Chapter 5

Gödel's First Incompleteness Theorem

5.1 Gödel Numbers

The idea is the following: intuitively, formulas from arithmetic talk about integers – no matter whether these are standard or not – we can turn them into formulas that talk about the arithmetic itself by encoding formulas, proofs, etc. by integers. This way a formula $\varphi(x)$ may say something like "*x is the code of a closed formula from our language* $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$" or $\psi(x, y)$ may eventually say "*x is the code of a closed formula θ from our language and* y *is the code of a proof of θ in Robinson arithmetic*".

As always in logic, we start with the terms: given term t we write $\ulcorner t \urcorner$ for its code.

Definition 1.1: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-terms

The Gödel numbering of the terms from the language whose signature is $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ is

$$\begin{array}{lllllll}
\circ & t & = & 0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(0,0,0) \\
\circ & t & = & x_n & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(n+1,0,0) \\
\circ & t & = & \mathrm{S}t_0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t \urcorner,0,1) \\
\circ & t & = & t_0 + t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner,2) \\
\circ & t & = & t_0 \cdot t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner,3)
\end{array}$$

Lemma 1.1

The set $\mathcal{T}$ of all codes of terms from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{T} = \left\{ \ulcorner t \urcorner \mid t \textit{ is a term from } \mathcal{L}_{\mathcal{A}} \right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

To almost immediately show this result we need to prove that some stronger form of construction by recursion still produces $\mathcal{P}rim.\ \mathcal{R}ec.$ functions.

Lemma 1.2

For all $\mathcal{P}rim.\ \mathcal{R}ec.$ functions $h \in \mathbb{N}^{(\mathbb{N}^{n+p+1})}, g \in \mathbb{N}^{(\mathbb{N}^{n})}, k_1, \ldots, k_p \in \mathbb{N}^{\mathbb{N}}$, such that every integer $y > 0$

$$\bigwedge_{0<i\leqslant p} k_i(y) < y$$

the function $f \in \mathbb{N}^{(\mathbb{N}^{n+1})}$ defined by

(1) $f(\vec{x}, 0) = g(\vec{x})$

(2) $f(\vec{x}, y+1) = h\big(f(\vec{x}, k_1(y)), \ldots, f(\vec{x}, k_p(y)), \vec{x}, y, f(\vec{x}, y)\big)$

is also $\mathcal{P}rim.\ \mathcal{R}ec.$

First, notice that the Ackermann function $A \in \mathbb{N}^{(\mathbb{N}^2)}$ defined by

$$A(m,n) = \begin{cases} n+1 & \textit{if} \quad m = 0, \\ A(m-1,1) & \textit{if} \quad m > 0 \textit{ and } n = 0, \\ A\big(m-1, A(m, n-1)\big) & \textit{if} \quad m > 0 \textit{ and } n > 0. \end{cases}$$

is not of this form for $A(m, n-1) < n$ certainly does not hold.

Proof of Lemma 1.1:

Its characteristic function $\chi_{\mathcal{T}} : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{lllllllllllllll}
\textit{if} & \beta_3^3(k) & = & 0 & \textit{and} & \beta_3^2(k) & = & 0 & \textit{and} & \beta_3^1(k) & = & 0 & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & 1 \\
\textit{if} & \beta_3^3(k) & = & 0 & \textit{and} & \beta_3^2(k) & = & 0 & \textit{and} & \beta_3^1(k) & > & 0 & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & 1 \\
\textit{if} & \beta_3^3(k) & = & 1 & \textit{and} & \beta_3^2(k) & = & 0 & \textit{and} & \beta_3^1(k) & > & 0 & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & \chi_{\mathcal{T}} \circ \beta_3^1(k) \\
\textit{if} & \beta_3^3(k) & = & 2 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & \chi_{\mathcal{T}}\big(\beta_3^1(k)\big) \cdot \chi_{\mathcal{T}}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 3 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & \chi_{\mathcal{T}}\big(\beta_3^1(k)\big) \cdot \chi_{\mathcal{T}}\big(\beta_3^2(k)\big) \\
\textit{else} & & & & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}}(k) & = & 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{P}rim.\ \mathcal{R}ec.$ function.

□

Proof of Lemma 1.2:

The idea of the proof is to define by recursion a function that carries all the datas $f(\overrightarrow{x}, z)$ for $z < y$ when defining $f(\overrightarrow{x}, y)$. To achieve this, we make use of the two $\mathcal{P}rim.\ \mathcal{R}ec.$ functions c and d that we defined in Example 4.1:

(1) the function $c : \mathbb{N}^{<\omega} \longrightarrow \mathbb{N}$ codes the finite sequences of integers and is defined by

$$\begin{cases} c(\varepsilon) & = & 1 \\ c(x_0, \ldots, x_p) & = & \Pi\,(0)^{x_0+1} \cdot \Pi\,(1)^{x_1+1} \cdots \Pi\,(p)^{x_p+1}. \end{cases}$$

(2) And the function $d \in \mathbb{N}^{(\mathbb{N}^2)}$ that allows, from any integer n, to recover every element of the sequence $\langle x_0, \ldots, x_p \rangle$ that c encodes (i.e., $c(x_0, \ldots, x_p) = n$) by

$$d(i, n) = \mu x \leqslant n \quad \Pi\,(i)^{x+1} \textit{ does not divide } n.$$

We want to define some $\mathcal{P}rim.\ \mathcal{R}ec.$ function $\theta \in \mathbb{N}^{(\mathbb{N}^{n+1})}$ such that

$$\theta(\overrightarrow{x}, y) = c\big(f(\overrightarrow{x}, 0), f(\overrightarrow{x}, 1), \ldots, f(\overrightarrow{x}, y)\big)$$

This is easily done by recursion:

(1) $\theta(\overrightarrow{x}, 0) = 2^{g(\overrightarrow{x})+1}$

(2) $\theta(\overrightarrow{x}, y+1) = \theta(\overrightarrow{x}, y) \cdot \Pi\,(y+1)^{h\big[d[k_1(y)),\theta(\overrightarrow{x},y)],\ldots,d[k_p(y)),\theta(\overrightarrow{x},y)],\overrightarrow{x},y,f(\overrightarrow{x},y)\big]+1}$.

Then, to show that f is also $\mathcal{P}rim.\ \mathcal{R}ec.$, it only remains to set

$$f(\overrightarrow{x}, y) = d\big[y, \theta(\overrightarrow{x}, y)\big] \dot{-} 1.$$

□

Definition 1.2: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-formulas

The Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-formulas is

$$\begin{array}{lllllll}
\varphi & = & t_0 = t_1 & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner t_0\urcorner, \ulcorner t_1\urcorner, 4) \\
\varphi & = & \neg\psi & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\psi\urcorner, 0, 5) \\
\varphi & = & (\varphi_0 \wedge \varphi_1) & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\varphi_0\urcorner, \ulcorner\varphi_1\urcorner, 6) \\
\varphi & = & (\varphi_0 \vee \varphi_1) & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\varphi_0\urcorner, \ulcorner\varphi_1\urcorner, 7) \\
\varphi & = & (\varphi_0 \longrightarrow \varphi_1) & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\varphi_0\urcorner, \ulcorner\varphi_1\urcorner, 8) \\
\varphi & = & (\varphi_0 \longleftrightarrow \varphi_1) & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\varphi_0\urcorner, \ulcorner\varphi_1\urcorner, 9) \\
\varphi & = & \forall x_n\ \psi & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\psi\urcorner, n, 10) \\
\varphi & = & \exists x_n\ \psi & \rightsquigarrow & \ulcorner\varphi\urcorner & = & \alpha_3(\ulcorner\psi\urcorner, n, 11).
\end{array}$$

Notice that for every formula φ, we have $\ulcorner\varphi\urcorner > 0$.

Lemma 1.3

The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$

$$\mathcal{F} = \left\{\ulcorner\varphi\urcorner \mid \varphi \textit{ is a formula from } \mathcal{L}_\mathcal{A}\right\} \in \mathcal{P}rim.\ \mathcal{R}ec.$$

Proof of Lemma 1.3:

Its characteristic function $\chi_\mathcal{F} : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{lllllllllll}
\textit{if} & \beta_3^3(k) & = & 4 & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F}\big(\beta_3^1(k)\big) \cdot \chi_\mathcal{F}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 5 & \textit{and} & \beta_3^2(k) & = 0 & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F} \circ \beta_3^1(k) \\
\textit{if} & \beta_3^3(k) & = & 6 & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F}\big(\beta_3^1(k)\big) \cdot \chi_\mathcal{F}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 7 & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F}\big(\beta_3^1(k)\big) \cdot \chi_\mathcal{F}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 8 & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F}\big(\beta_3^1(k)\big) \cdot \chi_\mathcal{F}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 9 & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F}\big(\beta_3^1(k)\big) \cdot \chi_\mathcal{F}\big(\beta_3^2(k)\big) \\
\textit{if} & \beta_3^3(k) & = & 10 & \textit{and} & \beta_3^2(k) & > 0 & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F} \circ \beta_3^1(k) \\
\textit{if} & \beta_3^3(k) & = & 11 & \textit{and} & \beta_3^2(k) & > 0 & \rightsquigarrow & \chi_\mathcal{F}(k) & = & \chi_\mathcal{F} \circ \beta_3^1(k) \\
\textit{else} & & & & & & & \rightsquigarrow & \chi_\mathcal{F}(k) & = & 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{P}rim.\ \mathcal{R}ec.$ function.

