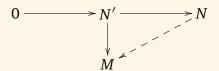
# 52 Injective modules

A right *R*-module *M* is **injective** if the functor  $\operatorname{Hom}_R(-,M)\operatorname{Mod}-R \longrightarrow \mathscr{A}b$ , is exact.

## Proposition. 52.1.

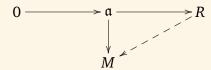
For any ring R and any right R-module M the following statements are equivalent:

- (a) M is injective.
- (b) Every diagram with exact row



can be completed to a commutative one.

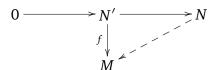
(c) (Baer's Lemma) For any right ideal a every diagram with exact row



can be completed to a commutative one.

PROOF. (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c). They are obvious.

(c)  $\Rightarrow$  (b). Let us consider the following diagram with exact row



We define  $\Gamma = \{(H,h) \mid N' \subseteq H \subseteq N, h : H \to M, \text{ and } h_{|N'} = f\}$ . This set is inductive, i.e., it is nonempty and, with the partial order;  $(H_1,h_1) \leq (H_2,h_2)$  if  $H_1 \subseteq H_2$  and  $h_{2|H_1} = h_1$ , every ascending chain has a upper bound. By Zorn's lemma there exists a maximal element in  $\Gamma$ , say (H,h).

If  $H \neq N$  there exists  $x \in N \setminus H$ . We consider H + Rx, and define  $\mathfrak{a} = (H : x) = \{a \in R \mid ax \in H\}$  and  $h' : \mathfrak{a} \longrightarrow M$  as h'(a) = h(ax). It is obvious that h' is a homomorphism, hence it can be extended to  $h'' : R \longrightarrow M$ . We take m = h''(1).

Now we define  $h_x: H + Rx \longrightarrow M$  as  $h_x(y + ax) = h(y) + am$ , for any  $y \in H$  and  $a \in R$ . This map is well define and a homomorphism. Indeed, if y + am = y' + a'm, then h(y) - h(y') = h(y - y') = h(y - y')

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h((a'-a)x) = h'(a'-a) = (a'-a)m, hence h(y) + am = h(y') + a'm. The existence of the pari (H + Rx, h'') contradices the maximality of (H, h).

## Proposition. 52.2.

For any ring R, and any family of right R–modules  $\{M_i \mid i \in I\}$ , the following statements are equivalent:

- (a)  $\prod_i M_i$  is injective.
- (b) Every  $M_i$  is injective.

PROOF. First we observe that for any right *R*–module *X* there exists a natural isomorphism

$$\operatorname{Hom}_A(X, \prod_i M_i) \cong \prod_i \operatorname{Hom}_A(X, M_i).$$

In addition, for any family of homomorphisms  $\{f_i: X_i \longrightarrow Y_i \mid i \in I\}$  we have  $\prod_i f_i: \prod_i X_i \longrightarrow \prod_i Y_i$  is surjective if, and only if, every  $f_i$  is surjective. The result follows from these two observations.

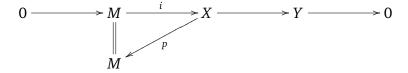
There are many different characterizations of injective modules, in the next proposition we collect some of them.

#### Proposition. 52.3.

Let M be a right R-modulo, the following statements are equivalent:

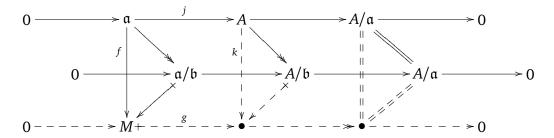
- (a) M is injective.
- (b) Every short exact sequence  $0 \to M \to X \to Y \to 0$  splits.
- (c) Every short exact sequence  $0 \to M \to X \to Y \to 0$ , with Y a cyclic right R-module, splits.

PROOF. (a)  $\Rightarrow$  (b). It is enough to consider the diagram with exact row



By the hypothesis there exists a map p such that  $pi = \mathrm{id}_M$ , hence M is a direct summand. (b)  $\Rightarrow$  (c). It is obvious.

(c)  $\Rightarrow$  (a). For any right ideal  $j : \mathfrak{a} \subseteq R$ , and any map  $f : \mathfrak{a} \longrightarrow M$ , with kernel  $\mathfrak{b} = \operatorname{Ker}(f)$ , we may build the following commutative diagram with exact rows



By the hypothesis g has an inverse h on the right:  $hg = \mathrm{id}_M$ . Therefore f = hkj as we have gf = ghgf = ghkj and g is monomorphism.

In particular, from this characterization we obtain:

## Corollary. 52.4.

For any ring R the following statements are equivalent:

- (a) Every short exact sequence of right R-modules splits.
- (b) Every right R-module is injective.

Observe that the conditions in Corollary (52.4.) characterize (artinian) semisimple rings. Compare with Corollary (49.4.).

# 53 Divisible abelian groups

We shall study the behaviour of injective abelian groups. First we point out that injective abelian groups have more interesting properties than injective modules over a general ring R.

Let R be a ring, a right R-module M is **divisible** if for any non-zero element  $r \in R$  and any  $m \in M$ , such that  $m \operatorname{Ann}(r) = 0$ , there exists  $m' \in M$  such that m = m'r. It is clear that every injective right R-module is divisible. The converse is true in some special cases, for instance, if R is a right principal ideal ring. We are interested in divisible modules over  $\mathbb{Z}$ , hence we restrict ourselves to domains. If R is not a domain some of the following results are not true.

#### Proposition. 53.1.

For any abelian group M the following statements are equivalent:

- (a) M is injective.
- (b) M is divisible.

PROOF. Since  $\mathbb{Z}$  is a principal ideal domain, every ideal is principal; by Baer's lemma we have the result.

## Proposition. 53.2.

Every homomorphic image of a divisible abelian group is divisible.

PROOF. Let M be a divisible abelian group, and  $N \subseteq M$ , for any  $m \in M$ , and any  $r \in R$  there exists  $m' \in M$  such that m = m'r, hence m + N = (m' + N)r.

## Proposition. 53.3.

For any family of abelian groups  $\{M_i \mid i \in I\}$  the following statements are equivalent:

- (a)  $\bigoplus_i M_i$  is divisible (= injective).
- (b) Every  $M_i$  is divisible (= injective).

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PROOF. Let  $(m_i)_i \in \bigoplus_i M_i$ , and  $0 \neq n \in \mathbb{Z}$ . For any index i there exists  $m_i' \in M_i$  such that  $m_i = nm_i'$ . Therefore we have  $(m_i)_i = n(m_i')_i$ , and  $\bigoplus_i M_i$  is divisible. The converse holds from the previous proposition.

## Example. 53.4.

Every field of characteristic zero is a divisible abelian group. Indeed, if K is a field of characteristic zero, for any  $k \in K$ , and any  $0 \ne n \in \mathbb{Z}$  we can take  $k' = \frac{k}{n}$ .

## Example. 53.5.

The abelian group  $\mathbb{Q}/\mathbb{Z}$  is a divisible, hence injective.

#### Remark. 53.6.

Observe that if K is a field of non–zero characteristic p, then K is not a divisible abelian group. In this case if we take  $m = 1 \in K$  and  $n = p \in \mathbb{Z}$ , there does not exist any  $k \in K$  such that 1 = pk, as the right part in the above identity is zero.

#### Lemma. 53.7.

 $\mathbb{Q}/\mathbb{Z}$  is an injective cogenerator of  $\mathcal{A}b$ .

