Chapter 4

Representing Functions

4.1 Robinson Arithmetic

We start by describing a first order theory called *Robinson Arithmetic*.

Definition 1.1: Robinson Arithmetic

The signature of the language is $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ where

- 0 is a constant symbol,
- S is a unary function symbol[a],
- $+$ and $\cdot$ are binary function symbols[b].

The axioms are

axiom 1. $\forall x\ Sx \neq 0$

axiom 2. $\forall x\ \exists y\ (x \neq 0 \rightarrow Sy = x)$

axiom 3. $\forall x\ \forall y\ (Sx = Sy \rightarrow x = y)$

axiom 4. $\forall x\ x{+}0 = x$

axiom 5. $\forall x\ \forall y\ \Big(x{+}Sy = S(x{+}y)\Big)$

axiom 6. $\forall x\ x{\cdot}0 = 0$

axiom 7. $\forall x\ \forall y\ \Big(x{\cdot}Sy = (x{\cdot}y){+}x\Big)$

[a] for any terms of $\mathcal{L}_{\mathcal{A}}$ t, we use the notation St instead of $S(t)$.

[b] for any terms of $\mathcal{L}_{\mathcal{A}}$ t_0, t_1, we use the notation $t_0{+}t_1$ (respectively $t_0{\cdot}t_1$) instead of $+(t_0, t_1)$ (respectively $\cdot(t_0, t_1)$).

Example 1.1

The *standard model* of Robinson Arithmetic is

$$\langle \mathbb{N}, 0, S, +, \cdot \rangle$$

where S is the successor function, $+$ is the usual addition, and $\cdot$ is the customary multiplication.
By abuse of notation we identify $\mathbb{N}$ with the model $\langle \mathbb{N}, 0, S, +, \cdot \rangle$. So that, for instance, we will write

$$\mathbb{N} \models \varphi$$

instead of the correct notation

$$\langle \mathbb{N}, 0, S, +, \cdot \rangle \models \varphi.$$

Example 1.2

A very simple non standard model of Robinson arithmetic $\mathcal{M}$ in which

- $S^{\mathcal{M}}$ admits a fixed point.
- $\cdot^{\mathcal{M}}$ is not commutative.

$$\mathcal{M} = \langle \mathbb{N} \cup \{a\}, 0^{\mathcal{M}}, S^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}} \rangle$$

Where $0^{\mathcal{M}} = 0$ and the operations $S^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}$ are defined the usual way on the integers. i.e

- $S^{\mathcal{M}} \upharpoonright \mathbb{N} = S$
- $+^{\mathcal{M}} \upharpoonright \mathbb{N} = +$
- $\cdot^{\mathcal{M}} \upharpoonright \mathbb{N} = \cdot$

And when the unique non standard integer a is involved:

- $S^{\mathcal{M}} a = a$
- $\alpha \cdot^{\mathcal{M}} a = a$ (any $\alpha \in \mathbb{N} \cup \{a\}$)
- $a +^{\mathcal{M}} \alpha = \alpha +^{\mathcal{M}} a = a$ (any $\alpha \in \mathbb{N} \cup \{a\}$)
- $a \cdot^{\mathcal{M}} 0 = 0$
- $a \cdot^{\mathcal{M}} \alpha = a$ (any $\alpha \in \big(\mathbb{N} \smallsetminus \{0\}\big) \cup \{a\}$)

We verify that

$$\mathcal{M} \models \mathcal{R}ob.$$

axiom 1. since the only non standard integer verifies $S^{\mathcal{M}}a = a$ we have

$$\mathcal{M} \models \forall x \; Sx \neq 0$$

axiom 2. since every standard integer different from 0 has a predecessor, and $S^{\mathcal{M}}a = a$, we have

$$\mathcal{M} \models \forall x \; \exists y \; (x \neq 0 \to Sy = x)$$

axiom 3. holds for standard integers, and for every $n \in \mathbb{N}$ we have $S^{\mathcal{M}}a \neq S^{\mathcal{M}}n$, thus

$$\mathcal{M} \models \forall x \; \forall y \; (Sx = Sy \to x = y)$$

axiom 4. holds for standard integers, and we have $a +^{\mathcal{M}} 0 = a$, hence

$$\mathcal{M} \models \forall x \; x + 0 = x$$

axiom 5. if $k, n \in \mathbb{N}$, then $k +^{\mathcal{M}} S^{\mathcal{M}}n = S^{\mathcal{M}}(k +^{\mathcal{M}} n)$ holds. If $\alpha \in \mathbb{N} \cup \{a\}$, we have

- $a +^{\mathcal{M}} S^{\mathcal{M}}\alpha = a$
- $S^{\mathcal{M}}(a +^{\mathcal{M}} \alpha) = S^{\mathcal{M}}a = a$
- $\alpha +^{\mathcal{M}} S^{\mathcal{M}}a = \alpha +^{\mathcal{M}} a = a$
- $S^{\mathcal{M}}(\alpha +^{\mathcal{M}} a) = S^{\mathcal{M}}a = a$

therefore we have

$$\mathcal{M} \models \forall x \; \forall y \; \Big(x + Sy = S(x + y)\Big).$$

axiom 6. if $\alpha \in \mathbb{N} \cup \{a\}$, we have $\alpha \cdot^{\mathcal{M}} 0 = 0$. Thus

$$\mathcal{M} \models \forall x \; x \cdot 0 = 0$$

axiom 7. if $k, n \in \mathbb{N}$ and $\alpha \in \mathbb{N} \cup \{a\}$, then

- $a \cdot^{\mathcal{M}} S^{\mathcal{M}}\alpha = a$
- $\alpha \cdot^{\mathcal{M}} S^{\mathcal{M}}a = a$
- $(a \cdot^{\mathcal{M}} \alpha) +^{\mathcal{M}} a = a$
- $(\alpha \cdot^{\mathcal{M}} a) +^{\mathcal{M}} \alpha = a$
- $k \cdot^{\mathcal{M}} S^{\mathcal{M}}n = (k \cdot^{\mathcal{M}} n) +^{\mathcal{M}} k$

so we have

$$\mathcal{M} \models \forall x \; \forall y \; \Big(x \cdot Sy = (x \cdot y) + x\Big).$$

Notation 1.1

For any integer n, we write n for the $\mathcal{L}_{\mathcal{A}}$-term $\underbrace{S\ldots S}_{n}0$.

