1 Rings

A **ring** is a quartet $R = (R, +, \times, 1)$ where:
(i) $(R, +)$ is an additive abelian group;
(ii) $(R, \times, 1)$ is a multiplicative monoid;
(iii) $\times$ is distributive with respect to $+$; this means that

$$r_1 \times (r_2 + r_3) = (r_1 \times r_2) + (r_1 \times r_3) \text{ and } (r_2 + r_3) \times r_1 = (r_2 \times r_1) + (r_3 \times r_1),$$

for any $r_1, r_2, r_3 \in R$.

R is a **commutative ring** whenever $\times$ is commutative.
In general, for any $r_1, r_2 \in R$ the element $r_1 \times r_2$ is called the **product** or **multiplication** of r_1 by r_2 and it will be simply represented as $r_1 r_2$. The neutral element for the addition is called **zero** and is represented by 0, and the element 1 is called the **unity** of the ring R.

Lemma. 1.1.
Let R be a ring and $r, r_1, r_2 \in R$ elements of R, the following statements hold.
(1) $0\,r = 0 = r\,0$.
(2) $r_1(-r_2) = -r_1 r_2 = (-r_1) r_2$.
(3) $r(r_1 - r_2) = r r_1 - r r_2$.

PROOF. (1). For any $r \in R$ we have: $r = 1r = (1+0)r = 1r + 0r = r + 0r$, hence $0r = 0$. □

Exercise. 1.2.
Show that in any ring R the commutativity of the addition is a consequence of the other axioms.
Ref.: 2101e_011 *SOLUTION*

SOLUTION. Exercise (1.2.)
If we develop the following expression $(1+1)(r_1 + r_2)$, using distributivity, we obtain:

$$\begin{aligned}(1+1)(r_1 + r_2) &= (1+1)r_1 + (1+1)r_2 = r_1 + r_1 + r_2 + r_2;\\ (1+1)(r_1 + r_2) &= 1(r_1 + r_2) + 1(r_1 + r_2) = r_1 + r_2 + r_1 + r_2.\end{aligned}$$

Hence we obtain: $r_1 + r_2 = r_2 + r_1$. □

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An element $d \in R$ is a **left zero-divisor** if there exists $0 \neq r \in R$ such that $dr = 0$; a non left zero-divisor is called a **left regular** element of R; this means, if $dr = 0$, then $r = 0$ for any $r \in R$. In a similar way we may define **right zero divisor** and **right regular** element.

A **zero divisor** is a left or right zero divisor; a **regular element** is a left and right regular element. A ring without non-zero zero divisor is an **integral ring**. Commutative integral rings are called **integral domains**.

An element $u \in R$ is **left invertible** if there exists $r \in R$ such that $ru = 1$. In this case we say that r is a **left inverse** of u and u a **right inverse** of r. In a similar way we may define **right invertible elements**. An **invertible element** is a left and right invertible element. Rings in which every non-zero element is invertible are called **división rings**. Commutative división rings are called **fields**.

Exercise. 1.3.
Show that if an element r in a ring R has a left inverse r_1 and a right inverse r_2, then $r_1 = r_2$.
Ref.: 2101e_013 *SOLUTION*

SOLUTION. Exercise (1.3.)
Indeed, we have:

$$r_1 = r_1 1 = r_1(rr_2) = (r_1 r)r_2 = 1r_2 = r_2.$$

□

Exercise. 1.4.
Let R be a ring in which every nonzero element x has a left inverse, say x', then every nonzero element y has a right inverse, say y''. By the above exercise we have for any nonzero element x we have $x' = x''$.
Ref.: 2101e_014 *SOLUTION*

SOLUTION. Exercise (1.4.)
Let $0 \neq x \in R$, since $x'x = 1$, then $x'xx' = x'$, and $x'(xx' - 1) = 0$, hence $(x')'x'(xx' - 1) = 0$, and $xx' = 1$. □

Exercise. 1.5.
Let R be a ring and $a \in R$ and element with a left inverse, i.e., there exists $b \in R$ such that $ba = 1$. Show that the following statements are equivalent:

(a) a is a right zero–divisor ($= a$ has a left zero–divisor, $0 \neq c$ such that $ca = 0$).
(b) a is not right invertible.cero a al

Ref.: 2101e_012 *SOLUTION*

SOLUTION. Exercise (1.5.)
(a) $\Rightarrow$ (b). If there exists a right inverse of a, let b, then $c = cab = 0b = 0$, which is a contradiction.
(b) $\Rightarrow$ (a). Since a is not right invertible, then $ab \neq 1$, hence $(ab-1)a = aba - a = 0$, and b has a left zero divisor. □

Exercise. 1.6.
There is a ring R and an element $0 \neq x \in R$ with a left inverse and no right inverse. In this case x has two different left inverses.
Ref.: 2101e_015 *SOLUTION*

SOLUTION. Exercise (1.6.)
Let us consider a field K and the vector space $V = K^{\mathbb{N}}$. In the ring $R = \mathrm{End}_K(V)$ we consider the maps $f : V \longrightarrow V$ defined $f(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$, and $g : V \longrightarrow V$ defined $g(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$. You can check that $gf = \mathrm{id}_V$, and there is not $h \in \mathrm{End}_K(V)$ such that $hg = \mathrm{id}_V$.
Since $gf = \mathrm{id}_V$, hence $fgf = f$, $(fg - \mathrm{id}_V)f = 0$, and $fg - \mathrm{id}_V \neq 0$. Therefore $g + fg - \mathrm{id}_V \neq g$ and

$$(g + fg - \mathrm{id}_V)f = gf + fgf - f = gf = \mathrm{id}_V.$$

□

Ring maps

Let R, S be rings, a **ring map** f from R to S is a map $f : R \to S$ which is homomorphism for $+$, $\times$ and 1. This means that

$$\begin{aligned} &f(r_1 + r_2) = f(r_1) + f(r_2), \\ &f(r_1 r_2) = f(r_1) f(r_2), \\ &f(1) = 1, \end{aligned}$$

for any $r_1, r_2 \in R$.

Lemma. 1.7.
The following statements hold:

(1) Let $f : R \to S$ and $g : S \to T$ be ring maps, then the composition $g \circ f : R \to T$ is a ring map.
(2) For any ring R the identity map $\mathrm{id}_R : R \to R$ is a ring map.

Let $f : R \to S$ be a ring map, the **image** of f is a subset $\mathrm{Im}(f) \subseteq S$ which satisfies the axioms of ring whenever we consider the restriction of the operations in S. Such subsets are called **subrings**.

Lemma. 1.8.
Let R be a ring and $S \subseteq R$ be a subset. The following statements are equivalent:

(a) S is a subring of R.
(b) S satisfies the following properties:
  (i) $s_1 - s_2 \in S$ for any $s_1, s_2 \in S$.
  (ii) $s_1 s_2 \in S$ for any $s_1, s_2 \in S$.
  (iii) $1 \in S$.

Lemma. 1.9.
Let $\{S_i \mid i \in I\}$ be a family of subrings of a ring R, then:

(1) $\cap\{S_i \mid i \in I\}$ is a subring of R.
(2) The relation $S \leq T$, if $S \subseteq T$, is a partial orden in the set of all subrings of R.
(3) The supremum of the family $\{S_i \mid i \in I\}$ is:

$$\bigvee\{S_i \mid i \in I\} = \bigcap\{S \subseteq R \mid S \text{ is a subring of } R \text{ containnig every } S_i\}.$$

(4) For any subset $X \subseteq R$ there exists a smallest subring $[X]$ of R containing X; it is described as:

$$[X] = \bigcap\{S \subseteq R \mid S \text{ is a subring of } R \text{ containnig } X\}.$$

Exercise. 1.10.
Show that the elements of $[X]$ can be described as:

$$[X] = \left\{\sum n_{i_1,\ldots,i_t} x_{i_1} \ldots x_{i_t} \mid x_{i_1}, \ldots, x_{i_t} \in X \cup \{1\},\, n_{i_1,\ldots,i_t} \in \mathbb{Z}\right\}.$$

Ref.: 2101e_018 *SOLUTION*

SOLUTION. Exercise (1.10.)
HACER □

Exercise. 1.11.
Let R be a ring, $S \subseteq R$ be a subring and $X \subseteq R$ be a subset, the elements of the smallest subring of R containing S and X can be described as:

$$S[X] = \left\{ \sum s_{i_1} x_{i_1} \ldots s_{i_t} x_{i_t} s_{i_{t+1}} \mid s_{i_1}, \ldots, s_{i_{t+1}} \in S,\ x_{i_1}, \ldots, x_{i_t} \in X \cup \{1\} \right\}.$$

Ref.: 2101e_019 *SOLUTION*

SOLUTION. Exercise (1.11.)
HACER □

Lemma. 1.12.
Let $f : R \to S$ be a ring map.
(1) If $R' \subseteq R$ is a subring, then $f(R') \subseteq S$ is a subring.
(2) If $S' \subseteq S$ is a subring, then $f^{-1}(S') \subseteq R$ is a subring.

The **kernel** of a ring map $f : R \to S$ is the subset $\mathrm{Ker}(f) \subseteq R$. It satisfies the following properties:
(i) $k_1 - k_2 \in \mathrm{Ker}(f)$ for any $k_1, k_2 \in \mathrm{Ker}(f)$.
(ii) $rk, kr \in \mathrm{Ker}(f)$ for any $k \in \mathrm{Ker}(f)$ and any $r \in R$.

A subset $\mathfrak{A} \subseteq R$ satisfying these two properties is called an **ideal** of R. Sometimes ideals are referred as **two-sided ideals**.

Lemma. 1.13.
Let $f : R \to S$ be a ring map
(1) f is injective if, and only if, $\mathrm{Ker}(f) = 0$.
(2) f is surjective if, and only if, $\mathrm{Im}(f) = S$.
(3) f is bijective (injective and surjective) if, and only if, there exists a ring map $g : S \to R$ such that $f \circ g = id_S$ and $g \circ f = id_R$.

A bijective ring map is called a **ring isomorphism**. A ring map $f : R \to R$ is called an **endomorphism**; a ring isomorphism $f : R \to R$ is called an **automorphism**.