PROOF. Let M be an abelian group, for any  $0 \neq m \in M$  the subgroup  $\langle m \rangle$  has a homomorphic image isomorphic to a cyclic group, hence to a finite cyclic group, and there exists a nonzero map  $f: \langle m \rangle \longrightarrow \mathbb{Q}/\mathbb{Z}$  which can be extended to a map  $f': M \longrightarrow \mathbb{Q}/\mathbb{Z}$  satisfying  $f'(m) \neq 0$ .

#### Corollary. 53.8.

Every abelian group is isomorphic to a submodule of an injective abelian group.

Indeed, every abelian group is isomorphic to a submodule of a direct product of copies of  $\mathbb{Q}/\mathbb{Z}$ .

#### Exercise. 53.9.

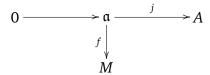
Let *D* be a commutative integral domain. For any torsionfree *D*–module *M* the following statements are equivalent:

- (a) M is injective.
- (b) M is divisible.

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SOLUTION. Ejercicio (53.9.)

Let us consider a diagram with exact row



We define a family  $\Gamma = \{(\mathfrak{b},g) \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq D, \text{ and } g_{|\mathfrak{a}} = f\}$ . In  $\Gamma$  we define the partial order  $(\mathfrak{b}_1,g_1) \leq (\mathfrak{b}_2,g_2)$  whenever  $\mathfrak{b}_1 \subseteq \mathfrak{b}_2$  and  $g_{2|\mathfrak{b}_1} = g_1$ . Since  $\Gamma$  is a non–empty inductive set, by Zorn's lemma, it has maximal elements. Let  $(\mathfrak{b},g) \in \Gamma$  maximal. If  $\mathfrak{b} \neq D$ , there exists  $r \in R \setminus \mathfrak{b}$ . If  $Dr \cap \mathfrak{b} = 0$ , then  $(\mathfrak{b},g)$  is not maximal, which is a contradiction.

If  $Dr \cap b \neq 0$  there exists  $s \in R$  such that  $0 \neq sr \in b$ , hence we may define a map  $h : Ds \longrightarrow M$  by h(ds) = g(dsr), for any  $d \in D$ , which can be extended to a map  $h' : D \longrightarrow M$ ; thus there exists  $m \in M$  such that m = h'(1), and g(sr) = h(s) = h'(s) = sm. Now we define a new map  $g' : b + Dr \longrightarrow M$  as follows:

$$g'(b) = g(b)$$
, for any  $b \in \mathfrak{b}$ ,  $g'(dr) = dm$ , for any  $d \in D$ .

It is necessary to check that g' is well defined and that it is a homomorphism.

If  $d_1r = d_2r$ , then  $(d_1 - d_2)r = 0$ , then  $d_1 - d_2 = 0$ , as D is an integral domain, and  $d_1 = d_2$ , hence  $g'(d_1r) = d_1m = d_2m = g'(d_2r)$ . If  $0 \neq dr \in \mathfrak{b}$ , we proceed as before to find an element  $m' \in M$  such that g(dr) = dm'; now, we may consider  $0 \neq sdr \in \mathfrak{b}$ , which satisfies:

$$sdm = dg(sr) = g(dsr) = sg(dr) = sdm',$$

hence sd(m-m')=0, and m=m', as M is torsionfree.

# 54 Existence of enough injective objects in module categories

## Proposition. 54.1.

For any ring R the right R-module  $\operatorname{Hom}(R,\mathbb{Q}/\mathbb{Z})$  is an injective cogenerator of **Mod**-R.

PROOF. Let *M* be a right *R*–module, then there are isomorphism:

$$\operatorname{Hom}_R(M, \operatorname{Hom}(R, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}(M \otimes_R R, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z}).$$

Now, since  $\mathbb{Q}/\mathbb{Z}$  is an injective cogenerator of  $\mathcal{A}b$ , it is clear that  $\operatorname{Hom}(R,\mathbb{Q}/\mathbb{Z})$  is an injective cogenerator of  $\operatorname{\mathbf{Mod}}-R$ .

To prove that  $\operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z})$  is injective, let us consider a diagram with exact row

There exists a natural commutative diagram

The result holds as  $\mathbb{Q}/\mathbb{Z}$  is an injective abelian group.

PROOF. We may also consider the adjunction

$$Mod-R$$
 $\mathscr{U} \downarrow hom_R(R,-)$ 
 $\mathscr{A} b$ 

The unity is  $\varepsilon_M: M \longrightarrow \operatorname{Hom}(R, \mathscr{U}(M))$ , defined  $\varepsilon_M(m)(r) = mr$ . It is a monomorphism. If D is a divisible abelian group such that  $0 \to \mathscr{U}(M) \to D$ , then there is a monomorphism  $M \xrightarrow{\varepsilon_M} \operatorname{Hom}(R, \mathscr{U}(M)) \xrightarrow{\eta_*} \operatorname{Hom}(R, D)$ . Since D is a divisible abelian group. then  $\operatorname{Hom}(R, D)$  is an injective right R-module.  $\square$ 

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## Corollary. 54.2.

For any ring R every right R-module is a submodule of an injective right R-module.

PROOF. This result holds as  $\operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z})$  is an injective cogenerator, hence every right R-module is a submodule of a direct product of copies of it.

As every right R-module is a submodule of an injective module, we say that the category  $\mathbf{Mod}$ -R has **enough injective objects**.

## **Appendix**

Study of the adjunction

$$\mathbf{Mod}-R$$

$$\mathbf{V} \mid \mathsf{Hom}(R,-)$$

$$\mathbf{A}b$$

There is a bijection

$$\theta = \theta_{A,B} : \text{Hom}(\mathcal{U}(A), B) \cong \text{Hom}_R(A, \text{Hom}(R, B))$$

defined

$$\theta(\psi)(a)(r) = \psi(ar) = [(\theta(\psi))(a)](r)$$

with inverse

$$\theta^{-1}(\varphi)(a) = \varphi(a)(1) = [\theta^{-1}(\varphi)](a).$$

Let us show that  $\theta$  is natural in A and in B. This means that for every pair of homomorphisms:  $f: A_2 \longrightarrow A_1$  and  $g: B_1 \longrightarrow B_2$  we have a commutative diagram

$$\operatorname{Hom}(\mathscr{U}(A_1), B_1) \xrightarrow{\theta_1} \operatorname{Hom}_R(A, \operatorname{Hom}(R, B))$$

$$\mathscr{U}(f)^* g_* \Big|_{f^* \operatorname{Hom}(R, g)_*} \Big|_{\theta_2} \operatorname{Hom}_R(A, \operatorname{Hom}(R, B))$$

Let us write  $\mathcal{U}(f)$  as f, and Hom(R,g) as  $g_*$ .