$\square$

Lemma 1.4

The occurence relation of variables in terms from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{T}_{\checkmark x} = \left\{(\ulcorner t \urcorner, n) \mid t \textit{ is a term from } \mathcal{L}_{\mathcal{A}} \textit{ and } t \textit{ contains } x_n\right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.4:

Its characteristic function $\chi_{\mathcal{T}_{\checkmark x_n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{llllllllllll}
\textit{if} & \beta_3^3(k) & = & 0 & \textit{and} & \beta_3^2(k) & = & 0 & \textit{and} & \beta_3^1(k) & = & n+1 & \rightsquigarrow & \chi_{\mathcal{T}_{\checkmark x_n}}(k,n) & = & 1 \\
\textit{if} & \beta_3^3(k) & = & 1 & \textit{and} & \beta_3^2(k) & = & 0 & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\checkmark x_n}}(k,n) & = & \chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^1(k),n) \\
\textit{if} & \beta_3^3(k) & = & 2 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\checkmark x_n}}(k,n) & = & \max\Big(\chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^1(k),n), \chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 3 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\checkmark x_n}}(k,n) & = & \max\Big(\chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^1(k),n), \chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^2(k),n)\Big) \\
\textit{else} & & & & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\checkmark x_n}}(k,n) & = & 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{P}rim.\ \mathcal{R}ec.$ function.

□

Lemma 1.5

The set

$$\mathcal{T}_{\times x} = \left\{(\ulcorner t \urcorner, n) \mid t \textit{ is a term from } \mathcal{L}_{\mathcal{A}} \textit{ and } t \textit{ does not contain } x_n\right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.5:

Its characteristic function $\chi_{\mathcal{T}_{\times x_n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by :

$$\begin{array}{llllllllllll}
\textit{if} & \beta_3^3(k) & = & 0 & \textit{and} & \beta_3^2(k) & = & 0 & \textit{and} & \beta_3^1(k) & \neq & n+1 & \rightsquigarrow & \chi_{\mathcal{T}_{\times x_n}}(k,n) & = & 1 \\
\textit{if} & \beta_3^3(k) & = & 1 & \textit{and} & \beta_3^2(k) & = & 0 & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\times x_n}}(k,n) & = & \chi_{\mathcal{T}_{\times x_n}}(\beta_3^1(k),n) \\
\textit{if} & \beta_3^3(k) & = & 2 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\times x_n}}(k,n) & = & \chi_{\mathcal{T}_{\times x_n}}(\beta_3^1(k),n) \cdot \chi_{\mathcal{T}_{\times x_n}}(\beta_3^2(k),n) \\
\textit{if} & \beta_3^3(k) & = & 3 & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\times x_n}}(k,n) & = & \chi_{\mathcal{T}_{\times x_n}}(\beta_3^1(k),n) \cdot \chi_{\mathcal{T}_{\times x_n}}(\beta_3^2(k),n) \\
\textit{else} & & & & & & & & & & & & \rightsquigarrow & \chi_{\mathcal{T}_{\times x_n}}(k,n) & = & 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{P}rim.\ \mathcal{R}ec.$ function.

$\square$

Lemma 1.6

The set

$$\mathcal{F}_{\checkmark x} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathcal{A}} \textit{ and } \varphi \textit{ contains } x_n \right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.6:

Its characteristic function $\chi_{\mathcal{F}_{\checkmark x n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{lllllllllll}
\textit{if} & \beta_3^3(k) & = & 4 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \max\Big(\chi_{\mathcal{T}_{\checkmark x n}}\big(\beta_3^1(k),n\big), \chi_{\mathcal{T}_{\checkmark x n}}\big(\beta_3^2(k),n\big)\Big) \\
\textit{if} & \beta_3^3(k) & = & 5 & \textit{and} & \beta_3^2(k) & = 0 & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big) \\
\textit{if} & \beta_3^3(k) & = & 6 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \max\Big(\chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big), \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^2(k),n\big)\Big) \\
\textit{if} & \beta_3^3(k) & = & 7 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \max\Big(\chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big), \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^2(k),n\big)\Big) \\
\textit{if} & \beta_3^3(k) & = & 8 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \max\Big(\chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big), \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^2(k),n\big)\Big) \\
\textit{if} & \beta_3^3(k) & = & 9 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \max\Big(\chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big), \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^2(k),n\big)\Big) \\
\textit{if} & \beta_3^3(k) & = & 10 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big) \\
\textit{if} & \beta_3^3(k) & = & 11 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & \chi_{\mathcal{F}_{\checkmark x n}}\big(\beta_3^1(k),n\big) \\
\textit{else} & & & & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x n}}(k,n) & = & 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{P}rim.\ \mathcal{R}ec.$ function.

$\square$

Lemma 1.7

The set

$$\mathcal{F}_{\times x} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathcal{A}} \textit{ and } \varphi \textit{ does not contain } x_n \right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.7:

Its characteristic function $\chi_{\mathcal{F}_{\text{✗}x_n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by :

$$
\begin{array}{llllllllll}
\textit{if} & \beta_3^3(k) & = & 4 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{T}_{\text{✗}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{T}_{\text{✗}x_n}} \circ \beta_3^2(k) \\
\textit{if} & \beta_3^3(k) & = & 5 & \textit{and} & \beta_3^2(k) & = 0 & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \\
\textit{if} & \beta_3^3(k) & = & 6 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^2(k) \\
\textit{if} & \beta_3^3(k) & = & 7 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^2(k) \\
\textit{if} & \beta_3^3(k) & = & 8 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^2(k) \\
\textit{if} & \beta_3^3(k) & = & 9 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^2(k) \\
\textit{if} & \beta_3^3(k) & = & 10 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \\
\textit{if} & \beta_3^3(k) & = & 11 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = \chi_{\mathcal{F}_{\text{✗}x_n}} \circ \beta_3^1(k) \\
\textit{else} & & & & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\text{✗}x_n}}(k,n) & = 0.
\end{array}
$$

By Lemma 1.2 this definition by case study yields a $\mathcal{Prim.\ Rec.}$ function.

□

Lemma 1.8

The set

$$\mathcal{F}_{\text{✓}x\ \textit{free}} = \{(\ulcorner\varphi\urcorner, n) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathcal{A}} \textit{ and } x_n \textit{ is free in } \varphi\}$$

is $\mathcal{Prim.\ Rec.}$

Proof of Lemma 1.8:

Its characteristic function $\chi_{\mathcal{F}_{\text{✓}x\ \textit{free}}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{llllllllll}
\textit{if} & \beta_3^3(k) & = & 4 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \max\Big(\chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n), \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 5 & \textit{and} & \beta_3^2(k) & = 0 & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n) \\
\textit{if} & \beta_3^3(k) & = & 6 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \max\Big(\chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n), \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 7 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \max\Big(\chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n), \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 8 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \max\Big(\chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n), \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 9 & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \max\Big(\chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n), \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^2(k),n)\Big) \\
\textit{if} & \beta_3^3(k) & = & 10 & \textit{and} & \beta_3^2(k) & \neq n & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n) \\
\textit{if} & \beta_3^3(k) & = & 11 & \textit{and} & \beta_3^2(k) & \neq n & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = \chi_{\mathcal{F}_{\checkmark x\,free}}(\beta_3^1(k),n) \\
\textit{else} & & & & & & & \rightsquigarrow & \chi_{\mathcal{F}_{\checkmark x\,free}}(k,n) & = 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a $\mathcal{Prim.\ Rec.}$ function.

□

Lemma 1.9

The set

$$\mathcal{F}_{\checkmark x\ bound} = \Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathcal{A}} \textit{ and } x_n \textit{ is bound in } \varphi\Big\}$$

is $\mathcal{Prim.\ Rec.}$

Proof of Lemma 1.9:

we have $\mathcal{F}_{\checkmark x\ bound} = \mathcal{F}_{\checkmark x} \setminus \mathcal{F}_{\checkmark x\ free}$.

□

Lemma 1.10

The set of all codes of closed formulas from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{F}_{\checkmark closed} = \Big\{\ulcorner\varphi\urcorner \mid \varphi \textit{ is a closed formula from } \mathcal{L}_{\mathcal{A}}\Big\}$$

is $\mathcal{Prim.\ Rec.}$

Proof of Lemma 1.10:

We have

$$k \in \mathcal{F}_{\checkmark closed} \Longleftrightarrow k \in \mathcal{F} \text{ and } \forall n \leqslant k \quad (k, n) \notin \mathcal{F}_{\checkmark x\, free}.$$

□

Lemma 1.11

The function $\mathcal{S}^{\mathcal{T}}_{ub.} \in \mathbb{N}^{(\mathbb{N}^3)}$ defined below is $\mathcal{P}rim.\ \mathcal{R}ec.$

$$\mathcal{S}^{\mathcal{T}}_{ub.}(n_u, n_t, n) = \begin{cases} \ulcorner u_{[t/x_n]} \urcorner & \text{if } n_u \in \mathcal{T},\ n_t \in \mathcal{T} \text{ and } n_u = \ulcorner u \urcorner, n_t = \ulcorner t \urcorner \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 1.11:

We first recall the definition of $\ulcorner t \urcorner$:

$$\begin{array}{rclcrcl} t & = & 0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(0,0,0) \\ t & = & x_n & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(n+1,0,0) \\ t & = & \mathsf{S}t_0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t \urcorner, 0, 1) \\ t & = & t_0 + t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 2) \\ t & = & t_0 \cdot t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 3) \end{array}$$

