

Quantitative Verification

Chapter 7: Continuous-time Markov chains

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Links:

http://cgi.di.uoa.gr/~istavrak/courses/06_pms524_CTMC.pdf

Continuous-time Markov chains

CTMC

CTMC – Motivation

Discrete-time abstraction is fitting for situations where

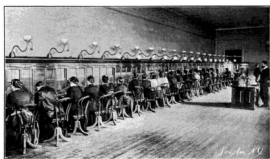
- ▶ flow of time **irrelevant**: execution steps, steps of a game, ...
- ▶ flow of time **important** but timing within the steps **irrelevant**: discretized time (e.g. one day per step) without loss of precision.

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For some problems, the discrete-time abstraction is not appropriate.



How long do I need to wait (in 1910') on average for a telephone connection?



How many pump machines a gas station needs to satisfy the peak demand?



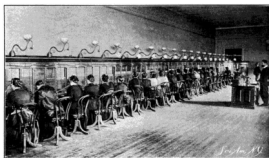
How often do safety functions in a nuclear power plant fail (at the same time)?

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How often do safety functions in a nuclear power plant fail (at the same time)?

- ▶ One could address these questions by discretizing time.
- ▶ However, continuous-time models are more suitable.
- ▶ Continuous-time models are actually also easy to solve!

CTMC – Stochastic Process Definition

CTMC - Math / Statistics Definition (1)

Recall

A discrete-time stochastic process $\{X_n \mid n \in \mathbb{N}\}$ over state space S is:

- ▶ **Markov** if for all $n > 1$ and s_0, \dots, s_n with $P(X_{n-1} = s_{n-1}) > 0$:
$$P(X_n = s_n \mid X_{n-1} = s_{n-1}, \dots, X_0 = s_0) = P(X_n = s_n \mid X_{n-1} = s_{n-1}).$$
- ▶ **homogeneous** if for all $n > 1$ and $s, s' \in S$ with $P(X_0 = s) > 0$:
$$P(X_{n+1} = s' \mid X_n = s) = P(X_1 = s' \mid X_0 = s)$$

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Definition:

A continuous-time stochastic process $\{X_t \mid t \in \mathbb{R}_{\geq 0}\}$ over states S is

- ▶ **Markov** if for all $n > 1$, $0 = t_0 < t_1 < \dots < t_n$ and s_0, \dots, s_n with $P(X_{t_{n-1}} = s_{n-1}) > 0$:
$$P(X_{t_n} = s_n \mid X_{t_{n-1}} = s_{n-1}, \dots, X_{t_0} = s_0) = P(X_{t_n} = s_n \mid X_{t_{n-1}} = s_{n-1}).$$
- ▶ **homogeneous** if for all $t, t' \in \mathbb{R}$ and $s, s' \in S$ with $P(X_0 = s) > 0$:
$$P(X_{t+t'} = s' \mid X_t = s) = P(X_{t'} = s' \mid X_0 = s).$$

We consider only discrete-space homogeneous Markov processes, that we call **continuous-time Markov chains** (CTMC).



CTMC - Math / Statistics definition (2)

Sojourn time:

Let $\{X_t \mid t \in \mathbb{R}_{\geq 0}\}$ be a continuous-time Markov chain. We define for each state s and $i \in \mathbb{N}$

- ▶ random variables $A_{s,i}$, $B_{s,i}$ denoting time of entering and leaving s for the i -th time, respectively i.e. $A_{s,1} = \inf\{t \geq 0 \mid X_t = s\}$, and $B_{s,i} = \inf\{t > A_{s,i} \mid X_t \neq s\}$ and $A_{s,i+1} = \inf\{t > B_{s,i} \mid X_t = s\}$;
- ▶ random variable $T_{s,i}$ denoting the sojourn time upon the i -th visit to s , i.e. $T_{s,i} = B_{s,i} - A_{s,i}$.

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Observation:

For any i and t, t' we have from the two properties

$$P(T_{s,i} \leq t + t' \mid T_{s,i} > t) = P(T_{s,i} \leq t').$$

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Proposition:

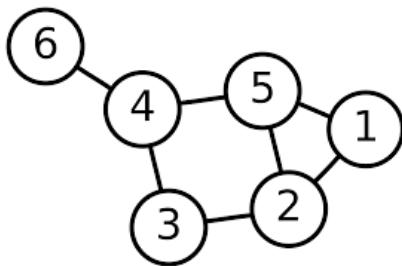
Each $T_{s,i}$ is **exponentially distributed** with some rate λ_s , i.e.

$$F_{T_{s,i}}(x) = 1 - e^{-\lambda_s x} \qquad f_{T_{s,i}}(x) = \lambda_s e^{-\lambda_s x}$$

with the expected value given by $E[T_{s,i}] = \frac{1}{\lambda_s}$.



CTMC – Graph Based Definition



CTMC [Graph] - Definition (1)

Definition: CTMC

A **continuous-time Markov chain** is a tuple (S, R, π_0) where

- ▶ S is the set of states,
- ▶ $R : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition rate matrix, and
- ▶ π_0 is the initial distribution.

