Exercise Sheet n°9

Let $\mathcal{L}_\mathcal{A}$ be the language of arithmetic consisting of $\{0, S, +, \cdot\}$.

Exercise 1:

Using the compactness theorem for first order logic, prove that there exists a model of Robinson Arithmetic $\mathcal{R}ob$ which is not isomorphic to the standard model $\mathbb{N}$.

Exercise 2:

Let $\mathrm{Th}(\mathbb{N})$ be the theory consisting of all closed first order formulas of $\mathcal{L}_\mathcal{A}$ satisfied by the standard model $\mathbb{N}$.

$$\mathrm{Th}(\mathbb{N}) = \{\varphi \mid \varphi \text{ is a closed } \mathcal{L}_\mathcal{A}\text{-formula and } \mathbb{N} \models \varphi\}$$

1. Using the compactness theorem for first order logic and the downward Löwenheim-Skolem theorem, show that there exist exactly $2^{\aleph_0}$ countable models of $\mathrm{Th}(\mathbb{N})$ which are pairwise non isomorphic.

   Hint: First show that there exist at most $2^{\aleph_0}$ countable $\mathcal{L}_\mathcal{A}$-structure up to ismorphism. Second show that for any set (finite or infinite) P of prime numbers there exists a model $\mathcal{M}_P$ of $\mathrm{Th}(\mathbb{N})$ in which there is an element which divisible by exactly all "prime numbers in P" and only by these prime. Show also that if $\mathfrak{M}$ and $\mathfrak{M}'$ are two countable isomorphic models of $\mathrm{Th}(\mathbb{N})$ then there exists $a \in \mathfrak{M}$ whose set of prime divisors (in $\mathfrak{M}$) is exactly P if and only if there exists $b \in \mathfrak{M}'$ whose set of prime divisors (in $\mathfrak{M}'$) is exactly P. Conclude.

2. Conclude that there are exactly $2^{\aleph_0}$ pairwise non isomorphic countable models of the theory of first order Robinson Arithmetic $\mathcal{R}ob$.

Exercise 3: This exercise is taken from *Logique mathématique, vol. 2*, Cori, R. and Lascar, D., 1993, Masson.

Let X be a non empty set and f be a function from $X \cdot X$ to X. We consider the $\mathcal{L}_\mathcal{A}$-structure $\mathfrak{M}$ whose domain is $M = \mathbb{N} \cup (X \cdot \mathbb{Z})$ and where the symbols $S, +, \cdot$ are interpreted as the functions $S, +,$ and $\cdot$ defined as follows:

- $\mathfrak{M}$ is an extension of $\mathbb{N}$, in particular $0^{\mathfrak{M}} = 0 \in \mathbb{N}$;

- if $a = (x, n) \in M \backslash \mathbb{N}$, then $S(a) = (x, n+1)$;

- if $a = (x, n) \in M \backslash \mathbb{N}$ and $m \in \mathbb{N}$, then $a + m = m + a = (x, n+m)$;

- if $a = (x, n)$ and $b = (y, m)$ belong to $M \backslash \mathbb{N}$, then $(x, n) + (y, m) = (x, n+m)$;

- if $a = (x, n) \in M \backslash \mathbb{N}$ and $m \in \mathbb{N}$, then $(x, n) \cdot m = (x, n \cdot m)$ if $m \neq 0$, and $(x, n) \cdot 0 = 0$;

- if $a = (x, n) \in M \backslash \mathbb{N}$ and $m \in \mathbb{N}$, then $m \cdot (x, n) = (x, m \cdot n)$;

- if $a = (x, n)$ and $b = (y, m)$ belong to $M \backslash \mathbb{N}$, then $(x, n) \cdot (y, m) = (f(x, y), n \cdot m)$.

1. Show that $\mathfrak{M}$ is a model of the theory $\mathcal{R}ob$ of Robinson arithmetic.

2. Show that the following formulas are not provable from $\mathcal{R}ob$:

    (a) $\forall v_0 \forall v_1 (v_0 + v_1 = v_1 + v_0)$;
    (b) $\forall v_0 \forall v_1 \forall v_2 (v_0 \cdot (v_1 \cdot v_2) = (v_0 \cdot v_1) \cdot v_2)$;
    (c) $\forall v_0 \forall v_1 ((v_0 \leq v_1 \wedge v_1 \leq v_0) \Rightarrow v_0 = v_1)$;
    (d) $\forall v_0 (0 \cdot v_0 = 0)$.

3. Construct a model of $\mathcal{R}ob$ in which the addition is not associative.