CATEGORY THEORETIC INTERPRETATION OF RINGS

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Abstract. We enhance the category of rings and the category of idempotented rings to 2-categories. After doing this, we prove an equivalence of 1-categories and 2-categories between the category of rings and the category of small preadditive categories with one object and between the category of idempotented rings and the category of small preadditive categories with finitely many objects. Under these equivalences, we demonstrate some analogues between notions in category theory and ring theory.

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1. Introduction

One of the most famous problems in mathematics was the proof for Fermat's Last Theorem, dating back to 1637, that states that there do not exist three positive integers a, b, c such that $a^n + b^n = c^n$ for an integer n greater than two. Through attempts to prove this theorem, the concept of a ring was introduced by Richard Dedekind in the 1800's which provided a generalization of arithmetic. However, it was not until the 1920's that rings were axiomatically defined by Emmy Noether and Wolfgang Krull in their theory of ideals. Ring theory has since grown to be an active field of research with interesting connections to algebraic number theory and algebraic geometry.

In comparison to a ring, the concept of a category is much younger with category theory being a field of mathematics introduced by Samuel Eilenberg and Saunders Mac Lane in 1945 as part of their work in topology. However, applications to other fields of mathematics have since grown tremendously. Notably, Alexander Grothendieck almost single-handedly shaped modern algebraic geometry with the use of category theory theory whereas William Lawvere applied category theory to logic to develop the field of categorical logic. While there

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are many diverse uses of category theory, applications to abstract algebra are especially interesting since one can interpret various algebraic structures such as sets, monoids and groups as categories and vice-versa in an effort to study them in an uniform fashion. By doing so, one can translate propositions proven in categories into results in their respective algebraic structures.

In the literature, especially in the field of categorification, one often views a ring together with a collection of idempotents as a category with an object for each idempotent. This is the point of view we follow in the current paper. Our goal is to make the connection between rings (with idempotents) and categories as precise as possible, and create a dictionary between the two points of view. In particular, we show how we can view a ring as a small preadditive category with one object and an idempotented ring as a small preadditive category with finitely many objects. We then prove an equivalence of 1-categories and 2-categories between the category of rings and the category of small preadditive categories with one object and between the category of idempotented rings and the category of small preadditive categories with finitely many objects. Under these equivalences, we show in Proposition 6.3 that two functors between two small preadditive categories with one object form an adjunction if and only if there exist specific 2-morphisms of rings and mutually inverse bijections of sets. Finally, we conclude with a proof in Proposition 6.6 that an idempotented ring that contains no zero divisors and whose characteristic is not two has a complete set of primitive orthogonal idempotents if and only if its corresponding category is idempotent complete.

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2. Background on Ring Theory

In this section, we recall the definition of a ring as well as different types of idempotents and systems of such idempotents within a ring. Furthermore, the notion of an idempotented ring is introduced and the example of a matrix ring is given. Lastly, we give a proof that every idempotented ring is isomorphic to a matrix ring.

Definition 2.1 (Ring)**.** A *ring* is a set R equipped with two binary operations (denoted by addition and multiplication) satisfying the following axioms:

- (Commutativity of addition) For all $a, b \in R$, $a + b = b + a$.
- (Associativity of addition) For all $a, b, c \in R$, $a + (b + c) = (a + b) + c$.
- (Additive identity) There exists an *additive identity* $0_R \in R$ such that $0_R + a = a = a + 0_R$ for all $a \in R$.
- (Additive inverse) For all $a \in R$, there exists an element $-a \in R$ such that $a+(-a) = 0_R$.
- (Associativity of multiplication) For all $a, b, c \in R$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- (Multiplicative identity) There exists a *multiplicative identity* $1_R \in R$ such that $1_R \cdot a = a = a \cdot 1_R$ for all $a \in R$.
- (Left distributive property) For all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
- (Right distributive property) For all $a, b, c \in R$, $(b + c) \cdot a = b \cdot a + c \cdot a$.

Definition 2.2 (Idempotent)**.** An element e in a ring R is *idempotent* if $e^2 = e$.

Definition 2.3 (Trivial idempotent, primitive idempotent)**.** The additive and multiplicative identities of a ring R are called *trivial idempotents*. If an idempotent e in R cannot be written as the sum of two non-zero idempotents, then it is a *primitive idempotent*.

Definition 2.4 (Orthogonal idempotents)**.** Idempotents e and e' in a ring R are a pair of *orthogonal idempotents* if $e \cdot e' = e' \cdot e = 0$.

Definition 2.5 (Complete set of orthogonal idempotents)**.** Let R be a ring and with a subset $I = \{e_1, e_2, \cdots, e_n\}$. We call I is a *complete set of orthogonal idempotents* if

- Every pair of distinct e_i, e_j in I is a pair of orthogonal idempotents.
- $0_R \in I$.
- $1_R = e_1 + e_2 + \cdots + e_n$.

If every non-zero idempotent in I is primitive, then it is a *complete set of primitive orthogonal idempotents*.

Definition 2.6 (Idempotented ring)**.** An *idempotented ring* is a pair (R, I), where R is a ring and I a complete set of orthogonal idempotents.

Example 2.7. (Matrix ring). Suppose (R, I) is an idempotented ring with $I = \{e_1, e_2, \cdots, e_n\}$. We define a ring $M_n(R)$ with multiplicative and additive identities $1_{M_n(R)}$ and $0_{M_n(R)}$ with the following:

$$M_n(R) = \left\{ \begin{pmatrix} e_1re_1 & e_1re_2 & \cdots & e_1re_n \\ e_2re_1 & e_2re_2 & \cdots & e_2re_n \\ \vdots & \vdots & \ddots & \vdots \\ e_nre_1 & e_nre_2 & \cdots & e_nre_n \end{pmatrix} : r \in R \right\},$$

$$1_{M_n(R)} = \begin{pmatrix} e_1 & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix} + \begin{pmatrix} 0_R & 0_R & \cdots & 0_R \\ 0_R & e_2 & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix} + \cdots + \begin{pmatrix} 0_R & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & e_n \end{pmatrix},$$

$$0_{M_n(R)} = \begin{pmatrix} 0_R & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix}.$$

Next we define a subset $I_{M_n(R)} \subseteq M_n(R)$ in this fashion,

$$I_{M_n(R)} = \left\{ \begin{pmatrix} e_1 & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix}, \begin{pmatrix} 0_R & 0_R & \cdots & 0_R \\ 0_R & e_2 & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix}, \cdots, \begin{pmatrix} 0_R & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & e_n \end{pmatrix} \right\} \cup \{0_{M_n(R)}\}.$$

It is fairly simple to prove that $M_n(R)$ is a ring since most of the ring axioms follow from the properties of matrices and so we will omit the proof. It is also straightforward to prove that every pair of $M, N \in I_{M_n(R)}$ is pairwise orthogonal and that the sum of all $M \in M_n(I) = 1_{M_n(R)}$. Combining these facts along with our definition that $0_{M_n(R)} \in I_{M_n(R)}$, we have that $I_{M_n(R)}$ is a complete set of orthogonal idempotents.

Proposition 2.8 (Every idempotented ring is isomorphic to a matrix ring)**.**

Proof. Let (R, I) be an idempotented ring and $(M_n(R), I_{M_n(I)})$ be the matrix ring. We define $f\colon (R, I) \to (M_n(R), I_{M_n(R)})$ to be the map $r \mapsto (e_j r e_i)$ where $(e_j r e_i)$ denotes the $n \times n$ matrix whose (i, j) entry is $e_j r e_i$. We claim that f is a ring isomorphism.

- (Injective) Assume that $f(r) = f(r')$ for some $r, r' \in R$. Then for any index (i, j),

$$
\begin{aligned}
r = (e_1 + \cdots + e_n) r (e_1 + \cdots + e_n) = \sum_{i,j=1}^{n} e_i r e_J = \sum_{i,j=1}^{n} &e_i r' e_j \\
= (e_1 + \cdots + e_n) &r' (e_1 + \cdots + e_n) = r'.
\end{aligned}
$$

- (Surjective) By the definition of $M_n(R)$, for any matrix $M \in M_n(R)$, $f(r) = M$ for some $r \in R$.
- (Preserves Sums) Let $r, r' \in R$, then we have the following set of equalities

$$
\begin{aligned}
f(r + r') &= \begin{pmatrix} e_1(r+r')e_1 & e_1(r+r')e_2 & \cdots & e_1(r+r')e_n \\ e_2(r+r')e_1 & e_2(r+r')e_2 & \cdots & e_2(r+r')e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n(r+r')e_1 & e_n(r+r')e_2 & \cdots & e_n(r+r')e_n \end{pmatrix} \\
&= \begin{pmatrix} e_1 r e_1 & e_1 r e_2 & \cdots & e_1 r e_n \\ e_2 r e_1 & e_2 r e_2 & \cdots & e_2 r e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n r e_1 & e_n r e_2 & \cdots & e_n r e_n \end{pmatrix} + \begin{pmatrix} e_1 r' e_1 & e_1 r' e_2 & \cdots & e_1 r' e_n \\ e_2 r' e_1 & e_2 r' e_2 & \cdots & e_2 r' e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n r' e_1 & e_n r' e_2 & \cdots & e_n r' e_n \end{pmatrix} \\
&= f(r) + f(r').
\end{aligned}
$$

- (Preserves Products) Let $r, r' \in R$, then we have the following set of equalities.

$$
\begin{aligned}
f(r) \cdot f(r') &= \begin{pmatrix} e_1 r e_1 & e_1 r e_2 & \cdots & e_1 r e_n \\ e_2 r e_1 & e_2 r e_2 & \cdots & e_2 r e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n r e_1 & e_n r e_2 & \cdots & e_n r e_n \end{pmatrix} \begin{pmatrix} e_1 r' e_1 & e_1 r' e_2 & \cdots & e_1 r' e_n \\ e_2 r' e_1 & e_2 r' e_2 & \cdots & e_2 r' e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n r' e_1 & e_n r' e_2 & \cdots & e_n r' e_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n} (e_1 r e_i)(e_i r' e_1) & \sum_{i=1}^{n} (e_1 r e_i)(e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_1 r e_i)(e_i r' e_n) \\ \sum_{i=1}^{n} (e_2 r e_i)(e_i r' e_1) & \sum_{i=1}^{n} (e_2 r e_i)(e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_2 r e_i)(e_i r' e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} (e_n r e_i)(e_i r' e_1) & \sum_{i=1}^{n} (e_n r e_i)(e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_n r e_i)(e_i r' e_n) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n} (e_1 r e_i r' e_1) & \sum_{i=1}^{n} (e_1 r e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_1 r e_i r' e_n) \\ \sum_{i=1}^{n} (e_2 r e_i r' e_1) & \sum_{i=1}^{n} (e_2 r e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_2 r e_i r' e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} (e_n r e_i r' e_1) & \sum_{i=1}^{n} (e_n r e_i r' e_2) & \cdots & \sum_{i=1}^{n} (e_n r e_i r' e_n) \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} e_1(r \cdot r')e_1 & e_1(r \cdot r')e_2 & \cdots & e_1(r \cdot r')e_n \\ e_2(r \cdot r')e_1 & e_2(r \cdot r')e_2 & \cdots & e_2(r \cdot r')e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n(r \cdot r')e_1 & e_n(r \cdot r')e_2 & \cdots & e_n(r \cdot r')e_n \end{pmatrix}$$
$$= f(r \cdot r')$$

- (Preserves Identity)

$$f(1_R) = \begin{pmatrix} e_1 \cdot 1_R \cdot e_1 & e_1 \cdot 1_R \cdot e_2 & \cdots & e_1 \cdot 1_R \cdot e_n \\ e_2 \cdot 1_R \cdot e_1 & e_2 \cdot 1_R \cdot e_2 & \cdots & e_2 \cdot 1_R \cdot e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n \cdot 1_R \cdot e_1 & e_n \cdot 1_R \cdot e_2 & \cdots & e_n \cdot 1_R \cdot e_n \end{pmatrix} = \begin{pmatrix} e_1 & 0_R & \cdots & 0_R \\ 0_R & e_2 & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & e_n \end{pmatrix}.$$

□

3. Background on Category Theory

In this section, we recall the notion of a category and give some examples involving algebraic structures. We then consider mappings between categories with the definition of a functor and consider relations between functors through natural transformations and adjunctions. Lastly, we give the definition of an equivalence of categories which we will use to great effect in later sections.