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Definition: Embedded DTMC

We define the **exit rate** of a state $s \in S$ as

$$E(s) = \sum_{s' \in S} R(s, s') \quad \leftarrow \text{including the self-loop!}$$

The **embedded DTMC** coincides on S and π_0 but has the transition probability matrix E defined by

$$E(s, s') = \begin{cases} \frac{R(s, s')}{E(s)} & \text{if } E(s) > 0, \\ 0 & \text{if } E(s) = 0 \wedge s \neq s', \text{ and} \\ 1 & \text{if } E(s) = 0 \wedge s = s'. \end{cases}$$

CTMC [Graph] - Definition (2)

Definition:

The **probability measure** for a CTMC (S, R, π_0) is induced by the measure for cylinder sets $P(C(s_0 I_0 \dots s_n))$ defined as

$$\pi_0(s) \prod_{0 \leq i < n} E(s_i, s_{i+1}) \left(e^{-E(s_i) \inf I_i} - e^{-E(s_i) \sup I_i} \right).$$



CTMC – Solution Techniques



CTMC - Transient Analysis (1)

Symbolic Solution:


For a CTMC $\mathcal{C} = (S, Q, \pi_0)$, the transient probability distribution π_t at time t satisfies

$$\frac{d\pi_t}{dt} = \pi_t Q.$$


We can **symbolically** solve this system of differential equations by

$$\pi_t = \pi_0 e^{Qt}$$

where

$$e^{Qt} = \sum_{i=0}^{\infty} \frac{(Qt)^i}{i!}.$$


But unfortunately $(Qt)^i$ is **unstable** to compute and the infinite sum is **not easy** to truncate.

CTMC - Transient Analysis (2)

Another approach?

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Separate the discrete and continuous randomness!



CTMC - Uniformization (1)

Definition: Uniformization Rate

For a CTMC $\mathcal{C} = (S, R, \pi_0)$ the **uniformization rate** q is defined as

$$q \geq \max_{s \in S} E(s).$$

Definition: Uniformized DTMC

For a CTMC $\mathcal{C} = (S, R, \pi_0)$, the **uniformized DTMC** (with uniformization rate q), denoted by $uni(\mathcal{C})$ is defined as the DTMC (S, P, π_0) with

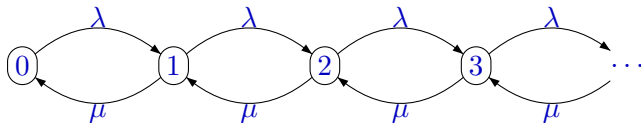
$$R'(i, j) = \begin{cases} R(i, j) & \text{for } i \neq j \\ q - \sum_k R(i, k) & \text{for } i = j \end{cases}.$$



CTMC - Uniformization (2)

Example:

CTMC:

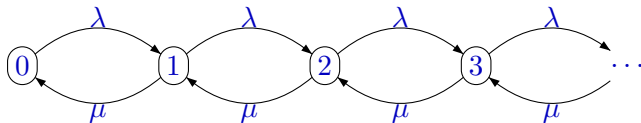


Uniformized DTMC with $q = \lambda + \mu$:

CTMC - Uniformization (2)

Example:

CTMC:



Uniformized DTMC with $q = \lambda + \mu$: add a self-loop with rate μ on 0



CTMC - Transient Analysis - By Uniformization

Lemma:

Let $\mathcal{C} = (S, R, \pi_0)$ be a CTMC and $\text{uni}(\mathcal{C}) = (S, P, \pi_0)$ its uniformized DTMC with uniformization rate q and $t > 0$.

Let $\psi(i, qt)$ denote the Poisson probability at i (with parameter qt) and π'_i the transient probability distribution of the uniformized DTMC at time step i . Then

$$\pi_t = \sum_{i=0}^{\infty} \psi(i, qt) \pi'_i.$$



CTMC - Transient Analysis - Algorithm

Computation:

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- ▶ For each $\epsilon > 0$, the Fox-Glynn algorithm provides bounds L, R such that

$$\sum_{i=L}^R \psi(i, qt) \geq 1 - \epsilon.$$

Moreover, $\psi(i, qt)$ for $L \leq i \leq R$ can be computed efficiently.

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- ▶ Computation of π'_i via the known algorithms for DTMCs.

CTMC - Steady-State

Theorem:

Steady state of a CTMC exists iff steady state of its **uniformized DTMC** exists. In this case, they equal.

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Steady state of a CTMC exists iff steady state of its **uniformized DTMC** exists. In this case, they equal.

By taking higher uniformization rate than $\max_s E(s)$, we have self-loops in every state; hence the uniformized DTMC is aperiodic (other conditions on existence of steady-state were equivalent).

Continuous Stochastic Logic (CSL)



CSL - Labelled CTMC and Syntax

Definition: Labelled CTMC

A **labelled CTMC** is a tuple $\mathcal{C} = (S, R, \pi_0, L)$ with labelling function $L : S \rightarrow 2^{AP}$ where AP is a set of **atomic propositions**.

