Concrete Semantics

with Isabelle/HOL

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Part II

Semantics

Chapter 7

IMP: A Simple Imperative Language

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

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2. Big-Step Semantics
3. Small-Step Semantics

Terminology

Statement: declaration of fact or claim

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Study the book until you have understood it.

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Expressions are *evaluated*, commands are *executed*

Commands

Concrete syntax:

$$
\begin{array}{rcl}
\mathit{com} & ::= & \texttt{SKIP} \\
& | & \mathit{string}\ \texttt{::=}\ \mathit{aexp} \\
& | & \mathit{com}\ \texttt{;;}\ \mathit{com} \\
& | & \texttt{IF}\ \mathit{bexp}\ \texttt{THEN}\ \mathit{com}\ \texttt{ELSE}\ \mathit{com} \\
& | & \texttt{WHILE}\ \mathit{bexp}\ \texttt{DO}\ \mathit{com}
\end{array}
$$

Commands

Abstract syntax:

$$
\begin{array}{rcl}
\textbf{datatype}\ com & = & SKIP \\
& \mid & Assign\ string\ aexp \\
& \mid & Seq\ com\ com \\
& \mid & If\ bexp\ com\ com \\
& \mid & While\ bexp\ com
\end{array}
$$

Com.thy

1. IMP Commands

2. Big-Step Semantics

3. Small-Step Semantics

Big-step semantics

Concrete syntax:

$$(\mathit{com},\ \mathit{initial\text{-}state}) \Rightarrow \mathit{final\text{-}state}$$

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"$\Rightarrow$" here not type!

Big-step rules

$$(SKIP,\ s) \Rightarrow\ s$$

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$$(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}$$

Big-step rules

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}$$

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$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}$$

$$\frac{bval\ b\ s_1 \quad (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}$$

Examples: derivation trees

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$$\frac{\vdots}{(''x'' ::= N\,5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{array}{rcl} w & = & WHILE\ b\ DO\ c \\ b & = & NotEq\ (V\ ''x'')\ (N\,2) \\ c & = & ''x'' ::= Plus\ (V\ ''x'')\ (N\,1) \\ s_i & = & s(''x'' := i) \end{array}$$

Examples: derivation trees

$$\frac{\vdots}{(''x'' ::= N\ 5;;\ ''y'' ::= V\ ''x'',\ s) \Rightarrow\ ?} \qquad \frac{\vdots}{(w,\ s_i) \Rightarrow\ ?}$$

where
$$\begin{aligned} w &= \mathit{WHILE}\ b\ \mathit{DO}\ c \\ b &= \mathit{NotEq}\ (V\ ''x'')\ (N\ 2) \\ c &= ''x'' ::= \mathit{Plus}\ (V\ ''x'')\ (N\ 1) \\ s_i &= s(''x'' := i) \end{aligned}$$

$$\mathit{NotEq}\ a_1\ a_2 =$$
$$\mathit{Not}(\mathit{And}\ (\mathit{Not}(\mathit{Less}\ a_1\ a_2))\ (\mathit{Not}(\mathit{Less}\ a_2\ a_1)))$$

Logically speaking

$$(c,\ s) \Rightarrow\ t$$

is just infix syntax for

$$big\_step\ (c{,}s)\ \ t$$

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is just infix syntax for

$$big\_step\ (c{,}s)\ t$$

where

$$big\_step :: com \times state \Rightarrow state \Rightarrow bool$$

is an inductively defined predicate.

Big_Step.thy

Semantics

Rule inversion

What can we deduce from

- $(SKIP,\, s) \Rightarrow t$?

Rule inversion

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- $(w,\ s) \Rightarrow t$ where $w = WHILE\ b\ DO\ c$?
  $\neg\ bval\ b\ s \wedge t = s\ \vee$
  $bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$

Automating rule inversion

Isabelle command **inductive_cases** produces theorems that perform rule inversions automatically.

We reformulate the inverted rules. Example:

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}{\exists\, s_2.\ (c_1,\ s_1) \Rightarrow s_2 \wedge (c_2,\ s_2) \Rightarrow s_3}$$

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is logically equivalent to

$$\frac{(c_1;;\ c_2,\ s_1) \Rightarrow s_3 \qquad \bigwedge s_2.\ [\![(c_1,\ s_1) \Rightarrow s_2;\ (c_2,\ s_2) \Rightarrow s_3]\!] \Longrightarrow P}{P}$$

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Replaces assm $(c_1;;\ c_2,\ s_1) \Rightarrow s_3$ by two assms $(c_1,\ s_1) \Rightarrow s_2$ and $(c_2,\ s_2) \Rightarrow s_3$

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No $\exists$ and $\wedge$!

