Part II

Relativization and Absoluteness

Chapter 6

From Inside a Class

6.1 Relativization

For each formula and class we define what this formula becomes when the sets involved are the ones that belong to the class. For this purpose, we recall[1] that a class $\mathbf{C}$ is nothing but a formula with one free variable — that may or may not have other free variables that behave as parameters — $\varphi_{\mathbf{C}}$.

Definition 164. *Let* $\mathbf{M}$ *be any class and* φ *any formula. The formula* $(\varphi)^{\mathbf{M}}$ *is defined by induction on* $ht(\varphi)$ *by:*

$$x = y^{\mathbf{M}} := x = y$$

$$x \in y^{\mathbf{M}} := x \in y$$

$$(\neg\varphi)^{\mathbf{M}} := \neg(\varphi)^{\mathbf{M}}$$

$$(\varphi_0 \wedge \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \wedge (\varphi_1)^{\mathbf{M}}$$

$$(\varphi_0 \vee \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \vee (\varphi_1)^{\mathbf{M}}$$

$$(\varphi_0 \rightarrow \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \rightarrow (\varphi_1)^{\mathbf{M}}$$

$$(\varphi_0 \leftrightarrow \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \leftrightarrow (\varphi_1)^{\mathbf{M}}$$

$$(\exists x\, \varphi)^{\mathbf{M}} := \exists x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$$

$$(\forall x\, \varphi)^{\mathbf{M}} := \forall x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$$

So, assuming that the class $\mathbf{M}$ is described by the formula $\psi_{\mathbf{M}}(x)$, we see that $(\exists x\, \varphi)^{\mathbf{M}}$ stands for $\exists x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$ which really is $\exists x \left(x \in \mathbf{M}\ \wedge\ (\varphi)^{\mathbf{M}}\right)$, i.e., $\exists x \left(\psi_{\mathbf{M}}(x)\ \wedge\ (\varphi)^{\mathbf{M}}\right)$. *Idem* with

[1] This can be found in Section 2.4.

the universal quantifier: the formula $\big(\forall x\, \varphi\big)^{\mathbf{M}}$ really is $\forall x \left(\psi_{\mathbf{M}}(x) \;\to\; (\varphi)^{\mathbf{M}}\right)$.

Remark 165. Notice that the relativization of various notions that we introduced requires to go back to the original definition. For instance,

(1) $\big(x \subseteq y\big)^{\mathbf{M}} \iff \big(x \cap \mathbf{M}\big) \subseteq y$ holds since

- $x \subseteq y \iff \forall z\; (z \in x \longrightarrow z \in y)$
- $\big(\forall z\; (z \in x \longrightarrow z \in y)\big)^{\mathbf{M}} := \forall z \in \mathbf{M}\; (z \in x \longrightarrow z \in y)$
- $\forall z \in \mathbf{M}\; (z \in x \longrightarrow z \in y) \iff x \cap \mathbf{M} \subseteq y.$

(2) $\big(\mathcal{P}(x)\big)^{\mathbf{M}} = \{z \in \mathbf{M} \mid z \cap \mathbf{M} \subseteq x\}$ holds — and in case $\mathbf{M}$ is transitive, $\big(\mathcal{P}(x)\big)^{\mathbf{M}} = \mathcal{P}(x) \cap \mathbf{M}$ holds too — since

- $y = \mathcal{P}(x) \iff \forall z\; (z \in y \longleftrightarrow z \subseteq x)$
- $\big(\forall z\; (z \in y \longleftrightarrow z \subseteq x)\big)^{\mathbf{M}} := \forall z \in \mathbf{M} \left(z \in y \longleftrightarrow \big(z \subseteq x\big)^{\mathbf{M}}\right)$
- $\forall z \in \mathbf{M} \left(z \in y \longleftrightarrow \big(z \subseteq x\big)^{\mathbf{M}}\right) \iff \forall z \in \mathbf{M}\; \big(z \in y \longleftrightarrow z \cap \mathbf{M} \subseteq x\big).$

From now on, we may use expressions such as "**ZF** *proves that* φ *holds true in* **M**" or "**ZFC** *proves that the theory* $\mathcal{T}$ *holds true in* **M**", where each time, being true in **M** refers to the relativized formula. So,

Definition 166. *Given any formula* φ *and any theory* $\mathcal{T}$*,*

(1) "φ holds true in **M**" *or* "$\mathbf{M} \models \varphi$" *stands for* "$(\varphi)^{\mathbf{M}}$"

(2) $\left.\begin{array}{c}\text{“}\mathcal{T}\textit{ holds true in }\mathbf{M}\text{”}\\ \textit{or equivalently}\\ \text{“}\mathbf{M}\textit{ is a model of }\mathcal{T}\text{”}\end{array}\right\}$ *stands for the assumption that for every* $\varphi \in \mathcal{T}$*,* "$(\varphi)^{\mathbf{M}}$"*.*

This means that when we say, for a given class **M**, that

$$\text{“}\mathbf{ZFC}\text{ proves }\mathbf{M} \models 2^{\aleph_0} = \aleph_2\text{”},$$

what we really mean is the statement:

$$\mathbf{ZFC} \vdash_c \big(2^{\aleph_0} = \aleph_2\big)^{\mathbf{M}}.$$

For instance, we will see that "**ZF** proves $\mathbf{L} \models \forall\alpha \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1}$", meaning that $\mathbf{ZF} \vdash_c \left(\forall\alpha \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1}\right)^{\mathbf{L}}$.
We will also say, working with **ZF**, that "$\mathbf{L} \models \mathbf{ZFC}$"; where what we mean is that for every axiom $\varphi \in \mathbf{ZFC}$, one has $\mathbf{ZF} \vdash_c (\varphi)^{\mathbf{L}}$.

Lemma 167. *Let φ be any closed formula and $\mathbf{M}$ be any non-empty class.*

$$\vdash_c \varphi \implies \vdash_c (\varphi)^{\mathbf{M}}.$$

Proof of Lemma 167: By the completeness Theorem, the statement comes down to

$$\models \varphi \implies \models (\varphi)^{\mathbf{M}}.$$

But, since $\models \varphi$ holds true, it follows that in any model $\mathcal{M} = \left\langle |\mathcal{M}|, \in_{\restriction|\mathcal{M}|}\right\rangle$ one has $\mathcal{M} \models \varphi$ — meaning in any set $|\mathcal{M}|$ equipped with the membership relation φ holds. So, in particular, for every set $|\mathcal{M}| \cap \mathbf{M}$, we have

$$\left\langle |\mathcal{M}| \cap \mathbf{M}, \in_{\restriction|\mathcal{M}| \cap \mathbf{M}}\right\rangle \models \varphi.$$

$\square$ 167

6.2 Consistency and Model Existence

Lemma 168. *Let $\mathcal{S}, \mathcal{T}$ be any $\mathcal{L}$[2]-theory, and $\mathbf{M}$ any non-empty class.*

If both $\mathcal{S} \vdash_c$ "$\mathbf{M}$ is a model of $\mathcal{T}$" and $\mathcal{S} \nvdash \bot$, then $\mathcal{T} \nvdash \bot$.