Given  $\psi \in \text{Hom}(\mathcal{U}(A_1), B_1)$ , we have:

$$(\theta_2 f^* g_*(\psi))(a_2)(r) = (\theta_2 f^*(g\psi))(a_2)(r) = (\theta_2(g\psi f))(a_2)(r) = (g\psi f)(a_2 r).$$

$$(f^*(g_*)_*\theta_1(\psi))(a_2)(r) = (g_*\theta_1(\psi)f)(a_2)(r) = (g_*\theta_1(\psi))(f(a_2))(r)$$

$$= (g_*(\theta_1(\psi)(f(a_2))))(r) = g(\theta_1(\psi)(f(a_2))(r))$$

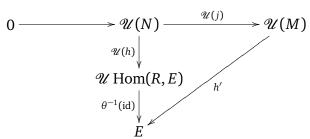
$$= g(\psi(f(a_2)r))$$

Let E be an injective abelian group, then  $\operatorname{Hom}(R, E)$  is an injective right R-module. We consider the diagram

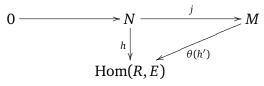
$$0 \xrightarrow{\qquad \qquad N \xrightarrow{\qquad \qquad j \qquad \qquad } M$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

Applying  $\mathscr{U}$ , and the isomorphism  $\theta = \theta_{\operatorname{Hom}(R,E),E} : \operatorname{Hom}(\mathscr{U} \operatorname{Hom}(R,E),E) \cong \operatorname{Hom}_R(\operatorname{Hom}(R,E),\operatorname{Hom}(R,B))$ , we have:



Since E is an injective abelian group, there exists h' completing the diagram:  $\theta^{-1}(\mathrm{id})\mathscr{U} = h'\mathscr{U}(j)$ . Using the isomorphism  $\theta = \theta_{M,E} : \mathrm{Hom}(\mathscr{U}(M), E) \cong \mathrm{Hom}_R(M, \mathrm{Hom}(R, E))$ , we have  $\theta(h') : M \longrightarrow \mathrm{Hom}(R, E)$ .



To show that this diagram is commutative, we may use the isomorphism  $\theta = \theta_{N,E}$ : Hom $(\mathcal{U}(N), E) \cong \text{Hom}_R(N, \text{Hom}(R, E))$ , to see that  $\theta^{-1}(h) = \theta^{-1}(\theta(h')j)$ . Indeed, for every  $n \in N$  we have:

$$\theta^{-1}(h)(n) = h(n)(1).$$

$$\theta^{-1}(\theta(h')j)(n) = (\theta(h')j)(n)(1) = \theta(h')(j(n))(1) = h'(j(n)) = (h'j)(n)$$

$$= (\theta^{-1}(\mathrm{id})h)(n) = \theta^{-1}(\mathrm{id})(h(n)) = \mathrm{id}(h(n))(1) = h(n)(1).$$

Thus,  $\operatorname{Hom}(R, E)$  is an injective right R-module derecha invectivo. In particular,  $\operatorname{Hom}(R, \mathbb{Q}/\mathbb{Z})$  is an injective right R-module.

To finish, let us see that every right R-module is contained in an injective right R-module. Let M be a right R-module, the abelian group (U)(M) is contained in a direct product of copies of  $\mathbb{Q}/\mathbb{Z}$ , let  $\mathscr{U}(M) \stackrel{j}{\longrightarrow} \prod \mathbb{Q}/\mathbb{Z}$ . Using the isomorphism  $\theta = \theta_{M,\mathscr{U}(M)} : \operatorname{Hom}(\mathscr{U}(M),\mathscr{U}(M)E) \cong \operatorname{Hom}_R(M,\operatorname{Hom}(R,\mathscr{U}(M)))$ , we have a sequence of homomorphisms

$$M \xrightarrow{\theta(\mathrm{id})} \mathrm{Hom}(R, \mathscr{U}(M)) \xrightarrow{j_*} \mathrm{Hom}(R, \prod \mathbb{Q}/\mathbb{Z})$$

each one of which is an injective map. For instance, we have  $\theta(id)(m)(1) = id(m)$ , hence  $Ker(\theta(id)) = \{0\}$ .

# 55 Essential extensions

Let  $N \subseteq M$  be a submodule of a right R-module M, we say N is **essential** in M or that M is an **essential extension** of N if for any non-zero submodule  $H \subseteq M$  we have  $N \cap H \neq 0$ .

Instead of an inclusion we may consider a monomorphism  $j: N \longrightarrow M$ ; if the image of j is essential in M it is called an **essential monomorphism**; we also say that j is an **essential extension**; an essential extension j in **proper** if j is not an isomorphism.

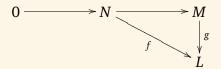
We write  $N \subseteq^e M$  to represent that N is an essential submodule of M.

A non-zero right *R*–module *M* is called **uniform** if every non-zero submodule is essential.

#### Lemma. 55.1.

Let  $N \subseteq M$  a submodule, the following statements are equivalent:

- (a) N is an essential submodule.
- (b) For every submodule  $H \subseteq M$ , if  $N \cap H = 0$ , then H = 0.
- (c) For every  $0 \neq m \in M$  there exists  $r \in R$  such that  $0 \neq mr \in N$ .
- (d) For every right R-module L and any pair of homomorphisms f, g



such that  $g_{|N} = f$ , if f is a monomorphism, then g is a monomorphism.

#### Proposition. 55.2.

Let M be a right R-module, the following statements hold:

- (1) If  $N \subseteq H \subseteq M$  are submodules the statements are equivalent:
  - (a)  $N \subseteq^e M$  is essential.
  - (b)  $N \subseteq^e H \subseteq^e M$  are essential.
- (2) If  $N_i \subseteq^e H_i \subseteq M$ , for i = 1, 2, then  $N_1 \cap N_2 \subseteq^e H_1 \cap H_2$ .
- (3) If  $f: M \longrightarrow M'$  is a homomorphism, for any essential submodule  $N' \subseteq^e M'$  we have  $f^{-1}(N') \subseteq^e M$  is essential.
- (4) For  $\{N_i \mid i \in I\}$  and  $\{H_i \mid i \in I\}$ , independent families of submodules of M, such  $N_i \subseteq^e H_i$  is essential for every index  $i \in I$ , we have  $\bigoplus_i N_i \subseteq \bigoplus_i H_i$  is essential.

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#### Remark. 55.3.

Observe that in (2) the intersection necessarily is finite as the following example shows. Let us consider  $\{n\mathbb{Z} \mid n \in \mathbb{N}^*\}$  be the family of all non–zero subgroups of  $\mathbb{Z}$ ; each of them is essential in  $\mathbb{Z}$ , and its intersection  $\cap_{n\neq 0} n\mathbb{Z} = 0$  is not essential in  $\mathbb{Z}$ .

#### Remark. 55.4.

Observe that in (4) the families must be independent; let us consider the abelian group  $M = \mathbb{Z} \oplus \mathbb{Z}_2$  and the submodules:

$$N_1 = \langle (2,0) \rangle \subseteq^e H_1 = \langle (1,0) \rangle,$$
  

$$N_2 = \langle (2,0) \rangle \subseteq^e H_2 = \langle (1,1) \rangle.$$

Then  $N_1 + N_2 = N_1$  and  $H_1 + H_1 = M$ , and  $((2,0)) \subseteq M$  is not essential.

The most important result in this section is the following.

### Proposition. 55.5.

Let M be a right R-module, for any submodule  $N \subseteq M$  there exists a submodule  $N \subseteq H \subseteq M$  maximal among those satisfying  $N \subseteq^e H$ .

PROOF. We define  $\Gamma = \{ H \subseteq M \mid N \subseteq^e H \subseteq M \}$ . Since  $N \in \Gamma$ , then  $\Gamma \neq \emptyset$ . On the other hand, every chain of elements in  $\Gamma$  has a upper bound in  $\Gamma$ . Therefore, by Zorn's lemma, there are maximal elements in  $\Gamma$ .

Our interest now is to study these maximal essential extensions.

If M is a right R-module and  $M = N_1 \oplus N_2$ , we say  $N_2$  is a **complement of**  $N_1$  in M; i.e.,  $N_1 \cap N_2 = 0$  and  $N_1 + N_2 = M$ .

