52 Injective modules

A right R–module M is **injective** if the functor $\mathrm{Hom}_R(-,M)\,\mathbf{Mod}\text{–}R \longrightarrow \mathscr{A}b$, is exact.

Proposition. 52.1.
For any ring R and any right R–module M the following statements are equivalent:
(a) M is injective.
(b) Every diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

can be completed to a commutative one.
*(c) (**Baer's Lemma**) For any right ideal $\mathfrak{a}$ every diagram with exact row*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

can be completed to a commutative one.

PROOF. (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). They are obvious.
(c) $\Rightarrow$ (b). Let us consider the following diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] \arrow[d, "f"'] & N \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

We define $\Gamma = \{(H,h) \mid\ N' \subseteq H \subseteq N, h : H \to M, \text{ and } h_{|N'} = f\}$. This set is inductive, i.e., it is nonempty and, with the partial order; $(H_1,h_1) \leq (H_2,h_2)$ if $H_1 \subseteq H_2$ and $h_{2|H_1} = h_1$, every ascending chain has a upper bound. By Zorn's lemma there exists a maximal element in Γ, say (H,h).
If $H \neq N$ there exists $x \in N \setminus H$. We consider $H+Rx$, and define $\mathfrak{a} = (H : x) = \{a \in R \mid\ ax \in H\}$ and $h' : \mathfrak{a} \longrightarrow M$ as $h'(a) = h(ax)$. It is obvious that h' is a homomorphism, hence it can be extended to $h'' : R \longrightarrow M$. We take $m = h''(1)$.
Now we define $h_x : H+Rx \longrightarrow M$ as $h_x(y+ax) = h(y)+am$, for any $y \in H$ and $a \in R$. This map is well define and a homomorphism. Indeed, if $y+am = y'+a'm$, then $h(y)-h(y') = h(y-y') =$

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$h((a'-a)x) = h'(a'-a) = (a'-a)m$, hence $h(y)+am = h(y')+a'm$. The existence of the pari $(H+Rx, h'')$ contradices the maximality of (H,h). □

Proposition. 52.2.
For any ring R, and any family of right R–modules $\{M_i \mid i \in I\}$, the following statements are equivalent:
(a) $\prod_i M_i$ is injective.
(b) Every M_i is injective.

PROOF. First we observe that for any right R–module X there exists a natural isomorphism

$$\mathrm{Hom}_A(X, \prod_i M_i) \cong \prod_i \mathrm{Hom}_A(X, M_i).$$

In addition, for any family of homomorphisms $\{f_i : X_i \longrightarrow Y_i \mid i \in I\}$ we have $\prod_i f_i : \prod_i X_i \longrightarrow \prod_i Y_i$ is surjective if, and only if, every f_i is surjective. The result follows from these two observations. □

There are many different characterizations of injective modules, in the next proposition we collect some of them.

Proposition. 52.3.
Let M be a right R–modulo, the following statements are equivalent:
(a) M is injective.
(b) Every short exact sequence $0 \to M \to X \to Y \to 0$ splits.
(c) Every short exact sequence $0 \to M \to X \to Y \to 0$, with Y a cyclic right R–module, splits.

PROOF. (a) $\Rightarrow$ (b). It is enough to consider the diagram with exact row

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & X \arrow[r] \arrow[dl, "p"] & Y \arrow[r] & 0 \\
 & M & & &
\end{tikzcd}$$

By the hypothesis there exists a map p such that $pi = \mathrm{id}_M$, hence M is a direct summand.
(b) $\Rightarrow$ (c). It is obvious.

(c) ⇒ (a). For any right ideal $j : \mathfrak{a} \subseteq R$, and any map $f : \mathfrak{a} \longrightarrow M$, with kernel $\mathfrak{b} = \operatorname{Ker}(f)$, we may build the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[rr] & & \mathfrak{a} \arrow[rr, "j"] \arrow[dr] \arrow[dd, "f"'] & & A \arrow[rr] \arrow[dr] \arrow[dd, dashed, "k"'] & & A/\mathfrak{a} \arrow[rr] \arrow[dr, equal] \arrow[dd, dashed, equal] & & 0 & \\
& 0 \arrow[rr] & & \mathfrak{a}/\mathfrak{b} \arrow[rr] \arrow[dl, tail] & & A/\mathfrak{b} \arrow[rr] \arrow[dl, dashed, tail] & & A/\mathfrak{a} \arrow[rr] \arrow[dl, dashed, equal] & & 0 \\
0 \arrow[rr, dashed] & & M \arrow[rr, dashed, tail, "g"] & & \bullet \arrow[rr, dashed, two heads] & & \bullet \arrow[rr, dashed] & & 0 &
\end{tikzcd}$$

By the hypothesis g has an inverse h on the right: $hg = \mathrm{id}_M$. Therefore $f = hkj$ as we have $gf = ghgf = ghkj$ and g is monomorphism. □

In particular, from this characterization we obtain:

Corollary. 52.4.
For any ring R the following statements are equivalent:

(a) Every short exact sequence of right R–modules splits.
(b) Every right R–module is injective.

Observe that the conditions in Corollary (52.4.) characterize (artinian) semisimple rings. Compare with Corollary (49.4.).

53 Divisible abelian groups

We shall study the behaviour of injective abelian groups. First we point out that injective abelian groups have more interesting properties than injective modules over a general ring R.

Let R be a ring, a right R–module M is **divisible** if for any non-zero element $r \in R$ and any $m \in M$, such that $m\operatorname{Ann}(r) = 0$, there exists $m' \in M$ such that $m = m'r$. It is clear that every injective right R–module is divisible. The converse is true in some special cases, for instance, if R is a right principal ideal ring. We are interested in divisible modules over $\mathbb{Z}$, hence we restrict ourselves to domains. If R is not a domain some of the following results are not true.

Proposition. 53.1.
For any abelian group M the following statements are equivalent:

(a) M is injective.
(b) M is divisible.

PROOF. Since $\mathbb{Z}$ is a principal ideal domain, every ideal is principal; by Baer's lemma we have the result. □

Proposition. 53.2.
Every homomorphic image of a divisible abelian group is divisible.

PROOF. Let M be a divisible abelian group, and $N \subseteq M$, for any $m \in M$, and any $r \in R$ there exists $m' \in M$ such that $m = m'r$, hence $m + N = (m' + N)r$. □

Proposition. 53.3.
For any family of abelian groups $\{M_i \mid i \in I\}$ the following statements are equivalent:

(a) $\oplus_i M_i$ is divisible (= injective).
(b) Every M_i is divisible (= injective).

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PROOF. Let $(m_i)_i \in \oplus_i M_i$, and $0 \neq n \in \mathbb{Z}$. For any index i there exists $m'_i \in M_i$ such that $m_i = nm'_i$. Therefore we have $(m_i)_i = n(m'_i)_i$, and $\oplus_i M_i$ is divisible. The converse holds from the previous proposition. □

Example. 53.4.
Every field of characteristic zero is a divisible abelian group. Indeed, if K is a field of characteristic zero, for any $k \in K$, and any $0 \neq n \in \mathbb{Z}$ we can take $k' = \frac{k}{n}$.

Example. 53.5.
The abelian group $\mathbb{Q}/\mathbb{Z}$ is a divisible, hence injective.

Remark. 53.6.
Observe that if K is a field of non–zero characteristic p, then K is not a divisible abelian group. In this case if we take $m = 1 \in K$ and $n = p \in \mathbb{Z}$, there does not exist any $k \in K$ such that $1 = pk$, as the right part in the above identity is zero.

