Part IV

Gödel's Constructible Universe

Chapter 11

The Constructible Sets

11.1 Definability

Definition 253 (Definability)**.** *Given any set Y, we say that $X \subseteq Y$ is definable over Y if there exists some $\mathcal{L}_{\mathrm{ST}}$-formula $\varphi := \varphi(x, x_1, \ldots, x_n)$ whose free variables are among $x, x_1, \ldots, x_n$ and parameters $a_1, \ldots, a_n \in Y$ such that*

$$X = \left\{ x \in Y \mid \Big(\varphi(x, a_1/x_1, \ldots, a_n/x_n) \Big)^{Y} \right\}.$$

Definition 254 (Definable subsets)**.** *Let Y be any set. The set of the definable subsets of Y is defined as*

$$\{X \subseteq Y \mid X \text{ is definable over } Y\}.$$

Notice that this definition does not fall under the strict framework of set theory. As such it quantifies over first order formulas which are not members of set theory. So, there are two options here in order to properly define it.
We can easily define a way of coding $\mathcal{L}_{\mathrm{ST}}$-formulas and proofs within **ZF** (or **ZFC**, or etc.) such that — among others — the following sets are $\mathcal{P}rim.\ \mathcal{R}ec.$:

- The set of all codes of $\mathcal{L}_{\mathrm{ST}}$-formulas

$$\left\{ \ulcorner \varphi \urcorner \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathrm{ST}} \right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\mathrm{ST}}$ that contain the variable x_n

$$\mathcal{F}_{\checkmark x} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathrm{ST}} \textit{ and } \varphi \textit{ contains } x_n\right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\mathrm{ST}}$ that do not contain the variable x_n

$$\mathcal{F}_{\times x} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathrm{ST}} \textit{ and } \varphi \textit{ does not contain } x_n\right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\mathrm{ST}}$ that contain x_n as a free variable

$$\mathcal{F}_{\checkmark x \textit{ free}} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathrm{ST}} \textit{ and } x_n \textit{ is free in } \varphi\right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\mathrm{ST}}$ that contain x_n as a bound variable

$$\mathcal{F}_{\checkmark x \textit{ bound}} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\mathrm{ST}} \textit{ and } x_n \textit{ is bound in } \varphi\right\}$$

- The set of all codes of closed formulas from $\mathcal{L}_{\mathrm{ST}}$

$$\mathcal{F}_{\checkmark \textit{closed}} = \left\{\ulcorner\varphi\urcorner \mid \varphi \textit{ is a closed formula from } \mathcal{L}_{\mathrm{ST}}\right\}$$

We define a relational $\texttt{Correct} \subseteq \omega \times \mathbf{V} \times \mathbf{V}$ by

$$\texttt{Correct}\,(k, S, Y) \iff \begin{cases} \text{“}k = \ulcorner\varphi\urcorner \text{ codes an } \mathcal{L}_{\mathrm{ST}}\text{-formula } \varphi\text{”} \\ \wedge \\ \text{“}S \text{ is a mapping from some (finite) set of integers to } Y\text{”} \\ \wedge \\ \text{“ for every integer } n \text{ s.t. } x_n \text{ is free in } \varphi,\ n \in \textit{dom}\,(S)\text{”.} \end{cases}$$

We also define a relational $\texttt{Holds} \subseteq \omega \times \mathbf{V} \times \mathbf{V}$ by induction on the integers by

$$\texttt{Holds}\,(k,S,Y) \iff \left\{ \begin{array}{c} k = \ulcorner\varphi\urcorner \;\wedge\; \texttt{Correct}\,(\ulcorner\varphi\urcorner, S, Y) \\ \bigwedge \\ \left(\begin{array}{c} \ulcorner\varphi\urcorner = \ulcorner x_i = x_j \urcorner \;\wedge\; S(i) = S(j) \\ \vee \\ \ulcorner\varphi\urcorner = \ulcorner x_i \in x_j \urcorner \;\wedge\; S(i) \in S(j) \\ \vee \\ \left(\ulcorner\varphi\urcorner = \ulcorner \neg\psi \urcorner \;\wedge\; \neg\texttt{Holds}\,(\ulcorner\psi\urcorner, S, Y)\right) \\ \vee \\ \left(\ulcorner\varphi\urcorner = \ulcorner (\psi \wedge \theta) \urcorner \;\wedge\; \texttt{Holds}\,(\ulcorner\psi\urcorner, S, Y) \;\wedge\; \texttt{Holds}\,(\ulcorner\theta\urcorner, S, Y)\right) \\ \vee \\ \ulcorner\varphi\urcorner = \ulcorner \exists x_i\ \psi \urcorner \;\wedge\; \exists y \in Y\ \texttt{Holds}\left(\ulcorner\psi\urcorner, (i,y) \cup S \restriction \left(dom\,(S) \setminus \{i\}\right), Y\right) \end{array} \right) \end{array} \right.$$

Definition 255 (Definability defined inside set theory)**.** *We define a "set-like" relational* $\mathtt{Definable\_Over} \subseteq \mathbf{V} \times \mathbf{V}$

$$\mathtt{Definable\_Over}\,(X,Y)$$
$$\iff$$
$$\exists n \in \omega\ \exists \ulcorner\varphi\urcorner \left(\begin{array}{c} \text{“}\varphi \textit{ has exactly } x_0, x_1, \ldots, x_n \textit{ as free variables ”} \\ \wedge \\ \exists S \text{ “}S \textit{ is a mapping from } \{1, \ldots, n\} \textit{ to } Y\text{”} \\ \forall x_0 \in Y \quad \left(x_0 \in X \longleftrightarrow \mathtt{Holds}\,(\ulcorner\varphi\urcorner, S \cup \{(0, x_0)\}, Y)\right). \end{array} \right)$$

Definition 256 (Definable subsets defined inside set theory)**.** *Let* Y *be any set The set* $\mathtt{Def}\,(Y)$ *of the definable subsets of* Y *is defined as*

$$\mathtt{Def}\,(Y) = \{X \subseteq Y \mid \mathtt{Definable\_Over}\,(X,Y)\}\,.$$

Definition 257 (Definability defined outside set theory)**.** *Given any set Y, we say that $X \subseteq Y$ is definable over Y if there exists some $\mathcal{L}_{\text{ST}}$-formula $\varphi := \varphi(x, x_1, \ldots, x_n)$ whose free variables are among $x, x_1, \ldots, x_n$ and parameters $a_1, \ldots, a_n \in Y$ such that*

$$X = \left\{x \in Y \mid \varphi(x, a_1/x_1, \ldots, a_n/x_n)^Y\right\}.$$

$$X \text{ is definable over } Y$$

$$\Longleftrightarrow$$

$$\exists n \in \omega\ \exists \ulcorner\varphi\urcorner \left(\begin{array}{c} \text{“}\varphi \text{ has exactly } x_0, x_1, \ldots, x_n \text{ as free variables ”} \\ \wedge \\ \exists S \text{ “}S \text{ is a mapping from } \{1, \ldots, n\} \text{ to } Y\text{”} \\ \forall x_0 \in Y \ \left(x_0 \in X \longleftrightarrow \texttt{Holds}\left(\ulcorner\varphi\urcorner, S \cup \{(0, x_0)\}, Y\right)\right). \end{array}\right)$$

Notation 258. *Given any set A, we denote by $\mathscr{P}_{\text{fin.}}(A)$ the set of all finite subsets of A.*

Lemma 259 (**ZF**)**.** *Let Y be any set.*

(1) $Y \in \texttt{Def}(Y)$

(2) $\mathscr{P}_{\text{fin.}}(Y) \subseteq \texttt{Def}(Y) \subseteq \mathscr{P}(Y)$

(3) Y *transitive* $\Longrightarrow Y \subseteq \texttt{Def}(Y)$

(4) *(**AC**)* $|Y| \geqslant \aleph_0 \Longrightarrow |\texttt{Def}(Y)| = |Y|$.

