

Quantitative Verification

Chapter 3: Markov chains

Jan Křetínský

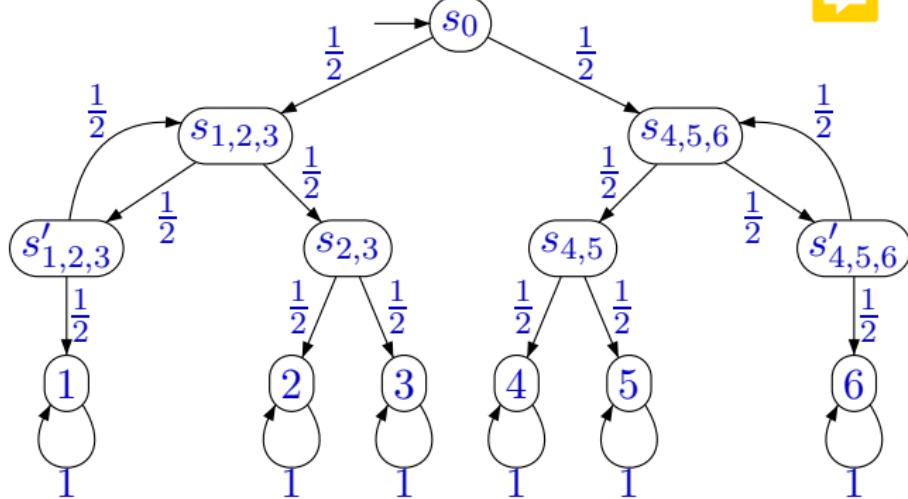
Technical University of Munich

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Motivation

Example: Simulation of a die by coins

Knuth & Yao die



Question:

- ▶ What is the probability of obtaining 2?

DTMC - Graph-based Definition

Definition: 

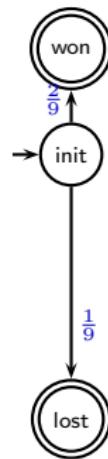
A discrete-time Markov chain (DTMC) is a tuple (S, P, π_0) where

- ▶ S is the set of states,
- ▶ $P : S \times S \rightarrow [0, 1]$ with $\sum_{s' \in S} P(s, s') = 1$ is the transitions matrix, and
- ▶ $\pi_0 \in [0, 1]^{|S|}$ with $\sum_{s \in S} \pi_0(s) = 1$ is the initial distribution.

Example: Craps

Two dice game:

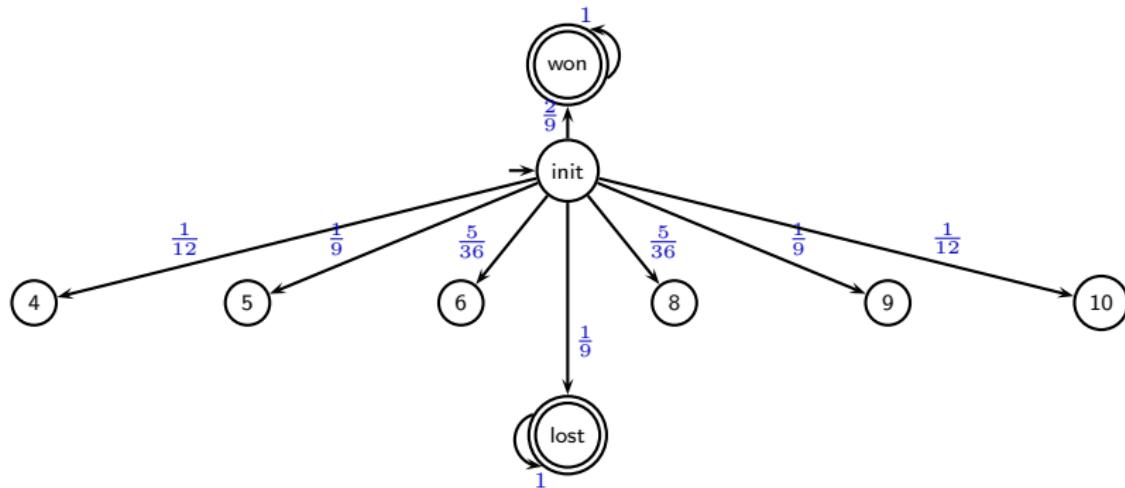
- ▶ First: $\sum \in \{7, 11\} \Rightarrow \text{win}$, $\sum \in \{2, 3, 12\} \Rightarrow \text{lose}$, else $s = \sum$
- ▶ Next rolls: $\sum = s \Rightarrow \text{win}$, $\sum = 7 \Rightarrow \text{lose}$, else iterate



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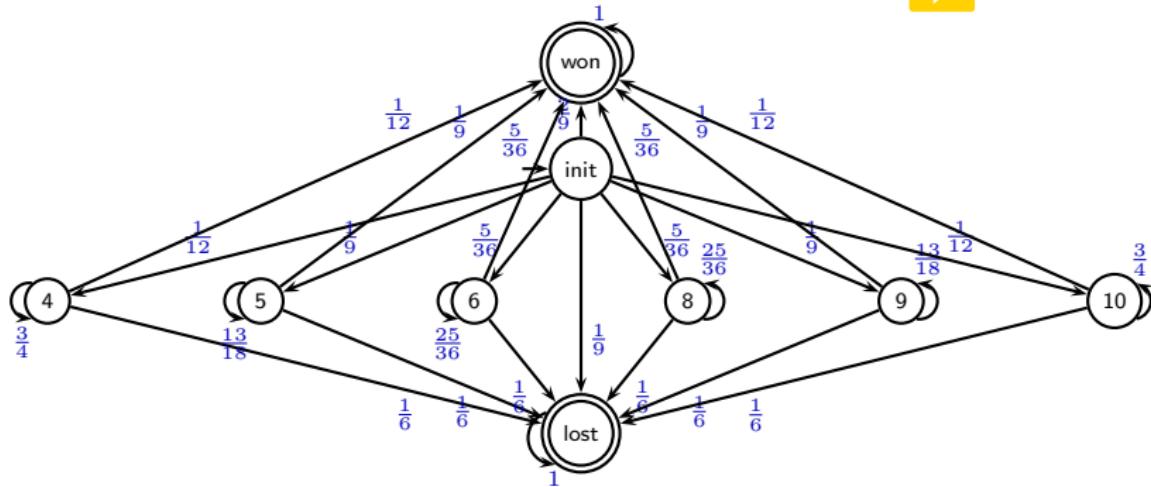
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Example: Zero Configuration Networking (Zeroconf)

- ▶ Previously: **Manual** assignment of IP addresses
- ▶ Zeroconf: **Dynamic** configuration of local IPv4 addresses
- ▶ Advantage: **Simple** devices able to communicate automatically

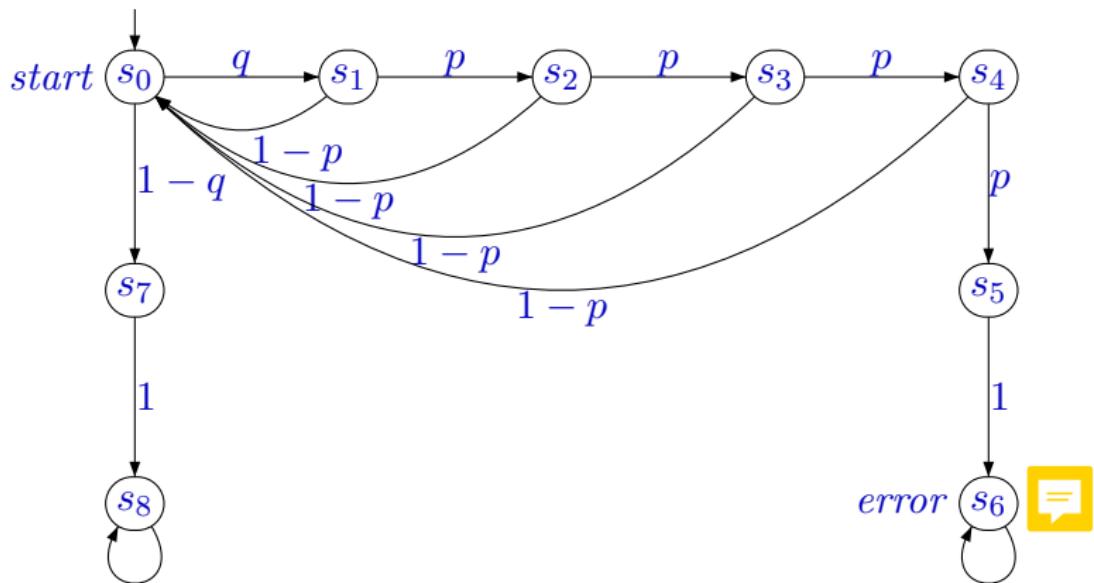
Automatic Private IP Addressing (APIPA) – RFC 3927

- ▶ Used when DHCP is **configured** but **unavailable**
- ▶ Pick randomly an address from 169.254.1.0 – 169.254.254.255
- ▶ Find out whether anybody else uses this address (by sending several ARP requests)

Model:

- ▶ Randomly pick an address among the K (65024) addresses.
- ▶ With m hosts in the network, collision probability is $q = \frac{m}{K}$.
- ▶ Send 4 ARP requests.
- ▶ In case of collision, the probability of no answer to the ARP request is p (due to the **lossy channel**)

Example: Zero Configuration Networking (Zeroconf)



For 100 hosts and $p = 0.001$, the probability of error is $\approx 1.55 \cdot 10^{-15}$.

Probabilistic Model Checking

What is probabilistic model checking?

- ▶ **Probabilistic** specifications, e.g. probability of reaching bad states shall be smaller than **0.01**.
- ▶ Probabilistic model checking is an automatic verification technique for this purpose.

Why quantities?

