Example. 4.20. (Direct product of modules)
Let I be a set, we consider the category $\mathscr{I}$ whose objects are the elements of I and the only morphisms the identities. For any functor $F : \mathscr{I} \longrightarrow \mathbf{Mod}\text{–}R$, the limit is a pair $(\prod_i F(i), \{p_i\}_i)$, of a module $\prod_i F(i)$, the cartesian product with pointwise operation, and the family of projectios $\{p_i : \prod_i F(i) \longrightarrow F(i) \mid i \in I\}$, it is the direct product of the family of modules $\{F(i) \mid i \in I\}$.

Example. 4.21. (Kernel in modules)
Let us consider the category $\mathscr{C}$ with two objects $X \overset{a}{\underset{b}{\rightrightarrows}} Y$, the identities and two morphisms: a y b. For any functor $F : \mathscr{C} \longrightarrow \mathbf{Mod}\text{–}R$ such that $F(b) = 0$ a limit is a pair $(L, \{p_X, p_Y\})$ that makes a commutative diagram

$$\begin{tikzcd}
 & & F(X) \arrow[d, shift right, "F(a)"'] \arrow[d, shift left, "0"] \\
L' \arrow[urr, "p'_X"] \arrow[drr, "p'_Y"'] \arrow[r, dashed, "\exists_1 f"] & L \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\
 & & F(Y)
\end{tikzcd}$$

therefore $F(a)p_X = 0$, and for any module map $p'_X : L' \longrightarrow F(X)$ such that $F(a)p'_X = 0$, there is a unique module map $f : L' \longrightarrow L$ such that $p'_X = p_X f$. Thus is, we have that (L, f) is the **kernel** of $F(a)$.

Example. 4.22. (Product in Fields)
In the category $\mathscr{C}$ of fields and field homomorphisms does not exist the product of some families of fields. Let us consider a field extension F/K, where K is the prime subfield of F, i.e., K is either $\mathbb{Q}$ or $\mathbb{F}_p$, with Galois group isomorphic to C_2, for instance $F = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}$. Then $\mathrm{Gal}(F/K) = \{1, \varphi\}$ Let us consider the product of F by F; it is a pair $(E, \{p_1, p_2\})$, and it satisfies a universal property; for any field L and homomorphisms $f_1 : K \longrightarrow F_1$, $f_2 : L \longrightarrow F_2$ there is a unique field homomorphism $f : L \longrightarrow E$ such that $p_1 f = f_1$ and $p_2 f = f_2$.

$$\begin{tikzcd}
 & L \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists_1 f"] \arrow[dr, "f_2"] & \\
F_1 & E \arrow[l, "p_1"] \arrow[r, "p_2"'] & F_2
\end{tikzcd}$$

We take several pairs $(L, \{f_1, f_2\})$ to check the direct producto.
Let take the pair $(F, \{\mathrm{id}, \mathrm{id}\})$, then there is a unique map, say $f : F \longrightarrow F$ such that $p_1 f = \mathrm{id}$, $p_2 f = \mathrm{id}$, i.e., $f = p_1^{-1} = p_2^{-1}$.
If we take the pair $(F, \{\varphi, \mathrm{id}\})$, there exists a unique map $g : F \longrightarrow f$ such that $p_1 g = \varphi$, $p_2 g = \mathrm{id}$. As a consequence, $p_1^{-1}\varphi = p_2^{-1} = p_1^{-1}$, and $\varphi = \mathrm{id}$, which is a contradiction.
Therefore, product of F by F does not exist.

Colimits

Let $\mathscr{P}$ be a small category (a diagram), its class of morphism is a set, and let $F : \mathscr{P} \longrightarrow \mathbf{Mod}\text{–}R$ be a functor. A **colimit** of F is a pair $(\{q_X\}_X, C)$, consitutes by a module C and a family of module maps