31 Módulos semisimples

Un módulo no nulo M es **simple** si no tiene submódulos propios no nulos.

Proposition. 31.1.
Sea M un módulo a derecha, $\{N_i \mid i \in I\}$ una familia de submódulos simples tales que $\sum_i N_i = M$, para cada submódulo propio $N \subseteq M$ existe $J \subseteq I$ tal que $M = N + (\oplus_{j\in J} N_j)$.

PROOF. Si $N \neq M$, existe N_i tal que $N_i \nsubseteq N$, luego $N \cap N_i = 0$. Consideramos familias $\Gamma_J = \{N_j \mid i \in J \subseteq I\}$ que son independientes y $N \cap (\oplus_{j\in J} N_j) = 0$. Consideramos la familia $\Gamma = \{\Gamma_J \mid$ verifican la propiedad anterior$\}$. Esta familia es inductiva y no vacía, ya que $\{N_i\} \in \Gamma$. Aplicando el lemma de Zorn, existe $\Gamma_J \in \Gamma$ maximal para la inclusión. Se tiene entonces $N \oplus (\oplus_{j\in J} N_j) \subseteq M$; si no son iguales, existe $N_i \nsubseteq N \oplus (\oplus_{j\in J} N_j)$, y por tanto $\Gamma_J \cup \{N_i\} \in \Gamma$, lo que es una contradicción. □

Corollary. 31.2.
Sea M un módulo a derecha, $\{N_i \mid i \in I\}$ una familia de submódulos simples tales que $\sum_i N_i = M$, existe un subconjunto $J \subseteq I$ tal que $M = \oplus_{j\in J} N_j$.

PROOF. Basta tomar $N = 0$ en la proposición anterior. □

Un módulo a derecha M se llama **semisimple** si M es una suma de submódulos simples. Según la definición el módulo cero es semisimple.

Proposition. 31.3.
(1) Todo submódulo de un módulo semisimple es un sumando directo.
(2) Todo cociente de un módulo semisimple es semisimple.
(3) Todo submódulo de un módulo semisimple es semisimple.
(4) La suma directa de módulos semisimples es un módulo semisimple.

Dos módulos simples S_1 y S_2 son **isotípicos** si $S_1 \cong S_2$. Dado un módulo semisimple $M = \oplus_i S_i$, para cada submódulo simple $S \subseteq M$ existe un índice $i \in I$ tal que $S \cong S_i$, la componente isotípica de S en M es $\oplus_j S_j$, donde $S \cong S_j$, la representamos por M_S.

2104-05.tex

Proposition. 31.4.
Cada módulo semisimple es la suma directa de sus componentes isotípicas, y para cada submódulo simple $S \subseteq M$ tenemos $S \subseteq M_S$.

PROOF. Dado $S \subseteq M$ simple existe $J \subseteq I$ tal que $M = S \oplus (\oplus j \in JS_j$; entonces $S \cong \oplus_{i\in I\setminus J} S_i$, y por tanto $I \setminus J$ es unitario, esto es, S es isomorfo a uno de los S_i.
Dado $T \subseteq M$ simple, tenemos $T \subseteq \oplus_S M_S$, y no todas las proyecciones a los M_S son nulas. Si $T \ncong S$, y $p_S(T) \neq 0$, entonces $T \cong p_S(T)$, esto es, T isomorfo a un submódulo de M_S, lo que es una contradicción, ya que T es necesariamente isomorfo a un sumando directo de M_S. Tenemos entonces $p_(T) = 0$ si $S \ncong T$, y por tanto $T \subseteq_{M_T}$. □

Corollary. 31.5.
Para cada módulo semisimple M, si $M = \oplus_S M_S$ es la descomposición en componentes isotípicas, entonces $\mathrm{End}_R(M) = \prod_S \mathrm{End}_R(M_S)$.

El problema ahora es determinar la estructura de $\mathrm{End}_R(M)$, cuando M tiene una sola componente isotípica. Supongamos que $M = \oplus_i S_i$, con $S_i \cong S_j$ para todos i, j. Es claro que

$$\mathrm{End}_R(\oplus_i S_i) \cong \prod_i \mathrm{End}_R(S_i, \oplus_j S_j)$$

Identificamos S con $R/\mathfrak{m}$, con $\mathfrak{m} \subseteq R$ un ideal derecha maximal, y sea $e = 1 + \mathfrak{m}$. Un sistema de generadores de $\oplus_i S_i$ es $e_i = (e\delta_{i,j})_j$.
Para cada homomorfismo $f : S \longrightarrow S$ se tiene $f(e) = ea$, con a determinado módulo $\mathfrak{m}$. Tenemos que $D = \mathrm{End}_R(S)$ es un anillo de división, S es un D–módulo izquierda, y existe una aplicación $\nu : \mathrm{End}_R(S) \longrightarrow R/\mathfrak{m}$, definida $\nu(f) = \overline{a}$, siendo $f(e) = ea$, que verifica:

$$\begin{aligned} \nu(f_1 + f_2) &= \overline{a_1} + \overline{a_2}, \\ \nu(f_1 f_2) &= \overline{a_1 a_2}. \end{aligned}$$

