Solution Sheet n°9

Solution of exercise 1:

Let n stand for $\overbrace{S\cdots S(0)}^{n \text{ times}}$. Consider the language $\mathcal{L}'_{\mathcal{A}} = \mathcal{L}_{\mathcal{A}} \cup \{c\}$, where c is a constant symbol and consider the theory $\mathcal{T} = \mathcal{R}ob \cup \{c \neq n \mid n \in \mathbb{N}\}$ in the language $\mathcal{L}'_{\mathcal{A}}$. Then $\mathbb{N}$ is a model of each finite subset of $\mathcal{T}$, since it is a model of each finite subset of $\mathcal{R}ob$ and, for each finite subset $A \subset \mathbb{N}$, $\mathbb{N}$ models $\mathcal{R}ob \cup \{c \neq n \mid n \in A\}$ by interpreting c as an integer greater than the maximum of A. By the compactness theorem, $\mathcal{T}$ has a model $\mathfrak{M}$. The interpretation of c in $\mathfrak{M}$ is an element which is not the successor of any integer, so $\mathfrak{M}$ is not isomorphic to $\mathbb{N}$, and neither is its restriction to $\mathcal{L}_{\mathcal{A}}$.

Solution of exercise 2:

1. We let κ be the number of countable models of $\mathrm{Th}(\mathbb{N})$ up to isomorphism. We first observe that up to ismorphism any countable $\mathcal{L}_{\mathcal{A}}$-structure consists in choosing one element of $\mathbb{N}$, a function from $\mathbb{N}$ to $\mathbb{N}$ and two functions from $\mathbb{N}^2$ into $\mathbb{N}$. Hence, $\kappa \leq 2^{\aleph_0}$.

   We next show that there are exactly $2^{\aleph_0}$ such models. Let $\mathbb{P}$ be the set of all prime numbers of $\mathbb{N}$. We denote by $x|y$ the formula $\exists z(x \cdot z = y)$. For all countable models $\mathfrak{M}$ of $\mathrm{Th}(\mathbb{N})$ and all elements a in the domain of $\mathfrak{M}$, we denote by $\mathrm{Div}_{\mathfrak{M}}(a)$ the set of prime numbers dividing a in $\mathfrak{M}$, that is

   $$\mathrm{Div}_{\mathfrak{M}}(a) = \{p \in \mathbb{P} \mid \mathfrak{M} \models p|a\}$$

   where p stands for $\overbrace{S\cdots S(0)}^{p \text{ times}}$. We then let

   $$D = \left\{ P \subset \mathbb{P} \;\middle|\; \begin{array}{l} \text{there exists a countable model } \mathfrak{M} \text{ of } \mathrm{Th}(\mathbb{N}) \\ \text{and } a \in \mathfrak{M} \text{ such that } P = \mathrm{Div}_{\mathfrak{M}}(a) \end{array} \right\}.$$

   We now show that two isomorphic countable models have the same contributions to D. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two countable models of $\mathrm{Th}(\mathbb{N})$ and f be an isomorphism form $\mathfrak{M}$ onto $\mathfrak{M}'$. Observe that the formulas $\exists! x \forall y \neg(S(y) = x)$ and $\forall y \neg(S(y) = 0)$ are satisfied in $\mathbb{N}$ and thus in $\mathfrak{M}$ and $\mathfrak{M}'$. It follows that 0 must be interpreted by the unique element of the domain not in the range of the interpretation of S. Hence necessarily f maps $0^{\mathfrak{M}}$ to $0^{\mathfrak{M}'}$. It follows by induction that f sends $n^{\mathfrak{M}}$ to $n^{\mathfrak{M}'}$ for all natural numbers n. Next, since f is an isomorphism we have that for all $a \in \mathfrak{M}$ and all $p \in \mathbb{P}$ that

   $$\mathfrak{M} \models p|a \quad \text{iff} \quad \mathfrak{M}' \models p|f(a)$$

   and thus

   $$\mathrm{Div}_{\mathfrak{M}}(a) = \mathrm{Div}_{\mathfrak{M}'}(f(a)).$$

   Consequently, since f is a bijection, the contribution of $\mathfrak{M}$ and $\mathfrak{M}'$ to D are identical. Since a countable model can contribute only to (at most) $\aleph_0$ elements of D, we have $\mathrm{Card}(D) \leq \kappa \cdot \aleph_0$.

We finally show that D is equal to the whole powerset of $\mathbb{P}$. This conludes the proof since,

$$2^{\aleph_0} = \mathrm{Card}(\mathcal{P}(\mathbb{P})) = \mathrm{Card}(D) \leq \kappa \cdot \aleph_0 = \max\{\kappa, \aleph_0\}$$

and $\kappa \leq 2^{\aleph_0}$ implies $\kappa = 2^{\aleph_0}$.

Let c be a new constant symbol and let $\mathcal{L}'_{\mathcal{A}} = \mathcal{L}_{\mathcal{A}} \cup \{c\}$ be the extended language. For each subset P of $\mathbb{P}$, we define the $\mathcal{L}'_{\mathcal{A}}$-theory

$$T_P = \mathrm{Th}(\mathbb{N}) \cup \{p|c \mid p \in P\} \cup \{\neg(p|c) \mid p \in \mathbb{P}\backslash P\},$$

For each such P, T_P is finitely satisfiable since for any finite subset F of P, we can interpret c in $\mathbb{N}$ as the product of the prime numbers in F. By the compactness theorem, T_P is satisfiable. Furthermore, by the downward Löwenheim-Skolem theorem T_P admits a countable model $\mathfrak{M}_P$. But this means that the symbol of constant c is interpreted as an $a \in \mathfrak{M}_P$ such that $\mathrm{Div}_{\mathfrak{M}_P}(a) = P$. Since $\mathfrak{M}_P$ is a countable model of $\mathrm{Th}(\mathbb{N})$, it follows that $P \in D$. Therefore $D = \mathcal{P}(\mathbb{P})$. This concludes the proof.

2. As we have seen above above, there cannot be more than $2^{\aleph_0}$ pairwise non isomorphic countable models of the language $\mathcal{L}_{\mathcal{A}}$, so in particular this holds for models of $\mathcal{R}ob$. Since $\mathcal{R}ob \subset \mathrm{Th}(\mathbb{N})$, all models of $\mathrm{Th}(\mathbb{N})$ are models of $\mathcal{R}ob$, so the cardinality of the set of countable models of $\mathcal{R}ob$ up to isomorphism is exactly $2^{\aleph_0}$.

Solution of exercise 3:

See the solution of exercise 1 of Chapter 6 p. 267 in *Logique mathématique, vol. 2*, Cori, R. and Lascar, D., 1993, Masson.