$\mathcal{S}^{\mathcal{T}}_{ub.} \in \mathbb{N}^{(\mathbb{N}^3)}$ is defined by

$$\mathcal{S}^{\mathcal{T}}_{ub.}(n_u, n_t, n) = \begin{cases} 0 & \text{if } \quad n_u \notin \mathcal{T} \quad \text{or} \quad n_t \notin \mathcal{T} \\ n_t & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 0 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) = n+1 \\ n_u & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 0 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) \neq n+1 \\ \alpha_3(\mathcal{S}^{\mathcal{T}}_{ub.}(\beta_3^1(n_u), n_t, n), 0, 1) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 1 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) \in \mathcal{T} \\ \alpha_3(\mathcal{S}^{\mathcal{T}}_{ub.}(\beta_3^1(n_u), n_t, n), \mathcal{S}^{\mathcal{T}}_{ub.}(\beta_3^2(n_u), n_t, n), 2) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 2 \text{ and } \beta_3^2(n_u) \in \mathcal{T} \text{ and } \beta_3^1(n_u) \in \mathcal{T} \\ \alpha_3(\mathcal{S}^{\mathcal{T}}_{ub.}(\beta_3^1(n_u), n_t, n), \mathcal{S}^{\mathcal{T}}_{ub.}(\beta_3^2(n_u), n_t, n), 3) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 3 \text{ and } \beta_3^2(n_u) \in \mathcal{T} \text{ and } \beta_3^1(n_u) \in \mathcal{T}. \end{cases}$$

By Lemma 1.2 $\mathcal{S}^{\mathcal{T}}_{ub.}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$

□

Lemma 1.12

The function $\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}} \in \mathbb{N}^{(\mathbb{N}^3)}$ defined below is $\mathcal{P}rim.\ \mathcal{R}ec.$

$$\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(n_\varphi, n_t, n\right) = \begin{cases} \ulcorner \varphi[t/x_n] \urcorner & \text{if } \; n_\varphi = \ulcorner \varphi \urcorner \in \mathcal{F}, \; n_t = \ulcorner t \urcorner \in \mathcal{T} \\ 0 & \text{otherwise}\ . \end{cases}$$

Proof of Lemma 1.12:

We first recall the definition of $\ulcorner \varphi \urcorner$:

$$\begin{array}{lclcccl}
\varphi & = & t_0 = t_1 & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 4) \\
\varphi & = & \neg\psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, 0, 5) \\
\varphi & = & (\varphi_0 \wedge \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 6) \\
\varphi & = & (\varphi_0 \vee \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 7) \\
\varphi & = & (\varphi_0 \longrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 8) \\
\varphi & = & (\varphi_0 \longleftrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 9) \\
\varphi & = & \forall x_n \; \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, n, 10) \\
\varphi & = & \exists x_n \; \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, n, 11).
\end{array}$$

$\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}} \in \mathbb{N}^{(\mathbb{N}^3)}$ is defined by

$$\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(n_\varphi, n_t, n\right) = \begin{cases}
0 & \text{if } \; n_\varphi \notin \mathcal{F} \;\; \text{or} \;\; n_t \notin \mathcal{T} \\
\alpha_3(\mathcal{S}^{\mathcal{T}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \mathcal{S}^{\mathcal{T}}_{\mathit{ub.}}\left(\beta^2_3(n_\varphi), n_t, n\right), 4) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 4 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), 0, 5) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 5 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^2_3(n_\varphi), n_t, n\right), 6) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 6 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^2_3(n_\varphi), n_t, n\right), 7) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 7 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^2_3(n_\varphi), n_t, n\right), 8) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 8 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^2_3(n_\varphi), n_t, n\right), 9) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 9 \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \beta^2_3(n_\varphi), 10) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 10 \;\; \text{and} \;\; \beta^2_3(n_\varphi) \neq n \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}\left(\beta^1_3(n_\varphi), n_t, n\right), \beta^2_3(n_\varphi), 11) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 11 \;\; \text{and} \;\; \beta^2_3(n_\varphi) \neq n \\
\beta^1_3(n_\varphi) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 10 \;\; \text{and} \;\; \beta^2_3(n_\varphi) = n \\
\beta^1_3(n_\varphi) & \text{if } \; n_\varphi \in \mathcal{F} \;\; \text{and} \;\; n_t \in \mathcal{T} \;\; \text{and} \;\; \beta^3_3(n_\varphi) = 11 \;\; \text{and} \;\; \beta^2_3(n_\varphi) = n
\end{cases}$$

By Lemma 1.2 $\mathcal{S}^{\mathcal{F}}_{\mathit{ub.}}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$

□

We will now define a way of coding (finite) sets of formulas. We will not really encode the set, but some finite sequence of formulas, because we will not care about the ordering of such a sequence, even if what we really encode is the sequence, we will handle it as if it were a set.

Definition 1.3: Coding and decoding sequences

We define both $\ulcorner\ \urcorner : \mathbb{N}^{<\omega} \longrightarrow \mathbb{N}$ and $\llcorner\ \urcorner : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by

$$\begin{cases} \ulcorner \varepsilon \urcorner & = & 0 \\ \ulcorner k_0, \ldots, k_p \urcorner & = & \Pi(0)^{k_0} \cdot \Pi(1)^{k_1} \cdots \Pi(p)^{k_p}. \end{cases}$$

Where $\Pi(i)$ enumerates the prime numbers[a].
And

$$\llcorner n \urcorner^i = \mu x \leqslant n \quad \Pi(i)^{x+1} \textit{ does not divide } n.$$

[a] $\Pi(0) = 2; \Pi(1) = 3; \Pi(2) = 5$; etc.

Notice that for all $i \leqslant p$ we have $\llcorner \ulcorner k_0, \ldots, k_p \urcorner \urcorner^i = k_i$. Furthermore, for every formula φ, the integer $\ulcorner \varphi \urcorner$ is strictly positive. Therefore, given any sequence $\langle k_0, \ldots, k_p \rangle \in \mathbb{N}^{<\omega}$ if $\llcorner \ulcorner k_0, \ldots, k_p \urcorner \urcorner^i = 0$ then we know for sure that k_i does not code a formula.

We will say that the integer 1 codes the empty set – which is also an empty set of formulas – and another integer codes the set $\Delta = \{\varphi_0, \varphi_1, \ldots, \varphi_p\}$ if this integer is of the form $\Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \cdots \Pi(i_p)^{\ulcorner \varphi_p \urcorner}$.

Definition 1.4: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-finite sets of formulas

The Gödel numbering of any set $\Delta = \{\varphi_0, \varphi_1, \ldots, \varphi_p\}$ of $\mathcal{L}_{\mathcal{A}}$-formulas is any integer of the form

$$\begin{array}{rcll} \ulcorner \Delta \urcorner & = & 1 & \text{if } \Delta = \varnothing, \\ & = & \Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \cdots \Pi(i_p)^{\ulcorner \varphi_p \urcorner} & \text{otherwise.} \end{array}$$

with $\{i_0, \ldots, i_p\}$ and Δ having the same cardinality[a].

We denote $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ the set of codes of finite sets of formulas:

$$\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} = \{\ulcorner \Delta \urcorner \mid \Delta \textit{ is any finite set of } \mathcal{L}_{\mathcal{A}} \textit{ formulas}\}.$$

[a] This means $\forall j, k \leqslant p \ \ (j \neq k \rightarrow i_j \neq i_k)$.

Lemma 1.13

The set $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ of codes of finite sets of formulas is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.13:

$$\chi_{\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}}(n) = \begin{cases} 1 & \textit{if} \quad n = 1 \\ 1 & \textit{if} \quad n \neq 1 \quad \textit{and} \quad \forall i \leqslant n \ \left[\llcorner n \urcorner^i > 0 \longrightarrow \llcorner n \urcorner^i \in \mathcal{F} \right] \\ 0 & \textit{else.} \end{cases}$$

□

Lemma 1.14

There exist two $\mathcal{P}rim.\ \mathcal{R}ec.$ functions $\mathcal{R}_{em.} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ and $\mathcal{A}_{dd.} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that

$$\mathcal{A}_{dd.}(n,m) = \begin{cases} \ulcorner \Delta \cup \{\varphi\} \urcorner & \textit{if} \quad n = \ulcorner \varphi \urcorner \in \mathcal{F} \quad \textit{and} \quad m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \end{cases}$$

$$\mathcal{R}_{em.}(n,m) = \begin{cases} \ulcorner \Delta \setminus \{\varphi\} \urcorner & \textit{if} \quad n = \ulcorner \varphi \urcorner \in \mathcal{F} \quad \textit{and} \quad m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}. \end{cases}$$

Proof of Lemma 1.14:

We have both

$$\mathcal{A}_{dd.}(n,m) = \begin{cases} 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ m \cdot \Pi \left(\mu i \leqslant m \ \llcorner m \urcorner^i = 0\right)^n & \textit{if} \quad \ulcorner \varphi \urcorner = n \in \mathcal{F} \quad \textit{and} \quad \ulcorner \Delta \urcorner = m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}. \end{cases}$$

and

$$\mathcal{R}_{em.}(n,m) = \begin{cases} 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ m & \textit{if} \quad n \in \mathcal{F} \quad \textit{and} \quad m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad \forall i \leqslant m \ \llcorner m \urcorner^i \neq n \\ \left[\dfrac{m}{\Pi \left(\mu i \leqslant m \ \llcorner m \urcorner^i = n\right)^n} \right] & \textit{if} \quad n \in \mathcal{F} \quad \textit{and} \quad m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad \exists i \leqslant m \ \llcorner m \urcorner^i = n \end{cases}$$

□

Lemma 1.15

The set
$$\mathcal{I}_{ns.} = \Big\{(\ulcorner\varphi\urcorner, \ulcorner\Delta\urcorner) \in \mathbb{N}^2 \mid \ulcorner\varphi\urcorner \in \mathcal{F},\ \ulcorner\Delta\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ and \ \ \varphi \in \Delta\Big\}$$
is $\mathcal{P}rim.\ \mathcal{R}ec.$.