Proof of Lemma 168: Towards a contradiction, we assume $\mathcal{T} \vdash_c \bot$. One has there exists some closed formula φ such that

$$\mathcal{T} \vdash_c (\varphi \wedge \neg\varphi)$$

and by the compactness Theorem there exist finitely many formulas $\varphi_0, \ldots, \varphi_n \in \mathcal{T}$ such that

$$\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \vdash_c (\varphi \wedge \neg\varphi)$$

which gives

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \longrightarrow (\varphi \wedge \neg\varphi) \right)$$

[2] $\mathcal{L}$ stands for the language of set theory. i.e., its signature is $\{\in, =\}$.

hence, by Lemma 167,

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \longrightarrow (\varphi \wedge \neg\varphi) \right)^{\mathbf{M}}$$

which yields

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} (\varphi_i)^{\mathbf{M}} \longrightarrow \big((\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}} \big) \right).$$

$\mathcal{S} \vdash_c$ " $\mathbf{M}$ is a model of $\mathcal{T}$ " yields $\mathcal{S} \vdash_c (\varphi_i)^{\mathbf{M}}$ (any $i \leqslant n$), hence

$$\mathcal{S} \vdash_c \bigwedge_{0 \leqslant i \leqslant n} (\varphi_i)^{\mathbf{M}}$$

and by *modus ponens*:

$$\mathcal{S} \vdash_c \big((\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}} \big)$$

contradicting the fact that $\mathcal{S}$ is consistent.

$\square$ 168

Chapter 7

The Mostowski Collapse

7.1 Recursion on Well-founded and "Set-Like" Relations

This section is concerned with collapsing certain classes in order to render them transitive. Morever, the collapsed class and the original one are then isomorphic and the isomorphism is unique. This gives a very natural way of transforming a class — that possibly satisfies various axioms of **ZFC** — into a transitive one which is often easier to handle.

Definition 169 (**ZF** $\smallsetminus$ {**AF**})**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$.

$$\mathbf{R} \textit{ is "set-like" on } \mathbf{M} \iff \forall y \in \mathbf{M} \;\; \mathbf{R}^{-1}[y] = \{x \in \mathbf{M} \mid x\mathbf{R}y\} \textit{ is a set.}$$

A relational is "set-like" on a class if the inverse image of every element is a set. In other words, $\mathbf{M}$ can be presented as a collection of sets which are the equivalent classes of the equivalence relation $x \sim y \iff \exists z \in \mathbf{M} \;\; \big(x\mathbf{R}z \wedge y\mathbf{R}z\big)$.

We consider the closure of taking the predecessors of an element x along $\mathbf{R}$.

Definition 170 (**ZF** $\smallsetminus$ {**AF**})**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{X} \subseteq \mathbf{M}$ *and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$. *We define* $cl\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ *by:*

- $cl^0\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}} = \mathbf{X} \cap \mathbf{R}^{-1}[x] = \{z \in \mathbf{X} \mid z\mathbf{R}x\}$
- $cl^{n+1}\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup\left\{\lceil\underline{z}\rceil^{\mathbf{R}}_{\mathbf{X}} \mid z \in cl^n\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right)\right\}$
- $cl\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup\left\{cl^n\left(\lceil\underline{x}\rceil^{\mathbf{R}}_{\mathbf{X}}\right) \mid n \in \omega\right\}$

Remark 171. If $\mathbf{R}$ is "set-like" on $\mathbf{M}$, then one can easily show by induction on the integers, that for each integer n,

$$cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \text{ is a set.}$$

Hence,

$$cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup \left\{ cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \mid n \in \omega \right\} \text{ is a set.}$$

Definition 172 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$*.*

$$\mathbf{R} \textit{ is "well-founded" on } \mathbf{M} \iff \forall X \subseteq \mathbf{M} \left(X \neq \varnothing \longrightarrow \exists y \in X \, \forall x \in X \;\; \neg x \mathbf{R} y \right).$$

So, a relational $\mathbf{R}$ on a class $\mathbf{M}$ is well-founded if every non-empty ***subset*** of $\mathbf{M}$ has a minimal element.

Remark 173. If $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ is both well-founded and "set-like" on $\mathbf{M}$, then for each $x \in \mathbf{M}$, the graph of $\mathbf{R}$ on x is a set $\mathcal{G} = (V, E)$ defined by:

$$V = \{x\} \cup cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right) \quad \text{and} \quad E = \left\{(a,b) \in E \times E \mid b \mathbf{R} a\right\}.$$

This directed graph is acyclic[1].

Theorem 174 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*.*

$$\forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X} \, \forall x \in \mathbf{X} \;\; \neg x \mathbf{R} y \right).$$

So, this theorem claims that the property that defines a well-founded relational $\mathbf{R}$ on a class $\mathbf{M}$, can be *lifted from non-empty subsets to non-empty classes* provided that the relational $\mathbf{R}$ be "set-like" on $\mathbf{M}$ in addition to being well-founded.

[1] By well-foundedness of $\mathbf{R}$.

Proof of Theorem 174: Take any $x \in \mathbf{X}$. If x is $\mathbf{R}$-minimal in $\mathbf{X}$ we are done. Otherwise, we consider $cl\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ which is a set since $\mathbf{R}$ is "set-like" on $\mathbf{M}$. Since $\mathbf{R}$ is well-founded on $\mathbf{M}$ and $cl\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ is a subset of $\mathbf{M}$, it follows that $cl\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ admits some $\mathbf{R}$-minimal element y. We show that y is also $\mathbf{R}$-minimal in $\mathbf{X}$. Indeed, since $y \in cl\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ there exists some integer k such that $y \in cl^k\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ and any $z \in \mathbf{X}$ that would satisfy $z\mathbf{R}y$ would belong to $cl^{k+1}\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right) \subseteq cl\left(\lceil \underline{x} \rceil_{\mathbf{X}}^{\mathbf{R}}\right)$ which would contradict the $\mathbf{R}$-minimality of y.

$\square$ 174

We saw on page 37 that one can define a functional by transfinite induction on the ordinals. This result can easily be extended from the ordinals to any well-founded and "set-like" relation.

Theorem 175 ([**ZF** $\smallsetminus$ {**AF**}). *Transfinite recursion along well-founded set-like relation] Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, and* $\mathbf{F} : \mathbf{M} \times \mathbf{V} \to \mathbf{V}$ *be any functional.*
There exists some unique $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *such that such that*

$$\forall x \in \mathbf{M} \quad \mathbf{G}(x) = \mathbf{F}\Big(x\,, \mathbf{G} \restriction_{\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}}\Big).$$

Proof of Theorem 175:

Uniqueness: Assume there exist two different functionals $\mathbf{G_1}$ and $\mathbf{G_2}$. By Theorem 174 the non-empty class $\{x \in \mathbf{M} \mid \mathbf{G_1}(x) \neq \mathbf{G_2}(x)\}$ has an $\mathbf{R}$-least element y. By construction, one comes to the following contradiction:

$$\mathbf{G_1}(y) = \mathbf{F}\Big(y\,, \mathbf{G_1} \restriction_{\lceil \underline{y} \rceil_{\mathbf{M}}^{\mathbf{R}}}\Big) = \mathbf{F}\Big(y\,, \mathbf{G_2} \restriction_{\lceil \underline{y} \rceil_{\mathbf{M}}^{\mathbf{R}}}\Big) = \mathbf{G_2}(y).$$

Existence: we construct functions that are approximations of $\mathbf{G}$ on some proper initial segment of the ordinals.

i.e., for each $x \in \mathbf{M}$, we construct a function $g_x : cl\left(\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}\right) \to \mathbf{V}$ such that

$$\forall z \in cl\left(\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}\right) \quad g_x(z) = \mathbf{F}\Big(z\,, \mathbf{G} \restriction_{\lceil \underline{z} \rceil_{\mathbf{M}}^{\mathbf{R}}}\Big).$$

So, since $\mathbf{R}$ is "set-like" on $\mathbf{M}$, it follows that $cl\left(\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}\right)$ is a set, hence g_x is a function with $dom\,(g_x) = cl\left(\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}\right)$ and $ran\,(g_x) = g_x\left[cl\left(\lceil \underline{x} \rceil_{\mathbf{M}}^{\mathbf{R}}\right)\right]$ is a set obtained by an instance of the **Replacement Schema**.

