Solution Sheet n°12

Solution of exercise 1:

Let T be a theory on the language of arithmetic such that the set $\#T = \{\ulcorner\varphi\urcorner \mid \varphi \in T\}$ is a recursively enumerable subset of $\mathbb{N}$. If $\#T$ is empty then it is recursive. So suppose it is not empty, then by Exercise 3 of Sheet 7, there is a primitive recursive function f whose range is exactly $\#T$. For all $n \in \mathbb{N}$ we let φ_n be the unique formula in T such that $f(n) = \ulcorner\varphi_n\urcorner$. Now consider the theory

$$T' = \left\{ \bigwedge_{i=0}^{n} \varphi_i \;\middle|\; n \in \mathbb{N} \right\}$$

It is clear that T and T' are equivalent. Moreover let $g : \mathbb{N} \to \mathbb{N}$ be the function $n \mapsto \ulcorner\varphi_0 \wedge \cdots \wedge \varphi_n\urcorner$. It is defined by induction as

$$\begin{aligned} g(0) &= \ulcorner\varphi_0\urcorner = f(0) \\ g(n+1) &= \alpha_3(g(n), f(n+1), 6), \end{aligned}$$

according to the coding of formulas used in the Lecture. Since f is total recursive, so is g. Moreover g is strictly increasing, so using Exercise 4 of Sheet 7 we conclude that T' is recursive.

Solution of exercise 2:

Suppose towards a contradiction that $\#\,\mathrm{Th}_1(\mathcal{N})$ is recursively enumerable. Then by Exercise 1, there is a recursive theory T' which is equivalent to $\mathrm{Th}_1(\mathcal{N})$. Moreover for each of the finitely many formulas φ of $\mathcal{R}ob$, since $\mathcal{N} \models \varphi$ we have $T' \vdash \varphi$. Therefore $T'' = T' \cup \mathcal{R}ob$ is consistent, recursive and equivalent to $\mathrm{Th}_1(\mathcal{N})$.

In particular since $\mathrm{Th}_1(\mathcal{N})$ is complete so is T''. As a recursive, consistent and complete theory, T'' is decidable. But since T'' extends $\mathcal{R}ob$ we have a contradiction with the Undecidability Theorem. We conclude that $\#\,\mathrm{Th}_1(\mathcal{N})$ is not recursively enumerable.

Solution of exercise 3:

Proposition. *The only model (up to isomorphism) of* PA^2 *is the standard model* $(\mathbb{N}, 0, S, +, \times, \leq)$.

Proof. Suppose that a $\mathcal{L}_{\text{Arithm}}$-structure $\mathcal{M}$ models the second order theory PA^2. Then in particular $\mathcal{M}$ models the first order theory $\mathcal{P}_0$. One easily verifies that the substructure of $\mathcal{M}$ whose domain is

$$\mathbb{N}_{\mathcal{M}} = \left\{ a \in |\mathcal{M}| \;\middle|\; \begin{array}{l} \text{there exists } n \in \mathbb{N} \text{ such that} \\ a \text{ is the interpretation of } n \text{ in } \mathcal{M} \end{array} \right\}$$

is isomorphic to $(\mathbb{N}, 0, S, +, \times, \leq)$, via $n \mapsto n^{\mathcal{M}}$.

Now, by hypothesis, $\mathcal{M}$ satisfies the induction principle consisting in the second order formula (IP). So from the fact that $\mathbb{N}_{\mathcal{M}}$ contains the interpretation of 0 and is closed under the interpretation of the symbol of function S, we can conclude that $\mathbb{N}_{\mathcal{M}}$ is the whole domain of $\mathcal{M}$. Consequently, $\mathcal{M}$ is isomorphic to $(\mathbb{N}, 0, S, +, \times, \leq)$. $\square$

Theorem. *There is no deductive system $\vdash$ for second order logic with the standard semantic satisfying the three desired attributes*

(Soundness) *Every provable formula is valid, i.e. for any sentence φ, if $\vdash \varphi$ then $\mathcal{M} \models \varphi$ for any structure (or model) $\mathcal{M}$;*

(Completeness) *Every valid formula is provable, i.e. for any sentence φ, if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$, then $\vdash \varphi$.*

(Effectiveness) *The set of provable formulas is recursively enumerable, i.e. the set $\{\ulcorner \varphi \urcorner \mid \vdash \varphi\}$ is recursively enumerable.*

Proof. First observe that PA^2 consists of a finite set of second order formulas. The second order theory PA^2 is thus equivalent to a unique second order formula P, the conjunction of formulas of PA^2.

Next, by the previous Proposition, the unique model of P is the standard model $(\mathbb{N}, 0, S, +, \times, \leq)$. Hence for any closed formula φ of second order arithmetic,

$$(\mathbb{N}, 0, S, +, \times, \leq) \models \varphi \quad \text{iff} \quad P \to \varphi \text{ is valid.}$$

Now suppose that there exist a deductive system $\vdash$ for second order logic with standard semantic satisfying the three conditions above. By soundness and completeness of $\vdash$, the above equivalence yields that for any closed formula φ of second order arithmetic,

$$(\mathbb{N}, 0, S, +, \times, \leq) \models \varphi \quad \text{iff} \quad \vdash P \to \varphi.$$

By the effectiveness assumption on $\vdash$, it follows that the of set

$$\#\mathrm{Th}_2(\mathbb{N}, 0, S, +, \times, \leq)$$

of codes of closed formulas of second order logic arithmetic satisfied in $(\mathbb{N}, 0, S, +, \times, \leq)$ is recursively enumerable. But this implies that the first order theory of arithmetic $\mathrm{Th}_1(\mathbb{N}, 0, S, +, \times, \leq)$ is recursively enumerable, a contradiction with Exercise 2. We must conclude that such a deductive system cannot exist. $\square$