

Quantitative Verification 7 - Solutions

Ex 1: Steady State

First, we need the transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Now, we solve the equation $\pi = \pi P$ to obtain the steady state distribution. This equation is equivalent to $\pi P - \pi = \vec{0}$, i.e. $\pi(P - I_3) = \vec{0}$, where I_3 is the 3-dimensional unit matrix. We have that

$$P - I_3 = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.4 & -1 & 0.6 \\ 0 & 0.2 & -0.2 \end{bmatrix}$$

and can solve the resulting equation system using, e.g., Gauss elimination:

$$\begin{aligned} -0.1x_1 + 0.4x_2 &= 0 \\ 0.1x_1 - x_2 + 0.2x_3 &= 0 \\ 0.6x_2 - 0.2x_3 &= 0 \end{aligned}$$

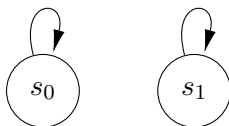
This gives, for example:

$$\begin{aligned} x_1 &= 4x_2 \\ x_3 &= 3x_2 \end{aligned}$$

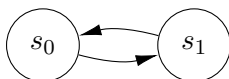
Since π has to be a distribution, we get the unique solution $\pi = [\frac{4}{9}, \frac{1}{9}, \frac{3}{9}]$.

Ex 2: Proof

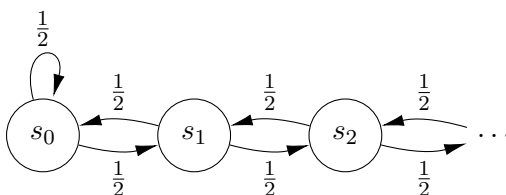
- For any state, we have $p_{ii}^1 \geq \frac{1}{2}$, hence $d_i = 1$.
- See the Markov Chain from Ex 2 with initial distribution $\pi_0 = [1, 0]$. Its stationary distribution equals $\pi^* = [0, 1]$, but we have $\pi_n = [0.999^n, 1 - 0.999^n]$.
- **Irreducible:** Each π_0 gives a distinct stationary distribution, namely itself.



Aperiodic: Take $\pi_0 = [1, 0]$, then $\pi_1 = [0, 1]$, $\pi_2 = [1, 0]$, ...



Recurrent non-null: No state is recurrent non-null, all recurrent null. No limiting distribution exists, since π_n converges to 0 point-wise.



Ex 2: Average Reward

Eventually, any run ends up in s_3 , hence the average reward will be 3 in all cases.

To compute the average n -step reward, we can compute the total expected reward and divide by n . To obtain the total expected reward, we compute the distribution for each step:

$$\begin{aligned}\pi_0 &= \{s_1 \mapsto 1\} \\ \pi_1 &= \{s_1 \mapsto 0.3, s_2 \mapsto 0.6, s_3 \mapsto 0.1\} \\ \pi_2 &= \{s_1 \mapsto 0.21, s_2 \mapsto 0.66, s_3 \mapsto 0.13\}\end{aligned}$$

Hence, the rewards obtained in each step are:

$$\begin{aligned}r_0 &= 1 \cdot 2 \\ r_1 &= 0.3 \cdot 2 + 0.6 \cdot 5 + 0.1 \cdot 3 = 3.9 \\ r_2 &= 0.21 \cdot 2 + 0.66 \cdot 5 + 0.13 \cdot 3 = 4.11\end{aligned}$$

In total, we have a 2-step expected reward of $2 + 3.9 + 4.11 = 10.1$, and the average reward equals $3.\overline{36}$.