Exercise Sheet n°3

Exercise 1: We consider a Turing machine

$$M = (\{q_0, q_1, q_2, q_{acc}, q_{rej}\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, q_0, q_{acc}, q_{rej}).$$

1. Describe the language of M if δ consists of the following set of rules[1]:
   (a) $\delta(q_0, 0) = (q_1, 1, R)$; $\delta(q_1, 1) = (q_0, 0, R)$; $\delta(q_1, \sqcup) = (q_{acc}, \sqcup, R)$.
   (b) $\delta(q_0, 0) = (q_1, 1, R)$; $\delta(q_1, 1) = (q_2, 0, L)$; $\delta(q_2, 1) = (q_0, 1, R)$; $\delta(q_1, \sqcup) = (q_{acc}, \sqcup, R)$.
2. Give a transition function δ such that M computes $n \mapsto n + 1$ in binary. We take the convention that for example $0101 \sqcup \sqcup \sqcup \cdots$ represents the decimal number 10.

Exercise 2: Show that for every bi-infinite tape Turing machine M there exists an equivalent Turing machine $\overline{M}$, that is such that $\mathcal{L}(\overline{M}) = \mathcal{L}(M)$.

Exercise 3: In this exercise we consider the following variant of a Turing machine. Let an n-state Turing machine be

- A Turing machine with n states labelled A, B, C... plus a Halt state H. A is the starting state;
- with a single **bi-infinite** tape;
- with $\{0, 1\}$ as tape alphabet, where 0 also serves as the blank symbol;
- whose head must move to the left or to the right at each step of the configuration;
- whose transition relation is deterministic but entering the Halt state takes immediate effect, that is, no transitions need to be defined from the Halt state.

1. How many one state Turing machines do not loop on the empty word?
2. How many 1s can a one state Turing machine write starting on the empty word before halting?
3. Find a two states Turing machine which halts when starting on the empty word and end up with four 1s written on its tape.

Remark. To go a little further, we let

- E_n be the finite set of n-state Turing machines which halt on the empty word;

[1] Undefined transitions lead to the reject state by convention.

- $\sigma(M)$ for each $M \in E_n$ is the number of 1s on the tape at the end of the computation;
- $\Sigma(n) = \max\{\sigma(M) \mid M \in E_n\}$;

Since $\{\sigma(M) \mid M \in E_n\}$ is a finite set of natural numbers, the maximum is well defined. A n-state Turing machine which realises $\Sigma(n)$ is called a Busy Beaver, a term due to Tibor Radó, who first introduced this concept.

The language $\{(\ulcorner n \urcorner, \ulcorner \Sigma(n) \urcorner) \mid n \in \omega\}$ is not Turing decidable. The known values of Σ are $\Sigma(0) = 0$, $\Sigma(1) = 1$, $\Sigma(2) = 4$, $\Sigma(3) = 6$ and $\Sigma(4) = 13$. It is also known that $\Sigma(5) \geq 4098$ and that $\Sigma(6) \geq 10^{18267}$.

Exercise 4:

1. Show that Turing decidable languages are closed under complementation, union, and intersection.
2. Show that Turing recognisable languages are closed under union and intersection.