Lemma. 1.14.
Let $f : R \to S$ be a ring map.
(1) If $\mathfrak{A} \subseteq R$ is an ideal, then $f(\mathfrak{A}) \subseteq S$ is not necessarily an ideal of S; if f is surjective, then $f(\mathfrak{A}) \subseteq S$ is an ideal.
(2) If $\mathfrak{B} \subseteq S$ is an ideal, then $f^{-1}(\mathfrak{B}) \subseteq R$ is an ideal.

Proposition. 1.15. (Intersection of ideals)
Let $\{\mathfrak{A}_i \mid i \in I\}$ be a family of ideals of a ring R, the intersection of this family is an ideal of R.

In the set of all ideals of a ring R the relation $\mathfrak{A} \leq \mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ defines a partial order. The intersection of a family of ideals is the infimum, and the supremum is computed as follows.

Proposition. 1.16.
The following statements hold:
(1) For any subset $X \subseteq R$ there is a smallest ideal containing X. Therefore, this ideal is

$$(X) = \cap\{\mathfrak{A} \subseteq R \mid \mathfrak{A} \text{ is an ideal of } R \text{ and } X \subseteq \mathfrak{A}\}.$$

(2) Let $\{\mathfrak{A}_i \mid i \in I\}$ be a family of ideals of R, there exists the supremum of this family. (See exercise (3.14.) below.)

Lemma. 1.17. (Sum of ideals)
Let $\{\mathfrak{A}_i \mid i \in i\}$ a family of ideals of R, then the smallest ideal containing every $\mathfrak{A}_i$ is denoted as $\sum_i \mathfrak{A}_i$, and its elements are

$$\left\{ \sum_{j=1}^{t} x_{i_j} \mid x_{i_j} \in \mathfrak{A}_{i_j} \right\}.$$

In particular, for two ideals $\mathfrak{A}, \mathfrak{B} \subseteq R$ the ideal $\mathfrak{A} + \mathfrak{B}$ is $\{a + b \mid a \in \mathfrak{A},\ b \in \mathfrak{B}\}$.

Lemma. 1.18. (Product of ideals)
Let $\mathfrak{A}, \mathfrak{B} \subseteq R$ ideals, the smallest ideal containing all product ab, where $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ is denoted by $\mathfrak{A}\mathfrak{B}$, and its elements are

$$\left\{ \sum_i a_i b_i \mid a_i \in \mathfrak{A},\ b_i \in \mathfrak{B} \right\}.$$

Examples

Example. 1.19.
The set $\mathbb{Z}$ of all **integer numbers** is an integral domain. For any ring R there is a unique ring map $\mathbb{Z} \to R$; this map is defined by $n \mapsto n \cdot 1$.

Example. 1.20.
The ideals of $\mathbb{Z}$ are the subsets $n\mathbb{Z} = \{nz \in \mathbb{Z} \mid z \in \mathbb{Z}\}$, and $\mathbb{Z}$ has no proper subrings.

Example. 1.21.
The rings $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of all **rational numbers**, **real numbers** and **complex numbers**, respectively, are fields. The ring of **quaternions**

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with operations defined from

$$ii = jj = kk = ijk = -1,$$

is a non–commutative division ring (hence it is not a field).

A division ring has no proper ideals. A ring R without proper ideals is called a **simple ring**. Therefore every division ring is a simple ring, but not every simple ring is a división ring.

Example. 1.22. (Matrix ring)
Let R be a ring, for instance $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then $M_2(R)$, the ring of 2×2 square matrices, is a non commutative ring,

$$M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$$

with addition defined componentwise and product defined as

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + a_2 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

The unity of $M_2(R)$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

This construction can be perform for any positive integer number n and any ring R to obtain the matrix ring $M_n(R)$. The ring $M_n(R)$ is not commutative whenever $n \geq 2$, even if R is.

Exercise. 1.23.
Shows that if R is a ring, the ideals of $M_2(R)$ are the subsets $M_2(\mathfrak{A})$, where $\mathfrak{A} \subseteq R$ is an ideal. (This result holds if $n \geq 2$).

Example. 1.24.
As a consequence, if D is a division ring, the ring $M_2(D)$ is a simple ring and is not a division ring.

Example. 1.25. (Triangular matrix rings)
Other interesting examples of rings are given by the triangular matrices $\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ and $\begin{pmatrix} R & \mathfrak{A} \\ 0 & R \end{pmatrix}$, where $\mathfrak{A} \subseteq R$ is an ideal of R.

Example. 1.26. (Infinite ring of matrices of finite rows or finite columns)
Let R be a ring, for any set I we consider the set square matrices $M_I(R)$ with rows and columns indexed by I. This set is an abelian group with addition defined componentwise, but in general, using matrix multiplication, we can not define a product.

In the set $FRM_I(R)$ of all matrices in $M_I(R)$ with finite rows (each row only contains finitely many non-zero entries) we may define a product in the usual way. With this product and unity the matrix $(\delta_{i,j})_{i,j}$, the set $FRM_I(R)$ is a ring.

In the same way we may define the ring $FCM_I(R)$ of all square matrices indexed in I with finite columns.

Let K be a field, I a set and V the vector space $V = K^{(I)}$. If B is a basis of V, each endomorphism $f : V \longrightarrow V$ can be represented by an element of $FCM_I(K)$, if we write elements in V by column, or by an element of $FRM_I(K)$, if they are written by row.

Example. 1.27. (Polynomial ring)
Let R be a ring, and $R[X]$ the **polynomial ring** in the indeterminate X (observe that elements of R commute with powers of X). In a similar way, we may define, for $n \geq 2$, the polynomial ring.

$$R[X_1, \ldots, X_n] = R[X_1, \ldots, X_{n-1}][X_n].$$

Example. 1.28. (Noncommutative polynomial ring)
Let R be a ring, and $X_1, \ldots, X_n$ indeterminates over R. We may define a new ring, denoted as $R\langle X_1, \ldots, X_n\rangle$ as the set of all formal expressions

$$\sum_{i_1,\ldots,i_s} r_{i_1,\ldots,i_s} X_{i_1} \cdots X_{i_s},$$

where $r_{i_1,\ldots,i_s} \in R$ and $X_{i_j} \in \{1, X_1, \ldots, X_n\}$. The set of these formal expressions with sum defined term to term, and multiplication as

$$r_{i_1,\ldots,i_s} X_{i_1} \cdots X_{i_s} t_{j_1,\ldots,j_u} X_{j_1} \cdots X_{j_u} = r_{i_1,\ldots,i_s} t_{j_1,\ldots,j_u} X_{i_1} \cdots X_{i_s} X_{j_1} \cdots X_{j_u},$$

is a ring with unity element 1.

Example. 1.29. (Weyl algebra)
The **first Weyl algebra** is defined as the quotient of the noncommutative polynomial ring $K\langle X,Y\rangle$ by the ideal generated by $YX-XY-1$, or equivalently the set $K[X,Y]$ satisfying the relationship $XY-XY=1$. It is represented as $A_1(K)$. The ring $A_1(K)$ is not a commutative ring, and it may be identified with the ring of operators on $K[X]$ generated by X and $\frac{\partial}{\partial X}$.
Let K be a field and R be a K–algebra, an operator φ of R is a linear map $\varphi : R \longrightarrow R$; we denote by $\mathscr{O}(R)=\operatorname{End}_K(R)$ the set of all operators of R and define two operations:
Sum: $(\varphi_1+\varphi_2)(x)=\varphi_1(x)+\varphi_2(x)$, for any $x\in R$.
Product: $\varphi_1\varphi_2(x)=\varphi_1(\varphi_2(x))$, for any $x\in R$.
Unity: id_R is the unity of $\mathscr{O}(R)$.

Observe that the multiplication by X is an operator on $K[X]$ as it is $\frac{\partial}{\partial X}$.

Example. 1.30. (Monoid ring)
Let M be a multiplicative monoid, we define $R[M]$ the set of all formal expressions:

$$\sum_i r_i m_i,$$

where $r_i\in R$, and $m_i\in M$. In $R[M]$ we consider two operations:
Sum: $\sum_i r_i m_i+\sum_i r_i' m_i=\sum_i (r_i+r_i')m_i$, the componentwise addition.
Product: $\sum_i r_i m_i \sum_j r_j' m_j=\sum_k \left(\sum_{m_i m_j=m_k} r_i r_j'\right) m_k$.
Unity: $1e$ is the unity of $R[M]$, where $e\in M$ is the neutral element of M.

For any monoid map $f : M_1 \longrightarrow M_2$ there exists a ring map $f : R[M_1]\longrightarrow R[M_2]$ defined as $f(\sum_i r_i m_i)=\sum_i r_i f(m_i)$.
If R is a ring then $\{1\}$ is a monoid, and the monoid ring $R[\{1\}]$ is isomorphic to R.
For any monoid M there is only a monoid map $f : M \longrightarrow \{1\}$, hence a ring map $f : R[S]\longrightarrow R$ defined $f(\sum_i r_i m_i)=\sum_i r_i$.

Example. 1.31. (The free monoid)
Let I be a nonempty set; we may consider a set A_I indexed in I, i.e., $A_I=\{a_i \mid i\in I\}$.

(1) Consider a new set $W(I)$ whose elements are words in A_I, i.e., finite sequences $a_{i_1}\cdots a_{i_t}$ where $a_{i_j}\in A$. If $t=0$, then we have the empty word, denoted by $\varnothing$.
(2) In $W(I)$ we define a binary operation, the concatenation:

$$(a_{i_1}\cdots a_{i_t})*(b_{j_1}\cdots b_{j_s})=a_{i_1}\cdots a_{i_t}b_{j_1}\cdots b_{j_s}.$$

The triple $(W(I),*,\varnothing)$ is a monoid.
(3) For any monoid M and any map $f : I\longrightarrow M$ there is a unique monoid map $f' : W(I)\longrightarrow M$ such that $f'(a_i)=f(i)$, for any $i\in I$, i.e., the following diagram commutes

$$\begin{tikzcd} I \arrow[r, "a_{(-)}"] \arrow[dr, "f"'] & W(A) \arrow[d, dashed, "\exists_1 f'"] \\ & M \end{tikzcd}$$

Lemma. 1.32.
Let R be a ring and M a monoid, for any ring S, any monoid map $f_1 : M \longrightarrow (S, \times, 1)$, and any ring map $f_0 : R \longrightarrow S$ such that

$$f_0(r)f_1(m) = f_1(m)f_0(r) \text{ for any } r \in R \text{ and } m \in M,$$

there exists a unique ring map $(f_0, f_1) : R[M] \longrightarrow S$ such that $(f_0, f_1)(rm) = f_0(r)f_1(m)$ for any $m \in M$ and any $r \in R$, i.e., the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "incl"] \arrow[dr, "f_1"'] & R[M] \arrow[d, dashed, "\exists_1 (f_0, f_1)"'] & R \arrow[l, "incl"'] \arrow[dl, "f_0"] \\
& S &
\end{tikzcd}$$

Lemma. 1.33.
For any ring R, and any nonempty set I there is a ring isomorphism $R[X_i \mid i \in I] \cong R[W(I)]$.