Lemma. 53.7.
$\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathscr{A}b$.

PROOF. Let M be an abelian group, for any $0 \neq m \in M$ the subgroup $\langle m \rangle$ has a homomorphic image isomorphic to a cyclic group, hence to a finite cyclic group, and there exists a nonzero map $f : \langle m \rangle \longrightarrow \mathbb{Q}/\mathbb{Z}$ which can be extended to a map $f' : M \longrightarrow \mathbb{Q}/\mathbb{Z}$ satisfying $f'(m) \neq 0$. □

Corollary. 53.8.
Every abelian group is isomorphic to a submodule of an injective abelian group.

Indeed, every abelian group is isomorphic to a submodule of a direct product of copies of $\mathbb{Q}/\mathbb{Z}$.

Exercise. 53.9.
Let D be a commutative integral domain. For any torsionfree D–module M the following statements are equivalent:

(a) M is injective.
(b) M is divisible.

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SOLUTION. Ejercicio (53.9.)
Let us consider a diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "j"] \arrow[d, "f"'] & A \\ & M & \end{tikzcd}$$

We define a family $\Gamma = \{(\mathfrak{b}, g) \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq D, \text{ and } g_{|\mathfrak{a}} = f\}$. In Γ we define the partial order $(\mathfrak{b}_1, g_1) \leq (\mathfrak{b}_2, g_2)$ whenever $\mathfrak{b}_1 \subseteq \mathfrak{b}_2$ and $g_{2|\mathfrak{b}_1} = g_1$. Since Γ is a non–empty inductive set, by Zorn's lemma, it has maximal elements. Let $(\mathfrak{b}, g) \in \Gamma$ maximal. If $\mathfrak{b} \neq D$, there exists $r \in R \setminus \mathfrak{b}$. If $Dr \cap \mathfrak{b} = 0$, then $(\mathfrak{b}, g)$ is not maximal, which is a contradiction.
If $Dr \cap \mathfrak{b} \neq 0$ there exists $s \in R$ such that $0 \neq sr \in \mathfrak{b}$, hence we may define a map $h : Ds \longrightarrow M$ by $h(ds) = g(dsr)$, for any $d \in D$, which can be extended to a map $h' : D \longrightarrow M$; thus there exists $m \in M$ such that $m = h'(1)$, and $g(sr) = h(s) = h'(s) = sm$. Now we define a new map $g' : \mathfrak{b} + Dr \longrightarrow M$ as follows:

$$\begin{array}{ll} g'(b) = g(b), & \text{for any } b \in \mathfrak{b}, \\ g'(dr) = dm, & \text{for any } d \in D. \end{array}$$

It is necessary to check that g' is well defined and that it is a homomorphism.
If $d_1 r = d_2 r$, then $(d_1 - d_2)r = 0$, then $d_1 - d_2 = 0$, as D is an integral domain, and $d_1 = d_2$, hence $g'(d_1 r) = d_1 m = d_2 m = g'(d_2 r)$. If $0 \neq dr \in \mathfrak{b}$, we proceed as before to find an element $m' \in M$ such that $g(dr) = dm'$; now, we may consider $0 \neq sdr \in \mathfrak{b}$, which satisfies:

$$sdm = dg(sr) = g(dsr) = sg(dr) = sdm',$$

hence $sd(m - m') = 0$, and $m = m'$, as M is torsionfree. □

54 Existence of enough injective objects in module categories

Proposition. 54.1.
For any ring R the right R–module $\operatorname{Hom}(R,\mathbb{Q}/\mathbb{Z})$ *is an injective cogenerator of* $\mathbf{Mod}$–R.

PROOF. Let M be a right R–module, then there are isomorphism:

$$\operatorname{Hom}_R(M,\operatorname{Hom}(R,\mathbb{Q}/\mathbb{Z}))\cong \operatorname{Hom}(M\otimes_R R,\mathbb{Q}/\mathbb{Z})\cong \operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z}).$$

Now, since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathscr{A}b$, it is clear that $\operatorname{Hom}(R,\mathbb{Q}/\mathbb{Z})$ is an injective cogenerator of $\mathbf{Mod}$–R.
To prove that $\operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z})$ is injective, let us consider a diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "f"'] & M \\ & \operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z}) & \end{tikzcd}$$

There exists a natural commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_A(M,\operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z})) \arrow[r, "i^*"] \arrow[d, "\cong"] & \operatorname{Hom}_A(N,\operatorname{Hom}_R(R,\mathbb{Q}/\mathbb{Z})) \arrow[d, "\cong"] \\ \operatorname{Hom}(M\otimes_R R,\mathbb{Q}/\mathbb{Z}) \arrow[r, "(i\otimes_R R)^*"] \arrow[d, "\cong"] & \operatorname{Hom}(N\otimes_R R,\mathbb{Q}/\mathbb{Z}) \arrow[d, "\cong"] \\ \operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z}) \arrow[r, "i^*"] & \operatorname{Hom}(N,\mathbb{Q}/\mathbb{Z}) \end{tikzcd}$$

The result holds as $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. □

PROOF. We may also consider the adjunction

$$\begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "\mathscr{U}"'] \\ \mathscr{A}b \arrow[u, shift right, "\operatorname{Hom}_R(R,-)"'] \end{tikzcd}$$

The unity is $\varepsilon_M : M \longrightarrow \operatorname{Hom}(R,\mathscr{U}(M))$, defined $\varepsilon_M(m)(r)=mr$. It is a monomorphism. If D is a divisible abelian group such that $0\to\mathscr{U}(M)\to D$, then there is a monomorphism $M\xrightarrow{\varepsilon_M}\operatorname{Hom}(R,\mathscr{U}(M))\xrightarrow{\eta_*}\operatorname{Hom}(R,D)$. Since D is a divisible abelian group. then $\operatorname{Hom}(R,D)$ is an injective right R–module. □

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Corollary. 54.2.
For any ring R every right R–module is a submodule of an injective right R-module.

PROOF. This result holds as $\mathrm{Hom}_R(R, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator, hence every right R–module is a submodule of a direct product of copies of it. □

As every right R–module is a submodule of an injective module, we say that the category **Mod**–R has **enough injective objects**.

Appendix

Study of the adjunction

$$\begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "\mathscr{U}"'] \\ \mathscr{A}b \arrow[u, shift right, "\mathrm{Hom}(R,-)"'] \end{tikzcd}$$

There is a bijection

$$\theta = \theta_{A,B} : \mathrm{Hom}(\mathscr{U}(A), B) \cong \mathrm{Hom}_R(A, \mathrm{Hom}(R, B))$$

defined

$$\theta(\psi)(a)(r) = \psi(ar) = [(\theta(\psi))(a)](r)$$

with inverse

$$\theta^{-1}(\varphi)(a) = \varphi(a)(1) = \left[\theta^{-1}(\varphi\right](a).$$

Let us show that θ is natural in A and in B. This means that for every pair of homomorphisms: $f : A_2 \longrightarrow A_1$ and $g : B_1 \longrightarrow B_2$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathscr{U}(A_1), B_1) \arrow[r, "\theta_1"] \arrow[d, "\mathscr{U}(f)^* g_*"'] & \mathrm{Hom}_R(A, \mathrm{Hom}(R, B)) \arrow[d, "f^* \mathrm{Hom}(R,g)_*"] \\ \mathrm{Hom}(\mathscr{U}(A_2), B_2) \arrow[r, "\theta_2"] & \mathrm{Hom}_R(A, \mathrm{Hom}(R, B)) \end{tikzcd}$$

