Solution Sheet n°7

Solution of exercise 1:

1. A seen during the lecture the relation

$$R_{\exists}(\vec{n}, z) \quad \longleftrightarrow \quad \exists i \leq z \quad R(\vec{n}, i)$$

is primitive recursive as long as $R \subseteq \mathbb{N}^p$ is. Now if $h : \mathbb{N}^p \to \mathbb{N}$ is primitive recursive the characteristic function of $R^h_{\exists}$ is primitive recursive since it can be defined by

$$\chi_{R^h_{\exists}} = \chi_{R_{\exists}}(\mathsf{proj}_{p,1}, \ldots, \mathsf{proj}_{p,p}, h(\mathsf{proj}_{p,1}, \ldots, \mathsf{proj}_{p,p})).$$

2. First we observe that if $f : \mathbb{N}^{p+1} \to \mathbb{N}$ is primitive recursive, then so is the function $\sum_f(\vec{n}, y) = \sum_{i<y} f(\vec{n}, i)$. Indeed it is obtained by recursion on primitive recursive functions:

$$\sum_f(\vec{n}, 0) = 0$$
$$\sum_f(\vec{n}, y+1) = f(\vec{n}, y) + \sum_f(\vec{n}, y).$$

Now let $f : \mathbb{N}^p \to \mathbb{N}$ be defined by

$$f(\vec{n}, i) = \begin{cases} 1 & \text{if } \forall j \leq i \ \neg R(\vec{n}, j), \\ 0 & \text{if } \exists j \leq i \ R(\vec{n}, j). \end{cases}$$

Since R and $\neg R$ are assumed to be primitive recursive, $\forall j \leq i \ \neg R(\vec{n}, j)$ and $\exists j \leq i \ R(\vec{n}, j)$ are primitive recursive too. Hence f is primitive recursive as it is defined by constants on primitive recursive sets (seen during the lecture). Therefore $\mu m < h(\vec{n}) \quad R(\vec{n}, m)$ is primitive recursive as it is equal to $\sum_f(\vec{n}, h(\vec{n}))$.

Solution of exercise 2: For $A \subseteq \mathbb{N}$:

1. $\to$ 2. If A is recursively enumerable, then as seen during the lecture there is a primitive recursive relation $B \subseteq \mathbb{N}^2$ such that $A = \{m \mid \exists n \ (m, n) \in B\}$. Now since A is non empty fix some $k \in A$. It is easy to see that A is the range of the primitive recursive function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$\begin{aligned} g(x) &= \begin{cases} \beta^1_2(x) & \text{if } (\beta^1_2(x), \beta^2_2(x)) \in B \\ k & \text{otherwise.} \end{cases} \\ &= \beta^1_2(x) \cdot \chi_B(\beta^1_2(x), \beta^2_2(x)) + \big(1 \dot{-} \chi_B(\beta^1_2(x), \beta^2_2(x))\big) \cdot k \end{aligned}$$

2. $\to$ 3. Any primitive function is a partial recursive function.

3. $\to$ 1. As seen during the lecture a partial function $f : \mathbb{N} \to \mathbb{N}$ is Turing computable iff $\{(x, f(x)) \mid f \text{ is defined at } x\}$ is Turing recognisable. Therefore a partial function f is recursive iff its graph is recursively enumerable. So if A is the image of a partial recursive function f then $A = \{m \mid \exists n\ f(n) = m\}$ is recursively enumerable since iti is the graph of f.

Solution of exercise 3:

1. Let $f : \mathbb{N} \to \mathbb{N}$ be total recursive and such that $n < f(n)$ for all $n \in \mathbb{N}$. By the previous exercise, the range A of f is recursively enumerable. Moreover we see that so is its complement. Indeed for all $m \in \mathbb{N}$ we have

$$m \notin A \quad \longleftrightarrow \quad \forall j < m\ f(j) \neq m.$$

2. Let $f : \mathbb{N} \to \mathbb{N}$ be total recursive and strictly increasing. We know by exercise 3 that $A = \{m \mid \exists n f(n) = m\}$ is recursively enumerable. We show its complement is also recursively enumerable. Since f is strictly increasing, for all $m \in \mathbb{N}$

$$m \notin A \quad \longleftrightarrow \quad \exists k\big(f(k) > m \text{ and } \forall i < k(f(i) \neq m)\big).$$

3. Let A be an infinite recursive subset of $\mathbb{N}$. Then we can define by induction a total recursive and strictly increasing function $g_A : \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned} g_A(0) &= \mu z\ \chi_A(z) = 1 \\ g_A(n+1) &= \mu z\,(\chi_A(z) = 1 \text{ and } z > g_A(n)). \end{aligned}$$

Clearly, the range of g_A is A. As for the second part, notice that if $f : \mathbb{N} \to \mathbb{N}$ is a total recursive strictly increasing function then $g_{f(\mathbb{N})} = f$. Thus if f is not primitive recursive, then so is g. For example, take $\xi(x, 2x)$, where ξ is the Ackerman function. In the last sheet we have seen it is recursive but non primitive recursive and it is easy to show that it is strictly increasing.