Example. 1.34. (Matrix ring)
Let R be a ring, we consider $M_n(R)$ the ring of all $n \times n$ matrices with coefficients in R, and in $M_n(R)$ the elements

$$e_{hk} = (a_{ij})_{ij}, \text{ being } a_{ij} = \begin{cases} 1 & \text{if } (i,j) = (h,k) \\ 0 & \text{if } (i,j) \neq (h,k) \end{cases}$$

It is easy to find the following relationships: $e_{ij}e_{hk} = \delta_{jh}e_{ik}$, and it is obvious that every element in $M_n(R)$ can be written, in a unique way, as $\sum_{ij} a_{ij}e_{ij}$, with $a_{ij} \in R$.
With this notation the product of two matrices is:

$$\sum_{ij} a_{ij}e_{ij} \sum_{ij} b_{ij}e_{ij} = \sum_{ij}\left(\sum_k a_{ik}b_{kj}\right)e_{ij}.$$

We have another relationship: $\sum_i e_{ii} = 1$, hence there is a ring map $\eta : R \longrightarrow M_n(R)$ defined $\eta(r) = \sum_i re_{ii}$, the diagonal matrix.
Since the elements $\{e_{ij} \mid i,j = 1,\ldots,n\}$, generate $M_n(R)$, we have a surjective ring map

$$\theta : R\langle X_{ij} \mid i,j = 1,\ldots,n\rangle \longrightarrow M_r(R), \text{ defined } \theta(X_{ij}) = e_{ij};$$

the kernel of θ is the ideal generated by the elements $\{X_{ij}X_{hk} - \delta_{jh}X_{ik}, \sum_i X_{ii} - 1 \mid i,j,h,k\}$.
We can point out the more interesting properties of the elements e_{ij}:

(1) The element e_{ii} is **idempotent**, i.e., $e_{ii}^2 = e_{ii}$.
(2) They are orthogonal, i.e., $e_{ii}e_{jj} = 0$ whenever $i \neq j$.
(3) The element e_{ij} is **nilpotent** whenever $i \neq j$; indeed, $e_{ij}^2 = 0$.
(4) If $i \neq j$, then $(1-e_{ij})(1+e_{ij}) = 1-e_{ij}^2 = 1$, hence they are invertible elements.

A set of idempotent elements $\{e_1, \ldots, e_n\}$ satisfying properties (1) and (2) is called a **complete set of orthogonal idempotents**.

A direct consequence of this description of the matrix ring $M_n(R)$ and its elements is the following.

Proposition. 1.35.
There is a lattice isomorphism between the lattice of ideals of R and the lattice of ideals of $M_n(R)$ given by $\mathfrak{A} \mapsto M_n(\mathfrak{A})$.

PROOF. First we observe that if $\mathfrak{A} \subseteq R$ is an ideal, then $M_n(\mathfrak{A})$ is the ideal of all expressions $\sum_{ij} a_{ij}e_{ij}$ with $a_{ij} \in \mathfrak{A}$. Otherwise, if $\mathfrak{B} \subseteq M_n(R)$ is an ideal, we define $\mathfrak{A} = \{r \in R \mid re_{11} \in \mathfrak{B}\}$; it is an ideal of R. We will relate $\mathfrak{B}$ and $M_n(\mathfrak{A})$.

- If $r \in \mathfrak{A}$ then $re_{ij} = e_{i1}(re_{11})e_{1j} \in \mathfrak{B}$, as r commutes with e_{**}. Hence $M_n(\mathfrak{A}) \subseteq \mathfrak{B}$.
- If $b = \sum_{ij} r_{ij}e_{ij} \in \mathfrak{B}$ then $r_{ij}e_{11} = e_{1i}be_{j1} \in \mathfrak{B}$, hence $r_{ij} \in \mathfrak{A}$. Hence $\mathfrak{B} \subseteq M_n(\mathfrak{A})$.

□

This means, for instance, that if K is a field, or more in general a division ring, the matrix ring $M_n(K)$ has no non–zero proper ideals; we shall say that $M_n(K)$ is a **simple ring**.

2 Construction of rings

Quotient ring

Let R be a ring, and $f : R \to S$ a surjective ring map. There exists a equivalence relation $\sim$ in R such that $x \sim y$ if $f(x) = f(y)$, for any $x, y \in R$. The relation $\sim$ satisfies:

(i) If $x \sim y$ and $z \in R$, then $(x+z) \sim (y+z)$.
(ii) If $x \sim y$ and $z \in R$, then $xz \sim yz$.

Equivalence relations $\sim$ satisfying (i–ii) are called **compatible** with the ring structure of R. On the other hand, if $\sim$ is an equivalence relation in R satisfying (i–ii), then in the quotient set $R/\sim$ we may define two binary operations $+$ and $\times$ in such a way that $(R/\sim, +, \times, \overline{1})$ is a ring and the canonical projection $p : R \to R/\sim$ is a ring map.
Therefore, $\sim$ defines an ideal $\mathfrak{A} = \{x \in R \mid \overline{x} = \overline{0}\}$, and conversely, if $\mathfrak{A}$ is an ideal the equivalence relation $\sim_{\mathfrak{A}}$ defined as $x \sim_{\mathfrak{A}} y$ if $x - y \in \mathfrak{A}$ satisfies (i–ii).
There is a bijective correspondence between equivalence relations $\sim$ satisfying (i–ii) and ideals of R.

Proposition. 2.1.
Let $\mathfrak{A}$ be an ideal of a ring R, in the quotient group $R/\mathfrak{A}$ there is a unique ring structure such that the projection $p : R \longrightarrow R/\mathfrak{A}$ is a ring map. This structure is defined as follows:

- $(r_1 + \mathfrak{A}) + (r_2 + \mathfrak{A}) = (r_1 + r_2) + \mathfrak{A}$.
- $(r_1 + \mathfrak{A})(r_2 + \mathfrak{A}) = (r_1 r_2) + \mathfrak{A}$.
- *The unity is* $1 + \mathfrak{A}$,

for any $r_1, r_2 \in R$.

Proposition. 2.2. (First isomorphism theorem)
Let $f : R \longrightarrow S$ be a ring map, there is a factorization of f

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "p"'] & S \\ R/\operatorname{Ker}(f) \arrow[r, "b"] & \operatorname{Im}(f) \arrow[u, "i"'] \end{tikzcd}$$

such that

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- *$p : R \to R/\operatorname{Ker}(f)$ is the projection,*
- *$i : \operatorname{Im}(f) \to S$ is the inclusión and*
- *$b : R/\operatorname{Ker}(f) \to \operatorname{Im}(f)$ is the isomorphism defined by $b(r + \operatorname{Ker}(f)) = f(r)$ for any $r \in R$.*

In particular, there is a bijection between ideals of $\operatorname{Im}(f)$ and ideals of R containing $\operatorname{Ker}(f)$.

Proposition. 2.3. (Quotient ring universal property)
Let $f : R \to S$ be a ring map and $\mathfrak{A} \subseteq R$ be an ideal such that $\mathfrak{A} \subseteq \operatorname{Ker}(f)$, i.e., $fj = 0$, there is a unique ring map $\overline{f} : R/\mathfrak{A} \to S$ such that $f = \overline{f} \circ p$, i.e., the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "j"] & R \arrow[r, "p"] \arrow[dr, "f"'] & R/\mathfrak{A} \arrow[d, dashed, "\exists_1 \overline{f}"] \\ & & S \end{tikzcd}$$

is commutative.

Proposition. 2.4. (Third isomorphism theorem)
Let $\mathfrak{A} \subseteq \mathfrak{B}$ ideals of a ring R, then $\mathfrak{B}/\mathfrak{A} := \{b + \mathfrak{A} \in R/\mathfrak{A} \mid b \in \mathfrak{B}\}$ is an ideal of $R/\mathfrak{A}$ and there is an isomorphism

$$\frac{R}{\mathfrak{B}} \cong \frac{R/\mathfrak{A}}{\mathfrak{B}/\mathfrak{A}}.$$

Therefore, the ideals of R containing $\mathfrak{B}$ are in one–to–one correspondence, which preserves the order, with ideals of $R/\mathfrak{A}$. This correspondence is given by $\mathfrak{C} \mapsto \frac{c}{a}$.

There is a relation between ideals of two quotient rings.

Proposition. 2.5. (Second isomorphism theorem)
Let $\mathfrak{A}, \mathfrak{B} \subseteq R$ be ideals, there is an isomorphism

$$\frac{\mathfrak{A} + \mathfrak{B}}{\mathfrak{A}} \cong \frac{\mathfrak{B}}{\mathfrak{A} \cap \mathfrak{B}}.$$

Direct product

Proposition. 2.6. (Direct product universal property)
Let $\{R_i \mid i \in I\}$ be a family of rings, then in $\prod\{R_i \mid i \in I\}$ there is a unique ring structure such that each projection $p_j : \prod_i R_i \to R_j$ is a ring map. The operations are defined componentwise.
In particular, for any ring S and any family $\{f_i : S \to R_i \mid i \in I\}$ there is a unique ring map $f : S \to \prod_i R_i$ such that $f_i = p_i \circ f$ for any index $i \in I$.

$$\begin{tikzcd} S \arrow[d, dashed, "\exists_1 f"'] \arrow[dr, "f_i"] & \\ \prod_i R_i \arrow[r, "p_i"'] & R_i \end{tikzcd}$$

An element $r \in R$ is **idempotent** if $r^2 = r$. If $R = R_1 \times R_2$, then $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are idempotent elements of R. In addition

(1) e_i is a **central element**, i.e., it commutes with all elements in R,
(2) $e_1 + e_2 = 1$ and
(3) $e_1 e_2 = 0 = e_2 e_1$.

