31 Módulos semisimples

Un módulo no nulo M es **simple** si no tiene submódulos propios no nulos.

Proposition. 31.1.

Sea M un módulo a derecha, $\{N_i \mid i \in I\}$ una familia de submódulos simples tales que $\sum_i N_i = M$, para cada submódulo propio $N \subseteq M$ existe $J \subseteq I$ tal que $M = N + (\bigoplus_{i \in J} N_i)$.

PROOF. Si $N \neq M$, existe N_i tal que $N_i \nsubseteq N$, luego $N \cap N_i = 0$. Consideramos familias $\Gamma_J = \{N_j \mid i \in J \subseteq I\}$ que son independientes y $N \cap (\bigoplus_{j \in J} N_j) = 0$. Consideramos la familia $\Gamma = \{\Gamma_J \mid \text{verifican la propiedad anterior}\}$. Esta familia es inductiva y no vacía, ya que $\{N_i\} \in \Gamma$. Aplicando el lemma de Zorn, existe $\Gamma_J \in \Gamma$ maximal para la inclusión. Se tiene entonces $N \oplus (\bigoplus_{j \in J} N_j) \subseteq M$; si no son iguales, existe $N_i \nsubseteq N \oplus (\bigoplus_{j \in J} N_i)$, y por tanto $\Gamma_J \cup \{N_i\} \in \Gamma$, lo que es una contradicción. \square

Corollary. 31.2.

Sea M un módulo a derecha, $\{N_i \mid i \in I\}$ una familia de submódulos simples tales que $\sum_i N_i = M$, existe un subconjunto $J \subseteq I$ tal que $M = \bigoplus_{i \in J} N_i$.

PROOF. Basta tomar N = 0 en la proposición anterior.

Un módulo a derecha M se llama **semisimple** si M es una suma de submódulos simples. Según la definición el módulo cero es semisimple.

Proposition. 31.3.

- (1) Todo submódulo de un módulo semisimple es un sumando directo.
- (2) Todo cociente de un módulo semisimple es semisimple.
- (3) Todo submódulo de un módulo semisimple es semisimple.
- (4) La suma directa de módulos semisimples es un módulo semisimple.

Dos módulos simples S_1 y S_2 son **isotípicos** si $S_1 \cong S_2$. Dado un módulo semisimple $M = \bigoplus_i S_i$, para cada submódulo simple $S \subseteq M$ existe un índice $i \in I$ tal que $S \cong S_i$, la componente isotípica de S en M es $\bigoplus_i S_i$, donde $S \cong S_i$, la representamos por M_S .

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Proposition. 31.4.

Cada módulo semisimple es la suma directa de sus componentes isotípicas, y para cada submódulo simple $S \subseteq M$ tenemos $S \subseteq M_S$.

PROOF. Dado $S \subseteq M$ simple existe $J \subseteq I$ tal que $M = S \oplus (\oplus j \in JS_j;$ entonces $S \cong \bigoplus_{i \in I \setminus J} S_i$, y por tanto $I \setminus J$ es unitario, esto es, S es isomorfo a uno de los S_i .

Dado $T \subseteq M$ simple, tenemos $T \subseteq \bigoplus_S M_S$, y no todas las proyecciones a los M_S son nulas. Si $T \ncong S$, y $p_S(T) \ne 0$, entonces $T \cong p_S(T)$, esto es, T isomorfo a un submódulo de M_S , lo que es una contradicción, ya que T es necesariamente isomorfo a un sumando directo de M_S . Tenemos entonces p(T) = 0 si $S \ncong T$, y por tanto $T \subseteq_{M_T}$.

Corollary. 31.5.

Para cada módulo semisimple M, si $M = \bigoplus_S M_S$ es la descomposición en componentes isotípicas, entonces $\operatorname{End}_R(M) = \prod_S \operatorname{End}_R(M_S)$.

El problema ahora es determinar la estructura de $\operatorname{End}_R(M)$, cuando M tiene una sola componente isotípica. Supongamos que $M=\oplus_i S_i$, con $S_i\cong S_j$ para todos i,j. Es claro que

$$\operatorname{End}_R(\oplus_i S_i) \cong \prod_i \operatorname{End}_R(S_i, \oplus_j S_j)$$

Identificamos S con R/\mathfrak{m} , con $\mathfrak{m} \subseteq R$ un ideal derecha maximal, y sea $e = 1 + \mathfrak{m}$. Un sistema de generadores de $\bigoplus_i S_i$ es $e_i = (e\delta_{i,j})_j$.

