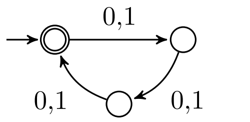
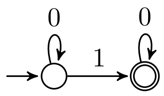
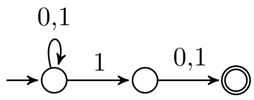
Solution Sheet n°1

Solution of exercise 1:

1. For example:

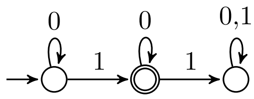
   (a) 

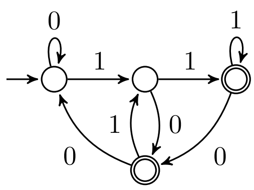
   (b) 

   (c) 

2. For example:

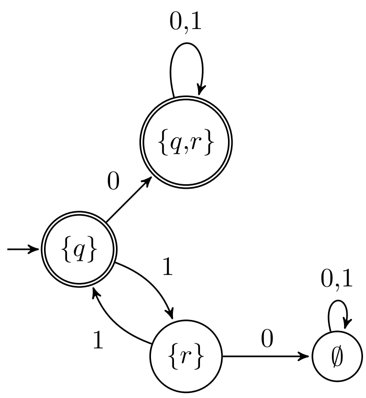
   (a) 1a is deterministic.

   (b) 

   (c) 

3. Let $N = (Q,\Sigma,\delta,q_0,F)$ be a NFA recognising some language L. We define a DFA $D = (Q',\Sigma,\delta',q_0',F')$ by letting $Q' = \mathcal{P}(Q)$, $q_0' = \{q_0\}$, $F' = \{R \subseteq Q \mid R \cap F \neq \emptyset\}$ and
$$\delta'(R,a) = \{q \in Q \mid \exists r \in R\,(r,a,q) \in \delta\} \quad \forall R \subseteq Q,\ \forall a \in \Sigma.$$
   Then D recognises L.

4. For example:



Solution of exercise 2:

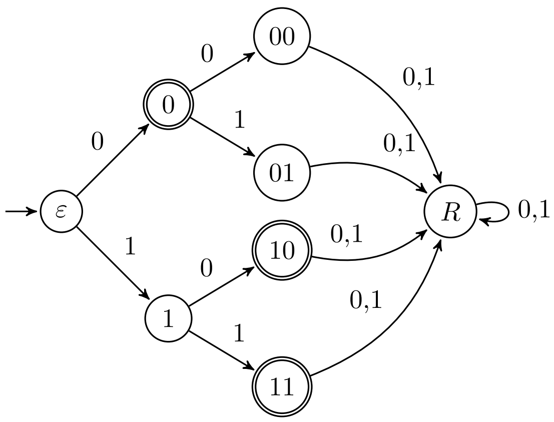
1. Let $L \subseteq \Sigma^*$ be a finite language. Let $l = \max\{length(w) \mid w \in L\}$ and let $D = (Q, \Sigma, \delta, q_0, F)$ be the DFA in which:

   - $Q = \Sigma^{\leq l} \cup \{\varepsilon, R\}$;
   - $q_0 = \varepsilon$;
   - $\delta(R, a) = R$ for all $a \in \Sigma$ and

   $$\delta(w, a) = \begin{cases} wa & \text{for } w \in \Sigma^{<l} \cup \{\varepsilon\} \\ R & \text{for } w \in \Sigma^{l} \end{cases}$$

   - $F = \{w \in \Sigma^{\leq l} \mid w \in L\}$.

   Then D recognises exactly the words of L. For example, the following automata recognises the finite language $\{0, 10{,}11\}$ on the alphabet $\{0{,}1\}$.

2. (a) A language L which is recognised by a NFA is also recognised by a DFA $D = (Q, \Sigma, \delta, q_0, F)$. Then the DFA $\tilde{D} = (Q, \Sigma, \delta, q_0, Q \setminus F)$ recognises $L^{\complement} = \Sigma^{<\omega} \setminus L$.

   (b) Until the end of the exercise, let $L, K \in \mathcal{L}(\Sigma)$ be recognised by the NFAs N_1 and N_2, respectively, and suppose that their sets of states are disjoint. To recognise the union $L \cup K$, construct a NFA N which is the union of a copy of N_1 and a copy of N_2 plus a new initial state, that ε-transitions to both the (old) initial states of N_1 and N_2.