- ▶ **Randomized** algorithms
- ▶ **Faults** e.g. due to the environment, lossy channels
- ▶ **Performance** analysis, e.g. reliability, availability



Basics of Probability Theory (Recap)

What are probabilities? - Intuition

Throwing a fair coin:

- ▶ The outcome **head** has a probability of **0.5**.
- ▶ The outcome **tail** has a probability of **0.5**.

What are probabilities? - Intuition

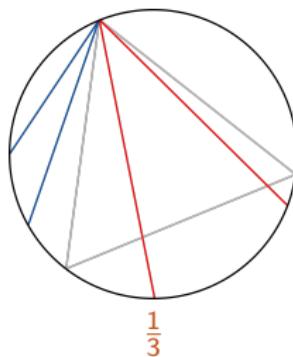
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But ...[Bertrand's Paradox]



Draw a random chord on the unit circle. What is the probability that its length exceeds the length of a side of the equilateral triangle in the circle?



$$\frac{1}{3}$$

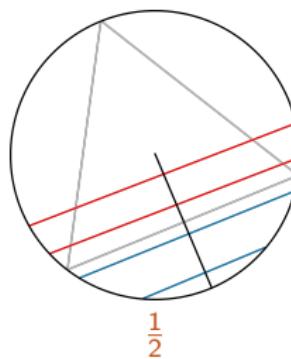
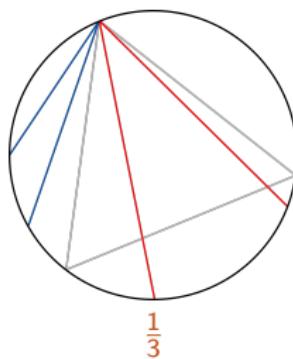
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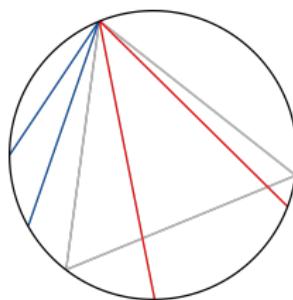
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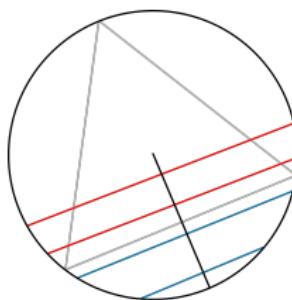
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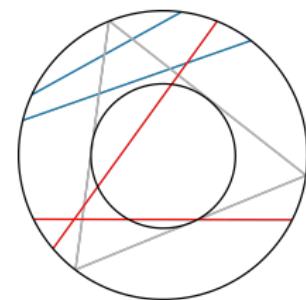
Draw a random chord on the unit circle. What is the probability that its length exceeds the length of a side of the equilateral triangle in the circle?



$$\frac{1}{3}$$



$$\frac{1}{2}$$



$$\frac{1}{4}$$

Probability Theory - Probability Space

Definition: Probability Function

Given sample space Ω and σ -algebra \mathcal{F} , a **probability function** $P : \mathcal{F} \rightarrow [0, 1]$ satisfies:

- ▶ $P(A) \geq 0$ for $A \in \mathcal{F}$,
- ▶ $P(\Omega) = 1$, and
- ▶ $P(\dot{\cup}_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for pairwise disjoint $A_i \in \mathcal{F}$



Definition: Probability Space

A **probability space** is a tuple (Ω, \mathcal{F}, P) with a sample space Ω , σ -algebra $\mathcal{F} \subseteq 2^{\Omega}$ and probability function P .

Example

A random real number taken uniformly from the interval $[0, 1]$.

- ▶ Sample space: $\Omega = [0, 1]$.

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Example

A random real number taken uniformly from the interval $[0, 1]$.

- ▶ Sample space: $\Omega = [0, 1]$.
- ▶ σ -algebra: \mathcal{F} is the minimal superset of $\{[a, b] \mid 0 \leq a \leq b \leq 1\}$ closed under complementation and countable union.
- ▶ Probability function: $P([a, b]) = (b - a)$, by Carathéodory's extension theorem there is a unique way how to extend it to all elements of \mathcal{F} .

Random Variables

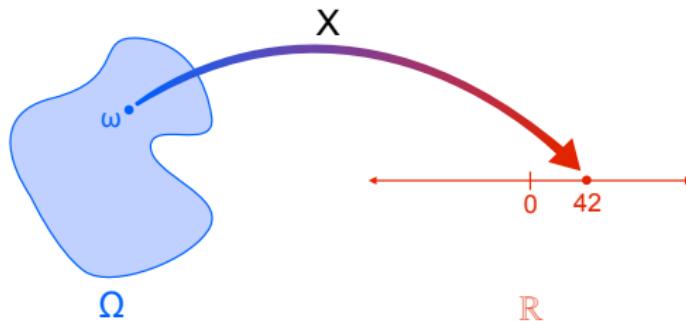
```
int getRandomNumber()
{
    return 4; // chosen by fair dice roll.
              // guaranteed to be random.
}
```

Random Variables - Introduction

Definition: Random Variable

A **random variable** X is a **measurable** function $X : \Omega \rightarrow I$ to some I .

Elements of I are called **random elements**. Often $I = \mathbb{R}$:



Example (Bernoulli Trials)

Throwing a coin 3 times: $\Omega_3 = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$.

We define 3 random variables $X_i : \Omega \rightarrow \{h, t\}$. For all $x, y, z \in \{h, t\}$,

- ▶ $X_1(xyz) = x$,
- ▶ $X_2(xyz) = y$,
- ▶ $X_3(xyz) = z$.

Stochastic Processes and Markov Chains

Stochastic Processes - Definition

Definition:

Given a probability space (Ω, \mathcal{F}, P) , a **stochastic process** is a family of random variables

$$\{X_t \mid t \in T\}$$

defined on (Ω, \mathcal{F}, P) . For each X_t we assume

$$X_t : \Omega \rightarrow S$$

where $S = \{s_1, s_2, \dots\}$ is a **finite** or **countable** set called **state space**.

A stochastic process $\{X_t \mid t \in T\}$ is called

- ▶ **discrete-time** if $T = \mathbb{N}$ or
- ▶ **continuous-time** if $T = \mathbb{R}_{\geq 0}$.

For the following lectures we focus on discrete time.

Discrete-time Stochastic Processes - Construction (1)

Example: Weather Forecast

- ▶ $S = \{sun, rain\}$,
- ▶ we model time as discrete – a random variable for each day:
 - ▶ X_0 is the weather today,
 - ▶ X_i is the weather in i days.
- ▶ how can we set up the probability space to measure e.g. $P(X_i = sun)$?



Discrete-time Stochastic Processes - Construction (2)

Let us fix a state space S . How can we construct the probability space (Ω, \mathcal{F}, P) ?

Definition: Sample Space Ω

We define $\Omega = S^\infty$. Then, each X_n maps a sample $\omega = \omega_0\omega_1\dots$ onto the respective state at time n , i.e.,

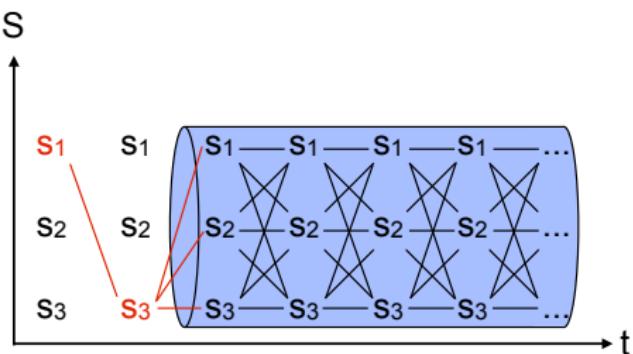
$$(X_n)(\omega) = \omega_n \in S.$$

Discrete-time Stochastic Processes - Construction (3)

Definition: Cylinder Set

For $s_0 \dots s_n \in S^{n+1}$, we set the cylinder $C(s_0 \dots s_n) = \{s_0 \dots s_n \omega \in \Omega\}$.

Example:
 $S = \{s_1, s_2, s_3\}$ and $C(s_1 s_3)$



Definition: σ -algebra \mathcal{F}

We define \mathcal{F} to be the smallest σ -Algebra that contains all cylinder sets, i.e.,

$$\{C(s_0 \dots s_n) \mid n \in \mathbb{N}, s_i \in S\} \subseteq \mathcal{F}.$$

Check: Is each X_i measurable?

(on the discrete set S we assume the full σ -algebra 2^S).

Discrete-time Stochastic Processes - Construction (4)

How to specify the probability Function P ?

We only needs to specify for each $s_0 \dots s_n \in S^n$

$$P(C(s_0 \dots s_n)).$$

This amounts to specifying

1. $P(C(s_0))$ for each $s_0 \in S$, and
2. $P(C(s_0 \dots s_i) | C(s_0 \dots s_{i-1}))$ for each $s_0 \dots s_i \in S^i$

since



$$\begin{aligned} P(C(s_0 \dots s_n)) &= P(C(s_0 \dots s_n) | C(s_0 \dots s_{n-1})) \cdot P(C(s_0 \dots s_{n-1})) \\ &= P(C(s_0)) \cdot \prod_{i=1}^n P(C(s_0 \dots s_i) | C(s_0 \dots s_{i-1})) \end{aligned}$$

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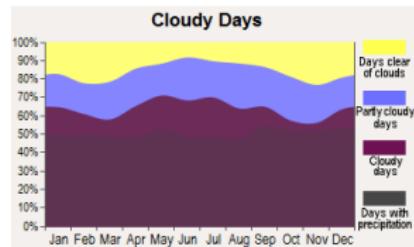
Still, lots of possibilities...