Later on, we will use a relation-like notation and write $\ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ \ulcorner\Delta\urcorner$ instead of $(\ulcorner\varphi\urcorner, \ulcorner\Delta\urcorner) \in \mathcal{I}_{ns.}$.

Proof of Lemma 1.15:

We have

$$\chi_{\mathcal{I}_{ns.}}(n,m) = \begin{cases} 1 & if \quad n \in \mathcal{F} \ \ and \ \ m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ and \ \ \exists i \leqslant m \begin{cases} \Pi\,(i)^n \ \ divides\ m \\ \ and \\ \Pi\,(i)^{n+1} \ \ does\ not\ divides\ m \end{cases} \\ 0 & else. \end{cases}$$

□

Lemma 1.16

The following set
$$\mathcal{E}_{qu.} = \Big\{(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner) \in \mathbb{N}^2 \mid \ulcorner\Gamma\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})},\ \ulcorner\Delta\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ and \ \ \Gamma = \Delta\Big\}.$$
is $\mathcal{P}rim.\ \mathcal{R}ec.$.

Notice that the equality “ $\Gamma = \Delta$” is between two sets, therefore it relies on extensionality. We will use a relation-like notation and write $\ulcorner\varphi\urcorner\ \mathcal{E}_{qu.}\ \ulcorner\Delta\urcorner$ instead of $(\ulcorner\varphi\urcorner, \ulcorner\Delta\urcorner) \in \mathcal{E}_{qu.}$.

Proof of Lemma 1.16:

We have
$$\chi_{\mathcal{E}_{qu.}}(n,m) = \begin{cases} 1 & if \quad n,m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ and \ \ \forall i \leqslant \max(n,m) \ \ \big(i \in \mathcal{F} \to (i\ \mathcal{I}_{ns.}\ n \leftrightarrow i\ \mathcal{I}_{ns.}\ m)\big) \\ 0 & else. \end{cases}$$

□

Lemma 1.17

There exists a $\mathcal{P}rim.\ \mathcal{R}ec.$ function $\mathcal{U}_{nion} \in \mathbb{N}^{(\mathbb{N}^2)}$ such that

$$\mathcal{U}_{nion}(n,m) = \begin{cases} 0 & \textit{if} \quad n \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ \ulcorner\Theta\urcorner & \textit{if} \quad n = \ulcorner\Gamma\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad m = \ulcorner\Delta\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad \ulcorner\Theta\urcorner\ \mathcal{E}_{qu.}\ \ulcorner\Gamma \cup \Delta\urcorner. \end{cases}$$

Proof of Lemma 1.17:

We first define $f \in \mathbb{N}^{(\mathbb{N}^3)}$ by recursion:

$$\begin{cases} f(\gamma,\delta,0) & = & \Pi(0)^{\llcorner\gamma\urcorner_0} \cdot \Pi(1)^{\llcorner\delta\urcorner_0} \\ f(\gamma,\delta,n+1) & = & \Pi(2n)^{\llcorner\gamma\urcorner_n} \cdot \Pi(2n+1)^{\llcorner\delta\urcorner_n} \cdot f(\gamma,\delta,n). \end{cases}$$

Then we set

$$\begin{array}{rcccc} \mathcal{U}_{nion}(n,m) & = & f(n,m,\max(n,m)) & \textit{if} \quad n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} & \textit{and} \quad m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ & = & 0 & \textit{otherwise.} & \end{array}$$

□

Definition 1.5: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-sequents from *sequent calculus*

The Gödel numbering of any sequent $\Gamma \vdash \Delta$ is

$$\ulcorner\Gamma \vdash \Delta\urcorner = \alpha_2(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner)$$

We denote $\mathcal{SQ}$ the set of codes of sequents:

$$\mathcal{SQ} = \{\ulcorner\Gamma \vdash \Delta\urcorner \mid \Gamma,\ \Delta \textit{ finite sets of } \mathcal{L}_{\mathcal{A}} \textit{ formulas}\}.$$

Given any integer n we use the notation ${}^{l}n$ for $\beta_2^1(n)$ and ${}^{r}n$ for $\beta_2^2(n)$. This way,

$$\text{if } n = \ulcorner\Gamma \vdash \Delta\urcorner, \text{ then } {}^{l}n = \ulcorner\Gamma\urcorner \text{ and } {}^{r}n = \ulcorner\Delta\urcorner.$$

Lemma 1.18

The set $\mathcal{SQ}$ of codes of sequents of *sequent calculus* is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.18:

$$\chi_{\mathcal{SQ}}(n) = \begin{cases} 1 & \text{if} \quad {}^{l}n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \text{and} \quad {}^{r}n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \text{else} \end{cases}$$

□

We will now denote $\mathcal{AX}$ the set of codes of axioms of sequent calculus which are not to be mistaken for the axiom of Robinson arithmetic.

Definition 1.6: Gödel numbering of the axioms of *sequent calculus*

$$\mathcal{AX} = \left\{ \alpha_2\left(2^{\ulcorner\varphi\urcorner}, 2^{\ulcorner\varphi\urcorner}\right) \mid \ulcorner\varphi\urcorner \in \mathcal{F} \right\}.$$

Lemma 1.19

The set $\mathcal{AX}$ of codes of axioms of *sequent calculus* is $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 1.19:

$$\chi_{\mathcal{AX}}(n) = \begin{cases} 1 & \text{if} \quad {}^{l}n = {}^{r}n \ \text{and} \ \llcorner {}^{l}n \urcorner^{0} \in \mathcal{F} \ \text{and} \ \forall i \leqslant n \ \ \llcorner {}^{l}n \urcorner^{i+1} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

□

5.2 Coding the Proofs

We recall that a proof in Sequent Calculus is a tree of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \vdash \psi}\,ax}{\varphi, \psi \vdash \psi}\,\mathrm{wkn}_l}{\varphi \vdash \neg\psi, \psi}\,\neg_r \qquad \dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,ax}{\varphi \vdash \varphi, \psi}\,\mathrm{wkn}_r}{\varphi, \neg\varphi \vdash \psi}\,\neg_l}{\varphi, \neg\psi \to \neg\varphi \vdash \psi}\,\to_l}{\neg\psi \to \neg\varphi \vdash \varphi \to \psi}\,\to_r$$

where the shape of the tree is controlled by the rules of Sequent Calculus.

We are now ready to define for each rule of the Sequent Calculus, a set of tuples of codes of sequents that satisfy the property that the rule defines.
We will successively define

Definition 2.1

(1)
- $\mathcal{R}_{ax} \subseteq \mathbb{N}$

(2)
- $\mathcal{R}_{\wedge l1} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\wedge l2} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\neg_l} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\forall_l} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\exists_l} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\vee r1} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\vee r2} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\neg_r} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\forall_r} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{\exists_r} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{wkn_l} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{wkn_r} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{ctr\ l\&r} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{cut} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{Rep} \subseteq \mathbb{N}^2$
- $\mathcal{R}_{Ref} \subseteq \mathbb{N}^2$

(3)
- $\mathcal{R}_{\vee_l} \subseteq \mathbb{N}^3$
- $\mathcal{R}_{\to_l} \subseteq \mathbb{N}^3$
- $\mathcal{R}_{\wedge_r} \subseteq \mathbb{N}^3$

and for each of them, the fact that it is $\mathcal{P}rim.\ \mathcal{R}ec.$ will derive from its definition. We first recall what the rules are.

Sequent Calculus

Axioms

$$\frac{}{\varphi \vdash \varphi}\ ax$$

Logical Rules

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ {\wedge_{l1}} \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ {\wedge_{l2}} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ {\wedge_r}$$

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ {\vee_l} \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\vee_{r1}} \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\vee_{r2}}$$

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ {\to_l} \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ {\to_r}$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ {\neg_l} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ {\neg_r}$$

$$\frac{\Gamma, \varphi_{[t/x]} \vdash \Delta\ ^1}{\Gamma, \forall x\ \varphi \vdash \Delta}\ {\forall_l} \qquad \frac{\Gamma \vdash \varphi_{[y/x]}, \Delta}{\Gamma \vdash \forall x\ \varphi, \Delta\ ^2}\ {\forall_r}$$

$$\frac{\Gamma, \varphi_{[y/x]} \vdash \Delta}{\Gamma, \exists x\ \varphi \vdash \Delta\ ^2}\ {\exists_l} \qquad \frac{\Gamma \vdash \varphi_{[t/x]}, \Delta\ ^1}{\Gamma \vdash \exists x\ \varphi, \Delta}\ {\exists_r}$$

$$\frac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta}\ Ref \qquad \frac{\Gamma, t = s, \varphi_{[s/x]}, \varphi_{[t/x]} \vdash \Delta}{\Gamma, s = t, \varphi_{[t/x]} \vdash \Delta}\ Rep$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ wkn_l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\ wkn_r$$

$$\frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ ctr_l \qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\ ctr_r$$