Proof of Lemma 259:

(1) Clearly,

$$Y = \{x \in Y \mid x = x\} = \left\{x \in Y \mid (x = x)^Y\right\}.$$

(2) Clearly $\varnothing \in \texttt{Def}(Y)$. If $\varnothing \neq X \in \mathscr{P}_{\text{fin.}}(Y)$, then there exists $a_1, \ldots, a_n$ such that $X = \{a_1, \ldots, a_n\}$. One has

$$X = \left\{x \in Y \mid \bigvee_{1 \leqslant i \leqslant n} x = a_i\right\}.$$

(3) Take any $y \in Y$. Since Y is transitive, it follows $y \subseteq Y$, hence

$$y = \{x \in Y \mid x \in y\}.$$

(4) One has $\mathscr{P}_{\text{fin.}}(Y) \subseteq \mathtt{Def}(Y)$, hence $|Y| = |\mathscr{P}_{\text{fin.}}(Y)| \leqslant |\mathtt{Def}(Y)|$. Moreover, since there are countably many $\mathcal{L}_{\text{ST}}$-formulas and $|Y^{<\omega}| = |Y|$, one has

$$|\mathtt{Def}(Y)| \leqslant \aleph_0 \cdot |Y^{<\omega}| = \aleph_0 \cdot |Y| = |Y|.$$

$\square$ 259

11.2 The Constructible Sets

Definition 260 (Gödel's Constructible Universe)**.** *By transfinite recursion on $\alpha \in \mathbf{On}$ we define the sets $\mathbf{L}(\alpha)$ by:*

- $\mathbf{L}(0) = \varnothing$
- $\mathbf{L}(\alpha + 1) = \mathtt{Def}(\mathbf{L}(\alpha))$
- $\mathbf{L}(\alpha) = \bigcup_{\xi<\alpha} \mathbf{L}(\xi)$ *(α a limit ordinal).*

We also define Gödel's Constructible Universe as the class

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{L}(\alpha).$$

Definition 261. *If $x \in \mathbf{L}$, then*

$$rk_{\mathbf{L}}(x) = \text{ the least } \alpha \in \mathbf{On} \text{ s.t. } x \in \mathbf{L}(\alpha + 1).$$

We list a few properties of the constructible hierarchy that will prove helpful.

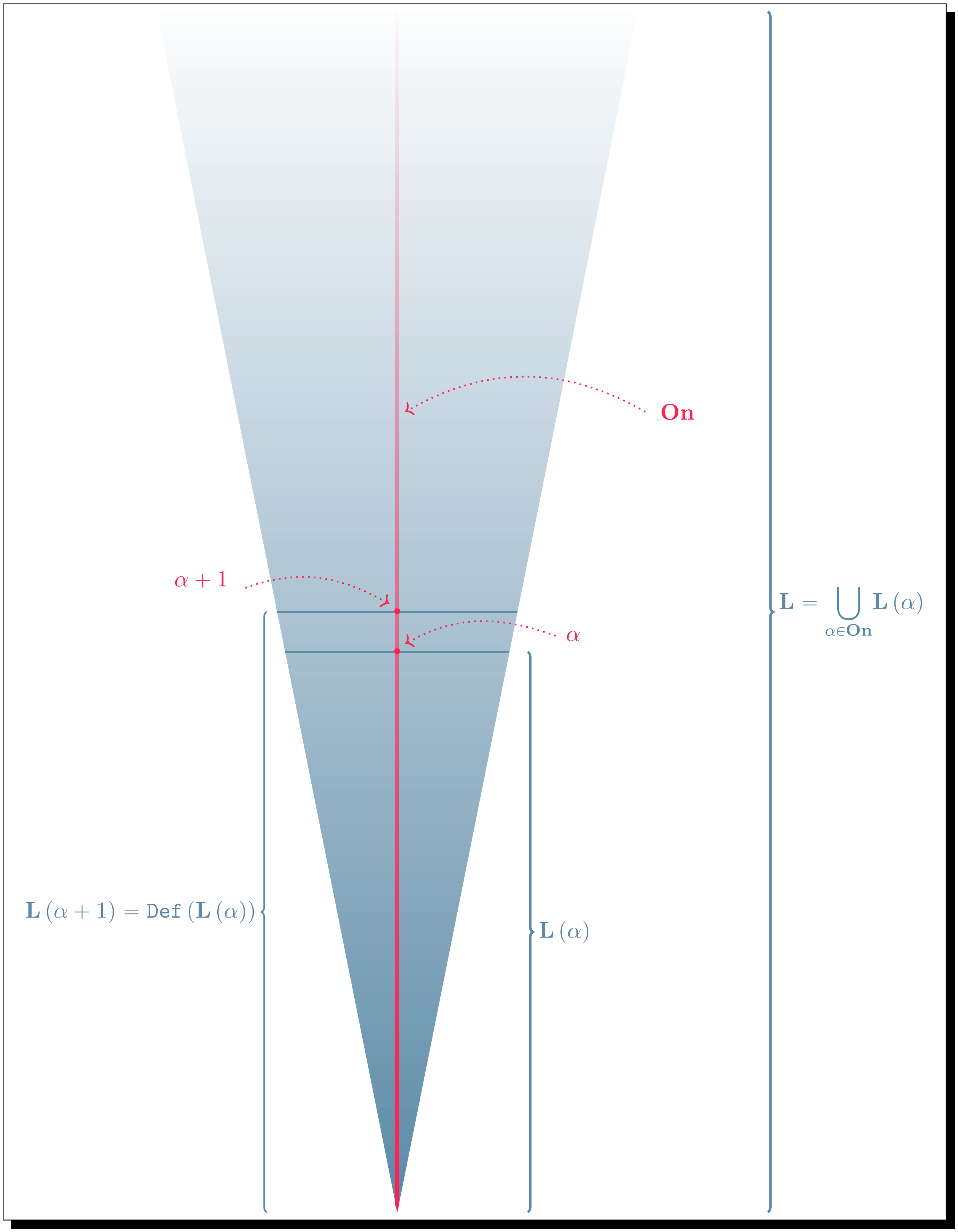

Figure 11.1: The Universe $\mathbf{L} = \bigcup_{\alpha\in\mathbf{On}} \mathbf{L}(\alpha)$.