Definition 3.1 (Category)**.** A category $\mathcal{C}$ consists of a class of *objects* $\operatorname{Ob}\mathcal{C}$ and for every pair of objects X, Y, a class of *morphisms* $\operatorname{Mor}_{\mathcal{C}}(X, Y)$, writing $f\colon X \to Y$ to denote a morphism in $\operatorname{Mor}_{\mathcal{C}}(X, Y)$. These classes must satisfy the following:

- (Identity) For any object $X \in \operatorname{Ob}\mathcal{C}$, there exists an *identity morphism of* X, $\mathrm{id}_X\colon X \to X$, such that for any morphism $f\colon X \to Y$, $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$.
- (Composition) For any $X, Y, Z \in \operatorname{Ob}\mathcal{C}$, there is a map of composition $\operatorname{Mor}_{\mathcal{C}}(X, Y) \times \operatorname{Mor}_{\mathcal{C}}(Y, Z) \to \operatorname{Mor}_{\mathcal{C}}(X, Z)$.
- (Associativity) Given $f \in \operatorname{Mor}_{\mathcal{C}}(X, X')$, $g \in \operatorname{Mor}_{\mathcal{C}}(X', Y)$ and $h \in \operatorname{Mor}_{\mathcal{C}}(Y, Y')$, $(h \circ g) \circ f = h \circ (g \circ f) \in \operatorname{Mor}_{\mathcal{C}}(X, Y')$.

Example 3.2 (Category of of finite-dimensional vector spaces). Let $\mathrm{FinVect}_{\mathbb{F}}$ denote the category of finite-dimensional vector spaces over a fixed field $\mathbb{F}$. The objects of $\mathrm{FinVect}_{\mathbb{F}}$ are finite-dimensional vector spaces over $\mathbb{F}$ and for any $V, W \in \operatorname{Ob}\mathrm{FinVect}_{\mathbb{F}}$, $\operatorname{Mor}_{\mathrm{FinVect}_{\mathbb{F}}}(V, W)$ is the class of linear maps from V to W. The axioms of a category are satisfied with the following data.

- (Identity) For any $V \in \operatorname{Ob}\mathrm{FinVect}_{\mathbb{F}}$, id_V is the identity map from V onto itself.
- (Composition) The composition of morphisms in $\mathrm{FinVect}_{\mathbb{F}}$ is the usual composition of linear maps.
- (Associativity) Associativity holds because the composition of linear maps is associative.

Example 3.3 (Category of matrices). Let $\mathrm{Mat}(\mathbb{F})$ denote the category of matrices over over a fixed field $\mathbb{F}$. The objects of $\mathrm{Mat}(\mathbb{F})$ are the natural numbers, and for any $m, n \in \operatorname{Ob}\mathrm{Mat}(\mathbb{F})$, $\operatorname{Mor}_{\mathrm{Mat}(\mathbb{F})}(m, n)$ is the class of $n \times m$ matrices. The axioms of a category are satisfied with the data below.

- (Identity) For any $m \in \operatorname{Ob}\operatorname{Mat}(\mathbb{F})$, id_m is the identity $m \times m$ matrix.
- (Composition) The composition of morphisms in $\operatorname{Mat}(\mathbb{F})$ is the multiplication of matrices.
- (Associativity) Associativity holds because the multiplication of matrices is associative.

Definition 3.4 (Small category)**.** A category $\mathcal{C}$ is *small* the class of objects and the class of morphisms are both sets.

Definition 3.5 (Isomorphism)**.** Suppose $\mathcal{C}$ is a category. A morphism $f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ is an *isomorphism* between X and Y if there exists a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then objects X and Y are *isomorphic*, which is symbolized as $X \cong Y$.

Definition 3.6 (Idempotent, split idempotent)**.** Much like the definition of an idempotent in a ring, a morphism $e \in \operatorname{Mor}_{\mathcal{C}}(X, X)$ is *idempotent* if $e \circ e = e$. An idempotent e is a *split idempotent* if there exists morphisms $f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(Y, X)$ such that $g \circ f = e$ and $f \circ g = \mathrm{id}_Y$.

Definition 3.7 (Idempotent complete)**.** A category $\mathcal{C}$ is *idempotent complete* if every idempotent morphism in $\mathcal{C}$ is a split idempotent.

Definition 3.8 (Preadditive category)**.** A category $\mathcal{C}$ is *preadditive* if for any $X, Y \in \operatorname{Ob}\mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ has the structure of an abelian group, which we write additively. Furthermore, we require that composition is distributive over this addition. So for any $f, f' \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and $g, g' \in \operatorname{Mor}_{\mathcal{C}}(Y, Z)$,

- $(g + g') \circ f = g \circ f + g' \circ f \in \operatorname{Mor}_{\mathcal{C}}(X, Z)$,
- $g \circ (f + f') = g \circ f + g \circ f' \in \operatorname{Mor}_{\mathcal{C}}(X, Z)$.

Remark 3.9. We will refer to the identity with respect to the addition operation as the *zero morphism* from X to Y, denoting it as $0_{X,Y}$.

Example 3.10. The category $\mathrm{FinVect}_{\mathbb{F}}$ is preadditive because for any $V, W \in \operatorname{Ob}\mathrm{FinVect}_{\mathbb{F}}$, $\operatorname{Mor}_{\mathrm{FinVect}_{\mathbb{F}}}(V, W)$ is a class of linear maps equipped with a commutative addition operation (addition of linear maps) with $0_{V,W}$ being linear map $v \mapsto 0_W$. Furthermore, the composition of linear maps is distributive over addition.

Example 3.11. The category $\operatorname{Mat}(\mathbb{F})$ is preadditive since for any $m, n \in \operatorname{Ob}\operatorname{Mat}(\mathbb{F})$, we have that $\operatorname{Mor}_{\operatorname{Mat}(\mathbb{F})}(m, n)$ is an abelian group with respect to the addition operation with $0_{m,n}$ being $n \times m$ matrix. The distributive property then follows from the distributive property of matrices.

Definition 3.12 (Category of rings, category of idempotented rings)**.** Let Ring denote the *category of rings*, where the objects are rings and the morphisms are ring homomorphisms. We let $\mathrm{Ring}_{\perp}$ denote the *category of idempotented rings*. In this case, the objects of $\mathrm{Ring}_{\perp}$ are idempotented rings and $\operatorname{Mor}_{\mathrm{Ring}_{\perp}}((R, I), (S, J))$ consists of homomorphisms $h\colon R \to S$ such that $h(e) \in J$ for all $e \in I$.

Definition 3.13 (Functor)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ (written as $F\colon \mathcal{C} \to \mathcal{D}$) consists of the following:

- F maps every object $X \in \operatorname{Ob}\mathcal{C}$ to an object $F(X) \in \operatorname{Ob}\mathcal{D}$ and every morphism $f\colon X \to Y$ to a morphism $F(f)\colon F(X) \to F(Y) \in \operatorname{Mor}_{\mathcal{D}}(F(X), F(Y))$.
- (Preservation of identity) For any $X \in \operatorname{Ob}\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \in \operatorname{Ob}\mathcal{D}$.
- (Preservation of composition) Given $f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(Y, Z)$, $F(g \circ f) = F(g) \circ F(f) \in \operatorname{Mor}_{\mathcal{D}}(F(X), F(Z))$.

Example 3.14 (Identity functor). Just like how there exists an identity mapping for any set, there exists an *identity functor* $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ for any category $\mathcal{C}$ such that $\mathrm{id}_{\mathcal{C}}(X) = X$ for all $X \in \operatorname{Ob}\mathcal{C}$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all $f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$.

Example 3.15 (Double dual functor). An example of a functor from linear algebra is the *double dual functor* $F\colon \mathrm{FinVect}_{\mathbb{F}} \to \mathrm{FinVect}_{\mathbb{F}}$. For $V \in \operatorname{Ob}\mathrm{FinVect}_{\mathbb{F}}$, $F(V) = V^{**}$, and for $f \in \operatorname{Mor}_{\mathrm{FinVect}_{\mathbb{F}}}(V, W)$, $F(f) = f^{**}\colon V^{**} \to W^{**}$. That is, F maps a vector space to its double dual and a linear map to its double transpose. It is straightforward to verify that F is a functor.

Definition 3.16 (Additive functor)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if for all $X, Y \in \operatorname{Ob}\mathcal{C}$, $F\colon \operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_{\mathcal{D}}(F(X), F(Y))$ has the structure of a group homomorphism with respect to addition. That is for any $f, g \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$, $F(f + g) = F(f) + F(g) \in \operatorname{Mor}_{\mathcal{D}}(F(X), F(Y))$.

Definition 3.17 (Category of small preadditive categories with one object)**.** Let PreCat_1 denote the category of small preadditive categories with one object. The objects are small preadditive categories with one object and $\operatorname{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$ is class of additive functors from $\mathcal{C}$ to $\mathcal{D}$.

Definition 3.18 (Category of small preadditive categories with finitely many objects)**.** Let $\mathrm{PreCat}_{\mathrm{Fin}}$ denote the category of small preadditive categories with finitely many objects. The objects are small preadditive categories with finitely many objects and $\operatorname{Mor}_{\mathrm{PreCat}_{\mathrm{Fin}}}(\mathcal{C}, \mathcal{D})$ is the class of additive functors from $\mathcal{C}$ to $\mathcal{D}$.

Definition 3.19 (Full functor, faithful functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ gives rise to map $F_{X,Y}\colon \operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_{\mathcal{D}}(F(X), F(Y))$, for all $X, Y \in \operatorname{Ob}\mathcal{C}$. The functor F is *full* on morphisms if $F_{X,Y}$ is injective and *faithful* on morphisms if $F_{X,Y}$ is surjective for all $X, Y \in \operatorname{Ob}\mathcal{C}$. If F is both full and faithful, it is called *fully faithful*.

Definition 3.20 (Essentially surjective)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* on objects if for all $Y \in \operatorname{Ob}\mathcal{D}$, there exists a $X \in \operatorname{Ob}\mathcal{C}$ such that $F(X) \cong Y$.

Definition 3.21 (Natural transformation, natural isomorphism)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ functors. A *natural transformation* η from F to G, denoted as $\eta\colon F \Rightarrow G$, is a mapping from $\operatorname{Ob}\mathcal{C}$ to $\operatorname{Mor}\mathcal{D}$ that associates every $X \in \operatorname{Ob}\mathcal{C}$ to a morphism

$\eta_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X)$ such that the following diagram commutes for any $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

If $\eta_X : F(X) \to G(X)$ is an isomorphism for all $X \in \mathrm{Ob}\,\mathcal{C}$, then η is a *natural isomorphism.*

Example 3.22. (Identity natural transformation) Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The *identity natural transformation* on F, $\mathrm{id}_F : F \Rightarrow F$, is the map $X \mapsto \mathrm{id}_{F(X)} \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(X))$ for all $X \in \mathrm{Ob}\,\mathcal{C}$.

Example 3.23. Let $\mathrm{id}_{\mathrm{FinVect}_{\mathbb{F}}} : \mathrm{FinVect}_{\mathbb{F}} \to \mathrm{FinVect}_{\mathbb{F}}$ be the identity functor and recall the double dual functor $F : \mathrm{FinVect}_{\mathbb{F}} \to \mathrm{FinVect}_{\mathbb{F}}$ from Example 3.15. We define a natural isomorphism $\eta : \mathrm{id}_{\mathrm{FinVect}_{\mathbb{F}}} \Rightarrow F$ like so: For any $V \in \mathrm{Ob}\,\mathrm{FinVect}_{\mathbb{F}}$, $\eta_V \in \mathrm{Mor}_{\mathrm{FinVect}_{\mathbb{F}}}(V, V^{**})$, where η_V is the isomorphism defined as $\eta_V(v) : V^* \to \mathbb{F}$ for all $v \in V$ and where $\eta_V(v)(f) = f(v)$, for all $f \in V^*$. It is then straightforward to check that the following diagram commutes for any $V, W \in \mathrm{Ob}\,\mathrm{FinVect}_{\mathbb{F}}$ and $f : V \to W$.

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\eta_W"] & W^{**} \end{tikzcd}$$

Definition 3.24 (Equivalence of categories)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* (written as $\mathcal{C} \simeq \mathcal{D}$) if there exists a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and a pair of natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$, $\epsilon \Rightarrow \mathrm{id}_{\mathcal{D}} \to F \circ G$.