Example 1.3

Let us show that the following holds for all integers k:

$$\mathcal{R}ob. \vdash_c \forall x \left(0+x = k \longrightarrow x = k\right) \tag{4.1}$$

We will make use of the excluded middle:

$$\vdash_c \forall x \quad \left(x = 0 \ \lor \ x \neq 0\right)$$

and distinguish between the two cases.
The proof is by induction on k:

if $\mathbf{k = 0}$:

if $\mathbf{x = 0}$: since $\vdash_c 0 = 0$ holds, we obtain

$$\mathcal{R}ob. \vdash_c 0+0 = 0 \longrightarrow 0 = 0$$

if $\mathbf{x \neq 0}$: by $\boxed{2 \quad \forall x\, \exists y\, (x \neq 0 \rightarrow Sy = x)}$ it is enough to show

$$\mathcal{R}ob. \vdash_c \forall y \left(0+Sy = 0 \longrightarrow Sy = 0\right)$$

by $\boxed{5 \quad \forall x\, \forall y \left(x+Sy = S(x+y)\right)}$ this comes down to establishing

$$\mathcal{R}ob. \vdash_c \forall y \left(S(0+y) = 0 \longrightarrow Sy = 0\right)$$

which is immediate by $\boxed{1 \quad \forall x\, Sx \neq 0}$.

if $\mathbf{k = n + 1}$:

if $\mathbf{x = 0}$: we need to show

$$\mathcal{R}ob. \vdash_c 0+0 = n+1 \longrightarrow 0 = n+1.$$

By $\boxed{④ \quad \forall x\ x{+}0 = x}$ this comes down to showing

$$\mathcal{R}ob.\ \vdash_c\ 0 = n+1 \ \longrightarrow\ 0 = n+1.$$

because of $\boxed{① \quad \forall x\ Sx \neq 0}$ we have

$$\mathcal{R}ob.\ \vdash_c\ 0 \neq n+1$$

which yields the result.

if x $\neq$ 0: by $\boxed{② \quad \forall x\ \exists y\ (x \neq 0 \rightarrow Sy = x)}$ it is enough to show

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}Sy = n+1 \ \longrightarrow\ Sy = n+1\big)$$

by $\boxed{⑤ \quad \forall x\ \forall y\ \big(x{+}Sy = S(x{+}y)\big)}$ this comes down to establishing

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(S(0{+}y) = n+1 \ \longrightarrow\ Sy = n+1\big)$$

which is exactly

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(S(0{+}y) = Sn \ \longrightarrow\ Sy = Sn\big)$$

by $\boxed{③ \quad \forall x\ \forall y\ (Sx = Sy \rightarrow x = y)}$ this amounts to showing

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}y = n \ \longrightarrow\ Sy = Sn\big).$$

The induction hypothesis gives

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}y = n \ \longrightarrow\ y = n\big)$$

from where we immediately get what we want.

Example 1.4

Let us show that the following holds for all integers k:

$$\mathcal{R}ob.\ \vdash_c\ \forall x \forall y\ \big(Sy{+}x = k \ \longrightarrow\ S(y{+}x) = k\big). \tag{4.2}$$

We make use of the excluded middle:

$$\vdash_c\ \forall x\ \ \big(x = 0\ \vee\ x \neq 0\big)$$

and distinguish between the two cases.

if $\mathbf{x} = \mathbf{0}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \left(Sy{+}0 = k \longrightarrow S(y{+}0) = k\right)$$

which is immediate by $\boxed{④ \quad \forall x\ x{+}0 = x}$.

if $\mathbf{x} \neq \mathbf{0}$: it is enough to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy{+}Sz = k \longrightarrow S(y{+}Sz) = k\right).$$

The proof goes by induction on k:

if $\mathbf{k} = \mathbf{0}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy{+}Sz = 0 \longrightarrow S(y{+}Sz) = 0\right).$$

By $\boxed{⑤ \quad \forall x\ \forall y\ \left(x{+}Sy = S(x{+}y)\right)}$ this comes down to

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(S(Sy{+}z) = 0 \longrightarrow S(y{+}Sz) = 0\right)$$

which trivially holds by $\boxed{① \quad \forall x\ Sx \neq 0}$.

if $\mathbf{k} = \mathbf{n} + \mathbf{1}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy{+}Sz = n + 1 \longrightarrow S(y{+}Sz) = n + 1\right).$$

By $\boxed{⑤ \quad \forall x\ \forall y\ \left(x{+}Sy = S(x{+}y)\right)}$ this comes down to

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(S(Sy{+}z) = Sn \longrightarrow S(y{+}Sz) = n + 1\right).$$

By $\boxed{③ \quad \forall x\ \forall y\ (Sx = Sy \to x = y)}$ this amounts to proving

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy{+}z = n \longrightarrow S(y{+}Sz) = n + 1\right).$$

if $\mathbf{z} = \mathbf{0}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall y\left(Sy{+}0 = n \longrightarrow S(y{+}S0) = n + 1\right).$$

By $\boxed{④ \quad \forall x\ x{+}0 = x}$ and $\boxed{⑤ \quad \forall x\ \forall y\ \left(x{+}Sy = S(x{+}y)\right)}$ this comes down to showing

$$\mathcal{R}ob. \vdash_c \forall y\left(Sy = n \longrightarrow SSy = n + 1\right).$$

which holds by definition.

if $\mathbf{z} \neq \mathbf{0}$: what we need to prove is equivalent to

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow S(y+SSz') = n+1\right).$$

By $\boxed{③ \quad \forall x\, \forall y\, (Sx = Sy \to x = y)}$ this comes down to showing

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow y+SSz' = n\right).$$

The induction hypothesis yields

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow S(y+Sz') = n\right).$$

from where we easily get the result by $\boxed{⑤ \quad \forall x\, \forall y\, \left(x+Sy = S(x+y)\right)}$.

4.2 Representable Functions

Definition 2.1

Let $f \in \mathbb{N}^{(\mathbb{N}^n)}$ and $\varphi(x_0, x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathcal{A}}$-formula whose free variables are among $\{x_0, x_1, \ldots, x_n\}$.
$\varphi(x_0, x_1, \ldots, x_n)$ represents the function f if for all $i_1, \ldots, i_n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f(i_1, \ldots, i_n) = x_0 \longleftrightarrow \varphi(x_0, i_1, \ldots, i_n)\Big).$$

Definition 2.2

Let $A \subseteq \mathbb{N}^n$ and $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathcal{A}}$-formula whose free variables are among $\{x_1, \ldots, x_n\}$.
$\varphi(x_1, \ldots, x_n)$ represents the set A if for all $i_1, \ldots, i_n \in \mathbb{N}$ we have:

- if $(i_1, \ldots, i_n) \in A$, then $\mathcal{R}ob. \vdash_c \varphi(i_1, \ldots, i_n)$;
- if $(i_1, \ldots, i_n) \notin A$, then $\mathcal{R}ob. \vdash_c \neg\varphi(i_1, \ldots, i_n)$.

Proposition 2.1

For any $A \subseteq \mathbb{N}^n$,

A is representable if and only if χ_A is representable.