Discrete-time Stochastic Processes - Construction (5)

Weather Example: Option 1 - statistics of days of a year

- ▶ the forecast starts on Jan 01,
- ▶ a distribution p_j over $\{\text{sun}, \text{rain}\}$ for each $1 \leq j \leq 365$,
- ▶ for each $i \in \mathbb{N}$ and $s_0 \dots s_i \in S^{i+1}$

$$P(C(s_0 \dots s_i) | C(s_0 \dots s_{i-1})) = p_i \% 365(s_i)$$



Weather Example: Option 2 - two past days

- ▶ a distribution $p_{s's''}$ over $\{\text{sun}, \text{rain}\}$ for each $s', s'' \in S$,
- ▶ for each $i \geq 2$ and $s_0 \dots s_i \in S^{i+1}$

$$P(C(s_0 \dots s_i) | C(s_0 \dots s_{i-1})) = p_{s_{i-2}s_{i-1}}(s_i)$$



29° 24° 29° 22° 28° 18°

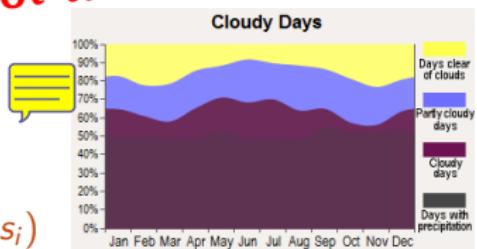
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Not time-homogeneous.



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Not Markovian.



29° 24° 29° 22° 28° 18°

Here: time-homogeneous Markovian stochastic processes

Stochastic Processes – Restrictions

Definition: Markov

A discrete-time stochastic process

$\{X_n \mid n \in \mathbb{N}\}$ is **Markov** if

$$\begin{aligned} P(X_n = s_n \mid X_{n-1} = s_{n-1}, \dots, X_0 = s_0) \\ = P(X_n = s_n \mid X_{n-1} = s_{n-1}) \end{aligned}$$

for all $n > 1$ and $s_0, \dots, s_n \in S$ with

$$P(X_{n-1} = s_{n-1}) > 0.$$

Definition: Time-homogeneous

A discrete-time Markov process $\{X_n \mid n \in \mathbb{N}\}$

is **time-homogeneous** if

$$P(X_{n+1} = s' \mid X_n = s) = P(X_1 = s' \mid X_0 = s)$$

for all $n > 1$ and $s, s' \in S$ with $P(X_0 = s) > 0$.

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for all $n > 1$ and $s, s' \in S$ with $P(X_0 = s) > 0$.



A. A. Марков (1886).



Discrete-time Stochastic Processes - Construction (6)

Weather Example: Option 3 - one past day

- ▶ a distribution $p_{s'}$ over $\{\text{sun}, \text{rain}\}$ for each $s' \in S$,
- ▶ for each $i \geq 1$ and $s_0 \dots s_i \in S^{i+1}$

$$P(C(s_0 \dots s_i) | C(s_0 \dots s_{i-1})) = p_{s_{i-1}}(s_i)$$



- ▶ a distribution π over $\{\text{sun}, \text{rain}\}$ such that $P(C(s_0)) = \pi(s_0)$.

Discrete-time Stochastic Processes - Construction (6)

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Overly restrictive, isn't it?

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- ▶ a distribution π over $\{\text{sun}, \text{rain}\}$ such that $P(C(s_0)) = \pi(s_0)$.

Overly restrictive, isn't it?

Not really – one only needs to extend the state space

- ▶ $S = \{1, \dots, 365\} \times \{\text{sun}, \text{rain}\} \times \{\text{sun}, \text{rain}\}$, 
- ▶ now each state encodes current day of the year, current weather, and weather yesterday,
- ▶ we can define over S a time-homogeneous Markov process based on both Options 1 & 2 given earlier.

Discrete-time Markov Chains

DTMC

DTMC – Relation of Definitions

Stochastic process → Graph based

Given a discrete-time homogeneous Markov process $\{X(n) \mid n \in \mathbb{N}\}$

- ▶ with state space S ,
- ▶ defined on a probability space (Ω, \mathcal{F}, P)

we take over the state space S and define

- ▶ $P(s, s') = P(X_n = s' \mid X_{n-1} = s)$ for an arbitrary $n \in \mathbb{N}$ and
- ▶ $\pi_0(s) = P(X_0 = s)$.

Graph based → stochastic process

Given a DTMC (S, P, π_0) , we set Ω to S^∞ , \mathcal{F} to the smallest σ -Algebra containing all cylinder sets and

$$P(C(s_0 \dots s_n)) = \pi_0(s_0) \cdot \prod_{1 \leq i \leq n} P(s_{i-1}, s_i)$$



which uniquely defines the probability function P on \mathcal{F} .

DTMC - Conditional Probability and Expectation

Let (S, P, π_0) be a DTMC. We denote by

- ▶ P_s the probability function of DTMC (S, P, δ_s) where


$$\delta_s(s') = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases}$$

- ▶ E_s the expectation with respect to P_s

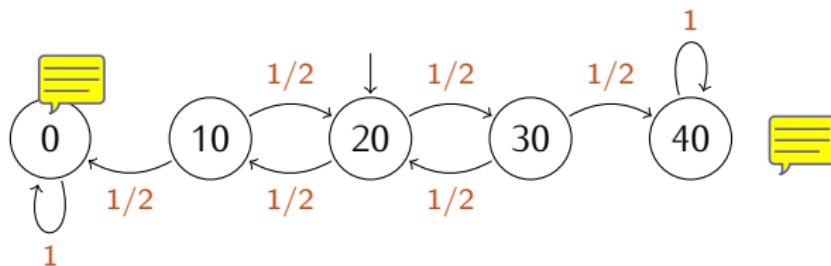
Analysis questions

- ▶ Transient analysis 
- ▶ Steady-state analysis 
- ▶ Rewards
- ▶ Reachability
- ▶ Probabilistic logics

DTMC - Transient Analysis

DTMC - Transient Analysis - Example (1)

Example: Gambling with a Limit



What is the probability of being in state 0 after 3 steps?

DTMC - Transient Analysis - n -step Probabilities

Definition:

Given a DTMC (S, P, π_0) , we assume w.l.o.g. $S = \{0, 1, \dots\}$ and write $p_{ij} = P(i, j)$. Further, we have

- ▶ $P^{(1)} = P = (p_{ij})$ is the **1-step transition matrix**
- ▶ $P^{(n)} = (p_{ij}^{(n)})$ denotes the **n -step transition matrix** with

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \quad (= P(X_{k+n} = j \mid X_k = i)).$$

How can we compute these probabilities?

Definition: Chapman-Kolmogorov Equation

Application of the **law of total probability** to the n -step transition probabilities $p_{ij}^{(n)}$ results in the **Chapman-Kolmogorov Equation**

$$p_{ij}^{(n)} = \sum_{h \in S} p_{ih}^{(m)} p_{hj}^{(n-m)} \quad \forall 0 < m < n.$$

Consequently, we have $P^{(n)} = PP^{(n-1)} = \dots = P^n$.

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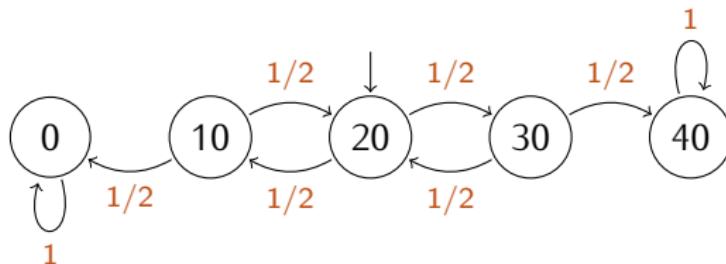
Definition: Transient Probability Distribution

The transient probability distribution at time $n > 0$ is defined by

$$\pi_n = \pi_0 P^n = \pi_{n-1} P.$$



DTMC - Transient Analysis - Example (2)

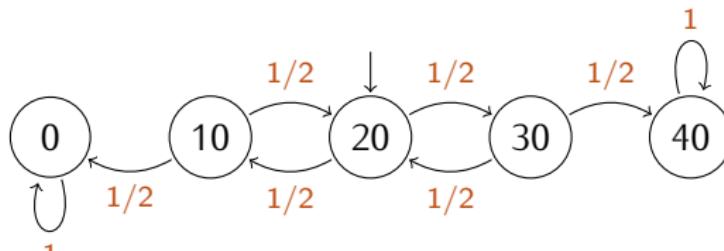


Example:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- For $\pi_0 = [0 \ 0 \ 1 \ 0 \ 0]$, $\pi_2 = \pi_0 P^2 = [0.25 \ 0 \ 0.5 \ 0 \ 0.25]$.
- For, $\pi_0 = [0.4 \ 0 \ 0 \ 0 \ 0.6]$, $\pi_2 = \pi_0 P^2 = [0.4 \ 0 \ 0 \ 0 \ 0.6]$.
Actually, $\pi_n = [0.4 \ 0 \ 0 \ 0 \ 0.6]$ for all $n \in \mathbb{N}$!

DTMC - Transient Analysis - Example (2)



Example: 0 1 2 3 4

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- For $\pi_0 = [0 \ 0 \ 1 \ 0 \ 0]$, $\pi_2 = \pi_0 P^2 = [0.25 \ 0 \ 0.5 \ 0 \ 0.25]$.
- For, $\pi_0 = [0.4 \ 0 \ 0 \ 0 \ 0.6]$, $\pi_2 = \pi_0 P^2 = [0.4 \ 0 \ 0 \ 0 \ 0.6]$.
Actually, $\pi_n = [0.4 \ 0 \ 0 \ 0 \ 0.6]$ for all $n \in \mathbb{N}$!

Are there other “stable” distributions?

DTMC - Steady State Analysis

DTMC - Steady State Analysis - Definitions

Definition: Stationary Distribution

A distribution π is **stationary** if

$$\pi = \pi P.$$

Stationary distribution is generally **not unique**.

DTMC - Steady State Analysis - Definitions

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Stationary distribution is generally **not unique**.