Lemma 3.25. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The functor F yields an equivalence of categories if and only if F is fully faithful on morphisms and essentially surjective on objects.*

Proof. A proof is available on page 93 of [ML98]. □

Example 3.26. Recall the category of matrices $\mathrm{Mat}(\mathbb{F})$ from Example 3.11 and the category of finite-dimensional vector spaces $\mathrm{FinVect}_{\mathbb{F}}$ from Example 3.10. Consider the functor $F : \mathrm{Mat}(\mathbb{F}) \to \mathrm{FinVect}_{\mathbb{F}}$ that maps any $m \in \mathrm{Ob}\,\mathrm{Mat}(\mathbb{F})$ to $\mathbb{R}^m$ and any matrix to its corresponding linear map with respect to the standard bases. Then for any $V \in \mathrm{Ob}\,\mathrm{FinVect}_{\mathbb{F}}$, there exists a $m \in \mathrm{Ob}\,\mathrm{Mat}(\mathbb{F})$ such that $\dim(\mathbb{R}^m) = \dim(V)$ and so $F(m) = \mathbb{R}^m \cong V$. Thus F is essentially surjective on objects and fully faithful on morphisms by basic results in linear algebra on the correspondences between linear maps and matrices, and so $\mathrm{Mat}(\mathbb{F}) \simeq \mathrm{FinVect}_{\mathbb{F}}$.

Definition 3.27 (Adjoint functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $H : \mathcal{C} \to \mathcal{D}$, $J : \mathcal{D} \to \mathcal{C}$ functors. The functor H is *left adjoint* to J, denoted as $H \dashv J$, if there exist natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow J \circ H$ and $\epsilon : H \circ J \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that the following diagrams commute for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $Y \in \mathrm{Ob}\,\mathcal{D}$.

$$\begin{tikzcd} H(X) \arrow[r, "H(\eta_X)"] \arrow[dr, "\mathrm{id}_{H(X)}"'] & HJH(X) \arrow[d, "\epsilon_{H(X)}"] \\ & H(X) \end{tikzcd} \qquad \begin{tikzcd} J(Y) \arrow[r, "\eta_{J(Y)}"] \arrow[dr, "\mathrm{id}_{J(Y)}"'] & JHJ(Y) \arrow[d, "J(\epsilon_Y)"] \\ & J(Y) \end{tikzcd}$$

The commutative diagrams above are referred to as the *trangle identities*.

4. Higher Category Theory

In this section, we extend some earlier definitions from category theory into higher category theory, namely categories and functors. We will show how the earlier examples of the category of rings and category of idempotented rings can be enhanced to 2-categories. Lastly, the definition of an equivalence of 2-categories is given.

Definition 4.1 (Strict 2-category)**.** A *strict 2-category* $\mathcal{C}$ is a category where for every $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is a category whose objects are the morphisms from X to Y together with the axioms below. To avoid confusion, we will refer to morphisms between objects of $\mathcal{C}$ as *1-morphisms* and denote their composition with the usual $\circ$. We will write $\alpha\colon f \Rightarrow g$ to denote α is a 2-morphism from $\mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Y)}(f,g)$.

- (Vertical composition) Let $f, g, h \in \mathrm{Ob}\,\mathrm{Mor}_{\mathcal{C}}(X,Y)$. There is a map of vertical composition $\mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Y)}(f,g) \times \mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Y)}(g,h) \to \mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Y)}(f,h)$ and we denote this composition as $\circ_v$. That is, for 2-morphisms $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, there is a 2-morphism $\beta \circ_v \alpha\colon f \Rightarrow h$. Furthermore, this vertical composition is associative.
- (Horizontal composition) Let $f, g \in \mathrm{Ob}\,\mathrm{Mor}_{\mathcal{C}}(X,Y)$, $f', g' \in \mathrm{Ob}\,\mathrm{Mor}_{\mathcal{C}}(Y,Z)$. There is a map of horizontal composition
$$\mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Y)}(f,g) \times \mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(Y,Z)}(f',g') \to \mathrm{Mor}_{\mathrm{Mor}_{\mathcal{C}}(X,Z)}(f' \circ f, g' \circ g),$$
and we denote this composition as $\circ_h$. That is, for 2-morphisms $\alpha\colon f \Rightarrow g$ and $\alpha'\colon f' \Rightarrow g'$, there is a 2-morphism $\alpha' \circ_h \alpha\colon f' \circ f \Rightarrow g' \circ g$. Furthermore, this horizontal composition is associative.
- (Interchange law) Let $f, g, h \in \mathrm{Ob}\,\mathrm{Mor}_{\mathcal{C}}(X,Y)$, $f', g', h' \in \mathrm{Ob}\,\mathrm{Mor}_{\mathcal{C}}(Y,Z)$. Given 2-morphisms $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$, $\alpha'\colon f' \Rightarrow g'$ and $\beta'\colon g' \Rightarrow h'$, $(\beta' \circ_v \alpha') \circ_h (\beta \circ_v \alpha) = (\beta' \circ_h \beta) \circ_v (\alpha' \circ_h \alpha)$.
- (Identity) For any $X \in \mathrm{Ob}\,\mathcal{C}$, there exists an identity 1-morphism id_X and an identity 2-morphism $\mathrm{id}_{\mathrm{id}_X}\colon \mathrm{id}_X \Rightarrow \mathrm{id}_X$. The 2-morphism $\mathrm{id}_{\mathrm{id}_X}$ serves as an identity for both vertical and horizontal composition. That is, for any $\alpha\colon \mathrm{id}_X \Rightarrow \mathrm{id}_X$, $\mathrm{id}_X \circ_v \alpha = \alpha = \alpha \circ_v \mathrm{id}_X$. For any 2-morphism $\beta\colon f \Rightarrow g$ where f and g are 1-morphisms in $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, $\beta \circ_h \mathrm{id}_{\mathrm{id}_X} = \beta = \mathrm{id}_{\mathrm{id}_Y} \circ_h \beta$.

Remark 4.2. Throughout this document, we will assume that all 2-categories are strict and so we will omit the term "strict" and refer to them just as 2-categories.

Example 4.3 (Category of small categories). The category of small categories is an example of a 2-category and we will simply denote it by Cat. As the name suggests, objects of Cat are small categories and the morphisms between these categories are functors. It is straightforward to verify that Cat is truly a category since the axioms of a functor satisfies the domain, codomain, identity, composition and associativity axioms of a morphism in a category. Then Cat can be enhanced to a 2-category with the 2-morphisms being natural transformations.

Proposition 4.4. *Recall the category of rings,* Ring*, whose objects are rings and whose morphisms are homomorphisms. Then* Ring *can enhanced to a 2-category with 2-morphisms being defined as follows: For rings R, S and homomorphisms $f\colon R \to S$ and $g\colon R \to S$, a 2-morphism $\alpha\colon f \Rightarrow g$ is an element $\alpha \in S$ such that $\alpha f(r) = g(r)\alpha$ for all $r \in R$.*

Proof.

- (Vertical composition) Let $R, S \in \operatorname{Ob}\mathrm{Ring}$ and $f, g, h \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$. Given 2-morphisms $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, we want to show that there exists the vertical composition $\beta \circ_v \alpha$ such that $(\beta \circ_v \alpha)f(r) = g(r)(\beta \circ_v \alpha)$ for all $r \in R$. In this case, vertical composition corresponds to the multiplication operation $\cdot$ in S, so we will often abuse notation and omit $\circ_v$. Note that associativity of $\circ_v$ follows from the associativity of multiplication in a ring. Since α and β are 2-morphisms, our definition tells us that $\alpha f(r) = g(r)\alpha$ and $\beta g(r) = h(r)\beta$ for all $r \in R$. Combining these two facts, we get:

$$\begin{aligned} \alpha f(r) &= g(r)\alpha \\ \implies \beta\alpha f(r) &= \beta g(r)\alpha \\ \implies \beta\alpha f(r) &= h(r)\beta\alpha \end{aligned}$$

So $\beta\alpha$ is precisely an element of S such that $\beta\alpha f(r) = h(r)\beta\alpha$ for all $r \in R$.

- (Horizontal composition) Let $R, S, T \in \operatorname{Ob}\mathrm{Ring}$, $f, g \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$ and $f', g' \in \mathrm{Mor}_{\mathrm{Ring}}(S, T)$. Given 2-morphisms $\alpha\colon f \Rightarrow g$ and $\alpha'\colon f' \Rightarrow g'$, we define the horizontal composition $\alpha' \circ_h \alpha$ to be the element $\alpha' f'(\alpha)$ in T. This gives the following equalities:

$$\begin{aligned} \alpha f(r) &= g(r)\alpha \\ \implies f'(\alpha f(r)) &= f'(g(r)\alpha) \\ \implies f'(\alpha)f'(f(r)) &= f'(g(r))f'(\alpha) \\ \implies \alpha' f'(\alpha)f'(f(r)) &= \alpha' f'(g(r))f'(\alpha) \\ \implies \alpha' f'(\alpha)f'(f(r)) &= g'(g(r))\alpha' f'(\alpha) \end{aligned}$$

Given another element $\alpha''\colon f'' \Rightarrow g''$ where $f'', g'' \in \mathrm{Mor}_{\mathrm{Ring}}(T, U)$,

$$\begin{aligned} (\alpha'' \circ_h \alpha) \circ_h \alpha &= (\alpha'' f''(\alpha')) \circ_h \alpha \\ &= \alpha'' f''(\alpha')f''f'(\alpha) \\ &= \alpha''(f''(\alpha' f'(\alpha)) \\ &= \alpha''(f''(\alpha' \circ_h \alpha)) \\ &= \alpha'' \circ_h (\alpha' \circ_h \alpha) \end{aligned}$$

and so horizontal composition is associative.

- (Interchange law) Let $R, S, T \in \operatorname{Ob}\mathrm{Ring}$ with 1-morphisms $f, g, h \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$, $f', g', h' \in \mathrm{Mor}_{\mathrm{Ring}}(S, T)$. Given 2-morphisms $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$, $\alpha'\colon f' \Rightarrow g'$ and $\beta'\colon g' \Rightarrow h'$ we have the following equalities:

$$\begin{aligned} (\beta'\alpha') \circ_h (\beta\alpha) &= \beta'\alpha' f'(\beta\alpha), \\ \beta' \circ_h \beta &= \beta' g'(\beta), \end{aligned}$$

$$\alpha' \circ_h \alpha = \alpha' f'(\alpha).$$

Then we know that:

$$\begin{aligned}(\beta' \circ_h \beta) \circ_v (\alpha' \circ_h \alpha) &= \beta' g'(\beta)\alpha' f'(\alpha)\\ &= \beta'\alpha' f'(\beta) f'(\alpha)\\ &= \beta'\alpha' f'(\beta\alpha)\\ &= (\beta' \circ_v \alpha') \circ_h (\beta \circ_v \alpha)\end{aligned}$$

- (Identity) Assume that $R, S \in \operatorname{Ob}\mathrm{Ring}$ with 1-morphisms $f, g \in \operatorname{Mor}_{\mathrm{Ring}}(R, S)$ and $\mathrm{id}_R \in \operatorname{Mor}_{\mathrm{Ring}}(R, R)$. We claim that the multiplicative ring identity $1_R \colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$ is an identity for vertical composition from id_R to id_R and horizontal composition from R to S. Recall that vertical composition corresponds to ring multiplication, so for all $\alpha \colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$, $1_R \circ_v \alpha = \alpha = \alpha \circ_v 1_R$. Moreover, for all $\beta \colon f \Rightarrow g$,

$$\beta \circ_h 1_R = \beta \circ_v f(1_R) = \beta = 1_S \circ_v \mathrm{id}_S(\beta) = 1_S \circ_h \beta. \qquad \square$$

Proposition 4.5. *Recall that the category of idempotented rings is a category whose objects are rings with a complete set of orthogonal idempotents and whose morphisms are idempotent preserving ring homomorphisms. Then* $\mathrm{Ring}_\perp$ *can be enhanced to a 2-category with 2-morphisms being defined as follows: Given* (R, I) *and* (S, J) *in* $\operatorname{Ob}\mathrm{Ring}_\perp$ *and* $f, g \in \operatorname{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$*, a 2-morphism* $\alpha \colon f \Rightarrow g$ *in* $\mathrm{Ring}_\perp$ *is a function* $\alpha \colon I \to S$*. This function has the property that for any* $e \in I$*,* $\alpha(e)$ *is an element of* $g(e)Sf(e) \subseteq S$ *and for any* $r \in R$ *and* $e' \in I$*,* $\alpha(e')f(e're) = g(e're)\alpha(e)$*.*

Proof.