We say $\{e_1, e_2\}$ is **complete set of orthogonal central idempotent** elements of R.

Proposition. 2.7.
A ring R is a direct product of finitely many rings, $R_1, \ldots, R_t$ if, and only if, there exists a complete set of orthogonal central idempotent elements $\{e_1, \ldots, e_t\}$ in R. In this case $R_i = e_i R$ for any index $i = 1, \ldots, t$.

Opposite ring

For any ring R we define a new ring, the **opposite ring**, as $(R, +, \times^{opp}, 1)$, where $\times^{opp}$ is defined: $r_1 \times^{opp} r_2 = r_2 \times r_1$ for any $r_1, r_2 \in R$. This ring is simply denoted by R^{opp}. Observe that if R is a commutative ring, then $R^{opp} = R$.

Proposition. 2.8.
Let R be a ring, the set

$$\mathrm{Cen}(R) = \{x \in R \mid rx = xr \text{ for any } r \in R\}$$

*is a subring of R, it is called the **center** of R.*

Commutative polynomial ring

Let R be a ring, M a monoid with unity element e, $u : R \longrightarrow R[M]$ the ring map $u(r) = re$ and $v : M \longrightarrow R[M]$ the monoid map $v(m) = 1m$.

Proposition. 2.9. (Monoid ring universal property)
Let M be a multiplicative monoid and R be a ring. For

- *any ring S,*
- *any ring map $f : R \to S$ and*
- *any multiplicative monoid map $g : M \to S$, such that*
- *$f(r)g(m) = g(m)f(r)$, for every $r \in R$ and any $m \in M$,*

there is a unique ring map $(f,g) : R[M] \to S$ such that $g = u \circ (f,g)$.

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[dr, "f"'] & R[M] \arrow[d, dashed, "(f,g)"] & M \arrow[l, "v"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

and $(f,g)(rm) = f(r)g(m)$ for every $m \in M$.

See Lemma (1.32.).

Proposition. 2.10. (Generalized commutative polynomial ring)
*Let R be a ring and $\{X_i \mid i \in I\}$ be a set of indeterminates. If we consider the free commutative monoid M on $\{X_i \mid i \in I\}$, i.e., all expressions $X_{i_1}^{a_1} \cdots X_{i_t}^{a_t}$ where $a_i \in \mathbb{N}$ and $i_1 < \cdots < i_t$ for some well ordering in I, the monoid ring $R[M]$ is called the **polynomial ring** on the commutative indeterminates $\{X_i \mid i \in I\}$.*
This polynomial ring is represented by $R[X_i \mid i \in I]$.

Proposition. 2.11. (Universal property of the commutative polynomial ring)
Let R be a ring and $\{X_i \mid i \in I\}$ be a set of indeterminates. For

- *any ring S,*
- *any ring map $f : R \to S$ and*
- *any map $g : \{X_i \mid i \in I\} \to S$, such that*
- *$g(X_i)g(X_j) = g(X_j)g(X_i)$ and $f(r)g(X_i) = g(X_i)f(r)$ for any $i, j \in I$, and any $r \in R$,*

there is a unique ring map $(f, g) : R[X_i \mid i \in I] \to S$ such that $g = u \circ (f, g)$.

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[dr, "f"'] & R[X_i \mid i \in I] \arrow[d, dashed, "(f,g)"] \\ & S \end{tikzcd}$$

and $(f, g)(X_i) = g(X_i)$ for every $i \in I$.

In the same way we may develop the same theory for

- polynomial rings of non–commutative indeterminates and
- polynomial rings of non–commutative indeterminates which do not commute with coefficients.

Let A be a commutative ring, an **algebra** over A is a ring R together a ring map $A \longrightarrow \mathrm{Cen}(R) \subseteq R$.

3 Modules

For any abelian group M the set $\mathrm{End}(M)$ of all abelian group endomorphisms of M has structure of ring with operations:

- $(f+g)(m) = f(m) + g(m)$, for any $m \in M$,
- $(f \circ g)(m) = f(g(m))$, for any $m \in M$, and
- $1 = \mathrm{id}_M$.

Thus we have the usual examples of ring: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the finite rings: $\mathbb{Z}_n$. Since $\mathrm{End}(M)$ is ubiquitous, we are interesting in relating these rings with $\mathrm{End}(M)$, thus we may consider a ring map $\theta : R \longrightarrow \mathrm{End}(M)$; in a natural way we have a left actinon of R on M, defined as $r \cdot m = \theta(r)(m)$ which allow us to study R through these endomorphism rings, and varying M obtain some properties of R. In this way we get the notion of left R–module, but, for technical reasons we prefer to consider right R–modules.

Let R be a ring. A **right R–module** is an abelian group M together with a **right action** $\alpha : M \times R \to M$ of R over M, such that, if we represent $\alpha(m,r)$ by mr, it satisfies the following properties:

(i) $(m_1 + m_2)r = m_1 r + m_2 r$, for any $r \in R$ and $m_1, m_2 \in M$,
(ii) $m(r_1 + r_2) = mr_1 + mr_2$, for any $r_1, r_2 \in R$ and $m \in M$,
(iii) $m(r_1 r_2) = (mr_1)r_2$, for any $r_1, r_2 \in R$ and $m \in M$,
(iv) $m1 = m$, for any $m \in M$.

This map α defines a ring map $\beta : R^{opp} \to \mathrm{End}(M)$ as follows:

$$\beta(r)(m) = mr.$$

Lemma. 3.1.
There is a bijective correspondence between maps α satisfying (i-iv) above and ring maps $\beta : R^{opp} \to \mathrm{End}(M)$.

As we point out before, we may define **left R–modules**. In this case we have a map $\alpha : R \times M \to M$ and a ring map $\beta : R \to \mathrm{End}(M)$. Observe that if M is a right R–module, then it is a left R^{opp}–module and viceversa. We write ${}_RM$, whenever M is a left R–module, and M_R if it is a right R–module.

Let R, S be rings and M a left R-module and a right S–module, hence we have product rm and ms for $r \in R$, $s \in S$ and $m \in M$. In some cases these two structures are compatible in the sense that $r(ms) = (rm)s$ for any r, s and m; in this case we say that M has a structure of $(R;S)$–module or a structure of two–sided $(R;S)$–module, and write ${}_RM_S$.

Let M, N be right R–modules, a **module map** $g : M \to N$ from M to N is an application which is an abelian group homomorphism and a homomorphism for the action of R over M and N, i.e., it satisfies:

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(i) $g(m_1+m_2)=g(m_1)+g(m_2)$, for any $m_1,m_2\in M$ and
(ii) $g(mr)=g(m)r$, for any $m\in M$ and $r\in R$.

A subset $N\subseteq M$ is a **submodule** if, with the operations in N defined by restriction of the operations in M, the inclusión $N\to M$ is a module map.

Lemma. 3.2.
Let M be a right R–module and $N\subseteq M$ a non-empty subset, the following statements are equivalent:
(a) N is a submodule of M.
(b) $n_1r_1+n_2r_2\in N$ for any $n_1,n_2\in N$ and any $r_1,r_2\in R$.

Example. 3.3.
Let $f:M\to N$ be a module map, then $\mathrm{Im}(f)$ is a submodule of N and $\mathrm{Ker}(f)$ is a submodule of M.

Example. 3.4.
We may consider R as right R–module via the canonical map $\rho:R\longrightarrow \mathrm{End}(R)$, defined $\rho(x)(r)=xr$, and as left R–module via the canonical map $\lambda:R\longrightarrow \mathrm{End}(R)$, defined $\lambda(x)(r)=xr$. Sometimes, these module structures on R are called the regular structures; R_R is the **regular right R–module**, and ${}_RR$ is the **regular left R–module**.

If we consider R as a right R–module, submodules of R are called **right ideals**. In the same way are defined **left ideals**. It is clear that every ideal is a right and a left ideal.

Kernel

Let $f:M\longrightarrow N$ be a module map, the inclusion $i:\mathrm{Ker}(f)\subseteq M$ is a submodule of M and satisfies a universal property with respect to homomorphisms that vanishes f; thus we have:

Lemma. 3.5. (Kernel universal property)
For any module map $h:X\longrightarrow M$ such that $fh=0$, there is a unique module map $h':X\longrightarrow \mathrm{Ker}(f)$ such that $h=ih'$, i.e., the following diagram commutes.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists_1 h'"'] \arrow[dr, "h"] & & \\ \mathrm{Ker}(f) \arrow[r, "i"] & M \arrow[r, "f"] & N \end{tikzcd}$$

Quotient module

Let $N \subseteq M$ be a submodule, we define an equivalence relation $\mathscr{R}_N$ in M as follows:

$$m_1 \mathscr{R}_N m_2 \text{ if } m_1 - m_2 \in N$$

An equivalence relation $\mathscr{R}$ in M is **compatible**, with the module structure of M, if it satisfies:

(i) If $m_1 \mathscr{R} m_2$ for any $m \in M$ we have $m_1 + m \mathscr{R} m_2 + m$.
(ii) If $m_1 \mathscr{R} m_2$ for any $r \in R$ we have $m_1 r \mathscr{R} m_2 r$.

The main result is the following:

Proposition. 3.6.
Let M be a right R–module, an equivalence relation $\mathscr{R}$ in M is compatible if and only if there exists a submodule $N \subseteq M$ such that $\mathscr{R} = \mathscr{R}_N$

It is obvious that if $\mathscr{R}$ is a compatible equivalence relation,the submodule N is defined as $N = \{m \in M \mid m\mathscr{R}0\}$.

Lemma. 3.7. (Quotient module)
Let $N \subseteq M$ be a submodule of right R–module M, in the quotient group M/N there is a unique module structure such that the projection $p : M \to M/N$ is a module map. This structure is defined as follows:

$$(m+N)r = (mr) + N$$

for any $m \in M$ and any $r \in R$.

Proposition. 3.8. (Quotient module universal property)
Let $N \subseteq M$ be a submodule, for any module map $f : M \to X$ such that $N \subseteq \mathrm{Ker}(f)$, i.e., $f(N) = 0$, there is a unique module map $f' : M/N \to X$ such that $f = f' \circ p$.

$$\begin{tikzcd}
N \arrow[r, "j"] & M \arrow[r, "p"] \arrow[dr, "f"'] & M/N \arrow[d, "\exists_1 f'"] \\
 & & X
\end{tikzcd}$$

For any module map $f : M \longrightarrow N$ the quotient $N/\operatorname{Im}(f)$ is called the **cokernel** of f, and the projection $p : N \longrightarrow M/\operatorname{Im}(f)$ satisfies a similar property to property in lemma (3.5.) of the kernel.

