

Exercise Sheet n°10

Exercise 1:

This exercise illustrates the idea of the Gödel first incompleteness theorem¹ in an informal fashion.

By an *expression* we mean a non empty finite string on the alphabet

$$\{\neg, P, N, (,)\}.$$

The *norm* of an expression X is defined as the expression $X(X)$. A *sentence* is any expression of one of the following four forms (where X is any expression):

- (1) $P(X);$
- (2) $PN(X);$
- (3) $\neg P(X);$
- (4) $\neg PN(X);$

We now consider a computing machine working as a printer or an enumerator on the alphabet $\{\neg, P, N, (,)\}$. We say that an expression X is *printable* if the machine can print it, that is, if sooner or later X is printed by the machine. We define a sentence of the form $P(X)$ to be *true* if (and only if) X is printable. We define $PN(X)$ to be *true* iff the norm of X is printable. Also the symbol \neg stands for the negation, so that $\neg P(X)$ is said to be *true* iff X is not printable and $\neg PN(X)$ is said to be *true* iff the norm of X is not printable.

Now we suppose that the machine is completely accurate in that all sentences printed by the machine are true. For example, if the machine ever prints the sentence $P(X)$, then X is printable, i.e. X will be printed by the machine sooner or later. Also if $\neg PN(X)$ is printable, then $X(X)$ will not be printed by the machine at any time.

1. Suppose that for some expression X , X is printable. Does it necessarily follow that $P(X)$ is printable?
2. Is it possible that the machine actually prints all true sentences?

Now we turn to a variant of the previous problem. We consider a machine which prints out non empty finite strings on the alphabet $\{\neg, P, N, 1, 0\}$ called *expressions*. To each expressions we assign a number in binary notation which we call the *Gödel number* of the expression. We do this by assigning to each individual symbol $\neg, P, N, 1, 0$ the respective Gödel numbers 10, 100, 1000, 10000, 100000. Then, the Gödel number of an expression is just the binary expression obtained by concatenation of the Gödel numbers of its symbols. For example, the Gödel number of the expression $PN10$ is 100100010000100000.

We redefine the *norm* of an expression X as the expression obtained by concatenation of X with its Gödel number. For example the norm of $PN10$ is $PN10100100010000100000$. A *sentence* is now defined as any expression of one of the four forms:

$$PX, PNX, \neg PX, \neg PNX,$$

where X is any number in binary notation. We say that a sentence of the form PX is *true* iff X is the Gödel number of a printable expression. We call PNX *true* iff X is the Gödel number of an expression whose norm is printable. Also, $\neg PX$ is said to be *true* exactly when PX is not true (i.e. when X is not the Gödel number of a printable expression) and $\neg PNX$ is said to be *true* exactly when PNX is not true.

Again we suppose that the printable sentences are true, that is the machine never prints false sentences.

3. Find a true sentence that the machine cannot print.

¹This exercise is taken from R.M. Smullyan, *Gödel's Incompleteness Theorems*, Oxford University Press, 1992.

4. Find a sentence that is not printable and whose negation is not printable.

Exercise 2:

This exercise is a more formal version of the previous exercise².

For this exercise, we consider an ‘abstract system’ \mathcal{L} consisting of the following items.

1. A infinite **countable** set \mathcal{E} whose elements are called *expressions* of \mathcal{L} ;
2. A subset \mathcal{S} of \mathcal{E} whose elements are called the *sentences* of \mathcal{L} ;
3. A subset \mathcal{P} of \mathcal{S} whose elements are called the *provable* sentences of \mathcal{L} ;
4. A subset \mathcal{H} of \mathcal{E} whose elements are called the *predicates* of \mathcal{L} ;
5. A function that assigns to every expression E and every natural number n an expression $E(n)$. The function is required to obey the condition that for every predicate H and every natural number n , the expression $H(n)$ is a sentence.
6. A subset \mathcal{T} of \mathcal{S} whose elements are called the *true* sentences of \mathcal{L} .

We say that a predicate H is *true for* a natural number n or that n *satisfies* H , if $H(n)$ is true, i.e. $H(n) \in \mathcal{T}$. By the set *expressed* by a predicate H , we mean the set of all natural numbers n that satisfy H . Thus for any set A of natural numbers, H expresses A if and only if for every number n :

$$H(n) \in \mathcal{T} \Leftrightarrow n \in A.$$

Definition. A set A of natural numbers is *expressible* or *nameable* in \mathcal{L} if A is expressed by some predicate of \mathcal{L} .

Of course, since the set of expression is countable and the powerset of the natural numbers is not, uncountably many sets of natural numbers are not expressible in \mathcal{L} .

Definition. The abstract system \mathcal{L} is *correct* if all provable sentences are true, i.e. \mathcal{P} is a subset of \mathcal{T} .

We now consider a bijective function g from \mathcal{E} to the natural numbers. For each n , we let E_n be the unique expression such that $g(E_n) = n$. We call $g(E)$ the *Gödel number* of the expression E .

The *diagonalisation* of the expression E_n is defined as the expression $E_n(n)$ as provided by the function described by 5. If E_n is a predicate, then its diagonalisation is a sentence. This sentence is true iff the predicate E_n is true of its own Gödel number n .

We define the *diagonal function* d from the natural numbers to the natural numbers by letting for all n , $d(n) = g(E_n(n))$, that is $d(n)$ is the Gödel number of the diagonalisation of E_n .

For a set A of natural numbers, we let $d^{-1}(A)$ be the set of all natural numbers n such that $d(n) \in A$, i.e. the inverse image of A by the diagonal function. For any subset A of the natural numbers, A^C denote the complement of A in the set of natural numbers.

We define the set of the Gödel numbers of the provable sentences:

$$P = \{n \in \mathbb{N} \mid E_n \in \mathcal{P}\}.$$

Prove the following:

Theorem (After Gödel with shades of Tarski). *If the set $d^{-1}(P^C)$ is expressible in \mathcal{L} and \mathcal{L} is correct, then there is a true sentence of \mathcal{L} not provable in \mathcal{L} .*

A *Gödel sentence* for a set A of natural numbers is defined as a sentence S such that the following condition holds

$$S \in \mathcal{T} \quad \text{iff} \quad g(S) \in A,$$

that is S is true iff $g(S)$ belongs to A .

Prove the following lemma:

Lemma (Diagonal lemma). *For any set A of natural numbers, if $d^{-1}(A)$ is expressible in \mathcal{L} , then there is a Gödel sentence for A .*

Finally use this diagonal lemma to give another proof of the previous theorem.

²This exercise is taken from R.M. Smullyan, *Gödel's Incompleteness Theorems*, Oxford University Press, 1992.