Definition: Limiting Distribution

$$\pi^* := \lim_{n \rightarrow \infty} \pi_n = \lim_{n \rightarrow \infty} \pi_0 P^n = \pi_0 \lim_{n \rightarrow \infty} P^n = \pi_0 P^*.$$

The limit **can** depend on π_0 and does **not** need to exist.

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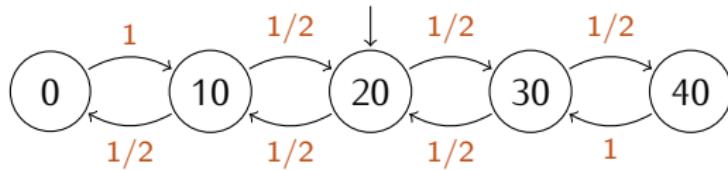
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Connection between stationary and limiting?



DTMC - Steady-State Analysis - Periodicity

Example: Gambling with Social Guarantees

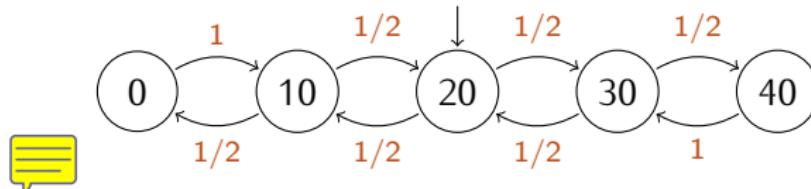


What are the stationary and limiting distributions?



DTMC - Steady-State Analysis - Periodicity

Example: Gambling with Social Guarantees



What are the stationary and limiting distributions?

Definition: Periodicity

The **period** of a state i is defined as

$$d_i = \gcd\{n \mid p_{ii}^n > 0\}.$$

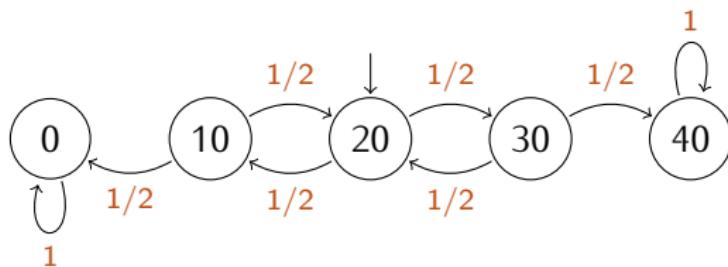
A state i is called **aperiodic** if $d_i = 1$ and **periodic** with period d_i otherwise. A Markov chain is **aperiodic** if all states are aperiodic.

Lemma

In a finite **aperiodic** Markov chain, the **limiting distribution exists**.

DTMC - Steady-State Analysis - Irreducibility (1)

Example



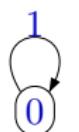
DTMC - Steady-State Analysis - Irreducibility (2)

Definition:

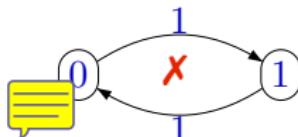
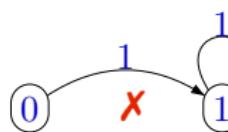
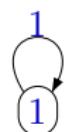
A DTMC is called **irreducible** if for all states $i, j \in S$ we have $P_{ij}^n > 0$ for some $n \geq 1$.

Lemma

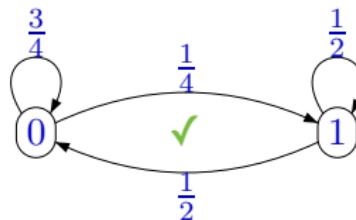
In an **aperiodic** and **irreducible** Markov chain, the *limiting distribution exists and does not depend on π_0* .



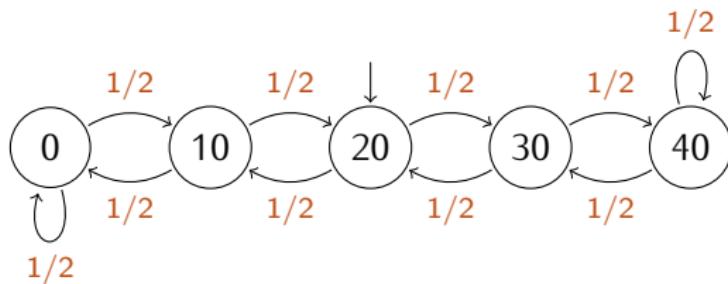
X



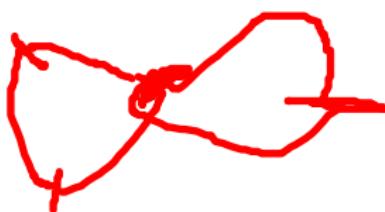
✓



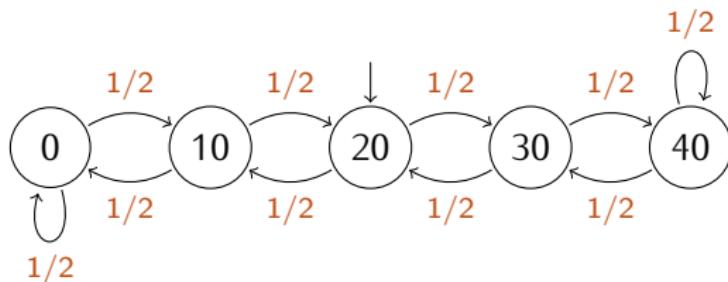
DTMC - Steady-State Analysis - Irreducibility (3)



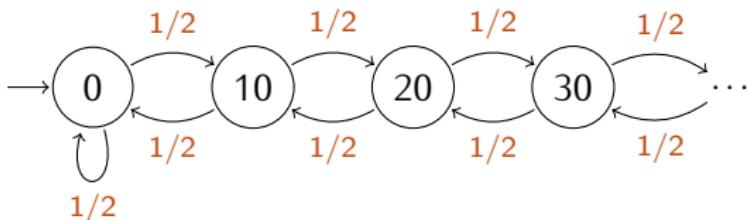
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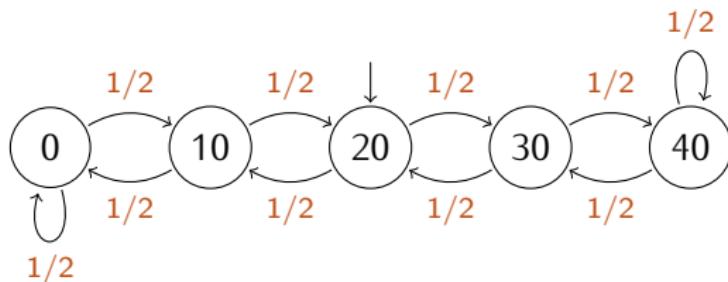
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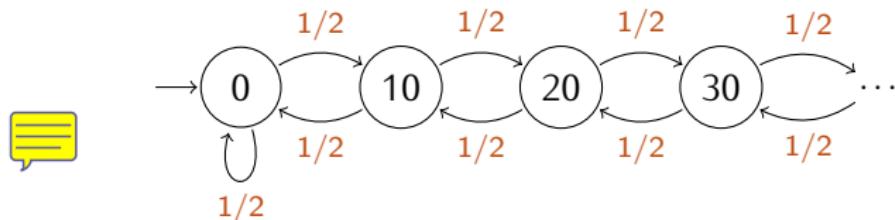
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DTMC - Steady-State Analysis - Irreducibility (3)



What is the stationary / limiting distribution?



Lemma

In a **finite aperiodic** and **irreducible** Markov chain, the **limiting distribution exists, does not depend on π_0 , and equals the unique stationary distribution.**



DTMC - Steady-State Analysis - Recurrence (1)



Definition:

Let $f_{ij}^{(n)} = P(X_n = j \wedge \forall 1 \leq k < n : X_k \neq j \mid X_0 = i)$ for $n \geq 1$ be the n -step hitting probability. The **hitting probability** is defined as

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

and a state i is called

- ▶ transient if $f_{ii} < 1$ and
- ▶ recurrent if $f_{ii} = 1$.



DTMC - Steady-State Analysis - Recurrence (2)

Definition:

Denoting expectation $m_{ij} = \sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)}$, a recurrent state i is called

- ▶ positive recurrent or recurrent non-null if $m_{ii} < \infty$ and
- ▶ recurrent null if $m_{ii} = \infty$.

Lemma

The states of an irreducible DTMC are all of the same type, i.e.,

- ▶ all periodic or
- ▶ all aperiodic and transient or
- ▶ all aperiodic and recurrent null or
- ▶ all aperiodic and recurrent non-null.



DTMC - Steady-State Analysis - Ergodicity

Definition: Ergodicity

A DTMC is **ergodic** if all its states are **irreducible**, **aperiodic** and **recurrent non-null**.

Theorem

In an ergodic Markov chain, the limiting distribution exists, does not depend on π_0 , and equals the unique stationary distribution.

As a consequence, the steady-state distribution can be computed by solving the equation system

$$\pi = \pi P, \sum_{s \in S} \pi_s = 1.$$

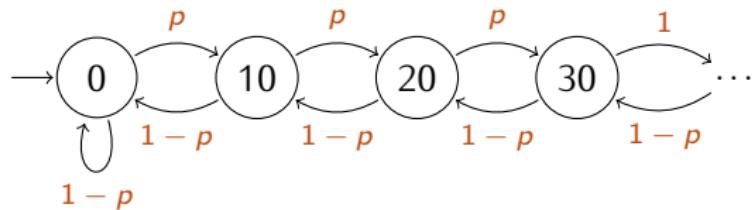


Note: The Lemma for finite DTMC follows from the theorem as every irreducible finite DTMC is positive recurrent.



DTMC - Steady-State Analysis - Ergodicity

Example: Unbounded Gambling with House Edge



The DTMC is only ergodic for $p \in [0, 0.5)$.



DTMC - Rewards



DTMC - Rewards - Definitions

Definition

A **reward Markov chain** is a tuple (S, P, π_0, r) where (S, P, π_0) is a Markov chain and $r : S \rightarrow \mathbb{Z}$ is a reward function.