Para cada homomorfismo $f: S \longrightarrow S$ se tiene f(e) = ea, con a determinado módulo m. Tenemos que $D = \operatorname{End}_R(S)$ es un anillo de división, S es un D-módulo izquierda, y existe una aplicación $v: \operatorname{End}_R(S) \longrightarrow R/\mathfrak{m}$, definida $v(f) = \overline{a}$, siendo f(e) = ea, que verifica:

$$\nu(f_1 + f_2) = \overline{a_1} + \overline{a_2},$$

$$\nu(f_1 f_2) = \overline{a_1 a_2}.$$

Tenemos que ν es un homomorfismo de grupos abelianos, pero no de anillos, ya que R/m no lo es. Para cada $f \in \operatorname{End}_R(M)$ y cada e_i , se tiene $f(e_i) = \sum_j e_j a_{j,i} = \sum_j f_{j,i}(e_j)$, siendo los $a_{j,i}$ determinados unívocamente módulo m, no así los $f_{j,i}$ que están unívocamente determinados. Tenemos un homomorfismo de anillos $\operatorname{End}_R(M) \longrightarrow MCF_I(D)$, el anillo de las matrices con columnas finitas, que envía cada $f \in \operatorname{End}_R(M)$ a la matriz con columnas finitas $M(f) = (f_{j,i})_{j,i}$ cuya columna i es $(f_{j,i})_j$.

Vamos a particularizar el caso en el que M es una suma directa finita de módulos simples. En este caso tenemos:

Theorem. 31.6.

Si M es una suma directa finita de n copias de un módulo simple S, entonces $\operatorname{End}_R(M) \cong M_n(\operatorname{End}_R(S))$, es un anillo de matrices sobre un anillo de división.

Corollary. 31.7.

Si M es un módulo semisimple finitamente generado, se tiene que $\operatorname{End}_R(M)$ es un producto finito de anillos de matrices sobre anillos de división.

Un anillo R es un **anillo semisimple** (derecha) si R_R es un R-módulo derecha semisimple.

Lemma. 31.8.

Sea R un anillo, son equivalentes:

- (a) R_R es un módulo semisimple
- (b) $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, donde cada D_i es un anillo de división.
- (c) _RR es un módulo semisimple.

En este punto hay dos propiedades de los anillos semisimples que conviene destacar:

- (1) Todo anillo semisimple es un anillo artiniano derecha.
- (2) El único ideal derecha nilpotente de *R* es el ideal cero (diremos que *R* es un anillo **semiprimo**).

Tenemos que estas dos propiedades, como veremos más adelante, caracterizan los anillos semisimples.

10 Tensor product

Let M be a right R-module and N be a left R-module. A group map $\varphi: M \times N \to A$ is called R-bilinear if it satisfies the following conditions:

- (i) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$,
- (ii) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$, and
- (iii) $\varphi(mr, n) = \varphi(m, rn)$.

for any $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

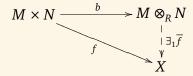
We are interesting in building an abelian group satisfying a universal property with respect to R-bilinear maps. Thus we build the free abelian group G on the set $\{(m,n) \mid m \in M, n \in N\}$, consider the subgroup G of G generated by the elements:

- $(m_1 + m_2, n) (m_1, n) (m_2, n)$, for all $m_1, m_2 \in M$ and $n \in N$,
- $(m, n_1 + n_2) (m, n_1) (m, n_2)$, for all $m \in M$ and $n_1, n_2 \in N$,
- (mr, n) (m, rn), for all $m \in M$, $n \in N$ and $r \in R$,

and the quotient group G/B. Hence there is a natural map $M \times N \stackrel{b}{\longrightarrow} G/B$ defined by b(m,n) = (m,n) + B. For simplicity we represent the element (m,n) + B by $m \otimes n$, and the quotient group G/B by $M \otimes_R N$. Thus every element of $M \otimes_R N$ has an expression as $\sum_i m_i \otimes n_i$. Observe that this expression is not unique. Finally, we have that the map $b: M \times N \to M \otimes_R N$ is an R-bilinear map. We call $M \otimes_R N$ the **tensor product** over R of M and N.

Proposition. 10.1. (Tensor product universal property)

Let M_R and $_RN$ be modules and $f: M \times N \to X$ be a R-bilinear map, there exists a unique group map $\overline{f}: M \otimes_R N \to X$ such that $f = \overline{f} \circ b$.



The following result is elemental from the above definition.

Proposition. 10.2.