The general format: *elimination rules*

$$\frac{asm \quad asm_1 \Longrightarrow P \quad \ldots \quad asm_n \Longrightarrow P}{P}$$

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Reading:

To prove a goal P with assumption asm,
prove all $asm_i \Longrightarrow P$

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Reading:

> To prove a goal P with assumption asm,
> prove all $asm_i \Longrightarrow P$

Example:

$$\frac{F \ \vee \ G \quad F \ \Longrightarrow \ P \quad G \ \Longrightarrow \ P}{P}$$

elim attribute

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- Can also be added locally, eg *(blast elim: . . .)*
- Variant: *elim!* applies elim-rules eagerly.

Big_Step.thy

Rule inversion

Command equivalence

Two commands have the same input/output behaviour:

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Example

$$w \sim w'$$

where $w = \mathit{WHILE}\ b\ \mathit{DO}\ c$

$w' = \mathit{IF}\ b\ \mathit{THEN}\ c;;\ w\ \mathit{ELSE}\ \mathit{SKIP}$

Equivalence proof

$$(w,\ s) \Rightarrow\ t$$

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$$\longleftrightarrow$$

$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow\ s' \wedge (w,\ s') \Rightarrow\ t)$$
$$\vee$$
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Equivalence proof

$$(w,\ s) \Rightarrow t$$

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$$bval\ b\ s \wedge (\exists\, s'.\ (c,\ s) \Rightarrow s' \wedge (w,\ s') \Rightarrow t)$$
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$$(w',\ s) \Rightarrow t$$

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$$\neg\ bval\ b\ s \wedge t = s$$

$$\longleftrightarrow$$

$$(w',\ s) \Rightarrow t$$

Using the rules and rule inversions for $\Rightarrow$.

Big_Step.thy

Command equivalence

Execution is deterministic

Any two executions of the same command in the same start state lead to the same final state:

$$(c,\ s) \Rightarrow t \implies (c,\ s) \Rightarrow t' \implies t = t'$$

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Proof by rule induction, for arbitrary t'.

`Big_Step.thy`

Execution is deterministic

The boon and bane of big steps

We cannot observe intermediate states/steps

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Example problem:

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(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

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$$(c,s) \text{ does not terminate iff } \nexists\, t.\ (c,\ s) \Rightarrow\ t\ ?$$

Needs a formal notion of nontermination to prove it.

The boon and bane of big steps

We cannot observe intermediate states/steps

Example problem:

(c,s) does not terminate iff $\nexists\, t.\ (c,\ s) \Rightarrow\ t$?

Needs a formal notion of nontermination to prove it.
Could be wrong if we have forgotten a $\Rightarrow$ rule.

Big-step semantics cannot directly describe

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We need a finer grained semantics!

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Small-step semantics

Concrete syntax:

$$(com, state) \rightarrow (com, state)$$

Small-step semantics

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Intended meaning of $(c,\ s) \rightarrow (c',\ s')$:

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$$(com, state) \rightarrow (com, state)$$

Intended meaning of $(c,\ s) \rightarrow (c',\ s')$:

> *The first step in the execution of* c *in state* s
> *leaves a "remainder" command* c'
> *to be executed in state* s'*.*

Small-step semantics

Concrete syntax:

$$(com, state) \to (com, state)$$

Intended meaning of $(c,\ s) \to (c',\ s')$:

> *The first step in the execution of c in state s*
> *leaves a "remainder" command c'*
> *to be executed in state s'.*

Execution as finite or infinite reduction:

$$(c_1, s_1) \to (c_2, s_2) \to (c_3, s_3) \to \ldots$$

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- A pair (c,s) is called a *configuration*.
- If $cs \to cs'$ we say that cs *reduces* to cs'.
- A configuration cs is *final* iff $\nexists\, cs'.\ cs \to cs'$

The intention:

$$(SKIP,\ s) \text{ is final}$$

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Why?

$SKIP$ is the empty program.

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$$(SKIP,\ s) \text{ is final}$$

Why?

$SKIP$ is the empty program. Nothing more to be done.

Small-step rules

$$(x ::= a,\ s)\ \rightarrow$$

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$$\frac{(c_1, s) \ \rightarrow\ (c_1', s')}{(c_1;;c_2, s) \ \rightarrow}$$

Small-step rules

$$(x ::= a,\ s) \ \rightarrow\ (SKIP,\ s(x := \ aval\ a\ s))$$

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Small-step rules

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$$(WHILE\ b\ DO\ c,\ s) \ \rightarrow$$
$$(IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)$$

Fact $(SKIP, s)$ is a final configuration.

