Solution Sheet n°5

Solution of exercise 1: We suppose we are given a reasonable coding in the alphabet $\{0,1\}$ of the Turing machines with $\Sigma = \{0,1\}$ and $\Gamma = \{0,1,\sqcup\}$ (cf. Lecture). We consider the following languages

$$\begin{aligned}\text{Acc} &= \{\langle \mathcal{M}, w\rangle \mid \mathcal{M} \text{ is a Turing machine which accepts } w.\}\\ \text{Halt} &= \{\langle \mathcal{M}, w\rangle \mid \mathcal{M} \text{ is a Turing machine which halts on } w.\}\\ \text{Halt}_\emptyset &= \{\mathcal{M} \mid \mathcal{M} \text{ is a Turing machine which halts on the empty word.}\}\\ \text{MaxStep} &= \left\{ \langle n, S(n)\rangle \,\middle|\, \begin{array}{l} S(n) \text{ is the maximal number of steps that} \\ \text{an } n\text{-state Turing machine can do before halting} \\ \text{when starting on the empty word.}\end{array}\right\}.\end{aligned}$$

The language Acc was proved to be undecidable during the lecture. We first show that the (un)decidablity of Acc is equivalent to the (un)decidability of Halt.

First suppose that we can decide Halt. To decide Acc do as follows.

On input $(\mathcal{M}, w)$:

1. Decide whether $\mathcal{M}$ halts on w. If not, reject. Otherwise go to step 2.
2. Run $\mathcal{M}$ on w. If $\mathcal{M}$ rejects w then reject. If $\mathcal{M}$ accepts w, then accept.

Conversely suppose that we can decide Acc. We can decide Halt as follows:

On input $(\mathcal{M}, w)$:

1. Modify $\mathcal{M}$ into a machine $\widetilde{\mathcal{M}}$ by replacing each transition to the rejecting state for a transition to the accepting state.
2. Decide whether $\widetilde{\mathcal{M}}$ accepts w. If $\mathcal{M}$ accepts w, then accept. Otherwise reject.

Now for the questions of the exercise sheet, we have:

1. Suppose towards a contradiction that $\text{Halt}_\emptyset$ is Turing decidable. Then we can decide Halt as follows.

   On input $(\mathcal{M}, w)$:

   (a) Write on a second tape the code of a Turing machine $\mathcal{M}_w$ which, on the empty input, acts as follows:
      i) writes w and then goes back to the left hand end of the tape in the initial state;
      ii) then works just as $\mathcal{M}$.

   (b) Decide whether $\mathcal{M}_w$ halts or not on the empty word. If $\mathcal{M}_w$ halts on the empty word, then accept. Otherwise reject.

2. Suppose towards a contradiction that MaxStep is decidable. Then we can decide $\text{Halt}_\emptyset$ as follows:

   On input $\mathcal{M}$:

   (a) Modify $\mathcal{M}$ into a machine $\widetilde{\mathcal{M}}$ by replacing each transition to the rejecting state for a transition to the accepting state.

   (b) For n the number of states of $\widetilde{\mathcal{M}}$, compute $S(n)$. This can be achieved by testing for $i = 0$ to ∞ whether $(n, i) \in \text{MaxStep}$; since MaxStep is decidable, the computation will eventually halt.

   (c) Simulate the first steps of $\widetilde{\mathcal{M}}$ on the empty word. If a halting state is reached, accept. If no halting state is reached after $S(n)$ steps, reject.

Solution of exercise 2: This solution is based on Sipser, M. (2006). *Introduction to the Theory of Computation*, pp. 203–209.

1. **Reduction of PCP to MPCP:** Notice that there exists a match for an instance $I = \langle p_1, \ldots, p_m\rangle$ of PCP if and only if there exists a match for one of the instances $I_1, I_2, \ldots, I_n$ of MPCP where $I_i = \langle p_i, p_{i+1}, \ldots, p_n, p_0, \ldots, p_{i-1}\rangle$ for $i = 1, \ldots m$.

   Reduction of MPCP to PCP Let $\bigstar$ and $\diamond$ be symbols which do not belong to A and let us denote by $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$ the set of non empty words on a finite alphabet Σ. We define two Turing computable functions $p, s : A^+ \to (A \cup \{\bigstar\})^+$ by

$$\begin{aligned} p(a_1 a_2 \cdots a_n) &= \bigstar a_1 \bigstar a_2 \cdots \bigstar a_n \\ s(a_1 a_2 \cdots a_n) &= a_1 \bigstar a_2 \bigstar \cdots a_n \bigstar. \end{aligned}$$

Notice that for all $v, w \in A^+$, $p(vw) = p(v)p(w)$ and $s(vw) = s(v)s(w)$. Moreover for each $v \in A^+$ we have $p(v)\bigstar = \bigstar s(v)$. Now consider the instance of MPCP on the alphabet A

$$I = \left\langle \frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n} \right\rangle$$

