isabelle

external
ATPs[1]

[1] Automatic Theorem Provers

Architecture:

Isabelle

Goal
& filtered library $\downarrow$

external
ATPs[1]

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Architecture:

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Goal & filtered library $\downarrow$ $\uparrow$ Proof

external **ATPs**[1]

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Architecture:

$$\begin{array}{ccc} & \textbf{Isabelle} & \\ \begin{array}{c}\text{Goal}\\ \text{\& filtered library}\end{array} & \downarrow\ \uparrow & \color{blue}{\text{Proof}} \\ & \begin{array}{c}\text{external}\\ \textbf{ATPs}^{1}\end{array} & \end{array}$$

Characteristics:

- Sometimes it works,

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Do you feel lucky?

[1] Automatic Theorem Provers

by$(\textit{proof-method})$

$$\approx$$

apply$(\textit{proof-method})$
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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  sets $?P$ to $a{=}b$ and $?Q$ to $False$.

Rule application

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Example: rule: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$

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Rule application

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"Backchaining"

Typical backwards rules

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They are known as introduction rules
because they *introduce* a particular connective.

Automating intro rules

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Also works for *auto* and *fastforce*

Can greatly increase the search space!

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```
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$$\leadsto$$
$$?Q \Longrightarrow a = a \land ?Q$$

The general case:

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If r is a theorem $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$
and $r_1, \ldots, r_m$ ($m \le n$) are theorems then

$$r[OF\ r_1\ \ldots\ r_m]$$

is the theorem obtained
by proving $A_1 \ldots A_m$ with $r_1 \ldots r_m$.

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```
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```

$$\rightsquigarrow$$

$$a = a \wedge b = b$$

From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

$\Longrightarrow$ versus $\longrightarrow$

$\Longrightarrow$ is part of the Isabelle framework. It structures theorems and proof states: $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$

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$\longrightarrow$ is part of HOL and can occur inside the logical formulas A_i and A.

Phrase theorems like this $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$
not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

Example: even numbers

Informally:

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Informally:

- 0 is even

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- 0 is even
- If n is even, so is $n + 2$

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- 0 is even
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In Isabelle/HOL:

inductive $ev :: nat \Rightarrow bool$
where
$\quad ev\ 0 \quad |$
$\quad ev\ n \Longrightarrow ev\ (n + 2)$

An easy proof: $ev\ 4$

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

fun $evn :: nat \Rightarrow bool$ **where**
$evn\ 0 = True\ |$
$evn\ (Suc\ 0) = False\ |$
$evn\ (Suc\ (Suc\ n)) = evn\ n$

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Two cases: $ev\ m$ is proved by

- rule $ev\ 0$
  $\Longrightarrow m = 0 \Longrightarrow evn\ m = True$
- rule $ev\ n \Longrightarrow ev\ (n{+}2)$
  $\Longrightarrow m = n{+}2$ and $evn\ n$ (IH)
  $\Longrightarrow evn\ m = evn\ (n{+}2) = evn\ n = True$

Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

Rule induction for ev

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- $P\ 0$
- $P\ n \Longrightarrow P(n{+}2)$

Rule induction for ev

To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

- $P\ 0$
- $P\ n \Longrightarrow P(n+2)$

Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n+2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

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Note:

- I may have multiple arguments.

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

$\llbracket\ I\ a_1; \ \ldots\ ;\ I\ a_n\ \rrbracket \Longrightarrow I\ a \ \ |$

$\vdots$

Note:

- I may have multiple arguments.
- Each rule may also contain *side conditions* not involving I.

Rule induction in general

To prove

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by *rule induction* on $I\ x$

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that P is preserved:

Rule induction in general

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that P is preserved:

$$[\![\ I\ a_1;\ P\ a_1;\ \dots\ ;\ I\ a_n;\ P\ a_n\ ]\!] \Longrightarrow P\ a$$

! Rule induction is absolutely central to (operational) semantics and the rest of this lecture course !