Clearly, by the same argument as above, g_x is unique for any given $x \in \mathbf{M}$. So, it is enough then to define $\mathbf{G}(x)$ by:

- If there exists some $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \frac{y}{} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$, then

$$\mathbf{G}(x) = g_y(x)$$

for some (any) $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \frac{y}{} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$.
- If there exists no $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \frac{y}{} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$, then

$$\mathbf{G}(x) = \mathbf{F}\Big(x\,, g_x \restriction_{\lceil \frac{x}{} \rceil^{\mathbf{R}}_{\mathbf{M}}}\Big).$$

□ 175

7.2 The Mostowski Collapsing Functional

We define a functional which will help us turn Emmentaler into Gruyère, by removing all the holes!

Definition 176 (ZF ∖ {AF})**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, the Mostowski collapsing functional* $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *is defined by*

$$\begin{aligned} \mathbf{G}(x) &= \Big\{\mathbf{G}(y) \mid y \in \mathbf{R}^{-1}[x]\Big\} && (7.1)\\ &= \Big\{\mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\Big\} && (7.2)\\ &= \Big\{\mathbf{G}(y) \in \mathbf{G}\Big[\mathbf{R}^{-1}[x]\Big] \mid y \in \mathbf{R}^{-1}[x]\Big\}. && (7.3)\\ & && (7.4) \end{aligned}$$

The class $\mathbf{N} = \mathbf{G}[\mathbf{M}]$ *is called the* ***Mostowski collapse*** *of* $\mathbf{M}$.

Remark 177. As already mentioned above, if $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ is both well-founded and "set-like" on $\mathbf{M}$, then for each $x \in \mathbf{M}$, the graph of $\mathbf{R}$ on x is a set $\mathcal{G} = (E, V)$ defined by:

$$E = \{x\} \cup cl\left(\lceil \frac{x}{} \rceil^{\mathbf{R}}_{\mathbf{M}}\right) \quad \text{and} \quad V = \big\{(a, b) \in E \times E \mid b\mathbf{R}a\big\}.$$

Not only, this directed graph is acyclic, but if we associate to each node $\mathtt{n}$ the set $\widehat{\mathtt{n}} = \{\widehat{\mathtt{c}} \mid \mathtt{c} \text{ is a child of } \mathtt{n}\}$ — as we did for instance in Remark 150 and Examples 151 and 152 — we obtain exactly at the root $\mathtt{r}$, the set $\widehat{\mathtt{r}}$ to which x is mapped to through the Mostowski collapsing functional as described in Definition 176, i.e., $\widehat{\mathtt{r}} = \mathbf{G}(x)$.

Lemma 178 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, and* $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *the Mostowski collapsing functional.*

(1) $\forall x, y \in \mathbf{M} \quad \big(y\mathbf{R}x \longrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\big)$

(2) $\mathbf{N} = \mathbf{G}[\mathbf{M}]$ *is transitive*

(3) $\mathbf{N} \subseteq \mathbf{WF}$.

Proof of Lemma 178:

(1) $\forall x, y \in \mathbf{M} \ \big(x\mathbf{R}y \longrightarrow \mathbf{G}(x) \in \mathbf{G}(y)\big)$ is immediate by definition of the Mostowski collapsing functional $\mathbf{G}$.

(2) Take any $v \in w \in \mathbf{N} = \mathbf{G}[\mathbf{M}]$. By definition of Mostowski collapsing functional $\mathbf{G}$, there exist $x, y \in \mathbf{M}$ such that $w = \mathbf{G}(x)$ and $v = \mathbf{G}(y) \in \mathbf{G}(x)$, hence $v \in \mathbf{N}$.

(3) Since $\mathbf{R}$ is well-founded, we show, by induction on $\mathbf{R}$, that $\mathbf{G}(x) \in \mathbf{WF}$ holds for every $x \in \mathbf{M}$.

- If $x \in \mathbf{M}$ is $\mathbf{R}$-minimal: $\mathbf{G}(x) = \Big\{\mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\Big\} = \varnothing \in \mathbf{W}(1) \subseteq \mathbf{WF}$.
- If $x \in \mathbf{M}$ is not $\mathbf{R}$-minimal: $\mathbf{G}(x) = \Big\{\mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\Big\}$. Since R is "set-like" on $\mathbf{M}$, $\mathbf{R}^{-1}[x]$ is a set. By induction hypothesis, $\mathbf{G}(y) \in \mathbf{WF}$ holds for each $y\mathbf{R}x$. Hence, by an instance of the **Replacement Schema**, one has

$$\Big\{rk\big(\mathbf{G}(y)\big) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\Big\}$$

is a set of ordinals. Therefore

$$\alpha = \sup\Big\{rk\big(\mathbf{G}(y)\big) + 1 \mid y \in \mathbf{M} \wedge y\mathbf{R}x\Big\}$$

is well defined and $x \in \mathbf{W}(\alpha) \subseteq \mathbf{WF}$.

$\square$ 178

Definition 179 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *is extensional on* $\mathbf{M}$ *if*

$$\forall x \in \mathbf{M} \ \forall y \in \mathbf{M} \quad \Big(\forall z \in \mathbf{M} \ \big(z\mathbf{R}x \longleftrightarrow z\mathbf{R}y\big) \longrightarrow x = y\Big).$$

i.e.,

$$\forall x \in \mathbf{M} \ \forall y \in \mathbf{M} \quad \Big(x \neq y \longrightarrow \lceil \tfrac{x}{-} \rceil^{\mathbf{R}}_{\mathbf{M}} \neq \lceil \tfrac{y}{-} \rceil^{\mathbf{R}}_{\mathbf{M}}\Big).$$

Remark 180. Notice, that when one replaces the relational $\mathbf{R}$ by the membership relation $\in$, this assertion becomes

$$\forall x \in \mathbf{M}\ y \in \mathbf{M}\ \ \Big(\forall z \in \mathbf{M}\ (z \in x \longleftrightarrow z \in y) \longrightarrow x = y\Big)$$

which is exactly

$$\Big(\forall x\ \forall y\ (\forall z\ (z \in x \longleftrightarrow z \in y) \to x = y))\Big)^{\mathbf{M}}$$

in other words:

$$(\textbf{Extensionality})^{\mathbf{M}}.$$

So, essentially, one requires a relational to be extensional when one wants the Mostowski collapse to satisfy the Axiom of **Extensionality**.

