To do this, we employ the proof given for Exercise 26 which uses only the fixed point theorems and can therefore be applied to the family ψ.

(d) We will use the functions δ and γ mentioned in part (c). We are going to construct two sequences of functions f_n and g_n for $n \geq -1$ that will be approximations for the functions ε and ε^{-1}, respectively, that we are trying to construct. More precisely, we will notice, once the construction is completed, that, for $n \in \mathbb{N}$,

$$f_n(p) = \varepsilon(p) \quad \text{and} \quad g_n(p) = \varepsilon^{-1}(p) \qquad \text{if } p \leq n,$$
$$f_n(p) = g_n(p) = 0 \quad \text{if } p > n.$$

We will arrange things so that, in addition, for all p less than or equal to n, $\phi_p = \theta^1_{f_n(p)}$ and $\theta_p = \phi^1_{g_n(p)}$. These functions f_n and g_n are defined simultaneously by induction on n. For f_{-1} and g_{-1}, we set both equal to the function that is constantly equal to 0. Let us examine the case $n + 1$:

- $f_{n+1}(p) = f_n(p)$ except if $p = n + 1$;
- if there exists $a \leq n$ such that $g_n(a) = n + 1$, then $f_{n+1}(n + 1) = a$;
- otherwise, $f_{n+1}(n + 1)$ is the least integer m that does not belong to the (finite) set $\{1, 2, \ldots, n\} \cup \{f_n(0), f_n(1), \ldots, f_n(n)\}$ (this condition is to be ignored if $n = -1$) and is such that m equals $\gamma(k, \beta(n + 1))$ for some element k.

The definition of g is analogous.

- $g_{n+1}(p) = g_n(p)$ except if $p = n + 1$;
- if there exists $a \leq n + 1$ such that $f_{n+1}(a) = n + 1$, then $g_{n+1}(n + 1) = a$;
- otherwise, $g_{n+1}(n+1)$ is the least non-zero integer m that does not belong to the set $\{1, 2, \ldots, n + 1\} \cup \{g_n(0), g_n(1), \ldots, g_n(n)\}$ and is such that m equals $\delta(k, \alpha(n + 1))$ for some element k.

We leave it to the reader to verify that the functions $\lambda nx.f_n(x)$ and $\lambda nx.g_n(x)$ are recursive, as is the function $\varepsilon = \lambda x.f_x(x)$; the function $\lambda x.g_x(x)$ is the inverse of ε, which is therefore bijective and has the desired properties.

Solutions to the exercises for Chapter 6

1. (a) It suffices to verify axioms A_1, A_2, ... and A_7; this does not present any special difficulty. We will treat A_7 for the sake of example. Let a and b belong to M and let us show that

$$a \times Sb = (a \times b) + a. \tag{$*$}$$

We must distinguish several cases:

(i) a and b are both in $\mathbb{N}$; then $(*)$ is obvious since $\mathcal{M}$ is an extension of $\mathbb{N}$.

(ii) $a \in X \times \mathbb{Z}$, say $a = (x, n)$, and $b \in \mathbb{N}$; then $Sb = b + 1$, $a \times Sb = (x, n \times (b + 1))$.

If $b = 0$, $a \times Sb = (x, n) = a$ and $a \times b = 0$, so we do have $(a \times b) + a = a \times Sb$.

If $b \neq 0$, $a \times b = (x, n \times b)$ and $(a \times b) + a = (x, (n \times b) + n) = a \times Sb$.

(iii) $a \in \mathbb{N}$ and $b \in X \times \mathbb{Z}$, say $b = (y, m)$; then $Sb = (y, m + 1)$ and $a \times Sb = (y, a \times (m + 1))$. Also, $a \times b = (y, a \times m)$ and $(a \times b) + a = (y, (a + m) + a)$.

(iv) $a \in X \times \mathbb{Z}$ and $b \in X \times \mathbb{Z}$, say $a = (x, n)$ and $b = (y, m)$; then $Sb = (y, m + 1)$, $a \times Sb = (f(x, y), n \times (m + 1))$; on the other hand, $a \times b = (f(x, y), n \times m)$ and $(a \times b) + a = (f(x, y), (n + m) + n)$.

(b) We will make use of (a) to construct a model of $\mathcal{P}_0$ in which none of the given formulas is true. It is sufficient to take any X that has at least two elements, for example $X = \mathbb{N}$, and to take, for f, any non-associative function, for example $f(x, y) = x + 2y$. In the model $\mathcal{M}$ built from this data according to (a), we have, for example,

$$(1, 1) + (2, 0) = (1, 1) \quad \text{and} \quad (2, 0) + (1, 1) = (2, 1)$$

which shows that addition is not commutative, and

$$((1, 1) \times (2, 2)) \times (3, 3) = (5, 2) \times (3, 3) = (11, 6), \qquad \text{and}$$
$$(1, 1) \times ((2, 2) \times (3, 3)) = (1, 1) \times (8, 6) = (17, 6)$$

which shows that multiplication is not associative. For the third formula, we see, for example, that $(1, 0) \leq (1, 1)$ [because $(1, 1) + (1, 0) = (1, 1)$] and $(1, 1) \leq (1, 0)$ [because $(1, -1) + (1, 1) = (1, 0)$]. The fourth formula is not satisfied because, for example, $0 \times (1, 0) = (1, 0)$.

