

## Solution Sheet n°8

- First observe that the interpretation in  $\mathcal{N}$  of each term  $t(x_1, \dots, x_p)$  is a primitive recursive function  $h_t : \mathbb{N}^p \rightarrow \mathbb{N}$ . This is shown by induction. The interpretations of variables and constant are respectively the projection  $\text{proj}_1^1 : \mathbb{N} \rightarrow \mathbb{N}$  and the constant functions  $\text{const}_0^1, \text{const}_1^1 : \mathbb{N} \rightarrow \mathbb{N}$  which are by definition primitive recursive. Then for terms  $t$  and  $s$  whose interpretation are the primitive recursive functions  $h_t$  and  $h_s$ , the interpretation of the terms  $t+s$  and  $t \cdot s$  are respectively the primitive recursive functions  $\text{add}(h_t, h_s)$  and  $\text{mult}(h_t, h_s)$ .

Atomic formulas: Since the usual relations  $=$  and  $\leq$  on  $\mathbb{N}$  are primitive recursive binary relations (as seen during the lecture), for each couple of terms  $s$  and  $t$  the atomic formulas  $t = s$  and  $t \leq s$  define the primitive recursive relation whose characteristic functions are  $\chi_=(h_s, h_t)$  and  $\chi_{\leq}(h_s, h_t)$ , respectively. Thus sets which are arithmetically defined by atomic formulas are primitive recursive.

$\Delta_0^0$ -rudimentary formulas: Suppose now that  $\Delta_0^0$ -rudimentary formulas  $\varphi$  and  $\psi$  (arithmetically) define sets which are primitive recursive. Then the same is true of the formulas  $\neg\varphi$ ,  $\varphi \wedge \psi$ ,  $\varphi \vee \psi$  by the fact that primitive recursive sets are closed under complementation, intersection and union. Also, a set which is defined by the formulas  $\forall x < t \varphi$  or  $\exists x < t \varphi$  is primitive recursive since it is obtained by bounded quantification

$$\exists i \leq h_t(\vec{n}) \ R(\vec{n}, i) \quad \text{or} \quad \forall i \leq h_t(\vec{n}) \ R(\vec{n}, i)$$

where  $h_t$  is the interpretation of the term  $t$  and  $R$  is a relation defined by  $\varphi$ . We proved in Sheet 7 that such relations are primitive recursive when both  $h_t$  and  $R$  are.

- During the lecture we saw that a set  $B \subseteq \mathbb{N}^p$  is recursively enumerable iff there exists  $A \subseteq \mathbb{N}^{p+1}$  primitive recursive such that  $B = \{\vec{x} \in \mathbb{N}^p \mid \exists y \in \mathbb{N} (\vec{x}, y) \in A\}$ . Thus by 1. any set which is definable by a formula of the form  $\exists x \varphi(x)$  where  $\varphi$  is a  $\Delta_0^0$ -rudimentary formulas is recursively enumerable.
- This function is arithmetically defined by the  $\Delta_0^0$ -rudimentary formula:

$$\text{quot}(x_1, x_2, y) : (x_2 = 0 \wedge y = 0) \vee \exists u < x_2 (x_1 = y \cdot x_2 + u).$$

- This function is arithmetically defined by the  $\Delta_0^0$ -rudimentary formula:

$$\text{rest}(x_1, x_2, y) : (x_2 = 0 \wedge y = x_1) \vee (y < x_2 \wedge \exists u \leq x_1 (x_1 = u \cdot x_2 + y)).$$