Lemma 181. *If $\mathbf{M}$ is any transitive class, then the membership relation $\in$ is extensional on $\mathbf{M}$.*

Proof of Lemma 181: Take any $x, y \in \mathbf{M}$ with $x \neq y$. By symmetry, there exists $z \in x \smallsetminus y$. Since $z \in x \in \mathbf{M}$ and $\mathbf{M}$ is transitive, one has $z \in \mathbf{M}$, which shows

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ (x \neq y \to \exists z \in \mathbf{M}\ (z \in x \longleftrightarrow z \notin y))$$

i.e.,

$$\Big(\forall x\ \forall y\ (x \neq y \to \exists z\ (z \in x \longleftrightarrow z \notin y))\Big)^{\mathbf{M}}$$

i.e.,

$$\Big(\forall x\ \forall y\ (\forall z\ (z \in x \longleftrightarrow z \in y) \to x = y))\Big)^{\mathbf{M}}$$

in other words:

$$(\textbf{Extensionality})^{\mathbf{M}}.$$

$\square$ 181

Lemma 182 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let $\mathbf{M}$ be any class, $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ be any well-founded and "set-like" relation on $\mathbf{M}$, and $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ the Mostowski collapsing functional.*
If $\mathbf{R}$ is extensional on $\mathbf{M}$, then

$$\mathbf{G} : \mathbf{M} \longrightarrow \mathbf{N} = \mathbf{G}[\mathbf{M}] \text{ is an isomorphism from } (\mathbf{M}, \mathbf{R}) \text{ to } (\mathbf{N}, \in).$$

Proof of Lemma 182:

- We show that $\mathbf{G}:\mathbf{M}\rightarrow\mathbf{N}$ is $1-1$. Otherwise, we consider any x that is $\mathbf{R}$-minimal in

$$\Big\{x\in\mathbf{M}\mid\exists y\in\mathbf{M}\ \big(\mathbf{G}(x)=\mathbf{G}(y)\ \wedge\ x\neq y\big)\Big\}$$

and for this x a fixed y.

Since $\mathbf{R}$ is extensional on $\mathbf{M}$ and $x\neq y$ holds, one has $\lceil\frac{x}{}\rceil^{\mathbf{R}}_{\mathbf{M}}\neq\lceil\frac{y}{}\rceil^{\mathbf{R}}_{\mathbf{M}}$. By symmetry, we assume $\lceil\frac{x}{}\rceil^{\mathbf{R}}_{\mathbf{M}}\smallsetminus\lceil\frac{y}{}\rceil^{\mathbf{R}}_{\mathbf{M}}\neq\varnothing$ and pick any $z\in\big(\lceil\frac{x}{}\rceil^{\mathbf{R}}_{\mathbf{M}}\smallsetminus\lceil\frac{y}{}\rceil^{\mathbf{R}}_{\mathbf{M}}\big)$. One has

$$\mathbf{G}(z)\in\mathbf{G}(x)\ \text{ since }\ z\mathbf{R}x\quad\text{and}\quad\mathbf{G}(z)\in\mathbf{G}(y)\ \text{ since }\ \mathbf{G}(x)=\mathbf{G}(y).$$

Therefore, there exists some $z'\in\mathbf{M}$ such that

$$z'\mathbf{R}y\ \text{ and }\ \mathbf{G}(z)=\mathbf{G}(z')\in\mathbf{G}(y).$$

Thus, we have found some $z\neq z'$ such that $\mathbf{G}(z)=\mathbf{G}(z')$ and $z\mathbf{R}x$, which contradicts the $\mathbf{R}$-minimality of x.

- Since $\mathbf{G}:\mathbf{M}\rightarrow\mathbf{N}$ is 1-1 and $\mathbf{N}=\mathbf{G}[\mathbf{M}]$, $\mathbf{G}:\mathbf{M}\xleftrightarrow{bij.}\mathbf{N}$. Thus, $\mathbf{G}$ being a bijection, it follows that $\mathbf{G}$ is an isomorphism since we have:

  - $x\mathbf{R}y\implies\mathbf{G}(x)\in\mathbf{G}(y)$ holds by definition of $\mathbf{G}$,
  - $\mathbf{G}(x)\in\mathbf{G}(y)\implies x\mathbf{R}y$ holds since, from $\mathbf{G}(x)\in\mathbf{G}(y)=\Big\{\mathbf{G}(z)\mid z\in\mathbf{M}\ \wedge\ z\mathbf{R}y\Big\}$, pick some $z\in\mathbf{M}$, such that $\mathbf{G}(x)=\mathbf{G}(z)$ and $z\mathbf{R}y$ and notice that $\mathbf{G}$ being $1-1$, one has $z=x$ and $x\mathbf{R}y$.

$\square$ 182

Mostowski Collapsing Theorem ($\mathbf{ZF}\smallsetminus\{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R}\subseteq\mathbf{M}\times\mathbf{M}$ *be any well-founded, "set-like", and extensional relation on* $\mathbf{M}$*.*

(1) there exists a transitive class $\mathbf{N}$*, and*

(2) an isomorphism $\mathbf{G}:(\mathbf{M},\mathbf{R})\xleftrightarrow{isom.}(\mathbf{N},\in)$*;*

(3) moreover, the isomorphism is unique.

Proof of the Mostowski Collapsing Theorem:

(1) The existence of $\mathbf{N}$ and the fact it is transitive is Lemma 178

(2) The fact $\mathbf{G}:(\mathbf{M},\mathbf{R})\xleftrightarrow{isom.}(\mathbf{N},\in)$ is an isomorphism is Lemma 182.

(3) Towards a contradiction, assume there exist isomorphisms $\mathbf{G} : \mathbf{M} \to \mathbf{N} = \mathbf{G}[\mathbf{M}]$ and $\mathbf{G}' : \mathbf{M} \to \mathbf{N}' = \mathbf{G}'[\mathbf{M}]$ between respectively $(\mathbf{M}, \mathbf{R})$ and $(\mathbf{N}, \in)$ and $(\mathbf{M}, \mathbf{R})$ and $(\mathbf{N}', \in)$. By induction on $\mathbf{R}$, we show that $\forall x \in \mathbf{M} \;\; \mathbf{G}(x) = \mathbf{G}'(x)$.

- If $\mathbf{x}$ is $\mathbf{R}$-minimal, then $\mathbf{G}, \mathbf{G}'$ being isomorphisms gives

$$\forall y \in \mathbf{M} \left(\neg y\mathbf{R}x \longrightarrow \left(\mathbf{G}(y) \notin \mathbf{G}(x) \;\wedge\; \mathbf{G}'(y) \notin \mathbf{G}'(x)\right)\right).$$

Hence (since $\mathbf{G}, \mathbf{G}'$ are bijective and $\mathbf{N}, \mathbf{N}'$ are transitive)

$$\forall z \in \mathbf{N} \; z \notin \mathbf{G}(x) \;\wedge\; \forall z \in \mathbf{N}' \; z \notin \mathbf{G}'(x).$$

Which yields $\mathbf{G}(x) = \mathbf{G}'(x) = \varnothing$.