- (Vertical composition) Let $(R, I), (S, J) \in \operatorname{Ob}\mathrm{Ring}_\perp$ with 1-morphisms

$$f, g, h \in \operatorname{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J)).$$

Given 2-morphisms $\alpha \colon f \Rightarrow g$ and $\beta \colon g \Rightarrow h$, we define $\beta \circ_v \alpha$ to be the function from I to S such that for any $e \in I$, $\beta \circ_v \alpha(e) = \beta(e) \cdot \alpha(e)$ where $\cdot$ is the usual multiplication in the ring S, which we will sometimes omit. Since $\alpha(e')f(e're) = g(e're)\alpha(e)$ and $\beta(e')g(e're) = h(e're)\beta(e)$, the equalities below hold.

$$\begin{aligned}&\alpha(e')f(e're) = g(e're)\alpha(e)\\ \implies\ &\beta(e')\alpha(e')f(e're) = \beta(e')g(e're)\alpha(e)\\ \implies\ &\beta(e')\alpha(e')f(e're) = h(e're)\beta(e)\alpha(e)\end{aligned}$$

So $\beta \circ_v \alpha$ is indeed a 2-morphism from f to h. Furthermore, when given another 2-morphism $\gamma \colon h \Rightarrow j$, where $j \in \operatorname{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$, vertical composition is associative since

$$\begin{aligned}((\gamma \circ_v \beta) \circ_v \alpha)(e) &= ((\gamma \circ_v \beta)(e)\alpha(e)\\ &= \gamma(e)\beta(e)\alpha(e)\\ &= \gamma(e) \circ (\beta \circ_v \alpha)(e)\\ &= (\gamma \circ_v (\beta \circ_v \alpha))(e).\end{aligned}$$

- (Horizontal composition) Let $(R, I), (S, J), (T, K) \in \operatorname{Ob}\mathrm{Ring}_\perp$ and $f, g, \colon (R, I) \to (S, J)$ and $f', g' \colon (S, J) \to (T, K)$. Given 2 morphisms $\alpha \colon f \Rightarrow g$ and $\alpha' \colon f' \Rightarrow g'$, we define

$\alpha' \circ_h \alpha$ to be the following function from I to T: For any $e \in I$, $\alpha' \circ_h \alpha(e) = \alpha'(g(e))f'(\alpha(e))$. Then we have the following equalities:

$$\begin{aligned} \alpha(e')f(e're) &= g(e're)\alpha(e) \\ \implies f'(\alpha(e')f(e're)) &= f'(g(e're)\alpha(e)) \\ \implies f'(\alpha(e'))f'(f(e're)) &= f'(g(e're))f'(\alpha(e)) \\ \implies \alpha'(g(e'))f'(\alpha(e'))f'(f(e're)) &= \alpha'(g(e'))f'(g(e're))f'(\alpha(e)) \\ \implies \alpha'(g(e'))f'(\alpha(e'))f'(f(e're)) &= g'(g(e're))\alpha'(g(e))f'(\alpha(e)) \end{aligned}$$

and so $\beta \circ_h \alpha$ is indeed a 2-morphism from $f' \circ f$ to $g' \circ g$. Given another object (U, L) in $\mathrm{Ring}_\perp$ with 1-morphisms $f'', g'' \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((T, K), (U, L))$ and 2-morphism $\alpha'' \colon f'' \Rightarrow g''$, we have that for any $e \in I$,

$$\begin{aligned} (\alpha'' \circ_h (\alpha' \circ_h \alpha))(e) &= \alpha''(g'g(e))f''((\alpha' \circ_h \alpha)(e)) \\ &= \alpha''(g'g(e))f''(\alpha'((g(e))f'(\alpha(e))) \\ &= \alpha''(g'g(e))f''(\alpha'(g(e)))f''f(\alpha(e)) \\ &= (\alpha'' \circ_h \alpha'(g(e))f''f(\alpha(e)) \\ &= ((\alpha'' \circ_h \alpha') \circ \alpha)(e) \end{aligned}$$

and so this horizontal composition is associative.

- (Interchange law) Suppose $(R, I), (S, J), (T, K) \in \mathrm{Ob}\,\mathrm{Ring}_\perp$ with 1-morphisms $f, g, h \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$, $f', g', h' \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((S, J), (T, K))$. Let $\alpha \colon f \Rightarrow g$, $\beta \colon g \Rightarrow h$, $\alpha' \colon f' \Rightarrow g'$ and $\beta' \colon g' \Rightarrow h'$ be 2-morphisms. We know that $\beta(e) \in h(e)Sg(e) \subseteq S$ and that $g(e)$ and $h(e)$ are in J. Then for any idempotent $h(e) \in T$, there exists element $\alpha'(g(e)) \in T$ such that $\alpha'(h(e))f'(h(e)sg(e)) = g'(h(e)sg(e))\alpha'(g(e))$. If we let that $h(e)sg(e) = \beta(e)$, we get that $\alpha'(h(e))f'(\beta(e)) = g'(\beta(e))\alpha'(g(e))$. Thus,

$$\begin{aligned} ((\beta' \circ_h \beta) \circ_v (\alpha' \circ_h \alpha))(e) &= (\beta' \circ_h \beta)(e) \circ_v (\alpha' \circ_h \alpha)(e) \\ &= \beta'(h(e))g'(\beta(e)) \circ_v \alpha'(g(e))f'(\alpha(e)) \\ &= \beta'(h(e))g'(\beta(e))\alpha'(g(e))f'(\alpha(e)) \\ &= \beta'(h(e))\alpha'(h(e))f'(\beta(e))f'(\alpha(e)) \\ &= (\beta' \circ_v \alpha')(h(e))f'((\beta \circ_v \alpha)(e)) \\ &= ((\beta' \circ_v \alpha') \circ_h (\beta \circ_v \alpha))(e). \end{aligned}$$

- (Identity) Let $(R, I) \in \mathrm{Ob}\,\mathrm{Ring}_\perp$ with $f, g \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$ and

$$\mathrm{id}_R \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((R, I), (R, I)),$$

the identity homomorphism from R to R. We define $\mathrm{id}_{\mathrm{id}_R} \colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$ to be the inclusion map of I into R. Then for any $\alpha \colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$ and $e \in I$, $\alpha(e) = ere \in eRe$ for some $r \in R$, and so

$$\begin{aligned} (\alpha \circ_v \mathrm{id}_{\mathrm{id}_R})(e) = \alpha(e)\mathrm{id}_{\mathrm{id}_R}(e) = eree = ere = eere& \\ = e\alpha(e) = \mathrm{id}_{\mathrm{id}_R}(e)\alpha(e) = (\mathrm{id}_{\mathrm{id}_R} \circ_v \alpha)(e).& \end{aligned}$$

Furthermore, let $(S, J) \in \mathrm{Ob}\,\mathrm{Ring}_\perp$ with $f, g \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$ and $\beta \colon f \Rightarrow g$. For any $e \in I$, $\beta(e) = g(e)sf(e) \in g(e)Sf(e)$ for some $s \in S$. Then we have the following:

$$(\beta \circ_h \mathrm{id}_{\mathrm{id}_R})(e) = \beta(\mathrm{id}_R(e))f(\mathrm{id}_{\mathrm{id}_R}(e)) = g(e)sf(e)f(e) = g(e)sf(e) = \beta(e)$$
$$= g(e)g(e)sf(e) = \mathrm{id}_{\mathrm{id}_S}(g(e))\mathrm{id}_S(\beta(e)) = (\mathrm{id}_{\mathrm{id}_S} \circ_h \beta)(e) \quad \square$$

Definition 4.6 (2-Functor)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are 2-categories. A map $F\colon \mathcal{C} \to \mathcal{D}$ is a *2-functor* if it satisfies the following properties:

- F maps every object $X \in \mathcal{C}$ to an object $F(X) \in \mathcal{D}$, every 1-morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ to a 1-morphism $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ and every 2-morphism $\alpha\colon f \Rightarrow g$ to a 2-morphism $F(\alpha)\colon F(f) \Rightarrow F(g)$.
- (Preservation of identity) For any $X \in \mathrm{Ob}\,\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(X))$. For any $\mathrm{id}_{\mathrm{id}_X} : \mathrm{id}_X \Rightarrow \mathrm{id}_X$, $F(\mathrm{id}_{\mathrm{id}_X}) = \mathrm{id}_{\mathrm{id}_{F(X)}}$, $\mathrm{id}_{\mathrm{id}_{F(X)}} \colon \mathrm{id}_{F(X)} \Rightarrow \mathrm{id}_{F(X)}$.
- (Preservation of composition) Given $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$, $F(g \circ f) = F(g) \circ F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Z))$.
- (Preservation of vertical composition) Given 1-morphisms $f, g, h \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and 2-morphisms $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)\colon F(f) \Rightarrow F(f'')$.
- (Preservation of horizontal composition) Given 1-morphisms $f, g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, $f', g' \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ and 2-morphisms $\alpha\colon f \Rightarrow g$, $\alpha'\colon f' \Rightarrow g'$, $F(\alpha' \circ_h \alpha) = F(\alpha') \circ_h F(\alpha)\colon F(f') \circ F(f) \Rightarrow F(g') \circ F(g)$.

Definition 4.7 (Equivalence of 2-categories)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are 2-categories. Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* as 2-categories if there exists a 2-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Lemma 4.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and $F\colon \mathcal{C} \to \mathcal{D}$ a 2-functor. Then F yields an equivalence of 2-categories if and only if F is essentially surjective on objects and fully faithful on both 1-morphisms and 2-morphisms.*

Proof. The 2-functor F is fully faithful on 1-morphisms and 2-morphisms if, for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, it induces an isomorphism of categories $\mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$. Then the statement follows from the argument found on [Kel05, p. 24] where one takes $\mathcal{V}$ to be the category of small categories. $\square$

5. Rings and Categories

Many familiar algebraic structures with an algebraic definition also have a category theoretic one. For example, a ring R can be viewed as a preadditive category with one object by defining the *ring category* associated to the ring R to be the category $\mathcal{C}_R$ such that $\mathrm{Ob}\,\mathcal{C}_R = \{*\}$ and $\mathrm{Mor}\,\mathcal{C}_R = R$. So the object of $\mathcal{C}_R$ is a formal one and the morphisms of $\mathcal{C}_R$ are elements of R. We let the composition and addition of morphisms be the respective multiplication and addition operation in R. It follows that the identity and zero morphism are the multiplicative and additive identities respectively in R. Likewise, to any preadditive category $\mathcal{C}$ with one object $*$, we can associate the ring $R_{\mathcal{C}} = \mathrm{Mor}_{\mathcal{C}}(*, *)$. So the multiplication and addition in the ring is the respective composition and addition in the class of morphisms of a category. We complete our definition of $R_{\mathcal{C}}$ by letting the additive and multiplicative identities of the ring be the zero and identity morphisms of $\mathrm{Mor}_{\mathcal{C}}(*, *)$. From a category theorist's point of view, a ring and a preadditive category with one object are the same thing seen through two different lenses.

Lemma 5.1. *If $\mathcal{C}$ is a preadditive category, then for all $X \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is a ring.*

Proof. For any $X \in \operatorname{Ob}\mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(X,X)$ has the structure of an abelian group with respect to addition and the definition of a category gives us the associative composition operation with an identity morphism id_X. Finally, the axioms of a preadditive category gives us the distributive property. □

Lemma 5.2. *If $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories and $F\colon \mathcal{C} \to \mathcal{D}$ an additive functor, then for all $X \in \operatorname{Ob}\mathcal{C}$, $F_{X,X}\colon \operatorname{Mor}_{\mathcal{C}}(X,X) \to \operatorname{Mor}_{\mathcal{D}}(F(X),F(X))$ is a ring homomorphism.*

Proof.

- (Preserves identity) The map $F_{X,X}\colon \operatorname{Mor}_{\mathcal{C}}(X,X) \to \operatorname{Mor}_{\mathcal{D}}(F(X),F(X))$ preserves the identity morphism which we have taken to be the identity element of our ring. Thus we have that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- (Preserves products) Recall that we took our multiplication operation to be the composition of morphisms. The axioms of a functor gives us that for all $f,g \in \operatorname{Mor}_{\mathcal{C}}(X,X)$, we have that $F_{X,X}(f \circ g) = F_{X,X}(f) \circ F_{X,X}(g)$.
- (Preserves sums) Since F is an additive functor, it acts like a group homomorphism on $\operatorname{Mor}_{\mathcal{C}}(X,X)$ with respect to addition. So for $f,g \in \operatorname{Mor}_{\mathcal{C}}(X,X)$, $F_{X,X}(f+g) = F_{X,X}(f) + F_{X,X}(g)$.