- This follows from 4. and the fact that  $(t \cdot (1+i)) + 1$  is a term of the language  $\mathcal{A}$ .
- Let  $k \in \mathbb{N}$  and  $(n_0, \dots, n_k) \in \mathbb{N}^{k+1}$ . We set  $m = \max\{n_0, \dots, n_k, k\}$  and  $t = m!$ . We show that for  $i$  and  $j$  with  $0 \leq i < j \leq k$  the natural numbers  $t(i+1)+1$  and  $t(j+1)+1$  are coprime (i.e. their greatest common divisor

is 1). To this end, suppose that a natural number  $r$  divides both  $t(i+1)+1$  and  $t(j+1)+1$ . Then it must divide their difference  $t(j-i)$ . Thus  $r$  divides  $j-i$  or  $t = m!$ . Since  $j-i \leq m$ , trivially  $j-i$  divides  $t = m!$ , necessarily  $r$  divides  $t$ . But  $t$  and  $t(i+1)+1$  are coprime since, if  $t = qr$  and  $qr(i+1)+1 = qr'$  then  $q(r(i+1)+1 - r') = -1$  so  $q = \pm 1$ . Consequently,  $r$  must equal 1 and  $t(i+1)+1$  and  $t(j+1)+1$  are coprime as desired.

We have thus obtained that the sequence of natural numbers  $t+1, 2t+1, \dots, t(k+1)+1$  is pairwise coprime and thus by the Chinese remainder theorem there exists a natural number  $s$  such that for all  $i$  with  $0 \leq i \leq k$  we have  $a_i = \text{rest}(s, t(i+1)+1) = \beta(s, t, i)$ .

We can thus use Gödel's  $\beta$  function to code sequences of natural numbers of arbitrary length using just two natural numbers,  $s$  and  $t$  in the above formulation.

7. Basic recursive function: the constant functions, the projections and the successor function are respectively defined by the  $\Delta_0^0$ -rudimentary formulas:

$$\begin{aligned} \text{const}_n^p(x_1, \dots, x_p, y) &: y = n \\ \text{proj}_j^p(x_1, \dots, x_p, y) &: y = x_j \\ \text{succ}(x, y) &: y = x + 1. \end{aligned}$$

Composition: Now suppose that  $g : \mathbb{N}^m \rightarrow \mathbb{N}$  and  $f_1, \dots, f_m : \mathbb{N}^p \rightarrow \mathbb{N}$  are (partial) recursive functions defined by generalised existential  $\Delta_0^0$ -rudimentary formulas (gen- $\exists$ - $\Delta_0^0$ -rud)  $\varphi_g(x_1, \dots, x_m, y)$  and  $\varphi_{f_i}(x_1, \dots, x_p, y)$  respectively. Then the partial recursive function  $g(f_1, \dots, f_m)$  is defined by the gen- $\exists$ - $\Delta_0^0$ -rud formula:

$$\psi(x_1, \dots, x_p, y) : \exists y_1 \exists y_2 \cdots \exists y_m \left( \bigwedge_{i=1}^m \varphi_{f_i}(x_1, \dots, x_p, y_i) \wedge \varphi_g(y_1, \dots, y_m, y) \right).$$

Induction: Suppose that  $g : \mathbb{N}^p \rightarrow \mathbb{N}$  and  $h : \mathbb{N}^{p+2} \rightarrow \mathbb{N}$  are (partial) recursive functions defined by gen- $\exists$ - $\Delta_0^0$ -rud formulas  $\varphi_g(\vec{x}, y)$  and  $\varphi_h(\vec{x}, y)$  respectively. The function  $f$  defined by induction from  $g$  and  $h$  is defined by the gen- $\exists$ - $\Delta_0^0$ -rud formula:

$$\begin{aligned} \psi(\vec{x}, y, z) : \exists s \exists t \Big( &\exists y_0 (\beta(s, t, 0, y_0) \wedge \varphi_g(\vec{x}, y_0)) \\ &\wedge \\ &\forall w < y \exists y_1 \exists y_2 (\beta(s, t, w, y_1) \wedge \beta(s, t, w+1, y_2) \wedge \varphi_h(\vec{x}, w, y_1, y_2)) \\ &\wedge \\ &\beta(s, t, y, z) \Big). \end{aligned}$$

What we have done is find  $s, t$  which code the sequence

$$(f(\vec{x}, 0), f(\vec{x}, 1), \dots, f(\vec{x}, y)) = (g(\vec{x}), h(\vec{x}, 0, g(\vec{x})), \dots, h(\vec{x}, y-1, f(\vec{x}, y-1)))$$

and finally check that  $z$  is equal to the last element of the sequence, that is  $z = f(\vec{x}, y)$ .