- If $\mathbf{x}$ is not $\mathbf{R}$-minimal, then one has

$$\left(\forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\right) \;\wedge\; \forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}'(y) \in \mathbf{G}'(x)\right)\right).$$

which leads to

$$\forall y \in \mathbf{M} \left(\left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\right) \;\wedge\; \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}'(y) \in \mathbf{G}'(x)\right)\right).$$

i.e.,

$$\forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x) \;\;\longleftrightarrow\;\; \mathbf{G}'(y) \in \mathbf{G}'(x)\right).$$

By induction hypothesis, this gives

$$\forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x) \;\;\longleftrightarrow\;\; \mathbf{G}(y) \in \mathbf{G}'(x)\right).$$

So, in particular,

$$\forall y \in \mathbf{M} \left(\mathbf{G}(y) \in \mathbf{G}(x) \;\;\longleftrightarrow\;\; \mathbf{G}(y) \in \mathbf{G}'(x)\right).$$

Therefore we obtain (since $\mathbf{G}, \mathbf{G}'$ are bijective and $\mathbf{N}, \mathbf{N}'$ are transitive)

$$\mathbf{G}(x) = \mathbf{G}'(x).$$

$\square$ Mostowski Collapsing Theorem

In particular, if we work with the axiom of **Foundation**, the relational $\in \subseteq \mathbf{V} \times \mathbf{V}$ is

- well-founded,
- "set-like" (for any set x, $\in^{-1}[x] = \{y \in \mathbf{V} \mid y \in x\} = x$),
- extensional.

Therefore, the Mostowski Collapsing Theorem immediately yields

Corollary 184 (ZF). *If* $\mathbf{M}$ *is any class,* $\in$ *is extensional on* $\mathbf{M}$*. Then there exist*

- *a transitive class* $\mathbf{N}$*,*
- *an isomorphism* $\mathbf{G} : (\mathbf{M}, \in) \xleftrightarrow{\textit{isom.}} (\mathbf{N}, \in)$*; i.e.,*

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \Big(x \in y \longleftrightarrow \mathbf{G}(x) \in \mathbf{G}(y)\Big);$$

- *moreover, the isomorphism is unique.*

Proof of Corollary 184: It is enough to notice that $\mathbf{ZF} \vdash_c$ " $\in$ is well-founded on $\mathbf{M}$ ", $\mathbf{ZF} \vdash_c$ " $\in$ is set-like on $\mathbf{M}$ ", $\mathbf{ZF} \vdash_c$ " $\in$ is extensional on $\mathbf{M}$ " and apply the Mostowski Collapsing Theorem (page 111).

$\square$ 184

Chapter 8

Preservation under Relativization

8.1 Relativization of ZF

In this section, we concentrate on the various properties that a class should satisfy in order to satisfy some of the axioms of **ZFC**.

Lemma 185. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\textit{If } \mathbf{M} \textit{ is transitive, then } (\textit{Extensionality})^{\mathbf{M}}.$$

i.e.,

$$\forall x \in \mathbf{M}\ x \subseteq \mathbf{M} \longrightarrow \Big(\forall x\ \forall y\ (\forall z\,(z \in x \longleftrightarrow z \in y) \to x = y))\Big)^{\mathbf{M}}.$$

Proof of Lemma 185: This was Lemma 181.

$\square$ 185

Lemma 186. *Let* $\mathbf{M}$ *be any non-empty class.*

- *If for each* $\varphi_{(x,y,z_1,\ldots,z_k)}$ *with free variables among* $\{x, y, z_1, \ldots, z_k\}$*, one has*

  $$\forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \ldots \forall z_k \in \mathbf{M}\ \ \Big\{x \in y \mid (\varphi)^{\mathbf{M}}{}_{(x,y,z_1,\ldots,z_k)}\Big\} \in \mathbf{M}.$$

  Then $(\textit{Comprehension Schema})^{\mathbf{M}}$*.*

- *In particular, if* $\mathbf{M}$ *is closed under* $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ *that maps* x *to* $\mathscr{P}(x)$ *(i.e.,* $\mathscr{P}[\mathbf{M}] \subseteq \mathbf{M}$*) then* $(\textit{Comprehension Schema})^{\mathbf{M}}$*.*

Proof of Lemma 186:

$$\begin{aligned}(\textbf{Comprehension Schema})^{\mathbf{M}} &= \Big(\forall y \forall z_1 \ldots \forall z_n \exists X \forall x \big(x \in X \longleftrightarrow (x \in y \in \mathbf{M} \ \wedge \ \varphi)\big)\Big)^{\mathbf{M}} \\ &= \forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \ldots \forall z_n \in \mathbf{M}\ \exists X \in \mathbf{M}\ \forall x \in \mathbf{M} \\ &\quad \big(x \in X \longleftrightarrow (x \in y \ \wedge \ \varphi)\big)^{\mathbf{M}} \\ &= \forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \ldots \forall z_n \in \mathbf{M}\ \exists X \in \mathbf{M}\ \forall x \in \mathbf{M} \\ &\quad \big(x \in X \longleftrightarrow (x \in y \ \wedge \ (\varphi)^{\mathbf{M}})\big).\end{aligned}$$

So, taking $X = \left\{x \in y \mid (\varphi)^{\mathbf{M}}{}_{(x,y,z_1,\ldots,z_k)}\right\}$ works since this set belongs to $\mathbf{M}$ by assumption.

□ 186

Lemma 187. *Let* $\mathbf{M}$ *be any non-empty class. If* $\mathbf{M}$ *is transitive, then*

- $$\big(\textbf{\textit{Power Set}}\big)^{\mathbf{M}}$$ $$\Longleftrightarrow$$ $$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \big(\mathscr{P}(x) \cap \mathbf{M}\big) \subseteq y.$$
- *In particular, if* $\mathbf{M}$ *is closed under* $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ *that maps* x *to* $\mathscr{P}(x)$ *(i.e.,* $\mathscr{P}[\mathbf{M}] \subseteq \mathbf{M}$*) then* $\big(\textbf{\textit{Power Set}}\big)^{\mathbf{M}}$.

Proof of Lemma 187: We first notice that

$$\forall u \in \mathbf{M}\ \ (u \in z \to u \in x) \iff z \cap \mathbf{M} \subseteq x.$$

Also, if $\mathbf{M}$ is transitive and $z \in \mathbf{M}$, then $z \cap \mathbf{M} = z$ because $z \subseteq \mathbf{M}$.
We have:

$$\begin{aligned}\big(\textbf{Power Set}\big)^{\mathbf{M}} &= \Big(\forall x\, \exists y\, \forall z\, \big(\forall u\ \ (u \in z \to u \in x) \to z \in y\big)\Big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(\forall u\ \ (u \in z \to u \in x) \to z \in y\big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(\forall u \in \mathbf{M}\ \ (u \in z \to u \in x) \to z \in y\big) \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(z \cap \mathbf{M} \subseteq x \to z \in y\big) \\ &\iff \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(z \subseteq x \to z \in y\big) \\ &\iff \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \big(\mathscr{P}(x) \cap \mathbf{M}\big) \subseteq y.\end{aligned}$$

$\square$ 187

Lemma 188. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ \big(x \in z\ \wedge\ y \in z\big).$$

$$\Longleftrightarrow$$

$$\big(\textit{Pairing}\big)^{\mathbf{M}}.$$

Proof of Lemma 188: We have

$$\begin{aligned}
\big(\textbf{Pairing}\big)^{\mathbf{M}} &= \Big(\forall x \forall y \exists z \big(x \in z\ \wedge\ y \in z\big)\Big)^{\mathbf{M}} \\
&= \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ \big(x \in z\ \wedge\ y \in z\big)^{\mathbf{M}} \\
&= \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ \big(x \in z\ \wedge\ y \in z\big).
\end{aligned}$$