Tenemos que ν es un homomorfismo de grupos abelianos, pero no de anillos, ya que $R/\mathfrak{m}$ no lo es.
Para cada $f \in \mathrm{End}_R(M)$ y cada e_i, se tiene $f(e_i) = \sum_j e_j a_{j,i} = \sum_j f_{j,i}(e_j)$, siendo los $a_{j,i}$ determinados unívocamente módulo $\mathfrak{m}$, no así los $f_{j,i}$ que están unívocamente determinados. Tenemos un homomorfismo de anillos $\mathrm{End}_R(M) \longrightarrow MCF_I(D)$, el anillo de las matrices con columnas finitas, que envía cada $f \in \mathrm{End}_R(M)$ a la matriz con columnas finitas $M(f) = (f_{j,i})_{j,i}$ cuya columna i es $(f_{j,i})_j$.
Vamos a particularizar el caso en el que M es una suma directa finita de módulos simples. En este caso tenemos:

Theorem. 31.6.
Si M es una suma directa finita de n copias de un módulo simple S, entonces $\operatorname{End}_R(M) \cong M_n(\operatorname{End}_R(S))$*, es un anillo de matrices sobre un anillo de división.*

Corollary. 31.7.
Si M es un módulo semisimple finitamente generado, se tiene que $\operatorname{End}_R(M)$ *es un producto finito de anillos de matrices sobre anillos de división.*

Un anillo R es un **anillo semisimple** (derecha) si R_R es un R–módulo derecha semisimple.

Lemma. 31.8.
Sea R un anillo, son equivalentes:

(a) R_R es un módulo semisimple
(b) $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, donde cada D_i es un anillo de división.
(c) ${}_RR$ es un módulo semisimple.

En este punto hay dos propiedades de los anillos semisimples que conviene destacar:

(1) Todo anillo semisimple es un anillo artiniano derecha.
(2) El único ideal derecha nilpotente de R es el ideal cero (diremos que R es un anillo **semiprimo**).

Tenemos que estas dos propiedades, como veremos más adelante, caracterizan los anillos semisimples.

10 Tensor product

Let M be a right R–module and N be a left R–module. A group map $\varphi : M \times N \to A$ is called R**–bilinear** if it satisfies the following conditions:

(i) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$,
(ii) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$, and
(iii) $\varphi(mr, n) = \varphi(m, rn)$.

for any $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.
We are interesting in building an abelian group satisfying a universal property with respect to R–bilinear maps. Thus we build the free abelian group G on the set $\{(m,n) \mid m \in M, n \in N\}$, consider the subgroup B of G generated by the elements:

- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$, for all $m_1, m_2 \in M$ and $n \in N$,
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$, for all $m \in M$ and $n_1, n_2 \in N$,
- $(mr, n) - (m, rn)$, for all $m \in M$, $n \in N$ and $r \in R$,

and the quotient group G/B. Hence there is a natural map $M \times N \xrightarrow{b} G/B$ defined by $b(m,n) = (m,n) + B$. For simplicity we represent the element $(m,n)+B$ by $m \otimes n$, and the quotient group G/B by $M \otimes_R N$. Thus every element of $M \otimes_R N$ has an expression as $\sum_i m_i \otimes n_i$. Observe that this expresión is not unique. Finally, we have that the map $b : M \times N \to M \otimes_R N$ is an R–bilinear map. We call $M \otimes_R N$ the **tensor product** over R of M and N.

Proposition. 10.1. (Tensor product universal property)
Let M_R and $_RN$ be modules and $f : M \times N \to X$ be a R–bilinear map, there exists a unique group map $\overline{f} : M \otimes_R N \to X$ such that $f = \overline{f} \circ b$.

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, dashed, "\exists_1 \overline{f}"] \\ & X \end{tikzcd}$$

The following result is elemental from the above definition.

Proposition. 10.2.
Let M_R and $_RN$ be modules, then $R \otimes_R N \cong N$, and $M \otimes_R R \cong M$.