Let M_R and $_RN$ be modules, then $R \otimes_R N \cong N$, and $M \otimes_R R \cong M$.

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Proposition. 10.3.

Let A be a commutative ring and let M, N and P be A-modules, the followings statements hold:

- $(1) \ M \otimes_A N \cong N \otimes_A M.$
- (2) $M \otimes_A N$ is an A-module with action defined by $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.
- $(3) (M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P).$

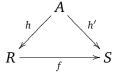
Tensor product of algebras

Sea A un anillo conmutativo y R un anillo. Dar una estructura de A-álgebra en R es dar un homomorfismo de anillos $h:A\longrightarrow R$ tal que $h(A)\subseteq \operatorname{Cen}(R)$. Decimos entonces que R es una A-álgebra. Observa que R es un A-módulo a la derecha y a la izquierda con acción dada por

$$ar = h(a)r = rh(a) = ra$$
,

para cada $a \in A$ y cada $r \in R$.

Dadas dos A-álgebras R y S, un **homomorfismo de** A-**álgebras** es un homomorfismo de anillos $f: R \longrightarrow S$ tal que f(ar) = af(r). Esto significa que si $h': A \longrightarrow S$ es el homomorfismo de estructura de S, entonces el siguiente diagrama de homomorfismos de anillos es conmutativo:



Se tiene:

$$hf(a) = f(a1) = af(1) = a1 = h'(a).$$

Si R y S son dos A-álgebras, y consideramos el producto tensor $R \otimes_A S$, tenemos una estructura de grupo abeliano en $R \otimes_A S$, que podemos extender a una estructura de A-módulo, ver Proposición (10.2.), definiendo

$$a(r \otimes s) = (ar) \otimes s = r \otimes (as).$$

Además, $R \otimes_A S$ es un anillo con producto definido por

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2),$$

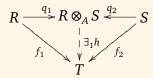
y extendiendo por distributividad a todos los elementos de $R \otimes_A S$. En este caso existen homomorfismos de A-álgebras $q_1: R \longrightarrow R \otimes_A S$ y $q_2: S \longrightarrow R \otimes_A S$ definidos

$$q_1(r) = r \otimes 1$$

 $q_2(s) = 1 \otimes s$.

Lemma. 10.4.

Sean R, S y T tres A-álgebras, para dos homomorfismos de A-álgebras $f_1: R \longrightarrow T$ y $f_2: S \longrightarrow T$ tales que $f_1(r)f_2(s) = f_2(s)f_1(r)$, para cada $r \in R$ y $s \in S$, existe un único homomorfismo de A-álgebras $h: R \otimes_A S \longrightarrow T$ tal que



PROOF. Consideramos la aplicación $\varphi: R \times S \longrightarrow T$ definida por $\varphi(r,s) = f_1(r)f_2(s)$, es claro que φ es A-bilineal, y existe un único homomorfismo de grupos abelianos $h: R \otimes_A S \longrightarrow T$ definido $h(r \otimes s) = f_1(r)f_2(s)$. Falta comprobar que h es un homomorfismo de A-álgebras.

Corollary. 10.5.

 SiR_1,R_2,S_1,S_2 son A-álgebras $yf:R_1\longrightarrow R_2,g:S_1\longrightarrow S_2$ son homomorfismos de A-álgebras, existe un único homomorfismo de A-álgebras, $h:R_1\otimes_A S_1\longrightarrow R_2\otimes_A S_2$, que hace conmutar el siguiente diagrama

Corollary. 10.6.

En la categoría de A-álgebras conmutativas el producto tensor de dos A-álgebras, junto con los homomorfismos q_1 y q_2 , es un coproducto.

Corollary. 10.7.

Si A es un anillo conmutativo, para indeterminadas X e Y se tiene un isomorfismo de A-álgebras $A[X] \otimes_A [Y] \cong A[X,Y]$.

Lemma. 10.8.

Sean R y S dos A-álgebras, se tiene

- (1) $\operatorname{Cen}(R) \xrightarrow{\operatorname{incl}} R$ es un homomorfismo de A-álgebras.
- (2) Si R es un anillo simple (no tiene ideales biláteros propios no nulos), entonces Cen(R) es un cuerpo.
- (3) Existe un homomorfismo de A-álgebras $h: \operatorname{Cen}(R) \otimes_A \operatorname{Cen}(S) \longrightarrow R \otimes_A S$ cuya imagen está contenida en $\operatorname{Cen}(R \otimes_A S)$.