Proof of Proposition 2.1:

(⇒) If A is represented by $\varphi(x_1, \dots, x_n)$, then χ_A is represented by

$$\big(x_0 = 1 \wedge \varphi(x_1, \dots, x_n)\big) \vee \big(x_0 = 0 \wedge \neg\varphi(x_1, \dots, x_n)\big).$$

(⇐) If χ_A is represented by $\varphi(x_0, x_1, \dots, x_n)$, then A is represented by

$$\varphi(S0, x_1, \dots, x_n).$$

□

Example 2.1

The constant function $f \in \mathbb{N}^{(\mathbb{N}^n)}$ defined by $f(i_1, \dots, i_n) = k$ (any $i_1, \dots, i_n \in \mathbb{N}$) is represented by the following formula of the form $\varphi(x_0, i_1, \dots, i_n)$:

$$x_0 = k.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f(i_1, \dots, i_n) = x_0 \longleftrightarrow x_0 = k\Big).$$

which is exactly

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(k = x_0 \longleftrightarrow x_0 = k\Big).$$

Example 2.2

The projection $\pi_j^n \in \mathbb{N}^{(\mathbb{N}^n)}$is represented by the formula:

$$x_0 = i_j.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(\pi_j^n(i_1, \dots, i_n) = x_0 \longleftrightarrow x_0 = i_j\Big).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_j = x_0 \longleftrightarrow x_0 = i_j \right).$$

Example 2.3

The successor $S \in \mathbb{N}^{\mathbb{N}}$ is represented by the formula:

$$x_0 = Sx_1.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S(i) = x_0 \longleftrightarrow x_0 = Si \right).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S \overbrace{S \dots S}^{i} 0 = x_0 \longleftrightarrow x_0 = \overbrace{S \dots S}^{i+1} 0 \right).$$

Example 2.4

The addition $+ \in \mathbb{N}^{(\mathbb{N}^2)}$ is represented by the formula:

$$x_0 = x_1 + x_2.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + i_2 = x_0 \longleftrightarrow x_0 = i_1 + i_2 \right) \tag{4.3}$$

The proof is by induction on i_2:

$\mathbf{i_2 = 0}$ because of $\boxed{4 \quad \forall x \; x+0 = x}$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 = x_0 \longleftrightarrow x_0 = i_1 + 0 \right)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + 0 = x_0 \longleftrightarrow x_0 = i_1 + 0 \right).$$

$\mathbf{i_2 = i+1}$ by $\boxed{(5)\quad \forall x\ \forall y\ \big(x+Sy = S(x+y)\big)}$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(S(i_1+i) = x_0 \longleftrightarrow x_0 = i_1+Si\Big)$$

The induction hypothesis yields

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 + i = x_0 \longleftrightarrow x_0 = i_1+i\Big)$$

hence we obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(S(i_1 + i) = x_0 \longleftrightarrow x_0 = i_1+Si\Big)$$

by the very definition of the terms involved we finally have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 + (i+1) = x_0 \longleftrightarrow x_0 = i_1+(i+1)\Big).$$

Example 2.5

The multiplication $\cdot \in \mathbb{N}^{(\mathbb{N}^2)}$ is represented by the formula:

$$x_0 = x_1 \cdot x_2.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot i_2 = x_0 \longleftrightarrow x_0 = i_1 \cdot i_2\Big). \tag{4.4}$$

The proof is by induction on i_2:

$\mathbf{i_2 = 0}$

because of $\boxed{(6)\quad \forall x\ x\cdot 0 = 0}$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(0 = x_0 \longleftrightarrow x_0 = i_1 \cdot 0\Big)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot 0 = x_0 \longleftrightarrow x_0 = i_1 \cdot 0\Big).$$

$\mathbf{i_2 = i+1}$

by ⑦ $\forall x\ \forall y\ \big(x \cdot Sy = (x \cdot y) + x\big)$ we have

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot Si = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

which is exactly

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

The induction hypothesis yields

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot i = x_0 \longleftrightarrow x_0 = i_1 \cdot i\Big)$$

so we have

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

by ⑦ $\forall x\ \forall y\ \big(x \cdot Sy = (x \cdot y) + x\big)$ and (4.3) we have

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big((i_1 \cdot i) + i_1 = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

which is exactly

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

and we finally obtain

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = i_1 \cdot (i+1)\Big).$$

Lemma 2.1

The set of representable functions is closed under composition.

Proof of Lemma 2.1:

Assume $f_1, \ldots, f_n \in \mathbb{N}^{(\mathbb{N}^p)}$ and $g \in \mathbb{N}^{(\mathbb{N}^n)}$ are represented respectively by

$$\varphi_{f_1}(x_0, x_1, \ldots, x_p), \ldots, \varphi_{f_n}(x_0, x_1, \ldots, x_p)$$

and $\varphi_g(x_0, x_1, \ldots, x_n)$. i.e., we have for all integers $i_1, \ldots, i_p, k_1, \ldots, k_n$ and $1 \leqslant j \leqslant n$:

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f_j(i_1, \ldots, i_p) = x_0 \longleftrightarrow \varphi_{f_j}(x_0, i_1, \ldots, i_p) \Big)$$

and

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(g(k_1, \ldots, k_n) = x_0 \longleftrightarrow \varphi_g(x_0, k_1, \ldots, k_n) \Big).$$

The function

$$h = g(f_1, \ldots, f_n) \in \mathbb{N}^{(\mathbb{N}^p)}$$

defined by

$$h(i_1, \ldots, i_p) = g\big(f_1(i_1, \ldots, i_p), \ldots, f_n(i_1, \ldots, i_p)\big)$$

is represented by

$$\varphi_h(x_0, x_1, \ldots, x_p) \;=\; \exists y_1\, \exists y_2\, \ldots \exists y_n \Big(\bigwedge_{1 \leqslant j \leqslant n} \varphi_{f_j}(y_j, x_1, \ldots, x_p) \;\wedge\; \varphi_g(x_0, y_1, \ldots, y_n) \Big).$$

Indeed, by the very definition of h for every $i_1, \ldots, i_p \in \mathbb{N}$ we have

$$\vdash_c \forall x_0 \Big(h(i_1, \ldots, i_p) = x_0 \;\longleftrightarrow\; \exists y_1\, \exists y_2\, \ldots \exists y_n \Big(\bigwedge\nolimits_{1 \leqslant j \leqslant n} f_j(i_1, \ldots, i_p) = y_j \;\wedge\; g(y_1, \ldots, y_n) = x_0 \Big) \Big).$$

Therefore

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(h(i_1, \ldots, i_1) = x_0 \;\longleftrightarrow\; \exists y_1\, \exists y_2\, \ldots \exists y_n \Big(\bigwedge\nolimits_{1 \leqslant j \leqslant n} \varphi_{f_j}(y_j, i_1, \ldots, i_1) \;\wedge\; \varphi_g(x_0, y_1, \ldots, y_n) \Big) \Big).$$

□

We now turn to minimisation. We need to prove that if $A \subseteq \mathbb{N}^{n+1}$ is representable and $f \in \mathbb{N}^{(\mathbb{N}^n)}$ is some **total** function defined by minimisation the following way:

$$f(i_1, \ldots, i_n) = \mu k \;\; (k, i_1, \ldots, i_n) \in A,$$

then f is representable.

This proof requires some good amount of preliminary work.

Example 2.6

We first notice that

- for all *non-zero* integer i the following holds

$$\mathcal{R}ob. \vdash_c i \neq 0. \tag{4.5}$$

To see this, let $i = j + 1$, by the very definition of the terms involved we have

$$\vdash_c \; i = Sj$$

hence by $\boxed{(1) \quad \forall x \; Sx \neq 0}$ we obtain

$$\mathcal{R}ob. \; \vdash_c \; Sj \neq 0$$

which gives the result.