$\square$ 188

Lemma 189. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ \big(\bigcup x \subseteq y\big).$$

$$\Longrightarrow$$

$$\big(\textit{Union}\big)^{\mathbf{M}}.$$

Proof of Lemma 189: We have

$$\begin{aligned}
\big(\textbf{Union}\big)^{\mathbf{M}} &= \Big(\forall x\, \exists y\, \forall a\, \forall b\, \big((a \in b\ \wedge\ b \in x) \to a \in y\big)\Big)^{\mathbf{M}} \\
&= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a \in \mathbf{M}\ \forall b \in \mathbf{M}\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big)^{\mathbf{M}} \\
&= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a \in \mathbf{M}\ \forall b \in \mathbf{M}\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big) \\
&\Longleftarrow \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a\ \forall b\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big) \\
&= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ \big(\bigcup x \subseteq y\big)
\end{aligned}$$

$\square$ 189

Lemma 190. *Let* $\mathbf{M}$ *be any non-empty class,* $\varphi := \varphi_{(x,y,A,w_1,\ldots,w_n)}$ *be any formula with free variables among* $x, y, A, w_1, \ldots, w_n$*.*

$\forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M}$

$$\Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \Big\{y \in \mathbf{M} \mid \exists x \in A \cap \mathbf{M}\ (\varphi)^{\mathbf{M}}\Big\} \subseteq B\Big)$$

$$\Longleftrightarrow$$

$$\big(\textit{Instance of Replacement Schema for } \varphi\big)^{\mathbf{M}}$$

We recall $\exists! y\, \varphi$ abbreviates $\exists y \Big(\varphi_{(x,y,A,w_1,\ldots,w_n)} \wedge \forall z \left(\varphi(x,z,A,w_1,\ldots,w_n) \longrightarrow z = y\right)\Big)$.
Proof of Lemma 190: We have

$$\forall A\ \forall w_1 \ldots \forall w_n \Big[\forall x \left(x \in A \longrightarrow \exists! y\ \varphi\right) \longrightarrow \exists B\ \forall x \left(x \in A \longrightarrow \exists y \left(y \in B \wedge \varphi\right)\right)\Big]$$

where

$$\begin{aligned}
& \big(\textbf{Instance of Replacement Schema for } \varphi\big)^{\mathbf{M}} \\
= \ & \Big(\forall A\ \forall w_1 \ldots \forall w_n \Big(\forall x \left(x \in A \longrightarrow \exists! y\ \varphi\right) \longrightarrow \exists B\ \forall x \left(x \in A \longrightarrow \exists y \left(y \in B \wedge \varphi\right)\right)\Big)\Big)^{\mathbf{M}} \\
= \ & \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in \mathbf{M}\ \left(x \in A \longrightarrow \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}}\right) \longrightarrow \exists B \in \mathbf{M}\ \forall x \in \mathbf{M}\ \left(x \in A \longrightarrow \exists y \in \mathbf{M}\ \left(y \in B \wedge (\varphi)^{\mathbf{M}}\right)\right)\Big) \\
= \ & \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \forall x \in A \cap \mathbf{M}\ \exists y \in B \cap \mathbf{M}\ \ (\varphi)^{\mathbf{M}}\Big) \\
\Longleftrightarrow \ & \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \Big\{y \in \mathbf{M} \mid \exists x \in A \cap \mathbf{M}\ (\varphi)^{\mathbf{M}}\Big\} \subseteq B\Big)
\end{aligned}$$

$\square$ 190

Lemma 191 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any non-empty class.*

$$\mathbf{M} \subseteq \mathbf{WF}$$

$$\Longrightarrow$$

$$\big(\textit{Foundation}\big)^{\mathbf{M}}.$$

Proof of Lemma 191: Assuming $\mathbf{M} \subseteq \mathbf{WF}$, one has $\in\restriction_{\mathbf{M}\times\mathbf{M}} \subseteq \mathbf{M} \times \mathbf{M}$ is well-founded and "set-like" on $\mathbf{M}$. So, by Theorem 174 one also has

$$\forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X}\, \forall x \in \mathbf{X}\;\; x \notin y \right).$$

$$\begin{aligned}
(\textbf{Foundation})^{\mathbf{M}} &= \left(\forall X \left(\exists y\, y \in X \to \exists y \left(y \in X \;\wedge\; \neg\exists x \left(x \in X \;\wedge\; x \in y\right)\right)\right)\right)^{\mathbf{M}} \\
&= \forall X \in \mathbf{M} \left(\exists y \in \mathbf{M}\, y \in X \to \exists y \in \mathbf{M} \left(y \in X \;\wedge\; \neg\exists x \in \mathbf{M}\, (x \in X \;\wedge\; x \in y)\right)\right) \\
&= \forall X \in \mathbf{M} \left(\exists y \in \mathbf{M} \cap X \to \exists y \in \mathbf{M} \cap X\, \forall x \in \mathbf{M} \cap X\;\; x \notin y\right) \\
&\Longleftarrow \forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X}\, \forall x \in \mathbf{X}\;\; x \notin y\right).
\end{aligned}$$

$\square$ 191

Lemma 192 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.**

(1) $\mathbf{ZF} \smallsetminus \{\mathbf{AF}\} \vdash_c \left(\mathbf{ZF} \smallsetminus \{\textbf{\textit{Infinity}}\}\right)^{\mathbf{W}(\omega)}$

(2) $\mathbf{ZF} \smallsetminus \{\mathbf{AF}\} \vdash_c \left(\mathbf{ZF} \smallsetminus \{\textbf{\textit{Infinity}}\}\right)^{\mathbf{WF}}$*.*

(3) $\mathbf{ZF} \smallsetminus \{\mathbf{AF}\} \vdash_c (\mathbf{ZF})^{\mathbf{WF}}$*.*

Proof of Lemma 192: Both $\mathbf{W}(\omega)$ and $\mathbf{WF}$ are transitive classes closed under $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ that maps x to $\mathscr{P}(x)$. So, one has

- $\textbf{Extensionality}^{\mathbf{W}(\omega)}$
- $\textbf{Comprehension Schema}^{\mathbf{W}(\omega)}$
- $(\textbf{Power Set})^{\mathbf{W}(\omega)}$
- $(\textbf{Pairing})^{\mathbf{W}(\omega)}$
- $(\textbf{Union})^{\mathbf{W}(\omega)}$
- $\textbf{Extensionality}^{\mathbf{WF}}$
- $\textbf{Comprehension Schema}^{\mathbf{WF}}$
- $(\textbf{Power Set})^{\mathbf{WF}}$
- $(\textbf{Pairing})^{\mathbf{WF}}$
- $(\textbf{Union})^{\mathbf{WF}}$

For the **Replacement Schema**, we consider any formula $\varphi := \varphi_{(x,y,A,w_1,\dots,w_n)}$ with free variables among $x, y, A, w_1, \dots, w_n$.