Let us write $\mathscr{U}(f)$ as f, and $\mathrm{Hom}(R, g)$ as g_*.
Given $\psi \in \mathrm{Hom}(\mathscr{U}(A_1), B_1)$, we have:

$$\begin{aligned}(\theta_2 f^* g_*(\psi))(a_2)(r) &= (\theta_2 f^*(g\psi))(a_2)(r) \qquad = (\theta_2(g\psi f))(a_2)(r)\\ &= (g\psi f)(a_2 r).\end{aligned}$$

$$\begin{aligned}(f^*(g_*)_*\theta_1(\psi))(a_2)(r) &= (g_*\theta_1(\psi)f)(a_2)(r) \qquad = (g_*\theta_1(\psi))(f(a_2))(r)\\ &= (g_*(\theta_1(\psi)(f(a_2))))(r) = g(\theta_1(\psi)(f(a_2))(r))\\ &= g(\psi(f(a_2)r))\end{aligned}$$

Let E be an injective abelian group, then $\mathrm{Hom}(R,E)$ is an injective right R–module.
We consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "h"'] & M \\
 & \mathrm{Hom}(R,E) &
\end{tikzcd}$$

Applying $\mathscr{U}$, and the isomorphism $\theta = \theta_{\mathrm{Hom}(R,E),E} : \mathrm{Hom}(\mathscr{U}\,\mathrm{Hom}(R,E),E) \cong \mathrm{Hom}_R(\mathrm{Hom}(R,E),\mathrm{Hom}(R,B))$, we have:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{U}(N) \arrow[r, "\mathscr{U}(j)"] \arrow[d, "\mathscr{U}(h)"'] & \mathscr{U}(M) \arrow[ddl, "h'"] \\
 & \mathscr{U}\,\mathrm{Hom}(R,E) \arrow[d, "\theta^{-1}(\mathrm{id})"'] & \\
 & E &
\end{tikzcd}$$

Since E is an injective abelian group, there exists h' completing the diagram: $\theta^{-1}(\mathrm{id})\mathscr{U} = h'\mathscr{U}(j)$. Using the isomorphism $\theta = \theta_{M,E} : \mathrm{Hom}(\mathscr{U}(M),E) \cong \mathrm{Hom}_R(M,\mathrm{Hom}(R,E))$, we have $\theta(h') : M \longrightarrow \mathrm{Hom}(R,E)$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "h"'] & M \arrow[dl, "\theta(h')"] \\
 & \mathrm{Hom}(R,E) &
\end{tikzcd}$$

To show that this diagram is commutative, we may use the isomorphism $\theta = \theta_{N,E} : \mathrm{Hom}(\mathscr{U}(N),E) \cong \mathrm{Hom}_R(N,\mathrm{Hom}(R,E))$, to see that $\theta^{-1}(h) = \theta^{-1}(\theta(h')j)$. Indeed, for every $n \in N$ we have:

$$\theta^{-1}(h)(n) \qquad = h(n)(1).$$

$$\begin{aligned}
\theta^{-1}(\theta(h')j)(n) &= (\theta(h')j)(n)(1) = \theta(h')(j(n))(1) = h'(j(n)) = (h'j)(n) \\
&= (\theta^{-1}(\mathrm{id})h)(n) \ = \theta^{-1}(id)(h(n)) = \mathrm{id}(h(n))(1) = h(n)(1).
\end{aligned}$$

Thus, $\mathrm{Hom}(R,E)$ is an injective right R–module derecha inyectivo. In particular, $\mathrm{Hom}(R,\mathbb{Q}/\mathbb{Z})$ is an injective right R–module.
To finish, let us see that every right R–module is contained in an injective right R–module.
Let M be a right R–module, the abelian group $(U)(M)$ is contained in a direct product of copies of $\mathbb{Q}/\mathbb{Z}$, let $\mathscr{U}(M) \stackrel{j}{\longrightarrow} \prod \mathbb{Q}/\mathbb{Z}$. Using the isomorphism $\theta = \theta_{M,\mathscr{U}(M)} : \mathrm{Hom}(\mathscr{U}(M),\mathscr{U}(M)E) \cong \mathrm{Hom}_R(M,\mathrm{Hom}(R,\mathscr{U}(M)))$, we have a sequence of homomorphisms

$$M \xrightarrow{\theta(\mathrm{id})} \mathrm{Hom}(R,\mathscr{U}(M)) \xrightarrow{j_*} \mathrm{Hom}(R,\prod \mathbb{Q}/\mathbb{Z})$$

each one of which is an injective map. For instance, we have $\theta(\mathrm{id})(m)(1) = \mathrm{id}(m)$, hence $\mathrm{Ker}(\theta(\mathrm{id})) = \{0\}$.

55 Essential extensions

Let $N \subseteq M$ be a submodule of a right R–module M, we say N is **essential** in M or that M is an **essential extensión** of N if for any non-zero submodule $H \subseteq M$ we have $N \cap H \neq 0$.

Instead of an inclusion we may consider a monomorphism $j : N \longrightarrow M$; if the image of j is essential in M it is called an **essential monomorphism**; we also say that j is an **essential extension**; an essential extension j in **proper** if j is not an isomorphism.

We write $N \subseteq^e M$ to represent that N is an essential submodule of M.

A non-zero right R–module M is called **uniform** if every non-zero submodule is essential.

Lemma. 55.1.
Let $N \subseteq M$ a submodule, the following statements are equivalent:

(a) N is an essential submodule.
(b) For every submodule $H \subseteq M$, if $N \cap H = 0$, then $H = 0$.
(c) For every $0 \neq m \in M$ there exists $r \in R$ such that $0 \neq mr \in N$.
(d) For every right R–module L and any pair of homomorphisms f, g

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[dr, "f"'] & M \arrow[d, "g"] \\ & & L \end{tikzcd}$$

such that $g_{|N} = f$, if f is a monomorphism, then g is a monomorphism.

Proposition. 55.2.
Let M be a right R–module, the following statements hold:

(1) If $N \subseteq H \subseteq M$ are submodules the statements are equivalent:
  (a) $N \subseteq^e M$ is essential.
  (b) $N \subseteq^e H \subseteq^e M$ are essential.
(2) If $N_i \subseteq^e H_i \subseteq M$, for $i = 1, 2$, then $N_1 \cap N_2 \subseteq^e H_1 \cap H_2$.
(3) If $f : M \longrightarrow M'$ is a homomorphism, for any essential submodule $N' \subseteq^e M'$ we have $f^{-1}(N') \subseteq^e M$ is essential.
(4) For $\{N_i \mid i \in I\}$ and $\{H_i \mid i \in I\}$, independent families of submodules of M, such $N_i \subseteq^e H_i$ is essential for every index $i \in I$, we have $\oplus_i N_i \subseteq \oplus_i H_i$ is essential.

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Remark. 55.3.
Observe that in (2) the intersection necessarily is finite as the following example shows. Let us consider $\{n\mathbb{Z} \mid n \in \mathbb{N}^*\}$ be the family of all non–zero subgroups of $\mathbb{Z}$; each of them is essential in $\mathbb{Z}$, and its intersection $\cap_{n\neq 0} n\mathbb{Z} = 0$ is not essential in $\mathbb{Z}$.

