Solution Sheet n°11

Solution of exercise 1: Let

$$A = \{n \mid \varphi_n(n) = 0\} \quad \text{and} \quad B = \{n \mid \varphi_n(n) = 1\}.$$

Since $(\varphi_n)_{n\in\mathbb{N}}$ is a recursive enumeration, these two sets are recursively enumerable. Indeed there is a TM which given n and k computes $\varphi_n(k)$. Hence we can construct a TM which when given n computes $\varphi_n(n)$ and accepts whenever the computation terminates and yields 0 or 1 accordingly.

These sets are disjoint and we show they are not recursively separable. Suppose to the contrary that $X \subseteq \mathbb{N}$ is a recursive set with $A \subset X$ and $B \cap X = \emptyset$. Then its characteristic function is a total unary recursive function and hence there exists an index $x \in \mathbb{N}$ such that φ_x is this function. Now if $x \in X$ it means that $\varphi_x(x) = 1$ and so that $x \in B$ contrary to our assumption on X. But if $x \notin X$ then $\varphi_x(x) = 0$ and then $x \in A$, a contradiction again.

We must conclude that such an X does not exist.

Solution of exercise 2: Suppose to the contrary that $\mathcal{M}$ is a non standard model of PA where $\varphi(x)$ holds only at each standard element n. Then clearly both $\varphi(0)$ and $\forall x(\varphi(x) \to \varphi(x+1))$ hold in $\mathcal{M}$. By the induction scheme which holds in $\mathcal{M}$, we must then have $\forall x \varphi(x)$. Therefore we obtain that $\varphi(x)$ also holds at every non standard element of $\mathcal{M}$, a contradiction. Hence $\varphi(x)$ must hold at some non standard element of $\mathcal{M}$.

Solution of exercise 3: Fix a Δ_0 formula $A(x, y)$.

1. We assume *without proof* that the following simple arithmetical truth is provable in PA, for every Δ_0 formula $A(x, y)$ and any $n \in \mathbb{N}$:

$$\forall b\ \exists a\ \forall u{<}n\ \big(\exists k{<}b\ A(k, u)\ \leftrightarrow\ \exists z\ (\pi(u){\cdot}z = a)\big).$$

2. If the previous formula is provable in PA for all $n \in \mathbb{N}$, then it holds in particular in $\mathcal{M}$. Now by Theorem 2 we obtain that the formula holds for some non standard element of $\mathcal{M}$ as desired.

3. By the previous step we obtain the existence of a non standard $e \in |\mathcal{M}|$ such that for all b there exists a and

$$\mathcal{M} \models \forall u \oslash [e]\ \big(\exists k \oslash [b]\ A(k, u)\ \leftrightarrow\ \exists z(\pi(u) \otimes z = [a])\big).$$

But any non standard element stands above every standard element, that is $\mathcal{M} \models n \oslash [e]$ for every $n \in \mathbb{N}$. We therefore have in particular for all $n \in \mathbb{N}$

$$\mathcal{M} \models \exists k \oslash [b]\ A(k, n)\ \leftrightarrow\ \exists z(\pi(n) \otimes z = [a]).$$

Solution of exercise 4: By Exercise 1 there exist recursively enumerable sets A and B which are recursively inseparable.

1. By a result of Exercise Sheet 6, any recursively enumerable set is definable by a $\exists\Delta_0$ formula. Thus there exist two Δ_0 formulas $A(x, y)$ and $B(x, y)$ such that

$$A = \{m \mid \exists n\ A(m, n)\} \quad \text{and} \quad B = \{m \mid \exists n\ B(m, n)\}.$$

2. Since A and B are disjoint, for any n,

$$\mathcal{N} \models \forall x < n\ \forall y < n\ \forall z < n\ \neg(A(x, y) \wedge B(x, z)).$$

But this is still a Δ_0 formula for every n. Hence by Theorem 4 it is provable in PA and thus holds in $\mathcal{M}$: for every n

$$\mathcal{M} \models \forall x \oslash n\ \forall y \oslash n\ \forall z \oslash n\ \neg(A(x, y) \wedge B(x, z)).$$

Now by Theorem 2 there is a non standard $e \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x \oslash [e]\ \forall y \oslash [e]\ \forall z \oslash [e]\ \neg(A(x, y) \wedge B(x, z)). \qquad (\bigstar)$$

Now consider the set

$$X = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists y \oslash [e]\ A(n, y)\}.$$

3. $A \subseteq X$: let $m \in A$ then for some n the statement $\mathcal{N} \models A(m, n)$ holds. Thus by Theorem 4 we have $\mathcal{M} \models A(m, n)$. Therefore by the fact that any standard element is less than a non standard element we get $\mathcal{M} \models \exists y \oslash [e]\ A(m, y)$.

4. $B \cap X = \emptyset$: Let $m \in B$. Then for some n we have $\mathcal{N} \models B(m, n)$. Arguing similarly as above we get $\mathcal{M} \models \exists z \oslash [e]\ B(m, z)$. By ($\bigstar$) we have

$$\mathcal{M} \not\models \exists y \oslash [e]\ A(m, y),$$

so $n \notin X$.

5. By the two previous points, the set X separates A and B and thus cannot be recursive by our hypothesis on A and B. By Theorem 3 there exists $b \in \mathcal{M}$ such that

$$X = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists z(\pi(n) \otimes z = [b])\}$$

so X is canonically coded in $\mathcal{M}$ by the element b.

Solution of exercise 5:

Since the Euclidean division holds in $\mathcal{M}$, for every n there exist a unique c and $r \oslash \pi(n)^{\mathcal{M}}$ such that $(\pi(n)^{\mathcal{M}} \otimes c) \oplus r = a$. Moreover these c and r are the unique elements such that

$$\underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{r\text{ times}} = a.$$

Therefore there exists a unique $c \in |\mathcal{M}|$ such that one (and exactly one) of the following holds:

$$\begin{aligned} \underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} &= a \\ \underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} \oplus 1 &= a \\ &\vdots \\ \underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{\pi(n)-1\text{ times}} &= a. \end{aligned}$$

Assume that $\oplus$ is recursive. We can define a decision procedure for

$$X = \{n \mid \mathcal{M} \models \exists z(\pi(n) \times z = [a])\}$$

as follows

On input n:

- compute $\pi(n)$;
- for each $c = 0, 1, 2, 3, \ldots$

  (S0) compute $\underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}}$. If it equals a, then **accept**, else

  (S1) compute $\underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} \oplus 1$. If it equals a, then **reject**, else

  $\vdots$

  (S($\pi(n) - 1$)) compute $\underbrace{c \oplus \cdots \oplus c}_{\pi(n)\text{ times}} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{\pi(n)-1\text{ times}}$. If it is equal to a then **reject**, else

- increase c by one.