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**

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Difference to **inductive**:

- arguments of I are tupled, not curried

Inductively defined sets

inductive_set $I :: \tau \ set$ **where**

$[\![\ a_1 \in I; \ \ldots \ ; a_n \in I \ ]\!] \Longrightarrow a \in I \ \ |$

$\vdots$

Difference to **inductive**:

- arguments of I are tupled, not curried
- I can later be used with set theoretic operators, eg $I \cup \ldots$

Chapter 5

Isar: A Language for Structured Proofs

Apply scripts

- unreadable

Apply scripts

- unreadable
- hard to maintain

Apply scripts

- unreadable
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- do not scale

Apply scripts

- unreadable
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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

A typical Isar proof

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  assume $formula_0$
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  show $formula_{n+1}$ **by** $\ldots$
qed

proves $formula_0 \Longrightarrow formula_{n+1}$

Isar core syntax

$$
\begin{array}{lcl}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\
 & \mid & \textbf{by}\ \text{method}
\end{array}
$$

Isar core syntax

proof = **proof** [method] step* **qed**
  | **by** method

method = $(\mathit{simp}\ \ldots) \mid (\mathit{blast}\ \ldots) \mid (\mathit{induction}\ \ldots) \mid \ldots$

Isar core syntax

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\text{method} \ = \ (\mathit{simp}\ \ldots) \mid (\mathit{blast}\ \ldots) \mid (\mathit{induction}\ \ldots) \mid \ldots
$$

$$
\begin{array}{lcll}
\text{step} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
 & | & \textbf{assume}\ \text{prop} & (\Longrightarrow) \\
 & | & [\textbf{from}\ \text{fact}^+]\ \ (\textbf{have} \mid \textbf{show})\ \text{prop}\ \ \text{proof} &
\end{array}
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Isar core syntax

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\end{array}
$$

$$
\text{prop} \;=\; [\text{name:}]\ \text{"formula"}
$$

Isar core syntax

proof	=	**proof** [method] step* **qed**
	$\mid$	**by** method
method	=	$(simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$
step	=	**fix** variables $(\bigwedge)$
	$\mid$	**assume** prop $(\Longrightarrow)$
	$\mid$	[**from** fact$^+$] (**have** $\mid$ **show**) prop proof
prop	=	[name:] "formula"
fact	=	name $\mid \ldots$

12 Isar by example

13 Proof patterns

14 Streamlining Proofs

15 Proof by Cases and Induction

Example: Cantor's theorem

lemma $\neg\ surj(f :: {}'a \Rightarrow {}'a\ set)$

Example: Cantor's theorem

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assume a: $surj\ f$

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by $blast$

Example: Cantor's theorem

lemma $\neg$ $surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume $surj$, show $False$
assume a: $surj\ f$
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    by $blast$
  from c **show** $False$
    by $blast$
qed

Isar_Demo.thy

Cantor and abbreviations

Abbreviations

$this$	=	the previous proposition proved or assumed
then	=	**from** $this$
thus	=	**then show**
hence	=	**then have**

using and **with**

(**have**|**show**) prop **using** facts

using and **with**

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

using and **with**

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

with facts

=

from facts $this$

Structured lemma statement

lemma
fixes $f :: 'a \Rightarrow 'a\ set$
assumes s: $surj\ f$
shows $False$

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Proves $surj\ f \Longrightarrow False$

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  thus $False$ **by** $blast$
qed

Proves $surj\ f \Longrightarrow False$
but $surj\ f$ *becomes local fact* s *in proof.*

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \ \ldots$
assumes *a:* P **and** *b:* $Q \ \ldots$
shows R

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- **fixes** and **assumes** sections optional

Structured lemma statements

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assumes *a:* P **and** *b:* $Q \ldots$
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

Case distinction

show R
proof $cases$
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg$ P
  $\vdots$
  show R $\langle proof \rangle$
qed

Case distinction

show R
proof cases
  assume P
  $\vdots$
  show R $\langle \mathit{proof} \rangle$
next
  assume $\neg\ P$
  $\vdots$
  show R $\langle \mathit{proof} \rangle$
qed

have $P \lor Q$ $\langle \mathit{proof} \rangle$
then show R
proof
  assume P
  $\vdots$
  show R $\langle \mathit{proof} \rangle$
next
  assume Q
  $\vdots$
  show R $\langle \mathit{proof} \rangle$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False$ $\langle proof \rangle$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False\ \langle proof\rangle$
qed

show P
proof $(rule\ ccontr)$
  assume $\neg P$
  $\vdots$
  show $False\ \langle proof\rangle$
qed