Remark. 55.4.
Observe that in (4) the families must be independent; let us consider the abelian group $M = \mathbb{Z} \oplus \mathbb{Z}_2$ and the submodules:

$$N_1 = \langle(2,0)\rangle \subseteq^e H_1 = \langle(1,0)\rangle,$$
$$N_2 = \langle(2,0)\rangle \subseteq^e H_2 = \langle(1,1)\rangle.$$

Then $N_1 + N_2 = N_1$ and $H_1 + H_1 = M$, and $\langle(2,0)\rangle \subseteq M$ is not essential.

The most important result in this section is the following.

Proposition. 55.5.
Let M be a right R–module, for any submodule $N \subseteq M$ there exists a submodule $N \subseteq H \subseteq M$ maximal among those satisfying $N \subseteq^e H$.

PROOF. We define $\Gamma = \{H \subseteq M \mid N \subseteq^e H \subseteq M\}$. Since $N \in \Gamma$, then $\Gamma \neq \emptyset$. On the other hand, every chain of elements in Γ has a upper bound in Γ. Therefore, by Zorn's lemma, there are maximal elements in Γ. □

Our interest now is to study these maximal essential extensions.
If M is a right R–module and $M = N_1 \oplus N_2$, we say N_2 is a **complement of** N_1 in M; i.e., $N_1 \cap N_2 = 0$ and $N_1 + N_2 = M$.
For any submodule $N \subseteq M$ of a right R–module M; a **pseudo–complement of** N in M is any submodule $L \subseteq M$ maximal among those satisfying $N \cap L = 0$.

Remark. 55.6.
Observe that for any submodule $N \subseteq M$ we have:
(1) Any complement of N in M is a pseudo–complement.
(2) Complements are not uniquely determined.
(3) Pseudo–complements are not uniquely determined.

Lemma. 55.7.
Let $N \subseteq M$ be a submodule of a right R–module M with pseudo–complement H, then
(1) $N \oplus H \subseteq^e M$ is an essential submodule.
(2) $\frac{N+H}{H} \subseteq^e \frac{M}{H}$ is essential.

PROOF. (1). Let $0 \neq X \subseteq M$ such that $(N+H)\cap X = 0$, if $N\cap(H+X) \neq 0$, there exist $n = h+x \neq 0$, hence $x = n-h$, and n, h, x are zero, which is a contradiction.
(2). Let $0 \neq \frac{L}{H} \subseteq \frac{M}{H}$, is $\frac{N+H}{H} \cap \frac{L}{H} = 0$, then $(N+H)\cap L = H$. For any $x \in N\cap L \setminus H$ we have $x \in (N+H)\cap L = H$, which is a contradiction. □

In some texts a submodule $H \subseteq M$ of a right R–module M is called a **complement submodule** if it is a pseudo–complement of some submodule of M. We shall give them a different name.

Lemma. 55.8.
Let $N \subseteq M$ be a submodule of a right R–module M, and $H \subseteq M$ a pseudo–complement of N in M, then

(1) There is a pseudo–complement L of H in M such that $N \subseteq L$.
(2) In addition, L can be build as a maximal essential extension of N in M.

PROOF. (1). We only need to consider the family $\{L \subseteq M \mid L\cap H = 0 \text{ and } L \supseteq N\}$.
(2). Let L be a pseudo–complement of H such that $L \supseteq N$, and let $L \subsetneq^e X \subseteq M$ be an essential extension, then $0 \neq H\cap X \subseteq X$, hence $0 \neq L\cap(H\cap X) \subseteq L\cap H = 0$, which is a contradiction. □

Theorem. 55.9.
Let $H \subseteq M$ be a submodule of a right R–module M, the following statements are equivalent:

(a) $H \subseteq M$ is a pseudo-complement of a submodule.
(b) H has no proper essential extensions in M.
(c) If L is a pseudo–complement of H in M, then H is a pseudo–complement of L in M.
(d) If $H \subseteq K \subseteq^e M$, then $\frac{K}{H} \subseteq^e \frac{M}{H}$.

PROOF. (a) ⇒ (b). Let $H \subsetneq^e X \subseteq M$ be an essential extension, then $0 \neq N\cap X \subseteq X$, hence $0 \neq (N\cap X)\cap H = N\cap H$, which is a contradiction.
(b) ⇒ (c). Let L be a pseudo–complement of H, then $H\cap L = 0$; if H' is a pseudo–complement of L such that $H \subsetneq H'$, then H is not essential in H', and there exists $0 \neq X \subseteq H'$ such that $H\cap X = 0$; it is easy to show that $H\cap(L+X) = 0$, which is a contradiction as L is a pseudo–complement of H.
(c) ⇒ (a). It is evident.
(a) ⇒ (d). Let $0 \neq \frac{L}{H} \subseteq \frac{M}{H}$ such that $0 = \frac{L}{H}\cap\frac{K}{H} = \frac{L\cap K}{H}$, then $(L\cap N)\cap K = H\cap N = 0$, hence $L\cap N = 0$, which is a contradiction as H is a pseudo–complement of N.
(d) ⇒ (a). Let L be a pseudo–complement of H, then $H \subseteq H+L \subseteq^e M$, and $\frac{H+L}{H} \subseteq^e \frac{M}{H}$. Let $H' \supseteq H$ be a pseudo–complement of L, then

$$\frac{H'}{H} \cap \frac{H+L}{H} = \frac{H'(H+L)}{H} = \frac{H+(H'+L)}{H} = 0.$$

Therefore, $\frac{H'}{H} = 0$, and H' is a pseudo–complement. □

A submodule $H \subseteq M$ of a right R–module M satisfying the equivalent conditions in the above theorem if also called an **essentially closed submodule** in M.

Proposition. 55.10.
Let $H \subseteq N \subseteq M$ be submodules such that $H \subseteq N$ and $N \subseteq M$ are essentially closed submodules, then $H \subseteq M$ is a essentially closed submodule.

Remark. 55.11.
As we saw before, essentially closed submodules are transitive, but they are not closed under intersections.
In the abelian group $M = \mathbb{Z} \oplus \mathbb{Z}_2$ the submodules

$$N_1 = \langle (1,0) \rangle \text{ and}$$
$$N_2 = \langle (1,1) \rangle,$$

are essentially closed submodules, but not its intersection: $N_1 \cap N_2 = \langle (2,0) \rangle \subseteq M$.

Remark. 55.12.
As a consequence of Lemma (55.8.), for any submodule $N \subseteq M$ there exists a essentially closed submodule $H \subseteq M$ such that $N \subseteq^e H \subseteq M$.

Exercises

Exercise. 55.13.
Give an example of families of right R–modules $\{N_i \mid i \in I\}$ and $\{H_i \mid i \in I\}$ such that $N_i \subseteq^e H_i$ for any index $i \in I$, and $\prod_i N_i \subseteq \prod_i H_i$ is not essential.
Ref.: 2108e_000 *SOLUCIÓN*

SOLUTION. Ejercicio (55.13.)
We consider $R = \digamma$, $I = \mathbb{N}^*$, $N_n = n\mathbb{Z}$, and $H_n = \mathbb{Z}$ for any $n \in I$. We have an inclusion $\prod_n N_n \subseteq \prod_n H_n$, but it is not essential. To prove that let us consider $x = (h_n)_n$, being $h_n = 1$ for every $n \in I$, then $(\prod_n N_n : x) = 0$, hence the inclusion is not essential. □

Exercise. 55.14.
Give an example of a family of right R–modules $\{N_i \mid i \in I\}$ such that $\oplus_i N_i \subseteq \prod_i N_i$ is not essential.
Ref.: 2108e_008 *SOLUCIÓN*

SOLUTION. Ejercicio (55.14.)
We consider $I = \mathbb{N}^*$, and $N_n = \mathbb{Z}$. We have an inclusion $\oplus_n N_n \subseteq \prod_n N_n$, but it is not essential. To prove that let us consider $x = (x_n)_n$, being $x_n = 1$ for every $n \in I$, then $(\oplus_n N_n : x) = 0$, hence the inclusion is not essential. □

Exercise. 55.15.
For any right R–module M are equivalent:
(a) M is noetherian.
(b) Every essential submodule is finitely generated.