Cut Rule

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

[1] t a term

[2] y with no free occurrence the sequent concluding the rule (not in $\Gamma, \exists x\ \varphi$ nor $\forall x\ \varphi$, nor Δ)

$$\boxed{\dfrac{}{\varphi \vdash \varphi}\ {\scriptstyle ax}}$$

$$D \in \mathcal{R}_{ax} \iff D \in \mathcal{AX}$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ {\scriptstyle \wedge l1}}$$

$$(U, D) \in \mathcal{R}_{\wedge l1}$$

$$\iff$$

$$\begin{cases} U \in \mathcal{SQ} \\ \quad \textit{and} \\ D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U \ \exists \ulcorner\psi\urcorner \leqslant {}^{l}D \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \ulcorner\varphi \wedge \psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\varphi \wedge \psi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

where

- " $\ulcorner\varphi \wedge \psi\urcorner$" stands for " $\alpha_3(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner, 6)$".
- " $\exists \ulcorner\varphi\urcorner \leqslant k \quad \theta_{[\ulcorner\varphi\urcorner/y]}$]" stands for " $\exists n \leqslant k \quad \big(n \in \mathcal{F} \ \wedge \ \theta_{[n/y]}\big)$" and more generally
- " $\exists \ulcorner\varphi_1\urcorner \leqslant k_1 \ \ldots \ldots \exists \ulcorner\varphi_n\urcorner \leqslant k_n \quad \theta_{[\ulcorner\varphi_1\urcorner/y_1, \ldots, \ulcorner\varphi_n\urcorner/y_n]}$" stands for " $\exists p \leqslant \alpha_n(k_1, \ldots, k_n) \quad \big(\bigwedge_{i \leqslant n} \big(\beta_n^i(p) \in \mathcal{F} \ \wedge \ \beta_n^i(p) \leqslant k_i\big) \quad \wedge \quad \theta_{[\beta_n^1(p)/y_1, \ldots, \beta_n^n(p)/y_n]}\big)$".

....

$$\boxed{\dfrac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \wedge_{l2}}$$

$$(U, D) \in \mathcal{R}_{\wedge_{l2}}$$

$$\Longleftrightarrow$$

$$\begin{cases} U \in \mathcal{SQ} \\ \textit{and} \\ D \in \mathcal{SQ} \\ \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \textit{and} \\ \exists \ulcorner\psi\urcorner \leqslant {}^{l}U \ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}D & \left(\begin{array}{c} \ulcorner\psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \ulcorner\varphi \wedge \psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^{l}U\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\varphi \wedge \psi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \vee_{l}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\vee_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \textit{and} \\ {}^{r}U_l \ \mathcal{E}_{qu.} \ {}^{r}U_r \ \mathcal{E}_{qu.} \ {}^{r}D \\ \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U_l \ \exists \ulcorner\psi\urcorner \leqslant {}^{l}U_r & \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U_l \ \textit{and} \ \ulcorner\psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U_r \ \textit{and} \ \ulcorner\varphi \wedge \psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U_l\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^{l}U_r\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\varphi \wedge \psi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta} \to_l}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\to_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \quad and \\ {}^rU_r \ \mathcal{E}_{qu.} \ {}^rD \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^rU_l \ \exists \ulcorner\psi\urcorner \leqslant {}^lU_r & \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^rU_l \quad and \quad \ulcorner\psi\urcorner \ \mathcal{I}_{ns.} \ {}^lU_r \quad and \quad \ulcorner\varphi \wedge \psi\urcorner \ \mathcal{I}_{ns.} \ {}^lD \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^rU_l\right) \ \mathcal{E}_{qu.} \ {}^rU_r \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^lU_r\right) \ \mathcal{E}_{qu.} \ {}^lU_l \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\varphi \to \psi\urcorner, {}^lD\right) \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^lU_r\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\varphi \to \psi\urcorner, {}^lD\right) \end{array} \right) \end{cases}$$

- where " $A \ \mathcal{E}_{qu.} \ B \ \mathcal{E}_{qu.} \ C$ " stands for " $A \ \mathcal{E}_{qu.} \ B$ *and* $B \ \mathcal{E}_{qu.} \ C$ "

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \neg_l}$$

$$(U, D) \in \mathcal{R}_{\neg_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^rU & \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^rU \qquad and \qquad \ulcorner\neg\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^lD \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^rU\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.}\left(\ulcorner\neg\varphi\urcorner, {}^lD\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi_{[t/x_n]} \vdash \Delta}{\Gamma, \forall x_n\ \varphi \vdash \Delta}\ \forall_l}$$

$$(U, D) \in \mathcal{R}_{\forall_l}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{cc} U, D \in \mathcal{SQ} & \\ \textit{and} & \\ {}^rU\ \mathcal{E}_{qu.}\ {}^rD & \\ \textit{and} & \\ \exists n \leqslant {}^lD\ \ \exists \ulcorner\forall x_n\varphi\urcorner \leqslant {}^lD\ \ \exists \ulcorner t\urcorner \leqslant {}^lU & \left(\begin{array}{c} \left(\begin{array}{c} \ulcorner\forall x_n\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lD \quad \textit{and} \quad \left(\ulcorner\varphi\urcorner, n\right) \in \mathcal{F}_{\checkmark x\ free} \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^lU \\ \textit{and} \\ \mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right), {}^lU\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\forall x_n\varphi\urcorner, {}^lD\right) \end{array}\right) \\ \\ \textit{or} \\ \\ \left(\begin{array}{c} \ulcorner\forall x_n\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lD \\ \textit{and} \\ \left(\ulcorner\varphi\urcorner, n\right) \in \left(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\right) \\ \textit{and} \\ \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lU \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^lU \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^lU\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\forall x_n\varphi\urcorner, {}^lD\right) \end{array}\right) \end{array}\right) \end{array}\right.$$

- " $\exists n \leqslant {}^rU\ \ \exists \ulcorner\forall x_n\varphi\urcorner \leqslant {}^rU\ \ \ldots$" stands for
  " $\exists n \leqslant {}^rU\ \ \exists m \leqslant {}^rU\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta^3_3(m) = 10\ \wedge\ \beta^2_3(m) = n\ \wedge\ \beta^1_3(m) = \ulcorner\varphi\urcorner\ \wedge\ \ \ldots\Big)$"
- " $\ulcorner\varphi\urcorner$ stands for " $\beta^1_3\big(\ulcorner\forall x_n\varphi\urcorner\big)$"
- " $\exists\ulcorner t\urcorner \leqslant {}^lU\ \ldots$" stands for " $\exists v \leqslant {}^lU\big(v \in \mathcal{T}\ \wedge\ \ \ldots\big)$"

....

$$\boxed{\dfrac{\Gamma, \varphi_{[x_k/x_n]} \vdash \Delta}{\Gamma, \exists x_n\ \varphi \vdash \Delta\ ^{2}}\ {\scriptstyle \exists_l}}$$

$$(U, D) \in \mathcal{R}_{\exists_l}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l}
U, D \in \mathcal{SQ} \\
\textit{and} \\
{}^{r}U\ \mathcal{E}_{qu.}\ {}^{r}D \\
\textit{and} \\
\exists \ulcorner x_n \urcorner \leqslant {}^{l}D\ \ \exists \ulcorner x_k \urcorner \leqslant {}^{l}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{l}D\ \ \exists \ulcorner \psi \urcorner \leqslant {}^{l}U \quad \left(\begin{array}{c}
\left(\begin{array}{c}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}U \\
\textit{and} \\
(\ulcorner \psi \urcorner, k) \in \mathcal{F}_{\checkmark x\ free} \ \ \textit{and}\ \ \Big(k \neq n \to (\ulcorner \psi \urcorner, n) \in (\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound})\Big) \\
\textit{and} \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{ub.}(\ulcorner \psi \urcorner, \ulcorner x_n \urcorner, k), n, 11) = \ulcorner \exists x_n \varphi \urcorner \\
\textit{and} \\
\mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{l}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^{l}D) \\
\textit{and} \\
\forall \ulcorner \theta \urcorner \leqslant {}^{l}D\ \ \Big(\ulcorner \theta \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \to (\ulcorner \theta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big) \\
\textit{and} \\
\forall \ulcorner \delta \urcorner \leqslant {}^{r}D\ \ \Big(\ulcorner \delta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \to (\ulcorner \delta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big)
\end{array}\right) \\
\\
\textit{or} \\
\\
\left(\begin{array}{c}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}U \quad \textit{and} \\
(\ulcorner \psi \urcorner, k) \in (\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}) \quad \textit{and} \quad (\ulcorner \psi \urcorner, n) \in (\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}) \\
\textit{and} \quad \alpha_3(\ulcorner \psi \urcorner, n, 11) = \ulcorner \exists x_n \varphi \urcorner \quad \textit{and} \\
\mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{l}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^{l}D)
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

where

- " $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \forall x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$" stands for
  " $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta^3_3(m) = 11\ \wedge\ \beta^2_3(m) = n\ \wedge\ \beta^1_3(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\Big)$"
- " $\ulcorner \varphi \urcorner$ stands for " $\beta^1_3(\ulcorner \forall x_n \varphi \urcorner)$"
- " $\exists \ulcorner x_k \urcorner \leqslant {}^{l}U\ \ldots$" stands for " $\exists k \leqslant {}^{l}U(\ulcorner x_k \urcorner = \alpha_3(k+1, 0, 0)\ \wedge\ \ldots)$"

....