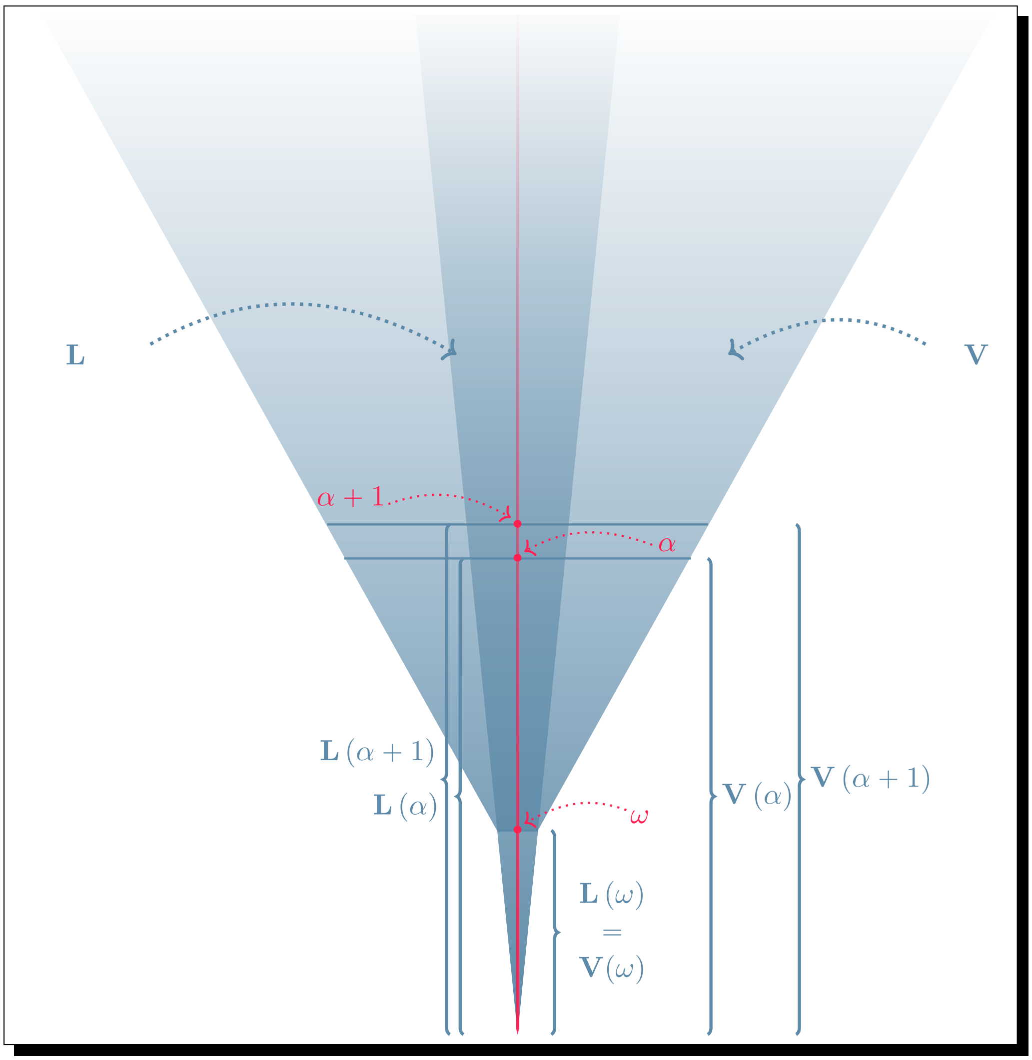

Figure 11.2: The Classes $\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{L}(\alpha)$ and $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{V}(\alpha)$.

Lemma 262 (**ZF**)**.** *Given any ordinals $\xi < \alpha$,*

(1) $\mathbf{L}(\alpha) \subseteq \mathbf{V}(\alpha)$

(2) For all $\alpha \leqslant \omega$, $\mathbf{L}(\alpha) = \mathbf{V}(\alpha)$

(3) $\mathbf{L}(\alpha)$ *is a transitive set*

(4) $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$

(5) $\mathbf{L}(\xi) \in \mathbf{L}(\alpha)$

(6) $\mathbf{L}(\alpha) = \{x \in \mathbf{L} \mid rk_{\mathbf{L}}(x) < \alpha\}$

(7) $\alpha \in \big(\mathbf{L}(\alpha+1) \smallsetminus \mathbf{L}(\alpha)\big)$.

Proof of Lemma 262: The different proofs all go by induction on α

(1) is obvious by definition of both $\mathbf{L}(\alpha)$ and $\mathbf{V}(\alpha)$.

(2) is immediate by Lemma 259 (2).

(3) $\boldsymbol{\alpha := 0}$ immediate since $\mathbf{L}(0) = \varnothing$;

$\boldsymbol{\alpha := \alpha + 1}$ if $x \in X \in \mathbf{L}(\alpha+1)$, then $x \in X \subseteq \mathbf{L}(\alpha)$. So, $x \in \mathbf{L}(\alpha)$ and also $x \subseteq \mathbf{L}(\alpha)$ since by induction hypothesis $\mathbf{L}(\alpha)$ is transitive. Then, one has

$$x = \{y \in \mathbf{L}(\alpha) \mid y \in x\} = \left\{y \in \mathbf{L}(\alpha) \mid (y \in x)^{\mathbf{L}(\alpha)}\right\} \in \mathbf{L}(\alpha+1).$$

$\boldsymbol{\alpha}$ **limit** immediate from Claim 130 where we showed that the union of any transitive set is transitive.

(4) By induction on α.

$\boldsymbol{\alpha := 0}$ immediate since there is no $\xi < \alpha$.

$\boldsymbol{\alpha := \alpha + 1}$ One has

(a) $\mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1)$ — by Lemma 259 (3) —

(b) $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$ — by induction hypothesis —

which yields

$$\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1),$$

hence, $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha+1)$.

$\boldsymbol{\alpha}$ **limit** Immediate since $\mathbf{L}(\alpha) = \bigcup_{\xi<\alpha} \mathbf{L}(\xi)$.

(5) One has

(a) $\mathbf{L}(\xi+1) \subseteq \mathbf{L}(\alpha)$ — by Lemma 262 (4)

(b) $\mathbf{L}(\xi) \in \mathbf{L}(\xi+1)$ — by Lemma 259 (1)

So, all together, one obtains $\mathbf{L}(\xi) \in \mathbf{L}(\alpha)$.