2102-01.tex

Proposition. 10.3.
Let A be a commutative ring and let M,N and P be A–modules, the followings statements hold:

(1) $M \otimes_A N \cong N \otimes_A M$.
(2) $M \otimes_A N$ *is an A–module with action defined by* $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.
(3) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

Tensor product of algebras

Sea A un anillo conmutativo y R un anillo. Dar una estructura de A–álgebra en R es dar un homomorfismo de anillos $h : A \longrightarrow R$ tal que $h(A) \subseteq \mathrm{Cen}(R)$. Decimos entonces que R es una **A–álgebra**. Observa que R es un A–módulo a la derecha y a la izquierda con acción dada por

$$ar = h(a)r = rh(a) = ra,$$

para cada $a \in A$ y cada $r \in R$.
Dadas dos A–álgebras R y S, un **homomorfismo de A–álgebras** es un homomorfismo de anillos $f : R \longrightarrow S$ tal que $f(ar) = af(r)$. Esto significa que si $h' : A \longrightarrow S$ es el homomorfismo de estructura de S, entonces el siguiente diagrama de homomorfismos de anillos es conmutativo:

$$\begin{tikzcd} & A \arrow[dl, "h"'] \arrow[dr, "h'"] & \\ R \arrow[rr, "f"'] & & S \end{tikzcd}$$

Se tiene:

$$hf(a) = f(a1) = af(1) = a1 = h'(a).$$

Si R y S son dos A–álgebras, y consideramos el producto tensor $R \otimes_A S$, tenemos una estructura de grupo abeliano en $R \otimes_A S$, que podemos extender a una estructura de A–módulo, ver Proposición (10.2.), definiendo

$$a(r \otimes s) = (ar) \otimes s = r \otimes (as).$$

Además, $R \otimes_A S$ es un anillo con producto definido por

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2),$$

y extendiendo por distributividad a todos los elementos de $R \otimes_A S$. En este caso existen homomorfismos de A–álgebras $q_1 : R \longrightarrow R \otimes_A S$ y $q_2 : S \longrightarrow R \otimes_A S$ definidos

$$q_1(r) = r \otimes 1$$
$$q_2(s) = 1 \otimes s.$$

Lemma. 10.4.
Sean R, S y T tres A–álgebras, para dos homomorfismos de A–álgebras $f_1 : R \longrightarrow T$ y $f_2 : S \longrightarrow T$ tales que $f_1(r)f_2(s) = f_2(s)f_1(r)$, para cada $r \in R$ y $s \in S$, existe un único homomorfismo de A–álgebras $h : R \otimes_A S \longrightarrow T$ tal que

$$\begin{tikzcd} R \arrow[r, "q_1"] \arrow[dr, "f_1"'] & R \otimes_A S \arrow[d, dashed, "\exists_1 h"] & S \arrow[l, "q_2"'] \arrow[dl, "f_2"] \\ & T & \end{tikzcd}$$

PROOF. Consideramos la aplicación $\varphi : R \times S \longrightarrow T$ definida por $\varphi(r,s) = f_1(r)f_2(s)$, es claro que φ es A–bilineal, y existe un único homomorfismo de grupos abelianos $h : R \otimes_A S \longrightarrow T$ definido $h(r \otimes s) = f_1(r)f_2(s)$. Falta comprobar que h es un homomorfismo de A–álgebras. □

Corollary. 10.5.
Si R_1, R_2, S_1, S_2 son A–álgebras y $f : R_1 \longrightarrow R_2$, $g : S_1 \longrightarrow S_2$ son homomorfismos de A–álgebras, existe un único homomorfismo de A–álgebras, $h : R_1 \otimes_A S_1 \longrightarrow R_2 \otimes_A S_2$, que hace conmutar el siguiente diagrama

$$\begin{tikzcd} R_1 \arrow[r, "q_{1,1}"] \arrow[d, "f"'] & R_1 \otimes_A S_1 \arrow[d, dashed, "\exists_1 h"] & S_1 \arrow[l, "q_{2,1}"'] \arrow[d, "g"] \\ R_2 \arrow[r, "q_{1,2}"] & R_2 \otimes_A S_2 & S_2 \arrow[l, "q_{2,2}"'] \end{tikzcd}$$

Corollary. 10.6.
En la categoría de A–álgebras conmutativas el producto tensor de dos A–álgebras, junto con los homomorfismos q_1 y q_2, es un coproducto.

Corollary. 10.7.
Si A es un anillo conmutativo, para indeterminadas X e Y se tiene un isomorfismo de A–álgebras $A[X] \otimes_A [Y] \cong A[X, Y]$.

Lemma. 10.8.
Sean R y S dos A–álgebras, se tiene

(1) $\mathrm{Cen}(R) \xrightarrow{incl} R$ *es un homomorfismo de A–álgebras.*
(2) Si R es un anillo simple (no tiene ideales biláteros propios no nulos), entonces $\mathrm{Cen}(R)$ *es un cuerpo.*
(3) Existe un homomorfismo de A–álgebras $h : \mathrm{Cen}(R) \otimes_A \mathrm{Cen}(S) \longrightarrow R \otimes_A S$ cuya imagen está contenida en $\mathrm{Cen}(R \otimes_A S)$.

PROOF. (1). Es inmediato.

(2). Sea $0 \neq a \in \mathrm{Cen}(R)$, entonces $Ra = aR$ es un ideal (bilátero) no nulo, por tanto $Ra = aR = R$, y a es un elemento invertible.