[2] x_k has no free occurrence in Γ, $\exists x_n\ \varphi$ and Δ

$$\boxed{\dfrac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta}\,{\scriptstyle Ref}}$$

$$(U, D) \in \mathcal{R}_{Ref}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner t \urcorner \leqslant {}^{l}U \quad \left(\begin{array}{c} \ulcorner t = t \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \mathcal{R}_{em.} \left(\ulcorner t = t \urcorner, {}^{l}U \right) \quad \mathcal{E}_{qu.} \ {}^{l}D \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, t = s, \varphi_{[s/x_n]}, \varphi_{[t/x_n]} \vdash \Delta}{\Gamma, s = t, \varphi_{[t/x_n]} \vdash \Delta}\,{\scriptstyle Rep}}$$

$$(U, D) \in \mathcal{R}_{Rep}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner t \urcorner \leqslant {}^{l}U \ \exists \ulcorner s \urcorner \leqslant {}^{l}U \ \exists n \leqslant U \ \exists \ulcorner \varphi \urcorner \leqslant U^{U} \quad \left(\begin{array}{c} \ulcorner t = s \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner s \urcorner, n \right) \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n \right) \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \ulcorner s = t \urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \quad \textit{and} \quad \mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n \right) \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.} \left(\mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n \right), \mathcal{R}_{em.} \left(\ulcorner s = t \urcorner, {}^{l}D \right) \right) \\ \mathcal{E}_{qu.} \\ \mathcal{R}_{em.} \left(\mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner s \urcorner, n \right), \mathcal{R}_{em.} \left(\mathcal{S}^{\mathcal{F}}_{ub.} \left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n \right), \mathcal{R}_{em.} \left(\ulcorner t = s \urcorner, {}^{l}U \right) \right) \right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ {\scriptstyle \wedge_r}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\wedge_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \quad and \\ {}^{l}U_l\ \mathcal{E}_{qu.}\ {}^{l}U_r\ \mathcal{E}_{qu.}\ {}^{l}D \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U_l\ \exists \ulcorner\psi\urcorner \leqslant {}^{r}U_r \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U_l \quad and \quad \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U_r \quad and \quad \ulcorner\varphi \wedge \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}U_l\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^{r}U_r\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\varphi \wedge \psi\urcorner, {}^{r}D\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\scriptstyle \vee_{r1}}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r1}}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad and \\ {}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U\ \exists \ulcorner\psi\urcorner \leqslant {}^{r}D \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \quad and \quad \ulcorner\varphi \vee \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}U\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\varphi \vee \psi\urcorner, {}^{r}D\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\scriptstyle \vee_{r2}}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r2}}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{c} U, D \in \mathcal{SQ} \\ \textit{and} \\ {}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\ \textit{and} \\ \exists \ulcorner\psi\urcorner \leqslant {}^{r}U\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}D \quad \left(\begin{array}{c} \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \quad \textit{and} \quad \ulcorner\varphi \vee \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \mathcal{R}_{em.}\,(\ulcorner\psi\urcorner, {}^{r}U) \quad \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner\varphi \vee \psi\urcorner, {}^{r}D) \end{array}\right) \end{array}\right.$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ {\scriptstyle \to_{r}}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\to_r}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{c} U_l, U_r, D \in \mathcal{SQ} \\ \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U_l\ \ \exists \ulcorner\psi\urcorner \leqslant {}^{r}U_r \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}U_l \quad \textit{and} \quad \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U_r \quad \textit{and} \quad \ulcorner\varphi \to \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \mathcal{R}_{em.}\,(\ulcorner\psi\urcorner, {}^{r}U_r) \quad \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner\varphi \to \psi\urcorner, {}^{r}D) \\ \textit{and} \\ \mathcal{R}_{em.}\,(\ulcorner\varphi\urcorner, {}^{l}U_l) \quad \mathcal{E}_{qu.}\ {}^{l}D \end{array}\right) \end{array}\right.$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ \neg_r}$$

$$(U, D) \in \mathcal{R}_{\neg_r}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l} U, D \in \mathcal{SQ} \\ \quad\quad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}U \quad \textit{and} \quad \ulcorner\neg\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U\right)\ \mathcal{E}_{qu.}\ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\neg\varphi\urcorner, {}^{r}D\right)\ \mathcal{E}_{qu.}\ {}^{r}U \end{array}\right) \end{array}\right.$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi_{[x_k/x_n]}, \Delta}{\Gamma \vdash \forall x_n\ \varphi, \Delta\ ^{2}}\ \forall_r}$$

$$(U, D) \in \mathcal{R}_{\forall_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \mathit{and} \\ {}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\ \mathit{and} \\ \exists \ulcorner x_n \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner x_k \urcorner \leqslant {}^{r}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner \psi \urcorner \leqslant {}^{r}U \left(\begin{array}{c} \left(\begin{array}{c} \ulcorner \forall x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \mathit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \\ \mathit{and} \\ (\ulcorner \psi \urcorner, k) \in \mathcal{F}_{\checkmark x\ free}\ \ \mathit{and}\ \ \Big(k \neq n \rightarrow (\ulcorner \psi \urcorner, n) \in \big(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\big)\Big) \\ \mathit{and} \\ \alpha_3(\mathcal{S}^{\mathcal{F}}_{ub.}(\ulcorner \psi \urcorner, \ulcorner x_n \urcorner, k), n, 11) = \ulcorner \forall x_n \varphi \urcorner \\ \mathit{and} \\ \mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{r}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \forall x_n \varphi \urcorner, {}^{r}D) \\ \mathit{and} \\ \forall \ulcorner \theta \urcorner \leqslant {}^{r}D\ \ \Big(\ulcorner \theta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \rightarrow (\ulcorner \theta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big) \\ \mathit{and} \\ \forall \ulcorner \delta \urcorner \leqslant {}^{r}D\ \ \Big(\ulcorner \delta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \rightarrow (\ulcorner \delta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big) \end{array} \right) \\ \\ \mathit{or} \\ \left(\begin{array}{c} \ulcorner \forall x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \mathit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \quad \mathit{and} \\ (\ulcorner \psi \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big) \quad \mathit{and} \quad (\ulcorner \psi \urcorner, n) \in \big(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\big) \\ \mathit{and} \quad \alpha_3(\ulcorner \psi \urcorner, n, 11) = \ulcorner \forall x_n \varphi \urcorner \quad \mathit{and} \\ \mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{r}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \forall x_n \varphi \urcorner, {}^{r}D) \end{array} \right) \end{array} \right) \end{cases}$$

where

- “ $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \forall x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$” stands for
  “ $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta^3_3(m) = 10\ \wedge\ \beta^2_3(m) = n\ \wedge\ \beta^1_3(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\Big)$”
- “ $\ulcorner \varphi \urcorner$ stands for “ $\beta^1_3\big(\ulcorner \forall x_n \varphi \urcorner\big)$”
- “ $\exists \ulcorner x_k \urcorner \leqslant {}^{r}U\ \ldots$” stands for “ $\exists k \leqslant {}^{r}U\big(\ulcorner x_k \urcorner = \alpha_3(k+1, 0, 0)\ \wedge\ \ldots\big)$”

....

[2] x_k has no free occurrence in Γ, $\forall x_n\ \varphi$ and Δ

$$\boxed{\dfrac{\Gamma \vdash \varphi_{[t/x_n]}, \Delta}{\Gamma \vdash \exists x_n\ \varphi, \Delta}\ \exists_r}$$

$$(U, D) \in \mathcal{R}_{\exists_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \textit{and} \\ {}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\ \textit{and} \\ \exists n \leqslant {}^{r}D\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner t \urcorner \leqslant {}^{r}U & \left(\begin{pmatrix} \ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \textit{and} \quad \left(\ulcorner \varphi \urcorner, n\right) \in \mathcal{F}_{\checkmark x\ free} \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^{r}U \\ \textit{and} \\ \mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right), {}^{r}U\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner \exists x_n \varphi \urcorner, {}^{r}D\right) \end{pmatrix} \\ \textit{or} \\ \begin{pmatrix} \ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \left(\ulcorner \varphi \urcorner, n\right) \in \left(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\right) \\ \textit{and} \\ \ulcorner \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^{r}U \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner \varphi \urcorner, {}^{r}U\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner \exists x_n \varphi \urcorner, {}^{r}D\right) \end{pmatrix} \right) \end{cases}$$

- " $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$" stands for
  " $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \left(m \in \mathcal{F}\ \wedge\ \beta^3_3(m) = 11\ \wedge\ \beta^2_3(m) = n\ \wedge\ \beta^1_3(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\right)$"
- " $\ulcorner \varphi \urcorner$ stands for " $\beta^1_3\left(\ulcorner \exists x_n \varphi \urcorner\right)$"
- " $\exists \ulcorner t \urcorner \leqslant {}^{r}U\ \ldots$" stands for " $\exists v \leqslant {}^{r}U\left(v \in \mathcal{T}\ \wedge\ \ldots\right)$"

....

$$\boxed{\dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,{\scriptstyle wkn_l}}$$

$$(U, D) \in \mathcal{R}_{wkn_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \qquad and \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \qquad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}D \qquad \begin{pmatrix} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}D\right) \ \mathcal{E}_{qu.} \ {}^{l}U \end{pmatrix} \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\,{\scriptstyle wkn_r}}$$

$$(U, D) \in \mathcal{R}_{wkn_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \qquad and \\ {}^{l}U \ \mathcal{E}_{qu.} \ {}^{l}D \\ \qquad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}D \qquad \begin{pmatrix} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{r}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}D\right) \ \mathcal{E}_{qu.} \ {}^{r}U \end{pmatrix} \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ {\scriptstyle ctr_l}} \qquad \boxed{\dfrac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\ {\scriptstyle ctr_r}}$$

$$(U, D) \in \mathcal{R}_{ctr\,{}_{l\&r}} \iff \begin{cases} U, D \in \mathcal{SQ} \\ \quad and \\ {}^{l}U \ \mathcal{E}_{qu.} \ {}^{l}D \\ \quad and \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ {\scriptstyle cut}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{cut}$$

$$\iff$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U_l \qquad \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{r}U_l \quad and \quad \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U_r \\ and \\ \mathcal{U}_{nion} \left(\mathcal{R}_{em.} \left(\ulcorner\varphi\urcorner, {}^{l}U_r \right), {}^{l}U_l \right) \quad \mathcal{E}_{qu.} \ {}^{l}D \\ and \\ \mathcal{U}_{nion} \left(\mathcal{R}_{em.} \left(\ulcorner\varphi\urcorner, {}^{r}U_l \right), {}^{r}U_r \right) \quad \mathcal{E}_{qu.} \ {}^{r}D \end{array} \right) \end{cases}$$

....