Ref.: 2108e_009 *SOLUCIÓN*

SOLUTION. Ejercicio (55.15.)
(b) ⇒ (a). Let $N \subseteq M$ be a submodule, the family $\{H \subseteq M \mid N \cap H = 0\}$ has maximal elements. If H is a maximal element, then $N \oplus H \subseteq^e M$, hence, by the hypothesis, $N \oplus H$ is finitely generated, and N is finitely generated. □

Exercise. 55.16.
Prove that for any right R–module E the following statements are equivalent:
(a) E is injective.
(b) $j^* : \mathrm{Hom}_A(R, E) \longrightarrow \mathrm{Hom}_R(\mathfrak{c}, E)$ is surjective for any essentially closed right ideal $j : \mathfrak{c} \subseteq R$.

Ref.: 2108e_010 *SOLUCIÓN*

SOLUTION. Ejercicio (55.16.)
Let us consider the diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "f"'] & R \\ & E & \end{tikzcd}$$

There exists a right ideal $\mathfrak{b} \subseteq R$ maximal satisfying $\mathfrak{a} \cap \mathfrak{b} = 0$, hence $\mathfrak{a} \oplus \mathfrak{b} \subseteq^e R$ is essential, and we may extend f to $\mathfrak{a} \oplus \mathfrak{b}$ defining the image of $\mathfrak{b}$ equal to zero. Therefore we may extend f to R, hence E is injective. □

Exercise. 55.17.
If R is a rings with a unique two–sided maximal ideal $\mathfrak{c}$, then $\mathfrak{c} \subseteq^e R$ is essential if, and only if, $\mathfrak{c} \neq 0$.
Ref.: 2108e_011 *SOLUCIÓN*

SOLUTION. Ejercicio (55.17.)
If $\mathfrak{c} \neq 0$, and there exists a right ideal $\mathfrak{a}$ such that $\mathfrak{a} \cap \mathfrak{c} = 0$, then $\mathfrak{a}\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{c} = 0$. If $R\mathfrak{a} \subseteq \mathfrak{c}$, then $\mathfrak{a} \subseteq R\mathfrak{a} \subseteq \mathfrak{c}$, and $\mathfrak{a} = 0$. If $R\mathfrak{a} \nsubseteq \mathfrak{c}$, then $R\mathfrak{a} = R$, and $\mathfrak{c} = R\mathfrak{c} = R\mathfrak{a}\mathfrak{c} = 0$, which is a contradiction. □

Exercise. 55.18.
Prove that every non–zero right R–module contains an essential submodule which is a direct sum of cyclic submodules.
Ref.: 2108e_012 *SOLUCIÓN*

SOLUTION. Ejercicio (55.18.)

Let us consider the family $\Gamma = \{\{C_i \mid i \in I\} \mid$ is a direct sum of cyclic submodules$\}$. Since $\{0\} \in \Gamma$, we have $\Gamma \neq \emptyset$. If we have a chain of elements in Γ, let $\{C_i \mid i \in I_1\} \subseteq \{C_i \mid i \in I_t\} \subseteq \cdots$, then we may assume $I_1 \subseteq \cdots \subseteq I_t \subseteq \cdots \subseteq I$ and take $I = \cup_t I_t$. Therefore, $\{C_i \mid i \in I\}$ is an upper bound of this family. Therefore Γ is inductive and by Zorn's lemma there is a maximal element $\{C_i \mid i \in I\}$ in Γ. Finally we may prove that $\oplus_i C_i \subseteq M$ is an essential submodule. Indeed, if $\oplus_i C_i \subseteq M$ is not essential, there is $0 \neq x \in M$ such that $xR \cap (\oplus_i C_i) = 0$, and the family $\{C_i \mid i \in I\} \cup \{xR\}$ belongs to Γ, which is a contradiction. $\square$

Exercise. 55.19.
Let A be a commutative ring. The following statements are equivalent:

(a) A is semiprime.
(b) The product of finitely many essential ideals is essential.

Ref.: 2108e_014 SOLUCIÓN

SOLUTION. Ejercicio (55.19.)
(a) $\Rightarrow$ (b). Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subseteq A$ be essential ideals, given $0 \neq x \in A$, there exists $a \in A$ such that $0 \neq xa \in \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_t$, hence $0 \neq (xa)^n \in \mathfrak{a}_1 \cdots \mathfrak{a}_t$.
(b) $\Rightarrow$ (a). Let $x \in A$ such that $x^2 = 0$, and define $\mathfrak{a} = xA$. Let $\mathfrak{b}$ be a pseudo–complement of $\mathfrak{a}$ in A. Since $\mathfrak{a} \oplus \mathfrak{b} \subseteq^e A$ is essential, then $(\mathfrak{a} \oplus \mathfrak{b})^2 \subseteq^e A$ is essential. Since $(\mathfrak{a} \oplus \mathfrak{b})^2 = \mathfrak{a}^2 + \mathfrak{a}\mathfrak{b} + \mathfrak{b}^2 = \mathfrak{a}\mathfrak{b} + \mathfrak{b}^2$, then $\mathfrak{a}\mathfrak{b} + \mathfrak{b}^2 \subseteq^e A$ is essential, hence $\mathfrak{b} \subseteq^e A$ is essential, and $\mathfrak{a} = 0$, so $x = 0$. $\square$

Exercise. 55.20.
Let $\mathfrak{a} \subseteq R$ be a two–sided nilpotent ideal. Prove that $\operatorname{Ann}(_R\mathfrak{a}) \subseteq R$ *is an essential right ideal of R.*
Ref.: 2108e_015 SOLUCIÓN

SOLUTION. Ejercicio (55.20.)
Let us assume $\mathfrak{a}^n = 0$ and $\mathfrak{a}^{n-1} \neq 0$. Let $\mathfrak{b} = \operatorname{Ann}(_R\mathfrak{a}) \subseteq R$, it is a two–sided ideal.
If $\mathfrak{c} \subseteq R$ is a right ideal and $\mathfrak{c} \cap \mathfrak{b} = 0$, since $0 \neq \mathfrak{a}^{n-1} \subseteq \mathfrak{b}$ then $\mathfrak{c}\mathfrak{b} \subseteq \mathfrak{c} \cap \mathfrak{b} = 0$, and $\mathfrak{c}\mathfrak{a}^{n-1} = 0$.
Therefore $\mathfrak{c}\mathfrak{a}^{n-2}\mathfrak{a} = 0$, and $\mathfrak{c}\mathfrak{a}^{n-2} \subseteq \mathfrak{c} \cap \mathfrak{b} = 0$. We may continue and obtain $0 = \mathfrak{c}\mathfrak{a}^{n-3} = \cdots = \mathfrak{c}\mathfrak{a}$. Hence $\mathfrak{c} \subseteq \mathfrak{c} \cap \mathfrak{b} = 0$, and $\mathfrak{c} = 0$. $\square$