(1)

$$(\textbf{Instance of Replacement Schema for } \varphi)^{\mathbf{W}(\omega)}$$

$$\iff$$

$$\forall A \in \mathbf{W}(\omega)\ \ \forall w_1 \in \mathbf{W}(\omega) \ldots \forall w_n \in \mathbf{W}(\omega)$$
$$\Big(\forall x \in A \cap \mathbf{W}(\omega)\ \exists! y \in \mathbf{W}(\omega)\ (\varphi)^{\mathbf{W}(\omega)} \longrightarrow \exists B \in \mathbf{W}(\omega)\ \Big\{y \in \mathbf{W}(\omega) \mid \exists x \in A \cap \mathbf{W}(\omega)\ (\varphi)^{\mathbf{W}(\omega)}\Big\} \subseteq B\Big)$$

$$\iff$$

$$\forall A \in \mathbf{W}(\omega)\ \ \forall w_1 \in \mathbf{W}(\omega) \ldots \forall w_n \in \mathbf{W}(\omega) \qquad \textit{(since } \mathbf{W}(\omega) \textit{ is transitive)}$$
$$\Big(\forall x \in A\ \exists! y \in \mathbf{W}(\omega)\ (\varphi)^{\mathbf{W}(\omega)} \longrightarrow \exists B \in \mathbf{W}(\omega)\ \Big\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\Big\} \subseteq B\Big)$$

which holds since $A \in \mathbf{W}(\omega)$ implies $A \in \mathbf{W}(n)$ holds for some integer n. Thus, both

$$A \text{ and } \Big\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\Big\}$$

are finite. We set

$$k = \sup\Big\{rk(y) + 1 \in \mathbf{On} \mid y \in \mathbf{W}(\omega)\ \ \wedge\ \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\Big\}$$

which leads to

$$\Big\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\Big\} \subseteq \mathbf{W}(k) \in \mathbf{W}(\omega).$$

(2)

$$\big(\textbf{Instance of Replacement Schema for } \varphi\big)^{\mathbf{WF}}$$

$$\iff$$

$$\forall A \in \mathbf{WF}\ \forall w_1 \in \mathbf{WF} \ldots \forall w_n \in \mathbf{WF}$$
$$\Big(\forall x \in A \cap \mathbf{WF}\ \exists! y \in \mathbf{WF}\ (\varphi)^{\mathbf{WF}} \longrightarrow \exists B \in \mathbf{WF}\ \Big\{y \in \mathbf{WF} \mid \exists x \in A \cap \mathbf{WF}\ (\varphi)^{\mathbf{WF}}\Big\} \subseteq B\Big)$$

$$\iff$$

$$\forall A \in \mathbf{WF}\ \forall w_1 \in \mathbf{WF} \ldots \forall w_n \in \mathbf{WF} \qquad \textit{(since } \mathbf{WF} \textit{ is transitive)}$$
$$\Big(\forall x \in A\ \exists! y \in \mathbf{WF}\ (\varphi)^{\mathbf{WF}} \longrightarrow \exists B \in \mathbf{WF}\ \Big\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\Big\} \subseteq B\Big)$$

which holds since $A \in \mathbf{WF}$ implies

$$\Big\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\Big\} \subseteq \mathbf{WF}$$

By Lemma 145 this leads to

$$\Big\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\Big\} \in \mathbf{WF}.$$

(3) ω belongs to **WF** since every ordinal α belongs to **WF** because it satisfies $\in$ is well-founded on $tc(\alpha) = \alpha$; hence by Theorem 161, $\alpha \in \mathbf{WF}$.

Lemma 191 takes care of the Axiom of **Foundation** since obviously both $\mathbf{W}(\omega) \subseteq \mathbf{WF}$ and $\mathbf{WF} \subseteq \mathbf{WF}$ hold.

□ 192

8.2 Absoluteness

Definition 193. *Let* $\mathbf{M}, \mathbf{N}$ *be non-empty classes, and* $\varphi_{(x_1,\dots,x_n)}$ *a formula with free variables among* $x_1, \dots, x_n$.

(1) If $\mathbf{M} \subseteq \mathbf{N}$,

$$\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N}$$

$$\Longleftrightarrow$$

$$\forall x_1 \in \mathbf{M} \dots \forall x_n \in \mathbf{M} \left((\varphi)^{\mathbf{M}} \longleftrightarrow (\varphi)^{\mathbf{N}} \right).$$

(2) φ *is absolute for* $\mathbf{M}$ *if* φ *is absolute for* $\mathbf{M}, \mathbf{V}$*. i.e.,*

$$\forall x_1 \in \mathbf{M} \dots \forall x_n \in \mathbf{M} \left((\varphi)^{\mathbf{M}} \longleftrightarrow \varphi \right).$$

Remark 194.

$$\left.\begin{array}{r} \mathbf{M} \subseteq \mathbf{N} \\ \varphi \text{ is absolute for } \mathbf{M} \\ \varphi \text{ is absolute for } \mathbf{N} \end{array}\right\} \Longrightarrow \varphi \text{ is absolute for } \mathbf{M}, \mathbf{N}.$$

Absolute formulas are closed under boolean operations.

Lemma 195. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *be non-empty classes, and* $\varphi_{(x_1,\dots,x_n)}$ *a formula with free variables among* $x_1, \dots, x_n$.

(1)

$$\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N} \Longrightarrow \neg\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N}.$$

(2)

$$\left.\begin{array}{r} \varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N} \\ \psi \textit{ is absolute for } \mathbf{M}, \mathbf{N} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} (\varphi \wedge \psi) \textit{ is absolute for } \mathbf{M}, \mathbf{N} \\ (\varphi \vee \psi) \textit{ is absolute for } \mathbf{M}, \mathbf{N} \\ (\varphi \rightarrow \psi) \textit{ is absolute for } \mathbf{M}, \mathbf{N} \\ (\varphi \leftrightarrow \psi) \textit{ is absolute for } \mathbf{M}, \mathbf{N}. \end{array}\right.$$

Proof of Lemma 195: Immediate from the definition of both relativization and absoluteness.

$\square$ 195

Definition 196. *φ is a Δ_0^{0-rud}-formula if*

- $\varphi := x = y$
- $\varphi := x \in y$

or

- *ψ is a Δ_0^{0-rud}-formula and*
  - $\varphi := \neg\psi$

or

- *φ_0, φ_1 are Δ_0^{0-rud}-formulas and*
  - $\varphi := \left(\varphi_0 \wedge \varphi_1\right)$
  - $\varphi := \left(\varphi_0 \vee \varphi_1\right)$
  - $\varphi := \left(\varphi_0 \longrightarrow \varphi_1\right)$
  - $\varphi := \left(\varphi_0 \longleftrightarrow \varphi_1\right)$

or

- *ψ is a Δ_0^{0-rud}-formula and*
  - $\varphi := \exists x \left(x \in y \wedge \psi\right)$
  - $\varphi := \forall x \left(x \in y \longrightarrow \psi\right)$.

Lemma 197. *Let $\mathbf{M}$ be any non-empty class, and φ any Δ_0^{0-rud}-formula.*

If $\mathbf{M}$ is transitive, then φ is absolute for $\mathbf{M}$.

Proof of Lemma 197: The proof is by induction on $ht(\varphi)$. The only case that matters is $\varphi := \exists x \left(x \in y \wedge \psi\right)$. Assuming that the free variables of ψ are among $x_1, \ldots, x_n$, one has

$$\begin{aligned}
&\forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M}\ \left((\varphi)^{\mathbf{M}} \leftrightarrow \varphi\right) \\
\Longleftrightarrow\ &\forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M}\ \left(\left(\exists x\ (x \in y \wedge \psi)\right)^{\mathbf{M}} \leftrightarrow \exists x \left(x \in y \wedge \psi\right)\right) \\
\Longleftrightarrow\ &\forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M}\ \left(\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \leftrightarrow \exists x \left(x \in y \wedge \psi\right)\right) \\
\Longleftrightarrow\ &\forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M}\ \left(\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \leftrightarrow \exists x \in \mathbf{M}\ \left(x \in y \wedge \psi\right)\right).
\end{aligned}$$

For the last two lines:

$$\Big(\exists x \in \mathbf{M}\left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \leftrightarrow \exists x\left(x \in y \wedge \psi\right)\Big)$$

- the ($\rightarrow$) direction comes from the induction hypothesis which yields $\left((\psi)^{\mathbf{M}} \rightarrow \psi\right)$.
- The ($\leftarrow$) comes from the induction hypothesis which yields $\left((\psi)^{\mathbf{M}} \leftarrow \psi\right)$ plus the transitivity of $\mathbf{M}$ which gives $x \in y \in \mathbf{M} \implies x \in \mathbf{M}$.