(6) Clearly, one has

(a) $\left(x \in \mathbf{L}(\alpha) \implies rk_{\mathbf{L}}(x) < \alpha\right)$: since $\mathbf{L}(\alpha) = \bigcup_{\xi \leqslant \alpha} \mathbf{L}(\xi)$ hods by Lemma 262 (4)

(b) $\left(rk_{\mathbf{L}}(x) < \alpha \implies x \in \mathbf{L}(\alpha)\right)$: since $rk_{\mathbf{L}}(x) = \xi < \alpha \implies x \in \mathbf{L}(\xi+1) \subseteq x \in \mathbf{L}(\alpha)$.

(7) First, notice that $rk(\alpha) = \alpha$, hence $\alpha \in \left(\mathbf{V}(\alpha+1) \smallsetminus \mathbf{V}(\alpha)\right)$.

(a) $\alpha \notin \mathbf{L}(\alpha)$ holds since $\mathbf{L}(\alpha) \supseteq \mathbf{V}(\alpha)$

(b) $\alpha \in \mathbf{L}(\alpha+1)$ is shown by induction on α.

$\boldsymbol{\alpha} := \mathbf{0}$ is immediate since $\mathbf{L}(1) = \{\varnothing\}$.

$\boldsymbol{\alpha} := \boldsymbol{\alpha} + \mathbf{1}$ for every ordinal $\xi \leqslant \alpha$

$$\xi \in \mathbf{L}(\alpha+1)$$

holds by induction hypothesis and Lemma 262 (4). So, one has both

- $\alpha \subseteq \mathbf{L}(\alpha+1)$
- $\alpha \in \mathbf{L}(\alpha+1)$

which yields

$$\begin{aligned}\alpha+1 &= \left\{y \in \mathbf{L}(\alpha+1) \mid y \in \alpha \vee y = \alpha\right\} \\ &= \left\{y \in \mathbf{L}(\alpha+1) \mid \left(y \in \alpha \vee y = \alpha\right)^{\mathbf{L}(\alpha+1)}\right\} \in \mathbf{L}(\alpha+2).\end{aligned}$$

$\boldsymbol{\alpha}$ **limit** for every $\xi < \alpha$, the induction hypothesis gives

$$\xi \in \mathbf{L}(\xi+1)$$

which yields

$$\mathbf{L}(\alpha) \cap \mathbf{On} = \alpha.$$

Using the fact that "x is an ordinal" is a Δ_0^{0-rud}-formula, hence absolute for transitive classes (see Lemma 197) we obtain

$$\begin{aligned}\alpha &= \left\{x \in \mathbf{L}(\alpha) \mid \text{“}x \text{ is an ordinal”}\right\} \\ &= \left\{x \in \mathbf{L}(\alpha) \mid \left(\text{“}x \text{ is an ordinal”}\right)^{\mathbf{L}(\alpha)}\right\} \in \mathbf{L}(\alpha+1).\end{aligned}$$

□ 262

Since for all integer n, we have $\mathbf{V}(n) = \mathbf{L}(n)$, we notice that $|\mathbf{L}(0)| = |\mathbf{V}(0)| = |\varnothing| = 0$ and for each n $|\mathbf{L}(n+1)| = |\mathbf{V}(n+1)| = 2^n$. Therefore $|\mathbf{V}(\omega)| = |\mathbf{L}(\omega)| = \aleph_0$ holds. But as soon as $\omega < \alpha$, the whole picture of the cardinality of $\mathbf{L}(\alpha)$ becomes very different from the one of $\mathbf{V}(\alpha)$. Indeed, assuming **AC**, the cardinality of $\mathbf{V}(\omega+\alpha)$ is $\beth_\alpha$ (see Definition 122) whereas we will see that the cardinality of $\mathbf{L}(\omega+\alpha)$ is simply the cardinality of α: compare $\beth_\alpha$ with $|\alpha|$!

Lemma 263 (**ZFC**)**.** *Given any $\omega \leqslant \alpha \in \mathbf{On}$,*

$$|\mathbf{L}(\alpha)| = |\alpha|.$$

Proof of Lemma 263: By induction on $\alpha \geqslant \omega$.

$\boldsymbol{\alpha := \omega}$ In this case, $\mathbf{L}(\omega) = \mathbf{V}(\omega)$, hence $|\mathbf{L}(\omega)| = |\mathbf{V}(\omega)| = \aleph_0 = |\omega|$.

$\boldsymbol{\alpha := \alpha + 1}$ since $|\alpha+1|_{|}\alpha|$, it is enough to show $|\mathbf{L}(\alpha)| = |\mathbf{L}(\alpha+1)|$.

(1) $|\mathbf{L}(\alpha)| \leqslant |\mathbf{L}(\alpha+1)|$: immediate from $\mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1)$.

(2) $|\mathbf{L}(\alpha)| \geqslant |\mathbf{L}(\alpha+1)|$:

$$\begin{aligned}\mathbf{L}(\alpha+1) &= \mathtt{Def}\big(\mathbf{L}(\alpha)\big)\\ &= \{X \subseteq \mathbf{L}(\alpha) \mid X \text{ is definable over } \mathbf{L}(\alpha)\}\\ &= \Big\{X \subseteq \mathbf{L}(\alpha) \mid X = \big\{x \in \mathbf{L}(\alpha) \mid \varphi_{(x, a_1/x_1, \ldots, a_n/x_n)}{}^{\mathbf{L}(\alpha)}\big\}\Big\}\\ &\quad \text{for some } \varphi_{(x,x_1,\ldots,x_n)} \text{ and } \langle a_1, \ldots, a_n\rangle \in \mathbf{L}(\alpha)^{<\omega}.\end{aligned}$$

Since there are $\aleph_0$-many $\mathcal{L}_{\text{ST}}$-formulas and $|\mathbf{L}(\alpha)^{<\omega}| = |\mathbf{L}(\alpha)|$, we obtain

$$|\mathbf{L}(\alpha+1)| \leqslant \aleph_0 \cdot |\mathbf{L}(\alpha)| = |\mathbf{L}(\alpha)|.$$

$\boldsymbol{\alpha}$ **limit** Since $\mathbf{L}(\alpha) = \bigcup_{\xi<\alpha} \mathbf{L}(\xi)$, the induction hypothesis and Lemma 104 yield

$$|\mathbf{L}(\alpha)| = \Big|\bigcup_{\xi<\alpha} \mathbf{L}(\xi)\Big| \leqslant |\alpha|.$$

□ 263

11.3 The Constructible Universe Satisfies ZF

This section is devoted to showing that $\mathbf{L} \models \mathbf{ZF}$. This means that for each formula[1] $\varphi \in \mathbf{ZF}$ we need to show that $\mathbf{L} \models \varphi$. The proof is done within $\mathbf{ZF}$, i.e., we show $\mathbf{ZF} \vdash_c (\varphi)^{\mathbf{L}}$.

Theorem 264.

$$\mathbf{ZF} \vdash_c (\mathbf{ZF})^{\mathbf{L}}.$$

Proof of Theorem 264:

(1) $(\textbf{Extensionality})^{\mathbf{L}}$ since $\mathbf{L}$ is transitive (see Lemma 185).