Exercise. 3.9. (Cokernel universal property)
For any module map $h : M \longrightarrow X$ such that $hf = 0$, there exists a unique modele maph$'$: $M/\operatorname{Im}(f) \longrightarrow X$ such that $h = h'p$, i.e., the following diagram commutes.

$$\begin{tikzcd}
N \arrow[r, "f"] & M \arrow[r, "p"] \arrow[dr, "h"'] & M/\operatorname{Im}(f) \arrow[d, dashed, "\exists_1 h'"] \\
 & & X
\end{tikzcd}$$

Submodule lattice

A **lattice** is a set L together with two binary operations $\wedge$ and $\vee$ satisfying the following properties:

(i) **Associative laws:** $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$.
(ii) **Commutative laws:** $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ for all $x, y \in L$.
(iii) **Absorption laws:** $x \wedge (y \vee x) = x$ and $x \vee (y \wedge x) = x$ for all $x, y \in L$.
(iv) **idempotent laws:** $x \wedge x = x$ and $x \vee x = x$ for all $x \in L$.

If $(L, \wedge, \vee)$ is a lattice we may define a partial order $x \leq y$ if $x = x \wedge y$, or equivalently $y = x \vee y$. In this case $x \wedge y$ is the infimum of $\{x, y\}$, and $x \vee y$ is the supremum of $\{x, y\}$. Thus, from a lattice we get a poset $(L, \leq$ in which every finite subset has supremum and infimum.

Conversely, if $(L, \leq)$ is a poset in which every finite subset has supremum and infimum and we define $x \wedge y = \inf\{x, y\}$, and $x \vee y = \sup\{x, y\}$, then $(L, \wedge, \vee)$ is a lattice.

Remark. 3.10.
Homomorphisms between lattices conserve the two operations: $\wedge$ and vee, and homomorphisms between posets are monotone maps. The followoing example shows that a monotone map is not necessarily a lattice map. Let $L_1 = \{a, b, c, d\}$, $L_2 = \{1, 2, 3, 4\}$ and $f : L_1 \longrightarrow L_2$ defined $f(a) = 4$, $f(b) = 3$, $f(c) = 2$ and $f(d) = 1$; we have that f is monotone but it is not a lattice map.

$$\begin{tikzcd}[column sep=small]
 & a \arrow[dl, no head] \arrow[dr, no head] & & & 4 \arrow[d, no head] \\
b \arrow[dr, no head] & & c \arrow[dl, no head] & & 3 \arrow[d, no head] \\
 & d & & & 2 \arrow[d, no head] \\
 & & & & 1
\end{tikzcd}$$

Let M be a right R–module, we denote by $\mathscr{L}(M)$ the set of all submodules of M and in it the partial order: $N_1 \leq N_2$ if $N_1 \subseteq N_2$; then $(\mathscr{L}(M), \leq)$ is a poset, hence a lattice. In addition, $\mathscr{L}(M)$ satisfies some other extra properties:

(v) $\mathscr{L}(M)$ has a **top**, M.
(vi) $\mathscr{L}(M)$ has a **bottom** $\{0\}$.
(vii) $\mathscr{L}(M)$ satisfies the **modular law**:

$$N_1 + (N_2 \cap N_3) = (N_1 + N_2) \cap N_3, \text{ whenever } N_1 \subseteq N_3.$$

Lemma. 3.11.
Let $\{N_i \mid i \in I\}$ be a family of submodules of a right R–module M, then $\cap\{N_i \mid i \in I\}$ is submodule.

In the set of all submodules of a right R–module M we define a partial order as follows: $N_1 \leq N_2$ if $N_1 \subseteq N_2$. Then the intersection is the infimum of a family. The supremum is the sum, and it is defined as the intersection of all submodules containing all members of the family.

Lemma. 3.12.
Let $\{N_i \mid i \in I\}$ be a family of submodules of a right R–module M, then $\sum\{N_i \mid i \in I\} = \cap\{N \subseteq M \mid N \supseteq N_i$ for all indices $i\}$ is a submodule of M.
There is a description of the elements of $\sum_i N_i$:

$$\sum_i N_i = \left\{ \sum_{i \in F \subseteq I} n_i \mid n_i \in N_i, F \text{ is finite} \right\}$$

Exercise. 3.13.
Let $X \subseteq M$ be a subset and let $\mathfrak{a} \subseteq R$ be a right ideal, we define

$$X\mathfrak{a} = \left\{ \sum x_i r_i \mid x_i \in X \text{ and } r_i \in \mathfrak{a} \right\}.$$

(1) $X\mathfrak{a}$ is a submodule of M.
(2) Let $\{N_\alpha \mid \alpha \in \Lambda\}$ be a family of submodules of a right R–module M and $\mathfrak{a}$ a right ideal of R, then $(\sum_\alpha N_\alpha)\mathfrak{a} = \sum_\alpha (N_\alpha \mathfrak{a})$.
(3) Let N be a submodule of a right R–module M and $\{\mathfrak{a}_\alpha \mid \alpha \in \Lambda\}$ be a family of right ideals of R, then $N(\sum_\alpha \mathfrak{a}_\alpha) = \sum_\alpha (N\mathfrak{a}_\alpha)$.

Ref.: 2101e_017 *SOLUTION*

SOLUTION. Exercise (3.13.)
HACER □

Exercise. 3.14.
Let M be a right R–module.

(1) Let $X \subseteq M$ be a subset, then XR is the smallest submodule containing X.
(2) Let $X \subseteq R$ be a subset, then RXR is the smallest ideal containing X.

Ref.: 2101e_016 *SOLUTION*

SOLUTION. Exercise (3.14.)
HACER □

A right R–module M is **finitely generated** is there exists a finite subset $X \subseteq M$ such that $M = XR$. If X is reduced to one element x, then M is called the **cyclic** right R–module generated by x.
A submodule $N \subseteq M$ is **maximal** if $N \neq M$ and for every submodule $N \subseteq X \subsetneq M$ we have $N = X$.

Lemma. 3.15.
Every non-zero finitely generated right R–module has a maximal submodule.

PROOF. Let $\Gamma = \{N \subseteq M \mid N \neq M\}$. Since $M \neq 0$, then $\Gamma \neq \varnothing$. It is clear that every chain in Γ has a upper bounded in Γ, hence Γ has maximal elements. Every maximal element in Γ is a maximal submodule. □

Corollary. 3.16.
Every ring R has a maximal right ideal.

PROOF. It is enough to consider that R is a cyclic right R–module. □

It is easy build module without maximal submodules. Let us show an example.

Example. 3.17.
Let $C = \mathbb{Q}/\mathbb{Z}$, and for any positive prime integer number p the submodule

$$M = \left\{ \frac{a}{b} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid \ b \text{ is a power of } p \right\}.$$

It satisfies:

(1) M contains a copy of $\mathbb{Z}_{p^t}$ for each $t \in \mathbb{N}$; it is enough to consider the subgroup generated by $\frac{1}{p^t} + \mathbb{Z}$.
(2) Each subgroup of M is one of the above.
(3) The proper subgroups of M form a chain without up bounded.

$$\{0\} \subsetneqq \left\langle \frac{1}{p} + \mathbb{Z} \right\rangle \subsetneqq \left\langle \frac{1}{p^2} + \mathbb{Z} \right\rangle \subsetneqq \left\langle \frac{1}{p^3} + \mathbb{Z} \right\rangle \subsetneqq \cdots$$

As a consequence, M has no maximal subgroups. The group M is represented by $\mathbb{Z}_{p^\infty}$, and it is called a **Prüfer group**.

The dual notion of maximal submodule is the notion of minimal submodule. A submodule $N \subseteq M$ is **minimal** if $N \neq 0$ and for any submodule $0 \neq X \subseteq N$ we have $X = N$.
Observe that the notion of minimal submodule can be applied to the whole module M, thus we have the notion of simple module, i.e., a module M is **simple** if $M \neq 0$ and it has no non–zero proper submodules.
In parallel to maximal submodules there are no non–zero modules which has no minimal submodules. An example is the abelian group $\mathbb{Z}$; observe that $\mathbb{Z}$ is even a finitely generated module.
one of the techniques to study rings and their modules consists in determining those modules which can be described by simple modules; in this line one of the necessary conditions will be to assure that these non–zero modules contain *enough* simple (minimal) submodules (Comment on the Prüfer group and finitely generated modules!). In this case the theory could be more easy as we have the following interesting result.

Proposition. 3.18. (Schur lema)
Let M be a simple right R–module, then $\mathrm{End}_R(M)$ *is a división ring.*

See also Lemma (16.16.).

4 The category Mod–R

The category **Mod**–R is the category of all right R–modules, thus its objects are the right R–modules and its morphisms are the module maps between right R–modules.

There are different types of module maps, and we can characterize them as follows.

Lemma. 4.1.
Let $f : M \to N$ be a module map between right R–modules,

(1) The following statements are equivalent:

- *(a) f is an injective map.*
- *(b) $\operatorname{Ker}(f) = 0$.*
- *(c) f is **left simplifiable**, i.e., if $g, h : N \to N'$ are module maps such that $f \circ g = f \circ h$, then $g = h$. (Module maps satisfying this property are called **monomorphisms**.)*

(2) The following statements are equivalent:

- *(a) f is a surjective map.*
- *(b) $\operatorname{Im}(f) = N$.*
- *(c) f is **right simplifiable**, i.e., if $g, h : M' \to M$ are module maps such that $g \circ f = h \circ f$, then $g = h$. (Module maps satisfying this property are called **epimorphisms**.)*

(3) The following statements are equivalent:

- *(a) f is a bijective map.*
- *(b) f has a (left and right) inverse map. (Module maps satisfying this property are called **isomorphisms**.)*

Exercise. 4.2.
Let

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "gf"'] & \bullet \arrow[d, "g"] \\ & \bullet \end{tikzcd}$$

be a commutative diagram of module maps.

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(1) If gf is monomorphism, show that f is monomorphism.
(2) If gf is epimorphism, show that g is epimorphism.