For any submodule  $N \subseteq M$  of a right R-module M; a **pseudo-complement of** N in M is any submodule  $L \subseteq M$  maximal among those satisfying  $N \cap L = 0$ .

## Remark. 55.6.

Observe that for any submodule  $N \subseteq M$  we have:

- (1) Any complement of *N* in *M* is a pseudo–complement.
- (2) Complements are not uniquely determined.
- (3) Pseudo-complements are not uniquely determined.

## Lemma. 55.7.

Let  $N \subseteq M$  be a submodule of a right R-module M with pseudo-complement H, then

- (1)  $N \oplus H \subseteq^{e} M$  is an essential submodule.
- (2)  $\frac{N+H}{H} \subseteq^{e} \frac{M}{H}$  is essential.

PROOF. (1). Let  $0 \neq X \subseteq M$  such that  $(N+H) \cap X = 0$ , if  $N \cap (H+X) \neq 0$ , there exist  $n = h + x \neq 0$ , hence x = n - h, and n, h, x are zero, which is a contradiction.

(2). Let  $0 \neq \frac{L}{H} \subseteq \frac{M}{H}$ , is  $\frac{N+H}{H} \cap \frac{L}{H} = 0$ , then  $(N+H) \cap L = H$ . For any  $x \in N \cap L \setminus H$  we have  $x \in (N+H) \cap L = H$ , which is a contradiction.

In some texts a submodule  $H \subseteq M$  of a right R-module M is called a **complement submodule** if it is a pseudo-complement of some submodule of M. We shall give them a different name.

#### Lemma. 55.8.

Let  $N \subseteq M$  be a submodule of a right R-module M, and  $H \subseteq M$  a pseudo-complement of N in M, then

- (1) There is a pseudo–complement L of H in M such that  $N \subseteq L$ .
- (2) In addition, L can be build as a maximal essential extension of N in M.

PROOF. (1). We only need to consider the family  $\{L \subseteq M \mid L \cap H = 0 \text{ and } L \supseteq N\}$ .

(2). Let L be a pseudo–complement of H such that  $L \supseteq N$ , and let  $L \subsetneq^e X \subseteq M$  be an essential extension, then  $0 \neq H \cap X \subseteq X$ , hence  $0 \neq L \cap (H \cap X) \subseteq L \cap H = 0$ , which is a contradiction.  $\square$ 

#### Theorem. 55.9.

Let  $H \subseteq M$  be a submodule of a right R-module M, the following statements are equivalent:

- (a)  $H \subseteq M$  is a pseudo-complement of a submodule.
- (b) H has no proper essential extensions in M.
- (c) If L is a pseudo-complement of H in M, then H is a pseudo-complement of L in M.
- (d) If  $H \subseteq K \subseteq^e M$ , then  $\frac{K}{H} \subseteq^e \frac{M}{H}$ .

PROOF. (a)  $\Rightarrow$  (b). Let  $H \subsetneq^e X \subseteq M$  be an essential extension, then  $0 \neq N \cap X \subseteq X$ , hence  $0 \neq (N \cap X) \cap H = N \cap H$ , which is a contradiction.

- (b)  $\Rightarrow$  (c). Let L be a pseudo–complement of H, then  $H \cap L = 0$ ; if H' is a pseudo–complement of L such that  $H \subsetneq H'$ , then H is not essential in H', and there exists  $0 \neq X \subseteq H'$  such that  $H \cap X = 0$ ; it is easy to show that  $H \cap (L + X) = 0$ , which is a contradiction as L is a pseudo–complement of H.
- (c)  $\Rightarrow$  (a). It is evident.
- (a)  $\Rightarrow$  (d). Let  $0 \neq \frac{L}{H} \subseteq \frac{M}{H}$  such that  $0 = \frac{L}{H} \cap \frac{K}{H} = \frac{L \cap K}{H}$ , then  $(L \cap N) \cap K = H \cap N = 0$ , hence  $L \cap N = 0$ , which is a contradiction as H is a pseudo–complement of N.
- (d)  $\Rightarrow$  (a). Let L be a pseudo–complement of H, then  $H \subseteq H + L \subseteq^e M$ , and  $\frac{H+L}{H} \subseteq^e \frac{M}{H}$ . Let  $H' \supseteq H$  be a pseudo–complement of L, then

$$\frac{H'}{H} \cap \frac{H+L}{H} = \frac{H'(H+L)}{H} = \frac{H+(H'+L)}{H} = 0.$$

Therefore,  $\frac{H'}{H} = 0$ , and H' is a pseudo–complement.

A submodule  $H \subseteq M$  of a right R-module M satisfying the equivalent conditions in the above theorem if also called an **essentially closed submodule** in M.

## Proposition. 55.10.

Let  $H \subseteq N \subseteq M$  be submodules such that  $H \subseteq N$  and  $N \subseteq M$  are essentially closed submodules, then  $H \subseteq M$  is a essentially closed submodule.

#### Remark. 55.11.

As we saw before, essentially closed submodules are transitive, but they are not closed under intersections.

In the abelian group  $M = \mathbb{Z} \oplus \mathbb{Z}_2$  the submodules

$$N_1 = \langle (1,0) \rangle$$
 and  $N_2 = \langle (1,1) \rangle$ ,

are essentially closed submodules, but not its intersection:  $N_1 \cap N_2 = \langle (2,0) \rangle \subseteq M$ .

#### Remark. 55.12.

As a consequence of Lemma (55.8.), for any submodule  $N \subseteq M$  there exists a essentially closed submodule  $H \subseteq M$  such that  $N \subseteq^e H \subseteq M$ .

## **Exercises**

#### Exercise. 55.13.

Give an example of families of right R-modules  $\{N_i \mid i \in I\}$  and  $\{H_i \mid i \in I\}$  such that  $N_i \subseteq^e H_i$  for any index  $i \in I$ , and  $\prod_i N_i \subseteq \prod_i H_i$  is not essential.

Ref.: 2108e 000 SOLUCIÓN

## SOLUTION. Ejercicio (55.13.)

We consider  $R = \varepsilon$ ,  $I = \mathbb{N}^*$ ,  $N_n = n\mathbb{Z}$ , and  $H_n = \mathbb{Z}$  for any  $n \in I$ . We have an inclusion  $\prod_n N_n \subseteq \prod_n H_n$ , but it is not essential. To prove that let us consider  $x = (h_n)_n$ , being  $h_n = 1$  for every  $n \in I$ , then  $(\prod_n N_n : x) = 0$ , hence the inclusion is not essential.

#### Exercise. 55.14.

Give an example of a family of right R-modules  $\{N_i \mid i \in I\}$  such that  $\bigoplus_i N_i \subseteq \prod_i N_i$  is not essential. Ref.: 2108e 008

## SOLUTION. Ejercicio (55.14.)

We consider  $I = \mathbb{N}^*$ , and  $N_n = \mathbb{Z}$ . We have an inclusion  $\bigoplus_n N_n \subseteq \prod_n N_n$ , but it is not essential. To prove that let us consider  $x = (x_n)_n$ , being  $x_n = 1$  for every  $n \in I$ , then  $(\bigoplus_n N_n : x) = 0$ , hence the inclusion is not essential.

### Exercise. 55.15.

For any right R-module M are equivalent:

- (a) M is noetherian.
- (b) Every essential submodule is finitely generated.