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Every run $\rho = s_0, s_1, \dots$ induces a sequence of values $r(s_0), r(s_1), \dots$

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$$\sum_{i=1}^{\infty} \lambda^i r(s_i) \quad \text{for some } 0 < \lambda < 1$$

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Definition

The **expected average reward** is

$$EAR := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \mathbb{E}[r(X_i)]$$

DTMC - Rewards - Solution Sketch

Definition: Time-average Distribution

$$\hat{\pi} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \pi_i.$$

$\hat{\pi}(s)$ expresses the ratio of time spent in s on the long run.

¹More details later for Markov decision processes.

DTMC - Rewards - Solution Sketch

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Lemma

1. $\mathbb{E}[r(X_i)] = \sum_{s \in S} \pi_i(s) \cdot r(s).$
2. If $\hat{\pi}$ exists then $EAR = \sum_{s \in S} \hat{\pi}(s) \cdot r(s).$
3. If limiting distribution exists, it coincides with $\hat{\pi}$.

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Algorithm



1. Compute $\hat{\pi}$ (or limiting distribution if possible).¹
2. Return $\sum_{s \in S} \hat{\pi}(s) \cdot r(s).$



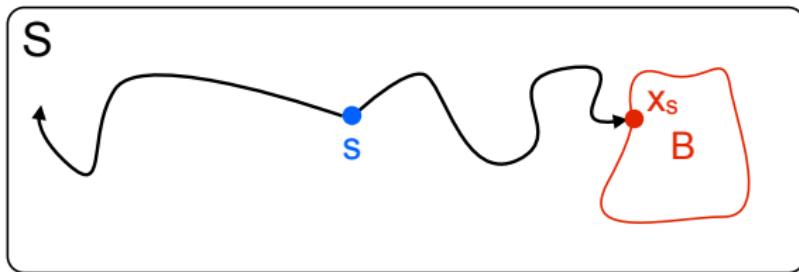
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DTMC - Reachability

DTMC – Reachability

Definition: Reachability

Given a DTMC (S, P, π_0) , what is the probability of **eventually reaching** a set of goal states $B \subseteq S$?



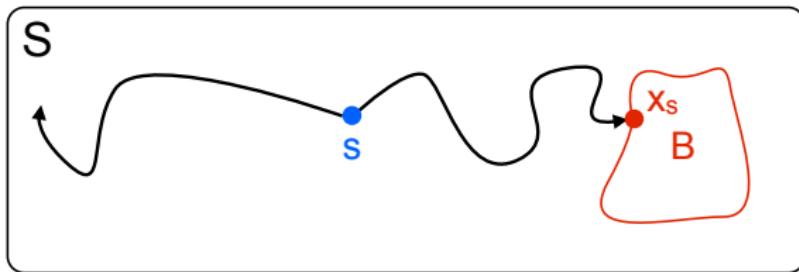
Let $x(s)$ denote $P_s(\diamond B)$ where $\diamond B = \{s_0 s_1 \dots \mid \exists i : s_i \in B\}$. Then

- ▶ $s \in B$: $x(s) =$
- ▶ $s \in S \setminus B$: $x(s) =$

DTMC – Reachability

Definition: Reachability

Given a DTMC (S, P, π_0) , what is the probability of **eventually reaching** a set of goal states $B \subseteq S$?



Let $x(s)$ denote $P_s(\diamond B)$ where $\diamond B = \{s_0 s_1 \dots \mid \exists i : s_i \in B\}$. Then

- ▶ $s \in B$: $x(s) = 1$
- ▶ $s \in S \setminus B$: $x(s) = \sum_{t \in S \setminus B} P(s, t)x(t) + \sum_{u \in B} P(s, u).$

DTMC – Reachability

Lemma (Reachability Matrix Form)

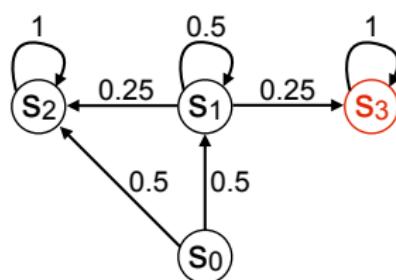
Given a DTMC (S, P, π_0) , the column vector $x = (x(s))_{s \in S \setminus B}$ of probabilities $x(s) = P_s(\diamond B)$ satisfies the constraint

$$x = Ax + b,$$

where matrix A is the submatrix of P for states $S \setminus B$ and $b = (b(s))_{s \in S \setminus B}$ is the column vector with $b(s) = \sum_{u \in B} P(s, u)$.

DTMC - Reachability

Example:



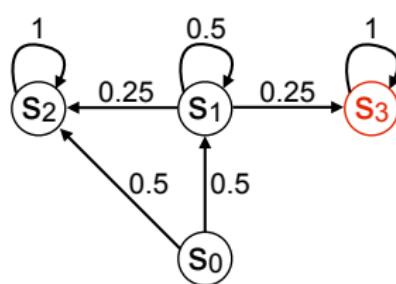
$$P = \begin{bmatrix} A & b \\ \begin{matrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.25 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0.25 \\ 0 \\ 1 \end{matrix} \end{bmatrix}$$

$$B = \{S_3\}$$

The vector $x = [x_0 \ x_1 \ x_2]^T = [0.25 \ 0.5 \ 0]^T$ satisfies the equation system $x = Ax + b$.

DTMC - Reachability

Example:



$$P = \begin{bmatrix} A & b \\ \begin{matrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.25 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0.25 \\ 0 \\ 1 \end{matrix} \end{bmatrix}$$

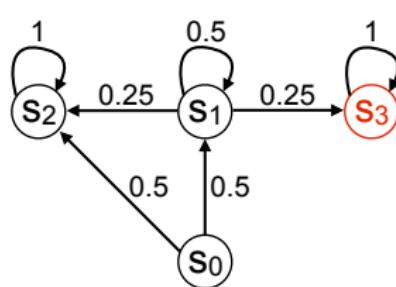
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Is it the only solution?

DTMC - Reachability

Example:



$$P = \begin{bmatrix} A & b \\ \begin{matrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.25 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0.25 \\ 0 \\ 1 \end{matrix} \end{bmatrix}$$

$$B = \{S_3\}$$

The vector $x = [x_0 \ x_1 \ x_2]^T = [0.25 \ 0.5 \ 0]^T$ satisfies the equation system $x = Ax + b$.

Is it the only solution?

- ▶ No! Consider, e.g., $[0.55 \ 0.7 \ 0.4]^T$ or $[1 \ 1 \ 1]^T$.
- ▶ While reaching the goal, such **bad** states need to be avoided.
- ▶ We first generalise such “avoiding”.

DTMC - Conditional Reachability

Definition:

Let $B, C \subseteq S$. The (unbounded) probability of reaching B from state s under the condition that C is not left before is defined as $P_s(C \cup B)$ where

$$C \cup B = \{s_0 s_1 \dots \mid \exists i : s_i \in B \wedge \forall j < i : s_j \in C\}.$$

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The probability of reaching B from state s within n steps under the condition that C is not left before is defined as $P_s(C \cup \leq^n B)$ where

$$C \cup \leq^n B = \{s_0 s_1 \dots \mid \exists i \leq n : s_i \in B \wedge \forall j < i : s_j \in C\}.$$

What is the equation system for these probabilities?

DTMC - Conditional Reachability - Solution

Let $S_{=0} = \{s \mid P_s(C \cap B) = 0\}$ and $S_? = S \setminus (S_{=0} \cup B)$.

DTMC - Conditional Reachability - Solution

Let $S_{=0} = \{s \mid P_s(C \cap B) = 0\}$ and $S_? = S \setminus (S_{=0} \cup B)$.

Theorem:

The column vector $x = (x(s))_{s \in S_?}$ of probabilities $x(s) = P_s(C \cap B)$ is the **unique solution** of the equation system

$$x = Ax + b,$$

where $A = (P(s, t))_{s, t \in S_?}$, $b = (b(s))_{s \in S_?}$ with $b(s) = \sum_{u \in B} P(s, u)$.

Furthermore, for $x_0 = (0)_{s \in S_?}$ and $x_i = Ax_{i-1} + b$ for any $i \geq 1$,

1. $x_n(s) = P_s(C \cap B^{\leq n})$ for $s \in S_?$,
2. x_i is increasing, and
3. $x = \lim_{n \rightarrow \infty} x_n$.

Proof Sketch:

- ▶ $(x_s)_{s \in S}$ is a solution: by inserting into definition.
- ▶ Unique solution: By contradiction. Assume y is another solution, then $x - y = A(x - y)$. One can show that $A - I$ is invertible, thus $(A - I)(x - y) = 0$ yields $x - y = (A - I)^{-1}0 = 0$ and finally $x = y$ ².

Furthermore,

1. From the definitions, by straightforward induction.
2. From 1. since $C \cup^{\leq n} B \subseteq C \cup^{\leq n+1} B$.
3. Since $C \cup B = \bigcup_{n \in \mathbb{N}} C \cup^{\leq n} B$.



²cf. page 766 of Principles of Model Checking

Algorithmic aspects

Algorithmic Aspects – Summary of Equation Systems

Equation Systems

- ▶ Transient analysis: $\pi_n = \pi_0 P^n = \pi_{n-1} P$
- ▶ Steady-state analysis: $\pi P = \pi, \pi \cdot 1 = \sum_{s \in S} \pi(s) = 1$ (ergodic)
- ▶ Reachability: $x = Ax + b$ (with $(x(s))_{s \in S?}$)

Solution Techniques

1. Analytic solution, e.g. by Gaussian elimination
2. Iterative power method ($\pi_n \rightarrow \pi$ and $x_n \rightarrow x$ for $n \rightarrow \infty$)
3. Iterative methods for solving large systems of linear equations, e.g. Jacobi, Gauss-Seidel

Missing pieces

- a. finding out whether a DTMC is ergodic,
- b. computing $S? = S \setminus \{s \mid P_s(\Diamond B) = 0\}$,
- c. efficient representation of P .