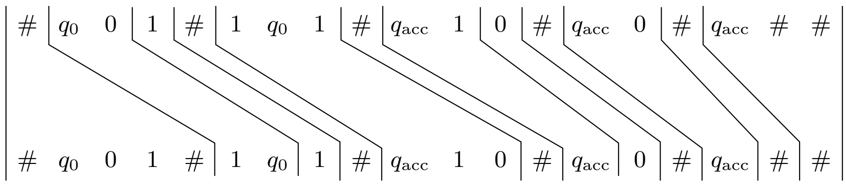
where $x_1, \ldots, x_n, y_1, \ldots, y_n \in A^+$ and a pair (x, y) is depicted as $\frac{x}{y}$. We define a corresponding instance of PCP with $m + 2$ pairs on the alphabet $A \cup \{\bigstar, \diamond\}$:

$$I' = \left\langle \frac{p(x_1)}{\bigstar s(y_1)}, \frac{p(x_1)}{s(y_1)}, \frac{p(x_2)}{s(y_2)}, \ldots, \frac{p(x_n)}{s(y_n)}, \frac{\bigstar\diamond}{\diamond} \right\rangle,$$

Notice that the only pair both whose strings start with the same symbol is $\frac{p(x_1)}{\bigstar s(y_1)}$, while the only pair for which both strings end with the same symbol is $\frac{\bigstar\diamond}{\diamond}$. Hence any match for I' is necessarily of the following form: it starts with the pair $\frac{p(x_1)}{\bigstar s(y_1)}$, followed by some pairs among the $\frac{p(x_1)}{s(y_1)}$ for $1 \leq i \leq n$ and then ends with the pair $\frac{\bigstar\diamond}{\diamond}$. We see that any match for I' therfore corresponds to a match for I which starts with the pair (x_1, y_1).
Conversely if $(i_1, \ldots, i_k)$ is a match for I, then we have $i_1 = 1$ and $x_1 x_{i_2} \ldots x_{i_k} = y_1 y_{i_2} \ldots y_{i_k}$. It follows that the following sequence constitues a match for I':

$$\frac{p(x_1)}{\bigstar s(y_1)}, \frac{p(x_{i_2})}{s(y_{i_2})}, \ldots, \frac{p(x_{i_k})}{s(y_{i_k})}, \frac{\bigstar\diamond}{\diamond}.$$

2. Let $M = (Q, \Sigma = \{0,1\}, \Gamma = \{0, 1, \sqcup\}, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$ be (the code of) a Turing machine and $w \in \Sigma^*$ be an input[1]. Assume that Q and Γ are disjoint an let $\#$ be a new symbol which does not belong to Γ. We define an instance $I_{M,w}$ of MPCP on the alphabet $A = Q \cup \Gamma \cup \{\#\}$ which is Turing computable from M, w. The idea is to define $I_{M,w}$ so that a match encodes the computation history of an accepting run of M on w. Let us first give an example. Suppose $M = (\{q_0, q_{\mathrm{acc}}\}, \{0,1\}, \{0,1,\sqcup\}, \delta, q_0, q_{\mathrm{acc}})$ with notably $\delta(q_0, 0) = (q_0, 1, R)$, $\delta(q_0, 1) = (q_{\mathrm{acc}}, 0, L)$, and $w = 01$ then $I_{M,w}$ is designed so that the following is a match:



In general, we define the first pair of $I_{M,w}$ (the starting pair for a match) to be:

r0) $\frac{\#}{\#q_0 w\#}$ if w is non empty and $\frac{\#}{\#q_0\sqcup\#}$ otherwise.

Other pairs are added to $I_{M,w}$ using the following rules:

r1) For each transition $\delta(q, a) = (r, b, R)$ with $q \neq q_{\mathrm{rej}}$, we add $\frac{qa}{br}$.

r2) For each transition $\delta(q, a) = (r, b, L)$ with $q \neq q_{\mathrm{rej}}$ and each tape symbol $c \in \Gamma$ we add $\frac{cqa}{rcb}$.

r3) For each $a \in \Gamma$ we add $\frac{a}{a}$ (this allows to copy unchanged symbols from one configuration to the next).

r4) We add $\frac{\#}{\#}$ and $\frac{\#}{\#\sqcup}$ (this allows to go from one configuration to the next by possibly adding a blank $\sqcup$, in case the head has reached the right end of the configuration).

r5) Finally for every $a \in \Gamma$ we add the pairs $\frac{a q_{\mathrm{acc}}}{q_{\mathrm{acc}}}$, $\frac{q_{\mathrm{acc}} a}{q_{\mathrm{acc}}}$ and $\frac{q_{\mathrm{acc}}\#\#}{\#}$ (this allows to conclude a match when the accepting state q_{acc} is reached).

Since applying rule r0) gives us a strictly longer string on the "bottom", and rules r1) to r4) add the same amount of symbols to both strings, a *match* must contain q_{acc} from rule r5). So it is the encoding of an accepting run of M on w.

3. Assume PCP is decidable. We construct a decider for the acceptance problem as follows.

   On input M, w:

   (a) compute the instance $I_{M,w}$ of MPCP corresponding to M, w as described in the previous item.

   (b) compute the corresponding instance $I'_{M,w}$ of PCP as in the first item.

   (c) Decide the existence of a match for $I'_{M,w}$.

[1] We assume for convenience that M never attempts to run off the left-hand of the tape while computing on w. This requires first modifying M to prevent this behaviour. Can you see how to perform this in a Turing computable way?