Solution Sheet n°2

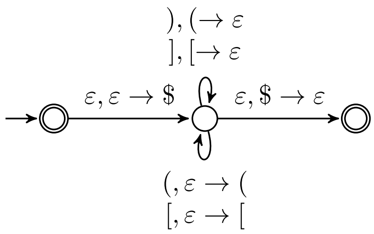
Solution of exercise 1:

1. For example, $G_1 = (V = \{S\}, \Sigma = \{(,)\}, R, S)$ with R being
$$S \to \varepsilon, \quad S \to SS, \quad S \to (S).$$
2. For example $G_2 = (V = \{E, T, F\}, \Sigma = \{(,), +, \cdot, a\}, R, E)$ where R consists of the following rules
$$E \to T,\ E \to E + T,\ T \to T \cdot F,\ T \to F,\ F \to (E),\ F \to a.$$
3. For example $G_3 = (V = \{A, F\}, \Sigma = \{\forall, \exists, (,), \wedge, \vee, \neg, =, x_1, \dots, x_n\}, R, F)$ with R contains the followings rules:
$$\begin{aligned} &F \to \forall x_i F,\ F \to \exists x_i F, \quad \text{for every } i \in \{1, \dots, n\} \\ &F \to \neg F,\ F \to (F \vee F),\ F \to (F \wedge F),\ F \to A \\ &A \to x_i = x_j \quad \text{for every } i, j \in \{1, \dots, n\}. \end{aligned}$$

Solution of exercise 2: A transition $a, b \to c$ can be used if you read "a" on the input (or whenever if $a = \varepsilon$) and b is on top of the stack (or whatever is on top of the stack if $b = \varepsilon$) and it means

- if $b = \varepsilon$, then write "c" on the top of the stack,
- if $b \neq \varepsilon$, then replace "b" by "c" (or simply erase "b" in case $c = \varepsilon$).

We can consider for example the following pushdown automaton:



Solution of exercise 3: We first observe that we can focus on even simpler grammars: A language is generated by a regular grammar if and only if it is generated by a *simply regular* grammar, namely a grammar (V, Σ, R, S) whose set R of rules satisfied
$$R \subseteq (V \times \Sigma V) \cup (V \times \Sigma) \cup (V \times \{\varepsilon\}),$$
i.e. the rules of which are all of one of the following form

- $A \to aB$ for some $A, B \in V$ and $a \in \Sigma$,
- $A \to a$ for some $A \in V$ and $a \in \Sigma$,

- $A \to \varepsilon$ for some $A \in V$.

Indeed to each rule of the form $A \to a_1 \cdots a_n B$ we can substitute the set of rules
$$A \to a_1 B_1,\ B_1 \to a_2 B_2,\ \dots,\ B_{n-1} \to a_n B,$$
where $B_1, \dots, B_{n-1}$ are new variables.

1. For $G = (V, \Sigma, R, S)$ a simply regular grammar we let $A = (Q, \Sigma, \Delta, q_0, \{q_F\})$ be the NFA defined by $Q = V \cup \{q_F\}$, $q_0 = S$ and
$$\begin{aligned}\Delta =& \{(p, a, q) \in V \times \Sigma_\varepsilon \times V \mid (p \to aq) \in R\}\\ &\cup \{(p, a, q_F) \mid\ p \in V,\ a \in \Sigma_\varepsilon \text{ and } (p \to a) \in R\},\end{aligned}$$
where $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$. Then A recognises the language generated by G.

2. For a DFA $A = (Q, \Sigma, \delta, q_0, F)$ we consider the (simply) regular grammar $G = (V, \Sigma, R, S)$ where $V = Q$, $S = q_0$ and
$$\begin{aligned}R =& \{p \to aq \mid p, q \in Q, a \in \Sigma, \text{ and } \delta(p, a) = q\}\\ &\cup \{p \to a \mid p \in Q, a \in \Sigma, \text{ and } \delta(p, a) \in F\}\\ &\big(\cup \{q_0 \to \varepsilon\}, \quad \text{if } q_0 \in F\big).\end{aligned}$$

Then the language generated by G is the language recognised by A.

Solution of exercise 4:

1. First notice that the language L_1 is generated by the context free grammar $G = (\{S, A, B\}, \{a, b, c\}, R, S)$ with R consisting in
$$S \to AB,\ A \to aA,\ A \to \varepsilon,\ B \to bBc,\ B \to \varepsilon.$$
Similarly, L_2 is context free. Now $L_1 \cap L_2 = \{a^n b^n c^n \mid n \in \omega\}$ is not context free. Hence context free languages are not closed under intersection.

2. Let $P = (Q_1, \Sigma, \Gamma_1, \Delta_1, s_1, F_1)$ be a pushdown automaton recognising C and $A = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be a DFA recognising R. We define a pushdown automaton $M = (Q, \Sigma, \Gamma, \Delta, s, F)$ by setting $Q = Q_1 \times Q_2$, $\Gamma = \Gamma_1$, $s = (s_1, s_2)$, $F = F_1 \times F_2$,
$$\begin{aligned}\Delta =& \{(((q_1, q_2), a, \beta), ((p_1, \delta(q_2, a)), \gamma)) \mid a \in \Sigma, ((q_1, a, \beta), (p_1, \gamma)) \in \Delta_1, q_2 \in Q_2\}\\ &\cup \{(((q_1, q_2), \varepsilon, \beta), ((p_1, q_2), \gamma)) \mid ((q_1, \varepsilon, \beta), (p_1, \gamma)) \in \Delta_1, q_2 \in Q_2\}\end{aligned}$$
The pushdown automaton M recognises the language $C \cap R$, which is therefore context free.

3. Suppose towards a contradiction that
$$L_3 = \big\{w \in \{a, b, c\}^* \big|\ w \text{ contains an equal number of } a\text{'s, } b\text{'s and } c\text{'s } \big\}$$
is context free. Since $a^*b^*c^*$ is regular (easy!), by the previous point it would follow that $L_3 \cap a^*b^*c^*$ is context free, a contradiction. Therefore L_3 is not context free.