Small-step examples

$$(''z'' ::= \ V \ ''x'';; \ ''x'' ::= \ V \ ''y'';; \ ''y'' ::= \ V \ ''z'', \ s) \rightarrow$$

$$\ldots$$

where $s = <''x'' := 3, \ ''y'' := 7, \ ''z'' := 5>$.

Small-step examples

$$("z" ::= V\ "x";;\ "x" ::= V\ "y";;\ "y" ::= V\ "z",\ s) \rightarrow$$
$$\ldots$$

where $s = <"x" := 3,\ "y" := 7,\ "z" := 5>$.

$$(w,\ s_0) \rightarrow \ldots$$

where
$$\begin{aligned} w &= \ WHILE\ b\ DO\ c \\ b &= \ Less\ (V\ "x")\ (N\ 1) \\ c &= \ "x" ::= Plus\ (V\ "x")\ (N\ 1) \\ s_n &= \ <"x" := n> \end{aligned}$$

`Small_Step.thy`

Semantics

Are big and small-step semantics equivalent?

From $\Rightarrow$ to $\rightarrow *$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP,\ t)$

From $\Rightarrow$ to $\rightarrow*$

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Proof by rule induction

From $\Rightarrow$ to $\rightarrow*$

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Proof by rule induction (of course on $cs \Rightarrow t$)

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In two cases a lemma is needed:

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (\mathit{SKIP}, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

Lemma
$(c_1, s) \rightarrow* (c_1', s') \implies (c_1;; c_2, s) \rightarrow* (c_1';; c_2, s')$

From $\Rightarrow$ to $\rightarrow*$

Theorem $cs \Rightarrow t \implies cs \rightarrow* (SKIP, t)$

Proof by rule induction (of course on $cs \Rightarrow t$)
In two cases a lemma is needed:

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Proof by rule induction.

From $\rightarrow*$ to $\Rightarrow$

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Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.

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In the induction step a lemma is needed:

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Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

From $\rightarrow*$ to $\Rightarrow$

Theorem $cs \rightarrow* (SKIP, t) \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow* (SKIP, t)$.
In the induction step a lemma is needed:

Lemma $cs \rightarrow cs' \implies cs' \Rightarrow t \implies cs \Rightarrow t$

Proof by rule induction on $cs \rightarrow cs'$.

Equivalence

Corollary $cs \Rightarrow t \;\longleftrightarrow\; cs \rightarrow* (SKIP,\ t)$

`Small_Step.thy`

Equivalence of big and small

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That is, are there any final configs except $(SKIP, s)$?

Lemma $\mathit{final}\ (c,\ s) \Longrightarrow c = SKIP$

We prove the contrapositive

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by induction on c.

Can execution stop prematurely?

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- Remaining cases: trivial or easy

By rule inversion: $(SKIP,\ s) \rightarrow ct \Longrightarrow False$

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Together:

Corollary $\mathit{final}\ (c,\ s) = (c = SKIP)$

Infinite executions

$\Rightarrow$ yields final state iff $\rightarrow$ terminates

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(by final $=$ SKIP)

Equivalent:

$\Rightarrow$ does not yield final state iff $\rightarrow$ does not terminate

May versus Must

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Therefore: $\Rightarrow$ correctly reflects termination behaviour.

With nondeterminism: may have both $cs \Rightarrow t$ and a nonterminating reduction $cs \rightarrow cs' \rightarrow \ldots$

Chapter 8

Hoare Logic

4 Weakest Preconditions

Introduction

We have proved functional programs correct

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We have modeled semantics of imperative languages

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But how do we prove imperative programs correct?

An example program:

$$
\begin{array}{l}
\mathit{program}\ \mathit{exp}\ \{ \\
\quad a := 1 \\
\quad \mathit{while}\ (0{<}n)\ \mathit{do}\ \{ \\
\quad\quad a := a + a; \\
\quad\quad n := n - 1 \\
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At the end of the execution, variable a should contain 2^n, where n is the original value of variable n!
and $0 \leq n$!

In general: If *precondition* holds for initial state
then, program terminates, and
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Weakest condition on state, such that program c will satisfy postcondition Q.

Some obvious facts:

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wp of equivalent programs is equal

$$c \sim c' \Longrightarrow wp\ c = wp\ c'$$

Correctness of exp

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$wp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s$

$\quad =$

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Reasoning along syntax of program!