Ref.: 2108e\_009 SOLUCIÓN

#### SOLUTION. Ejercicio (55.15.)

(b)  $\Rightarrow$  (a). Let  $N \subseteq M$  be a submodule, the family  $\{H \subseteq M \mid N \cap H = 0\}$  has maximal elements. If H is a maximal element, then  $N \oplus H \subseteq^e M$ , hence, by the hypothesis,  $N \oplus H$  is finitely generated, and N is finitely generated.

#### Exercise. 55.16.

Prove that for any right R-module E the following statements are equivalent:

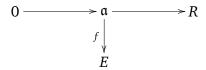
(a) E is injective.

(b)  $j^* : \operatorname{Hom}_A(R, E) \longrightarrow \operatorname{Hom}_R(\mathfrak{c}, E)$  is surjective for any essentially closed right ideal  $j : \mathfrak{c} \subseteq R$ .

Ref.: 2108e 010 SOLUCIÓN

## SOLUTION. Ejercicio (55.16.)

Let us consider the diagram with exact row



There exists a right ideal  $\mathfrak{b} \subseteq R$  maximal satisfying  $\mathfrak{a} \cap \mathfrak{b} = 0$ , hence  $\mathfrak{a} \oplus \mathfrak{b} \subseteq^e R$  is essential, and we may extend f to  $\mathfrak{a} \oplus \mathfrak{b}$  defining the image of  $\mathfrak{b}$  equal to zero. Therefore we may extend f to R, hence E is injective.

#### **Exercise. 55.17.**

If R is a rings with a unique two–sided maximal ideal  $\mathfrak{c}$ , then  $\mathfrak{c} \subseteq^e R$  is essential if, and only if,  $\mathfrak{c} \neq 0$ .

Ref.: 2108e 011

SOLUCIÓN

## SOLUTION. Ejercicio (55.17.)

If  $\mathfrak{c} \neq 0$ , and there exists a right ideal  $\mathfrak{a}$  such that  $\mathfrak{a} \cap \mathfrak{c} = 0$ , then  $\mathfrak{ac} \subseteq \mathfrak{a} \cap \mathfrak{c} = 0$ . If  $R\mathfrak{a} \subseteq \mathfrak{c}$ , then  $\mathfrak{a} \subseteq R\mathfrak{a} \subseteq \mathfrak{c}$ , and  $\mathfrak{a} = 0$ . If  $R\mathfrak{a} \not\subseteq \mathfrak{c}$ , then  $R\mathfrak{a} = R$ , and  $\mathfrak{c} = R\mathfrak{c} = R\mathfrak{ac} = 0$ , which is a contradiction.

#### Exercise. 55.18.

Prove that every non–zero right R–module contains an essential submodule which is a direct sum of cyclic submodules.

Ref.: 2108e 012 SOLUCIÓN

SOLUTION. Ejercicio (55.18.)

Let us consider the family  $\Gamma = \{\{C_i \mid i \in I\} \mid \text{ is a direct sum of cyclic submodules}\}$ . Since  $\{0\} \in \Gamma$ , we have  $\Gamma \neq \emptyset$ . If we have a chain of elements in  $\Gamma$ , let  $\{C_i \mid i \in I_1\} \subseteq \{C_i \mid i \in I_t\} \subseteq \cdots$ , then we may assume  $I_1 \subseteq \cdots \subseteq I_t \subseteq \cdots \subseteq I$  and take  $I = \cup_t I_t$ . Therefore,  $\{C_i \mid i \in I\}$  is an upper bound of this family. Therefore  $\Gamma$  is inductive and by Zorn's lemma there is a maximal element  $\{C_i \mid i \in I\}$  in  $\Gamma$ . Finally we may prove that  $\bigoplus_i C_i \subseteq M$  is an essential submodule. Indeed, if  $\bigoplus_i C_i \subseteq M$  is not essential, there is  $0 \neq x \in M$  such that  $xR \cap (\bigoplus_i C_i) = 0$ , and the family  $\{C_i \mid i \in I\} \cup \{xR\}$  belongs to  $\Gamma$ , which is a contradiction.

#### Exercise. 55.19.

Let A be a commutative ring. The following statements are equivalent:

- (a) A is semiprime.
- (b) The product of finitely many essential ideals is essential.

Ref.: 2108e 014 SOLUCIÓN

## SOLUTION. Ejercicio (55.19.)

- (a)  $\Rightarrow$  (b). Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_t \subseteq A$  be essential ideals, given  $0 \neq x \in A$ , there exists  $a \in A$  such that  $0 \neq xa \in \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t$ , hence  $0 \neq (xa)^n \in \mathfrak{a}_1 \cdots \mathfrak{a}_t$ .
- (b)  $\Rightarrow$  (a). Let  $x \in A$  such that  $x^2 = 0$ , and define  $\mathfrak{a} = xA$ . Let  $\mathfrak{b}$  be a pseudo-complement of  $\mathfrak{a}$  in A. Since  $\mathfrak{a} \oplus \mathfrak{b} \subseteq^e A$  is essential, then  $(\mathfrak{a} \oplus \mathfrak{b})^2 \subseteq^e A$  is essential. Since  $(\mathfrak{a} \oplus \mathfrak{b})^2 = \mathfrak{a}^2 + \mathfrak{a}\mathfrak{b} + \mathfrak{b}^2 = \mathfrak{a}\mathfrak{b} + \mathfrak{b}^2$ , then  $\mathfrak{a}\mathfrak{b} + \mathfrak{b}^2 \subseteq^e A$  is essential, hence  $\mathfrak{b} \subseteq^e A$  is essential, and  $\mathfrak{a} = 0$ , so x = 0.

## Exercise. 55.20.

Let  $\mathfrak{a} \subseteq R$  be a two–sided nilpotent ideal. Prove that  $\mathrm{Ann}(_R\mathfrak{a}) \subseteq R$  is an essential right ideal of R.

Ref.: 2108e 015 SOLUCIÓN

## SOLUTION. Ejercicio (55.20.)

Let us assume  $\mathfrak{a}^n = 0$  and  $\mathfrak{a}^{n-1} \neq 0$ . Let  $\mathfrak{b} = \mathrm{Ann}(_R\mathfrak{a}) \subseteq R$ , it is a two–sided ideal.

If  $\mathfrak{c} \subseteq R$  is a right ideal and  $\mathfrak{c} \cap \mathfrak{b} = 0$ , since  $0 \neq \mathfrak{a}^{n-1} \subseteq \mathfrak{b}$  then  $\mathfrak{cb} \subseteq \mathfrak{c} \cap \mathfrak{b} = 0$ , and  $\mathfrak{ca}^{n-1} = 0$ .

Therefore  $\mathfrak{ca}^{n-2}\mathfrak{a}=0$ , and  $\mathfrak{ca}^{n-2}\subseteq\mathfrak{c}\cap\mathfrak{b}=0$ . We may continue and obtain  $0=\mathfrak{ca}^{n-3}=\cdots=\mathfrak{ca}$ . Hence  $\mathfrak{c}\subseteq\mathfrak{c}\cap\mathfrak{b}=0$ , and  $\mathfrak{c}=0$ .

#### Exercise. 55.21.

Let  $N \subseteq H \subseteq M$  be right R-modules. If  $H \subseteq M$  is a complement submodule, then  $H/N \subseteq M/N$  is a complement submodule. The converse could be false.

Ref.: 2108e 016 SOLUCIÓN

## SOLUTION. Ejercicio (55.21.)

If  $H/N \subseteq^e L/N \subseteq M/N$  is an essential extension, for any  $l \in L \setminus N$  there exists  $r \in R$  such that  $ls \in H \setminus N$ , and for any  $0 \neq l \in N$  we have  $0 \neq l \in N \subseteq H$ .