$\longleftrightarrow$

show $P \longleftrightarrow Q$
proof
  assume P
  $\vdots$
  show Q $\langle proof \rangle$
next
  assume Q
  $\vdots$
  show P $\langle proof \rangle$
qed

$\forall$ and $\exists$ introduction

show $\forall\, x.\ P(x)$
proof
  fix x local fixed variable
  show $P(x)$ $\langle proof \rangle$
qed

$\forall$ and $\exists$ introduction

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  fix x local fixed variable
  show $P(x)$ $\langle proof \rangle$
qed

show $\exists\, x.\ P(x)$
proof
  $\vdots$
  show $P(witness)$ $\langle proof \rangle$
qed

$\exists$ elimination: **obtain**

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have $\exists\, x.\ P(x)$
then **obtain** x **where** *p:* $P(x)$ **by** *blast*
$\vdots$ $\quad x$ fixed local variable

$\exists$ elimination: **obtain**

have $\exists\, x.\ P(x)$
then obtain x **where** *p:* $P(x)$ **by** *blast*
$\vdots$ $\quad x$ fixed local variable

Works for one or more x

obtain example

lemma $\neg\ surj(f :: {'a} \Rightarrow {'a}\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\, x\} = f\ a$ **by**$(auto\ simp{:}\ surj\_def)$

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  then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$
  hence $a \notin f\ a \longleftrightarrow a \in f\ a$ **by** $blast$

obtain example

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\!: surj\_def)$
  then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$
  hence $a \notin f\ a \longleftrightarrow a \in f\ a$ **by** $blast$
  thus $False$ **by** $blast$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  $\vdots$
  show $x \in B$ $\langle proof \rangle$
qed

Isar_Demo.thy

Exercise

14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)
proof
  assume $?L$
  $\vdots$
  show $?R$ $\langle proof \rangle$
next
  assume $?R$
  $\vdots$
  show $?L$ $\langle proof \rangle$
qed

$?thesis$

show $formula$
proof -
  ⋮
  show $?thesis$ $\langle proof \rangle$
qed

$?thesis$

show $formula$ (is $?thesis$)
proof -
$\vdots$
show $?thesis$ $\langle proof \rangle$
qed

$?thesis$

show $formula$ (is $?thesis$)
proof -
$\vdots$
show $?thesis$ $\langle proof \rangle$
qed

Every show implicitly defines $?thesis$

let

Introducing local abbreviations in proofs:

let $?t$ = *"some-big-term"*

$\vdots$

have *"*$\ldots$ $?t$ $\ldots$*"*

Quoting facts by value

By name:

> **have** *x0*: $''x > 0''$ ...
>
> $\vdots$
>
> **from** *x0* ...

Quoting facts by value

By name:

have *x0:* $''x > 0''$ ...
$\vdots$
from *x0* ...

By value:

have $''x > 0''$ ...
$\vdots$
from `` `x>0` `` ...

Quoting facts by value

By name:

have *x0:* $''x > 0''$ $\ldots$
$\vdots$
from *x0* $\ldots$

By value:

have $''x > 0''$ $\ldots$
$\vdots$
from \`$x{>}0$\` $\ldots$

$\uparrow$ $\uparrow$
back quotes

Isar_Demo.thy

Pattern matching and quotations

14 Streamlining Proofs

- Pattern Matching and Quotations
- Top down proof development
- **moreover**
- Local lemmas

Example

lemma

$$\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge$$
$$(length\ ys = length\ zs \vee length\ ys = length\ zs + 1)$$

Example

lemma

$$\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge$$
$$(length\ ys = length\ zs \vee length\ ys = length\ zs + 1)$$

proof ???

`Isar_Demo.thy`

Top down proof development

When automation fails

Split proof up into smaller steps.

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Or explore by **apply**:

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Or explore by **apply**:

have . . . **using** . . .

