4 Simple and semisimple modules (Mars 14)

Proposition 4.1.
If M is semisimple then every submodule $N \subseteq M$ is a direct summand.

In other terms, there exists a submodule $H \subseteq M$ such that $N \cap H = \{0\}$ and $N + H = M$.

Considering the lattice of submodules $L(M)$ (where the infimum is $\cap$ and the supremum is $+$). We may consider what property can we impose on the module M so that $L(M)$ is complemented (for each N there exists H such that $N \cap H = \{0\}$ and $N + H = M$).

In view of the above proposition:

Corollary 4.2.
The lattice of submodules of a semisimple module is complemented.

One can further impose the complement to be unique. What property do I need on M so that $L(M)$ is uniquely complemented?

EXAMPLE 4.1:

- For instance, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has a lattice of submodules given by:

$$\begin{tikzcd}[arrows=dash] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[dl] \arrow[d] \arrow[dr] & \\ (0,1) \arrow[dr] & (1,0) \arrow[d] & (1,1) \arrow[dl] \\ & 0 & \end{tikzcd}$$

  so we don't have the unique complement property. This diagram is called diamond lattice.

- On the other hand, $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ has a lattice of submodules given by:

$$\begin{tikzcd}[arrows=dash] & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \arrow[dl] \arrow[dr] & \\ (1,0) \arrow[dr] & & (0,1) \arrow[dl] \\ & 0 & \end{tikzcd}$$

  which is uniquely complemented.

Definition 4.1 (Square-free semi-simple module)**.**
A semi-simple module is square-free if no two distinct simple submodules are isomorphic.

Notice that in the above situation $(0,1), (1,0)$ are isomorphic in the first case but not in the second.

Definition 4.2 (Distributive lattice)**.**
A lattice is distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$

We will show the following lemma:

Lemma 4.3.
Let M be a semisimple module. The following statements are equivalent:

1. *$L(M)$ is a distributive lattice.*
2. *M is square-free*

3. $L(M)$ is uniquely complemented.

Proof. We proof circularly the implications:

Distributive $\implies$ square free

If a semi-simple module isn't square-free, then we can construct a diamond or a pentagon lattice as in the example above and that makes it non-distributive. *

Square free $\implies$ uniquely complemented

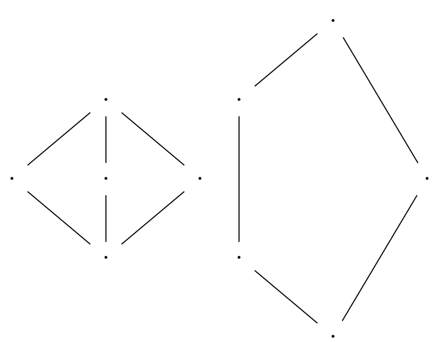
Given submodule N. The possible complements are isomorphic. If N_1, N_2 are two distinct complements, then $M = N \oplus N_1 = N \oplus N_2$. Then, taking quotient we would have $M/N \cong N_1 \cong N_2$ (to see this apply first isomorphism theorem to the projection going out of the sum).

Since module homomorphisms direct and inverse image preserve submodules, simple submodules are invariant under these homomorphisms which means that there is a bijection between simple submodules of N_1 and N_2, taking one simple submodule to an isomorphic pair. But since N is square free, bijection is actually and identity, so that N_1, N_2 have the same simple submodules.

Since, N_1, N_2 are semi-simple, they are the sum of their simple submodules and therefore $N_1 = N_2$. So complements of N are equal.

Uniquely complemented $\implies$ Distributive

If it is not distributive, then it contains a diamond or a pentagon as a sublattice.



In any of these two situations we would have that some element S has two complements S_1, S_2. This would give that sublattice M_1/N_1 does not enjoy the unique complement property (if M_1 represents the top and N_1 the bottom).

How can we deduce the failure of unique complement in the whole lattice?

$M_1/N_1 = N_2/N_1 \oplus N_3/N_1 = N_2/N_1 \oplus N_4/N_1$

$\implies \frac{M_1}{N_1} = \frac{N_2 \oplus N_3}{N_1} = \frac{N_2 \oplus N_4}{N_1}$

$\implies M_1 = N_2 \oplus N_3 = N_2 \oplus N_4$

From here it is easy to deduce that M would not enjoy the unique complement property by using different complements of submodule N_1 of M_1, we would get different complements in M.

* $\square$

We have solved the initial question we posed ourselves.