We consider  $2\mathbb{Z}_4 \subseteq 2\mathbb{Z}_4 \subseteq \mathbb{Z}$ , the  $2\mathbb{Z}_4 \subseteq \mathbb{Z}$  is not a complement submodule, but  $0 \subseteq \mathbb{Z}_2$  is a complement submodule.

#### Exercise. 55.22.

Let  $N \subseteq H \subseteq M$  right R-module. If there exists a pseudo-complement N' of N in M, then there exists a pseudo-complement H' of H in M such that  $H' \subseteq N'$ .

Ref.: 2108e\_017 SOLUCIÓN

#### SOLUTION. Ejercicio (55.22.)

We consider  $H \cap (N \oplus N') = H_0$ . In M/N we consider  $H_0/N \subseteq (N \oplus N')/N$  and  $H_0'/N$  a pseudocomplement of  $H_0/N$  in  $(N \oplus N')/N$ , then  $H_0 = N \oplus N_0$  with  $N_0 \subseteq N'$ .

$$H'_0 \cap H_0 = (N + N_0) \cap H \cap (N + N') = H \cap (N + N_0) = N + (H \cap N_0) \subseteq N.$$

Since  $N_0' \cap H_0 \subseteq H_0' \cap H_0 \subseteq N$ , then  $N_0' \cap H_0 = 0$ :

$$N_0' \cap H = N_0' \cap (N + N') \cap H = N_0' \cap H_0 = 0.$$

Now we see that  $N_0'$  is maximal. Let  $N_0' \subseteq L$  such that  $L \cap H = 0$ .

Let  $L \supseteq N_0'$  such that  $L \cap H = 0$ , then  $L \cap N = 0$ , and we may assume  $L \subseteq N'$ .

We have  $L \cap H_0 = L \cap H \cap (N+N') = 0$ , then  $\frac{N+L}{N} \cap \frac{H_0}{N} = \frac{(N+L) \cap H_0}{N} = \frac{N+(L \cap H_0)}{N} = 0$ , and  $\frac{N+L}{N} \subseteq \frac{H'_0}{N} = \frac{N+N'_0}{N}$ , hence  $\frac{N+L}{N} = \frac{N+N'_0}{N}$ , and  $N+L=N+N'_0$ . Therefore  $L=L \cap (N+L) = L \cap (N+N'_0) = (L \cap N) + N'_0 = 0 + N'_0 = N'_0$ .

## Exercise. 55.23.

If  $H \subseteq M$  is a complement submodule and  $N \subseteq^e M$  an essential extension, then  $H \cap N \subseteq N$  is a complement submodule.

Ref.: 2108e\_018 SOLUCIÓN

SOLUTION. Ejercicio (55.23.)

There exists  $H' \subseteq M$  such that H is a pseudo–complement of H' in M. Let us assume  $H' \neq 0$ . Since  $N \subseteq^e M$ , then  $H' \cap N \neq 0$ , and  $(H' \cap N) \cap (H \cap N) = 0$ ; si there exists  $H \cap N \subsetneq L \subseteq N$  such that  $(H' \cap N) \cap L = 0$ , then  $H' \cap L = 0$ , and  $N \cap H' \cap (H + L) = H' \cap ((N \cap H) + L) = H' \cap L = 0$ .

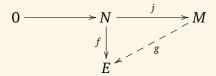
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# 56 Injective hulls

Let M be a right R-module, there is a monomorphism from M to an injective module E, we are interested in the existence of minimal monomorphisms to injective modules. In order to build one of them we proceed as follows.

#### Lemma. 56.1.

In any diagram of right R-modules with exact row



where *E* is an injective and *j* is an essential monomorphism there exists a monomorphism *g* such the diagram commutes.

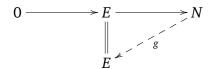
PROOF. We have  $Ker(g) \cap N \subseteq Ker(f) = 0$ , then Ker(g) = 0.

Lemma. 56.2. (Eckmann-Shopf's lemma)

For any right R-module E are equivalent:

- (a) E is injective.
- (b) E has no proper essential extensions.

PROOF. (a)  $\Rightarrow$  (b). Let  $E \subseteq N$  be an essential extension, then we may complete the diagram with a monomorphism g. Hence g is an isomorphism and E = N.



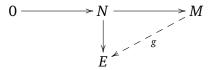
(b)  $\Rightarrow$  (a). If E is not injective, there exists a monomorphism from E to an injective module F. Observe that E is not a direct summand, hence there exists a pseudo–complement E of E in E, and we have an essential monomorphism  $E \cong \frac{E \oplus H}{H} \xrightarrow{ess} \frac{F}{H}$ . By the hypothesis  $\frac{E \oplus H}{H} = \frac{F}{H}$ , hence  $E \oplus H = F$ , which is a contradiction.

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## Corollary. 56.3.

If E is an injective right R-module, then each essentially closed submodule is an injective module. The converse also holds.

PROOF. Let  $N \subseteq E$  be a essentially closed submodule, and let us assume there exists an essential extension  $0 \to N \xrightarrow{ess} M$ . Since E is injective, we may complete the following diagram with a monomorphism g



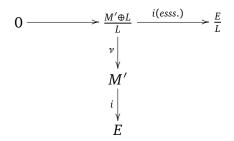
Therefore N has a nontrivial essential extension in E, which is a contradiction.

Let us to collect some results in the following proposition.

#### Proposition. 56.4.

Every right R-module M contained in an injective right R-module E, there is a submodule injective  $M \subseteq^e M' \subseteq E$ .

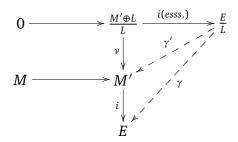
PROOF. Let  $M \subseteq E$  be a submodule of an injective right R-module E, and let  $M \subseteq E$  be a maximal essential extension. Let E be a pseudo-complement of E0 in E1, then there is a diagram



where  $v: M' \oplus l/L \longrightarrow M'$  is the canonical isomorphism, and  $i: M' \longrightarrow E$  is the inclusion. Since E in injective, there is a homomorphism  $\gamma: E(L \longrightarrow E \text{ such that } \gamma j = i v$ . Since j is essential and i v is an injective map, then  $\gamma$  is an injective map. On the other hand, we consider the inclusion  $M \subseteq M'$ , then  $M \subseteq M' \xrightarrow{v^{-1}} M' \oplus L/L \xrightarrow{j} E/L$  is an essential injective map, hence  $M \subseteq {}^e \gamma(E/L)$ , and

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 $M' \subseteq \gamma(E/L)$ ; by the maximality of M' we have  $M' = \gamma(E/L)$ , and  $\gamma$  factorizes through M'.



Therefore,  $\gamma'$  is an isomorphism and j also. In particular,  $M' \oplus L = E$ , and M' is a direct summand of E, hence an injective right R–module.

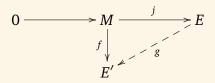
As a consequence, for any right R-module M there exists an injective module E and an essential monomorphism  $M \longrightarrow E$ .

A pair constituted by an injective module E and an essential monomorphism  $M \longrightarrow E$  is called an **injective hull** of M. Sometimes we identify M with its image in E, hence we do not mention the monomorphism when we deal with injective hulls.

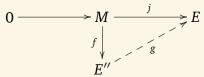
#### Theorem. 56.5.

