Exercise Sheet n°11

- Let $\mathcal{L}_0$ be the first order language of arithmetic with non logical symbols $0, 1, +, \cdot, <$.
- Given a formula $\varphi(x)$ with a free variable x, an $\mathcal{L}_0$-structure $\mathcal{M}$ and some $e \in \mathcal{M}$, we write $\mathcal{M} \models \varphi[e]$ to say that $\mathcal{M}$ models $\varphi(x)$ when x is interpreted as e.
- Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot, <)$ be the standard model of arithmetic.
- We are only interested in distinguishing countable models of PA 'up to isomorphism'. We can therefore restrict our attention to $\mathcal{L}_0$-structures $\mathcal{M}$ whose domain is the set of natural numbers $\mathbb{N}$ and we can further take the interpretation of 0 and 1 to be standard too. So, without loss of generality, by *a model of PA* we henceforth mean a structure $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \lessdot \rangle$ that satisfies the Peano axioms.

  A model $\mathcal{M}$ of PA is called non standard if it not isomorphic to $\mathcal{N}$. We have already seen that there are continuum many ($2^{\aleph_0}$) such models.
- For any $n \in \mathbb{N}$ we denote by $\mathbf{n}$ its syntactic counterpart $\overbrace{1+\cdots+1}^{n \text{ times}}$. In any model $\mathcal{M}$ of PA these terms are interpreted as 0, 1, $\mathbf{2}^{\mathcal{M}} = 1 \oplus 1$, $\mathbf{3}^{\mathcal{M}} = 1 \oplus 1 \oplus 1$. These elements are called standard elements. Notice that $\mathbf{n}^{\mathcal{M}} = n$ holds for every $n \in \mathbb{N}$ *only* in the standard model.

Consider the following

Definition. *A countable model $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \lessdot \rangle$ of PA is called recursive if both binary functions $\oplus : \mathbb{N}^2 \to \mathbb{N}$, $\otimes : \mathbb{N}^2 \to \mathbb{N}$ are recursive and the binary relation $\lessdot \subseteq \mathbb{N}^2$ is recursive.*

The aim of this exercise sheet is to show

Tennenbaum's Theorem.
The standard model is the only recursive model of PA.

Exercise 1: We consider a *recursive enumeration* of all Turing machines which compute partial functions from $\mathbb{N}$ to $\mathbb{N}$. According to this enumeration, we call $\mathcal{M}_k$ the k-th Turing machine. For all $k \in \mathbb{N}$, we let φ_k be the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by $\mathcal{M}_k$.
Show the following

Theorem 1. *There exist two nonempty recursively enumerable subsets A and B of $\mathbb{N}$ that are* recursively inseparable*, i.e. such that*

1. *A and B are disjoint, and*
2. *there is no recursive set X such that $A \subseteq X$ and $X \cap B = \emptyset$.*

Exercise 2:
Prove the following

Theorem 2 (Overspill)**.** *Let $\varphi(x)$ be an arithmetic formula whose only free variable is x. Assume that $\mathcal{M}$ is a non standard model of PA such that for all n we have $\mathcal{M} \models \varphi[\mathbf{n}]$. Then there is a non standard element $e \in |\mathcal{M}|$ such that $\mathcal{M} \models \varphi[e]$.*

Exercise 3:
Recall that a Δ_0 formula is a formula in which every quantification is bounded (these are the Δ^0_0-rud from Exercise Sheet 6).

The n-th prime is denoted by $\pi(n)$, that is $\pi(0) = 2$, $\pi(1) = 3\ldots$. We have seen that the function $\pi : n \to \pi(n)$ is primitive recursive, so it can be represented in PA by a Σ^0_1 formula $P(x, y)$.

We will abbreviate $\exists y \; (P(\mathbf{k}, y) \wedge y \otimes \mathbf{l} = \mathbf{m})$ with $\pi(\mathbf{k}) \otimes \mathbf{l} = \mathbf{m}$, meaning that the k-th prime multiplied by l equals m.