(2) $\big(\textbf{Comprehension Schema}\big)^{\mathbf{L}}$: at first glance, we may think of using the condition stated as a special case in Lemma 186 which assures that if $\mathbf{M}$ is closed under $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ that maps x to $\mathscr{P}(x)$, then $\big(\textbf{Comprehension Schema}\big)^{\mathbf{M}}$. But we cannot show that $\mathbf{L}$ is closed under the operation $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ that maps x to $\mathscr{P}(x)$. In fact, if it were the case then we would have in particular that for each ordinal α, $\mathbf{V}(\alpha) \subseteq \mathbf{L}$ would holds, which yields $\mathbf{V} = \mathbf{L}$.

We are then left with — the main condition of Lemma 186, i.e., — proving that for each $\varphi(x, X, z_1, \ldots, z_k)$ with free variables among $\{x, X, z_1, \ldots, z_k\}$, one has

$$\forall X \in \mathbf{L}\ \forall z_1 \in \mathbf{L}\ \ldots \forall z_k \in \mathbf{L}\ \left\{x \in X \mid \big(\varphi(x, X, z_1, \ldots, z_k)\big)^{\mathbf{L}}\right\} \in \mathbf{L}.$$

In order to complete the proof we need a very general result known as as a reflection principle.

to be continued...

11.4 A Reflection Principle for L

We first need to prove a *reflection principle for* $\mathbf{L}$ which is a copy of the reflection principle for $\mathbf{V}$ due to Azriel Lévy and Richard Montague [27].

Reflection Principle (Lévy & Montague)**.** *Let $\varphi_0, \ldots, \varphi_n$ be any $\mathcal{L}_{\mathrm{ST}}$-formulas.*

$$\mathbf{ZF} \vdash_c \forall \alpha \in \mathbf{On}\ \exists \beta > \alpha \quad \text{“}\varphi_0, \ldots, \varphi_n \quad \textit{are absolute for } \mathbf{V}(\beta),\ \mathbf{V}.\text{”}$$

Proof of the Reflection Principle: Identical to the proof of Theorem 266, *mutatis mutandis.*

[1] Each axiom or instance of axiom schema.

☐ Reflection Principle

In particular, the Reflection Principle states that given any finite subtheory Δ of **ZF** and any ordinal α, there exists some ordinal β (way larger than α) such that $\mathbf{V}(\beta) \models \Delta$. In particular, $\mathbf{V}(\beta)$ is a set which is a model of Δ. So, for every finite subtheory of $\Delta \subseteq \mathbf{ZF}$, we have $\mathbf{ZF} \vdash \Delta$. Since **ZFC** proves the compactness theorem which says that given any first order theory $\mathscr{T}$,

$$\mathscr{T} \text{ has a model if and only if every finite subtheory of } \mathscr{T} \text{ has a model.}$$

So, at first glance it seems that a consequence is that **ZF** has a model, which would contradict "$\mathbf{ZF} \nvdash_c cons(\mathbf{ZF})$". But what is required to be able to apply the compactness theorem is not just that for for every finite subtheory of $\Delta \subseteq \mathbf{ZF}$, $\mathbf{ZF} \vdash \Delta$ holds, but rather **ZF** proves that for all finite subtheory of $\Delta \subseteq \mathbf{ZF}$, $\mathbf{ZF} \vdash \Delta$. This is the difference between for each instance of a problem schema, proving that particular instance and proving the problem schema.

In particular, a consequence of the Reflection Principle is that, assuming **ZF** is consistent, **ZF** is not finitely axiomatizable. Otherwise, there would exist some formula $\varphi_{\mathbf{ZF}}$ such that

- $\mathbf{ZF} \vdash_c \varphi_{\mathbf{ZF}}$
- $\varphi_{\mathbf{ZF}} \vdash_c \psi$, holds for every $\psi \in \mathbf{ZF}$
- $\mathbf{V}(\beta) \models \varphi_{\mathbf{ZF}}$ holds for some (infinitely many indeed!) ordinal β.

Hence, $\mathbf{V}(\beta) \models \mathbf{ZF}$ would hold, contradicting Gödel's second incompleteness theorem.

Theorem 266 (**Reflection Principle for L**)**.** *Let $\varphi_0, \ldots, \varphi_n$ be any $\mathcal{L}_{\text{ST}}$-formulas.*

$$\mathbf{ZF} \vdash_c \forall \alpha \in \mathbf{On}\ \exists \beta > \alpha \quad \text{“}\varphi_0, \ldots, \varphi_n \quad \textit{are absolute for } \mathbf{L}(\beta), \mathbf{L}.\text{”}$$

Proof of Theorem 266: First, without loss of generality we may assume that the set of formulas $\{\varphi_0, \ldots, \varphi_n\}$ is closed under sub-formulas and only contains formulas using $\neg, \wedge$ as connectors and $\exists$ as quantifiers.
For each integer $i \leqslant n$ such that φ_i is of the form $\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})$, we define a functional $\mathbf{G}_i : \underbrace{\mathbf{L} \times \ldots \times \mathbf{L}}_{k_i} \to \mathbf{On}$ by

$$\begin{aligned} \mathbf{G}_i(y_1, \ldots, y_{k_i}) &= 0 \text{ if } \Big(\neg \exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})\Big)^{\mathbf{L}} \\ &= \text{least } \theta \text{ s.t. } \exists x \in \mathbf{L}(\theta)\ \ \big(\varphi_j(x, y_1, \ldots, y_{k_i})\big)^{\mathbf{L}} \text{ otherwise.} \end{aligned}$$

Then, for each integer $i \leqslant n$ we define a functional $\mathbf{F}_i : \mathbf{On} \to \mathbf{On}$ by

$$\mathbf{F}_i(\xi) = \sup \{\mathbf{G}_i(y_1, \ldots, y_{k_i}) \mid y_1, \ldots, y_{k_i} \in \mathbf{L}(\xi)\} \text{ if } \mathbf{G}_i \text{ is defined}$$

$$\mathbf{F}_i(\xi) = 0 \text{ otherwise.}$$

Given any ordinal α, one defines the strictly increasing sequence $(\beta_k)_{n\in\omega}$ and a limit ordinal β by:

- $\beta_0 = \alpha$
- $\beta_{k+1} = \sup\left\{\beta_k + 1, \mathbf{F}_1(\beta_k), \ldots, \mathbf{F}_n(\beta_k)\right\}$
- $\beta = \sup_{k\in\omega} \beta_k$

We show — by induction on the height of the formula — that for each integer $i \leqslant n$, one has

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\varphi_i(y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \longleftrightarrow \varphi_i(y_1, \ldots, y_{k_i})^{\mathbf{L}}\right)$$

It turns out that the only interesting case is when φ_i is of the form $\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})$. So, we have to check that

