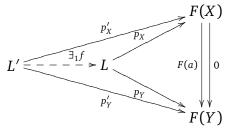
Example. 4.20. (Direct product of modules)

Let I be a set, we consider the category \mathscr{I} whose objects are the elements of I and the only morphisms the identities. For any functor $F: \mathscr{I} \longrightarrow \mathbf{Mod} - R$, the limit is a pair $(\prod_i F(i), \{p_i\}_i)$, of a module $\prod_i F(i)$, the cartesian product with pointwise operation, and the family of projectios $\{p_i: \prod_i F(i) \longrightarrow F(i) \mid i \in I\}$, it is the direct product of the family of modules $\{F(i) \mid i \in I\}$.

Example. 4.21. (Kernel in modules)

Let us consider the category $\mathscr C$ with two objects $X \xrightarrow{a \atop b} Y$, the identities and two morphisms: a y

b. For any functor $F: \mathscr{C} \longrightarrow \mathbf{Mod} - R$ such that F(b) = 0 a limit is a pair $(L, \{p_X, p_Y\})$ that makes a commutative diagram



therefore $F(a)p_X = 0$, and for any module map $p_X' : L' \longrightarrow F(X)$ such that $F(a)p_X' = 0$, there is a unique module map $f : L' \longrightarrow L$ such that $p_X' = p_X f$. Thus is, we have that (L, f) is the **kernel** of F(a).

Example. 4.22. (Product in Fields)

In the category $\mathscr C$ of fields and field homomorphisms does not exist the product of some families of fields. Let us consider a field extension F/K, where K is the prime subfield of F, i.e., K is either $\mathbb Q$ or $\mathbb F_p$, with Galois group isomorphic to C_2 , for instance $F=\mathbb Q(\sqrt{2})$ and $K=\mathbb Q$. Then $\mathrm{Gal}(F/K)=\{1,\varphi\}$ Let us consider the product of F by F; it is a pair $(E,\{p_1,p_2\})$, and it satisfies a universal property; for any field L and homomorphisms $f_1:K\longrightarrow F_1,\,f_2:L\longrightarrow F_2$ there is a unique field homomorphism $f:L\longrightarrow E$ such that $p_1f=f_1$ and $p_2f=f_2$.

$$F_{1} \stackrel{f_{1}}{\underset{|\exists_{1}f}{\longleftarrow}} F_{2}$$

$$F_{2} \stackrel{\forall}{\underset{p_{1}}{\longleftarrow}} F_{2}$$

We take several pairs $(L, \{f_1, f_2\})$ to check the direct producto.

Let take the pair $(F, \{id, id\})$, then there is a unique map, say $f: F \longrightarrow F$ such that $p_1 f = id$, $p_2 f = id$, i.e., $f = p_1^{-1} = p_2^{-1}$.

If we take the pair $(F, \{\varphi, \mathrm{id}\})$, there exists a unique map $g: F \longrightarrow f$ such that $p_1g = \varphi$, $p_2g = \mathrm{id}$. As a consequence, $p_1^{-1}\varphi = p_2^{-1} = p_1^{-1}$, and $\varphi = \mathrm{id}$, which is a contradiction.

Therefore, product of *F* by *F* does not exist.

Colimits

Let \mathscr{P} be a small category (a diagram), its class of morphism is a set, and let $F: \mathscr{P} \longrightarrow \mathbf{Mod} - R$ be a functor. A **colimit** of F is a pair $(\{q_X\}_X, C)$, consitutes by a module C and a family of module maps