□

Proposition 5.3. *Similar to how we can associative every ring with a preadditive category with one object, we can also associate an idempotented ring with with a category that preserves the idempotent structure. For an idempotented ring (R,I), we can construct a preadditive category $\mathcal{C}_{(R,I)}$ where $\operatorname{Ob}\mathcal{C}_{(R,I)} = I$ and $\operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j) = e_jRe_i = \{e_jre_i\colon r \in R\}$. Composition in the category is just multiplication in the ring and so the identity morphism is e_i for any $e_i \in \operatorname{Ob}\mathcal{C}_{(R,I)}$.*

Proof.

- (Identity) For any $e_jre_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$, $(e_jre_i)(e_i) = e_jre_ie_i = e_jre_i = e_je_jre_i = (e_j)(e_jre_i)$.
- (Composition) For any $e_jre_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$, $e_kse_j \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_j,e_k)$, $(e_kre_j)(e_jse_i) = e_kre_je_jse_i = e_kre_jse_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_k)$.
- (Associativity) Since the morphisms of $\mathcal{C}_{(R,I)}$ are just elements in R, associativity of morphisms follows from the associativity of multiplication in R.
- (Abelian group structure) For any $e_jre_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$, we have a morphism $e_jre_i + e_jse_i = e_j(r+s)e_i = e_j(s+r)e_i = e_jse_i + e_jre_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$.
- (Left distributive property) For any $e_jre_i, e_jse_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$ and $e_kte_j, e_kue_j \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_j,e_k)$, we have that:

$$\begin{aligned}
(e_kte_j)(e_jre_i + e_jse_i) &= (e_kte_j)(e_j(r+s)e_i) \\
&= (e_kte_j(r+s)e_j) \\
&= (e_kte_jre_i) + (e_kte_jse_i) \\
&= (e_kte_je_jre_i) + (e_kte_je_jse_i) \\
&= (e_kte_j)(e_jre_i) + (e_kte_j)(e_jse_i)
\end{aligned}$$

- (Right distributive property) For any $e_kte_j, e_kue_j \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_j,e_k)$ and $e_jre_i \in \operatorname{Mor}_{\mathcal{C}_{(R,I)}}(e_i,e_j)$,

$$\begin{aligned}
(e_kte_j + e_kue_j)(e_jre_i) &= (e_k(t+u)e_j)(e_jre_i) \\
&= (e_k(t+u)e_je_jre_i)
\end{aligned}$$

$$
\begin{aligned}
&= (e_k(t+u)e_jre_i) \\
&= (e_kte_jre_i + e_kue_jre_i) \\
&= (e_kte_jre_i) + (e_kue_jre_i) \\
&= (e_kte_j)(e_jre_i) + (e_kue_j)(e_jre_i)
\end{aligned}
$$

□

Remark 5.4. We will denote elements of a direct sum $\bigoplus_{G\in\Gamma} G$ of abelian groups by formal sums $\sum_{G\in\Gamma} g_G$, where $g_G \in G$ for all $G \in \Gamma$. By convention, we omit zero summands. For instance, any element g of some $G \in \Gamma$ can be thought of as an element of $\bigoplus_{G\in\Gamma} G$, where all other summands are zero.

Proposition 5.5. *Suppose $\mathcal{C}$ is a preadditive category. Let $R_\mathcal{C} = \bigoplus_{X,Y\in\operatorname{Ob}\mathcal{C}} \operatorname{Mor}_\mathcal{C}(X,Y)$ and $I_\mathcal{C} = \{\mathrm{id}_X : X \in \operatorname{Ob}\mathcal{C}\}$. Then $(R_\mathcal{C}, I_\mathcal{C})$ is an idempotented ring with addition and multiplication given by the addition and composition of morphisms.*

Proof. We let addition and multiplication in $R_\mathcal{C}$ be the component-wise addition of morphisms in $\mathcal{C}$ and the composition of morphisms in $\mathcal{C}$ respectively. That is:

$$
\begin{gathered}
\sum_{X,Y\in\operatorname{Ob}\mathcal{C}} f_{X,Y} + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} g_{X,Y} = \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (f_{X,Y} + g_{X,Y}), \\
\left(\sum_{Y,Y'\in\operatorname{Ob}\mathcal{C}} g_{Y,Y'}\right) \cdot \left(\sum_{X,X'\in\operatorname{Ob}\mathcal{C}} f_{X,X'}\right) = \sum_{X,X',Y'\in\operatorname{Ob}\mathcal{C}} (g_{X',Y'} \circ f_{X,X'}).
\end{gathered}
$$

Note that we will consider the composition of non-composable morphisms to be zero. Finally, we let the respective additive and multiplicative identities be

$$
0_{R_\mathcal{C}} = \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} 0_{X,Y}, \quad 1_{R_\mathcal{C}} = \sum_{X\in\operatorname{Ob}\mathcal{C}} \mathrm{id}_X.
$$

- (Commutativity of addition)

$$
\begin{aligned}
\sum_{X,Y\in\operatorname{Ob}\mathcal{C}} f_{X,Y} + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} g_{X,Y} &= \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (f_{X,Y} + g_{X,Y}) \\
&= \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (g_{X,Y} + f_{X,Y}) = \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} g_{X,Y} + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} f_{X,Y}
\end{aligned}
$$

- (Associativity of addition)

$$
\begin{aligned}
\left(\sum_{X,Y\in\operatorname{Ob}\mathcal{C}} f_{X,Y} + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} g_{X,Y}\right) + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} h_{X,Y} &= \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (f_{X,Y} + g_{X,Y}) + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} h_{X,Y} \\
&= \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (f_{X,Y} + g_{X,Y} + h_{X,Y}) \\
&= \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} f_{X,Y} + \sum_{X,Y\in\operatorname{Ob}\mathcal{C}} (g_{X,Y} + h_{X,Y})
\end{aligned}
$$

$$= \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y} + \left(\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} g_{X,Y} + \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} h_{X,Y} \right)$$

- (Additive identity)

$$\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} 0_{X,Y} + \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y} = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (0_{X,Y} + f_{X,Y}) = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y}$$
$$= \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (f_{X,Y} + 0_{X,Y}) = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y} + \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} 0_{X,Y}$$

- (Additive inverses) The existence of additive inverses follows from the fact that for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has the structure of an abelian group.

$$\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y} + \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (-f_{X,Y}) = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (f_{X,Y} - f_{X,Y}) = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} 0_{X,Y} = \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (-f_{X,Y} + f_{X,Y})$$
$$= \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} (-f_{X,Y}) + \sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y}$$

- (Associativity of multiplication)

$$\left(\left(\sum_{Z,Z' \in \mathrm{Ob}\,\mathcal{C}} h_{Z,Z'} \right) \cdot \left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} g_{Y,Y'} \right) \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right)$$
$$= \left(\sum_{Y,Y',Z,Z' \in \mathrm{Ob}\,\mathcal{C}} (h_{Z,Z'} \circ g_{Y,Y'}) \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right)$$
$$= \left(\sum_{Y,Y',Z' \in \mathrm{Ob}\,\mathcal{C}} (h_{Y',Z'} \circ g_{Y,Y'}) \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right)$$
$$= \left(\sum_{X,X',Y,Y',F \in \mathrm{Ob}\,\mathcal{C}} (h_{Y',Z'} \circ g_{Y,Y'} \circ f_{X,X'}) \right)$$
$$= \left(\sum_{X,Y',Y,Z' \in \mathrm{Ob}\,\mathcal{C}} (h_{Y',Z'} \circ g_{Y,Y'} \circ f_{X,Y}) \right)$$
$$= \left(\sum_{Y',Z' \in \mathrm{Ob}\,\mathcal{C}} h_{Y',Z'} \right) \cdot \left(\sum_{X,Y,Y' \in \mathrm{Ob}\,\mathcal{C}} (g_{Y,Y'} \circ f_{X,Y}) \right)$$
$$= \left(\sum_{Y',Z' \in \mathrm{Ob}\,\mathcal{C}} h_{Y',Z'} \right) \cdot \left(\left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} g_{Y,Y'} \right) \cdot \left(\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y} \right) \right)$$
$$= \left(\sum_{Z,Z' \in \mathrm{Ob}\,\mathcal{C}} h_{Z,Z'} \right) \cdot \left(\left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} g_{Y,Y'} \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right) \right)$$

- (Multiplicative identity)

$$\left(\sum_{X \in \mathrm{Ob}\,\mathcal{C}} \mathrm{id}_X \right) \cdot \left(\sum_{Y,Z \in \mathrm{Ob}\,\mathcal{C}} f_{Y,Z} \right) = \sum_{X,Y,Z \in \mathrm{Ob}\,\mathcal{C}} (\mathrm{id}_X \circ f_{Y,Z})$$

$$
\begin{aligned}
&= \sum_{Y,Z \in \mathrm{Ob}\,\mathcal{C}} (\mathrm{id}_Z \circ f_{Y,Z}) \\
&= \sum_{Y,Z \in \mathrm{Ob}\,\mathcal{C}} f_{Y,Z} \\
&= \sum_{Y,Z \in \mathrm{Ob}\,\mathcal{C}} (f_{Y,Z} \circ \mathrm{id}_Y) \\
&= \sum_{X,Y,Z \in \mathrm{Ob}\,\mathcal{C}} (f_{Y,Z} \circ \mathrm{id}_X) \\
&= \left(\sum_{Y,Z \in \mathrm{Ob}\,\mathcal{C}} f_{Y,Z} \right) \circ \left(\sum_{X \in \mathrm{Ob}\,\mathcal{C}} \mathrm{id}_X \right)
\end{aligned}
$$

- (Left distributive property)

$$
\begin{aligned}
&\left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \right) \cdot \left(\left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} g_{X,X'} \right) + \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right) \right) \\
&= \left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} (g_{X,X'} + f_{X,X'}) \right) \\
&= \left(\sum_{X,X',Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \circ (g_{X,X'} + f_{X,X'}) \right) \\
&= \left(\sum_{X,X',Y,Y' \in \mathrm{Ob}\,\mathcal{C}} (h_{Y,Y'} \circ g_{X,X'}) + (h_{Y,Y'} \circ f_{X,X'}) \right) \\
&= \left(\sum_{X,X',Y' \in \mathrm{Ob}\,\mathcal{C}} (h_{X',Y'} \circ g_{X,X'}) + (h_{X',Y'} \circ f_{X,X'}) \right) \\
&= \left(\sum_{X,X',Y' \in \mathrm{Ob}\,\mathcal{C}} (h_{X',Y'} \circ g_{X,X'}) \right) + \left(\sum_{X,X',Y' \in \mathrm{Ob}\,\mathcal{C}} (h_{X',Y'} \circ f_{X,X'}) \right) \\
&= \left(\sum_{Y,Y'} h_{Y,Y'} \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} g_{X,X'} \right) + \left(\sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \right) \cdot \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right)
\end{aligned}
$$

- (Right distributive property)

$$
\begin{aligned}
&\left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} g_{X,X'} + \sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} f_{X,X'} \right) \cdot \sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \\
&= \left(\sum_{X,X' \in \mathrm{Ob}\,\mathcal{C}} (g_{X,X'} + f_{X,X'}) \right) \cdot \sum_{Y,Y' \in \mathrm{Ob}\,\mathcal{C}} h_{Y,Y'} \\
&= \left(\sum_{X,X',Y,Y' \in \mathrm{Ob}\,\mathcal{C}} (g_{X,X'} + f_{X,X'}) \circ h_{Y,Y'} \right)
\end{aligned}
$$

$$= \left(\sum_{X,X',Y,Y' \in \operatorname{Ob} \mathcal{C}} g_{X,X'} \circ h_{Y,Y'} + f_{X,X'} \circ h_{Y,Y'} \right)$$
$$= \left(\sum_{X,X'Y \in \operatorname{Ob} \mathcal{C}} g_{X,X'} \circ h_{Y,X} + f_{X,X'} \circ h_{Y,X} \right)$$
$$= \left(\sum_{X,X',Y \in \operatorname{Ob} \mathcal{C}} g_{X,X'} \circ h_{Y,X} \right) + \left(\sum_{X,X',Y \in \operatorname{Ob} \mathcal{C}} f_{X,X'} \circ h_{Y,X} \right)$$
$$= \left(\sum_{X,X' \in \operatorname{Ob} \mathcal{C}} g_{X,X'} \right) \cdot \left(\sum_{Y,Y' \in \operatorname{Ob} \mathcal{C}} h_{Y,Y'} \right) + \left(\sum_{X,X' \in \operatorname{Ob} \mathcal{C}} f_{X,X'} \right) \cdot \left(\sum_{Y,Y' \in \operatorname{Ob} \mathcal{C}} h_{Y,Y'} \right)$$

It is straightforward to verify that $1_{R_{\mathcal{C}}}$ is the sum of all elements in $I_{\mathcal{C}}$. □

6. Equivalences

In this section, we prove an equivalence of categories between the category of rings and the category of small preadditive categories with one object starting with Proposition 6.1 and between the category of idempotented rings and the category of small preadditive categories with finitely many objects under Proposition 6.4. Under these equivalences, we demonstrate some correspondences between these categories in Propositions 6.3 and 6.6.