Algorithmic Aspects: a. Ergodicity of finite DTMC (1)

Ergodicity = Irreducibility + Aperiodicity + P. Recurrence

- ▶ A DTMC is called **irreducible** if for all states $i, j \in S$ we have $p_{ij}^n > 0$ for some $n \geq 1$.
- ▶ A state i is called **aperiodic** if $\gcd\{n \mid p_{ii}^n > 0\} = 1$.
- ▶ A state i is called **positive recurrent** if $f_{ii} = 1$ and $m_{ii} < \infty$.

How do we tell that a finite DTMC is **ergodic**?

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How do we tell that a finite DTMC is **ergodic**?

By analysis of the induced graph!

For a DTMC $(S, P, \pi(0))$ we define the **induced directed graph** (S, E) with $E = \{(s, s') \mid P(s, s') > 0\}$.

Recall:

- ▶ A **directed graph** is called **strongly connected** if there is a path from each vertex to every other vertex.
- ▶ **Strongly connected components (SCC)** are its **maximal** strongly connected subgraphs.
- ▶ A SCC T is **bottom (BSCC)** if no $s \notin T$ is reachable from T .

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For **finite** DTMCs, it holds that:

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For **finite** DTMCs, it holds that:

- ▶ The DTMC is **irreducible** iff the induced graph is strongly connected.

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Ergodicity = Irreducibility + Aperiodicity + P. Recurrence

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Theorem:

For **finite** DTMCs, it holds that:

- ▶ The DTMC is **irreducible** iff the induced graph is strongly connected.
- ▶ A state in a BSCC is **aperiodic** iff the BSCC is aperiodic, i.e. the greatest common divisor of the lengths of all its cycles is **1**.

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Ergodicity = Irreducibility + Aperiodicity + P. Recurrence

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Theorem:

For **finite** DTMCs, it holds that:

- ▶ The DTMC is **irreducible** iff the induced graph is strongly connected.
- ▶ A state in a BSCC is **aperiodic** iff the BSCC is aperiodic, i.e. the greatest common divisor of the lengths of all its cycles is **1**.
- ▶ A state is **positive recurrent** iff it belongs to a BSCC otherwise it is **transient**.

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

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- ▶ $\text{gcd}\{n \geq 1 \mid \exists s : P^n(s, s) > 0\} = 1$

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

- ▶ $\text{gcd}\{n \geq 1 \mid \exists s : P^n(s, s) > 0\} = 1$
- ▶ in time $\mathcal{O}(n + m)$?

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

- ▶ $\text{gcd}\{n \geq 1 \mid \exists s : P^n(s, s) > 0\} = 1$
- ▶ in time $\mathcal{O}(n + m)$? By the following DFS-based procedure:

Algorithm: PERIOD(vertex v , unsigned $level$: init 0)

```
1 global period : init 0;
2 if period = 1 then
3   | return
4 end
5 if v is unmarked then
6   | mark v;
7   | vlevel = level;
8   | for v' ∈ out(v) do
9     |   | PERIOD(v', level + 1)
10    | end
11 else
12   |   | period = gcd(period, level - vlevel);
13 end
```

Algorithmic Aspects: b. Computing the set $S_?$

We have $S_? = S \setminus (B \cup S_{=0})$ where $S_{=0} = \{s \mid P_s(\Diamond B) = 0\}$.
Hence,

$$s \in S_{=0} \quad \text{iff} \quad p_{ss'}^n = 0 \quad \text{for all } n \geq 1 \text{ and } s' \in B.$$

Algorithmic Aspects: b. Computing the set $S_?$

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Hence,

$$s \in S_= \quad \text{iff} \quad p_{ss'}^n = 0 \quad \text{for all } n \geq 1 \text{ and } s' \in B.$$

This can be again easily checked from the induced graph:

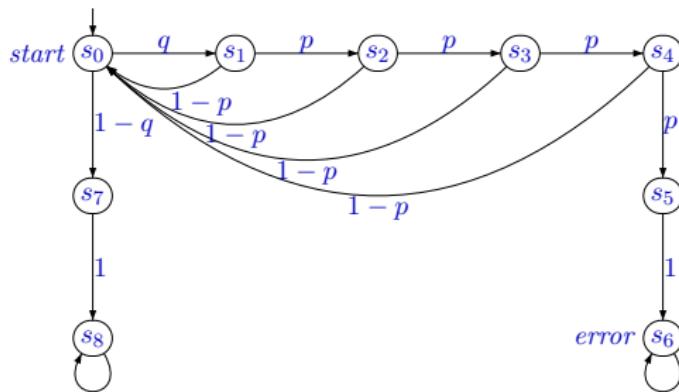
Lemma

We have $s \in S_=$ iff there is no path from s to any state from B .

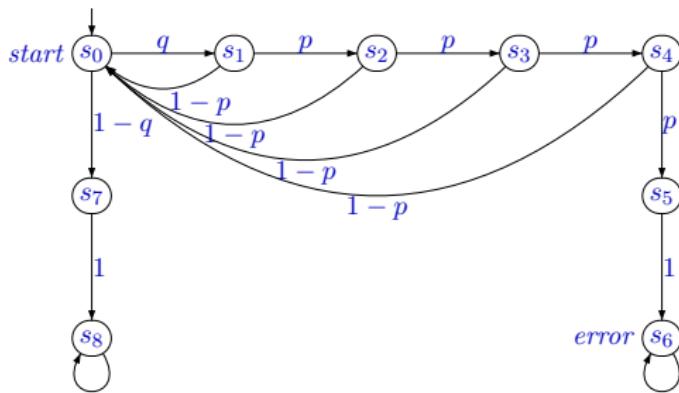
Proof.

Easy from the fact that $p_{ss'}^n > 0$ iff there is a path of length n to s' . \square

Algorithmic Aspects: c. Efficient Representations

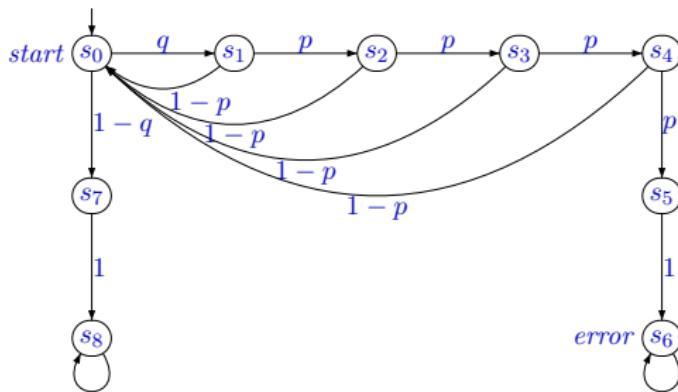


Algorithmic Aspects: c. Efficient Representations



1. There are many 0 entries in the transition matrix.
Sparse matrices offer a more concise storage.

Algorithmic Aspects: c. Efficient Representations



1. There are many **0** entries in the transition matrix.
Sparse matrices offer a more concise storage.
2. There are many similar entries in the transition matrix.
Multi-terminal binary decision diagrams offer a more concise storage, using automata theory.

DTMC - Probabilistic Temporal Logics for Specifying Complex Properties

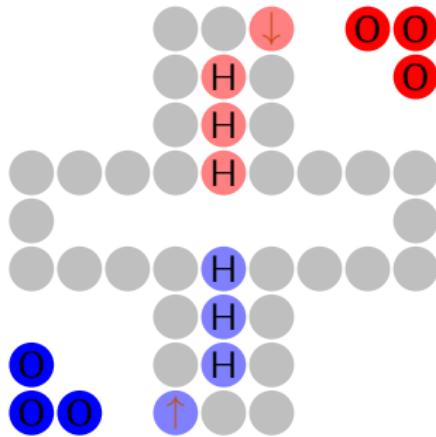
Logics - Adding Labels to DTMC

Definition:

A **labeled** DTMC is a tuple $\mathcal{D} = (S, P, \pi_0, L)$ with $L : S \rightarrow 2^{AP}$, where

- ▶ AP is a set of **atomic propositions** and
- ▶ L is a **labeling function**, where $L(s)$ specifies which properties hold in state $s \in S$.

Logics - Examples of Properties



States and transitions

state = configuration of the game;

transition = rolling the dice and acting (randomly) based on the result.

State labels

- ▶ init, rwin, bwin, rkicked, bkicked, ...
- ▶ r30, r21, ...,
- ▶ b30, b21, ...,

Examples of Properties

- ▶ the game cannot return back to start
- ▶ at any time, the game eventually ends with prob. 1
- ▶ at any time, the game ends within 100 dice rolls with prob. ≥ 0.5
- ▶ the probability of winning without ever being kicked out is ≤ 0.3

How to specify them formally?

Linear-time view

- ▶ corresponds to our (human) perception of time
- ▶ can specify properties of **one concrete** linear execution of the system

Example: eventually red player is kicked out followed immediately by blue player being kicked out.

Branching-time view

- ▶ views future as a set of all possibilities
- ▶ can specify properties of **all executions** from a given state – specifies execution trees

Example: in every computation it is always possible to return to the initial state.

Logics - Temporal Logics - non-probabilistic (2)

Linear Temporal Logic (LTL)

Syntax for formulae specifying executions:

$$\psi = \text{true} \mid a \mid \psi \wedge \psi \mid \neg \psi \mid \mathcal{X} \psi \mid \psi \mathcal{U} \psi \mid \mathcal{F} \psi \mid \mathcal{G} \psi$$

Example: eventually red player is kicked out followed immediately by blue player being kicked out: $\mathcal{F} (\text{rkicked} \wedge \mathcal{X} \text{bkicked})$

Question: do all executions satisfy the given LTL formula?