Exercise. 55.21.
Let $N \subseteq H \subseteq M$ be right R–modules. If $H \subseteq M$ is a complement submodule, then $H/N \subseteq M/N$ is a complement submodule. The converse could be false.
Ref.: 2108e_016 *SOLUCIÓN*

SOLUTION. Ejercicio (55.21.)
If $H/N \subseteq^e L/N \subseteq M/N$ is an essential extension, for any $l \in L \setminus N$ there exists $r \in R$ such that $ls \in H \setminus N$, and for any $0 \neq l \in N$ we have $0 \neq 1 \in N \subseteq H$.
We consider $2\mathbb{Z}_4 \subseteq 2\mathbb{Z}_4 \subseteq \mathbb{Z}$, the $2\mathbb{Z}_4 \subseteq \mathbb{Z}$ is not a complement submodule, but $0 \subseteq \mathbb{Z}_2$ is a complement submodule. □

Exercise. 55.22.
Let $N \subseteq H \subseteq M$ right R–module. If there exists a pseudo–complement N' of N in M, then there exists a pseudo–complement H' of H in M such that $H' \subseteq N'$.
Ref.: 2108e_017 *SOLUCIÓN*

SOLUTION. Ejercicio (55.22.)
We consider $H \cap (N \oplus N') = H_0$. In M/N we consider $H_0/N \subseteq (N \oplus N')/N$ and H_0'/N a pseudo–complement of H_0/N in $(N \oplus N')/N$, then $H_0 = N \oplus N_0$ with $N_0 \subseteq N'$.
$H_0' \cap H_0 = (N + N_0) \cap H \cap (N + N') = H \cap (N + N_0) = N + (H \cap N_0) \subseteq N$.
Since $N_0' \cap H_0 \subseteq H_0' \cap H_0 \subseteq N$, then $N_0' \cap H_0 = 0$:
$N_0' \cap H = N_0' \cap (N + N') \cap H = N_0' \cap H_0 = 0$.
Now we see that N_0' is maximal. Let $N_0' \subseteq L$ such that $L \cap H = 0$.
Let $L \supseteq N_0'$ such that $L \cap H = 0$, then $L \cap N = 0$, and we may assume $L \subseteq N'$.
We have $L \cap H_0 = L \cap H \cap (N + N') = 0$, then $\frac{N+L}{N} \cap \frac{H_0}{N} = \frac{(N+L)\cap H_0}{N} = \frac{N+(L\cap H_0)}{N} = 0$, and $\frac{N+L}{N} \subseteq \frac{H_0'}{N} = \frac{N+N_0'}{N}$, hence $\frac{N+L}{N} = \frac{N+N_0'}{N}$, and $N + L = N + N_0'$. Therefore $L = L \cap (N + L) = L \cap (N + N_0') = (L \cap N) + N_0' = 0 + N_0' = N_0'$. □

Exercise. 55.23.
If $H \subseteq M$ is a complement submodule and $N \subseteq^e M$ an essential extension, then $H \cap N \subseteq N$ is a complement submodule.
Ref.: 2108e_018 *SOLUCIÓN*

SOLUTION. Ejercicio (55.23.)

There exists $H' \subseteq M$ such that H is a pseudo–complement of H' in M. Let us assume $H' \neq 0$. Since $N \subseteq^e M$, then $H' \cap N \neq 0$, and $(H' \cap N) \cap (H \cap N) = 0$; si there exists $H \cap N \subsetneq L \subseteq N$ such that $(H' \cap N) \cap L = 0$, then $H' \cap L = 0$, and $N \cap H' \cap (H + L) = H' \cap ((N \cap H) + L) = H' \cap L = 0$. □

56 Injective hulls

Let M be a right R–module, there is a monomorphism from M to an injective module E, we are interested in the existence of minimal monomorphisms to injective modules. In order to build one of them we proceed as follows.

Lemma. 56.1.
In any diagram of right R–modules with exact row

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "f"'] & M \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

where E is an injective and j is an essential monomorphism there exists a monomorphism g such the diagram commutes.

PROOF. We have $\mathrm{Ker}(g) \cap N \subseteq \mathrm{Ker}(f) = 0$, then $\mathrm{Ker}(g) = 0$. □

Lemma. 56.2. (Eckmann-Shopf's lemma)
For any right R–module E are equivalent:

(a) E is injective.
(b) E has no proper essential extensions.

PROOF. (a) ⇒ (b). Let $E \subseteq N$ be an essential extension, then we may complete the diagram with a monomorphism g. Hence g is an isomorphism and $E = N$.

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, equal] & N \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

(b) ⇒ (a). If E is not injective, there exists a monomorphism from E to an injective module F. Observe that E is not a direct summand, hence there exists a pseudo–complement H of E in F, and we have an essential monomorphism $E \cong \frac{E\oplus H}{H} \xrightarrow{ess} \frac{F}{H}$. By the hypothesis $\frac{E\oplus H}{H} = \frac{F}{H}$, hence $E \oplus H = F$, which is a contradiction. □

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Corollary. 56.3.
If E is an injective right R–module, then each essentially closed submodule is an injective module. The converse also holds.

PROOF. Let $N \subseteq E$ be a essentially closed submodule, and let us assume there exists an essential extension $0 \to N \xrightarrow{ess} M$. Since E is injective, we may complete the following diagram with a monomorphism g

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Therefore N has a nontrivial essential extension in E, which is a contradiction. □

Let us to collect some results in the following proposition.

Proposition. 56.4.
Every right R–module M contained inan injective right R–module E, there is a submodule injective $M \subseteq^e M' \subseteq E$.

PROOF. Let $M \subseteq E$ be a submodule of an injective right R–module E, and let $M \subseteq^e M' \subseteq E$ be a maximal essential extension. Let L be a pseudo–complement of M' in E, then there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & \frac{M'\oplus L}{L} \arrow[r, "i(esss.)"] \arrow[d, "\nu"] & \frac{E}{L} \\ & M' \arrow[d, "i"] & \\ & E & \end{tikzcd}$$

where $\nu : M' \oplus l/L \longrightarrow M'$ is the canonical isomorphism, and $i : M' \longrightarrow E$ is the inclusion. Since E in injective, there is a homomorphism $\gamma : E(L \longrightarrow E$ such that $\gamma j = i\nu$. Since j is essential and $i\nu$ is an injective map, then γ is an injective map. On the other hand, we consider the inclusión $M \subseteq M'$, then $M \subseteq M' \xrightarrow{\nu^{-1}} M' \oplus L/L \xrightarrow{j} E/L$ is an essential injective map, hence $M \subseteq^e \gamma(E/L)$, and

$M' \subseteq \gamma(E/L)$; by the maximality of M' we have $M' = \gamma(E/L)$, and γ factorizes through M'.

$$\begin{tikzcd}
0 \arrow[r] & \frac{M'\oplus L}{L} \arrow[r, "i(esss.)"] \arrow[d, "\nu"'] & \frac{E}{L} \arrow[dl, dashed, "\gamma'"'] \arrow[ddl, dashed, "\gamma"] \\
M \arrow[r] & M' \arrow[d, "i"'] & \\
& E &
\end{tikzcd}$$

Therefore, γ' is an isomorphism and j also. In particular, $M' \oplus L = E$, and M' is a direct summand of E, hence an injective right R–module. □

As a consequence, for any right R–module M there exists an injective module E and an essential monomorphism $M \longrightarrow E$.