PROOF. (1). Es inmediato.

- (2). Sea $0 \neq a \in \text{Cen}(R)$, entonces Ra = aR es un ideal (bilátero) no nulo, por tanto Ra = aR = R, y a es un elemento invertible.
- (3). Definimos $f: \operatorname{Cen}(R) \longrightarrow R \otimes_A S$ mediante $f(r) = r \otimes 1$ y definimos $g: \operatorname{Cen}(S) \longrightarrow R \otimes_A S$ mediante $g(s) = 1 \otimes s$. Es claro que $f(r)g(s) = (r \otimes 1)(1 \otimes s) = (1 \otimes s)(r \otimes 1) = g(s)f(r)$, y por tanto existe un homomorfismo de A-álgebras $j: \operatorname{Cen}(R) \otimes_A \operatorname{Cen}(S) \longrightarrow R \otimes_A S$, definido $j(r \otimes s) = (r \otimes s)$. Es claro que $\operatorname{Im}(j) \subseteq \operatorname{Cen}(R \otimes_A S)$. Tenemos que la acción está dada por el homomorfismo composición siguiente:

$$A \longrightarrow A \otimes_A A \xrightarrow{h \otimes h'} \operatorname{Cen}(R) \otimes_A \operatorname{Cen}(S) \xrightarrow{j} R \otimes_A S$$

Example. 10.9. (Coproducto en la categoría de *A*-álgebras)

Sea A un anillo conmutativo y R, S dos A-álgebras, vamos a construir el coproducto de R y S en la categoría de A-álgebras y homomorfismos de A-álgebras.

Construimos una A-álgebra T de la siguiente forma:

- (1) Definimos $T_0 = A$,
- (2) Definimos $T_1 = R \oplus S$,
- (3) Si $t \ge 1$, definimos $T_{t+1} = (T_t \otimes_A R) \oplus (T_t \otimes_A S)$.
- (4) Se define $T = \bigoplus_{i=0}^{\infty} T_i$.

Es claro que T es un A-módulo, y que $q_1: R \longrightarrow T$, definido $q_1(r) = r$, y $q_2: S \longrightarrow T$, definido $q_2(s) = s$ son homomorfismos de A-módulos.

En T se define un producto mediante la concatenación de productos tensores, esto es, si $x_1 \otimes \cdots \otimes x_s, y_1 \otimes \cdots \otimes y_t \in T$, se define

$$(x_1 \otimes \cdots \otimes x_s) * (y_1 \otimes \cdots \otimes y_t) = x_1 \otimes \cdots \otimes x_s \otimes y_1 \otimes \cdots \otimes y_t.$$

y la extendemos por linealidad. Tenemos que T es una A-álgebra, con elemento uno ideal al elemento 1 de A, ya que $a*(x_1 \otimes \cdots \otimes x_s) = a \otimes (x_1 \otimes \cdots \otimes x_s) = (ax_1 \otimes \cdots \otimes x_s)$, para $a \in A$.

En T consideramos el ideal $\mathfrak A$ generado por los elementos:

$$r_1 \otimes r_2 - r_1 r_2,$$

 $s_2 \otimes s_2 - s_1 s_2,$
 $a - a_R,$
 $a - a_S,$

para $r_1, r_2 \in R$, $s_1s_2 \in S$ y $a \in A$, siendo $a_R \in R$ y $a_S \in S$ las imágenes de a en R y S, respectivamente, y consideramos el cociente T/\mathfrak{A} . Tenemos homomorfismos de A-álgebras:

$$q_1: R \xrightarrow{q_1} T \xrightarrow{proy.} T/\mathfrak{A},$$

$$q_2: S \xrightarrow{q_2} T \xrightarrow{proy.} T/\mathfrak{A}.$$

Veamos que el par $(T/\mathfrak{A}\{q_1,q_2\})$ es un coproducto de R y S. Dada una A-álgebra U y homomorfismos de A-álgebras $f_1:R\longrightarrow U$ y $f_2:S\longrightarrow U$, para cada lista de R's y S's, por ejemplo R, S, tenemos una aplicación bilineal $f_1\times f_2:R\times_A S\longrightarrow U$ definida $f_1\times f_2(r,s)=f_1(r)f_2(s)$, por tanto existe un homomorfismo de A-módulos $f_1\otimes f_2:R\otimes S\longrightarrow U$, definido $f_1\otimes f_2(r\otimes s)=f_1(r)f_2(s)$. Como existe un homomorfismo a U de cada sumando de T, existe un homomorfismo de A-módulos de $f:T\longrightarrow U$, definido $f(x_1\otimes \cdots \otimes x_s)=f_*(x_1)\cdots f_*(x_s)$, utilizando el f_* correspondiente al x_i . Es claro que f es un homomorfismo de A-álgebras, y que se anula sobre $\mathfrak A$, ya que se tiene

