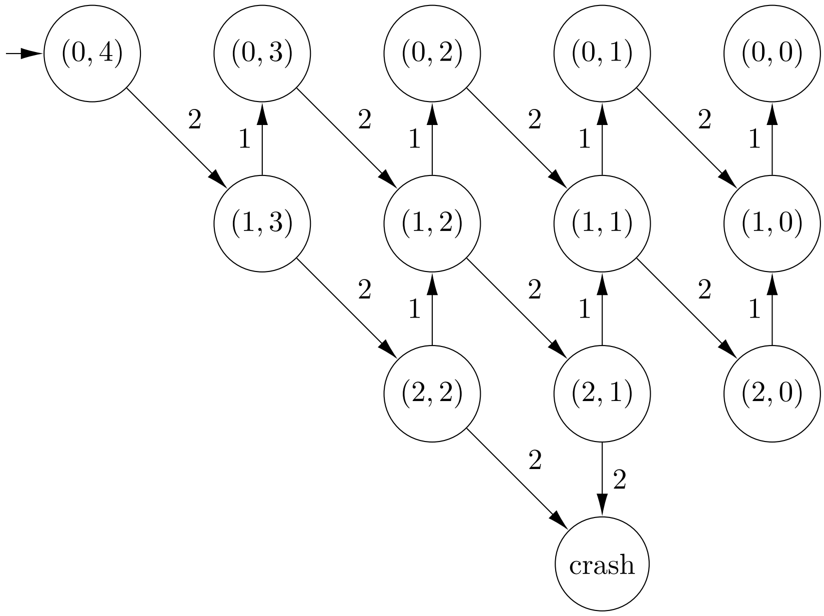
Quantitative Verification 13 - Solutions

Ex 1: Modelling a CTMC

Our state space has two variables, namely the printer's queue size and the number of remaining jobs. Furthermore, there is a special "crashed" state. Uniformizing with rate $q = 3$ divides all transitions by 3 and adds self-loops in states with $E(s) < 3$.



Ex 2: Two Views on CTMC

Let X, Y, and Z be as given. Recall that for exponential distributions the CDF is given by $F(x, \lambda) = 1 - e^{-\lambda x}$ for $x \geq 0$. Now, we derive the CDF for Z. The probability of $Z \leq x$ equals the probability of at least one of X or Y being $\leq x$, which in turn equals 1 minus the probability of both being larger, i.e.

$$F_Z(x) = 1 - (1 - F_X(x)) \cdot (1 - F_Y(x)) = 1 - e^{-(\lambda_X + \lambda_Y)x}.$$

Hence, Z is exponentially distributed with rate $\lambda_X + \lambda_Y$.

Now, we show that that the second view implies the first. In particular, let X_i be exponentially distributed with rate λ_i and define $Z := \min\{X_i\}$, which is distributed according to $\lambda = \sum_i \lambda_i$. We show that $\mathcal{P}[X_i = Z] = \frac{\lambda_i}{\lambda}$. N.B.: $\mathcal{P}[X_i = X_j] = 0$ for $i \neq j$.

Let $Z_i = \min_{j \neq i}\{X_j\}$. By above reasoning, Z_i is distributed according to $\mu_i := \sum_{j \neq i} \lambda_j$. We have that $\mathcal{P}[X_i = Z] = \mathcal{P}[X_i \leq Z_i]$. Observe that X_i and Z_i are independent. We can compute this value using integration:

$$\mathcal{P}[X_i \leq Z_i] = \int_0^\infty f_{X_i}(x) \cdot (1 - F_{Z_i}(x))\,\mathrm{d}x = \int_0^\infty \lambda_i e^{-\lambda_i x} \cdot e^{-\mu_i x}\,\mathrm{d}x = \int_0^\infty \lambda_i e^{\lambda x}\,\mathrm{d}x = \frac{\lambda_i}{\lambda}.$$

Ex 3: Self loops

Let X_i be some exponential distributions with rates λ_i and set $\lambda = \sum \lambda_i$. We now introduce a "self-loop", i.e. another exponential process X_0 with rate λ_0. By above reasoning, the "successors" are distributed according to $\frac{\lambda_i}{\lambda+\lambda_0}$. Since the 0 successor corresponds to repeating the experiment, we can view this as a coin flip on whether we repeat the successor choice or we actually pick a successor.

Let P_i be the chance of selecting successor i. We have that $P_i = \frac{\lambda_0}{\lambda+\lambda_0} \cdot P_i + \frac{\lambda_i}{\lambda+\lambda_0}$. Solving this equation gives $P_i = \lambda_i$, the original distribution.