A pair constituted by an injective module E and an essential monomorphism $M \longrightarrow E$ is called an **injective hull** of M. Sometimes we identify M with its image in E, hence we do not mention the monomorphism when we deal with injective hulls.

Theorem. 56.5.
Let M be a right R–module, and $j : M \longrightarrow E$ an injective hull, the following statements hold.

(1) *For any injective module E' and any monomorphism $f : M \longrightarrow E'$ there exists a monomorphism $g : E \longrightarrow E'$ such that $f = gj$.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "f"'] & E \arrow[dl, dashed, "g"] \\
& E' &
\end{tikzcd}$$

(2) *For any essential monomorphism $f : M \longrightarrow E''$ there exists a monomorphism $g : E'' \longrightarrow E$ such that $j = gf$.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "f"'] & E \\
& E'' \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Corollary. 56.6.
For any right R–module M, if $j_i : M \longrightarrow E_i$, $i = 1, 2$, are injective hulls, there exists an isomorphism

$b : E_1 \longrightarrow E_2$ *such that* $j_2 = bj_1$.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "j_1"] \arrow[d, "j_2"'] & E_1 \arrow[dl, dashed, "b"] \\ & E_2 & \end{tikzcd}$$

PROOF. Since j_1 is essential, then b is monomorphism. Since $j_2 = bj_1$ is essential, then b is essential, hence an isomorphism as E_1 is injective. □

In particular, the injective hull of M is determined up to isomorphism. We denote by $E(M)$ an injective hull of M. In general the map $M \mapsto E(M)$ is not functorial.

Example. 56.7.
Let us consider $R = \mathbb{Z}$, and $M = \mathbb{Z}_2$. An injective hull of M is $E(M) = \mathbb{Z}_{2^\infty} = \{\frac{a}{b} + \mathbb{Z} \in \frac{\mathbb{Q}}{\mathbb{Z}} \mid \ b = 2^t\}$. There exists a commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}_{2^\infty} \arrow[d, shift right, "id"'] \arrow[d, shift left, "g"] \\ \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_{2^\infty} \end{tikzcd}$$

where $g(x) = -x$. In this case, both, id and g, extends $\mathrm{id}_{\mathbb{Z}_2}$

Lemma. 56.8.
Let M be a right R–module, the following statements hold:
(1) M in injective if, and only if, $M = E(M)$.
(2) If $N \subseteq^e M$ is essential, then $E(N) = E(M)$.
(3) If E is an injective right R–module and $M \subseteq E$, there is a decomposition $E = E(M) \oplus X$ for some submodule $X \subseteq E$.

Lemma. 56.9.
For any finite family of right R–modules $\{N_i \mid i = 1, \ldots, t\}$ there exists an isomorphism $E(\oplus_{i=1}^t N_i) \cong \oplus_{i=1}^t E(N_i)$.

Lemma. 56.10.
Let $\{M_i \mid i \in I\}$ be an independent family of right R–modules. If $\oplus_i E(M_i)$ is injective, then $E(\oplus_i M_i) = \oplus_i E(M_i)$.

SOLUTION. We have that $\oplus_i M_i \subseteq^e \oplus_i E(M_i)$. □

Theorem. 56.11.
Let R be a ring, the following statements are equivalent:

(a) Every direct sum of injective right R–module is injective.
(b) For any independent family $\{M_i \mid i \in I\}$ of right R–modules we have $\oplus_i E(M_i) = E(\oplus_i M_i)$.
(c) R is right noetherian.

PROOF. (a) $\Leftrightarrow$ (b). It is consequence of the above lemma.
(a) $\Rightarrow$ (c). Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be an ascending chain of right ideals and $\mathfrak{a} = \cup_n \mathfrak{a}_n$. We define $f : \mathfrak{a} \longrightarrow \oplus_n E(R/\mathfrak{a}_n)$ as $f(a) = (a + \mathfrak{a}_n)_n$. Since $\oplus_n E(R/\mathfrak{a}_n)$ is injective, there exists an extension, say g, of f to R.

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dashed, "g"] \\ & \oplus_n E(R/\mathfrak{a}_n) & \end{tikzcd}$$

If $(x_n)_n = x = g(1)$, then we have $f(a) = xa$. Let $m \in \mathbb{N}$ such that $x_{m+k} = 0$ for all $k \in \mathbb{N}$, then for any $a \in \mathfrak{a}$ we have $a + \mathfrak{a}_m = (ax)_m = 0$; this means $a \in \mathfrak{a}_m$, i.e., $\mathfrak{a} = \mathfrak{a}_m$, and the chain is stationary.
(c) $\Rightarrow$ (a). Since R is right noetherian, every right ideal is finitely generated. For any independent family if injective right R–modules, any right ideal $\mathfrak{a} \subseteq R$ and any map $f : \mathfrak{a} \longrightarrow \oplus_i E_i$, the image of f is contained in a direct sum fo finitely many $E_i's$, hence it may be extended to R, and $\oplus_i E_i$ is injective. □

Theorem. 56.12.
Let E_1, E_2 be two injective right R–modules, if there are monomorphism $E_1 \longrightarrow E_2$ and $E_2 \longrightarrow E_1$, then $E_1 \cong E_2$.

Ver R. T. Bumby. *Modules which are isomorphic to submodules of each other*. Archiv der Mathematik, **16** (1965), 184–185.

PROOF. Let us assume $E_1 \subseteq E_2$ and there is a monomorphism $f : E_2 \to E_1$, then we have:

$$\begin{aligned} E_2 &= E_1 \oplus G \\ &= f(E_2) \oplus H \oplus G \\ &= f(E_1) \oplus f(G) \oplus G \oplus H \\ &= ff(F_2) \oplus f(H) \oplus f(G) \oplus G \oplus H \\ &= ff(E_1) \oplus f(G) \oplus G \oplus H \oplus f(H) \\ &\dots \\ &= f^{t+1}(E_1) \oplus (G \oplus f(G) \oplus \cdots) \oplus (H \oplus f(H) \oplus \cdots) \end{aligned}$$

If we call $X = G \oplus f(G) \oplus \cdots$, then $X \cap E_1 = f(G) \oplus ff(G) \oplus \cdots = f(X)$.
Let Y be the maximal essential extension of $X \cap E_1$ in E_1; since Y is injective, then $E_1 = Y \oplus Z$, and we have:

$$E_2 = E_1 \oplus G = Y \oplus Z \oplus G = (G \oplus Y) \oplus Z.$$

Hence $Y \oplus G$ is injective.
Since $X = G \oplus f(X) \subseteq G \oplus Y$, and $f(X) = X \cap E_1$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\cong"] \arrow[d, "f"', "\cong"] & G \oplus X \cap E_1 \arrow[r, "ess"] & G \oplus Y \arrow[ddll, dashed, "g"] \\ & X \cap E_1 \arrow[d, "ess"'] & & \\ & Y & & \end{tikzcd}$$

Then g is an isomorphism, and we have:

$$E_2 = G \oplus Y \oplus Z \xrightarrow{g \oplus Z} Y \oplus Z = E_1.$$

□

Let $N \subseteq M$ be a submodule of a right R–module, if we consider $E(N) \subseteq E(M)$ and $N' = M \cap E(N)$, then $N \subseteq^e N'$ is an essential extension of N in M, hence $N' \subseteq M$ is an essentially closed submodule. Thus $N \subseteq M$ is an essentially closed submodule if, and only if, $N = M \cap E(N)$.