Proposition 6.1. *The category* Ring *is equivalent to the category* PreCat_1.

Proof. We define a functor $F\colon \mathrm{Ring} \to \mathrm{PreCat}_1$ where on objects

$$F(R) = \mathcal{C}_R \in \operatorname{Ob} \mathrm{PreCat}_1 \quad \text{for all } R \in \operatorname{Ob} \mathrm{Ring}.$$

That is, F maps a ring to its corresponding preadditive category. On morphisms, F maps any $h \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$ to an additive functor $F(h)\colon \mathcal{C}_R \to \mathcal{C}_S$ such that

$$F(h)(*) = * \in \operatorname{Ob} \mathcal{C}_S,$$
$$F(h)(r) = h(r) \in S = \mathrm{Mor}_{\mathcal{C}_S}(*, *) \text{ for all } r \in R.$$

We will verify that $F\colon \mathrm{Ring} \to \mathrm{PreCat}_1$ is a functor below.

- (Preserves identity morphisms) Let $R \in \operatorname{Ob} \mathrm{Ring}$, then we have the functor $F(\mathrm{id}_R)\colon \mathcal{C}_R \to \mathcal{C}_R$ such that

$$F(\mathrm{id}_R)(*) = * \in \operatorname{Ob} \mathcal{C}_R,$$
$$F(\mathrm{id}_R)(r) = \mathrm{id}_R(r) = r \in R = \mathrm{Mor}_{\mathcal{C}_R}(*, *) \quad \text{for all } r \in R.$$

Thus $F(\mathrm{id}_R)$ is the identity functor from $\mathcal{C}_R$ to $\mathcal{C}_R$.

- (Preserves composition) Let $R, S, T \in \operatorname{Ob} \mathrm{Ring}$ with $h\colon R \to S$ and $h'\colon S \to T$. For all $r \in \mathrm{Mor}_{\mathcal{C}_R}(*, *)$,

$$F(h' \circ h)(r) = h' \circ h(r) = F(h') \circ F(h)(r) \in T = \mathrm{Mor}_{\mathcal{C}_T}(*, *),$$
$$F(h' \circ h)(*) = * = F(h') \circ F(h)(*) \in \operatorname{Ob} \mathcal{C}_T.$$

Similarly, we define a functor $G\colon \mathrm{PreCat}_1 \to \mathrm{Ring}$ as follows:

$$G(\mathcal{C}) = R_{\mathcal{C}} \in \operatorname{Ob} \mathrm{Ring} \quad \text{for all } \mathcal{C} \in \operatorname{Ob} \mathrm{PreCat}_1,$$
$$G(H)(r) = H(r) \in \mathrm{Mor}_{\mathcal{D}}(*, *) \quad \text{for all } r \in \mathrm{Mor}_{\mathcal{C}}(*, *) \text{ and } H \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D}).$$

That is, G maps a preadditive category to its corresponding ring. Let $H \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$, an additive functor between preadditive categories $\mathcal{C}$ and $\mathcal{D}$ containing one object. By Lemma 5.2, any $H \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$ is a ring homomorphism. So we define $G(H)$ to be this ring homomorphism. Then $G(H) \in \mathrm{Mor}_{\mathrm{Ring}}(G(\mathcal{C}), G(\mathcal{D})) = \mathrm{Mor}_{\mathrm{Ring}}(R_{\mathcal{C}}, R_{\mathcal{D}}) = \mathrm{Mor}_{\mathrm{Ring}}(\mathrm{Mor}_{\mathcal{C}}(*,*), \mathrm{Mor}_{\mathcal{D}}(*,*))$. It is straightforward to verify that $G \circ F = \mathrm{id}_{\mathrm{Ring}}$ and that G is a functor.

- (Preserves identity morphisms) Let $\mathcal{C} \in \mathrm{Ob}\,\mathrm{PreCat}_1$. Then for all $r \in \mathrm{Mor}_{\mathcal{C}}(*,*)$, $G(\mathrm{id}_{\mathcal{C}})(r) = \mathrm{id}_{\mathcal{C}}(r) = r$, and so $G(\mathrm{id}_{\mathcal{C}})$ is the identity ring homomorphism from $R_{\mathcal{C}}$ to $R_{\mathcal{C}}$.
- (Preserves composition) Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Ob}\,\mathrm{PreCat}_1$ with additive functors $H : \mathcal{C} \to \mathcal{D}$ and $H' : \mathcal{D} \to \mathcal{E}$. Then for all $r \in R$, $G(H' \circ H)(r) = H' \circ H(r) = G(H') \circ G(H)(r) \in \mathrm{Mor}_{\mathcal{E}}(*,*)$.

We claim that G yields an equivalence of Ring and PreCat_1.

- (Full) For any $h \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$, we have that $G(F(h)) = G \circ F(h) = \mathrm{id}_{\mathrm{Ring}}(h) = h$.
- (Faithful) Let $H, J \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$, then there exists maps $H_{*,*}, J_{*,*} : \mathrm{Mor}_{\mathcal{C}}(*,*) \to \mathrm{Mor}_{\mathcal{D}}(*,*)$. Assume that $G(H) = G(J)$. Since $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Ob}\,\mathcal{D}$ consists of only one object $*$, H and J must map them to each other, so H and J are the same on objects. Furthermore we have that

$$\begin{aligned} & G(H) = G(J) \\ \implies & G(H)(r) = G(J)(r) \quad \text{for all } r \in R \\ \implies & H(r) = J(r) \quad \text{for all } r \in R \\ \implies & H = J \end{aligned}$$

- (Essentially surjective) For all $R \in \mathrm{Ob}\,\mathrm{Ring}$, $G(F(R)) = G \circ F(R) = \mathrm{id}_{\mathrm{Ring}}(R) = R$.

□

The following proposition shows that the categories Ring and PreCat_1 are equivalent not only as 1-categories, but also as 2-categories.

Proposition 6.2. *The 2-category* Ring *is equivalent to the 2-category* PreCat_1 *as 2-categories.*

Proof. From Proposition 6.1, we defined functors $F : \mathrm{Ring} \to \mathrm{PreCat}_1$ and $G : \mathrm{PreCat}_1 \to \mathrm{Ring}$ to show that $\mathrm{Ring} \simeq \mathrm{PreCat}_1$ as 1-categories. To prove their equivalence as 2-categories, we will extend this functor $G : \mathrm{PreCat}_1 \to \mathrm{Ring}$ to a 2-functor by defining its behaviour on 2-morphisms. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}\,\mathrm{PreCat}_1$, $H, J \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$. Then for any $\alpha : H \Rightarrow J$,

$$G(\alpha) : G(H) \Rightarrow G(J) \quad G(\alpha) = \alpha_* \in R_{\mathcal{D}} = \mathrm{Mor}_{\mathcal{D}}(*,*).$$

The functor G is an extension of the 1-functor mentioned in Proposition 6.1 so it has been proven that is preserves identity 1-morphisms and the composition of 1-morphisms. We will verify that G is a 2-functor by the following:

- (Preserves vertical composition) For $\mathcal{C}$ and $\mathcal{D}$ in $\mathrm{Ob}\,\mathrm{PreCat}_1$, $H, J, K : \mathcal{C} \to \mathcal{D}$, $\alpha : H \Rightarrow J$ and $\beta : J \Rightarrow K$, we have that $G(\beta \circ_v \alpha) = (\beta \circ_v \alpha)_* = \beta_* \circ_v \alpha_* = G(\beta) \circ_v G(\alpha) : G(H) \Rightarrow G(K)$.
- (Preserves horizontal composition) For $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Ob}\,\mathrm{PreCat}_1$, $H, J : \mathcal{C} \to \mathcal{D}$, $H', J' : \mathcal{D} \to \mathcal{E}$, $\alpha : H \Rightarrow J$ and $\alpha' : H' \Rightarrow J'$, we have that $G(\alpha' \circ_h \alpha) = (\alpha' \circ_h \alpha)_* = \alpha'_* \circ_v G(J')(\alpha_*) = G(\alpha') \circ_h G(\alpha) : G(H' \circ H) \Rightarrow G(J' \circ J)$.
- (Preserves identity 2-Morphisms) For $\mathcal{C} \in \mathrm{Ob}\,\mathrm{PreCat}_1$, $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$, we have $G(\mathrm{id}_{\mathrm{id}_{\mathcal{C}}}) = \mathrm{id}_{\mathrm{id}_{\mathcal{C}_*}} = \mathrm{id}_* : \mathrm{id}_{\mathrm{id}_{\mathcal{C}_R}} \Rightarrow \mathrm{id}_{\mathrm{id}_{\mathcal{C}_R}}$.

We can extend $F\colon \mathrm{Ring} \to \mathrm{PreCat}_1$ to a 2-functor by the same reasoning. For $R, S \in \mathrm{Ob}\,\mathrm{Ring}$, $f, g\colon R \to S$ and $\alpha\colon f \Rightarrow g$, we define F's behaviour on 2-morphisms as follows:

$$F(\alpha)\colon F(f) \Rightarrow F(g) \quad F(\alpha)(*) = \alpha \in \mathrm{Mor}_{\mathcal{C}_S}(*,*)$$

The functor F has been proven to preserve identity 1-morphisms and the composition of 1-morphism so we will verify that it is truly a 2-functor as follows:

- (Preserves vertical composition) For $R, S \in \mathrm{Ob}\,\mathrm{Ring}$, $f, g, h\colon R \to S$, $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, we have that $F(\beta \circ_v \alpha)(*) = (\beta \circ \alpha)(*) = \beta \circ_v \alpha = F(\beta)(*) \circ_v F(\alpha)(*)$.
- (Preserves horizontal composition) For $R, S, T \in \mathrm{Ob}\,\mathrm{Ring}$, $f, g\colon R \to S$, $f', g'\colon S \to T$, $\alpha\colon f \Rightarrow g$ and $\alpha'\colon f' \Rightarrow g'$, we have $F(\alpha' \circ_h \alpha)(*) = (\alpha' \circ_h \alpha)(*) = \alpha' \circ_v (F(f')(\alpha)) = F(\alpha')(*) \circ_h F(\alpha)(*)$.
- (Preserves identity 2-morphisms) For $R \in \mathrm{Ob}\,\mathrm{Ring}$, $\mathrm{id}_R\colon R \to R$ and $1_R\colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$, we have $F(1_R)(*) = 1_R = \mathrm{id}_* = \mathrm{id}_{\mathrm{id}_{\mathcal{C}_R}}\colon \mathrm{id}_{\mathcal{C}_R} \Rightarrow \mathrm{id}_{\mathcal{C}_R}$.

Then it is straightforward that $G \circ F = \mathrm{id}_{\mathrm{Ring}}$. We claim that G yields an equivalence of Ring and PreCat_1 as 2-categories. We know that G is essentially surjective on objects and fully faithful on 1-morphisms from the proof that $\mathrm{Ring} \simeq \mathrm{PreCat}_1$ as 1-categories from Proposition 6.1, so we just need to show that G is fully faithful on 2-morphisms.