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\left(\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}(\beta)} \longleftrightarrow \left(\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}}\right)$$

i.e., Clearly, the direction

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\exists x \in \mathbf{L}(\beta)\ \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \longrightarrow \exists x \in \mathbf{L}\ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}}\right)$$

is taken care of by the induction hypothesis. So, we show

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\exists x \in \mathbf{L}\ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}} \longrightarrow \exists x \in \mathbf{L}(\beta)\ \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)}\right)$$

We fix $y_1 \in \mathbf{L}(\beta), \ldots, y_{k_i} \in \mathbf{L}(\beta)$. For some large enough integer p, one has

$$\{y_1, \ldots, y_{k_i}\} \subseteq \mathbf{L}(\beta_p).$$

By construction, there exists $x \in \mathbf{L}(\mathbf{G}_i(y_1, \ldots, y_{k_i}))$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}}$. Since $\mathbf{G}_i(y_1, \ldots, y_{k_i}) \leqslant \mathbf{F}_i(\beta_p) \leqslant \beta_{p+1}$, it follows that there exists $x \in \mathbf{L}(\beta_{p+1}) \subseteq \mathbf{L}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}}$. Finally, by induction hypothesis, there exists $x \in \mathbf{L}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}(\beta)}$.

$\square$ 266

We now resume the proof that the **Comprehension Schema** holds inside Gödel's constructible universe.

(2) ...continued. Proving $\left(\textbf{Comprehension Schema}\right)^{\mathbf{L}}$ comes down to showing

$$\forall X \in \mathbf{L}\ \forall z_1 \in \mathbf{L}\ \ldots \forall z_k \in \mathbf{L}\ \ \left\{x \in X \mid \left(\varphi(x, X, z_1, \ldots, z_k)\right)^{\mathbf{L}}\right\} \in \mathbf{L}.$$

So, we fix $\{X, z_1, \ldots, z_k\} \subseteq \mathbf{L}$ and consider any $\alpha > \sup\{rk_{\mathbf{L}}(X), rk_{\mathbf{L}}(z_1), \ldots, rk_{\mathbf{L}}(z_k)\}$. By the Reflection Principle for $\mathbf{L}$ (Theorem 266) there exists some $\beta > \alpha$ such that the formula $\left(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\right)$ is absolute for $\mathbf{L}(\beta), \mathbf{L}$.

Therefore,

$$\left\{x \in X \mid \varphi(x, X, z_1, \ldots, z_k)^{\mathbf{L}}\right\} = \left\{x \in \mathbf{L} \mid \Big(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\Big)^{\mathbf{L}}\right\}$$
$$= \left\{x \in \mathbf{L}\,(\beta) \mid \Big(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\Big)^{\mathbf{L}(\beta)}\right\} \in \mathbf{L}\,(\beta + 1)\,.$$

(3) $(\textbf{Pairing})^{\mathbf{L}}$ is almost immediate, since we only need to show

$$\forall x \in \mathbf{L}\ \forall y \in \mathbf{L}\ \exists z \in \mathbf{L}\ \ \big((x \in z\ \wedge\ y \in z)\big)^{\mathbf{L}}$$

i.e.,

$$\forall x \in \mathbf{L}\ \forall y \in \mathbf{L}\ \exists z \in \mathbf{L}\ \ (x \in z\ \wedge\ y \in z)$$

Take any $\alpha > \max\{rk_{\mathbf{L}}\,(x)\,, rk_{\mathbf{L}}\,(y)\}$. One has $x, y \in \mathbf{L}\,(\alpha)$, hence

$$\begin{aligned}\{x, y\} &= \big\{z \in \mathbf{L}\,(\alpha) \mid (x = z\ \vee\ y = z)\big\}\\ &= \left\{z \in \mathbf{L}\,(\alpha) \mid \big((x = z\ \vee\ y = z)\big)^{\mathbf{L}}\right\} \in \mathbf{L}\,(\alpha + 1) \subseteq \mathbf{L}.\end{aligned}$$

(4) $(\textbf{Union})^{\mathbf{L}}$ is easy. We simply show that given any $X \in \mathbf{L}$, the set $\bigcup X$ also belongs to $\mathbf{L}$. So, we assume $X \in \mathbf{L}\,(\alpha)$. Since $\mathbf{L}\,(\alpha)$ is transitive, $tc\,(X) \subseteq \mathbf{L}\,(\alpha)$ holds and

$$\begin{aligned}\bigcup X &= \big\{x \in tc\,(X) \mid \exists y\ (x \in y\ \wedge\ y \in X)\big\}\\ &= \left\{x \in tc\,(X) \mid \exists y\ \big((x \in y\ \wedge\ y \in X)\big)^{\mathbf{L}(\alpha)}\right\}\\ &= \left\{x \in tc\,(X) \mid \exists y \in \mathbf{L}\,(\alpha)\ \ \big((x \in y\ \wedge\ y \in X)\big)^{\mathbf{L}(\alpha)}\right\}\\ &= \left\{x \in \mathbf{L}\,(\alpha) \mid \exists y \in \mathbf{L}\,(\alpha)\ \ \big((x \in y\ \wedge\ y \in X)\big)^{\mathbf{L}(\alpha)}\right\}\\ &= \left\{x \in \mathbf{L}\,(\alpha) \mid \big(\exists y\ (x \in y\ \wedge\ y \in X)\big)^{\mathbf{L}(\alpha)}\right\} \in \mathbf{L}\,(\alpha + 1) \subseteq \mathbf{L}.\end{aligned}$$

(5) $(\textbf{Infinity})^{\mathbf{L}}$ is immediate since by Lemma 262 (7) $\mathbf{L} \cap \mathbf{On} = \mathbf{On}$; so in particular ω belongs to $\mathbf{L}$.

(6) $(\textbf{Power Set})^{\mathbf{L}}$ is proved by making use of the fact $\mathbf{L}$ is transitive and following Lemma 187 which states that it is enough to establish

$$\forall x \in \mathbf{L}\ \exists y \in \mathbf{L}\ \big(\mathscr{P}\,(x) \cap \mathbf{L}\big) \subseteq y.$$

Given any $x \in \mathbf{L}$ we set

$$\alpha = \sup\{rk_{\mathbf{L}}\,(z) + 1 \mid z \in \mathbf{L}\ \ \wedge\ \ z \subseteq x\}\,,$$

so we obtain $\Big(\mathscr{P}\,(x) \cap \mathbf{L}\Big) \subseteq \mathbf{L}\,(\alpha)$.

(7) (**Foundation**)$^{\mathbf{L}}$ is immediate because working within **ZF** every class is well-founded.

(8) (**Replacement Schema**)$^{\mathbf{L}}$ holds since, by Lemma 190, we need to show that given any formula $\varphi := \varphi(x, y, A, w_1, \ldots, w_n)$ whose free variables are among $x, y, A, w_1, ..., w_n$.

