Solution Sheet n°6

Solution of exercise 1: Recall from the lecture that the following functions are primitive recursive:

$$\begin{aligned} \mathsf{add} &: (n,m) \mapsto n+m & \mathsf{diff} &: (n,m) \mapsto n \dot{-} m \\ \mathsf{prod} &: (n,m) \mapsto n \cdot m & \mathsf{succ} &: n \mapsto n+1 \\ \mathsf{n} &: m \mapsto n, \text{ for every } n \in \mathbb{N} & \mathsf{proj}_{p,i} &: (n_1, \ldots, n_p) \mapsto n_i, , 1 \leq i \leq n. \end{aligned}$$

1. For each $n \in \mathbb{N}$, the characteristic function $\chi_{\{n\}}$ of $\{n\}$ is primtive recursive since
$$\chi_{\{n\}} = \mathsf{prod}(\mathsf{diff}(\mathsf{succ}(\mathsf{proj}_{1,1}), \mathsf{n}), \mathsf{diff}(\mathsf{succ}(\mathsf{n}), \mathsf{proj}_{1,1})),$$
or, with lighter notation, $\chi_{\{n\}}(m) = ((m+1) \dot{-} n) \cdot ((n+1) \dot{-} m)$. Also the characteristic function of the empty set is just the constant $\mathsf{0}$.
Moreover for each p-tuple $\vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$, the singleton $\{\vec{n}\}$ is recursive, since its characteristic function can be expressed by
$$\chi_{\{\vec{n}\}} = \mathsf{prod}_p(\chi_{\{n_1\}}(\mathsf{proj}_{p,1}), \ldots, \chi_{\{n_p\}}(\mathsf{proj}_{p,p})),$$
where by induction we let $\mathsf{prod}_2 = \mathsf{prod}$ and define $\mathsf{prod}_p \colon \mathbb{N}^p \to \mathbb{N}$ by
$$\mathsf{prod}_p = \mathsf{prod}(\mathsf{prod}_p(\mathsf{proj}_{p,1}, \ldots, \mathsf{proj}_{p,p-1}), \mathsf{proj}_{p,p}),$$
for $p > 2$. Also the empty subset of $\mathbb{N}^p$ admits the characteristic function $\mathsf{0}(\mathsf{proj}_{p,1})$.
Finally we show that if A and B are primitive recursive subsets of $\mathbb{N}^p$, then $A \cup B$ is primtive recursive, which concludes the claim. Indeed we have
$$\chi_{A \cup B} = \mathsf{diff}(1, \mathsf{diff}(1, \mathsf{add}(\chi_A, \chi_B))).$$
2. Let $g : \mathbb{N}^0 \to \mathbb{N}$ be the constant 1 and $h : \mathbb{N}^2 \to \mathbb{N}$ be the primtive recursive function $\mathsf{prod}(\mathsf{succ}(\mathsf{proj}_{2,1}), \mathsf{proj}_{2,2})$. Then factorial is obtained by recursion on g and h as
$$\begin{aligned} f(0) &= g = 1 \\ f(y+1) &= h(y, f(y)) = \mathsf{prod}(\mathsf{succ}(y), f(y)). \end{aligned}$$
3. We have $d(x,y) = (x \dot{-} y) + (y \dot{-} x)$, that is,
$$d = \mathsf{add}(\mathsf{diff}(\mathsf{proj}_{2,1}, \mathsf{proj}_{2,2}), \mathsf{diff}(\mathsf{proj}_{2,2}, \mathsf{proj}_{2,1})).$$
4. By induction on $k \in \mathbb{N}$. Observe that for $k = 0$, $p_0 = \mathsf{proj}_{2,1}$. Now supposing by induction that p_k is primtive recursive we have
$$\begin{aligned} p_{k+1} = \mathsf{add}(p_k(\mathsf{proj}_{k+3,1}, \ldots, \mathsf{proj}_{k+3,k+1}, \mathsf{proj}_{k+3,k+3}),& \\ \mathsf{prod}(\mathsf{proj}_{k+3,k+2}, \mathsf{exp}(\mathsf{proj}_{k+3,k+3}, k+1)),& \end{aligned}$$
where $\mathsf{exp}(m,n) = m^n$ is primtive recursive since it is defined by recursion on the constant 1 and prod as
$$\begin{aligned} \mathsf{exp}(n, 0) &= 1 \\ \mathsf{exp}(n, m+1) &= \mathsf{prod}(n, \mathsf{exp}(n,m)). \end{aligned}$$

Solution of exercise 2: See for example section 2 of Chapter 5 (on Moodle) in:

R. Cori, D. Lascar, and D.H. Pelletier. *Mathematical logic: a course with exercises. Part 2 : recursion theory, Gödel's theorems, set theory, model theory.* Mathematical Logic: A Course with Exercises. Oxford University Press, 2001.