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have ... **using** ...
apply -   to make incoming facts part of proof state

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When automation fails

Split proof up into smaller steps.

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At the end:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...
apply -   to make incoming facts part of proof state
apply *auto*   or whatever
apply ...

At the end:

- **done**

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...	
apply -	to make incoming facts part of proof state
apply $auto$	or whatever
apply ...	

At the end:

- **done**
- Better: convert to structured proof

14 Streamlining Proofs

- Pattern Matching and Quotations
- Top down proof development
- **moreover**
- Local lemmas

moreover—ultimately

have P_1 $\ldots$
moreover
have P_2 $\ldots$
moreover
$\vdots$
moreover
have P_n $\ldots$
ultimately
have P $\ldots$

moreover—ultimately

have P_1 $\ldots$ **moreover** **have** P_2 $\ldots$ **moreover** $\vdots$ **moreover** **have** P_n $\ldots$ **ultimately** **have** P $\ldots$	$\approx$	**have** lab_1: P_1 $\ldots$ **have** lab_2: P_2 $\ldots$ $\vdots$ **have** lab_n: P_n $\ldots$ **from** lab_1 lab_2 $\ldots$ **have** P $\ldots$

With names

14 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Local lemmas

have B **if** *name:* $A_1 \; \ldots \; A_m$ **for** $x_1 \; \ldots \; x_n$
$\langle proof \rangle$

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$\langle proof \rangle$

proves $\llbracket\ A_1; \ \ldots\ ;\ A_m\ \rrbracket \Longrightarrow B$

Local lemmas

have B **if** *name:* $A_1 \ \ldots \ A_m$ **for** $x_1 \ \ldots \ x_n$
$\langle proof \rangle$

proves $[\![\ A_1; \ \ldots \ ; \ A_m \ ]\!] \Longrightarrow B$
where all x_i have been replaced by $?x_i$.

Proof state and Isar text

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In general: **proof** *method*

Proof state and Isar text

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Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ldots x_n.\ \llbracket\ A_1; \ldots\ ; A_m\ \rrbracket \Longrightarrow B$$

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Proof state and Isar text

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$$\bigwedge x_1 \ \ldots \ x_n. \ \llbracket \ A_1; \ \ldots \ ; \ A_m \ \rrbracket \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ \ldots \ x_n$
assume $A_1 \ \ldots \ A_m$
$\vdots$
show B

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \dots \ x_n. \ [\![\ A_1; \ \dots \ ; \ A_m \ ]\!] \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ \dots \ x_n$
assume $A_1 \ \dots \ A_m$
$\vdots$
show B

Separated by **next**

12 Isar by example

13 Proof patterns

14 Streamlining Proofs

15 Proof by Cases and Induction

Isar_Induction_Demo.thy

Proof by cases

Datatype case analysis

$\textbf{datatype}\ t\ =\ C_1\ \vec{\tau}\ \mid\ \ldots$

Datatype case analysis

datatype $t \ = \ C_1 \ \vec{\tau} \ \mid \ \ldots$

proof $(cases$ "$term$"$)$
  case $(C_1 \ x_1 \ \ldots \ x_k)$
  $\ldots \ x_j \ \ldots$
next
$\vdots$
qed

Datatype case analysis

datatype $t = C_1\ \vec{\tau}\ \mid\ \ldots$

proof $(cases$ "$term$"$)$
  case $(C_1\ x_1\ \ldots\ x_k)$
  $\ldots\ x_j\ \ldots$
next
$\vdots$
qed

where **case** $(C_i\ x_1\ \ldots\ x_k)$ $\equiv$

fix $x_1\ \ldots\ x_k$
assume $\underbrace{C_i:}_{\text{label}}\ \underbrace{term = (C_i\ x_1\ \ldots\ x_k)}_{\text{formula}}$

Isar_Induction_Demo.thy

Structural induction for nat

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$
  $\vdots$
  show $?case$
qed

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **let** $?case = P(0)$
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$
  $\vdots$
  show $?case$
qed