- (Faithful on 2-morphisms) Let $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}\,\mathrm{PreCat}_1$, $H, J \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$ and $\alpha, \beta\colon H \Rightarrow J$. If we assume that $G(\alpha) = G(\beta)$, then we have that $\alpha_* = \beta_*$. This implies that $\alpha = \beta$ since $\mathrm{Ob}\,\mathcal{C}$ contains only one object $*$ and both natural transformations assign the same morphism to this object.
- (Full on 2-morphisms) Let $R, S \in \mathrm{Ob}\,\mathrm{Ring}$, $f, g \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$ and $\alpha\colon f \Rightarrow g$. Then we have that $G(F(\alpha)) = \mathrm{id}_{\mathrm{Ring}}(\alpha) = \alpha$.

□

The following proposition explains the analogue for rings of the notion of adjoint functors, under the correspondences given above.

Proposition 6.3. *For $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}\,\mathrm{PreCat}_1$, $H \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{C}, \mathcal{D})$ and $J \in \mathrm{Mor}_{\mathrm{PreCat}_1}(\mathcal{D}, \mathcal{C})$, let $G(\mathcal{C}) = R$, $G(\mathcal{D}) = S$, $G(H) = h \in \mathrm{Mor}_{\mathrm{Ring}}(R, S)$, $G(J) = j \in \mathrm{Mor}_{\mathrm{Ring}}(S, R)$. Then H is left adjoint to J if and only if there exist 2-morphisms $\eta\colon \mathrm{id}_R \Rightarrow j \circ h$ and $\epsilon\colon h \circ j \Rightarrow \mathrm{id}_S$ in* Ring *such that the maps $\phi\colon r \mapsto \epsilon h(r)$ and $\psi\colon s \mapsto j(s)\eta$ are mutually inverse bijections of sets.*

Proof. $\Longrightarrow$ Assume that $H \dashv J$. By the definition of adjointness, there exist natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow J \circ H$ and $\epsilon\colon H \circ J \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that $\epsilon H(\eta_*) = \mathrm{id}_{J(*)}$ and $J(\epsilon_*)\eta_* = \mathrm{id}_{H(*)}$. Under the correspondences given in Proposition 6.2, the natural transformations η and ϵ correspond to 2-morphisms in Ring such that the following equations hold:

$$\begin{aligned} \eta r &= jh(r)\eta, &&\text{for all } r \in R,\\ \epsilon hj(s) &= s\epsilon, &&\text{for all } s \in S. \end{aligned}$$

Furthermore, the triangle identities in PreCat_1 correspond to the equations $\epsilon h(\eta) = 1_S$ and $j(\epsilon)\eta = 1_R$ in Ring. Then, for any $r \in R$, we have

$$\psi(\phi(r)) = \psi(\epsilon h(r)) = j(\epsilon h(r))\eta = j(\epsilon)(jh(r))\eta = j(\epsilon)\eta r = r,$$

and for any $s \in S$,

$$\phi(\psi(s)) = \phi(j(s)\eta) = \epsilon h(j(s)\eta) = \epsilon(hj(s))h(\eta) = s\epsilon h(\eta) = s.$$

So ϕ and ψ are mutually inverse bijections of sets.

$\Longleftarrow$ Assume that $\eta\colon \mathrm{id}_R \Rightarrow j \circ h$ and $\epsilon\colon h \circ j \Rightarrow \mathrm{id}_S$ are 2-morphisms in Ring and that $\phi\colon r \mapsto \epsilon h(r)$ and $\psi\colon s \mapsto j(s)\eta$ are mutually inverse bijections of sets. By the definition of a 2-morphism in Ring, for any $r \in R$, we have that $\eta r = jh(r)\eta$. Likewise, for any $s \in S$, $\epsilon hj(s) = s\epsilon$. Since ψ is the left-inverse of ϕ, $\psi(\phi(r)) = \mathrm{id}_R(r)$ for all $r \in R$. Taking $r = 1_R$, we get that

$$\psi(\phi(1_R)) = \mathrm{id}_R(1_R) \implies j(\epsilon h(1_R))\eta = 1_R \implies j(\epsilon)j \circ h(1_R)\eta = 1_R \implies j(\epsilon)\eta = 1_R$$

Similarly, since ϕ is the left-inverse of ψ, $\phi(\psi(s)) = \mathrm{id}_S(s)$ for all $s \in S$. Taking $s = 1_S$, we get that

$$\phi(\psi(1_S)) = \mathrm{id}_S(1_S) \implies \epsilon h(j(1_S)\eta) = 1_S \implies \epsilon h \circ j(1_S)h(\eta) = 1_S \implies \epsilon h(\eta) = 1_S$$

Applying the functor G from Proposition 6.2 to the equations $j(\epsilon)\eta = 1_R$ and $\epsilon h(\eta) = 1_S$ correspond to the triangle identities in PreCat_1, so $G(h) = H$ is left adjoint to $G(j) = J$. $\square$

Proposition 6.4. *The category* $\mathrm{Ring}_\perp$ *is equivalent to the category* $\mathrm{PreCat}_{\mathrm{Fin}}$.

Proof. We define a functor $F\colon \mathrm{Ring}_\perp \to \mathrm{PreCat}_{\mathrm{Fin}}$ as follows: For any $(R, I) \in \mathrm{Ring}_\perp$ where $I = \{e_1, e_2, \cdots, e_n\}$, $F(R) = \mathcal{C}_{(R,I)}$. For any $h \in \mathrm{Mor}_{\mathrm{Ring}_\perp}((R, I), (S, J))$, we have an additive functor $F(h)\colon \mathcal{C}_R \to \mathcal{C}_S$ whose behaviour on any object $e_i \in \mathrm{Ob}\,\mathcal{C}_{(R,I)}$ is:

$$F(h)(e_i) = h(e_i) \in J = \mathrm{Ob}\,\mathcal{C}_{(S.J)}$$

and on any morphism $e_jre_i \in \mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e_i, e_j)$ is:

$$\begin{aligned} F(h)(e_jre_i) &= h(e_jre_i) \\ &= h(e_j)h(r)h(e_i) \in h(e_j)Sh(e_i) = \mathrm{Mor}_{\mathcal{C}_{(S,J)}}(h(e_i), h(e_j)). \end{aligned}$$

We will check that F is a functor directly below.

- (Preserves identity morphisms) Let $(R, I) \in \mathrm{Ob}\,\mathrm{Ring}_\perp$ and id_R be the identity ring homomorphism of R.

$$F(\mathrm{id}_R)(e_i) = \mathrm{id}_R(e_i) = e_i \in I = \mathrm{Ob}\,\mathcal{C}_{(R,I)} \quad \text{for all } e_i \in \mathrm{Ob}\,\mathcal{C}_{(R,I)},$$
$$F(\mathrm{id}_R)(e_ire_i) = \mathrm{id}_R(e_i)\mathrm{id}_R(r)\mathrm{id}_R(e_i) = e_ire_i \in e_iRe_i = \mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e_i, e_i)$$

Thus $F(\mathrm{id}_R)$ is the identity functor from $\mathcal{C}$ and so F preserves identity morphisms.

- (Preserves composition) Let $(R, I), (S, J), (T, K) \in \mathrm{Ob}\,\mathrm{Ring}_\perp$ with ring homomorphisms $h\colon (R, I) \to (S, J)$ and $h'\colon (S, J) \to (T, K)$. Then for all $e_i \in \mathrm{Ob}\,\mathcal{C}_{(R,I)}$,

$$F(h' \circ h)(e_i) = h' \circ h(e_i) = F(h') \circ F(h)(e_i).$$

For any $e_jre_i \in \mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e_i, e_j)$, $F(h' \circ h)(e_jre_i) = h' \circ h(e_jre_i) = h'h(e_j)h'h(r)h'h(e_i) = F(h') \circ F(h)(e_jre_i)$. So F preserves the composition of morphisms.

Similarly, we define a functor $G\colon \mathrm{PreCat}_{\mathrm{Fin}} \to \mathrm{Ring}_\perp$ as follows:

$$G(\mathcal{C}) = (R_\mathcal{C}, I_\mathcal{C}) \in \mathrm{Ob}\,\mathrm{Ring}_\perp \quad \text{for all } \mathcal{C} \in \mathrm{Ob}\,\mathrm{PreCat}_{\mathrm{Fin}},$$
$$G(H)\left(\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} f_{X,Y}\right) = \left(\sum_{X,Y \in \mathrm{Ob}\,\mathcal{C}} H(f_{X,Y})\right),$$

and verify that it is indeed a functor.

- (Preserves identity morphism) For any $\mathcal{C} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$,

$$G(\mathrm{id}_{\mathcal{C}})\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) = \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} \mathrm{id}_{\mathcal{C}}(f_{X,Y})\right) = \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) = \mathrm{id}_{G(\mathcal{C})}\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right).$$

- (Preserves Composition) Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$ with morphisms $H\colon \mathcal{C} \to \mathcal{D}$ and $H'\colon \mathcal{D} \to \mathcal{E}$. For all $\sum\limits_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}$,

$$G(H' \circ H)\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) = \sum_{X,Y \in \operatorname{Ob}\mathcal{C}} H' \circ H(f_{X,Y}) = G(H') \circ G(H)\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right)$$

We claim that G yields an equivalence of $\mathrm{Ring}_{\perp}$ and $\mathrm{PreCat}_{\mathrm{Fin}}$.

- (Full) Let $h \in \operatorname{Mor}_{\mathrm{Ring}_{\perp}}(G(\mathcal{C}), G(\mathcal{D})) = \operatorname{Mor}_{\mathrm{Ring}_{\perp}}((R_{\mathcal{C}}, I_{\mathcal{C}}), (R_{\mathcal{D}}, I_{\mathcal{D}}))$. Then we define a functor $H \in \operatorname{Mor}_{\mathrm{PreCat}_{\mathrm{Fin}}}(\mathcal{C}, \mathcal{D})$ as follows:

$$H(X) = \operatorname{Dom}(h(\mathrm{id}_X)) \quad \text{for all } X \in \operatorname{Ob}\mathcal{C},$$
$$H(f_{X,Y}) = h(f_{X,Y}) \quad \text{for all } f_{X,Y} \in \operatorname{Mor}_{\mathcal{C}}(X,Y) \text{ and } X, Y \in \operatorname{Ob}\mathcal{C}.$$

Then $H(f_{X,Y})$ is a morphism from $H(X)$ to $H(Y)$ since $H(f_{X,Y}) = H(\mathrm{id}_Y f_{X,Y} \mathrm{id}_X) = H(\mathrm{id}_Y)H(f_{X,Y})H(\mathrm{id}_X) = \mathrm{id}_{H(Y)}H(f_{X,Y})\mathrm{id}_{H(X)}$. So for all $\left(\sum\limits_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) \in (R_{\mathcal{C}}, I_{\mathcal{C}})$, we have that:

$$\begin{aligned} G(H)\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) &= \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} H(f_{X,Y})\right) \\ &= \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} h(f_{X,Y})\right) \\ &= h\left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} f_{X,Y}\right) \\ &\implies G(H) = h \end{aligned}$$

- (Faithful) Assume that $G(H) = G(J)$, $H, J \in \operatorname{Mor}_{\mathrm{PreCat}_{\mathrm{Fin}}}(\mathcal{C}, \mathcal{D})$. Since $G(H)$ and $G(J)$ are the same ring homomorphism, they are the same map on the idempotents which are the objects of $\mathcal{C}$, so H and J must be the same on objects. Then for all $(f_{X,Y}) \in R_{\mathcal{C}}$, we have that:

$$\begin{gathered} G(H)(f_{X,Y}) = G(J)(f_{X,Y}) \\ \implies \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} H(f_{X,Y})\right) = \left(\sum_{X,Y \in \operatorname{Ob}\mathcal{C}} J(f_{X,Y})\right) \\ \implies H(f_{X,Y}) = J(f_{X,Y}) \quad \text{for all } X, Y \in \operatorname{Ob}\mathcal{C},\ f_{X,Y} \in \operatorname{Mor}_{\mathcal{C}}(X,Y) \\ \implies H = J \end{gathered}$$

• (Essentially surjective) For all $(R, I) \in \operatorname{Ob} \mathrm{Ring}_{\perp}$ where $I = \{e_1, e_2, \cdots, e_n\}$, we claim that $G(F(R)) = (G \circ F(R)) \cong (M_n(R), I_{M_n(R)})$. We define a function,

$$\phi\colon (M_n(R), I_{M_n(R)}) \to G(F(R)), \quad \phi((e_j r e_i)) = \left(\sum_{i,j=1}^{n} e_j r e_i\right),$$

where $(e_j r e_i)$ is the $n \times n$ matrix whose (i, j) entry is $e_j r e_i$. Then the proof of most of the ring isomorphism axioms are essentially the same as the ones from Example 2.8. Since $G(F(R)) \cong (M_n(R), I_{M_n(R)})$ and Proposition 2.8 gives us that $(M_n(R), I_{M_n(R)}) \cong (R, I)$, so we have that $G(F(R)) \cong (R, I)$.