(3). Definimos $f : \mathrm{Cen}(R) \longrightarrow R \otimes_A S$ mediante $f(r) = r \otimes 1$ y definimos $g : \mathrm{Cen}(S) \longrightarrow R \otimes_A S$ mediante $g(s) = 1 \otimes s$. Es claro que $f(r)g(s) = (r \otimes 1)(1 \otimes s) = (1 \otimes s)(r \otimes 1) = g(s)f(r)$, y por tanto existe un homomorfismo de A–álgebras $j : \mathrm{Cen}(R) \otimes_A \mathrm{Cen}(S) \longrightarrow R \otimes_A S$, definido $j(r \otimes s) = (r \otimes s$. Es claro que $\mathrm{Im}(j) \subseteq \mathrm{Cen}(R \otimes_A S)$. Tenemos que la acción está dada por el homomorfismo composición siguiente:

$$A \longrightarrow A \otimes_A A \xrightarrow{h \otimes h'} \mathrm{Cen}(R) \otimes_A \mathrm{Cen}(S) \xrightarrow{j} R \otimes_A S$$

□

Example. 10.9. (Coproducto en la categoría de A–álgebras)
Sea A un anillo conmutativo y R, S dos A–álgebras, vamos a construir el coproducto de R y S en la categoría de A–álgebras y homomorfismos de A–álgebras.

Construimos una A–álgebra T de la siguiente forma:

(1) Definimos $T_0 = A$,
(2) Definimos $T_1 = R \oplus S$,
(3) Si $t \geq 1$, definimos $T_{t+1} = (T_t \otimes_A R) \oplus (T_t \otimes_A S)$.
(4) Se define $T = \oplus_{i=0}^{\infty} T_i$.

Es claro que T es un A–módulo, y que $q_1 : R \longrightarrow T$, definido $q_1(r) = r$, y $q_2 : S \longrightarrow T$, definido $q_2(s) = s$ son homomorfismos de A–módulos.

En T se define un producto mediante la concatenación de productos tensores, esto es, si $x_1 \otimes \cdots \otimes x_s, y_1 \otimes \cdots \otimes y_t \in T$, se define

$$(x_1 \otimes \cdots \otimes x_s) * (y_1 \otimes \cdots \otimes y_t) = x_1 \otimes \cdots \otimes x_s \otimes y_1 \otimes \cdots \otimes y_t.$$

y la extendemos por linealidad. Tenemos que T es una A–álgebra, con elemento uno ideal al elemento 1 de A, ya que $a * (x_1 \otimes \cdots \otimes x_s) = a \otimes (x_1 \otimes \cdots \otimes x_s) = (ax_1 \otimes \otimes \cdots \otimes x_s)$, para $a \in A$.

En T consideramos el ideal $\mathfrak{A}$ generado por los elementos:

$$\begin{array}{l} r_1 \otimes r_2 - r_1 r_2, \\ s_2 \otimes s_2 - s_1 s_2, \\ a - a_R, \\ a - a_S, \end{array}$$

para $r_1, r_2 \in R$, $s_1 s_2 \in S$ y $a \in A$, siendo $a_R \in R$ y $a_S \in S$ las imágenes de a en R y S, respectivamente, y consideramos el cociente $T/\mathfrak{A}$. Tenemos homomorfismos de A–álgebras:

$$\begin{array}{l} q_1 : R \xrightarrow{q_1} T \xrightarrow{proy.} T/\mathfrak{A}, \\ q_2 : S \xrightarrow{q_2} T \xrightarrow{proy.} T/\mathfrak{A}. \end{array}$$

Veamos que el par $(T/\mathfrak{A}\{q_1, q_2\})$ es un coproducto de R y S. Dada una A–álgebra U y homomorfismos de A–álgebras $f_1 : R \longrightarrow U$ y $f_2 : S \longrightarrow U$, para cada lista de $R's$ y $S's$, por ejemplo R, S, tenemos una aplicación bilineal $f_1 \times f_2 : R \times_A S \longrightarrow U$ definida $f_1 \times f_2(r,s) = f_1(r)f_2(s)$, por tanto existe un homomorfismo de A–módulos $f_1 \otimes f_2 : R \otimes S \longrightarrow U$, definido $f_1 \otimes f_2(r \otimes s) = f_1(r)f_2(s)$. Como existe un homomorfismo a U de cada sumando de T, existe un homomorfismo de A–módulos de $f : T \longrightarrow U$, definido $f(x_1 \otimes \cdots \otimes x_s) = f_*(x_1) \cdots f_*(x_s)$, utilizando el f_* correspondiente al x_i.
Es claro que f es un homomorfismo de A–álgebras, y que se anula sobre $\mathfrak{A}$, ya que se tiene

$$\begin{array}{l} f(r_1 \otimes r_2 - r_1 r_2) = f_1(r_1)f_2(r_2) - f_1(r_1 r_2) = 0, \\ f(s_1 \otimes s_2 - s_1 s_2) = f_s(s_1)f_2(s_2) - f_2(s_1 s_2) = 0, \\ f(a - a_R) = f_1(a) - f_1(a) = 0, \\ f(a - a_S) = f_2(a) - f_2(a) = 0. \end{array}$$