Ref.: 2101e_020 *SOLUTION*

SOLUTION. Exercise (4.2.)
HACER □

Proposition. 4.3. (First isomorphism theorem)
Let $f : M \to N$ be a module map between right R–modules, there is a factorization of f

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & N \\ M/\operatorname{Ker}(f) \arrow[r, "b"] & \operatorname{Im}(f) \arrow[u, "i"'] \end{tikzcd}$$

such that

- $p : M \to M/\operatorname{Ker}(f)$ *is the projection,*
- $i : \operatorname{Im}(f) \to N$ *is the inclusión and*
- $b : M/\operatorname{Ker}(f) \to \operatorname{Im}(f)$ *is an isomorphism defined by* $b(m + \operatorname{Ker}(f)) = f(m)$ *for any* $m \in M$.

In particular, there is a bijection between submodules of $\operatorname{Im}(f)$ *and submodules of* M *containing* $\operatorname{Ker}(f)$.

A sequence of module maps $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ is **exact** at M_2 if $\operatorname{Im}(f_1) = \operatorname{Ker}(f_2)$.
A sequence $\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$ is **exact** if it is exact at every M_i.
An exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is named a **short exact sequence**

Example. 4.4.
Let $N \subseteq M$ be a submodule, then $0 \longrightarrow N \xrightarrow{incl} M \xrightarrow{proy} M/N \longrightarrow 0$ is a short exact sequence.

Lemma. 4.5. (Cokernel universal property)
Let $0 \longrightarrow M_1 \stackrel{i}{\longrightarrow} M_2 \stackrel{p}{\longrightarrow} M_3 \longrightarrow 0$ *be a short exact sequence. For any module map* $f : M_2 \longrightarrow X$*, satisfying* $fi = 0$*, there is a unique module map* $f' : M_3 \longrightarrow X$ *such that* $f = f'p$*.*

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "i"] & M_2 \arrow[r, "p"] \arrow[dr, "f"'] & M_3 \arrow[r] \arrow[d, dashed, "\exists_1 f'"] & 0 \\
& & & X &
\end{tikzcd}$$

Lemma. 4.6. (Kernel universal property)
Let $0 \longrightarrow M_1 \stackrel{i}{\longrightarrow} M_2 \stackrel{p}{\longrightarrow} M_3 \longrightarrow 0$ *be a short exact sequence. For any module map* $f : X \longrightarrow M_2$*, satisfying* $pf = 0$*, there is a unique module map* $f' : X \longrightarrow M_1$ *such that* $f = if'$*.*

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "i"] & M_2 \arrow[r, "p"] & M_3 \arrow[r] & 0 \\
& X \arrow[u, dashed, "\exists_1 f'"] \arrow[ur, "f"'] & & &
\end{tikzcd}$$

Proposition. 4.7. (Second isomorphism theorem)
Let $N_1, N_2 \subseteq M$ *be submodules of a right R–module* M*, there exists an isomorphism*

$$\frac{N_1}{N_1 \cap N_2} \cong \frac{N_1 + N_2}{N_2}.$$

Proposition. 4.8. (Third isomorphism theorem)
Let $N_1 \subseteq N_2 \subseteq M$ *be submodules of a right R–module* M*. There exists an isomorphism*

$$\frac{M}{N_2} \cong \frac{M/N_1}{N_2/N_1}.$$

Let $f : M_1 \longrightarrow M_2$ be a module map, we have a commutative diagram with two short exact sequences, where b is the isomorphism map built in Proposition (4.3.).

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[dr, tail] & & & & & \frac{M_2}{\operatorname{Im}(f)} \\
& M_1 \arrow[rrr, "f"] \arrow[dr, two heads] & & & M_2 \arrow[ur, two heads] & \\
& & \frac{M_1}{\operatorname{Ker}(f)} \arrow[r, equal, "b"] & \operatorname{Im}(f) \arrow[ur, tail] & &
\end{tikzcd}$$

Sometimes, the quotient module $\frac{M_1}{\operatorname{Ker}(f)}$ es named the coimage of f, and $\frac{M_2}{\operatorname{Im}(f)}$ the cokernel of f.

Lemma. 4.9. (Kernel and cokernel lemma)
Let

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, "u"'] & M_2 \arrow[d, "v"] \\
N_1 \arrow[r, "g"'] & N_2
\end{tikzcd}$$

be a commutative diagram of modules, then we have:

(1) There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[dr, tail] \arrow[ddd, "u'"] & & & & & \frac{M_2}{\operatorname{Im}(f)} \arrow[ddd, "v''"] \\
& M_1 \arrow[rrr, "f"] \arrow[dr, two heads] \arrow[ddd, "u"] & & & M_2 \arrow[ur, two heads] \arrow[ddd, "v"] & \\
& & \frac{M_1}{\operatorname{Ker}(f)} \arrow[r, equal, "b"] \arrow[ddd, "u''"] & \operatorname{Im}(f) \arrow[ur, tail] \arrow[ddd, "v'"] & & \\
\operatorname{Ker}(g) \arrow[dr, tail] & & & & & \frac{N_2}{\operatorname{Im}(g)} \\
& N_1 \arrow[rrr, "g"] \arrow[dr, two heads] & & & N_2 \arrow[ur, two heads] & \\
& & \frac{N_1}{\operatorname{Ker}(g)} \arrow[r, equal, "b"] & \operatorname{Im}(g) \arrow[ur, tail] & &
\end{tikzcd}$$

(2) If u is monomorphism, then u' is monomorphism.

(3) If u is epimorphism, then u'' and v' are epimorphism.
(4) If v is monomorphism, then u'' and v' are monomorphism.
(5) If v is epimorphism, then v'' is epimorphism.

Lemma. 4.10. (Five lemma)
Let

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & M_5 \arrow[d, "h_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5
\end{tikzcd}$$

be a commutative diagrama with exact rows.

(1) If h_1 is epimorphism and h_2, h_4 are monomorphism, then h_3 is monomorphism.
(2) If h_5 is monomorphism and h_2, h_4 are epimorphism, then h_3 is epimorphism.

PROOF. (1). Let $m_3 \in M_3$ tal que $h_3(m_3) = 0$, then

$$h_4 f_3(m_3) = g_3 h_3(m_3) = 0, \text{ hence } f_3(m_3) = 0 \text{ and } m_3 \in \operatorname{Ker}(f_3) = \operatorname{Im}(f_2).$$

There exists $m_2 \in M_2$ such that $f_2(m_2) = m_3$. Then

$$g_2 h_2(m_2) = h_3 f_2(m_2) = h_3(m_3) = 0, \text{ hence } h_2(m_2) \in \operatorname{Ker}(g_2) = \operatorname{Im}(g_1).$$

There exists $n_1 \in N_1$ such that $g_1(n_1) = h_2(m_2)$, and there exists $m_1 \in M_1$ such that $h_1(m_1) = n_1$. Then

$$h_2 f_1(m_1) = g_1 h_1(m_1) = g_1(n_1) = h_2(m_2), \text{ hence } m_2 = f_1(m_1).$$

Thus, we obtain $m_3 = f_2(m_2) = f_2 f_1(m_1) = 0$, and h_3 is a monomorphism. □

Corollary. 4.11. (Short five lemma)
Ver álgebra básica.

Proposition. 4.12. (Snake lemma)
Ver álgebra básica.

Direct product and direct sum

Proposition. 4.13. (Direct product universal property)
Let $\{M_i \mid i \in I\}$ be a family of right R–modules, then in $\prod\{M_i \mid i \in I\}$ there is a unique right R–module structure such that each projection $p_j : \prod_i M_i \to M_j$ is a module map. The operations in $\prod_i M_i$ are defined componentwise.
In particular, for any right R–module N and any family $\{f_i : N \to M_i \mid i \in I\}$ there is a unique module map $f : N \to \prod_i M_i$ such that $f_i = p_i \circ f$ for any index $i \in I$.

$$\begin{tikzcd} N \arrow[d, dashed, "\exists_1 f"'] \arrow[dr, "f_i"] & \\ \prod_i M_i \arrow[r, "p_i"'] & M_i \end{tikzcd}$$

The dual construction of direct product is the direct sum.
Let $\{M_i \mid i \in I\}$ be a family of right R–módulos. For any element $(m_i)_i \in \prod_i M_i$ we define the **support** of $(m_i)_i$ as $\operatorname{Supp}((m_i)_i) = \{i \in I \mid m_i \neq 0\}$

Proposition. 4.14. (Direct sum universal property)
Let $\{M_i \mid i \in I\}$ be a family of right R–modules, then in the submodule

$$\begin{aligned} \oplus\{M_i \mid i \in I\} &= \left\{(m_i)_i \in \textstyle\prod_i M_i \mid \text{ almost all } m_i \text{ are zero}\right\} \\ &= \left\{(m_i)_i \in \textstyle\prod_i M_i \mid \operatorname{Supp}((m_i)_i) \text{ is finite}\right\} \end{aligned}$$

there is a unique right R–module structure such that each map $q_j : M_j \to \oplus_i M_i$, defined $q_j(x) = (x\delta_{i,j})_i$, is a module map.
In particular, for any right R–module N and any family $\{f_i : M_i \to N \mid i \in I\}$ there is a unique module map $f : \oplus_i M_i \to N$ such that $f_i = f \circ q_i$ for any index $i \in I$.

$$\begin{tikzcd} M_i \arrow[r, "q_i"] \arrow[dr, "f_i"'] & \oplus_i M_i \arrow[d, dashed, "\exists_1 f"] \\ & N \end{tikzcd}$$

Lemma. 4.15.
Let M, N be right R–modules, then $\mathrm{Hom}_R(M,N)$ has an abelian group structure, given by

$$(f+g)(m) = f(m) + g(m) \text{ for any } f, g \in \mathrm{Hom}_R(M,N) \text{ and any } m \in M.$$

Proposition. 4.16.
Let $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ be finite families of right R–modules, there exists an isomorphism of abelian groups

$$\mathrm{Hom}_R(\oplus_i M_i, \oplus_j N_j) \cong \oplus_{i,j} \mathrm{Hom}_R(M_i, N_j).$$

We may represent each element in $\prod_{i,j} \mathrm{Hom}_R(M_i, N_j)$ by a $J \times I$–matrix, being the element in the component (j,i) a map from $\mathrm{Hom}_R(M_i, N_j)$, the composition of homomorphisms corresponds to the matrix product.