- for all integers i, j such that $i \neq j$ the following holds

$$\mathcal{R}ob. \; \vdash_c \; i \neq j. \tag{4.6}$$

the proof is by induction on $\min\{i, j\}$:

$\min\{\mathbf{i}, \mathbf{j}\} = \mathbf{0}$: this is case $\boxed{(4.5) \quad \mathcal{R}ob. \; \vdash_c \; i \neq 0 \quad \textit{for all } i \in \mathbb{N}, \quad i \neq 0}$.

$\min\{\mathbf{i}, \mathbf{j}\} > \mathbf{0}$: set $k + 1 = i$ and $n + 1 = j$.

By $\boxed{(3) \quad \forall x \; \forall y \; (Sx = Sy \to x = y)}$ we have

$$\mathcal{R}ob. \; \vdash_c \; \forall x \; \forall y \; (x \neq y \to Sx \neq Sy).$$

so that we easily obtain

$$\mathcal{R}ob. \; \vdash_c \; k \neq n \to Sk \neq Sn$$

which is precisely

$$\mathcal{R}ob. \; \vdash_c \; k \neq n \to k + 1 \neq n + 1$$

i.e.,

$$\mathcal{R}ob. \; \vdash_c \; k \neq n \to i \neq j.$$

By induction hypothesis we have

$$\mathcal{R}ob. \; \vdash_c \; k \neq n,$$

therefore by *modus ponens* we finally get

$$\mathcal{R}ob. \; \vdash_c \; i \neq j.$$

- The following holds

$$\mathcal{R}ob. \; \vdash_c \; \forall x \forall y \; \big(y \neq 0 \longrightarrow x + y \neq 0\big). \tag{4.7}$$

By $\textcircled{2}\ \ \forall x\ \exists y\ (x \neq 0 \rightarrow Sy = x)$ and $\textcircled{5}\ \ \forall x\ \forall y\ \big(x{+}Sy = S(x{+}y)\big)$ and $\textcircled{1}\ \ \forall x\ Sx \neq 0$ we obtain

$$\mathcal{R}ob.\ \vdash_c\ \forall x \forall y \exists z\ \Big(y \neq 0 \longrightarrow (y = Sz\ \wedge\ x{+}Sz = S(x{+}z)\ \wedge\ S(x{+}z) \neq 0)\Big)$$

which immediately yields the result.

- The following holds

$$\mathcal{R}ob.\ \vdash_c\ \forall x \forall y\ \Big(x{+}y = 0 \longrightarrow (x = 0\ \wedge\ y = 0)\Big). \tag{4.8}$$

By previous result $\textcircled{4.7}\ \ \mathcal{R}ob.\ \vdash_c\ \forall x \forall y\ \big(y \neq 0 \longrightarrow x{+}y \neq 0\big)$ we see that

$$\mathcal{R}ob.\ \vdash_c\ \forall x \forall y\ \big(x{+}y = 0 \longrightarrow y = 0\big).$$

and by $\textcircled{4}\ \ \forall x\ x{+}0 = x$ we obtain immediately the result.

Notation 2.1

We introduce " $x \leqslant z$" to abbreviate the formula " $\exists y\ y{+}x = z$". We also introduce " $x < z$" for the formula " $\exists y\ \big(y{+}x = z\ \wedge\ x \neq z\big)$".

Example 2.7

We establish

$$\mathcal{R}ob.\ \vdash_c\ \forall x\ \ \neg x < 0 \tag{4.9}$$

We recall that $x < y$ stands for " $\exists z\ \big(z{+}x = y\ \wedge\ x \neq y\big)$".
So we need to prove

$$\mathcal{R}ob.\ \vdash_c\ \forall x\ \ \neg\exists z\ \big(z{+}x = 0\ \wedge\ x \neq 0\big)$$

which is

$$\mathcal{R}ob.\ \vdash_c\ \forall x\ \ \forall z\ \big(z{+}x \neq 0\ \vee\ x = 0\big).$$

This is logically equivalent to

$$\mathcal{R}ob.\ \vdash_c\ \forall x\ \ \forall z\ \big(z{+}x = 0\ \longrightarrow\ x = 0\big).$$

Which is also logically equivalent to 4.7 $\mathcal{R}ob. \vdash_c \forall x \forall y \ \left(y \neq 0 \longrightarrow x+y \neq 0\right)$.

Example 2.8

For all integers n the following holds:

$$\mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow \left(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\right)\right]. \tag{4.10}$$

The direction $(\longleftarrow)$

$$\mathcal{R}ob. \vdash_c \forall x \ \left(\left(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\right) \longrightarrow x \leqslant n\right)$$

First, by making use of 4 $\forall x \ x+0 = x$ and 5 $\forall x \ \forall y \ \left(x+Sy = S(x+y)\right)$, the very definition of $k = \underbrace{S\ldots S}_{k}0$ and "$x \leqslant z$":= "$\exists y \ \left(y+x = z \ \wedge \ x \neq z\right)$", it is straightforward to establish by induction on n

$$\mathcal{R}ob. \vdash_c \forall x \ \left[\left(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\right) \longrightarrow \exists y \ \left(y+x = n\right)\right].$$

So it only remains to prove the direction $(\longrightarrow)$

$$\mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longrightarrow \left(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\right)\right]$$

.

The proof is by induction on n:

$\mathbf{n = 0}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant 0 \longrightarrow x = 0\right)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x \ \left(\exists y \ y+x = 0 \longrightarrow x = 0\right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \left(\neg\exists y \ y+x = 0 \ \vee \ x = 0\right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \left(\forall y \ y+x \neq 0 \ \vee \ x = 0\right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \left(y+x \neq 0 \ \vee \ x = 0\right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \left(y+x = 0 \ \longrightarrow \ x = 0\right).$$

We easily obtain the result by $\boxed{4.8 \quad \mathcal{R}ob. \vdash_c \forall x \forall y \ \ (x+y=0 \longrightarrow (x=0 \ \wedge \ y=0))}$

$\mathbf{n = k+1}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\right).$$

which really is

$$\mathcal{R}ob. \vdash_c \forall x \ \left(\exists y \ y+x=k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\right).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \left(y+x=k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\right).$$

We make use of the excluded middle:

$$\vdash_c \forall y \ \ (y=0 \ \vee \ y \neq 0)$$

and distinguish between the two cases:

$\mathbf{y=0}$**:** by $\boxed{4.1 \quad \mathcal{R}ob. \vdash_c \forall x \ (0+x=k \ \longrightarrow \ x=k)}$ we obtain

$$\mathcal{R}ob. \vdash_c \forall x \ \ (0+x=k+1 \longrightarrow x=k+1).$$

$\mathbf{y \neq 0}$**:** by $\boxed{2 \quad \forall x \ \exists y \ (x \neq 0 \to Sy = x)}$ what we need to show comes down to

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \left(Sz+x=k+1 \longrightarrow (x=0 \vee x=S0 \vee \ \ldots \vee x=k \vee x=k+1)\right).$$

by $\boxed{4.2 \quad \mathcal{R}ob. \vdash_c \forall x \forall y \ (Sy+x=k \ \longrightarrow \ S(y+x)=k)}$ we already know that

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ (Sz+x=k \ \longrightarrow \ S(z+x)=k)$$

so that we need to prove

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \left(S(z+x)=k+1 \longrightarrow (x=0 \vee x=1 \vee \ \ldots \vee x=k \vee x=k+1)\right).$$

By $\boxed{3 \quad \forall x \ \forall y \ (Sx = Sy \to x=y)}$ we only need to prove

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \left(z+x=k \longrightarrow (x=0 \ \vee \ x=1 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\right)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(\exists z \ z{+}x = k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k \ \vee \ x = k+1\big)\Big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(x \leqslant k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k \ \vee \ x = k+1\big)\Big).$$

By induction hypothesis one has

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(x \leqslant k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k\big)\Big)$$

so that we obtain the result very easily.