Let M be a right R-module, and  $j: M \longrightarrow E$  an injective hull, the following statements hold.

(1) For any injective module E' and any monomorphism  $f: M \longrightarrow E'$  there exists a monomorphism  $g: E \longrightarrow E'$  such that f = gj.



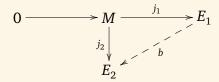
(2) For any essential monomorphism  $f: M \longrightarrow E''$  there exists a monomorphism  $g: E'' \longrightarrow E$  such that j = gf.



#### Corollary. 56.6.

For any right R–module M, if  $j_i: M \longrightarrow E_i$ , i = 1, 2, are injective hulls, there exists an isomorphism

 $b: E_1 \longrightarrow E_2$  such that  $j_2 = bj_1$ .



PROOF. Since  $j_1$  is essential, then b is monomorphism. Since  $j_2 = b j_1$  is essential, then b is essential, hence an isomorphism as  $E_1$  is injective.

In particular, the injective hull of M is determined up to isomorphism. We denote by E(M) an injective hull of M. In general the map  $M \mapsto E(M)$  is not functorial.

## Example. 56.7.

Let us consider  $R = \mathbb{Z}$ , and  $M = \mathbb{Z}_2$ . An injective hull of M is  $E(M) = \mathbb{Z}_{2^{\infty}} = \{\frac{a}{b} + \mathbb{Z} \in \frac{\mathbb{Q}}{\mathbb{Z}} \mid b = 2^t\}$ . There exists a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow \mathbb{Z}_{2^{\infty}} \\ & & & \text{id} \bigvee g \\ \mathbb{Z}_{2} & \longrightarrow \mathbb{Z}_{2^{\infty}} \end{array}$$

where g(x) = -x. In this case, both, id and g, extends  $id_{\mathbb{Z}_2}$ 

#### Lemma. 56.8.

Let M be a right R-module, the following statements hold:

- (1) M in injective if, and only if, M = E(M).
- (2) If  $N \subseteq^e M$  is essential, then E(N) = E(M).
- (3) If E is an injective right R-module and  $M \subseteq E$ , there is a decomposition  $E = E(M) \oplus X$  for some submodule  $X \subseteq E$ .

### Lemma. 56.9.

For any finite family of right R-modules  $\{N_i \mid i = 1,...,t\}$  there exists an isomorphism  $E(\bigoplus_{i=1}^t N_i) \cong \bigoplus_{i=1}^t E(\overline{N_i})$ .

#### Lemma. 56.10.

Let  $\{M_i \mid i \in I\}$  be an independent family of right R-modules. If  $\bigoplus_i E(M_i)$  is injective, then  $E(\bigoplus_i M_i) = \bigoplus_i E(M_i)$ .

SOLUTION. We have that  $\bigoplus_i M_i \subseteq^e \bigoplus_i E(M_i)$ .

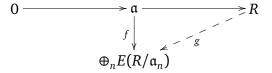
#### Theorem. 56.11.

Let R be a ring, the following statements are equivalent:

- (a) Every direct sum of injective right R-module is injective.
- (b) For any independent family  $\{M_i \mid i \in I\}$  of right R-modules we have  $\bigoplus_i E(M_i) = E(\bigoplus_i M_i)$ .
- (c) R is right noetherian.

PROOF. (a)  $\Leftrightarrow$  (b). It is consequence of the above lemma.

(a)  $\Rightarrow$  (c). Let  $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$  be an ascending chain of right ideals and  $\mathfrak{a} = \bigcup_n \mathfrak{a}_n$ . We define  $f : \mathfrak{a} \longrightarrow \bigoplus_n E(R/\mathfrak{a}_n)$  as  $f(a) = (a + \mathfrak{a}_n)_n$ . Since  $\bigoplus_n E(R/\mathfrak{a}_n)$  is injective, there exists an extension, say g, of f to R.



If  $(x_n)_n = x = g(1)$ , then we have f(a) = xa. Let  $m \in \mathbb{N}$  such that  $x_{m+k} = 0$  for all  $k \in \mathbb{N}$ , then for any  $a \in \mathfrak{a}$  we have  $a + \mathfrak{a}_m = (ax)_m = 0$ ; this means  $a \in \mathfrak{a}_m$ , i.e.,  $\mathfrak{a} = \mathfrak{a}_m$ , and the chain is stationary. (c)  $\Rightarrow$  (a). Since R is right noetherian, every right ideal is finitely generated. For any independent family if injective right R-modules, any right ideal  $\mathfrak{a} \subseteq R$  and any map  $f: \mathfrak{a} \longrightarrow \oplus_i E_i$ , the image of f is contained in a direct sum fo finitely many  $E_i's$ , hence it may be extended to R, and  $\bigoplus_i E_i$  is injective.

#### Theorem. 56.12.

Let  $E_1, E_2$  be two injective right R-modules, if there are monomorphism  $E_1 \longrightarrow E_2$  and  $E_2 \longrightarrow E_1$ , then  $E_1 \cong E_2$ .

Ver R. T. Bumby. *Modules which are isomorphic to submodules of each other*. Archiv der Mathematik, **16** (1965), 184–185.

PROOF. Let us assume  $E_1 \subseteq E_2$  and there is a monomorphism  $f: E_2 \to E_1$ , then we have:

$$E_{2} = E_{1} \oplus G$$

$$= f(E_{2}) \oplus H \oplus G$$

$$= f(E_{1}) \oplus f(G) \oplus G \oplus H$$

$$= ff(F_{2}) \oplus f(H) \oplus f(G) \oplus G \oplus H$$

$$= ff(E_{1}) \oplus f(G) \oplus G \oplus H \oplus f(H)$$
...
$$= f^{t+1}(E_{1}) \oplus (G \oplus f(G) \oplus \cdots) \oplus (H \oplus f(H) \oplus \cdots)$$

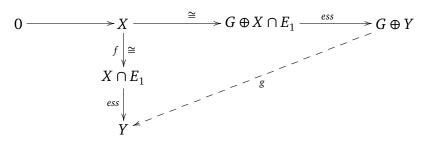
If we call  $X = G \oplus f(G) \oplus \cdots$ , then  $X \cap E_1 = f(G) \oplus f(G) \oplus \cdots = f(X)$ .

Let *Y* be the maximal essential extension of  $X \cap E_1$  in  $E_1$ ; since *Y* is injective, then  $E_1 = Y \oplus Z$ , and we have:

$$E_2 = E_1 \oplus G = Y \oplus Z \oplus G = (G \oplus Y) \oplus Z.$$

Hence  $Y \oplus G$  is injective.

Since  $X = G \oplus f(X) \subseteq G \oplus Y$ , and  $f(X) = X \cap E_1$ , we have a commutative diagram



Then g is an isomorphism, and we have:

$$E_2 = G \oplus Y \oplus Z \xrightarrow{g \oplus Z} Y \oplus Z = E_1.$$

Let  $N \subseteq M$  be a submodule of a right R-module, if we consider  $E(N) \subseteq E(M)$  and  $N' = M \cap E(N)$ , then  $N \subseteq^e N'$  is an essential extension of N in M, hence  $N' \subseteq M$  is an essentially closed submodule. Thus  $N \subseteq M$  is an essentially closed submodule if, and only if,  $N = M \cap E(N)$ .

### Exercise. 56.13.

Let R be a noetherian ring and let  $\{N_i \mid i \in I\}$  and  $\{H_i \mid i \in I\}$  families of right R-modules such that  $N_i \subseteq H_i$  is an essentially closed submodule for any index  $i \in I$ , then  $\bigoplus_i N_i \subseteq \bigoplus_i H_i$  is an essentially closed submodule.