Por lo tanto f se factoriza por $T/\mathfrak{A}$, esto es, existe un homomorfismo de A–álgebras f':

$$\begin{tikzcd} \mathfrak{a} \arrow[r] & T \arrow[r, "proy."] \arrow[dr, "f"'] & T/\mathfrak{A} \arrow[d, dashed, "f'"] \\ & & U \end{tikzcd}$$

Tenemos un diagrama conmutativo

$$\begin{tikzcd} R \arrow[r, "q_1"] \arrow[dr, "f_1"'] & T/\mathfrak{a} \arrow[d, dashed, "f'"] & S \arrow[l, "q_2"'] \arrow[dl, "f_2"] \\ & U & \end{tikzcd}$$

La unicidad de f' haciendo conmutar este diagrama es consecuencia de la unicidad en la construcción de los homomorfismos que hemos ido realizando a partir de f_1 y f_2.

11 Two-sided modules

Let M be a right R–module, then there exists a ring map $R^{op} \to \mathrm{End}(M)$, and we may define on M a structure of left module over the ring R^{op}. This process can be realized also for left modules. As a consequence we may restrict the study of modules to only consider either right or left modules.

There are examples in with there are more than one action acting on a module, one of this is the following: Let $f : T \to R$ be a ring map, then R has four module structure.

(1) R_R is the natural right action of R over itself.
(2) ${}_RR$ is the natural left action of R over itself.
(3) R_T is the right action defined: $r \cdot t = rf(t)$ induced by the multiplication.
(4) ${}_TR$ is the left action defined: $t \cdot r = f(t)r$ induced by the multiplication.

It is obvious that the actions ${}_TR$ and R_T satisfy the following identity:

$$t_1(rt_2) = (t_1r)t_2,$$

for any $t_1, t_2 \in T$ and any $r \in R$. And the same holds if we consider the actions ${}_RR$ and R_T. But with the actions R_R and R_T we have a problem, if we consider $r, r_1 \in R$ and $t_1 \in T$, then $(rr_1)t_1$, in general, is not equal to $(rt_1)r_1$, as $f(t_1)$ and r_1 necessarily do not commute.

It appears, in the way, some kind of compatibility or incompatibility when we have two actions on a module.

Let M be a left R–module and a right S–module, we say that M is a two-sided $(R;S)$-module, or simply $(R;S)$–module, if

$$r(ms) = (rm)s,$$

for any $r \in R$, $m \in M$ and $s \in S$. In this case we have used structures on the left and on the right for simplicity, observe that we may consider M only with right structures: as a right R^{op}–module and a right S–module, then the compatibility means

$$(ms)r = (mr)s.$$

Proposition. 11.1.
Let M be a left R–module and a right S–module. The following statements are equivalent:

(a) M is a (two-sided) $(R;S)$–module.
(b) The ring map $R \to \mathrm{End}(M)$ factorizes through $\mathrm{End}(M_S) := \mathrm{End}_S(M)$.
(c) The ring map $S^{op} \to \mathrm{End}(M)$ factorizes through $\mathrm{End}({}_RM) := \mathrm{End}_R(M)$.

2102-02.tex

Let M be a (two-sided) $(R;S)$–module, then it is a left module over R and over S^{op}, and there are ring maps $\beta_R : R \to \mathrm{End}(M)$ and $\beta_S : S^{op} \to \mathrm{End}(M)$, and elements in $\mathrm{Im}(\beta_R)$ commute with elements in $\mathrm{Im}(\beta_S)$, as a consequence of the compatibility of the two actions. If we consider the tensor product $R \otimes_{\mathbb{Z}} S^{op}$, there exists a ring map $R \otimes_{\mathbb{Z}} S^{op} \to \mathrm{End}(M)$ defined $r \otimes s \mapsto \beta_R(r)\beta_S(s)$, hence we have a new left action over M defined

$$(r \otimes s)m = rms.$$

When β_R and β_S are surjective maps we say M is a **balanced two–sided $(R;S)$–module**. If β_R and β_S are isomorphisms we says M is a **faithfully balanced two–sided $(R;S)$–module**.
Thus we may complete Proposition (11.1.) with a new equivalent statement:

Proposition. 11.2.
With the same notation as in Proposition (11.1.) the statements there are equivalent to

(d) M is a left $R \otimes_{\mathbb{Z}} S^{op}$–module.
(e) M is a right $R^{op} \otimes_{\mathbb{Z}} S$–module.