Definition: Syntax of CSL

State formulas:

$$\Phi = \text{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid \mathcal{P}_J(\phi) \mid \mathcal{S}_J(\Phi)$$


where $a \in AP$, $J \subseteq [0, 1]$ is an interval with rational bounds.

Path formulas:

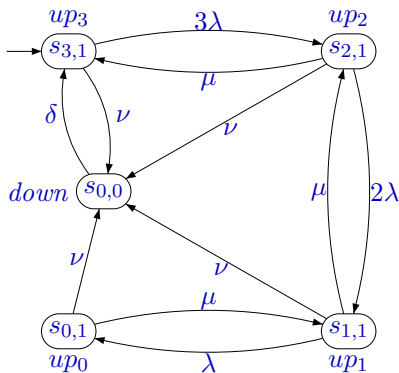
$$\phi = \mathcal{X}^I \Phi \mid \Phi_1 \mathcal{U}^I \Phi_2$$

where $I \subseteq \mathbb{R}_{\geq 0}$ denotes an interval.

CSL - Example

Example:

We consider a triple modular redundant system:



CSL - Example

Specifications:

Let $up = \neg down$, then we can specify the following performance properties:

- ▶ $S_J(up)$: steady state availability
- ▶ $\mathcal{P}_J(\mathcal{F}^{[t,t]} up)$: instantaneous availability at time t
- ▶ $\mathcal{P}_J(\Phi \mathcal{U}^{[t,t]} up)$: conditional instantaneous availability at time t
- ▶ $\mathcal{P}_J(\mathcal{G}^{[t,t']} up)$: interval availability

We can even nest \mathcal{P} and \mathcal{S} :

- ▶ $\mathcal{P}_{[0,0.01]}(up_2 \vee up_3 \mathcal{U}^{[0,10]} down)$: The probability of going down within 10 time units after having continuously operated with at least two processors is at most 0.01.
- ▶ $\mathcal{S}_{[0.9,1.0]}(\mathcal{P}_{[0.8,1.0]}(\mathcal{G}^{[0,10]} \neg down))$: In the long run, at least 90% of time is spent in states where the probability that the system will not go down within 10 time units is at least 0.8.

CSL - Reachability

Let us fix a labelled CTMC $\mathcal{C} = (S, R, \pi_0, L)$, goal states $B \subseteq S$, and interval of time $I \subseteq \mathbb{R}_{\geq 0}$. We write $Reach^I(B) := Paths(\mathcal{F}^I B)$.

Reachability

What is the probability $P(Reach^I(B))$?

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Lemma:

For any labelled CTMC $\mathcal{C} = (S, R, \pi_0, L)$, let $\mathcal{C}[B]$ be obtained by making states in B absorbing. Then, the reachability probability $P(Reach^I(B))$ does not change.



CSL - Reachability - Time-bounded

Theorem:

For $\mathcal{C} = (S, R, \pi_0, L)$ with an absorbing set of states B and $I = [0, t]$:

$$P(\text{Reach}^{[0,t]}(B)) \stackrel{1}{=} P(\text{Reach}^{[t,t]}(B)) \stackrel{2}{=} \sum_{s' \in B} \pi_t(s').$$

Proof Sketch:

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Proof Sketch:

1. Show that $\sigma \in \text{Reach}^{[0,t]}(B)$ iff $\sigma \in \text{Reach}^{[t,t]}(B)$.
2. Trivial.

CSL - Reachability - Interval Bounded (1)

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Assuming $I = [a, b]$ with $0 < a \leq b$. Note that

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Counterexample:



CSL - Reachability - Interval Bounded (2)

Theorem:

Let $\mathcal{C} = (S, R, \pi_0, L)$ with an absorbing set of states B and $I = [a, b]$ with $a > 0$, then,

$$P(\text{Reach}^{[a,b]}(B)) = \sum_{s \in S} \pi_a(s) \cdot P_s(\text{Reach}^{[0,b-a]}(B)).$$

Proof Sketch:

By the theorem of **total probability** we have:

$$\begin{aligned} P(\text{Reach}^{[a,b]}(B)) &= \sum_{s \in S} P_s(X_a = s) P(\text{Reach}^{[a,b]}(B) \mid X_a = s') \\ &= \sum_{s \in S} \pi_a(s) P(\text{Reach}^{[0,b-a]}(B) \mid X_0 = s) \\ &= \sum_{s' \in S} \pi_a(s) P_s(\text{Reach}^{[0,b-a]}(B)). \end{aligned}$$

CSL Model Checking



CSL - Model Checking

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Interval-bounded reachability - just discussed.

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$$P_s(\mathcal{X}^{[a,b]} \Phi) = (e^{-E(s)a} - e^{-E(s)b}) \sum_{s' \in \text{Sat}(\Phi)} E(s, s').$$

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Steady State $\mathcal{S}_{<p}(\Phi)$:

Can be **reduced** to the analogous PCTL model checking problem on the **uniformized** Markov chain $\text{uni}(\mathcal{C})$. **Intuition**: A CTMC and its uniformized chain have the same steady state distribution.