We use the following notations:

Notation 2.1

- $\mathcal{R}^0 = \mathcal{R}_{ax}$
- $\mathcal{R}^1 = \begin{cases} \mathcal{R}_{\wedge_{l1}} \cup \mathcal{R}_{\wedge_{l2}} \cup \mathcal{R}_{\neg_l} \cup \mathcal{R}_{\forall_l} \cup \mathcal{R}_{\exists_l} \cup \mathcal{R}_{\vee_{r1}} \cup \mathcal{R}_{\vee_{r2}} \cup \mathcal{R}_{\neg_r} \cup \mathcal{R}_{\forall_r} \\ \quad \cup \mathcal{R}_{\exists_r} \cup \mathcal{R}_{wkn_l} \cup \mathcal{R}_{wkn_r} \cup \mathcal{R}_{ctr\ l\&r} \cup \mathcal{R}_{Rep} \cup \mathcal{R}_{Ref} \cup \mathcal{R}_{cut} \end{cases}$
- $\mathcal{R}^2 = \mathcal{R}_{\vee_l} \cup \mathcal{R}_{\rightarrow_l} \cup \mathcal{R}_{\wedge_r}$.

We say an integer codes a proof if it is of the form

$$\alpha_4(\texttt{node}, \texttt{left proof-tree}, \texttt{right proof-tree}, \texttt{arity of the rule}).$$

Definition 2.2

The set $\mathcal{P}_{roofs}$ of the codes of all possible proofs is defined by

$$k = \alpha_4(n_1, n_2, n_3, n_4) \in \mathcal{P}_{roofs}$$

$$\Longleftrightarrow$$

$$\begin{cases} n_4 = 0 \quad \textit{and} \quad n_3 = 0 \quad \textit{and} \quad n_2 = 0 \quad \textit{and} \quad n_1 \in \mathcal{R}^0 \\ \qquad \textit{or} \\ n_4 = 1 \quad \textit{and} \quad n_3 = 0 \quad \textit{and} \quad n_2 \in \mathcal{P}_{roofs} \quad \textit{and} \quad \big(\beta_4^1(n_2), n_1\big) \in \mathcal{R}^1 \\ \qquad \textit{or} \\ n_4 = 2 \quad \textit{and} \quad n_3 \in \mathcal{P}_{roofs} \quad \textit{and} \quad n_2 \in \mathcal{P}_{roofs} \quad \textit{and} \quad \big(\beta_4^1(n_3), \beta_4^1(n_2), n_1\big) \in \mathcal{R}^2. \end{cases}$$

Notation 2.2

Given any proof P we write $\ulcorner P \urcorner$ for the integer described above that codes this proof.

Lemma 2.1

The set $\mathcal{P}_{roofs}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$.

Proof of Lemma 2.1:

$$\chi_{\mathcal{P}_{roofs}}(n) = \begin{cases} 1 & \textit{if} \quad \beta_4^4(n) = 0 \ \textit{and}\ \beta_4^3(n) = 0 \ \textit{and}\ \beta_4^2(n) = 0 \ \textit{and}\ \beta_4^1(n) \in \mathcal{R}^0 \\ 1 & \textit{if} \quad \beta_4^4(n) = 1 \ \textit{and}\ \beta_4^3(n) = 0 \ \textit{and}\ \chi_{\mathcal{P}_{roofs}}\big(\beta_4^2(n)\big) = 1 \ \textit{and}\ \big(\beta_4^1 \circ \beta_4^2(n), \beta_4^1(n)\big) \in \mathcal{R}^1 \\ 1 & \textit{if} \quad \beta_4^4(n) = 2 \ \textit{and}\ \chi_{\mathcal{P}_{roofs}}\big(\beta_4^3(n)\big) = \chi_{\mathcal{P}_{roofs}}\big(\beta_4^2(n)\big) = 1 \ \textit{and}\ \big(\beta_4^1 \circ \beta_4^3(n), \beta_4^1 \circ \beta_4^2(n), \beta_4^1(n)\big) \in \mathcal{R}^2 \\ 0 & \textit{otherwise.} \end{cases}$$

By Lemma 1.2, $\mathcal{P}_{roofs}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$

$\square$

5.3 Undecidability of Robinson Arithmetic

Since Robinson Arithmetic is some very weak theory, one might think that it should be easy to solve any question posed in such a theory. To the contrary, it turns out that Robinson Arithmetic is undecidable.

Definition 3.1

(1) A theory T is recursive if the following set is recursive:

$$\Big\{ \ulcorner \varphi \urcorner \mid \varphi \in T \Big\}.$$

(2) A theory T is decidable if the following set is recursive:

$$thms\,(T) = \Big\{ \ulcorner \varphi \urcorner \mid T \ \vdash_c \ \varphi \Big\}.$$

Informally, this means that a theory is decidable if one has an algorithm which on any input that represents a formula φ stops and accepts if T proves φ, and stops and rejects if T does not prove φ.

Theorem 3.1

Given any theory T, the set

$$\left\{\left(\ulcorner P \urcorner, \ulcorner \varphi \urcorner\right) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T \vdash_c \varphi\right\}$$

is

- primitive recursive if T is primitive recursive,
- recursive if T is recursive.

Proof of Theorem 3.1:

First, P is proof that $T \vdash_c \varphi$ if P is a proof-tree whose root is some sequent "$\Delta \vdash \varphi$" for some finite $\Delta \subseteq T$.
We recall Lemma 1.10 which stated that the following set is $\mathcal{P}rim.\ \mathcal{R}ec.$

$$\mathcal{F}_{\checkmark closed} = \left\{\ulcorner \varphi \urcorner \mid \varphi \textit{ is a closed formula from } \mathcal{L}_{\mathcal{A}}\right\}$$

Second, let $\chi_T \in \mathbb{N} \longrightarrow \mathbb{N}$ be the characteristic function of T. i.e.,

$$\chi_T(n) = \begin{cases} 1 & \textit{if} \quad n = \ulcorner \varphi \urcorner \in T \\ 0 & \textit{otherwise.} \end{cases}$$

The characteristic function of

$$A = \left\{\left(\ulcorner P \urcorner, \ulcorner \varphi \urcorner\right) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T \vdash_c \varphi\right\}$$

is

$$\chi_A(n,m) = \begin{cases} 1 & \textit{if} \quad \begin{cases} n \in \mathcal{P}_{roofs} \\ \textit{and} \\ m \in \mathcal{F} \\ \textit{and} \\ \forall i \leqslant {}^{l}\beta^1_4(n) \quad \llcorner {}^{l}\beta^1_4(n) \urcorner^{i} \neq 0 \longrightarrow \chi_T\left(\llcorner {}^{l}\beta^1_4(n) \urcorner^{i}\right) = 1 \\ \textit{and} \\ \forall j \leqslant {}^{r}\beta^1_4(n) \quad \left(\llcorner {}^{r}\beta^1_4(n) \urcorner^{j}\right) = m \end{cases} \\ 0 & \textit{otherwise.} \end{cases}$$

We see that this function is primitive recursive if χ_T is primitive recursive, and total recursive if χ_T is total recursive.

$\square$

Proposition 3.1

Given any theory T,

$$\left\{ \ulcorner\psi\urcorner \mid \psi \in T \right\} \textit{ is recursive } \Longrightarrow \left\{ \ulcorner\varphi\urcorner \mid T \vdash_c \varphi \right\} \textit{ is recursively enumerable.}$$

Proof of Proposition 3.1:

We set

$$A = \left\{ \left(\ulcorner P \urcorner, \ulcorner\varphi\urcorner \right) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T \vdash_c \varphi \right\} \quad \textit{and} \quad B = \left\{ \ulcorner\varphi\urcorner \mid T \vdash_c \varphi \right\}.$$

By Theorem 3.1, the set A is recursive. Hence the function

$$\chi_B^{part}(n) = 1 \dot{-} \Big(1 \dot{-} \big(\mu k \;\; \chi_A(k, n) = 1 \big) \Big)$$

is also $\mathcal{P}art.\ \mathcal{R}ec.$ and it satisfies $n \in B \iff \chi_B^{part}(n) = 1$. $\square$

We recall that a theory is **complete** if it is both *consistent* and *satisfies for each formula* φ *either*

$$T \vdash_c \varphi \quad \textit{or} \quad T \vdash_c \neg\varphi.$$

Corollary 3.1

Let T be any recursive theory.