Ref.: 2108e 006 SOLUCIÓN

SOLUTION. Ejercicio (56.13.)

We have  $N_i = H_i \cap E(N_i)$  for any index  $i \in I$ , and  $E(\bigoplus_i N_i) = \bigoplus_i E(N_i)$  as R is noetherian, then  $\bigoplus_i N_i = \bigoplus_i (H_i \cap E(N_i)) = (\bigoplus_i H_i) \cap (\bigoplus_i E(N_i)) = (\bigoplus_i H_i) \cap E(\bigoplus_i N_i)$ .

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# More on essentially closed submodules

We shall prove that every essentially closed submodule N in a right R-module M is the trace on M of the injective hull of E(N), hence the essentially closed submodules are the traces of injective submodules of E(M).

#### Lemma. 56.14.

Let  $N \subseteq^e M$  be an essential submodule of a right R-module M, for a submodule  $H \subseteq N$  the following statements are equivalent:

- (a)  $H \subseteq N$  is a essentially closed submodule.
- (b)  $H = N \cap H'$  for some  $H' \subseteq M$  essentially closed submodule.

PROOF. (a)  $\Rightarrow$  (b). We consider  $H' \subseteq M$ , a maximal essential extension of H in M, then  $H \subseteq N \cap H'$ , and  $H \subseteq^e H'$ , hence  $H = N \cap H \subseteq^e N \cap H' \subseteq N$ ; by the hypothesis we obtain  $H = N \cap H'$ .

(b)  $\Rightarrow$  (a). Let  $H' \subseteq M$  be an essentially closed submodule, and let  $N \cap H' \subseteq L \subseteq^e N$ , then  $L \subseteq^e M$ , and  $L + H' \subseteq^e M$ . By Theorem (55.9.) we have  $\frac{L+H'}{H'} \subseteq^e \frac{M}{H'}$ . On the other hand,  $\frac{L}{N \cap H'} \subseteq \frac{N}{N \cap H'} \cong \frac{N+H'}{H'}$ , and  $\frac{L}{N \cap H'}$  can be identity with  $\frac{L+H'}{H'} \subseteq \frac{N+H'}{H'} \subseteq \frac{M}{H'}$ , which id essential. By Theorem (55.9.)we have  $N \cap H' \subseteq N$  is an essentially closed submodule.

### Corollary. 56.15.

Let  $N \subseteq M$  be a submodule, the following statements are equivalent:

- (a)  $N \subseteq M$  is an essentially closed submodule.
- (b) For every  $X \subseteq N$  essentially closed submodules we have  $X \subseteq M$  is an essentially closed submodule.

PROOF. (a)  $\Rightarrow$  (b). From the inclusion  $X \subseteq N \subseteq M$  we have  $E(X) \subseteq E(N) \subseteq E(M)$ . Since  $X \subseteq N$  is an essentially closed submodule then  $X = N \cap E(X)$ , and since  $N \subseteq M$  is an essentially closed submodule, then  $N = M \cap E(N)$ . Then  $X = N \cap E(X) = M \cap E(N) \cap E(X) = M \cap E(X)$ , and  $X \subseteq M$  is an essentially closed submodule.

### Corollary. 56.16.

Let  $N \subseteq M$  be a submodule, the following statements are equivalent:

- (a)  $N \subseteq M$  is an essentially closed submodule.
- (b) For every  $E(N) \subseteq E(M)$  we have  $N = M \cap E(N)$ .

## Indecomposable injective modules

A right R-module M is **indecomposable** if for any decomposition  $M = N_1 \oplus N_2$  we have either  $N_1 = M$  or  $N_2 = M$ . A weakest notion is the following one; a right R-module M is **uniform** if and any nonzero submodules  $0 \neq N_1, N_2 \subseteq M$  we have  $N_1 \cap N_2 \neq 0$ , or equivalently, every nonzero submodule is essential. It is evident that every uniform right R-module is indecomposable. For injective modules both notions coincide.

## Lemma. 56.17.

If E is an injective right R-module, then E is indecomposable if, and only if, E is uniform; a a consequence, for any right R-module M, the following statements are equivalent:

- (a) M is uniform.
- (b) E(M) is indecomposable.

Our interest now is, given a right R-module M, decompose the injective hull of M as a direct sum of indecomposable injective submodules, hence we need to have **enough uniform submodules** in the following sense: every nonzero submodule of M contains a uniform submodule.

#### Lemma, 56.18.

Let R be a right noetherian ring, then any nonzero right R-module contains a uniform submodule.

PROOF. Let M be a nonzero right R-module, let  $0 \neq m \in M$ , then  $0 \neq mR \subseteq M$ . If mR is not uniform, there exist  $0 \neq N_1, N_1' \subseteq mR$  such that  $N_1 \cap N_1' \neq 0$ . If  $N_1'$  is not uniform, there exist  $0 \neq N_2, N_2' \subseteq N_1'$  such that  $N_2 \cap N_2' = 0$ . In this way, if we never find a uniform submodule  $N_i'$ , then we obtain a strictly ascending chain  $\{N_1 \oplus \cdots \oplus N_i \mid i \in \mathbb{N} \setminus \{0\}\}$  of submodules of mR, which is a contradiction as mR is noetherian.

#### Proposition. 56.19.

Let *R* be a right noetherian ring, then any injective right *R*–module *E* is a direct sum of indecomposable injective submodules.

PROOF. Let  $E \neq 0$  be a nonzero injective right R-module, there exists a uniform submodule  $U \subseteq E$ , hence  $E(U) \subseteq^{\oplus} E$  is a direct summand. Therefore, the family  $\Gamma$ , of independent families  $\{E_i \mid i \in I\}$  of indecomposable injective submodules, is nonempty and inductive whenever we consider the

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inclusion. By Zorn's lemma there exists a maximal element in  $\Gamma$ . Let  $\{E_i \mid i \in I\} \in \Gamma$  be a maximal element. If  $\bigoplus_i E_i \neq E$ , since  $\bigoplus_i E_i \subseteq^{\oplus} E$  is a direct summand, there exists  $0 \neq H \subseteq E$  such that  $E = (\bigoplus_i E_i) \bigoplus H$ , and there exists an indecomposable injective submodule  $E_0 \subseteq H$ , hence  $\{E_i \mid i \in I\} \cup \{E_0\} \in \Gamma$ , which is a contradiction.

#### Exercise. 56.20.

Se considera la categoría de grupos abelianos; en este caso  $R = \mathbb{Z}$ .

- (1) Prueba que  $\mathbb{Z}$  es un grupo abeliano uniforme. Determina todos los grupos cíclicos uniformes.
- (2) Prueba que el grupo  $\mathbb{Z}_{p^{\infty}}$  es un grupo uniforme y no es un grupo cíclico. Se consideran  $\mathbb{Q}$  y  $\mathbb{R}$ ; ¿es alguno uniforme?
- (3) Determina todos los grupos abelianos inyectivos indescomponibles.
- (4) Si M es un grupo abeliano finitamente generado sabemos que  $M \cong (\bigoplus_{i=1}^t \mathbb{Z}_{p_i^{n_i}}) \oplus \mathbb{Z}^n$ , para  $n, n_1, \ldots, n_t \in \mathbb{N}$ . ¿Cuál es la descomposición de E(M) como suma de inyectivos indescomponibles.

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SOLUTION. Ejercicio (56.20.) HACER