$\square$ 197

Definition 198. *Let $\mathbf{M} \subseteq \mathbf{N}$ be any non-empty class.*

(1) A relational $\mathbf{R} \subseteq \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n}$ is absolute for $\mathbf{M}, \mathbf{N}$ if there exists some formula $\varphi(x_1, \ldots, x_n)$ whose free variables are among $x_1, \ldots, x_n$ which is is absolute for $\mathbf{M}, \mathbf{N}$ and such that

$$\forall x_1 \ldots \forall x_n \quad \big((x_1, \ldots, x_n) \in \mathbf{R} \longleftrightarrow \varphi(x_1, \ldots, x_n)\big).$$

(2) A functional $\mathbf{F} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n} \rightarrow \mathbf{V}$ is absolute for $\mathbf{M}, \mathbf{N}$ if there exists some formula $\varphi(x_1, \ldots, x_n, y)$ whose free variables are among $x_1, \ldots, x_n, y$ which is is absolute for $\mathbf{M}, \mathbf{N}$ and such that

$$\forall x_1 \ldots \forall x_n \forall y \quad \big(\mathbf{F}(x_1, \ldots, x_n) = y \longleftrightarrow \varphi(x_1, \ldots, x_n, y)\big).$$

Formally, φ must also satisfy:

- $\forall x_1 \ldots \forall x_n \exists! y \quad \varphi(x_1, \ldots, x_n, y)$
- $\forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \exists! y \in \mathbf{M} \quad \varphi(x_1, \ldots, x_n, y)$
- $\forall x_1 \in \mathbf{N} \ldots \forall x_n \in \mathbf{N} \exists! y \in \mathbf{N} \quad \varphi(x_1, \ldots, x_n, y)$

Lemma 199. *Let $\mathbf{M} \subseteq \mathbf{N}$ be non-empty classes, $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathrm{ST}}$-formulas whose free variables are among $x_1, \ldots, x_n$, and $\mathcal{T}$ some $\mathcal{L}_{\mathrm{ST}}$-theory.*
If

(1) $\mathcal{T} \vdash_c \forall x_1 \ldots \forall x_n \quad \left(\varphi \longleftrightarrow \psi\right)$.

(2) $\mathbf{ZF} \vdash_c \left(\mathcal{T}\right)^{\mathbf{M}}$

(3) $\mathbf{ZF} \vdash_c \left(\mathcal{T}\right)^{\mathbf{N}}$.

Then

$$\mathbf{ZF} \vdash_c \text{“}\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N}\text{”} \longleftrightarrow \text{“}\psi \textit{ is absolute for } \mathbf{M}, \mathbf{N}\text{”}.$$

Proof of Remark 199: Exercise.

□ 199

In particular, if φ is equivalent to some Δ_0^{0-rud}-formula, then φ is absolute for transitive models.

Lemma 200. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *be non-empty classes,* $\varphi(x_1, \ldots, x_n)$ *any* $\mathcal{L}_{\mathrm{ST}}$*-formula whose free variables are among* $x_1, \ldots, x_n$*, and* $\mathbf{F} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n} \to \mathbf{V}$*,* $\mathbf{G_1} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{k} \to \mathbf{V}, \ldots, \mathbf{G_n} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{k}$ *any functionals.*

(1) If "$\varphi, \mathbf{G_1}, \ldots, \mathbf{G_n}$ *are absolute for* $\mathbf{M}, \mathbf{N}$*",*
then "$\varphi\Big(\mathbf{G_1}(z_1, \ldots, z_k), \ldots, \mathbf{G_n}(z_1, \ldots, z_k)\Big)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*".*

(2) If "$\mathbf{F}, \mathbf{G_1}, \ldots, \mathbf{G_n}$ *are absolute for* $\mathbf{M}, \mathbf{N}$*",*
then "$\mathbf{F}\Big(\mathbf{G_1}(z_1, \ldots, z_k), \ldots, \mathbf{G_n}(z_1, \ldots, z_k)\Big)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*".*

Proof of Remark 200: Exercise.

□ 200

Notation 201. *we write "***ZF***"(respectively "***ZFC***") for "finitely many axioms from* **ZF***" (respectively "finitely many axioms from* **ZFC***").*

A proof is something that only makes use of finitely many axioms or instances of axiom schemas. For instance, we showed that the empty set exists using the axiom of **Extensionality** and one instance of the **Comprehension Schema**. So, we could write "**ZF**"$\vdash_c$ "$\varnothing$ exists" to indicate both that **ZF** $\vdash_c$ "$\varnothing$ exists" and "**ZF**"refers to the axioms that were used during the proof. An other example, would be the proof of the existence of a functional as in Theorem 53 :

Given any $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, there exists a unique $\mathbf{G} : \mathbf{On} \to \mathbf{V}$ such that for each ordinal α

$$\forall \alpha \;\; \mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha).$$

Strictly speaking, this theorem is a theorem schema: there are infinitely many theorems, one for every functional $\mathbf{F}$. Indeed a functional $\mathbf{F}$ refers to some formula $\varphi_{\mathbf{F}}$, and the result consists in constructing another formula $\varphi_{\mathbf{G}}$ which satisfies the required property and showing that the functional $\mathbf{G}$ it represents is unique. Although the whole construction only requires finitely many axioms or instances of axiom schemas, but we do not bother precisely indicating which one we used, reason why we use the notation "**ZF**" for "these finitely many axioms that a hard work could precisely point out, but we don't really care as long as there are only finitely many of them".

Exercise 202. *The following notions are provably equivalent in "***ZFC***" to Δ_0^{0-rud}-formulas, hence they are absolute for transitive models of "***ZFC***".*

(1) $x = y$

(2) $x \in y$

(3) $x \subseteq y$

(4) $\varnothing$

(5) $\{x, y\}$

(6) $\{x\}$

(7) (x, y)

(8) $x \cup y$

(9) $x \cap y$

(10) $x \cup \{x\}$

(11) $x \smallsetminus y$

(12) $\bigcup x$

(13) $\bigcap x \quad (x \neq \varnothing)$

(14) $x \times y$

(15) *"x is a relation"*

(16) *"dom* (x)*"*

(17) *"ran* (x)*"*

(18) *"x is a function"*

(19) *"x is a 1-1 function"*

(20) *"x is transitive"*

(21) *"x is* $\in$*-well-ordered"*

(22) *"α is an ordinal"*

(23) *"α is a successor ordinal"*

(24) *"α is a limit ordinal"*

(25) *"ω"*

(26) *"α is finite"*

(27) *"$<_x$ well-orders x"*

(28) *"type* $(x, <_x)$*"*

(29) *"x is finite"*

(30) *"x^n"* *(any $n \in \omega$)*

(31) *"$x^{<\omega}$"*