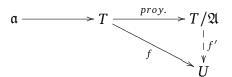
$$f(r_1 \otimes r_2 - r_1 r_2) = f_1(r_1) f_2(r_2) - f_1(r_1 r_2) = 0,$$

$$f(s_1 \otimes s_2 - s_1 s_2) = f_s(s_1) f_2(s_2) - f_2(s_1 s_2) = 0,$$

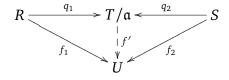
$$f(a - a_R) = f_1(a) - f_1(a) = 0,$$

$$f(a - a_S) = f_2(a) - f_2(a) = 0.$$

Por lo tanto f se factoriza por T/\mathfrak{A} , esto es, existe un homomorfismo de A-álgebras f':



Tenemos un diagrama conmutativo



La unicidad de f' haciendo conmutar este diagrama es consecuencia de la unicidad en la construcción de los homomorfismos que hemos ido realizando a partir de f_1 y f_2 .

11 Two-sided modules

Let M be a right R-module, then there exists a ring map $R^{op} \to \operatorname{End}(M)$, and we may define on M a structure of left module over the ring R^{op} . This process can be realized also for left modules. As a consequence we may restrict the study of modules to only consider either right or left modules.

There are examples in with there are more than one action acting on a module, one of this is the following: Let $f: T \to R$ be a ring map, then R has four module structure.

- (1) R_R is the natural right action of R over itself.
- (2) $_RR$ is the natural left action of R over itself.
- (3) R_T is the right action defined: $r \cdot t = rf(t)$ induced by the multiplication.
- (4) $_TR$ is the left action defined: $t \cdot r = f(t)r$ induced by the multiplication.

It is obvious that the actions $_TR$ and R_T satisfy the following identity:

$$t_1(rt_2) = (t_1r)t_2,$$

for any $t_1, t_2 \in T$ and any $r \in R$. And the same holds if we consider the actions R and R. But with the actions R and R we have a problem, if we consider $r, r_1 \in R$ and $t_1 \in T$, then $(rr_1)t_1$, in general, is not equal to $(rt_1)r_1$, as $f(t_1)$ and r_1 necessarily do not commute.

It appears, in the way, some kind of compatibility or incompatibility when we have two actions on a module.

Let M be a left R-module and a right S-module, we say that M is a two-sided (R; S)-module, or simply (R; S)-module, if

$$r(ms) = (rm)s$$
,

for any $r \in R$, $m \in M$ and $s \in S$. In this case we have used structures on the left and on the right for simplicity, observe that we may consider M only with right structures: as a right R^{op} —module and a right S—module, then the compatibility means

$$(ms)r = (mr)s$$
.

Proposition. 11.1.

Let M be a left R-module and a right S-module. The following statements are equivalent:

- (a) M is a (two-sided) (R; S)-module.
- (b) The ring map $R \to \text{End}(M)$ factorizes through $\text{End}(M_S) := \text{End}_S(M)$.
- (c) The ring map $S^{op} \to \text{End}(M)$ factorizes through $\text{End}_R(M) := \text{End}_R(M)$.

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Let M be a (two-sided) (R;S)-module, then it is a left module over R and over S^{op} , and there are ring maps $\beta_R: R \to \operatorname{End}(M)$ and $\beta_S: S^{op} \to \operatorname{End}(M)$, and elements in $\operatorname{Im}(\beta_R)$ commute with elements in $\operatorname{Im}(\beta_S)$, as a consequence of the compatibility of the two actions. If we consider the tensor product $R \otimes_{\mathbb{Z}} S^{op}$, there exists a ring map $R \otimes_{\mathbb{Z}} S^{op} \to \operatorname{End}(M)$ defined $r \otimes s \mapsto \beta_R(r)\beta_S(s)$, hence we have a new left action over M defined

$$(r \otimes s)m = rms$$
.

When β_R and β_S are surjective maps we say M is a **balanced two-sided** (R; S)-**module**. If β_R and β_S are isomorphisms we says M is a **faithfully balanced two-sided** (R; S)-**module**.

Thus we may complete Proposition (11.1.) with a new equivalent statement:

Proposition. 11.2.