That was easy!

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$$wp\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s$$
$$=$$

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Unfolding will continue forever!

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Unfolding will continue forever!

Obviously, need some inductive argument!

But, let's get less ambitious (for first)

Weakest liberal precondition

$$wlp\ c\ Q\ s \equiv \forall\, t.\ (c,\ s) \Rightarrow\ t \longrightarrow\ Q\ t$$

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Cannot reason about termination. This is called *partial correctness*.

Some obvious facts:

$$c \sim c' \Longrightarrow wlp\ c = wlp\ c'$$
$$[\![wlp\ c\ P\ s;\ \bigwedge s.\ P\ s \Longrightarrow Q\ s]\!] \Longrightarrow wlp\ c\ Q\ s$$

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Relation between wp and wlp

$$wp\ c\ Q\ s \Longrightarrow wlp\ c\ Q\ s$$
$$wlp\ c\ Q\ s \wedge (c,\ s) \Rightarrow t \Longrightarrow wp\ c\ Q\ s$$

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Unfold rules still hold:

$$wlp\ SKIP\ Q\ s = Q\ s$$
$$wlp\ (x ::= a)\ Q\ s = Q\ (s(x := aval\ a\ s))$$
$$wlp\ (c_1;;\ c_2)\ Q\ s = wlp\ c_1\ (wlp\ c_2\ Q)\ s$$
$$wlp\ (IF\ b\ THEN\ c_1\ ELSE\ c_2)\ Q\ s =$$
$$(\textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c_1\ Q\ s\ \textsf{\textit{else}}\ wlp\ c_2\ Q\ s)$$

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Let's try to find predicate I, such that

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop.

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Intuition: I holds initially, is preserved by iteration, and implies Q at end of loop. I is called *loop invariant*

While-rule for partial correctness

$$[\![I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s]\!]$$
$$\Longrightarrow wlp\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

Wp_Demo.thy

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Otherwise, use unfold rules.

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If $c = WHILE\ \_\ DO\ \_$, provide invariant and apply while rule

Otherwise, use unfold rules.

Iterate, until all wlps gone!

wlp_if_eq and wlp_whileI' produce $if{-}then{-}else$

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Combine rule with splitting!

`Wp_Demo.thy`

Proving Partial Correctness

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Well-foundedness implies induction principle

$$\frac{wf\ r \qquad \bigwedge x.\ \dfrac{\forall\, y.\ (y,\ x) \in r \longrightarrow P\ y}{P\ x}}{P\ a}$$

Wellfounded_Demo.thy

For while loop: Find wf relation $<$ such that state decreases in each iteration

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Then use wf-induction to prove:

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s) \in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

assumes WF: $wf\ R$
assumes $INIT$: $I\ s_0$
assumes $STEP$: $\bigwedge s.\ [\![\ I\ s;\ bval\ b\ s\ ]\!]$
$\Longrightarrow wp\ c\ (\lambda s'.\ I\ s' \wedge (s',s) {\in} R)\ s$
assumes $FINAL$: $\bigwedge s.\ [\![\ I\ s;\ \neg bval\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $wp\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Or, equivalently

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assumes INIT: $I\ s_0$
assumes STEP: $\bigwedge s.\ [\![\ I\ s;\ \mathit{bval}\ b\ s\ ]\!]$
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assumes FINAL: $\bigwedge s.\ [\![\ I\ s;\ \neg \mathit{bval}\ b\ s\ ]\!] \Longrightarrow Q\ s$
shows $\mathit{wp}\ (\mathit{WHILE}\ b\ \mathit{DO}\ c)\ Q\ s_0$

Now we can prove total correctness ...

Wp_Demo.thy

Total Correctness

lemma $ASSUME\_\Theta\_alt$:
$ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta = (\forall (f,(P,c,Q)){\in}\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0){\in}R \wedge\ P\ s)\ c\ Q)$

unfolding $ASSUME\_\Theta\_def\ HT'set\_r\_def$ **..**

Let's make our VCG more usable

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Add standard arithmetic operators to IMP

Let's make our VCG more usable

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Simplify specification of pre/postcondition, and invariants

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For example:

$Cmpop\ (\leq)\ (Binop\ (+)\ (Unop\ uminus\ (V\ ''x''))\ (N\ 42))\ (N\ 50)$

`IMP2/Introduction.thy`

Adding more Operators

C-like syntax

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Arith: $+, -, *, /$ with usual binding

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$while\ (bexp)\ c$

`IMP2/Introduction.thy`

Program Syntax

More Readable VCs

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If subgoal can still be proved for arbitrary (Isabelle) variable x, it can, in particular, be proved for $s\ ''x''$.