Minimisation: Suppose that  $g : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$  is a (partial) recursive function defined by a gen- $\exists$ - $\Delta_0^0$ -rud formula  $\varphi_g(\vec{x}, y, z)$ . The function  $f(\vec{x}) = \mu y g(\vec{x}, y) = 0$  is defined by the gen- $\exists$ - $\Delta_0^0$ -rud formula

$$\psi(\vec{x}, z) : \varphi_g(\vec{x}, z, 0) \wedge \forall y < z \exists u (\varphi_g(\vec{x}, y, u) \wedge 1 \leq u).$$

8. (a) A formula  $\varphi$  is logically equivalent to  $\exists w(w = w \wedge \varphi)$  for a variable  $w$  with no free occurrence in  $\varphi$ . Moreover if  $\varphi$  is  $\Delta_0^0$ -rud, then so is  $w = w \wedge \varphi$ .

- (b)  $\exists x \varphi(x) \wedge \exists y \psi(y)$  is arithmetically equivalent to

$$\exists w \exists x < w \exists y < w (\varphi(x) \wedge \psi(y))$$

for  $w$  with no free occurrence in  $\varphi$  and  $\psi$ . The backward direction is straightforward, while the forward one follows from the fact that in  $\mathcal{N}$  for any two natural numbers  $n$  and  $m$  there exists a natural number greater than both, that is  $\mathcal{N} \models \forall x \forall y \exists w (x < w \wedge y < w)$ . Moreover if  $\varphi$  and  $\psi$  are  $\Delta_0^0$ -rud, then so is  $\exists x < w \exists y < w (\varphi(x) \wedge \psi(y))$ .

- (c)  $\exists x \varphi(x) \vee \exists y \psi(y)$  is logically equivalent to  $\exists x (\varphi(x) \vee \psi(x))$ .

- (d)  $\forall z < t(x_1, \dots, x_p) \exists u \varphi(x_1, \dots, x_p, z, u)$  is arithmetically equivalent to

$$\exists w \forall z < t(x_1, \dots, x_p) \exists u < w \varphi(x_1, \dots, x_p, z, u).$$

where  $w$  has no free occurrence in  $\varphi$  and  $\psi$ . The backward direction is straightforward, while the forward one is based on the fact about  $\mathcal{N}$  according to which for every finitely many natural numbers  $u_0, \dots, u_{t-1}$  there exists a natural number greater than all these  $u_z$ .

- (e)  $\exists z < y \exists u \varphi(u, z)$  is logically equivalent to  $\exists u \exists z < y \varphi(u, z)$ .

- (f) For similar reasons as in (b),  $\exists u \exists v \varphi(u, v)$  is arithmetically equivalent to  $\exists w \exists u < w \exists v < w \varphi(u, v)$ .

9. First notice that atomic formulas are  $\exists \Delta_0^0$ -rud by (a) of the previous point. Next recall that gen- $\exists$ - $\Delta_0^0$ -rud formulas are built up from the atomic formulas by disjunctions, conjunctions, bounded quantifications and existential quantifications. Hence, by (b)-(f) of the previous point one can prove by induction on the height of the formulas that every gen- $\exists$ - $\Delta_0^0$ -rud is equivalent to an  $\exists \Delta_0^0$  formula.

10. By point 1. every set which is definable by a  $\exists \Delta_0^0$ -rud formula is recursively enumerable. Conversely, observe that a set is recursively enumerable iff it is the domain of a recursive function. Hence given a recursively enumerable set  $R$  there is a recursive function  $f$  whose domain is  $R$ . By 7. every recursive function is arithmetically definable by a gen- $\exists$ - $\Delta_0^0$ -rud formula  $\varphi_f(\vec{x}, y)$ . The domain  $R$  of  $f$  is thus arithmetically defined by the formula

$$\varphi_R(\vec{x}) : \exists y \varphi_f(\vec{x}, y)$$

which is arithmetically equivalent to a  $\exists \Delta_0^0$ -rud formula by 9. Consequently, every recursively enumerable set is arithmetically definable by  $\exists \Delta_0^0$ -rud formula.