With the same notation as in Proposition (11.1.) the statements there are equivalent to

- (d) M is a left $R \otimes_{\mathbb{Z}} S^{op}$ -module.
- (e) M is a right $R^{op} \otimes_{\mathbb{Z}} S$ -module.

Exercise, 11.3.

Let $_RM_S$, $_RN_T$ be two–sided modules, then $\operatorname{Hom}_R(_RM_S,_RN_T)$ has a natural structure of a two–sided (S;T)–module with structures given by:

- (sf)(m) = f(ms), for any $s \in S$, $f \in Hom_R(M, N)$ and $m \in M$.
- (f t)(m) = f(m)t, for any $t \in T$, $f \in \text{Hom}_{\mathbb{R}}(M, N)$ and $m \in M$.

Ref.: 2102e 017 SOLUTION

SOLUTION. Exercise (11.3.)

HACER \square

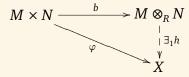
Two-sided modules are a useful tools in the tensor product theory.

Proposition. 11.4.

Let ${}_TM_R$ and ${}_RN_S$ be two–sided modules, then $M \otimes_R N$ is a (T;S)–module, and it satisfies the following universal property: Let X be a two–sided (T;S)–module and $\varphi: M \times N \to X$ a R–bilinear map satisfying

$$\varphi(tm, ns) = t\varphi(m, n)s$$
,

there exists a unique (T; S)-module map h such that $\varphi = h \circ b$.



Let ${}_SM_R$ be a (S;R)-module, and let ${}_RN_T$ be a (R;T)-module, then in the tensor product $M \otimes_R N$ we have a structure of (S;T)-module with actions defined by

$$s(m \otimes n) = (sm) \otimes n$$

 $(m \times n)t = m \otimes (nt).$

Let $_RM_S$ be a (R;S)-module and $_RN_T$ a (R;T)-module, in the abelian group $\operatorname{Hom}_R(M,N)$ of left R-module maps we define a structure of (S;T)-module as follows, see Exercise (11.3.):

$$(sf)(m) = f(ms)$$
$$(ft)(m) = f(m)t.$$

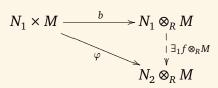
In particular, for any right R-module M the abelian group $\operatorname{Hom}_R(R,M)$ is a right R-module with action given by (rf)(x) = f(rx).

12 The functor tensor product; flat modules

Let $_RM$ be a module, for any right R-module N_R we have the tensor product abelian group $N \otimes_R M$.

Lemma. 12.1.

Let $_RM$ be a module, for any right R-module map $f: N_1 \longrightarrow N_2$ there is an abelian group map $f \otimes_R M: N_1 \otimes_R M \longrightarrow N_1 \otimes_R M$. The map $f \otimes_R M$ is defined by the R-bilineal map $\varphi: N_1 \times M \longrightarrow N_2 \otimes_R M$ defined $\varphi(n_1, m) = f(n_1) \otimes m$, hence $(f \otimes_R M)(n \otimes m) = f(n) \otimes m$.



Proposition. 12.2.

Let $_RM$ be a module, there is a functor $-\otimes_R M: Mod-R \longrightarrow \mathcal{A}b$.

If *M* is a (R; S)-module, then the codomain of the tensor product functor $- \otimes_R M$ is **Mod**-S.

Proposition. 12.3.

Let $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ be a short exact sequence of right R-modules and _RM be a left R-module, then

$$N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

is an exact sequence of abelian groups.

A left R-module $_RF$ is **flat** if for any short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ the above sequence is a short exact sequence, i.e., $N_1 \otimes_R F \to N_2 \otimes_R F$ always is a monomorphism.

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The functor Hom; projective and injective modules 13

Let M_R be a right R-module, for any right R-module N we have two abelian groups $\operatorname{Hom}_R(M,N)$ and $\operatorname{Hom}_R(N, M)$.

Lemma. 13.1.

Let M_R be a right R-module, for any right module map $f: N_1 \longrightarrow N_2$ there is an abelian group map $f_* = \operatorname{Hom}_R(M, f) : \operatorname{Hom}_R(M, N_1) \longrightarrow \operatorname{Hom}_R(M, N_2).$

Proposition. 13.2.

Let M_R be a right R-module, there is a functor $\operatorname{Hom}_R(M,-): \operatorname{Mod}\!-R \longrightarrow \mathscr{A}b$.

Lemma. 13.3.