$(\bigwedge x.\ P\ x) \Longrightarrow P\ (s\ ''x'')$

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More Readable VCs

More Readable Annotations

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Postcondition/Invariant: x as $s\ ''x''$, x_0 as $s_0\ ''x''$

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More Readable Annotations

6 Example Verifications

Loop Patterns

Euclid's Algorithm

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Applications: $*$ by $+$, exp, Fibonacci, factorial, ...

`IMP2/Examples.thy`

Count-up, Count-Down

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Applications: sqrt, log, ...

`IMP2/Examples.thy`

Approximate from Below

Bisection

We can compute sqrt more efficiently.

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l=0; h=n+1;
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  m = (l + h) / 2;
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This program is actually tricky to get right!

IMP2/Examples.thy

Bisection

6 Example Verifications

Loop Patterns

Euclid's Algorithm

Euclid Intro

Compute gcd of positive numbers a, b

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Compute gcd of positive numbers a, b

Reminder: Divides: $(b\ dvd\ a) = (\exists\, k.\ a = b * k)$
Greatest Common Divisor: $gcd{::}int \Rightarrow int \Rightarrow int$ such that
$gcd\ a\ b\ dvd\ a$ and $gcd\ a\ b\ dvd\ b$ and
$[\![a \neq 0;\ b \neq 0;\ c\ dvd\ a;\ c\ dvd\ b]\!] \Longrightarrow c \leq gcd\ a\ b$

Euclid Variants

By subtraction. Using $gcd\ (m - n)\ n = gcd\ m\ n$

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By modulo. Using: $gcd\ x\ y = gcd\ y\ (x\ mod\ y)$

`IMP2/Examples.thy`

Euclid

Modified Variables

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Program modifies at most variables it assigns to

π: $(c,\ s) \Rightarrow t \Longrightarrow \mathit{modifies}\ (\mathit{lhsv}\ \pi\ c)\ t\ s$

Modified Variables

We can strengthen correctness statement (automatically)

$wp\ \pi\ \ c\ Q\ s \Longrightarrow wp\ \pi\ \ c\ (\lambda s'.\ Q\ s' \wedge modifies\ (lhsv\ \pi\ \ c)\ s'\ s)\ s$

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$s'\ s)\ \ s$

For while-rule, we get

lemma wp_whileI_modset:
  fixes c
  defines $[simp]$: $modset \equiv lhsv\ c$
  assumes WF: $wf\ R$
  assumes $INIT$: $I\ \mathfrak{s}_0$
  assumes $STEP$: $\bigwedge\mathfrak{s}.\ \llbracket\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ bval\ b\ \mathfrak{s}\ \rrbracket$
$\Longrightarrow\ wp\ \ c\ (\lambda\mathfrak{s}'.\ I\ \mathfrak{s}' \wedge (\mathfrak{s}',\mathfrak{s}) {\in} R)\ \mathfrak{s}$
  assumes $FINAL$: $\bigwedge\mathfrak{s}.\ \llbracket\ modifies\ modset\ \mathfrak{s}\ \mathfrak{s}_0;\ I\ \mathfrak{s};\ \neg bval\ b\ \mathfrak{s}\ \rrbracket$
$\Longrightarrow\ Q\ \mathfrak{s}$
  shows $wp\ (WHILE\ b\ DO\ c)\ Q\ \mathfrak{s}_0$

Modified Variables

The VCG will automatically rewrite with rule

$$\llbracket modifies\ vs\ s\ s';\ x \notin vs \rrbracket \Longrightarrow s\ x = s'\ x$$

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program_spec computes lhs-variables:

$$\mathit{HT\_mods}\ \pi\ \mathit{mods}\ P\ c\ Q \equiv \mathit{HT}\ \pi\ P\ c\ (\lambda s_0\ s.\ \mathit{modifies}\ \mathit{mods}\ s\ s_0 \wedge Q\ s_0\ s)$$

`IMP2/Examples.thy`

Euclid – show modified sets

Modular Proofs

Consider program

$$
\begin{array}{l}
a{=}1; \\
\mathit{while}\ (m{>}0)\ \{ \\
\quad n{=}a;\ a = 1; \\
\quad \mathit{while}\ (n{>}0)\ \{ \\
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What does this compute

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What does this compute?