   (c) To recognise the concatenation LK, construct a NFA N which is the union of a copy of N_1 and a copy of N_2, whose initial state is the initial state of N_1, whose accepting states are just the ones of N_2, and such that it ε-transitions from the (old) accepting states of N_1 to the (old) initial state of N_2.

   (d) To recognise the star language L^*, add a new initial state, which is also accepting, and ε-transitions from the new initial state to the old one and from the accepting states to the old initial one.

   For details and a proof of correctness see Sipser, M. (2012) *Introduction to the Theory of Computation.* Cengage Learning, pages 58-62.

3. $\mathcal{L}(\Sigma)$ is at least countably infinite, since it contains all finite languages. On the other side, since each $L \in \mathcal{L}(\Sigma)$ is recognised by at least one DFA, the cardinality of $\mathcal{L}(\Sigma)$ is less or equal than the cardinality of the set of all DFAs on the alphabet Σ. Fix $n \in \mathbb{N}$, then the number of DFAs with n states is less or equal than:

   $$|\mathbb{N}^n \times {}^{(n \times \Sigma)}n \times n \times 2^n| = \aleph_0$$

Then the cardinality of the set of all DFAs is the cardinality of a countable union of countable sets, which is countable.

Solution of exercise 3:

1. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognising L. Let $p = |Q|$ be the number of states of D. Now, suppose $w \in L$ has length m greater or equal than p. Let us write $w = w_1 \cdots w_m$, and consider the sequence $q_0 q_1 \cdots q_m$ corresponding to the computation of D on w. Since $p = |Q|$, $q_0 q_1 \cdots q_p$ must contain some state at least twice, say $q_j = q_k$, for $j < k$. Let $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$ and $z = w_{k+1} \cdots w_m$. Now, $|y| > 0$ since $j < k$ and $|xy| \leq p$ since $k \leq p$. For the remaining property notice that the computation of D on $xy^n z$ will pass through $q_0 \cdots q_{j-1}$ then travel through the loop $q_j \cdots q_k$ n times and finally go through $q_{j+1} \cdots q_m$ and accept. So $xy^n z$ belongs to L for each $n \in \mathbb{N}$.

2. (a) Assume towards contradiction that $L = \{0^n 1^n \mid n \in \omega\}$ is recognised by some DFA D and let $p \in \mathbb{N}$ be given by the pumping lemma. Since $|0^p 1^p| \geq p$ and $0^p 1^p \in L$, by the pumping lemma, there exist k,l,$m \in \mathbb{N}$ such that $l > 0$, $k + l \leq p$ and $0^k 0^{nl} 0^m 1^p \in L$, $\forall n \in \mathbb{N}$. But for each $n \geq 2$, $k + nl + m > p$, so $0^k 0^{nl} 0^m 1^p$ cannot be in L, contradiction.

   (b) Assume towards contradiction that $L = \{ww \mid w \in \{0,1\}^{<\omega}\}$ is recognised by some DFA D and let $p \in \mathbb{N}$ be given by the pumping lemma. The word $0^p 1 0^p 1 \in L$ leads to a contradiction the same way as in 2a.

   (c) Assume towards contradiction that $L = \{0^n \mid n \text{ is prime }\}$ is recognised by some DFA D and let $p \in \mathbb{N}$ be given by the pumping lemma. Let $q > p$ be a prime and notice that we can write 0^q as $0^r 0^s 0^t$ with $s > 0$ and $r + s \leq p$. By the pumping lemma $0^{r+ns+t} \in L$ for all $n \in \mathbb{N}$, that is, $r + ns + t$ is prime for each n. But let $n = r + t + 2s + 2$, then $r + ns + t = (r + 2s + t)(s + 1)$, and $s + 1 > 1$, contradiction.

3. Assume towards contradiction that the language of well-bracketed words is recognised by some DFA D and let $p \in \mathbb{N}$ be given by the pumping lemma. Since $|(^p\ )^p| \geq p$ and $(^p\ )^p \in L$, by the pumping lemma, there exist k,l,$m \in \mathbb{N}$ such that $l > 0$, $k + l \leq p$ and $(^k (^{nl}\ (^m\ (^p \in L$, $\forall n \in \mathbb{N}$. But for each $n \geq 2$, $k + nl + m > p$, so $(^k (^{nl}\ (^m\ )^p$ cannot be in L, contradiction.

4. You need some sort of memory to store the information about how many parenthesis have remained open.