Exercise. 11.3.
Let ${}_RM_S$, ${}_RN_T$ be two–sided modules, then $\mathrm{Hom}_R({}_RM_S, {}_RN_T)$ has a natural structure of a two–sided $(S;T)$–module with structures given by:

- *$(sf)(m) = f(ms)$, for any $s \in S$, $f \in \mathrm{Hom}_R(M,N)$ and $m \in M$.*
- *$(ft)(m) = f(m)t$, for any $t \in T$, $f \in \mathrm{Hom}_R(M,N)$ and $m \in M$.*

Ref.: 2102e_017 *SOLUTION*

SOLUTION. Exercise (11.3.)
HACER □

Two–sided modules are a useful tools in the tensor product theory.

Proposition. 11.4.
Let ${}_TM_R$ and ${}_RN_S$ be two–sided modules, then $M \otimes_R N$ is a $(T;S)$–module, and it satisfies the following universal property: Let X be a two–sided $(T;S)$–module and $\varphi : M \times N \to X$ a R–bilinear map satisfying

$$\varphi(tm, ns) = t\varphi(m,n)s,$$

there exists a unique $(T;S)$*–module map* h *such that* $\varphi = h \circ b$.

$$
\begin{tikzcd}
M \times N \arrow[r, "b"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dashed, "\exists_1 h"] \\
& X
\end{tikzcd}
$$

Let ${}_SM_R$ be a $(S;R)$–module, and let ${}_RN_T$ be a $(R;T)$–module, then in the tensor product $M \otimes_R N$ we have a structure of $(S;T)$–module with actions defined by

$$
\begin{aligned}
s(m \otimes n) &= (sm) \otimes n \\
(m \times n)t &= m \otimes (nt).
\end{aligned}
$$

Let ${}_RM_S$ be a $(R;S)$–module and ${}_RN_T$ a $(R;T)$–module, in the abelian group $\mathrm{Hom}_R(M,N)$ of left R–module maps we define a structure of $(S;T)$–module as follows, see Exercise (11.3.):

$$
\begin{aligned}
(sf)(m) &= f(ms) \\
(ft)(m) &= f(m)t.
\end{aligned}
$$

In particular, for any right R–module M the abelian group $\mathrm{Hom}_R(R,M)$ is a right R–module with action given by $(rf)(x) = f(rx)$.

12 The functor tensor product; flat modules

Let ${}_RM$ be a module, for any right R–module N_R we have the tensor product abelian group $N \otimes_R M$.

Lemma. 12.1.
Let ${}_RM$ be a module, for any right R–module map $f : N_1 \longrightarrow N_2$ there is an abelian group map $f \otimes_R M : N_1 \otimes_R M \longrightarrow N_1 \otimes_R M$. The map $f \otimes_R M$ is defined by the R–bilineal map $\varphi : N_1 \times M \longrightarrow N_2 \otimes_R M$ defined $\varphi(n_1, m) = f(n_1) \otimes m$, hence $(f \otimes_R M)(n \otimes m) = f(n) \otimes m$.

$$\begin{tikzcd}
N_1 \times M \arrow[r, "b"] \arrow[dr, "\varphi"'] & N_1 \otimes_R M \arrow[d, dashed, "\exists_1 f \otimes_R M"] \\
 & N_2 \otimes_R M
\end{tikzcd}$$

Proposition. 12.2.
Let ${}_RM$ be a module, there is a functor $- \otimes_R M : \mathbf{Mod}\text{–}R \longrightarrow \mathscr{A}b$.

If M is a $(R;S)$–module, then the codomain of the tensor product functor $- \otimes_R M$ is **Mod**–S.

Proposition. 12.3.
Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of right R–modules and ${}_RM$ be a left R–module, then

$$N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

is an exact sequence of abelian groups.

A left R–module ${}_RF$ is **flat** if for any short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ the above sequence is a short exact sequence, i.e., $N_1 \otimes_R F \to N_2 \otimes_R F$ always is a monomorphism.

2102-03.tex

13 The functor Hom; projective and injective modules

Let M_R be a right R–module, for any right R–module N we have two abelian groups $\mathrm{Hom}_R(M,N)$ and $\mathrm{Hom}_R(N,M)$.

Lemma. 13.1.
Let M_R be a right R–module, for any right module map $f : N_1 \longrightarrow N_2$ there is an abelian group map $f_* = \mathrm{Hom}_R(M,f) : \mathrm{Hom}_R(M,N_1) \longrightarrow \mathrm{Hom}_R(M,N_2)$.

Proposition. 13.2.
Let M_R be a right R–module, there is a functor $\mathrm{Hom}_R(M,-) : \textbf{\textit{Mod}}\text{–}R \longrightarrow \mathscr{A}b$.

Lemma. 13.3.
Let M_R be a right R–module, for any module map $f : N_1 \longrightarrow N_2$ there is an abelian group map $f^* = \mathrm{Hom}_R(f,M) : \mathrm{Hom}_R(N_2,M) \longrightarrow \mathrm{Hom}_R(N_1,M)$.