Power-tower function: $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (m times)

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Still, we already have verified inner loop!

Idea: Split and verify separately!

Modular Proofs

$a{=}1;$
$while\ (m{>}0)\ \{$
$\quad n{=}a;$
$\quad inline\ exp\_count\_down;$
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Reuse existing proof of exp-count-down program!

Modular Proofs

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with modified sets:

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VCG will automatically use this rule.
If inlined program has been proved with **program_spec**

`IMP2/Examples.thy`

Power-Tower

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Every variable is of type $int \Rightarrow int$.

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$ArrayCpy :: char\ list \Rightarrow char\ list \Rightarrow com$

π: $(x[] ::= y,\ s) \Rightarrow s(x := s\ y)$

Arrays

Every variable is of type $int \Rightarrow int$.

Arithmetic Expressions:

$Vidx :: char\ list \Rightarrow aexp \Rightarrow aexp$

$aval\ (Vidx\ x\ i)\ s = s\ x\ (aval\ i\ s)$

Commands:

$AssignIdx :: char\ list \Rightarrow aexp \Rightarrow aexp \Rightarrow com$

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π: $(CLEAR\ x[],\ s) \Rightarrow s(x := \lambda\_.\ 0)$

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Only with index 0: Bind $\ VAR\ (s\ ''x''\ 0)\ (\lambda x.\ \ldots)$

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`IMP2/Examples.thy`

Array-Sum

Reasoning about Arrays

Usually, use function $int \Rightarrow int$ directly.

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Theory $IMP2/IMP2\_Aux\_Lemmas$ provides useful lemmas and definitions

`IMP2/Examples.thy`

Sortedness Check

Binary Search Algorithm

Find element in sorted array. In time $O(\log n)$.

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Find element in sorted array. In time $O(\log n)$.

Idea: Halve interval in each step.

This algorithm is tricky to implement correctly!

Although the basic idea of binary search is comparatively straightforward, the details can be surprisingly tricky ...

— Donald Knuth

Only 5 out of 20 surveyed textbooks had correct implementations

— Richard E. Pattis, 1988

Binary Search Algorithm

0	1	2	3	4	5	6	7	8
2	3	5	7	11	13	17	19	

$x = 13$

```
while (l < h) {
    m = (l + h) / 2;
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$\uparrow$								$\uparrow$
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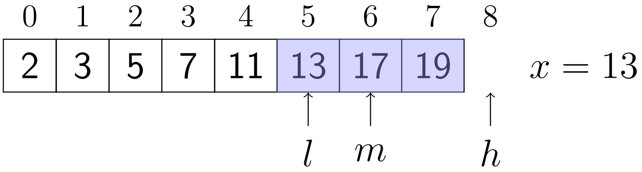


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while (l < h) {
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}
```

Returns smallest i with $x \le a[i]$

Notes on Binary Search

```
while (l < h) {
    m = (l + h) / 2;
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Otherwise, $m = (l + h)/2$ may overflow!

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Bug in Java Standard Library for > 9 years!

Proving Binary Search

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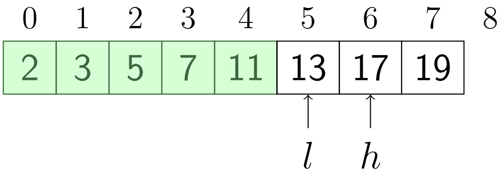
$x = 13$

Invariant:

Proving Binary Search



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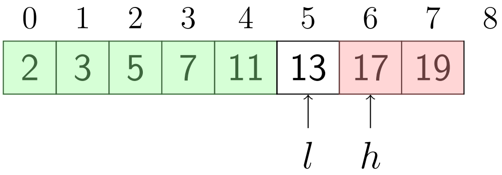

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- $i < l \implies a[i] < x$ (strictly smaller than x)
- $i \geq h \implies x \leq a[i]$ (greater or equal to x)
- and the usual bounds

IMP2/Examples.thy

Binary Search

Insertion Sort

```
j = l + 1;
while (j<h) {
   key = a[j];
   i = j-1;
   while (i>=l && a[i]>key) {
      a[i+1] = a[i];
      i=i-1
   };
   a[i+1] = key
   j=j+1
}
```

Idea: Build sorted array from start.
In each iteration, move next element to its position

Specifying Sorting Algorithms

Precondition: $l \leq h$

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$$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$$

Multisets in Isabelle

imports $HOL{-}Library.Multiset$

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$mset\_ran\ a\ r = (\sum i{\in}r.\ \{\#a\ i\#\})$

Multiset of elements at indexes in finite set r

Proving Insertion Sort

Separate proof for inner loop!

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ensures $ran\_sorted\ a\ l\ (j+1)$ and
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Invariant of outer loop:

$ran\_sorted\ a\ l\ j$
$\wedge\ mset\_ran\ a\ \{l..{<}h\} = mset\_ran\ a_0\ \{l..{<}h\}$

Insort: Inner Loop

```
key = a[j];
i = j-1;
while (i>=l && a[i]>key) {
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a[i+1] = key
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$a[j]$ is moved backwards

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Intuition: ?