If $M = N_1 \oplus N_2$ there is a ring isomorphism $\mathrm{End}_R(M) \cong \begin{pmatrix} \mathrm{Hom}_R(N_1,N_1) & \mathrm{Hom}_R(N_2,N_1) \\ \mathrm{Hom}_R(N_1,N_2) & \mathrm{Hom}_R(N_2,N_2) \end{pmatrix}$. In particular $\mathrm{End}_R(M)$ has two idempotent elements: $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The set $\{e_1, e_2\}$ is a complete set of orthogonal idempotent elements in $\mathrm{End}_R(M)$. The converse is also true, i.e., for any complete set $\{e_1, \ldots, e_t\}$ of orthogonal idempotent in $\mathrm{End}_R(M)$ there is a module isomorphism $M \cong e_1 M \oplus \cdots \oplus e_t M$.

Proposition. 4.17.
Let $\{N_1, \ldots, N_t\}$ be a finite family of submodules of a right R–module M, the following statements are equivalent:

(a) The map $\oplus_{i=1}^t N_i \to M$, defined $(n_1, \ldots, n_t) \mapsto \sum_{i=1}^t n_i$, is a module isomorphism.
*(b) $N_1 + \cdots + N_t = M$ and $N_i \cap (N_1 + \cdots + \widehat{N_i} + \cdots + N_t) = 0$ for every index $i = 1, \ldots, t$. (A family of submodules satisfying these properties is called **independent**.)*
(c) There is a complete set, $\{e_1, \ldots, e_t\}$, of orthogonal idempotent elements in $\mathrm{End}_R(M)$ such that $N_i = e_i M$ for every index $i = 1, \ldots, t$.

We may also consider in Proposition (4.16.) families with an infinite number of modules; in this case we have:

Proposition. 4.18.
Let $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ be families of right R–modules, there exists an isomorphism of abelian groups
$$\mathrm{Hom}_R(\oplus_i M_i, \prod_j N_j) \cong \prod_{i,j} \mathrm{Hom}_R(M_i, N_j).$$

Limits

Let $\mathscr{P}$ be a small category (a diagram), its class of morphism is a set, ans let $F : \mathscr{P} \longrightarrow$ **Mod**–R be a functor, a **limit** of F is a pair $(L, \{p_X\}_X)$, constituted by a module L and a family of morphisms $\{p_X : L \longrightarrow F(X) \mid X \in \mathscr{P}\}$ such that for any module map f in $\mathscr{P}$ the following diagrams commute

$$\begin{tikzcd} & F(X_1) \arrow[dd, "F(f)"] \\ L \arrow[ur, "p_{X_1}"] \arrow[dr, "p_{X_2}"'] & \\ & F(X_2) \end{tikzcd}$$

and for any other pair of the same type $(L', \{p'_X\}_X)$, there is a unique module map $f : L' \longrightarrow L$ making commutative the following diagrams, for any map f in $\mathscr{P}$

$$\begin{tikzcd} & & F(X_1) \arrow[dd, "F(f)"] \\ L' \arrow[r, dashed, "\exists_1 f"] \arrow[urr, "p'_{X_1}"] \arrow[drr, "p'_{X_2}"'] & L \arrow[ur, "p_{X_1}"] \arrow[dr, "p_{X_2}"'] & \\ & & F(X_2) \end{tikzcd}$$

The limit does not exist for any functor $F : \mathscr{P} \longrightarrow$ **Mod**–R, but, if it exists, it is unique up to isomorphism.

Lemma. 4.19.
*In the above situation, it there is a limit of $F : \mathscr{P} \longrightarrow$ **Mod**–R, if is unique up to isomorphism.*

Example. 4.20. (Direct product)
Let I be a set, we consider the category $\mathscr{I}$ whose objects are the elements of I and the only morphisms the identities. For any functor $F : \mathscr{I} \longrightarrow \mathbf{Mod}\text{–}R$, the limit is a pair $(\prod_i F(i), \{p_i\}_i)$, of a module $\prod_i F(i)$, the cartesian product with pointwise operation, and the family of projectios $\{p_i : \prod_i F(i) \longrightarrow F(i) \mid i \in I\}$, it is the direct product of the family of modules $\{F(i) \mid i \in I\}$.

Example. 4.21. (Kernel)
Let us consider the category $\mathscr{C}$ with two objects $X \underset{b}{\overset{a}{\rightrightarrows}} Y$, the identities and two morphisms: a y b. For any functor $F : \mathscr{C} \longrightarrow \mathbf{Mod}\text{–}R$ such that $F(b) = 0$ a limit is a pair $(L, \{p_X, p_Y\})$ that makes a commutative diagram

$$\begin{tikzcd}
& & F(X) \arrow[dd, shift right, "F(a)"'] \arrow[dd, shift left, "0"] \\
L' \arrow[r, dashed, "\exists_1 f"] \arrow[urr, "p'_X"] \arrow[drr, "p'_Y"'] & L \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\
& & F(Y)
\end{tikzcd}$$

therefore $F(a)p_X = 0$, and for any module map $p'_X : L' \longrightarrow F(X)$ such that $F(a)p'_X = 0$, there is a unique module map $f : L' \longrightarrow L$ such that $p'_X = p_X f$. Thus is, we have that (L, f) is the **kernel** of $F(a)$.

Colimits

Let $\mathscr{P}$ be a small category (a diagram), its class of morphism is a set, and let $F : \mathscr{P} \longrightarrow \mathbf{Mod}\text{–}R$ be a functor. A **colimit** of F is a pair $(\{q_X\}_X, C)$, consitutes by a module C and a family of module maps $\{q_X : F(X) \longrightarrow C \mid X \in \mathscr{P}\}$ such that for any map f in $\mathscr{P}$ the following diagram commutes

$$\begin{tikzcd}
F(X_1) \arrow[dd, "F(f)"'] \arrow[dr, "q_{X_1}"] & \\
& C \\
F(X_2) \arrow[ur, "q_{X_2}"'] &
\end{tikzcd}$$

and satisfying that for any other pair of the same type $(\{q'_X\}_X, C')$, there exists a unique module map $f : C \longrightarrow C'$ such that for any map f in $\mathscr{P}$, the diagram commutes

$$\begin{tikzcd}
F(X_1) \arrow[dd, "F(f)"'] \arrow[dr, "q_{X_1}"] \arrow[drr, "q'_{X_1}"] & & \\
& C \arrow[r, dashed, "\exists_1 f"] & C' \\
F(X_2) \arrow[ur, "q_{X_2}"'] \arrow[urr, "q'_{X_2}"'] & &
\end{tikzcd}$$

The colimit does not exist for any functor $F : \mathscr{P} \longrightarrow \mathbf{Mod}\text{–}R$, but, if it exists, it is unique up to isomorphism.

Lemma. 4.22.
In the above situation, if there exists a colimit of $F : \mathscr{P} \longrightarrow \mathbf{Mod}\text{–}R$, it is unique up to isomorphism.

Example. 4.23. (Direct sum)
Let I be a set, we consider the category $\mathscr{I}$ whose objects are the elements of I and the only morphisms the identities. For any functor $F : \mathscr{I} \longrightarrow \mathbf{Mod}\text{–}R$ the colimit is a pair $(\{q_i\}_i, \oplus_i F(i))$, of a module $\oplus_i F(i)$, the direct sum module of the family, and the family of inclusions $\{q_i : F(i) \longrightarrow \oplus_i F(i) \mid i \in I\}$; it is the direct sum of the family of modules $\{F(i) \mid i \in I\}$.

Example. 4.24. (Cokernel)
Let us consider the category $\mathscr{C}$ with two objects $X \overset{a}{\underset{b}{\rightrightarrows}} Y$, the identities and two morphisms: a y b. For any functor $F : \mathscr{C} \longrightarrow \mathbf{Mod}\text{–}R$ such that $F(b) = 0$, a colimit is a pair $(\{q_X, q_Y\}, C)$ that makes a commutative diagram

$$\begin{tikzcd}
F(X) \arrow[dr, "q_X"] \arrow[drr, "q'_X"] \arrow[dd, shift right, "F(a)"'] \arrow[dd, shift left, "0"] & & \\
& C \arrow[r, dashed, "\exists_1 f"] & C' \\
F(Y) \arrow[ur, "q_Y"'] \arrow[urr, "q'_Y"'] & &
\end{tikzcd}$$

therefore $q_Y F(a) = 0$, and for any module map $q'_Y : F(Y) \longrightarrow C'$ such that $q'_Y F(a) = 0$, there is a unique module map $f : C \longrightarrow C'$ such that $q'_Y = f q_Y$. Thus is, we have that (f, C) is the **cokernel** of $F(a)$.

Example. 4.25. (Direct limit)
Let $\mathscr{I}$ be a category, satisfying the following conditions:

- $\mathscr{I}$ is a small category.
- For any $i, j \in \mathrm{Obj}(\mathscr{I})$ we have $|\mathrm{Hom}_{\mathscr{I}}(i, j)| \leq 1$.
- For any $i, j \in \mathrm{Obj}(\mathscr{I})$, there exists $k \in \mathrm{Obj}(\mathscr{I})$ such that $|\mathrm{Hom}_{\mathscr{I}}(i, k)| = 1 |\mathrm{Hom}_{\mathscr{I}}(j, k)|$.

In this case we say $\mathscr{I}$ is a **upper directed small category**.

A **direct limit** in a category $\mathscr{C}$ is a colimit for a functor $F : \mathscr{I} \longrightarrow \mathscr{C}$, where $\mathscr{I}$ is a upper directed small category; the direct limit of F is represented by $\varinjlim F$.

If $\mathscr{C} = \mathbf{Mod}\text{–}R$, every functor $F : \mathscr{I} \longrightarrow \mathscr{C}$ determines a pair $(\{M_i\}_i, \{f_{i,j}\}_{i \leq j})$, where

- $i \in \mathrm{Obj}(\mathscr{I}) =: I$,
- $M_i = F(i)$, for any $i \in I$,
- $i \leq j$ if $|\mathrm{Hom}_{\mathscr{I}}(i, j)| = 1$,

- for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ y $j \leq k$,
- $f_{i,j}$ is the image of the only morphism in $\mathrm{Hom}_{\mathscr{I}}(i, j)$,.