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **let** $?case = P(0)$
  ⋮
  show $?case$
next
  case $(Suc\ n)$ $\equiv$ **fix** n **assume** Suc: $P(n)$
  ⋮ **let** $?case = P(Suc\ n)$
  ⋮
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$
  case 0
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **assume** $0\colon\ A(0)$
  $\vdots$ **let** $?case = P(0)$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$

case 0	$\equiv$	**assume** 0: $A(0)$
$\vdots$		**let** $?case = P(0)$
show $?case$		

next

case $(Suc\ n)$	$\equiv$	**fix** n
$\vdots$		**assume** Suc: $A(n) \Longrightarrow P(n)$ $A(Suc\ n)$
$\vdots$		**let** $?case = P(Suc\ n)$
show $?case$		

qed

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of

case C

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we have

$C.IH$ the induction hypotheses

Named assumptions

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In the context of

case C

we have

$C.IH$	the induction hypotheses
$C.prems$	the premises A_i

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:

In the context of

case C

we have

$C.IH$	the induction hypotheses
$C.prems$	the premises A_i
C	$C.IH + C.prems$

A remark on style

- **case** $(Suc\ n)$ ... **show** $?case$
  is easy to write and maintain

A remark on style

- **case** $(Suc\ n)$ ... **show** $?case$
  is easy to write and maintain
- **fix** n **assume** $formula$ ... **show** $formula'$
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. $?case$)

15 Proof by Cases and Induction

Rule Induction

Rule Inversion

`Isar_Induction_Demo.thy`

Rule induction

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: $\ldots$
$\vdots$
$rule_n$: $\ldots$

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: ...
$\vdots$
$rule_n$: ...

show $I\ x\ y \Longrightarrow P\ x\ y$

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: ...
$\vdots$
$rule_n$: ...

show $I\ x\ y \Longrightarrow P\ x\ y$
proof $(induction\ rule\colon I.induct)$

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: $\ldots$
$\vdots$
$rule_n$: $\ldots$

show $I\ x\ y \Longrightarrow P\ x\ y$
proof (*induction rule: I.induct*)
  case $rule_1$
  $\ldots$
  show *?case*
next
$\vdots$
next
  case $rule_n$
  $\ldots$
  show *?case*
qed

Fixing your own variable names

case $(rule_i\ x_1\ \ldots\ x_k)$

Renames the first k variables in $rule_i$ (from left to right) to $x_1\ \ldots\ x_k$.

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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In the context of

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Named assumptions

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$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

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we have

$R.IH$ the induction hypotheses

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i
R	$R.IH + R.hyps + R.prems$

15 Proof by Cases and Induction

Rule Induction

Rule Inversion

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
$evSS$: $ev\ n \Longrightarrow ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
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What can we deduce from $ev\ n$?
That it was proved by either $ev0$ or $evSS$!

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
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What can we deduce from $ev\ n$?
That it was proved by either $ev0$ or $evSS$!

$$ev\ n \Longrightarrow n = 0 \lor (\exists\, k.\ n = Suc\ (Suc\ k) \land ev\ k)$$

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
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What can we deduce from $ev\ n$?
That it was proved by either $ev0$ or $evSS$!

$$ev\ n \Longrightarrow n = 0 \lor (\exists\, k.\ n = Suc\ (Suc\ k) \land ev\ k)$$

Rule inversion = case distinction over rules

Isar_Induction_Demo.thy

Rule inversion

Rule inversion template

from $`ev\ n`$ **have** P
proof $cases$
  case $ev0$ $\qquad\qquad\qquad n = 0$
  $\vdots$
  show $?thesis$ $\ldots$
next
  case $(evSS\ k)$ $\qquad\qquad\qquad n = Suc\ (Suc\ k),\ ev\ k$
  $\vdots$
  show $?thesis$ $\ldots$
qed

Rule inversion template

from $`ev\ n`$ **have** P
proof $cases$
  case $ev0$ $\quad\quad\quad\quad n = 0$
  $\vdots$
  show $?thesis\ \ \ldots$
next
  case $(evSS\ k)$ $\quad\quad\quad\quad n = Suc\ (Suc\ k),\ ev\ k$
  $\vdots$
  show $?thesis\ \ \ldots$
qed

Impossible cases disappear automatically