Exercise Sheet n°5

Exercise 1:

We only consider Turing machines with $\Sigma = \{0, 1\}$ and $\Gamma = \{0, 1, \sqcup\}$.

1. Show that the problem of determining whether a given Turing machine M halts on the empty word is undecidable.

2. Show that the language $\{\langle n, S(n)\rangle \mid n \in \omega\}$ where for all $n \in \omega$, $S(n)$ is the maximal number of steps that an n-state Turing machine (see Exercise Sheet n°3) can do before halting when starting on the empty word, and $\langle\ ,\ \rangle : \mathbb{N}^2 \to \{0, 1\}^{<\omega}$, is undecidable.

Exercise 2:

Let A be a finite alphabet with at least two distinct symbols. The *Post Correspondence Problem* (PCP) is to decide if given a finite list

$$\langle (x_1, y_1), \ldots, (x_n, y_n)\rangle$$

of ordered pairs of non empty finite words $x_i, y_i \in A^+$ there exists a *match*, i.e. a finite sequence $(i_1, \ldots, i_m)$ of integers in $\{1, \ldots, n\}$ such that the two concatenations

$$x \coloneqq x_{i_1}{}^\frown x_{i_2}{}^\frown \cdots {}^\frown x_{i_m}$$
$$y \coloneqq y_{i_1}{}^\frown y_{i_2}{}^\frown \cdots {}^\frown y_{i_m}$$

are the same word, that is

$$x = y.$$

The decidability of the Post Correspondence Problem implies the decidability of the acceptance problem for Turing Machines. We have seen that the latter is undecidable.

1. The modified Post Correspondence problem (MPCP) is the variant of the Post Correspondence Problem where a match must start with the first pair of the list, that is requiring in the above notations that $i_1 = 1$. Show that MPCP reduces to PCP and vice versa.

2. Given a Turing machine M and an input w for M, explain how to construct an instance of MPCP whose matches witness an accepting run of M on w.

3. Explain how the undecidability of PCP can be concluded.