Computation Tree Logic (CTL)

Syntax for specifying states:

$$\phi = \text{true} \mid a \mid \phi \wedge \phi \mid \neg \phi \mid A \psi \mid E \psi$$

Syntax for specifying executions:

$$\psi = \mathcal{X} \phi \mid \phi \mathcal{U} \phi \mid \mathcal{F} \phi \mid \mathcal{G} \phi$$

Example: in all computations it is always possible to return to initial state: $A \mathcal{G} \ E \mathcal{F} \text{ init}$

Question: does the given state satisfy the given CTL state formula?

Logics - LTL

Syntax $\psi = \text{true} \mid a \mid \psi \wedge \psi \mid \neg \psi \mid \mathcal{X} \psi \mid \psi \mathcal{U} \psi.$

Semantics (for a path $\omega = s_0 s_1 \dots$)

- ▶ $\omega \models \text{true}$ (always),
- ▶ $\omega \models a$ iff $a \in L(s_0)$,
- ▶ $\omega \models \psi_1 \wedge \psi_2$ iff $\omega \models \psi_1$ and $\omega \models \psi_2$,
- ▶ $\omega \models \neg \psi$ iff $\omega \not\models \psi$,
- ▶ $\omega \models \mathcal{X} \psi$ iff $s_1 s_2 \dots \models \psi$,

		Ψ										
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- ▶ $\omega \models \psi_1 \mathcal{U} \psi_2$ iff $\exists i \geq 0 : s_i s_{i+1} \dots \models \psi_2$ and $\forall j < i : s_j s_{j+1} \dots \models \psi_1$.

Ψ_1	.	.	.	Ψ_1	Ψ_2							
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Syntactic sugar

- ▶ $\mathcal{F} \psi \equiv$
- ▶ $\mathcal{G} \psi \equiv$

Logics - LTL

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Syntactic sugar

- ▶ $\mathcal{F} \psi \equiv \text{true} \mathcal{U} \psi$
- ▶ $\mathcal{G} \psi \equiv \neg(\text{true} \mathcal{U} \neg\psi)$ ($\equiv \neg\mathcal{F} \neg\psi$)

Logics - CTL

Syntax

State formulae:

$\phi = \text{true} \mid a \mid \phi \wedge \phi \mid \neg \phi \mid A \psi \mid E \psi$

where ψ is a path formula.

Semantics

For a state s :

- ▶ $s \models \text{true}$ (always),
 - ▶ $s \models a$ iff $a \in L(s)$,
 - ▶ $s \models \phi_1 \wedge \phi_2$ iff $s \models \phi_1$ and
 $s \models \phi_2$,
 - ▶ $s \models \neg\phi$ iff $s \not\models \phi$,
 - ▶ $s \models A\psi$ iff $\omega \models \psi$ for all
 paths $\omega = s_0s_1 \dots$ with $s_0 = s$,
 - ▶ $s \models E\psi$ iff $\omega \models \psi$ for some
 path $\omega = s_0s_1 \dots$ with $s_0 = s$.

Path formulae:

$$\psi = \mathcal{X} \phi \mid \phi \mathcal{U} \phi$$

where ϕ is a state formula.

For a path $\omega = s_0 s_1 \dots$:

- $\omega \models \chi \phi$ iff
 $s_1 s_2 \dots$ satisfies ϕ



- $\omega \models \phi_1 \cup \phi_2$ iff $\exists i : s_i s_{i+1} \dots \models \phi_2$ and $\forall j < i : s_j s_{j+1} \dots \models \phi_1$.



Logics - Temporal Logics - non-probabilistic (2)

Linear Temporal Logic (LTL)

Syntax for formulae specifying executions:

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Example: eventually red player is kicked out followed immediately by blue player being kicked out: $\mathcal{F} (\text{rkicked} \wedge \mathcal{X} \text{bkicked})$

Question: do all executions satisfy the given LTL formula?

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Syntax for specifying states:

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Example: in all computations it is always possible to return to initial state: $A \mathcal{G} \ E \mathcal{F} \text{ init}$

Question: does the given state satisfy the given CTL state formula?

Logics – Temporal Logics – probabilistic

Linear Temporal Logic (LTL) + probabilities

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Example: with prob. ≥ 0.8 , eventually red player is kicked out followed immediately by blue player being kicked out:

$$P(\mathcal{F} (\text{rkicked} \wedge \mathcal{X} \text{bkicked})) \geq 0.8$$

Question: is the formula satisfied by executions of given probability?

Logics - Temporal Logics - probabilistic

Linear Temporal Logic (LTL) + probabilities

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$$P(\mathcal{F} (\text{rkicked} \wedge \mathcal{X} \text{bkicked})) \geq 0.8$$

Question: is the formula satisfied by executions of given probability?

Probabilistic Computation Tree Logic (PCTL)

Syntax for specifying states:

$$\phi = \text{true} \mid a \mid \phi \wedge \phi \mid \neg \phi \mid \mathcal{P}_J \psi$$

Syntax for specifying executions:

$$\psi = \mathcal{X} \phi \mid \phi \mathcal{U} \phi \mid \phi \mathcal{U}^{\leq k} \phi \mid \mathcal{F} \phi \mid \mathcal{G} \phi$$

Example: with prob. at least 0.5 the probability to return to initial state is always at least 0.1: $P_{\geq 0.5} \mathcal{G} P_{\geq 0.1} \mathcal{F} \text{init}$

Question: does the given state satisfy the given PCTL state formula?

Logics - PCTL - Examples

Syntactic sugar:

- ▶ $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$, $\phi_1 \Rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$, etc.
- ▶ ≤ 0.5 denotes the interval $[0, 0.5]$, $= 1$ denotes $[1, 1]$, etc.

Examples:

- ▶ A fair die:

$$\bigwedge_{i \in \{1, \dots, 6\}} \mathcal{P}_{=\frac{1}{6}}(\mathcal{F} i).$$

- ▶ The probability of winning "Who wants to be a millionaire" without using any joker should be negligible:

$$\mathcal{P}_{<1e-10}(\neg(J_{50\%} \vee J_{audience} \vee J_{telephone}) \text{ } \cup \text{ } win).$$

Logics - PCTL - Semantics

Semantics

For a state s :

- ▶ $s \models \text{true}$ (always),
- ▶ $s \models a$ iff $a \in L(s)$,
- ▶ $s \models \phi_1 \wedge \phi_2$ iff $s \models \phi_1$ and $s \models \phi_2$,
- ▶ $s \models \neg\phi$ iff $s \not\models \phi$,
- ▶ $s \models \mathcal{P}_J(\psi)$ iff $P_s(\text{Paths}(\psi)) \in J$

For a path $\omega = s_0s_1 \dots$:

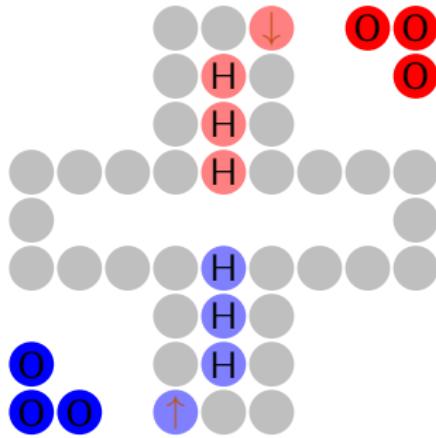
- ▶ $\omega \models \mathcal{X} \phi$ iff $s_1s_2 \dots$ satisfies ϕ ,

		Φ												
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Φ	1	Φ	1	Φ	2				
---	---	---	---	---	---	---	---	---	---	---	--	--	--	--
- ▶ $\omega \models \phi_1 \mathcal{U}^{\leq n} \phi_2$ iff $\exists i \leq n :$
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Φ	1	Φ	1	Φ	2				
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Logics - Examples of Properties

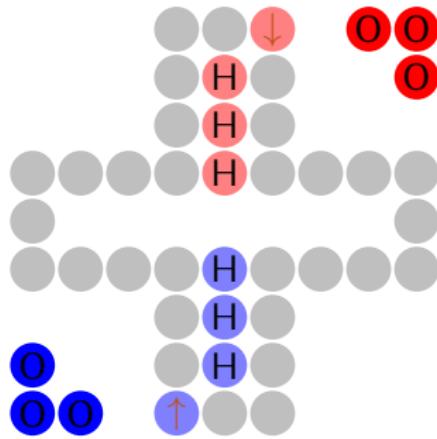


Examples of Properties

1. the game cannot return back to start
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Formally

Logics - Examples of Properties



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4. the probability of winning without ever being kicked out is ≤ 0.3

Formally

1. $P(\mathcal{X} \mathcal{G} \neg \text{init}) = 1$ (LTL + prob.)
 $P_{=1}(\mathcal{X} P_{=0}(\mathcal{G} \neg \text{init}))$ (PCTL)
2. $P_{=1}(\mathcal{G} P_{=1}(\mathcal{F} (\text{rwin} \vee \text{bwin})))$ (PCTL)
3. $P_{=1}(\mathcal{G} P_{\geq 0.5}(\mathcal{F}^{\leq 100}(\text{rwin} \vee \text{bwin})))$ (PCTL)
4. $P((\neg \text{rkicked} \wedge \neg \text{bkicked}) \mathcal{U} (\text{rwin} \vee \text{bwin})) \leq 0.3$ (LTL + prob.)

PCTL Model Checking Algorithm

PCTL Model Checking

Definition: PCTL Model Checking

Let $\mathcal{D} = (S, P, \pi_0, L)$ be a DTMC, Φ a PCTL state formula and $s \in S$.
The **model checking problem** is to decide whether $s \models \Phi$.

Theorem

The PCTL model checking problem can be decided in time polynomial in $|\mathcal{D}|$, linear in $|\Phi|$, and linear in the maximum step bound n .