Proposition. 13.4.
Let M_R be a module, there is a contravariant functor $\mathrm{Hom}_R(-,M) : \textbf{\textit{Mod}}\text{–}R \longrightarrow \mathscr{A}b$.

Proposition. 13.5.
Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of right R–modules and M_R be a right R–module, then

$$0 \to \mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2) \to \mathrm{Hom}_R(M,N_3) \text{ and}$$
$$0 \to \mathrm{Hom}_R(N_3,M) \to \mathrm{Hom}_R(N_2,M) \to \mathrm{Hom}_R(N_1,M)$$

are exact sequences.

2102-04.tex

A right R–module E is **injective** if for every short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \mathrm{Hom}_R(N_3, E) \to \mathrm{Hom}_R(N_2, E) \to \mathrm{Hom}_R(N_1, E) \to 0$ is a short exact sequence, i.e., $\mathrm{Hom}_R(N_2 E) \to \mathrm{Hom}_R(N_1, E)$ is an epimorphism, or equivalently, we can complete any diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[dl, dashed] \\ & E & \end{tikzcd}$$

A right R–module P is **projective** if for every short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \mathrm{Hom}_R(P, N_1)) \to \mathrm{Hom}_R(P, N_2) \to \mathrm{Hom}_R(P, N_3) \to 0$ is a short exact sequence, i.e., $\mathrm{Hom}_R(P, N_2) \to \mathrm{Hom}_R(P, N_3)$ is an epimorphism, or equivalently, we can complete any diagram with exact row

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ N_2 \arrow[r] & N_3 \arrow[r] & 0 \end{tikzcd}$$

Let M be a right R–module and ${}_S X_R$ a two–sided $(S;R)$–module. We have a left S–module estructure on $\mathrm{Hom}_R(M, X)$ defined by $sf(m) = f(m)s$. Thus we may consider the right R–module $\mathrm{Hom}_S(\mathrm{Hom}_R(M, X), X)$.

Let ${}_S X_R$ be a two–sided $(S;R)$–module. A right R–module M is X**–reflexive** if the canonical map $M \to \mathrm{Hom}_S({}_S\mathrm{Hom}_R(M_R, X_R)), {}_S X)$ is an isomorphism. In particular, the **reflexive** right R–modules are the R–reflexive modules, i.e., they are the right R–modules such that the homomorphism $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M_R, R_R), {}_R R)$ is an isomorphism.

Adjunctions

Let us consider an **adjunction** $\begin{tikzcd} \mathbf{Mod}\text{–}S \arrow[d, shift right, "F"'] \\ \mathbf{Mod}\text{–}R \arrow[u, shift right, "G"'] \end{tikzcd}$,

- $\eta_{N,M} : \mathrm{Hom}_R(FN, M) \cong \mathrm{Hom}_S(N, GM)$ the natural transformation,
- $\varepsilon_N : N \longrightarrow GFN$ the **unity**, $\varepsilon_N = \eta(\mathrm{id}_{FN})$, and
- $\delta_M : FGM \longrightarrow M$ the **counity**, $\delta_M = \eta^{-1}(\mathrm{id}_{GM})$.

Proposition. 13.6.
If $\begin{tikzcd} \mathbf{Mod}\text{–}S \arrow[d, shift right, "F"'] \\ \mathbf{Mod}\text{–}R \arrow[u, shift right, "G"'] \end{tikzcd}$ *is an adjunction and G preserves epimorphisms, then F preserves projectives.*

[26, pag. 162]

PROOF. Let P a projective right S–module; for any R–module epimorphism

$$\begin{tikzcd} & FP \arrow[d, "f"] & \\ M \arrow[r, "p"] & M'' \arrow[r] & 0 \end{tikzcd}$$

if we apply G, and complete the diagram

$$\begin{tikzcd} P \arrow[r, "\varepsilon_P"] \arrow[d, dashed, "g"'] & GFP \arrow[d, "Gf"] & \\ GM \arrow[r, "Gp"] & GM'' \arrow[r] & 0 \end{tikzcd}$$

Since P is projective, there exists $g\,\mathrm{Hom}_S(P, GM)$ such that $Gf \cdot \varepsilon_P = Gp \cdot g$. Applying the adjunction isomorphisms we have

$$\begin{tikzcd} \mathrm{Hom}_R(FP, M) \arrow[r, "\eta_{P,M}", "\cong"'] \arrow[d, "p_*"'] & \mathrm{Hom}_S(P, GM) \arrow[d, "(Gp)_*"] \\ \mathrm{Hom}_R(FP, M'') \arrow[r, "\cong", "\eta_{P,M''}"'] & \mathrm{Hom}_S(P, GM'') \end{tikzcd}$$

then

$$\eta_{P,M''}(p \cdot \eta_{P,M}^{-1}(g)) = Gp \cdot G\eta_{P,M}^{-1}(g) \cdot \varepsilon_P = Gp \cdot g = Gf \cdot \varepsilon_P$$

$$\eta_{P,M''}(f) = Gf \cdot \varepsilon_P$$

and $f = p \cdot \eta^{-1}(g)$, which shows that FP is a projective right R–module. □