Let M_R be a right R-module, for any module map $f: N_1 \longrightarrow N_2$ there is an abelian group map $f^* = \operatorname{Hom}_R(f, M) : \operatorname{Hom}_R(N_2, M) \longrightarrow \operatorname{Hom}_R(N_1, M).$

Proposition. 13.4.

Let M_R be a module, there is a contravariant functor $\operatorname{Hom}_R(-,M): \operatorname{Mod}\!-R \longrightarrow \mathscr{A}b$.

Proposition. 13.5.

Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of right R-modules and M_R be a right R-module, then

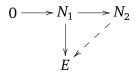
$$0 \to \operatorname{Hom}_R(M, N_1) \to \operatorname{Hom}_R(M, N_2) \to \operatorname{Hom}_R(M, N_3)$$
 and

 $0 \to \operatorname{Hom}_R(N_3, M) \to \operatorname{Hom}_R(N_2, M) \to \operatorname{Hom}_R(N_1, M)$

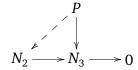
are exact sequences.

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A right R-module E is **injective** if for every short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \operatorname{Hom}_R(N_3, E) \to \operatorname{Hom}_R(N_2, E) \to \operatorname{Hom}_R(N_1, E) \to 0$ is a short exact sequence, i.e., $\operatorname{Hom}_R(N_2E) \to \operatorname{Hom}_R(N_1, E)$ is an epimorphism, or equivalently, we can complete any diagram with exact row



A right R-module P is **projective** if for every short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \operatorname{Hom}_R(P, N_1)) \to \operatorname{Hom}_R(P, N_2) \to \operatorname{Hom}_R(P, N_3) \to 0$ is a short exact sequence, i.e., $\operatorname{Hom}_R(P, N_2) \to \operatorname{Hom}_R(P, N_3)$ is an epimorphism, or equivalently, we can complete any diagram with exact row



Let M be a right R-module and ${}_{S}X_{R}$ a two-sided (S;R)-module. We have a left S-module estructure on $\operatorname{Hom}_{R}(M,X)$ defined by sf(m)=f(m)s. Thus we may consider the right R-module $\operatorname{Hom}_{S}(\operatorname{Hom}_{R}(M,X),X)$.

Let $_SX_R$ be a two–sided (S;R)–module. A right R–module M is X–**reflexive** if the canonical map $M \to \operatorname{Hom}_S(_S\operatorname{Hom}_R(M_R,X_R)),_SX)$ is an isomorphism. In particular, the **reflexive** right R–modules are the R–reflexive modules, i.e., they are the right R–modules such that the homomorphism $M \longrightarrow \operatorname{Hom}_R(M_R,R_R),_RR)$ is an isomorphism.

Adjunctions

Let us consider an **adjunction** Mod-S,



- $\eta_{N,M}$: $\operatorname{Hom}_R(FN,M) \cong \operatorname{Hom}_S(N,GM)$ the natural transformation,
- $\varepsilon_N: N \longrightarrow GFN$ the **unity**, $\varepsilon_N = \eta(\mathrm{id}_{FN})$, and
- $\delta_M : FGM \longrightarrow M$ the **counity**, $\delta_M = \eta^{-1}(\mathrm{id}_{GM})$.

Proposition. 13.6.

If **Mod**-S is an adjunction and G preserves epimorphisms, then F preserves projectives.



Mod-R

[26, pag. 162]

PROOF. Let *P* a projective right *S*–module; for any *R*–module epimorphism

$$\begin{array}{ccc}
& & & FP \\
& \downarrow f \\
M & & \longrightarrow M'' & \longrightarrow 0
\end{array}$$

if we apply *G*, and complete the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\varepsilon_{P}} & GFP \\
\downarrow g & & \downarrow Gf \\
\downarrow GM & \xrightarrow{Gp} & GM'' & \longrightarrow 0
\end{array}$$

Since *P* is projective, there exists $g \operatorname{Hom}_S(P, GM)$ such that $Gf \cdot \varepsilon_P = Gp \cdot g$. Applying the adjunction isomorphisms we have

$$\operatorname{Hom}_{R}(FP,M) \xrightarrow{\eta_{P,M}} \operatorname{Hom}_{S}(P,GM)$$

$$\downarrow^{(Gp)_{*}} \qquad \qquad \downarrow^{(Gp)_{*}}$$

$$\operatorname{Hom}_{R}(FP,M'') \xrightarrow{\eta_{P,M''}} \operatorname{Hom}_{S}(P,GM'')$$

then

$$\eta_{P,M''}(p \cdot \eta_{P,M}^{-1}(g)) = Gp \cdot G\eta_{P,M}^{-1}(g) \cdot \varepsilon_P = Gp \cdot g = Gf \cdot \varepsilon_P$$

$$\eta_{PM''}(f) = Gf \cdot \varepsilon_P$$

and $f = p \cdot \eta^{-1}(g)$, which shows that *FP* is a projective right *R*–module.