$\forall A \in \mathbf{L}\ \forall w_1 \in \mathbf{L} \ldots \forall w_n \in \mathbf{L}$

$$\left(\forall x \in A \cap \mathbf{L}\ \exists! y \in \mathbf{L}\ (\varphi)^{\mathbf{L}} \longrightarrow \exists B \in \mathbf{L}\ \left\{y \in \mathbf{L} \mid \exists x \in A \cap \mathbf{L}\ (\varphi)^{\mathbf{L}}\right\} \subseteq B\right)$$

Given any $A \in \mathbf{L}$ $w_1 \in \mathbf{L} \ldots w_n \in \mathbf{L}$ we set

$$\alpha = \sup \left\{ rk_{\mathbf{L}}\,(y) + 1 \mid \exists x \in A \ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}} \right\},$$

in order to get

$$\begin{aligned} &\left\{y \mid \exists x \in A \ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\right\} \subseteq \mathbf{L}\,(\alpha) \\ = &\left\{y \in \mathbf{L}\,(\alpha) \mid \left(\exists x \in A \ \ \varphi(x, y, A, w_1, \ldots, w_n)\right)^{\mathbf{L}}\right\} \in \ \mathbf{L}\,(\alpha + 1) \subseteq \mathbf{L}. \end{aligned}$$

$\square$ 264

Chapter 12

AC and CH inside Gödel's Constructible Universe

12.1 The Axiom of Constructibility and the Axiom of Choice

Not only the Constructible Universe satisfies the Axiom of **Choice** — since it satisfies the equivalent statement that every set can be well-ordered — but it satisfies that one can define a "set-like" relational on the whole universe of Constructible Sets that well-orders it.

Theorem 267.

$$\mathbf{ZF} \vdash_c (\mathbf{AC})^{\mathbf{L}}$$

Proof of Theorem 267: By induction on $\alpha \in \mathbf{On}$, one defines a (strict) well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$ such that for all ordinals $\beta < \alpha$, $\lhd_\alpha$ extends $\lhd_\beta$. i.e., one has

$$\lhd_\alpha \cap \mathbf{L}(\beta) \times \mathbf{L}(\beta) = \lhd_\beta \text{ and for every } x \in \mathbf{L}(\beta) \text{ and } y \in \mathbf{L}(\alpha) \smallsetminus \mathbf{L}(\beta),\ x \lhd_\alpha y.$$

Before we define $\lhd_\alpha$, we need some notation.
Assuming $X \subseteq Y \subseteq \mathbf{L}(\alpha)$ and $\lhd_\alpha$ is a well-ordering on $\mathbf{L}(\alpha)$, we define the well-ordering $\lessdot_\alpha$ on $\omega \times \mathbf{L}(\alpha)^{<\omega}$ as the lexicographic ordering induced by the usual ordering on the integers and the well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$:

$$(n,k,S) \lessdot_\alpha (n',k',S') \iff \left(\begin{array}{c} n < n' \\ \vee \\ n = n' \ \wedge \ k < k' \\ \vee \\ n = n' \ \wedge \ k = k' \ \wedge \ lh(S) < lh(S') \\ \vee \\ n = n' \ \wedge \ k = k' \ \wedge \ lh(S) = lh(S') \ \wedge \ S \lhd_{lex.\alpha} S' \end{array} \right)$$

where

$$\langle a_0, a_1, \ldots, a_m \rangle \lhd_{lex.\alpha} \langle a'_0, a'_1, \ldots, a'_m \rangle$$
$$\Longleftrightarrow$$
$$\exists\, p \leqslant m \;\left(\forall i < p \; a_i = a'_i \;\wedge\; a_p \lhd_\alpha a'_p\right)$$

Given any sets X, Y such that $\texttt{Definable\_Over}\,(X, Y)$, we denote by $\texttt{Wittn.\_Def.\_Over}\,(X, Y)$ the following set:

$$\texttt{Wittn.\_Def.\_Over}\,(X, Y)$$
$$=$$
$$\left\{ (\ulcorner\varphi\urcorner, n, S) \in \omega \times \omega \times Y^{<\omega} \;\middle|\; \left(\begin{array}{c} \text{“}\varphi \text{ has exactly } x_0, x_1, \ldots, x_n \text{ as free variables ”} \\ \wedge \\ \text{“}S \text{ is a mapping from } \{1, \ldots, n\} \text{ to } Y\text{”} \\ \forall x_0 \in Y \;\; \Big(x_0 \in X \longleftrightarrow \texttt{Holds}\,(\ulcorner\varphi\urcorner, S \cup \{(0, x_0)\}, Y) \Big). \end{array} \right) \right\}$$

Finally, we define the well-orderings $\lhd_\alpha$ on $\mathbf{L}\,(\alpha)$ by induction on α by:

$\boldsymbol{\alpha} := \mathbf{0}$ Obviously $\lhd_0 = \varnothing$

$\boldsymbol{\alpha} := \boldsymbol{\beta} + \mathbf{1}$ $\lhd_\alpha$ is defined by $\forall x, y \in \mathbf{L}\,(\alpha)$,

$$x \lhd_\alpha y$$
$$\Longleftrightarrow$$
$$\left(\begin{array}{c} x, y \in \mathbf{L}\,(\beta) \;\wedge\; x \lhd_\beta y \\ \vee \\ x \in \mathbf{L}\,(\beta) \;\wedge\; y \notin \mathbf{L}\,(\beta) \\ \vee \\ x, y \notin \mathbf{L}\,(\beta) \;\wedge\; \left(\begin{array}{c} \ll_\beta\text{-least } (\ulcorner\varphi\urcorner, n, S) \in \texttt{Wittn.\_Def.\_Over}\,(x, \mathbf{L}\,(\beta)) \\ \ll_\beta \\ \ll_\beta\text{-least } (\ulcorner\varphi'\urcorner, n', S') \in \texttt{Wittn.\_Def.\_Over}\,(y, \mathbf{L}\,(\beta)) \end{array} \right) \end{array} \right)$$

$\boldsymbol{\alpha}$ **limit** $\lhd_\alpha$ is defined by: $\forall x, y \in \mathbf{L}\,(\alpha)$, let $\beta = \sup\{rk_{\mathbf{L}}\,(x) + 1, rk_{\mathbf{L}}\,(y) + 1\}$. Notice that $\beta < \alpha$ and $x, y \in \mathbf{L}\,(\beta)$ and set

$$x \lhd_\alpha y \iff x \lhd_\beta y.$$

So far, we have constructed for each ordinal α, a well-ordering $\lhd_\alpha$ of $\mathbf{L}\,(\alpha)$. It remains to show that every set $X \in \mathbf{L}$ can be well-ordered. For this, it is enough to consider $\alpha = rk_{\mathbf{L}}\,(X)$ since both $X \subseteq \mathbf{L}\,(\alpha)$ and $(\mathbf{L}\,(\alpha), \lhd_\alpha)$ is a well-ordering, which yields $(X, \lhd_\alpha)$.

$\square$ 267

Since for every ordinal $\beta < \alpha$, the well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$ is an extension of the well-ordering $\lhd_\beta$ on $\mathbf{L}(\beta)$, we may easily define a relational $<_{\mathbf{L}}$ that well-orders the whole universe of constructible sets.

