Exercise Sheet n°12

Exercise 1: Let T be a recursively enumerable theory on the language of arithmetic, i.e. a set of arithmetic formulas T such that $\{\ulcorner\varphi\urcorner \mid \varphi \in T\}$ is recursively enumerable.

Show that there is a recursive theory T' that is equivalent to T, that is, $T \vdash \varphi$ if and only if $T' \vdash \varphi$ for every arithmetic formula φ.

Exercise 2: Show that the set of codes of first order arithmetical truths,

$$\#\,\mathrm{Th}_1(\mathcal{N}) = \left\{ \ulcorner\varphi\urcorner \;\middle|\; \begin{array}{l} \varphi \text{ is a closed formula of first order arithmetic} \\ \text{satisfied in the standard model } \mathcal{N}. \end{array} \right\},$$

is not recursively enumerable.

Exercise 3:

The aim of this exercise is to show that second order logic with standard semantics admits no "good" deductive system. The language of *second order logic* is defined similarly as in the case of first order logic:

- We consider an infinite countable set of first order variables $x_0, x_1, x_2, \ldots$ and for all $n > 0$ an infinite countable set of n-ary relation variables $X_0^n, X_1^n, X_2^n, \ldots$.
- As in the case of first order logic, a second order logic language is specified by a set $\mathcal{L}$ of non-logical symbols, that is symbols of constant, of function, or of relation.
- The set of terms on $\mathcal{L}$ is defined as in the case of first order logic: it is the smallest set containing the first order variables and the symbols of constant of $\mathcal{L}$ which is closed under the application of symbols of function in $\mathcal{L}$.
- The second order atomic formulas on $\mathcal{L}$ are the expressions of one of the three following forms

  $$t_1 = t_2 \qquad R(t_1, \ldots, t_n) \qquad X^n(t_1, \ldots t_n)$$

  where $t_1, \ldots, t_n$ are terms on $\mathcal{L}$, R is a symbol of n-ary relation in $\mathcal{L}$, and X^n is a n-ary relation variable.
- The set of second order formulas on $\mathcal{L}$ is the smallest set containing the atomic formulas on $\mathcal{L}$ which is closed under composition by logical connectives, quantification over first order variables and **quantification over n-ary relation variables.**

The **standard** (or **full**[1]) semantic of second order logic is very intuitive. A standard $\mathcal{L}$-structure for the language of second order is the same as a $\mathcal{L}$-structure in first order logic. The n-ary relation variables are interpreted in a $\mathcal{L}$-structure as the n-ary relations on the domain of the structure. The satisfaction by a $\mathcal{L}$-structure of a second order formula on $\mathcal{L}$ is extended in a straightforward manner from the semantic of first order logic.

We now consider the second order theory of Peano arithmetic denoted by PA^2. It consists of the first order axioms of $\mathcal{P}_0$ (that is, first order Peano arithmetics minus the induction scheme), the second order induction principle

$$\forall X^1 \big((X(0) \wedge \forall x (X(x) \to X(S(x)))) \to \forall x \; X(x) \big) \tag{IP}$$

and the first order definition of $\leq$

$$\forall x \forall y (x \leq y \leftrightarrow \exists z (x + z = y)).$$

Show the following proposition.

Proposition. *The only model (up to isomorphism) of* PA^2 *is the standard model* $(\mathbb{N}, 0, S, +, \times, \leq)$.

We now show that there is no good notion of proof for second order logic with the standard semantic. Precisely, show that

Theorem. *There is no deductive system $\vdash$ for second order logic with the standard semantic satisfying the three desired attributes*

(Soundness) *Every provable formula is valid, i.e. for any sentence φ, if $\vdash \varphi$ then $\mathcal{M} \models \varphi$ for any structure (or model) $\mathcal{M}$;*

(Completeness) *Every valid formula is provable, i.e. for any sentence φ, if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$, then $\vdash \varphi$.*

(Effectiveness) *The set of codes of provable formulas is recursively enumerable, i.e. the set $\{\ulcorner \varphi \urcorner \mid \; \vdash \varphi\}$ is recursively enumerable.*

Hint: Show that the existence of such a deductive system contradicts the fact that the set of codes of arithmetical truth is not recursively enumerable.

[1] as opposed to *Henkin semantics*