So we have proved the following two statements:

(1) $\mathcal{R}ob. \vdash_c \forall x \forall y \ \Big(y = 0 \longrightarrow \big(x{+}y = k+1 \longrightarrow x = k+1\big)\Big)$ and

(2) $\mathcal{R}ob. \vdash_c \forall x \forall y \Bigg(y \neq 0 \longrightarrow \Big(x{+}y = k+1 \longrightarrow \big(x = 0 \vee x = 1 \vee \ldots \vee x = k\big)\Big)\Bigg).$

Therefore, by an immediate application of the excluded middle we have proved the result.

Example 2.9

For all integers n the following holds:

$$\mathcal{R}ob. \vdash_c \forall x \ \big(x \leqslant n \ \vee \ n \leqslant x\big). \tag{4.11}$$

What we need to show is

$$\mathcal{R}ob. \vdash_c \forall x \ \big(\exists y \ \ y{+}x = n \ \vee \ \exists y \ \ y{+}n = x\big).$$

We make use of the excluded middle:

$$\vdash_c \forall x \ \big(x = 0 \ \vee \ x \neq 0\big)$$

and distinguish between the two cases.

if $\mathbf{x} = \mathbf{0}$: we need to show

$$\mathcal{R}ob. \vdash_c \exists y \ \ y{+}0 = n \ \vee \ \exists y \ \ y{+}n = 0$$

which is immediate by $\boxed{④ \quad \forall x\ x+0 = x}$.

if $\mathbf{x} \neq 0$: what wee need to prove comes down to

$$\mathcal{R}ob. \vdash_c \forall z \ \left(\exists y \ \ y+Sz = n \ \vee \ \exists y \ \ y+n = Sz\right).$$

The proof goes by induction on n:

if $\mathbf{n} = \mathbf{0}$: we need to prove

$$\mathcal{R}ob. \vdash_c \forall z \ \left(\exists y \ \ y+Sz = 0 \ \vee \ \exists y \ \ y+0 = Sz\right).$$

By $\boxed{④ \quad \forall x\ x+0 = x}$ this comes down to

$$\mathcal{R}ob. \vdash_c \forall z \ \left(\exists y \ \ y+Sz = 0 \ \vee \ \exists y \ \ y = Sz\right)$$

which is immediate.

if $\mathbf{n} = \mathbf{k}+\mathbf{1}$: we need to prove

$$\mathcal{R}ob. \vdash_c \forall z \ \left(\exists y \ \ y+Sz = k+1 \ \vee \ \exists y \ \ y+(k+1) = Sz\right).$$

By $\boxed{⑤ \quad \forall x\ \forall y\ \left(x+Sy = S(x+y)\right)}$ and $\boxed{③ \quad \forall x\ \forall y\ (Sx = Sy \rightarrow x = y)}$ this amounts to proving

$$\mathcal{R}ob. \vdash_c \forall z \ \left(\exists y \ \ y+z = k \ \vee \ \exists y \ \ y+k = z\right)$$

which is exactly the induction hypothesis.

We have already proved the following results:

Lemma 2.2

(4.1) $\mathcal{R}ob. \vdash_c \forall x \ \left(0+x = k \ \longrightarrow \ x = k\right)$

(4.2) $\mathcal{R}ob. \vdash_c \forall x \forall y \ \left(Sy+x = k \ \longrightarrow \ S(y+x) = k\right)$

(4.3) $\mathcal{R}ob. \vdash_c \forall x_0 \ \left(i_1 + i_2 = x_0 \longleftrightarrow x_0 = i_1+i_2\right)$

(4.4) $\mathcal{R}ob. \vdash_c \forall x_0 \ \left(i_1 \cdot i_2 = x_0 \longleftrightarrow x_0 = i_1 \cdot i_2\right)$

(4.5) $\mathcal{R}ob. \vdash_c i \neq 0 \quad$ *(any $i \in \mathbb{N}$, $\ i \neq 0$)*

(4.6) $\mathcal{R}ob. \vdash_c i \neq j \quad$ *(any $i, j \in \mathbb{N}$, $\ i \neq j$)*

(4.7) $\quad \mathcal{R}ob. \vdash_c \forall x \forall y \ \left(y \neq 0 \longrightarrow x+y \neq 0\right)$

(4.8) $\quad \mathcal{R}ob. \vdash_c \forall x \forall y \ \left(x+y = 0 \longrightarrow (x = 0 \ \wedge \ y = 0)\right)$

(4.9) $\quad \mathcal{R}ob. \vdash_c \forall x \ \neg x < 0$

(4.10) $\quad \mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right)$

(4.11) $\quad \mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant n \ \vee \ n \leqslant x\right).$

Proof of Lemma 2.2:

See Examples 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9.

$\square$

Lemma 2.3

Let $A \subseteq \mathbb{N}^{n+1}$ be representable. If the following function $f \in \mathbb{N}^{(\mathbb{N}^n)}$ is total, then f is representable.

$$f(i_1, \ldots, i_n) = \mu k \ \ (k, i_1, \ldots, i_n) \in A.$$

Proof of Lemma 2.3:

Assume $\varphi(x_0, x_1, \ldots, x_n)$ represents the set A. We claim the function f is represented by the formula

$$\varphi(x_0, x_1, \ldots, x_n) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, x_1, \ldots, x_n).$$

We need to show that for all $i_1, \ldots, i_n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(i_1, \ldots, i_n) = x_0 \longleftrightarrow \left(\varphi(x_0, i_1, \ldots, i_n) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, i_1, \ldots, i_n)\right)\right).$$

(We write $\overrightarrow{i}$ for $i_1, \ldots, i_n$ and $\overrightarrow{i}$ for $i_1, \ldots, i_n$)

($\Rightarrow$) (1) by the very definition of f and the fact that $\varphi(x_0, x_1, \ldots, x_n)$ represents A we trivially have :

$$\mathcal{R}ob. \vdash_c \varphi(f(\overrightarrow{i}), \overrightarrow{i})$$

which is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) = x_0 \longrightarrow \varphi(x_0, \overrightarrow{i})\right).$$

(2) To show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) = x_0 \longrightarrow \forall y < x_0 \neg\varphi(y, \overrightarrow{i})\right)$$

we show the equivalent

$$\mathcal{R}ob. \vdash_c \forall y \ \left(y < f(\overrightarrow{i}) \longrightarrow \ \neg\varphi(y, \overrightarrow{i})\right)$$

We have two cases

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{0}$: we make use of 4.9 $\mathcal{R}ob. \vdash_c \forall x \ \ \neg x < 0$ which settles this case.