In this case , $(\{M_i\}_i, \{f_{i,j}\}_{i,j})$ is called a **directed system of modules**, and the direct limit of $(\{M_i\}_i, \{f_{i,j}\}_{i,j})$ is denoted by $\varinjlim_i M_i$.

Lemma. 4.26.
For any directed system of modules there us a direct limit.

PROOF. First we build $\oplus_i M_i$, and the submodule N generated by the elements $(m_i)_i$ such that there exists two indices $i_1 \leq i_2$ such that $m_i = 0$ if $i \neq i_1, i_2$ and $j_{i_1,i_2}(m_{i_1}) = m_{i_2}$, i.e., the elements $(q_{i_2} f_{i_1,i_2} - q_{i_1})(x)$, with $x \in M_{i_1}$. For any pair of indices $i \leq j$ there exists a commutative diagram

$$\begin{tikzcd}
M_i \arrow[dd, "f_{i,j}"'] \arrow[dr, "q_i"] \arrow[drrr, "g_i"] & & & \\
& \oplus_i M_i \arrow[r, "p"] & \dfrac{\oplus_i M_i}{N} \arrow[r, "h"] & C \\
M_j \arrow[ur, "q_j"'] \arrow[urrr, "g_j"'] & & &
\end{tikzcd}$$

Let us assume there exists a pair $(\{g_i\}_j, C)$ making commutative the above diagram. By the universal property of the direct sum there exists $g : \oplus_i M_i \longrightarrow C$ such that $g_i = g q_i$. Since $g_i = g_j f_{i,j}$, we have $g q_i = g q_j f_{i,j}$, hence $g(q_i - g_j f_{i,j}) = 0$. Therefore, g factorizes thorough $\dfrac{\oplus_i M_i}{N}$, let $g = hp$.
We ha a commutative diagram

$$\begin{tikzcd}
M_i \arrow[dd, "f_{i,j}"'] \arrow[dr, "pq_i"] \arrow[drr, "g_i"] & & \\
& \dfrac{\oplus_i M_i}{N} \arrow[r, "h"] & C \\
M_j \arrow[ur, "pq_j"'] \arrow[urr, "g_j"'] & &
\end{tikzcd}$$

The uniqueness of h is obvious, hence $(\{pq_i\}_i, \frac{\oplus_i M_i}{N})$ is a direct limit of the directed system of modules $(\{M_i\}_i, \{f_{i,j}\}_{i \leq j})$. □

Example. 4.27. (Finitely generated submodules)
Every module M is the direct limit of the family of its finitely generated submodules.

Example. 4.28. (Prüfer group)
Let p be a positive prime integral number and $(\{\mathbb{Z}_{p^i}\}_{i\in\mathbb{N}}, \{f_{i,j}\}_{i\leq j})$ the directed system of abelian groups, where $f_{i,j} : \mathbb{Z}_{p^i} \longrightarrow \mathbb{Z}_{p^j}$ is defined as $f_{i,j}(1) = p^{j-i}$. The direct limit of this system is represented as $\varinjlim_n \mathbb{Z}_{p^n}$, and it can be identify with $\mathbb{Z}_{p^\infty} = \{\frac{a}{p^n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$.

Functors

Let M be a right R–module and $f : T \to R$ be a ring map, the composition

$$T \xrightarrow{f} R \longrightarrow \mathrm{End}(M)$$

induces a right T–module structure on M. Indeed, $mt = mf(t)$.

Thus we have a functor U_f : **Mod**–$R \longrightarrow$ **Mod**–T, the **change ring functor**. Since, for every rings R there is a ring map $\mathbb{Z} \longrightarrow R$, then there is a functor U : **Mod**–$R \longrightarrow \mathscr{A}b$ that sends every right R module M to the underlying abelian group of M, and every module map to the underlying abelian group map.

We may build many other functors, thus, for every right R–module M, there is a functor $\mathrm{Hom}_R(M,-)$: **Mod**–$R \longrightarrow \mathscr{A}b$, where, for any right R–module N the operation in $\mathrm{Hom}_R(M,N)$ is defined $(h_1 + h_2)(m) = h_1(m) + h_2(m)$. For any module map $f : N_1 \longrightarrow N_2$ there is an abelian group map $\mathrm{Hom}_R(M,f) = f_* : \mathrm{Hom}_R(M,N_1) \longrightarrow \mathrm{Hom}_R(M,N_2)$ defined by $f_*(h) = fh$, for any $h \in \mathrm{Hom}_R(M,N_1)$. We have that $\mathrm{Hom}_R(M,-)$ is a functor. Indeed,

(1) If $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ are module maps, we have $(f_2f_1)_* = (f_2)_*(f_1)_*$.
(2) For any right R–module N we have $(\mathrm{id}_N)_* = \mathrm{Hom}_R(M,\mathrm{id}_N) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$.

In the same way, for every right R–module M we can consider $\mathrm{Hom}_R(-,M)$ defined from **Mod**–R to $\mathscr{A}b$. But for every module map $f : N_1 \longrightarrow N_2$ there is an abelian group map $\mathrm{Hom}_R(f,M) = f^*$: $\mathrm{Hom}_R(N_2,M) \longrightarrow \mathrm{Hom}_R(N_2,M)$, defined as $f^*(h)) = hf$. Thus $\mathrm{Hom}_R(-,M)$ is not a functor as it reserves the maps. However, it is <u>contravariant functor</u> as it satisfies the following two properties:

(1) If $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ are module maps, we have $(f_2f_1)^* = (f_1)^*(f_2)^*$.
(2) For any right R–module N we have $(\mathrm{id}_N)^* = \mathrm{Hom}_R(\mathrm{id}_N, M) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$.

Let us consider a set X and the free right R–module $L(X) = R^{(X)}$, i.e., the right R–module of all finite formal R–lineal combinations $\sum_x a_x x$, where $a_x \in R$, with sum defined componentwise and action of R defined as $(\sum_x a_x x)a = \sum_x (a_x a)x$. For any map $f : X \longrightarrow Y$ there is a module map $\overline{f} : R^{(X)} \longrightarrow R^{(Y)}$ defined by $\overline{(}\sum_x a_x x) = \sum_x a_x f(x)$. In this way, we have functor $L : \mathscr{S}et \longrightarrow$ **Mod**–R which is named the **free module functor**.

Let us consider a new module associated to a set X, in this case we consider $R^X = \{h \mid h : X \longrightarrow R\}$ with sum defined componentwise, and action defined $(fa)(x) = f(x)a$. First observe that R^X is a right R–module. For any map $f : X \longrightarrow Y$ we may define a map $R^f : R^Y \longrightarrow R^X$ as $R^f(h) = hf$; it is a module map. Thus R^- defines a contravariant functor from $\mathscr{S}et$ to **Mod**–R.

Remark. 4.29.
Which is the difference between R^X and $R^{(X)}$? Remember that

$$R^{(X)} = \{h \mid h : X \longrightarrow R \text{ such that only finitely many of } f(x) \text{ are nonzero}\}.$$

Thus $R^{(X)}$ is a submodule of R^X. Actually, R^X is the **direct product** of X copies of R, and $R^{(X)}$ is the **direct sum** of X copies of R. You may check that $R^{(-)}$, defined in the obvious way, is nor a functor neither a contravariant functor from $\mathscr{S}et$ to **Mod**–R.

Change of ring

Let $f : R \longrightarrow S$ be a ring map, then every right S–module is a right R–module , and there exists a functor: $\mathscr{U} : \mathbf{Mod}\text{–}S \longrightarrow \mathbf{Mod}\text{–}R$. In this case there are two adjunctions:

$$\begin{tikzcd} \mathbf{Mod}\text{–}S \arrow[d, shift right, "-\otimes_R S"'] \\ \mathbf{Mod}\text{–}R \arrow[u, shift right, "\mathscr{U}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "\mathscr{U}"'] \\ \mathbf{Mod}\text{–}S \arrow[u, shift right, "\mathrm{Hom}_R(S,-)"'] \end{tikzcd}$$

In this case $- \otimes_R S$ preserves projective modules and $\mathrm{Hom}_R(S, -)$ preserves injective modules.
A right S–module P is projective if we can complete any diagram with exact row.

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed] & \\ M \arrow[r, "p"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

Dually, a right R–module E is injective if we can complete any diagram with exact row.

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "j"] \arrow[d, "f"'] & M \arrow[dl, dashed] \\ & E & \end{tikzcd}$$

Lemma. 4.30.
(1) Let $\begin{tikzcd} \mathbf{Mod}\text{–}S \arrow[d, shift right, "F"'] \\ \mathbf{Mod}\text{–}R \arrow[u, shift right, "U"'] \end{tikzcd}$ *be an adjunction such that U preserves epimorphisms, i.e., U is an exact functor, then F preserves projective modules.*
(2) Let $\begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "U"'] \\ \mathbf{Mod}\text{–}S \arrow[u, shift right, "G"'] \end{tikzcd}$ *be an adjunction such that U preserves monomorphism, i.e., U is an exact functor, then G preserves injective modules*

PROOF. Let P be a right R–module, for any diagram of right S–modules

$$\begin{tikzcd} & FP \arrow[d, "f"] & \\ M \arrow[r, "p"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

if we apply U we have a diagram of right R–modules and there is a module map, g, that completes it.

$$\begin{tikzcd} P \arrow[r, "\varepsilon_P"] \arrow[d, dashed, "g"'] & UFP \arrow[d, "Uf"] & \\ UM \arrow[r, "Up"'] & UM'' \arrow[r] & 0 \end{tikzcd}$$

this is, $Up \circ g = Uf \circ \varepsilon_P$.

By the adjunction isomorphism η, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(FP, M) \arrow[r, "\eta_{P,M}"] \arrow[d, "p_*"'] & \mathrm{Hom}_R(P, UM) \arrow[d, "(Up)_*"] \\ \mathrm{Hom}_S(FP, M'') \arrow[r, "\eta_{P,M''}"] & \mathrm{Hom}_R(P, M'') \end{tikzcd}$$

Hence,

$$\eta(p \circ \eta^{-1}(g)) = Up \circ U\eta^{-1}(g)\varepsilon_P = Up \circ \eta\eta^{-1}(g) = Up \circ g.$$

On the other hand,

$$\eta(f) = Uf \circ \varepsilon_P.$$

Thus, $f = p \circ \eta^{-1}(g)$, and FP is projective. □