□

The following proposition shows that equivalence of 1-categories between $\mathrm{Ring}_{\perp}$ and $\mathrm{PreCat}_{\mathrm{Fin}}$ can be extended to 2-categories.

Proposition 6.5. *The 2-category* $\mathrm{Ring}_{\perp}$ *is equivalent to the 2-category* $\mathrm{PreCat}_{\mathrm{Fin}}$ *as 2-categories.*

Proof. In Proposition 6.4, we defined 1-functors F and G to prove that $\mathrm{Ring}_{\perp}$ and $\mathrm{PreCat}_{\mathrm{Fin}}$ are equivalent as 1-categories. We can extend our 1-functor F into a 2-functor by defining its behaviour on 2-morphisms as so: Let $(R, I), (S, J) \in \operatorname{Ob} \mathrm{Cat}$ and $h, j\colon (R, I) \to (S, J)$. Given a 2-morphism $\alpha\colon h \Rightarrow j$, we define the 2-morphism $F(\alpha)\colon F(h) \Rightarrow F(j)$ to be the natural transformation $F(\alpha)\colon \operatorname{Ob} \mathcal{C}_R \to \operatorname{Mor}_{\mathcal{C}}(S, J)$ such that for any $e \in \operatorname{Ob} \mathcal{C}_{(R,I)}$,

$$F(\alpha)(e) = \alpha(e) \in \operatorname{Mor}_{\mathcal{C}}(F(h)(e), F(j)(e)).$$

Since F is an extension of the 1-functor used in Proposition 6.4 where we proved it preserves the identity 1-morphism and the composition of 1-morphisms, we can verify that it is a 2-functor with the following:

• (Preserves vertical composition) Let $(R, I), (S, J) \in \operatorname{Ob} \mathrm{Ring}_{\perp}$, $f, g, h\colon R \to S$, $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$. For any $e \in I$, $F(\beta \circ_v \alpha)(e) = (\beta \circ_v \alpha)(e) = \beta(e)\alpha(e) = F(\beta)(e) \circ_v F(\alpha)(e)$.

• (Preserves horizontal composition) Let $(R, I), (S, J), (T, K) \in \operatorname{Ob} \mathrm{Ring}_{\perp}$, $f, g\colon R \to S$, $f', g'\colon S \to T$, $\alpha\colon f \Rightarrow g$, $\alpha'\colon f' \Rightarrow g'$. For any $e \in I$, $F(\alpha' \circ_h \alpha)(e) = (\alpha' \circ_h \alpha)(e) = \alpha'(F(g)(e))F(f')(\alpha(e)) = F(\alpha')(e) \circ_h F(\alpha)(e)$.

• (Preserves identity 2-morphisms) Let $(R, I) \in \operatorname{Ob} \mathrm{Ring}_{\perp}$, $\mathrm{id}_R\colon R \to R$ and $\mathrm{id}_{\mathrm{id}_R}\colon \mathrm{id}_R \Rightarrow \mathrm{id}_R$, where $\mathrm{id}_{\mathrm{id}_R}$ is the inclusion map of I into R. Then for all $e \in I$, $F(\mathrm{id}_{\mathrm{id}_R})(e) = \mathrm{id}_{\mathrm{id}_R}(e) = e$.

Likewise, we can extend G into a 2-functor with the following: Let $\mathcal{C}, \mathcal{D} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $H, J\colon \mathcal{C} \to \mathcal{C}$. Then for any $\alpha\colon H \Rightarrow J$, the 2-morphism $G(\alpha)\colon G(H) \Rightarrow G(J)$ is the mapping from $I_{\mathcal{C}}$ to $R_{\mathcal{D}}$ such that for any $X \in \operatorname{Ob} \mathcal{C}$,

$$G(\alpha)(\mathrm{id}_X) = \alpha_X \in G(\mathcal{D}) = R_{\mathcal{D}}.$$

Likewise, G is also an extension of the 1-functor in Proposition 6.4 where it was proven that G preserves identity 1-morphisms and the composition of 1-morphisms. Thus we check that it is indeed a 2-functor with the following:

• (Preserves vertical composition) Let $\mathcal{C}, \mathcal{D} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $H, J, K\colon \mathcal{C} \to \mathcal{D}$, $\alpha\colon H \Rightarrow J$ and $\beta\colon J \Rightarrow K$. Then for any $X \in \operatorname{Ob} \mathcal{C}$, $G(\beta \circ_v \alpha)(\mathrm{id}_X) = (\beta \circ_v \alpha)(X) = \beta_X \circ \alpha_X = G(\beta)(\mathrm{id}_X) \circ_v G(\alpha)(\mathrm{id}_X)$.

- (Preserves horizontal composition) Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $H, J \colon \mathcal{C} \to \mathcal{D}$, $H', J' \colon \mathcal{D} \to \mathcal{E}$, $\alpha \colon H \Rightarrow J$ and $\alpha' \colon H' \Rightarrow J'$. Then for any $X \in \operatorname{Ob} \mathcal{C}$, $G(\alpha' \circ_h \alpha)(\mathrm{id}_X) = (\alpha' \circ_h \alpha)(\mathrm{id}_X) = \alpha'_{G(J)(X)} \circ G(J)(\alpha_e) = G(\alpha')(\mathrm{id}_X) \circ_h G(\alpha)(\mathrm{id}_X)$.
- (Preserves identity 2-morphisms) Let $\mathcal{C} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ and $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$. Then for all any $X \in \operatorname{Ob} \mathcal{C}$, $G(\mathrm{id}_{\mathrm{id}_{\mathcal{C}}})(\mathrm{id}_X) = \mathrm{id}_{\mathrm{id}_{\mathcal{C}} X} = \mathrm{id}_X$.

We claim that G yields an equivalence of 2-categories between the $\mathrm{Ring}_{\perp}$ and $\mathrm{PreCat}_{\mathrm{Fin}}$.

- (Faithful on 2-morphisms) Let $\mathcal{C}, \mathcal{D} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $h, j \in \mathrm{Mor}_{\mathrm{Ring}_{\perp}}(G(\mathcal{C}), G(\mathcal{D}))$ and $\alpha, \beta \colon h \Rightarrow j$. Assume that $G(\alpha) = G(\beta)$. Then for any $X \in \operatorname{Ob} \mathcal{C}$, we have that

$$G(\alpha) = G(\beta) \implies G(\alpha)(\mathrm{id}_X) = G(\beta)(\mathrm{id}_X) \implies \alpha_X = \beta_X \implies \alpha = \beta.$$

- (Full on 2-morphisms) Let $\mathcal{C}, \mathcal{D} \in \operatorname{Ob} \mathrm{PreCat}_{\mathrm{Fin}}$, $h, j \colon G(\mathcal{C}) \to G(\mathcal{D})$ and $\alpha \colon h \Rightarrow j$. The 2-morphism α has the property that for any id_X and id_Y in $I_{\mathcal{C}}$, $\alpha(\mathrm{id}_X)$ is an element in $R_{\mathcal{D}}$ such that

$$j\left(\mathrm{id}_Y\left(\sum_{X,Y \in \operatorname{Ob} \mathcal{C}} f_{X,Y}\right)\mathrm{id}_X\right)\alpha(\mathrm{id}_X) = \alpha(\mathrm{id}_Y) h\left(\mathrm{id}_Y\left(\sum_{X,Y \in \operatorname{Ob} \mathcal{C}} f_{X,Y}\right)\mathrm{id}_X\right)$$

for any $\sum\limits_{X,Y \in \operatorname{Ob} \mathcal{C}} f_{X,Y} \in R_{\mathcal{C}}$. The functor G is surjective on 1-morphisms, so let $H, J \colon \mathcal{C} \to \mathcal{D}$ such that $G(H) = h$ and $G(J) = j$. We construct a natural transformation $\beta \colon H \Rightarrow J$ as so: for any $X \in \operatorname{Ob} \mathcal{C}$, $\beta_X = \alpha(\mathrm{id}_X)$. Any $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an element in $R_{\mathcal{C}}$, so we have that $\beta_X H(f) = J(f)\beta_Y$, thus proving naturality of β. Then for all $\mathrm{id}_X \in R_{\mathcal{C}}$, $G(\beta)(\mathrm{id}_X) = \beta_X = \alpha(\mathrm{id}_X)$ and so $G(\beta) = \alpha$.

□

The following proposition shows that, under the correspondences described above, the notion of a category being idempotent complete corresponds to the set of idempotents of an idempotented ring being a complete set of primitive orthogonal idempotents.

Proposition 6.6. *Let (R, I) be an idempotented ring that contains no zero divisors and where* $\mathrm{Char}(R) \neq 2$. *Then $\mathcal{C}_{(R,I)}$ is idempotent complete if and only if I is a complete set of primitive orthogonal idempotents.*

Proof. $\implies$ Assume $\mathcal{C}_{(R,I)}$ is idempotent complete and suppose $e_i = e' + e'' \in I$, where e' and e'' are nonzero idempotents. We claim that e' and e'' are pairwise orthogonal.

$$\begin{aligned}
e_i &= e_i^2 \\
\implies e' + e'' &= (e' + e'')^2 \\
\implies e' + e'' &= (e' + e'')(e' + e'') \\
\implies e' + e'' &= e'e' + e'e'' + e''e' + e''e'' \\
\implies e' + e'' &= e' + e'e'' + e''e' + e'' \\
\implies 0_R &= e'e'' + e''e' \\
\implies e'e'' &= -e''e' \\
\implies e'e'e'' &= -e'e''e' \\
\implies e'e'' &= -e'e''e' \\
\implies e'e'' + e'e''e' &= 0
\end{aligned}$$

$$\implies e'e''(1+e') = 0_R$$

Since Char(R) $\neq 0$, the element -1 is not an idempotent. This implies that $e'e'' = 0$, proving our claim.

Note that because

$$e'e_ie' = e'(e'+e'')e' = e'e'e' + e'e''e' = e',$$

this means that $e_ie'e_i$ is an idempotent since $e_ie'e_i = (e'+e'')e'(e'+e'') = e'$. Moreover, $e_ie'e_i$ is a split idempotent since $\mathcal{C}_{(R,I)}$ is idempotent complete, so there exist morphisms $e_jre_i \in \mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e_i, e_j)$ and $e_ise_j \in \mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e_j, e_i)$ such that $e_j = (e_jre_i)(e_ise_j) = e_jre_ise_j$ and $e' = e_ie'e_i = (e_ise_j)(e_jre_i) = e_ise_jre_i$. Then we have that

$$\begin{aligned}
e' &= e_ise_jre_i \\
\implies e' &= (e'+e'')se_jr(e'+e'') \\
\implies e'e'e' &= e'(e'+e'')se_jr(e'+e'')e' \\
\implies e' &= e'se_jre' \\
\implies e'e_ie' &= e'se_jre' \\
\implies e'e_ie' - e'se_jre' &= 0_R \\
\implies e'(e_i - se_jr)e' &= 0_R
\end{aligned}$$

Since R has no zero divisors and e' is nonzero, we have that $e_i - se_jr = 0_R \implies e_i = se_jr$. Putting it all together gives us that

$$e' = e_ise_jre_i = e_ie_ie_i = e_i.$$

Finally, $e_i = e' + e'' \implies e' = e' + e'' \implies e'' = 0_R$, which contradicts our assumption that e' and e'' were nonzero idempotents.

$\Longleftarrow$ Assume I is a complete set of primitive orthogonal idempotents. Suppose ere is an idempotent morphism in $\mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e, e)$. Of course, we can write e as the sum: $e = ere + (e - ere)$. Then $(e - ere)$ is idempotent since

$$(e-ere)(e-ere) = ee - e(ere) - (ere)e + (ere)(ere) = (e - ere).$$

We have expressed the idempotent morphism e as the sum of two idempotents and since e is in I, it must be primitive, so one of ere and $(e - ere)$ is zero. Assume that $ere = 0$, then ere splits via maps to/from 0_R. On the other hand, assume that $(e - ere) = 0$, this implies that $e = ere$. Since our idempotent e is also the identity morphism of $\mathrm{Mor}_{\mathcal{C}_{(R,I)}}(e, e)$, we can spit e as the composition $e \circ e$ so it is indeed a split idempotent. □

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