If T is complete, then T is decidable.

Proof of Corollary 3.1:

By Proposition 3.1 both sets

$$\left\{ \ulcorner\varphi\urcorner \mid T \vdash_c \varphi \right\} \quad \textit{and} \quad \left\{ \ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi \right\}$$

are recursively enumerable. Since T is complete we have

$$\left\{\ulcorner\varphi\urcorner \mid T \nvdash_c \varphi\right\} = \left\{\ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi\right\}$$

Hence

$$\mathbb{N} \setminus \left\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\right\} = \left(\mathbb{N} \setminus \mathcal{F}\right) \cup \left\{\ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi\right\}$$

is recursively enumerable, which yields the result. □

Theorem 3.2

Let T be any $\mathcal{L}_{\mathcal{A}}$-theory. If T is consistent and extends Robinson Arithmetic ($\mathcal{R}ob. \subseteq T$), then

$$T \text{ is undecidable.}$$

Notice that it is equivalent to say that any $\mathcal{L}_{\mathcal{A}}$-theory which extends $\mathcal{R}ob.$ satisfies

$$T \text{ is consistent} \iff T \text{ is undecidable.}$$

Because if a theory T is inconsistent, then it proves everything, therefore we have

$$\left\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\right\} = \mathcal{F}.$$

which shows that T decidable.

Proof of Theorem 3.2:

Towards a contradiction, we assume that T is decidable. We then consider

$$\mathcal{F}_{\checkmark x_0 \text{ !free}} = \left\{\ulcorner\varphi\urcorner \mid \varphi \textit{ is a formula whose only free variable is } x_0\right\}.$$

Since we already know that the set

$$\mathcal{F}_{\checkmark x \text{ free}} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathcal{A}} \textit{ and } x_n \textit{ is free in } \varphi\right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$ (see Lemma 1.8) and we have

$$\ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \iff \left(\ulcorner\varphi\urcorner, 0\right) \in \mathcal{F}_{\checkmark x \text{ free}} \quad \textit{and} \quad \forall n \leqslant \ulcorner\varphi\urcorner \; \left(n \neq 0 \to \left(\ulcorner\varphi\urcorner, n\right) \notin \mathcal{F}_{\checkmark x \text{ free}}\right).$$

an immediate consequence is that $\mathcal{F}_{\checkmark x_0 \text{ !free}}$ is also $\mathcal{P}rim.\ \mathcal{R}ec.$

Then the set[a]

$$\left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \textit{ and } T \vdash_c \varphi[n/x_0]\right\}$$

$$=$$

$$\Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \quad and \quad \mathcal{S}^{\mathcal{F}}_{ub.}(\ulcorner\varphi\urcorner, \ulcorner n\urcorner, 0) \in \{\ulcorner\psi\urcorner \mid T \vdash_c \psi\}\Big\}$$

is recursive.
We then consider the following set

$$\ddot{\mathcal{D}}_{iag.} = \Big\{k \in \mathbb{N} \mid (k,k) \notin \Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \quad and \quad T \vdash_c \varphi_{[n/x_0]}\Big\}\Big\}.$$

$\ddot{\mathcal{D}}_{iag.}$ is clearly recursive, therefore there exists some formula $\varphi_{\ddot\smile}(x_0)$ that represents $\ddot{\mathcal{D}}_{iag.}$. This means that for all $k \in \mathbb{N}$ we have:

- $k \in \ddot{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \varphi_{\ddot\smile [k/x_0]}$
- $k \notin \ddot{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \neg\varphi_{\ddot\smile [k/x_0]}$.

It is enough to consider the closed formula $\varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]}$ where $\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil$ stands for the term $\overbrace{S\ldots S}^{\ulcorner\varphi_{\ddot\smile}\urcorner}0$.
As with the Halting problem where we asked the question whether our machine would stop on its own code as input, here we ask the question whether or not T proves $\varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]}$. This depends on whether $\ulcorner\varphi_{\ddot\smile}\urcorner$ belongs to $\ddot{\mathcal{D}}_{iag.}$ or not.

- $\ulcorner\varphi_{\ddot\smile}\urcorner \in \ddot{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]} \implies T \vdash_c \varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]} \implies \ulcorner\varphi_{\ddot\smile}\urcorner \notin \ddot{\mathcal{D}}_{iag.}$
- $\ulcorner\varphi_{\ddot\smile}\urcorner \notin \ddot{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \neg\varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]} \implies T \vdash_c \neg\varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]}$.

  Since T is consistent we cannot have both

$$T \vdash_c \neg\varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]} \qquad and \qquad T \vdash_c \varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]}.$$

  Therefore, we have $T \nvdash_c \varphi_{\ddot\smile [\lceil\ulcorner\varphi_{\ddot\smile}\urcorner\rceil/x_0]}$ which immediately implies $\ulcorner\varphi_{\ddot\smile}\urcorner \in \ddot{\mathcal{D}}_{iag.}$.

We obtain

$$\ulcorner\varphi_{\ddot\smile}\urcorner \in \ddot{\mathcal{D}}_{iag.} \iff \ulcorner\varphi_{\ddot\smile}\urcorner \notin \ddot{\mathcal{D}}_{iag.}.$$

This contradiction finishes the proof that T is undecidable.

□

[a] we recall that on page 145 we defined $\mathcal{S}^{\mathcal{F}}_{ub.}(n_u, n_t, n) = \begin{cases} \ulcorner\varphi[t/x_n]\urcorner & if\ n_\varphi = \ulcorner\varphi\urcorner \in \mathcal{F},\ n_t = \ulcorner t\urcorner \in \mathcal{T} \\ 0 & otherwise\ . \end{cases}$

This brings to the mind what we did in the proof of Proposition 7.5.
We propose again a picture that illustrates this diagonal argument. If $(\varphi_i)_{i\in\mathbb{N}}$ is a enumeration

of all the formulas with x_0 as one and only free variable, we make sure to define a formula which satisfies this requirement although it is none of them.

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5		φ_n	
$\ulcorner\varphi_0\urcorner$	0	1	1	0	1	0	$\ldots$	0	$\ldots$
$\ulcorner\varphi_1\urcorner$	1	1	1	0	0	0	$\ldots$	0	$\ldots$
$\ulcorner\varphi_2\urcorner$	1	0	1	0	0	0	$\ldots$	1	$\ldots$
$\ulcorner\varphi_3\urcorner$	0	0	1	0	1	0	$\ldots$	0	$\ldots$
$\ulcorner\varphi_4\urcorner$	0	1	0	1	1	1	$\ldots$	0	$\ldots$
$\ulcorner\varphi_5\urcorner$	1	1	0	0	0	0	$\ldots$	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$\ulcorner\varphi_n\urcorner$	1	0	0	0	1	1	$\ldots$	1	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

There is a 1 on the array – for instance on row 3 and column 2 – if $T \vdash_c \varphi_2(\lceil\ulcorner\varphi_3\urcorner\rceil)$, and there is a 0 – for instance on row 2 and column 5 – if $T \nvdash_c \varphi_5(\lceil\ulcorner\varphi_2\urcorner\rceil)$.

Now if T is decidable, the whole array is decidable. This means there is a Decider that on any input (n, m) accepts if there is a 1 on position (n, m), and rejects if there is a 0. Furthermore, for the whole array is decidable, its diagonal is also decidable. Hence the complement of the diagonal is decidable as well. Finally, since all recursive sets are representable, the complement of the diagonal is represented by some formula among the enumeration – say φ_n – which inevitably stumbles on $\lceil\ulcorner\varphi_n\urcorner\rceil$).

Theorem 3.3: Undecidability of first order logic

The following set is not recursive:

$$\left\{ \ulcorner\varphi\urcorner \mid \ \vdash_c \varphi \right\}.$$

Proof of Theorem 3.3:

Since $\mathcal{R}ob.$ is a finite theory, we let $\varphi_{\mathcal{R}ob.}$ be the conjunction of the seven axioms from $\mathcal{R}ob.$.
For any formula ψ we have

$$\mathcal{R}ob. \vdash_c \psi \iff \varphi_{\mathcal{R}ob.} \vdash_c \psi \iff \ \vdash_c \varphi_{\mathcal{R}ob.} \to \psi.$$

$$\ulcorner\psi\urcorner \in \left\{ \ulcorner\varphi\urcorner \mid \mathcal{R}ob. \vdash_c \varphi \right\} \iff \ulcorner\varphi_{\mathcal{R}ob.} \to \psi\urcorner \in \left\{ \ulcorner\varphi\urcorner \mid \ \vdash_c \varphi \right\}.$$

Therefore, if the set of codes of universally valid formulas were decidable, then Robinson arithmetic would also be decidable.

□

Theorem 3.4: Gödel's first incompleteness theorem

If $\mathcal{R}ob. \subseteq T$ is any $\mathcal{L}_{\mathcal{A}}$-theory both consistent and recursive, then

$$T \text{ is incomplete.}$$

Proof of Theorem 3.4:

By corollary 3.1 every recursive complete theory is decidable. By Theorem 3.2 the theory T is undecidable.

□

Another way of stating this theorem that one encounters very frequently among the philosophy community is the following:

There exists a true sentence that is not provable.

Or even,

There exists a sentence that is true although it is not provable.

"Provable" usually refers to Peano Arithmetic, and "true" means true in the standard model. And very often people add

Taking this true sentence as an axiom inevitably yields another one that is not provable.

Of course the understatement hidden behind is "the complete theory of the standard model is not recursive".