Exercise. 56.13.
Let R be a noetherian ring and let $\{N_i \mid i \in I\}$ and $\{H_i \mid i \in I\}$ families of right R–modules such that $N_i \subseteq H_i$ is an essentially closed submodule for any index $i \in I$, then $\oplus_i N_i \subseteq \oplus_i H_i$ is an essentially closed submodule.
Ref.: 2108e_006 *SOLUCIÓN*

SOLUTION. Ejercicio (56.13.)
We have $N_i = H_i \cap E(N_i)$ for any index $i \in I$, and $E(\oplus_i N_i) = \oplus_i E(N_i)$ as R is noetherian, then $\oplus_i N_i = \oplus_i (H_i \cap E(N_i)) = (\oplus_i H_i) \cap (\oplus_i E(N_i)) = (\oplus_i H_i) \cap E(\oplus_i N_i)$. □

More on essentially closed submodules

We shall prove that every essentially closed submodule N in a right R–module M is the trace on M of the injective hull of $E(N)$, hence the essentially closed submodules are the traces of injective submodules of $E(M)$.

Lemma. 56.14.
Let $N \subseteq^e M$ be an essential submodule of a right R–module M, for a submodule $H \subseteq N$ the following statements are equivalent:

(a) $H \subseteq N$ is a essentially closed submodule.
(b) $H = N \cap H'$ for some $H' \subseteq M$ essentially closed submodule.

PROOF. (a) ⇒ (b). We consider $H' \subseteq M$, a maximal essential extension of H in M, then $H \subseteq N \cap H'$, and $H \subseteq^e H'$, hence $H = N \cap H \subseteq^e N \cap H' \subseteq N$; by the hypothesis we obtain $H = N \cap H'$.
(b) ⇒ (a). Let $H' \subseteq M$ be an essentially closed submodule, and let $N \cap H' \subseteq L \subseteq^e N$, then $L \subseteq^e M$, and $L + H' \subseteq^e M$. By Theorem (55.9.) we have $\frac{L+H'}{H'} \subseteq^e \frac{M}{H'}$. On the other hand, $\frac{L}{N\cap H'} \subseteq \frac{N}{N\cap H'} \cong \frac{N+H'}{H'}$, and $\frac{L}{N\cap H'}$ can be identity with $\frac{L+H'}{H'} \subseteq \frac{N+H'}{H'} \subseteq \frac{M}{H'}$, which id essential. By Theorem (55.9.)we have $N \cap H' \subseteq N$ is an essentially closed submodule. □

Corollary. 56.15.
Let $N \subseteq M$ be a submodule, the following statements are equivalent:

(a) $N \subseteq M$ is an essentially closed submodule.
(b) For every $X \subseteq N$ essentially closed submodules we have $X \subseteq M$ is an essentially closed submodule.

PROOF. (a) ⇒ (b). From the inclusion $X \subseteq N \subseteq M$ we have $E(X) \subseteq E(N) \subseteq E(M)$. Since $X \subseteq N$ is an essentially closed submodule then $X = N \cap E(X)$, and since $N \subseteq M$ is an essentially closed submodule, then $N = M \cap E(N)$. Then $X = N \cap E(X) = M \cap E(N) \cap E(X) = M \cap E(X)$, and $X \subseteq M$ is an essentially closed submodule. □

Corollary. 56.16.
Let $N \subseteq M$ be a submodule, the following statements are equivalent:

(a) $N \subseteq M$ is an essentially closed submodule.
(b) For every $E(N) \subseteq E(M)$ we have $N = M \cap E(N)$.

Indecomposable injective modules

A right R–module M is **indecomposable** if for any decomposition $M = N_1 \oplus N_2$ we have either $N_1 = M$ or $N_2 = M$. A weakest notion is the following one; a right R–module M is **uniform** if and any nonzero submodules $0 \neq N_1, N_2 \subseteq M$ we have $N_1 \cap N_2 \neq 0$, or equivalently, every nonzero submodule is essential. It is evident that every uniform right R–module is indecomposable. For injective modules both notions coincide.

Lemma. 56.17.
If E is an injective right R–module, then E is indecomposable if, and only if, E is uniform; a a consequence, for any right R–module M, the following statements are equivalent:

(a) M is uniform.
(b) $E(M)$ is indecomposable.

Our interest now is, given a right R–module M, decompose the injective hull of M as a direct sum of indecomposable injective submodules, hence we need to have **enough uniform submodules** in the following sense: every nonzero submodule of M contains a uniform submodule.

Lemma. 56.18.
Let R be a right noetherian ring, then any nonzero right R–module contains a uniform submodule.

PROOF. Let M be a nonzero right R–module, let $0 \neq m \in M$, then $0 \neq mR \subseteq M$. If mR is not uniform, there exist $0 \neq N_1, N_1' \subseteq mR$ such that $N_1 \cap N_1' \neq 0$. If N_1' is not uniform, there exist $0 \neq N_2, N_2' \subseteq N_1'$ such that $N_2 \cap N_2' = 0$. In this way, if we never find a uniform submodule N_i', then we obtain a strictly ascending chain $\{N_1 \oplus \cdots \oplus N_i \mid i \in \mathbb{N} \setminus \{0\}\}$ of submodules of mR, which is a contradiction as mR is noetherian. □

Proposition. 56.19.
Let R be a right noetherian ring, then any injective right R–module E is a direct sum of indecomposable injective submodules.

PROOF. Let $E \neq 0$ be a nonzero injective right R–module, there exists a uniform submodule $U \subseteq E$, hence $E(U) \subseteq^{\oplus} E$ is a direct summand. Therefore, the family Γ, of independent families $\{E_i \mid i \in I\}$ of indecomposable injective submodules, is nonempty and inductive whenever we consider the inclusion. By Zorn's lemma there exists a maximal element in Γ. Let $\{E_i \mid i \in I\} \in \Gamma$ be a maximal element. If $\oplus_i E_i \neq E$, since $\oplus_i E_i \subseteq^{\oplus} E$ is a direct summand, there exists $0 \neq H \subseteq E$ such that $E = (\oplus_i E_i) \oplus H$, and there exists an indecomposable injective submodule $E_0 \subseteq H$, hence $\{E_i \mid i \in I\} \cup \{E_0\} \in \Gamma$, which is a contradiction. □

Exercise. 56.20.
Se considera la categoría de grupos abelianos; en este caso $R = \mathbb{Z}$.

(1) Prueba que $\mathbb{Z}$ es un grupo abeliano uniforme. Determina todos los grupos cíclicos uniformes.
(2) Prueba que el grupo $\mathbb{Z}_{p^\infty}$ es un grupo uniforme y no es un grupo cíclico. Se consideran $\mathbb{Q}$ y $\mathbb{R}$; ¿es alguno uniforme?
(3) Determina todos los grupos abelianos inyectivos indescomponibles.
(4) Si M es un grupo abeliano finitamente generado sabemos que $M \cong (\oplus_{i=1}^t \mathbb{Z}_{p_i^{n_i}}) \oplus \mathbb{Z}^n$, para $n, n_1, \ldots, n_t \in \mathbb{N}$. ¿Cuál es la descomposición de $E(M)$ como suma de inyectivos indescomponibles.

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SOLUTION. Ejercicio (56.20.)
HACER □