We make the following definition

Definition. *We say that a subset $A \subseteq \mathbb{N}$*

1. *is* coded *in a model $\mathcal{M}$ if there exists a formula $\varphi(x, y)$ and some $a \in \mathcal{M}$ such that $A = \{n \in \mathbb{N} \mid \mathcal{M} \models \phi(\mathbf{n}, [a])\}$,*

2. *is* canonically coded *in a model $\mathcal{M}$ if it is coded by the formula $\psi(x,y): \exists z\ \pi(x)\otimes z = y$, i.e there exists $a \in \mathcal{M}$ such that $A = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists z\ \pi(n)\otimes z = [a]\}$.*

We need the following.

Theorem 3. *Let $A(x,y)$ be a Δ_0 formula and $\mathcal{M}$ be a non standard model of PA. Then for all elements $b \in \mathcal{M}$ there exists an element $a \in \mathcal{M}$ such that for every $n \in \mathbb{N}$*

$$\mathcal{M} \models \quad \exists k \otimes [b]\ A(k,n)\ \leftrightarrow\ \exists z\ (\pi(n)\otimes z = [a])$$

We prove this in several steps:

1. We assume *without proof* that the following simple arithmetical truth is provable in PA, for every Δ_0 formula $A(x,y)$ and any $n \in \mathbb{N}$:
$$\forall b\ \exists a\ \forall u{<}n\ \big(\exists k{<}b\ A(k,u)\ \leftrightarrow\ \exists z\ (\pi(u)\cdot z = a)\big).$$
2. Show that for some non standard element e of $\mathcal{M}$ we have:
$$\mathcal{M} \models \forall b\ \exists a\ \forall u \otimes [e]\ \big(\exists k \otimes b\ A(k,u)\ \leftrightarrow \exists y\ (\pi(u)\otimes y = a)\big).$$
3. Conclude the proof of Theorem 3.

We recall the following basic fact

Theorem 4. *The theory PA is Δ_0 complete, that is every Δ_0 formula which is true in the standard model is provable in PA.*

We can now show that non standard elements can code non recursive sets.

Theorem 5. *In any non standard model of PA, there is a non standard element which canonically codes a non recursive set.*

Fix a non standard model $\mathcal{M}$ of PA. Here is how to proceed.

Exercise 4: By Theorem 1, let A and B be recursively enumerable sets that are recursively inseparable.

1. There exist two Δ_0 formulas $A(x,y)$ and $B(x,y)$ such that
$$A = \{m \mid \exists n\ A(m,n)\} \quad \text{and} \quad B = \{m \mid \exists n\ B(m,n)\}.$$
2. Show that there is a non standard $e \in \mathcal{M}$ such that
$$\mathcal{M} \models \forall x \otimes [e]\ \forall y \otimes [e]\ \forall z \otimes [e]\ \neg(A(x,y) \wedge B(x,z)).$$

Now consider the set
$$X = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists y \otimes [e]\ A(n,y)\}.$$

3. Show that $A \subseteq X$.
4. Show that $B \cap X = \emptyset$.
5. Conclude the proof of Theorem 5.

We can now prove Tennenbaum's Theorem.

Exercise 5: Let $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes \rangle$ be a non standard model of PA. Let X be a non recursive set canonically coded by some (non standard) element $a \in |\mathcal{M}| = \mathbb{N}$.

1. Observe that $\mathcal{M}$ satisfies the following formulas
   - $\forall x \forall y\big(y \neq 0 \to \exists! q\ \exists! r\ (r \otimes y \wedge x = (q \otimes y) \oplus r)\big)$ (Euclidean division);
   - for all n, $\forall x(\pi(n)\otimes x = \underbrace{x \oplus \cdots \oplus x}_{\pi(n)\text{ times}})$;
   - for all n, $\forall x(x \otimes \pi(n) \leftrightarrow x = 0 \vee x = 1 \vee \cdots \vee x = \underbrace{1 \oplus \cdots \oplus 1}_{\pi(n)\text{ times}})$.
2. Towards a contradiction suppose that $\oplus : \mathbb{N}^2 \to \mathbb{N}$ is recursive and design a decision procedure for X.
3. Conclude!