(c) In the models we have just constructed, addition is associative. We can use the same idea to show that the associativity of addition does not follow from $\mathcal{P}_0$. Here is a model of $\mathcal{P}_0$, among many others, in which addition is not associative. The base set is $\mathbb{N} \cup (\mathbb{N} \times \mathbb{Z})$ (so it is an extension of $\mathbb{N}$) and $\underline{S}$, $\underline{+}$, and $\underline{\times}$ are interpreted by

$$\begin{aligned}
S(n, a) &= (n, a + 1);\\
(n, a) + m &= (n, a + m) = m + (n, a);\\
(n, a) + (m, b) &= (n + 2m, a + b) \quad \text{if } n \neq m;\\
(n, a) + (n, b) &= (n, a + b);\\
(n, a) \times m &= (n, am) = m \times (n, a) \quad \text{if } m \neq 0;\\
(n, a) \times 0 &= 0 \times (n, a) = 0;\\
(n, a) \times (m, b) &= (2nb, ab).
\end{aligned}$$

Here, once more, it is not difficult to show that the seven axioms of $\mathcal{P}_0$ hold but that, for example,

$$((1,0)+(2,0))+(3,0)=(11,0);$$
$$(1,0)+((2,0)+(3,0))=(17,0).$$

2. (a) It is clear that the relation $\approx$ is symmetric; it is reflexive because of axiom A_4. Let us prove it is transitive. If x, y, and z are elements of $\mathcal{M}$ and if there exist integers n, m, p, and q such that

$$\mathcal{M} \models x \,\underline{+}\, \underline{n} \simeq y \,\underline{+}\, \underline{m} \quad \text{and} \quad \mathcal{M} \models y \,\underline{+}\, \underline{p} \simeq z \,\underline{+}\, \underline{q},$$

then, because addition is associative and commutative in any model of $\mathcal{P}$,

$$\mathcal{M} \models x \,\underline{+}\, \underline{n+p} \simeq z \,\underline{+}\, \underline{m+q}.$$

(b) By hypothesis, we have integers n, m, p, and q such that

$$\mathcal{M} \models a \,\underline{+}\, \underline{n} \simeq a' \,\underline{+}\, \underline{m} \quad \text{and} \quad \mathcal{M} \models b \,\underline{+}\, \underline{p} \simeq b' \,\underline{+}\, \underline{q}$$

and, because addition is associative and commutative in any model of $\mathcal{P}$,

$$\mathcal{M} \models (a \,\underline{+}\, b) \,\underline{+}\, \underline{n+p} \;\simeq\; (a' \,\underline{+}\, b') \,\underline{+}\, \underline{m+q}.$$

(c) Reflexivity is clear. Let us prove transitivity: so suppose x, y, and z are in E and that xRy and yRz. Thus there exist a in x, b and b' in y, and c in z such that $\mathcal{M} \models a \leq b \wedge b' \leq c$; also, there exists n in $\mathbb{N}$ such that

$$\mathcal{M} \models b \leq b' + n.$$

It follows that

$$\mathcal{M} \models a \leq c + n,$$

and hence that xRz because $c+n$ is also in z.

Let us now prove that R is antisymmetric: we assume there are points a and a' in $x \in E$ and b and b' in $y \in E$ such that

$$\mathcal{M} \models a \leq b \quad \text{and} \quad \mathcal{M} \models b' \leq a'.$$

We must show that $x = y$. The hypotheses translate as follows: there exist u and v in $\mathcal{M}$ and integers n, m, p, and q such that

$$\mathcal{M} \models a \,\underline{+}\, u \simeq b; \qquad \mathcal{M} \models b' \,\underline{+}\, v \simeq a';$$
$$\mathcal{M} \models a \,\underline{+}\, \underline{n} \simeq a' \,\underline{+}\, \underline{m}; \qquad \mathcal{M} \models b \,\underline{+}\, \underline{p} \simeq b' \,\underline{+}\, \underline{q}.$$

All this, together with the associativity and commutativity of addition, yields

$$\mathcal{M} \models a \,\underline{+}\, u \,\underline{+}\, v \,\underline{+}\, \underline{p+m} \simeq a \,\underline{+}\, \underline{n+q}.$$