Exercise Sheet n°6

Exercise 1:

Show that the following sets or functions are primitive recursive.

1. Any finite subset of $\mathbb{N}^p$, for $p \in \mathbb{N}$.
2. The factorial function $n \mapsto n!$.
3. The distance function $d : \mathbb{N}^2 \to \mathbb{N}$, $(m, n) \mapsto |m - n|$.
4. For any $k \in \mathbb{N}$, the function $p_k : \mathbb{N}^{k+1} \times \mathbb{N} \to \mathbb{N}$ defined by $p_k(a_0, \ldots, a_k, n) = \sum_{i=0}^{k} a_i n^i$.

Exercise 2:

The aim of this exercise is to show step by step that a certain total function is not primitive recursive while it is Turing computable. This is a function $\xi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ called the Ackermann function and defined by the following clauses

(i) $\forall x \in \mathbb{N}$, $\xi(0, x) = 2^x$;

(ii) $\forall y \in \mathbb{N}$, $\xi(y, 0) = 1$;

(iii) $\forall x, y \in \mathbb{N}$, $\xi(y + 1, x + 1) = \xi(y, \xi(y + 1, x))$.

For each $n \in \mathbb{N}$, we define the function $\xi_n : \mathbb{N} \to \mathbb{N}$ by $\xi_n(x) = \xi(n, x)$.

1. Show the following claims.
   (a) For all $n \geq 1$, the function ξ_n is defined by recursion on the constant 1 and the function ξ_{n-1};
   (b) For all $n \in \mathbb{N}$, the function ξ_n is primitive recursive;
   (c) For all $n \in \mathbb{N}$ and all $x \in \mathbb{N}$, $\xi_n(x) > x$;
   Hint: Proceed by induction on n and x simultaneously.
   (d) For all $n \in \mathbb{N}$, the function ξ_n is strictly increasing;
   (e) For all $n \geq 1$ and all $x \in \mathbb{N}$, $\xi_n(x) \geq \xi_{n-1}(x)$;
   (f) Let ξ_n^k be the k^{th} iteration of ξ_n, that is $\xi_n^0 = \mathrm{id}_{\mathbb{N}}$ and for all $k \in \mathbb{N}$, $\xi_n^{k+1} = \xi_n(\xi_n^k)$. For all m, n, k and x, $\xi_n^k(x) < \xi_n^{k+1}(x)$, $\xi_n^k(x) \geq x$, and if $m \leq n$, $\xi_m^k(x) \leq \xi_n^k(x)$.

2. For functions $g : \mathbb{N}^p \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ we say that f *dominates* g if there exists a natural number B such that for all $(n_1, \dots n_p) \in \mathbb{N}^p$

$$g(n_1, \dots n_p) \leq f(\sup(n_1, \dots n_p, B)).$$

   For each $n \in \mathbb{N}$, we let C_n be the set of all functions from $\mathbb{N}^p$, for some $p \in \mathbb{N}$, into $\mathbb{N}$ which are dominated by an iteration of ξ_n. Formally:

$$C_n = \{f : \mathbb{N}^p \to \mathbb{N} \mid p \in \mathbb{N} \text{ and } \exists k \in \mathbb{N}\ (\xi_n^k \text{ dominates } f)\}.$$

   (a) Show that the projections, the constants, the successor function and the function sup belong to C_0;

   (b) Show that for all $n \in \mathbb{N}$, C_n is closed under composition and contains ξ_n;

   (c) Show that for all $n, k, x \in \mathbb{N}$, $\xi_n^k(x) \leq \xi_{n+1}(x + k)$;

   (d) Suppose that $g : \mathbb{N}^p \to \mathbb{N}$ and $h : \mathbb{N}^{p+2} \to \mathbb{N}$ belong to C_n, for a fixed $n \in \mathbb{N}$. Show that the function f defined by recursion on g and h belongs to C_{n+1}.
   Hint: Show by induction that there is a relation of the form $f(x_1, \dots, x_p, y) \leq \xi_n^{k_1+yk_2}(\sup(x_1, \dots, x_p, y, B_1, B_2))$ *for all* $x_1, \dots, x_p, y$.

3. Show that $\bigcup_{n\in\mathbb{N}} C_n$ contains all primitive recursive functions.

4. Finally show that the Ackermann function ξ is not primitive recursive. *Hint: Towards a contradiction suppose that it is. Consider then the function* $x \mapsto \xi(x, 2x)$ *and show that there should exist* $x, n, k \in \mathbb{N}$ *such that* $\xi(x, 2x) \leq \xi_{n+1}(x + k) < \xi(x, 2x)$.

Remark. *It can be shown (not easily) that the graph* $G = \{(y, x, z) \in \mathbb{N}^3 \mid z = \xi(y, x)\}$ *is in fact primitive recursive (see for example R. Cori and D. Lascar,* Logique mathématique II*, Chapter 5, exercise 11, p. 57). The Ackerman function is thus recursive since it can be defined as*

$$\xi(y, x) = \mu z \quad (y, x, z) \in G.$$

In general, indeed, one can show (easily) that a total function is recursive if and only if its graph is recursive. Finally notice that the Ackermann function is an example of a total function whose graph is primitive recursive but which nonetheless is not primitive recursive.