Exercise Sheet n°7

Exercise 1:

1. Let R be a primitive recursive relation on $\mathbb{N}^{p+1}$ and h be a primitive recursive function on $\mathbb{N}^p$. Show that the relations $R^h_\exists$ and $R^h_\forall$ on $\mathbb{N}^p$ defined by

$$R^h_\exists(\vec{n}) \Leftrightarrow \exists i \leq h(\vec{n}) \quad R(\vec{n}, i) \quad \text{and} \quad R^h_\forall(\vec{n}) \Leftrightarrow \forall i \leq h(\vec{n}) \quad R(\vec{n}, i)$$

   are primitive recursive.

2. For a function $h : \mathbb{N}^p \to \mathbb{N}$ and a relation R on $\mathbb{N}^{p+1}$ we let

$$\mu m < h(\vec{n}) \quad R(\vec{n}, m) = \begin{cases} \text{the smallest } m < h(\vec{n}) \text{ such that } R(\vec{n}, m) & \text{if it exists,} \\ h(\vec{n}) & \text{otherwise.} \end{cases}$$

   Show that for h and R primitive recursive, the function $\min_<(h, R)$ on $\mathbb{N}^p$ defined by $\vec{n} \mapsto \mu m < h(\vec{n}) \quad R(\vec{n}, m)$ is primitive recursive.

Exercise 2: Let $A \subseteq \mathbb{N}$ be non empty. Show that the following conditions are equivalent

1. A is recursively enumerable;
2. A is the range of a primitive recursive function;
3. A is the range of a partial recursive function.

Exercise 3:

1. Let $f : \mathbb{N} \to \mathbb{N}$ be a total unary recursive function satisfying $n < f(n)$ for all n. Show that the range of f is recursive.
2. Show that the range of a strictly increasing total unary recursive function is recursive.
3. Show that any infinite recursive $A \subseteq \mathbb{N}$ is the range of a strictly increasing recursive function. Can one prove it is actually primitive recursive?