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{k+1}$: by the very definition of f and the fact that $\varphi(x_0, x_1, \ldots, x_n)$ represents A we trivially have :

$$\mathcal{R}ob. \vdash_c \neg\varphi(0, \overrightarrow{i}) \ \wedge \ \neg\varphi(S0, \overrightarrow{i}) \ \wedge \ \ldots \ \wedge \ \neg\varphi(k, \overrightarrow{i})$$

which sums up to

$$\mathcal{R}ob. \vdash_c \forall y \ \left((y = 0 \ \vee \ y = S0 \ \vee \ \ldots \ \vee \ y = k) \longrightarrow \neg\varphi(y, \overrightarrow{i})\right)$$

by 4.10 $\mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right]$ we obtain

$$\mathcal{R}ob. \vdash_c \forall y \ \ (y < k+1 \longrightarrow (y = 0 \ \vee \ y = S0 \ \vee \ \ldots \ \vee \ y = k))$$

which yields

$$\mathcal{R}ob. \vdash_c \forall y \ \left(y < k+1 \longrightarrow \neg\varphi(y, \overrightarrow{i})\right).$$

($\Leftarrow$) we need to show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \neg\varphi(y, \overrightarrow{i})\right) \longrightarrow f(\overrightarrow{i}) = x_0\right).$$

We prove the result by contraposition, which means we prove

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) \neq x_0 \longrightarrow \neg\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \neg\varphi(y, \overrightarrow{i})\right)\right).$$

By 4.11 $\mathcal{R}ob. \vdash_c \forall x \ \ (x \leqslant n \ \vee \ n \leqslant x)$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \ (x_0 \leqslant f(\overrightarrow{i}) \ \vee \ f(\overrightarrow{i}) \leqslant x_0)$$

Which is also

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(x_0 < f(\overrightarrow{i}) \ \vee \ f(\overrightarrow{i}) < x_0 \ \vee \ x_0 = f(\overrightarrow{i})\right)$$

so that we only need to prove

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\left(x_0 < f(\overrightarrow{i}) \ \vee \ f(\overrightarrow{i}) < x_0\right) \longrightarrow \neg\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i})\right)\right).$$

We will successively prove

(1) $\mathcal{R}ob. \vdash_c \forall x_0 \left(x_0 < f(\overrightarrow{i}) \ \longrightarrow \ \neg\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i})\right)\right).$

We have two cases:

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{0}$: we make use of

$$\boxed{4.9 \quad \mathcal{R}ob. \vdash_c \forall x \ \ \neg x < 0}$$

which settles this case.

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{k+1}$: by

$$\boxed{4.10 \quad \mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right]}$$

we obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \ \left(x_0 < k+1 \longrightarrow (x_0 = 0 \ \vee \ x_0 = S0 \ \vee \ \ldots \ \vee \ x_0 = k)\right)$$

and by the very definition of f:

$$\mathcal{R}ob. \vdash_c \neg\varphi(0, \overrightarrow{i}) \ \wedge \ \neg\varphi(S0, \overrightarrow{i}) \ \wedge \ \ldots \ \wedge \ \neg\varphi(k, \overrightarrow{i})$$

which gives

$$\mathcal{R}ob. \vdash_c \forall x_0 \ \left(x_0 < k+1 \longrightarrow \neg\varphi(x_0, \overrightarrow{i})\right)$$

which settles this case.

(2) $\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) < x_0 \ \longrightarrow \ \neg\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i})\right)\right).$

By the very definition of f:

$$\mathcal{R}ob. \vdash_c \varphi(f(\overrightarrow{i}), \overrightarrow{i})$$

Thus

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) < x_0 \longrightarrow \exists y < x_0\ \varphi(y, \overrightarrow{i})\right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) < x_0 \longrightarrow \neg\forall y < x_0\ \neg\varphi(y, \overrightarrow{i})\right)$$

which yields what we want.

(1) and (2) finish the proof.

$\square$

Theorem 2.1: Chinese remainder Theorem

Suppose $n_0, n_1, \ldots, n_k$ are positive integers which are pairwise co-prime. Then, for any given sequence of integers $a_0, a_1, \ldots, a_k$ there exists an integer x solving the system of simultaneous congruences

$$\begin{cases} x & \equiv & a_0 & \mod n_0 \\ x & \equiv & a_1 & \mod n_1 \\ & \vdots & & \\ x & \equiv & a_k & \mod n_k. \end{cases}$$

Proof of Theorem 2.1:

We set

$$\alpha = \prod_{i \leqslant k} n_i$$

and notice that for each $i \leqslant k$ the two integers n_i and $\frac{\alpha}{n_i}$ are co-prime. By Bézout, there exist coefficients $c_i, d_i \in \mathbb{Z}$ such that

$$c_i \cdot n_i + d_i \cdot \frac{\alpha}{n_i} = 1$$

if we set

$$e_i = d_i \cdot \frac{\alpha}{n_i}$$

we see that

$$\begin{cases} e_i & \equiv & 1 & \mod n_i & \\ e_i & \equiv & 0 & \mod n_j & \textit{(any } j \neq i\textit{)} \end{cases}$$

It follows immediately that

$$\beta = \sum_{i \leqslant k} a_i \cdot e_i$$

is a solution to the system.

$\square$

Lemma 2.4: Gödel's β-function

There exists some function $\beta \in \mathbb{N}^{(\mathbb{N}^3)}$ which is both representable and $\mathcal{P}rim.\ \mathcal{R}ec.$ such that for all $k \in \mathbb{N}$ and every sequence $n_0, n_1, \ldots, n_k$ there exist $a, b \in \mathbb{N}$ such that

$$\begin{cases} \beta(0, a, b) & = & n_0 \\ \beta(1, a, b) & = & n_1 \\ & \vdots & \\ \beta(k, a, b) & = & n_k. \end{cases}$$

Proof of Lemma 2.4:

The function is defined[a] by

$$\beta(i, a, b) = b \dot{-} \left(\left[\frac{b}{a(i+1)+1} \right] \cdot (a(i+1)+1) \right)$$

This shows that it is $\mathcal{P}rim.\ \mathcal{R}ec.$. To show that β is representable we consider the formula

$$x_0 < S(x_2 \cdot S x_1) \ \wedge\ \exists y \leqslant x_3 \quad \Big(y \cdot S(x_2 \cdot S x_1) \Big) + x_0 = x_3$$

To show that this formula represents the function β, we need to show that for all $i_1, i_2, i_3 \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \ \Big(\beta(i_1, i_2, i_3) = x_0 \longleftrightarrow x_0 < S(i_2 \cdot S i_1) \ \wedge\ \exists y \leqslant i_3 \quad \big(y \cdot S(i_2 \cdot S i_1) \big) + x_0 = i_3 \Big)$$

which is left as a tedious but straightforward exercise.

Now given $n_0, n_1, \ldots, n_k$, in order to find a and b, we consider any integer m that satisfies both

(1) $m \geqslant k+1$ (2) $m! \geqslant \max\{n_0, n_1, \ldots, n_k\}$.