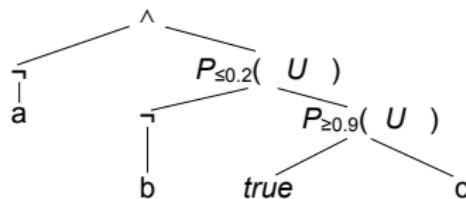
PCTL Model Checking - Algorithm - Outline (1)

Algorithm:

Consider the bottom-up traversal of the parse tree of Φ :

- ▶ The leaves are $a \in AP$ or *true* and
- ▶ the inner nodes are:
 - ▶ unary – labelled with the operator \neg or $\mathcal{P}_J(\mathcal{X})$;
 - ▶ binary – labelled with an operator \wedge , $\mathcal{P}_J(U)$, or $\mathcal{P}_J(U^{\leq n})$.

Example: $\neg a \wedge \mathcal{P}_{\leq 0.2}(\neg b \ U \ \mathcal{P}_{\geq 0.9}(\Diamond c))$



Compute $Sat(\Psi) = \{s \in S \mid s \models \Psi\}$ for each node Ψ of the tree in a bottom-up fashion. Then $s \models \Phi$ iff $s \in Sat(\Phi)$.

PCTL Model Checking - Algorithm - Outline (2)

“Base” of the algorithm:

We need a procedure to compute $Sat(\Psi)$ for Ψ of the form a or $true$:

PCTL Model Checking - Algorithm - Outline (2)

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We need a procedure to compute $\text{Sat}(\Psi)$ for Ψ of the form a or $true$:

Lemma

- ▶ $\text{Sat}(true) = S$,
- ▶ $\text{Sat}(a) = \{s \mid a \in L(s)\}$

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“Induction” step of the algorithm:

We need a procedure to compute $\text{Sat}(\Psi)$ for Ψ given the sets $\text{Sat}(\Psi')$ for all state sub-formulas Ψ' of Ψ :

Lemma

- ▶ $\text{Sat}(\Phi_1 \wedge \Phi_2) =$
- ▶ $\text{Sat}(\neg\Phi) =$

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We need a procedure to compute $Sat(\Psi)$ for Ψ given the sets $Sat(\Psi')$ for all state sub-formulas Ψ' of Ψ :

Lemma

- ▶ $Sat(\Phi_1 \wedge \Phi_2) = Sat(\Phi_1) \cap Sat(\Phi_2)$
- ▶ $Sat(\neg\Phi) = S \setminus Sat(\Phi)$

$Sat(\mathcal{P}_J(\Phi)) = \{s \mid P_s(Paths(\Phi)) \in J\}$ discussed on the next slide.

PCTL Model Checking - Algorithm - Path Operator

Lemma

- ▶ Next:

$$P_s(\text{Paths}(\mathcal{X} \Phi)) =$$

- ▶ Bounded Until:

$$P_s(\text{Paths}(\Phi_1 \ U^{\leq n} \Phi_2)) =$$

- ▶ Unbounded Until:

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PCTL Model Checking - Algorithm - Path Operator

Lemma

- ▶ Next:

$$P_s(Paths(\mathcal{X} \Phi)) = \sum_{s' \in Sat(\Phi)} P(s, s')$$

- ▶ Bounded Until:

$$P_s(Paths(\Phi_1 \ U^{\leq n} \Phi_2)) = P_s(Sat(\Phi_1) \ U^{\leq n} \ Sat(\Phi_2))$$

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- ▶ Unbounded Until:

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As before:

can be reduced to transient analysis and to unbounded reachability.

Precise algorithm

Computation for every node in the parse tree and for every state:

- ▶ All node types except for path operator – trivial.
- ▶ **Next**: Trivial.
- ▶ **Until**: Solving equation systems can be done by polynomially many elementary arithmetic operations.
- ▶ **Bounded until**: Matrix vector multiplications can be done by polynomial many elementary arithmetic operations as well.

Overall complexity:

Polynomial in $|\mathcal{D}|$, linear in $|\Phi|$ and the maximum step bound n .

In practice

The **until** and **bounded until** probabilities computed approximatively:

- ▶ rounding off probabilities in matrix-vector multiplication,
- ▶ using approximative iterative methods **without error guarantees**.

pLTL Model Checking Algorithm

LTL Model Checking – Overview

Definition: LTL Model Checking

Let $\mathcal{D} = (S, P, \pi_0, L)$ be a DTMC, Ψ a LTL formula, $s \in S$, and $p \in [0, 1]$. The **model checking problem** is to decide whether $s \models P_s^{\mathcal{D}}(\text{Paths}(\Psi)) \geq p$.

Theorem

The LTL model checking can be decided in time $\mathcal{O}(|\mathcal{D}| \cdot 2^{|\Psi|})$.

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Algorithm Outline

1. Construct from Ψ a **deterministic** Rabin automaton A recognizing words satisfying Ψ , i.e. $\text{Paths}(\Psi) := \{L(\omega) \in (2^A)^{\infty} \mid \omega \models \Psi\}$

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2. Construct a product DTMC $\mathcal{D} \times A$ that “embeds” the deterministic execution of A into the Markov chain.

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1. Construct from Ψ a **deterministic** Rabin automaton A recognizing words satisfying Ψ , i.e. $\text{Paths}(\Psi) := \{L(\omega) \in (2^{Ap})^\infty \mid \omega \models \Psi\}$
2. Construct a product DTMC $\mathcal{D} \times A$ that “embeds” the deterministic execution of A into the Markov chain.
3. Compute in $\mathcal{D} \times A$ the probability of paths where A satisfies the acceptance condition.

LTL Model Checking – ω -Automata (1.)

Deterministic Rabin automaton (DRA): $(Q, \Sigma, \delta, q_0, Acc)$

- ▶ a DFA with a different acceptance condition,
- ▶ $Acc = \{(E_i, F_i) \mid 1 \leq i \leq k\}$
- ▶ each accepting infinite path must visit for some i
 - ▶ all states of E_i at most finitely often and
 - ▶ some state of F_i infinitely often.

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Example

Give some automata recognizing the language of formulas

- ▶ $(a \wedge \chi b) \vee aUc$



- ▶ FGa

- ▶ GFa

LTL Model Checking – ω -Automata (1.)

Deterministic Rabin automaton (DRA): $(Q, \Sigma, \delta, q_0, Acc)$

- ▶ a DFA with a different acceptance condition,
- ▶ $Acc = \{(E_i, F_i) \mid 1 \leq i \leq k\}$
- ▶ each accepting infinite path must visit for some i
 - ▶ all states of E_i at most finitely often and
 - ▶ some state of F_i infinitely often.

Example

Give some automata recognizing the language of formulas

- ▶ $(a \wedge \mathcal{X} b) \vee aUc$
- ▶ FGa
- ▶ GFa

Lemma (Vardi&Wolper'86, Safra'88)

For any LTL formula Ψ there is a DRA A recognizing $Paths(\Psi)$ with $|A| \in 2^{\mathcal{O}(|\Psi|)}$.



LTL Model Checking – Product DTMC (2.)

For a labelled DTMC $\mathcal{D} = (S, P, \pi_0, L)$ and a DRA $A = (Q, 2^{Ap}, \delta, q_0, \{(E_i, F_i) \mid 1 \leq i \leq k\})$ we define

1. a DTMC $\mathcal{D} \times A = (S \times Q, P', \pi'_0)$:

- ▶ $P'((s, q), (s', q')) = P(s, s')$ if $\delta(q, L(s')) = q'$ and 0, otherwise;
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2. $\{(E'_i, F'_i) \mid 1 \leq i \leq k\}$ where for each i :
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Lemma

The construction preserves probability of accepting as

$$P_s^{\mathcal{D}}(\text{Lang}(A)) = P_{(s, q_s)}^{\mathcal{D} \times A}(\omega \mid \exists i : \inf(\omega) \cap E'_i = \emptyset, \inf(\omega) \cap F'_i \neq \emptyset)$$

where $\inf(\omega)$ is the set of states visited in ω infinitely often.

Proof sketch.

We have a one-to-one correspondence between executions of \mathcal{D} and $\mathcal{D} \times A$ (as A is deterministic), mapping $\text{Lang}(A)$ to $\{\dots\}$, and preserving probabilities.

LTL Model Checking – Computing Acceptance Pr. (3.)

How to check the probability of accepting in $\mathcal{D} \times A$?

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Identify the BSCCs $(C_j)_j$ of $\mathcal{D} \times A$ that for some $1 \leq i \leq k$,

1. contain no state from E'_i and
2. contain some state from F'_i .

Lemma

$$P_{(s, q_s)}^{\mathcal{D} \times A}(\{\omega \mid \exists i : \inf(\omega) \cap E'_i = \emptyset, \inf(\omega) \cap F'_i \neq \emptyset\}) = P_{(s, q_s)}^{\mathcal{D} \times A}(\Diamond \bigcup_j C_j).$$

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Proof sketch.

- ▶ Note that some BSCC of each finite DTMC is reached with probability 1 (*short paths with prob. bounded from below*), 
- ▶ Rabin acceptance condition does not depend on any finite prefix of the infinite word,
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Corollary

$$P_s^{\mathcal{D}}(\text{Lang}(A)) = P_{(s, q_s)}^{\mathcal{D} \times A}(\Diamond \bigcup_j C_j).$$



LTL Model Checking - Algorithm - Complexity

Doubly exponential in Ψ and polynomial in \mathcal{D}
(for the algorithm presented here):

1. $|A|$ and hence also $|\mathcal{D} \times A|$ is of size $2^{2^{\mathcal{O}(|\Psi|)}}$
2. BSCC computation: Tarjan algorithm - linear in $|\mathcal{D} \times A|$
(number of states + transitions)
3. Unbounded reachability: system of linear equations ($\leq |\mathcal{D} \times A|$):
 - ▶ exact solution: \approx cubic in the size of the system
 - ▶ approximative solution: efficient in practice