$a[j]$ is moved backwards until

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Intuition: ?

$a[j]$ is moved backwards until previous element is $\leq a[j]$

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Short: Move $a[j]$ backwards over greater elements.

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Invariants easier to find!

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Insort: Inner Loop

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assumes $l < j$, $let\ key = a_0\ j$
ensures $i \in \{l - 1..{<}j\}$
ensures $\forall\, k{\in}\{l..i\}.\ a\ k = a_0\ k$ and
$a\ (i + 1) = key$ and

Insort: Inner Loop

Move $a[j]$ backwards over greater elements.

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Insort: Finding Invariant

0	1	2	3	4	5	6	7
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$\uparrow$						$\uparrow$	$\uparrow$
l						i	j

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- $+$ the usual bounds: $l - 1 \leq i \wedge i < j$

`IMP2/Examples.thy`

Insertion Sort

Summary so Far

Understand what program does!

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Split program into handy parts

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Specify what parts do (independently of users)

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Specify what parts do (independently of users)

Prove that this implies expectations of users

Prove parts separately and assemble to bigger parts

7 Advanced Verification

Abstract View

Model $int \Rightarrow int$ not always appropriate

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`IMP2/Examples.thy`

Filter, Merge, dedup

7 Advanced Verification

Local Variables

Introduce local variables

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Don't worry about name-clashes with subroutine's auxiliary variables

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$$is\_local\ a = \neg is\_global\ a$$

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$SCOPE\ c$ — Execute c with fresh set of local variables. Restore original local variables afterwards

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Pass information over scope boundaries by globals

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Wrapping Specification

Given specification of body: $HT\ P\ body\ Q$
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Recall:

$HT\ \pi\ P\ c\ Q \equiv \forall\, s_0.\ P\ s_0 \longrightarrow wp\ \pi\ c\ (Q\ s_0)\ s_0$

Prologue

$$HT\ \pi\ P\ body\ Q \Longrightarrow$$
$$HT\ \pi\ (wp\ \pi\ prologue\ P)\ (prologue;;\ body)$$
$$(\lambda s_0\ s.\ wp\ \pi\ prologue\ (\lambda s_0.\ Q\ s_0\ s)\ s_0)$$

Intuition: Weakest precondition to enforce P after prologue

Epilogue

$$[\![HT\ \pi\ \ P\ body\ \ Q;\ \forall\, s.\ \exists\, t.\ \pi\colon\ (epilogue,\ s) \Rightarrow\ t]\!]$$
$$\Longrightarrow\ HT\ \pi\ \ P\ (body;;\ epilogue)\ (\lambda s_0.\ sp\ \pi\ \ (Q\ \ s_0)\ epilogue)$$

Intuition: Strongest postcondition we get from Q after epilogue

Strongest Postconditions

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Scope: $HT\ \pi\ P\ c\ Q \Longrightarrow$
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IMP2/Examples.thy

Merge as Procedure

Recursive Procedures

Program is map $pname \rightharpoonup com$

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Note: Gets stuck if procedure does not exist!
No problem when proving total correctness

Proof Rules for Recursion

Unfolding: $\pi\ p = Some\ c \Longrightarrow wp\ \pi\ (PCall\ p)\ Q\ s = wp\ \pi\ c\ Q\ s$

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assumes $wf\ R$

$\bigwedge s.\ \llbracket HT\ \pi\ (\lambda s'.\ (s',s){\in}R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ \rrbracket$

$\Longrightarrow wp\ \pi\ (PCall\ p)\ (Q\ s)\ s$

shows $HT\ \pi\ P\ (PCall\ p)\ Q$

Proof Rules for Recursion

Unfolding: $\pi\ p = Some\ c \Longrightarrow wp\ \pi\ (PCall\ p)\ Q\ s = wp\ \pi\ c\ Q\ s$

Idea: Well-Founded induction on state

assumes $wf\ R$
$\bigwedge s.\ [\![HT\ \pi\ (\lambda s'.\ (s',s) {\in} R \wedge P\ s')\ (PCall\ p)\ Q;\ P\ s\ ]\!]$
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Show specification for state s, assuming it holds for smaller states s'.