We set $a = m!$ so that we make sure that $a+1, a \cdot 2+1, \ldots, a \cdot k+1, a \cdot (k+1)+1$ are co-prime. To see this, we proceed by contradiction and consider there exists some prime number p that divides both $a(i+1)+1$ and $a(j+1)+1$ for some $0 \leqslant i < j \leqslant k$. Then p also divides

$$a(j+1)+1 \;-\; \big(a(i+1)+1\big) = a(j-i) = m!(j-i)$$

Since $m > (j-i)$ holds, p divides $m!$ which contradicts p divides $m!(i+1)+1$.

The Chinese Remainder Theorem (2.1) guarantees that there exists some integer b that satisfies

$$\begin{cases} b & \equiv & n_0 & \mod a+1 \\ b & \equiv & n_1 & \mod a \cdot 2+1 \\ & \vdots & & \\ b & \equiv & n_k & \mod a \cdot (k+1)+1. \end{cases}$$

We chose m such that $a = m! \geqslant \max\{n_0, n_1, \ldots, n_k\}$ in order to insure $n_i < a(i+1)+1$ for every integer $i \leqslant k$. This makes certain that for each $i \leqslant k$ we have $\beta(i,a,b) = n_i$.

□

[a] $\beta(i,a,b)$ is the remainder of the division of b by $a(i+1)+1$.

Lemma 2.5

If both functions $g \in \mathbb{N}^{(\mathbb{N}^p)}$ and $h \in \mathbb{N}^{(\mathbb{N}^{p+2})}$ are representable, then the function $f \in \mathbb{N}^{(\mathbb{N}^{p+1})}$ defined by recursion below is also representable.

$$\begin{cases} f(\overrightarrow{x}, 0) & = & g(\overrightarrow{x}) \\ f(\overrightarrow{x}, y+1) & = & h\big(\overrightarrow{x}, y, f(\overrightarrow{x}, y)\big). \end{cases}$$

Proof of Lemma 2.5:

We let $\overrightarrow{x}$ stand for $x_1, \ldots, x_p$ and assume $g \in \mathbb{N}^{(\mathbb{N}^p)}$ and $h \in \mathbb{N}^{(\mathbb{N}^{p+2})}$ are represented respectively by $\varphi_g(x_0, \overrightarrow{x})$ and $\varphi_h(x_0, \overrightarrow{x}, x_{p+1}, x_{p+2})$.
We also consider the following formula that represents the β-function [a] :

$$\varphi(x_0,x_1,x_2,x_3) := \;\; x_0 < S(x_2 \cdot S x_1) \;\wedge\; \exists y \leqslant x_3 \quad \Big(y \cdot S(x_2 \cdot S x_1)\Big) + x_0 = x_3$$

Instead of $\varphi(x_0,x_1,x_2,x_3)$ we prefer the formula $\varphi_\beta(x_0,x_1,x_2,x_3)$ below which also obviously represents β but *in a strong way*.

$$\varphi_\beta(x_0,x_1,x_2,x_3) := \;\; \varphi(x_0,x_1,x_2,x_3) \;\wedge\; \forall y < x_0 \; \neg\varphi(y,x_1,x_2,x_3)$$

because for any integers i, k we have

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ \varphi_\beta(x_0, i, \tilde{a}, \tilde{b})\right] \longrightarrow x_0 = k\right].$$

This holds because

$$\mathcal{R}ob. \vdash_c x_0 \left[x_0 \neq k \longrightarrow \neg(x_0 \leqslant k \vee k \leqslant x_0) \ \vee \ x_0 < k \ \vee \ k < x_0\right]$$

and by 4.11 $\mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant n \ \vee \ n \leqslant x\right)$, this comes down to

$$\mathcal{R}ob. \vdash_c x_0 \left[x_0 \neq k \longrightarrow x_0 < k \ \vee \ k < x_0\right]$$

and by the definition of both φ_β and k we have

(1) $\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ x_0 < k\right] \longrightarrow \neg\varphi_\beta(x_0, i, \tilde{a}, \forall\tilde{b})\right]$

(2) $\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ k < x_0\right] \longrightarrow \neg\varphi_\beta(x_0, i, \tilde{a}, \forall\tilde{b})\right].$

which yields the following which is also logically equivalent to what we want:

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[x_0 \neq k \ \wedge \ \varphi_\beta(k, i, \tilde{a}, \tilde{b})\right] \ \longrightarrow \ \neg\varphi_\beta(x_0, i, \tilde{a}, \tilde{b})\right].$$

The formula the we choose to represent f is the following formula $\varphi_f(x_0, x_1, \ldots, x_{p+1})$. We use the notation $\overrightarrow{x} = x_1, \ldots, x_p$ so that $\varphi_f(x_0, x_1, \ldots, x_{p+1}) = \varphi_f(x_0, \overrightarrow{x}, x_{p+1}) :=$

$$\exists\tilde{a}\exists\tilde{b}\forall\tilde{i} \leqslant x_{p+1} \ \exists y \ \exists z \ \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{x}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{x}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, x_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix}.$$

In order to show that this formula $\varphi_f(x_0, \overrightarrow{x}, x_{p+1})$ represents f, we need to prove that for all integers $i_1, \ldots, i_p, i_{p+1}$ (we write $\overrightarrow{i}$ for $i_1, \ldots, i_p$) we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1} \; \exists y \; \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \longleftrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

(⇐) We first prove

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1} \; \exists y \; \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \right).$$

We consider the sequence of integers $f(\overrightarrow{i}, 0), \ldots, f(\overrightarrow{i}, i_{p+1})$. Following the proof of Lemma 2.4, we obtain two integers a and b to make the β-function work. Since the formulas $\varphi_\beta, \varphi_g, \varphi_h$ respectively represent the functions β, g, h, we have

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(g(\overrightarrow{i}), 0, a, b\big) \ \wedge \ \varphi_g\big(g(\overrightarrow{i}), \overrightarrow{i}\big)$$

together with

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(f(\overrightarrow{i}, i_{p+1}), i_{p+1}, a, b\big)$$

and for each integer $n < i_{p+1}$

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(f(\overrightarrow{i}, n), n, a, b\big) \wedge \varphi_h\Big(f(\overrightarrow{i}, n+1), \overrightarrow{i}, n, f(\overrightarrow{i}, n)\Big) \wedge \varphi_\beta\big(f(\overrightarrow{i}, n+1), Sn, a, b\big).$$

hence we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \bigwedge_{k \leqslant i_{p+1}} \left(\begin{array}{c} \mathrm{k}= 0 \longrightarrow \varphi_g(g(\overrightarrow{i}), \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(f(\overrightarrow{i}, k), k, a, b) \\ \bigwedge \\ \varphi_h\big(f(\overrightarrow{i}, k+1), \overrightarrow{i}, k, f(\overrightarrow{i}, k)\big) \\ \bigwedge \\ \varphi_\beta(f(\overrightarrow{i}, k+1), Sk, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{array} \right) \right).$$

from which we logically derive

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \bigwedge_{k \leqslant i_{p+1}} \exists y\, \exists z \begin{pmatrix} \text{k}= 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, k, a, b) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, k, y) \\ \bigwedge \\ \varphi_\beta(z, Sk, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{pmatrix} \right).$$

Furthermore, we know from

$$\boxed{4.10 \quad \mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right]}$$

that

$$\mathcal{R}ob. \vdash_c \forall \tilde{i} \ \left(\tilde{i} \leqslant i_{p+1} \longleftrightarrow \left[\ \tilde{i} = 0 \ \vee \ \tilde{i} = 1 \ \vee \ \ldots \ \vee \ \tilde{i} = i_{p+1}\right]\right).$$

Thus we obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \forall \tilde{i} \leqslant i_{p+1} \ \exists y\, \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, a, b) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{pmatrix} \right)$$

and finally

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \exists \tilde{a}\ \exists \tilde{b}\ \forall \tilde{i}\ \leqslant i_{p+1}\ \exists y\ \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \right)$$

which completes the first part of the proof.