Dually we have:

Proposition. 13.7.

If **Mod**-S is an adjunction and F preserves monomorphisms, then G preserves injectives.



Mod–R

As an application let us consider, for a ring R, the ring map $\mathbb{Z} \longrightarrow R$, hence there exists an adjunction

$$\begin{array}{c|c}
\mathscr{A}b \\
-\otimes_{\mathbb{Z}^R} \middle| \uparrow_{\mathscr{U}} \\
\mathbf{Mod}-R
\end{array}$$

where \mathscr{U} is the forgetful functor. Since \mathscr{U} preserves epimorphisms, then $-\otimes_{\mathbb{Z}} R$ sends projective abelian groups to projective right *R*-modules. On the other hand, if $- \otimes_{\mathbb{Z}} R$ preserves monomorphisms, then \mathcal{U} preserves injectives; this happens if R is either a projective abelian group or a flat abelian group.

Another adjunction situation given by

$$Mod-R$$
 $\mathscr{U} \mid Hom(R,-)$
 $\mathscr{A}b$

In this case \mathscr{U} preserves monomorphisms, hence $\operatorname{Hom}(R,-)$ preserves injectives. Also, if $\operatorname{Hom}(R,-)$ preserves epimorphisms, then \mathcal{U} preserves projectives; in particular, this happens si R is a projective abelian group.

A generalization of this situation in given in the following adjunction: $\operatorname{\mathbf{Mod-}R}_{-\otimes_R M}$, where $_R M_S$

$$\stackrel{\cdot \otimes_R M}{\downarrow} \stackrel{\mathsf{Hom}_S(N)}{\mathsf{Mod}} = S$$

is any (R; S)-module. In this case the isomorphism of the adjunction is

$$\operatorname{Hom}_{S}(X \otimes_{R} M, Y) \stackrel{\eta}{\cong} \operatorname{Hom}_{R}(X, \operatorname{Hom}_{S}(M, Y)),$$

defined by

$$\eta(f)(x)(m) = f(x \otimes m)$$
, for every $f \in \text{Hom}_S(X \otimes_R M, Y)$,

and

$$\eta^{-1}(g)(x \otimes m) = g(x)(m)$$
, for every $g \in \operatorname{Hom}_R(X, \operatorname{Hom}_S(M, Y))$,

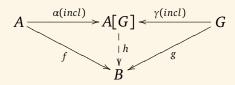
Group algebra

 $\mathcal{R}ing$. It expresses the universal property of the group Let us consider the following adjunction

ring R[G].

Proposition. 13.8.

Let A be a commutative ring, B an A-algebra, G a group, $f: A \longrightarrow B$ the structure algebra map, and $g: G \longrightarrow B^{\times}$ a group map in the group of all invertible elements of B, there exists an algebra map $h: A[G] \longrightarrow B$ extending f and g.



Lemma. 13.9.

Let $H \subseteq G$ be a subgroup, then A[G] is a free right (and left) A[H]-module.

PROOF. Let $\{x_iH \mid i \in I\}$ be a representative set of left classes of H in G. Hence G is a disjoint union $G = \stackrel{\bullet}{\cup_i} x_i H$, and for any left class $x_i H$ we have an isomorphism $\bigoplus_{g \in x_i H} Ag \cong \bigoplus_{h \in H} Ax_i h$.

There is a right A[H]—module isomorphism $v:A[H] \cong \bigoplus_{h\in H} Ax_ih$ defined as $v(c_hh) = c_h(x_ih)$. Therefore, there is a right A[H]—module isomorphism

$$\mu: \oplus_{i \in I} \oplus_{h \in H} A(x_i h) \cong A[G], \text{ defined } \mu \left(\sum_i \sum_h c_{i,h}(x_i h) \right) = \sum_{i,h} c_{i,h}(x_i h),$$

and the result follows.

To establish the result in the left module case we must consider right classes of *H* in *G*.

Corollary. 13.10.

For any subgroup $H \subseteq G$,

- (1) any projective right A[G]-module if a projective right A[H]-module
- (2) any injective right A[G]-module if an injective right A[H]-module