Mutual Recursion

Same idea, but for sets of specifications.

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$(\forall\,(f,\ P,\ c,\ Q){\in}\Theta.\ HT'\ \pi\ (\lambda s.\ (f\ s,\ f_0\ s_0) \in R \wedge P\ s)\ c\ Q)$
Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

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Hoare-triples valid for states less than $f_0\ s_0$. Annotation is variant.

$PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta \equiv$
$\forall\ P\ c\ Q.\ (f_0,\ P,\ c,\ Q) \in \Theta \wedge P\ s_0 \longrightarrow wp\ \pi\ (c\ s_0)\ (Q\ s_0)\ s_0$
Hoare-triples valid for fixed variant f_0 and state s_0.

Mutual Recursion

lemma $vcg\_HT'setI$:
assumes $wf\ R$
assumes RL: $\bigwedge f_0\ s_0.\ \llbracket\ ASSUME\_\Theta\ \pi\ f_0\ s_0\ R\ \Theta\ \rrbracket \Longrightarrow PROVE\_\Theta\ \pi\ f_0\ s_0\ \Theta$
shows $HT'set\ \pi\ \Theta$

Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

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Fix variant and state,
assume that Hoare-triples hold for smaller states
prove that Hoare-triples hold for this state.

$\llbracket \pi\ p = Some\ c;\ HT\_mods\ \pi\ mods\ P\ c\ Q \rrbracket \Longrightarrow HT\_mods\ \pi\ mods\ P\ (PCall\ p)\ Q$

Maps Hoare-Triples to procedure calls

Local Procedures

Idea: Recursive procedure names only valid locally!

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Call procedure with local procedure environment

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Call procedure with local procedure environment

$HT\_mods\ \pi\ \ mods\ P\ (PCall\ p)\ Q \Longrightarrow HT\_mods\ \pi'\ mods\ P\ (PScope\ \pi\ \ p)\ Q$

Wrap current procedure environment

Specification of Mutual Recursive Procedures

The IMP2 tools take care of

- wf-relation. Default $less\_than$

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- parameters and return values.
- variants: expression over parameters.
- localization of procedure environment.

IMP2/Examples.thy

Ackermann, Odd/Even, Merge Sort

Completeness

Consider program with $HT\,\pi\;\;P\;\;c\;\;Q$

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Can we always find annotations to get provable VCs?

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Consider program with $HT\,\pi\ P\ c\ Q$

Can we always find annotations to get provable VCs?

Only consider while-rule here

Partial Correctness

$$\llbracket I\ s_0;\ \bigwedge s.\ I\ s \Longrightarrow \textsf{\textit{if}}\ bval\ b\ s\ \textsf{\textit{then}}\ wlp\ \pi\ c\ I\ s\ \textsf{\textit{else}}\ Q\ s \rrbracket$$
$$\Longrightarrow wlp\ \pi\ (WHILE\ b\ DO\ c)\ Q\ s_0$$

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What invariant shall we use?

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What invariant shall we use?

$wlp\ \pi\ c\ Q$!

Total Correctness

$[\![wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ \pi\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s)$
$\in R)\ s$ *else* $Q\ s]\!]$
$\Longrightarrow wp\ \pi\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Invariant: $wp\ \pi\ c\ Q$

Total Correctness

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Invariant: $wp\ \pi\ c\ Q$

Variant?

Total Correctness

$\llbracket wf\ R;\ I\ s_0;$
$\bigwedge s.\ I\ s \Longrightarrow$ *if* $bval\ b\ s$ *then* $wp\ \pi\ c\ (\lambda s'.\ I\ s' \wedge (s',\ s)$
$\in R)\ s$ *else* $Q\ s\rrbracket$
$\Longrightarrow wp\ \pi\ (WHILE\ b\ DO\ c)\ Q\ s_0$

Invariant: $wp\ \pi\ c\ Q$

Variant?

Number of iterations until termination!

IMP2/Examples.thy

Completeness of While-Rule

Conclusions

IMP2: Verification of simple programs in Isabelle/HOL

- while-language, arrays, local-vars, recursive procedures
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IMP2: Verification of simple programs in Isabelle/HOL

- while-language, arrays, local-vars, recursive procedures
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Caveats:

- Procedures+Recursion tools not well-tested
- VCG is slow for many procedures/inlines