(⇒) We need to show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1}\ \exists y\ \exists z \begin{bmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

which is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \left[\tilde{i} \leqslant i_{p+1} \longrightarrow \exists y \, \exists z \begin{bmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

By 4.10 $\mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right]$ this is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \left[\left(\bigvee_{k \leqslant i_{p+1}} \tilde{i} = k \right) \rightarrow \exists y \exists z \begin{bmatrix} \tilde{i} = 0 \rightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \rightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

which is equivalent to [b]

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \left[\bigwedge_{k \leqslant i_{p+1}} \left(\tilde{i} = k \to \exists y \exists z \begin{bmatrix} \tilde{i} = 0 \to \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right) \right] \to f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

which again is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \left[\bigwedge_{k \leqslant i_{p+1}} \exists y \, \exists z \begin{bmatrix} \mathrm{k} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, k, y) \\ \bigwedge \\ \varphi_\beta(z, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

which is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \underbrace{\ldots \exists y_k \exists z_k \ldots}_{k \leqslant i_{p+1}} \left[\bigwedge_{k \leqslant i_{p+1}} \left[\begin{array}{c} \text{k} = 0 \rightarrow \varphi_g(y_k, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{array} \right] \right] \rightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

and also to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \underbrace{\ldots \exists y_k \exists z_k \ldots}_{k \leqslant i_{p+1}} \left[\bigwedge_{k \leqslant i_{p+1}} \left[\begin{array}{c} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{array} \right] \right] \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

and also to

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \underbrace{\ldots \forall y_k \forall z_k \ldots}_{k \leqslant i_{p+1}} \left(\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

Finally, making use of the following three facts:

(1) φ_β represents β *in a strong way* since we also have for all integers k, n

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \forall x_0 \left[[\varphi_\beta(n, k, \tilde{a}, \tilde{b}) \wedge \varphi_\beta(x_0, k, \tilde{a}, \tilde{b})] \longrightarrow x_0 = n \right]$$

(2) φ_g represents g

(3) φ_h represents h

At last, by induction on i_{p+1} we show

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \underbrace{\ldots \forall y_k \forall z_k \ldots}_{k \leqslant i_{p+1}} \left(\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \end{bmatrix} \rightarrow \bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} y_0 = g(\overrightarrow{i}) \\ \bigwedge \\ y_k = f(\overrightarrow{i}, k) \\ \bigwedge \\ z_k = f(\overrightarrow{i}, k+1) \end{bmatrix} \right).$$

$\mathbf{i_{p+1} = 0}$**:** we only need to prove

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall y_0\forall z_0 \left(\begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_0, 0, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_0, \overrightarrow{i}, 0, y_0) \\ \bigwedge \\ \varphi_\beta(z_0, S0, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow \begin{bmatrix} y_0 = g(\overrightarrow{i}) \\ \bigwedge \\ y_0 = f(\overrightarrow{i}, 0) \\ \bigwedge \\ z_0 = f(\overrightarrow{i}, 1) \end{bmatrix} \right)$$

which directly follows from the fact that φ_g and φ_h represent respectively g and h.

$\mathbf{i_{p+1} = n + 1}$**:** we assume

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\underbrace{\ldots\forall y_k\forall z_k\ldots}_{k\leqslant n} \left(\bigwedge_{k\leqslant n} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow \bigwedge_{k\leqslant n} \begin{bmatrix} y_0 = g(\overrightarrow{i}) \\ \bigwedge \\ y_k = f(\overrightarrow{i}, k) \\ \bigwedge \\ z_k = f(\overrightarrow{i}, k+1) \end{bmatrix} \right).$$

We only need to show

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \forall z_n \forall y_{n+1} \forall z_{n+1} \left(\begin{bmatrix} z_n = f(\overrightarrow{i}, n+1) \\ \bigwedge \\ \varphi_\beta(z_n, n+1, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(y_{n+1}, n+1, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_{n+1}, \overrightarrow{i}, n+1, y_{n+1}) \end{bmatrix} \longrightarrow \begin{bmatrix} y_{n+1} = f(\overrightarrow{i}, n+1) \\ \bigwedge \\ z_{n+1} = f(\overrightarrow{i}, n+2) \end{bmatrix} \right).$$

which holds because:

(1) since φ_β represents β *in a strong way* we have

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \forall y_{n+1} \Big[\big[\varphi_\beta(f(\overrightarrow{i}, n+1), n+1, \tilde{a}, \tilde{b}) \wedge \varphi_\beta(y_{n+1}, n+1, \tilde{a}, \tilde{b}) \big] \rightarrow y_{n+1} = f(\overrightarrow{i}, n+1) \Big]$$

(2) since φ_h represents h we have

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \forall y_{n+1} \forall z_{n+1} \Big[\big[\varphi_h(z_{n+1}, \overrightarrow{i}, n+1, y_{n+1}) \wedge y_{n+1} = f(\overrightarrow{i}, n+1) \big] \rightarrow z_{n+1} = f(\overrightarrow{i}, n+2) \Big]$$

To sum up things, we have obtained

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \underbrace{\ldots \forall y_k \forall z_k \ldots}_{k \leqslant i_{p+1}} \left(\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow \begin{bmatrix} y_{i_{p+1}} = f(\overrightarrow{i}, i_{p+1}) \\ \bigwedge \\ \varphi_\beta(y_{i_{p+1}}, i_{p+1}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right).$$

Once again, since φ_β *strongly* represents β we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \left[\left[\varphi_\beta(f(\overrightarrow{i}, i_{p+1}), i_{p+1}, \tilde{a}, \tilde{b}) \ \wedge \ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \right] \longrightarrow x_0 = f(\overrightarrow{i}, n+1) \right]$$

which finishes the proof.

□

[a] see the proof of Lemma 2.4.

[b] we have $(A \vee B) \longrightarrow C \equiv \neg(A \vee B) \vee C \equiv (\neg A \wedge \neg B) \vee C \equiv (\neg A \vee C) \wedge (\neg B \vee C) \equiv (A \longrightarrow C) \wedge (B \longrightarrow C)$.

Theorem 2.2

All total recursive functions are representable.

Proof of Theorem 2.2:

An immediate consequence of Examples 2.1, 2.2 and 2.3 and Lemmas 2.1 , 2.3 and 2.5.

□