Dually we have:

Proposition. 13.7.
If $\begin{tikzcd} \textbf{Mod}\text{–}S \arrow[d, shift right, "F"'] \\ \textbf{Mod}\text{–}R \arrow[u, shift right, "G"'] \end{tikzcd}$ *is an adjunction and F preserves monomorphisms, then G preserves injectives.*

As an application let us consider, for a ring R, the ring map $\mathbb{Z} \longrightarrow R$, hence there exists an adjunction

$$\begin{tikzcd} \mathscr{A}b \arrow[d, shift right, "-\otimes_{\mathbb{Z}} R"'] \\ \textbf{Mod}\text{–}R \arrow[u, shift right, "\mathscr{U}"'] \end{tikzcd}$$

where $\mathscr{U}$ is the forgetful functor. Since $\mathscr{U}$ preserves epimorphisms, then $-\otimes_{\mathbb{Z}} R$ sends projective abelian groups to projective right R–modules. On the other hand, if $-\otimes_{\mathbb{Z}} R$ preserves monomorphisms, then $\mathscr{U}$ preserves injectives; this happens if R is either a projective abelian group or a flat abelian group.

Another adjunction situation given by

$$\begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "\mathscr{U}"'] \\ \mathscr{A}b \arrow[u, shift right, "\mathrm{Hom}(R,-)"'] \end{tikzcd}$$

In this case $\mathscr{U}$ preserves monomorphisms, hence $\mathrm{Hom}(R,-)$ preserves injectives. Also, if $\mathrm{Hom}(R,-)$ preserves epimorphisms, then $\mathscr{U}$ preserves projectives; in particular, this happens si R is a projective abelian group.

A generalization of this situation in given in the following adjunction: $\begin{tikzcd} \mathbf{Mod}\text{–}R \arrow[d, shift right, "-\otimes_R M"'] \\ \mathbf{Mod}\text{–}S \arrow[u, shift right, "\mathrm{Hom}_S(M,-)"'] \end{tikzcd}$, where ${}_RM_S$ is any $(R;S)$–module. In this case the isomorphism of the adjunction is

$$\mathrm{Hom}_S(X\otimes_R M, Y) \overset{\eta}{\cong} \mathrm{Hom}_R(X, \mathrm{Hom}_S(M,Y)),$$

defined by

$$\eta(f)(x)(m) = f(x\otimes m), \text{ for every } f\in \mathrm{Hom}_S(X\otimes_R M, Y),$$

and

$$\eta^{-1}(g)(x\otimes m) = g(x)(m), \text{ for every } g\in \mathrm{Hom}_R(X, \mathrm{Hom}_S(M,Y)),$$

Group algebra

Let us consider the following adjunction $\begin{tikzcd} \mathscr{R}ing \arrow[d, shift right, "\mathscr{U}"'] \\ \mathscr{G}r \arrow[u, shift right, "A[-]"'] \end{tikzcd}$. It expresses the universal property of the group ring $R[G]$.

Proposition. 13.8.
Let A be a commutative ring, B an A–algebra, G a group, $f: A\longrightarrow B$ the structure algebra map, and $g: G\longrightarrow B^{\times}$ a group map in the group of all invertible elements of B, there exists an algebra map $h: A[G]\longrightarrow B$ extending f and g.

$$\begin{tikzcd} A \arrow[r, "\alpha(incl)"] \arrow[dr, "f"'] & A[G] \arrow[d, dashed, "h"] & G \arrow[l, "\gamma(incl)"'] \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Lemma. 13.9.
Let $H \subseteq G$ be a subgroup, then $A[G]$ is a free right (and left) $A[H]$–module.

PROOF. Let $\{x_iH \mid i \in I\}$ be a representative set of left classes of H in G. Hence G is a disjoint union $G = \dot{\cup}_i\, x_iH$, and for any left class x_iH we have an isomorphism $\oplus_{g \in x_iH} Ag \cong \oplus_{h \in H} Ax_ih$.
There is a right $A[H]$–module isomorphism $\nu : A[H] \cong \oplus_{h \in H} Ax_ih$ defined as $\nu(c_hh) = c_h(x_ih)$. Therefore, there is a right $A[H]$–module isomorphism

$$\mu : \oplus_{i \in I} \oplus_{h \in H} A(x_ih) \cong A[G], \text{ defined } \mu\left(\sum_i \sum_h c_{i,h}(x_ih)\right) = \sum_{i,h} c_{i,h}(x_ih),$$

and the result follows.
To establish the result in the left module case we must consider right classes of H in G. □

Corollary. 13.10.
For any subgroup $H \subseteq G$,

(1) any projective right $A[G]$–module if a projective right $A[H]$–module
(2) any injective right $A[G]$–module if an injective right $A[H]$–module