Definition 268 (Well-ordering of $\mathbf{L}$)**.** *We define a relational $<_{\mathbf{L}}$ that well-orders $\mathbf{L}$ by*

$$x <_{\mathbf{L}} y \iff \Big(x \lhd_\alpha y \text{ and } \alpha = \max\{rk_{\mathbf{L}}(x), rk_{\mathbf{L}}(y)\} + 1\Big).$$

Definition 269 (Axiom of Constructibility)**.**

$$\mathbf{V} = \mathbf{L} \textit{ is the statement } \text{“}\forall x\ \exists \alpha \in \mathbf{On}\ \ x \in \mathbf{L}(\alpha)\text{”}.$$

We have just proved

Theorem 270.

$$\mathbf{ZF} + \mathbf{V} = \mathbf{L} \vdash_c \mathbf{AC}.$$

12.2 The Axiom of Constructibility and the Generalized Continuum Hypothesis

We now come to **GCH** assuming $\mathbf{V} = \mathbf{L}$. This is slightly more complicated than **AC**. Our goal is to show that $\mathbf{L} \models \mathbf{GCH}$ or more precisely that **ZF** proves $(\mathbf{GCH})^{\mathbf{L}}$ or equivalently to prove the following theorem:

Theorem 271.

$$\mathbf{ZF} + \mathbf{V} = \mathbf{L} \vdash_c \mathbf{GCH}.$$

This main theorem is a direct consequence of the following lemma which shows that if a subset of a cardinal number κ appears somewhere [1] in the construction of the Constructible Universe, then it appears in less than κ^+ steps.

[1] As opposed to never appearing anywhere, for the reason that it is simply not constructible.

Lemma 272 (**ZF**)**.** *If* $\mathbf{V} = \mathbf{L}$*, then for every infinite cardinal* κ*,*

$$\mathscr{P}(\kappa) \subseteq \mathbf{L}\left(\kappa^{+}\right).$$

Proof of Theorem 271: The result follows easily from Lemma 272 and Lemma 263 which stated that given any infinite ordinal α, one has $|\mathbf{L}(\alpha)| = |\alpha|$. Indeed, we obtain for every infinite cardinal κ,

$$\kappa < 2^{\kappa} = |\mathscr{P}(\kappa)| \leqslant |\mathbf{L}\left(\kappa^{+}\right)| = |\kappa^{+}| = \kappa^{+}.$$

$\square$ 271

We now concentrate on proving the main lemma. For this purpose, we make use of the notion of an $\mathcal{L}_{\mathrm{ST}}$-*elementary submodel.*

Definition 273 (Elementary Submodel)**.** *Let* $X \subseteq Y$ *be sets.* X *is an elementary submodel of* Y *— denoted* $X \prec Y$ *— if and only if for all* $\mathcal{L}_{\mathrm{ST}}$*-formula* $\varphi(x_1, \ldots, x_n)$ *— whose free variables are among* $x_1, \ldots, x_n$ *— and all* $a_1 \in X, \ldots, a_n \in X$*,*

$$\left(\varphi(a_1/x_1, \ldots, a_n/x_n)\right)^{X} \longleftrightarrow \left(\varphi(a_1/x_1, \ldots, a_n/x_n)\right)^{Y}.$$

In words, X is an elmentary submodel of Y if $X \subseteq Y$ and both structures satisfy the same formulas whose parameters are taken from the smaller one.

Lemma 274 (**ZF**)**.** *Let* X *be any set and* $\omega < \alpha$ *any limit ordinal. If* $X \prec \mathbf{L}(\alpha)$*, then*

there exists $\beta \leqslant \alpha$*, and an isomorphism* $\pi : \left(X, \in\right) \approx \left(\mathbf{L}(\beta), \in\right)$*.*

Proof of Lemma 274: We simply consider the Mostowski collapse $\pi : \left(X, \in\right) \to \left(M, \in\right)$. We recall that for each $y \in X$, one has

$$\pi(y) = \left\{\pi(x) \mid x \in y\right\}.$$

Notice, that $X \prec \mathbf{L}(\alpha)$ implies $X \subseteq \mathbf{L}(\alpha)$. Moreover, since $\mathbf{L}(\alpha)$ is extensional[2], it follows that $\in$ is also extensional on X because from $X \prec \mathbf{L}(\alpha)$ we have

$$\left(\forall x\ \forall y\ \left(\forall z\left(z \in x \longleftrightarrow z \in y\right) \to x = y\right)\right)^{X} \longleftrightarrow \left(\forall x\ \forall y\ \left(\forall z\left(z \in x \longleftrightarrow z \in y\right) \to x = y\right)\right)^{\mathbf{L}(\alpha)}.$$

[2]We recall that $\in$ is extensional on "$\mathbf{L}(\alpha)$" means $\left(\forall x\ \forall y\ \left(\forall z\left(z \in x \longleftrightarrow z \in y\right) \to x = y\right)\right)^{\mathbf{L}(\alpha)}$

i.e.,

$$(\textbf{Extensionality})^X \longleftrightarrow (\textbf{Extensionality})^{\mathbf{L}(\alpha)}.$$

By Corollary 184, it follows that M is both extensional and transitive[3]. We are going to show that $M = \mathbf{L}(\beta)$ for some limit ordinal $\beta \leqslant \alpha$. For this, we first consider $\pi^{-1} : M \to X \subseteq \mathbf{L}(\alpha)$. Given any $\mathcal{L}_{\mathrm{ST}}$-formula $\varphi(x_1, \dots, x_n)$ whose free variables are among $x_1, \dots, x_n$, and any $a_1, \dots, a_n \in M$, since both $M \approx X$ and $X \prec \mathbf{L}(\alpha)$ hold, one has

$$\begin{aligned}\Big(\varphi(a_1/x_1, \dots, a_n/x_n)\Big)^M &\longleftrightarrow \Big(\varphi(\pi^{-1}(a_1)/x_1, \dots, \pi^{-1}(a_n)/x_n)\Big)^X \\ &\longleftrightarrow \Big(\varphi(\pi^{-1}(a_1)/x_1, \dots, \pi^{-1}(a_n)/x_n)\Big)^{\mathbf{L}(\alpha)}\end{aligned}$$

which shows that $\pi^{-1} : M \to \mathbf{L}(\alpha)$ is an elementary injection: both $\pi^{-1} : M \xrightarrow{1-1} \mathbf{L}(\alpha)$ is an injective homomorphism and $\pi^{-1}[M] \prec \mathbf{L}(\alpha)$.

In particular we have

$$\forall \xi < \alpha\ \exists x \in \mathbf{L}(\alpha) \qquad \underbrace{x = \mathbf{L}(\xi)}_{\exists y \in \mathbf{L}(\alpha)\ \underbrace{\psi(x, y, \xi)}_{\Delta_0^{0-rud}}}$$

where ψ is some Δ_0^{0-rud}-formula (hence absolute for all transitive classes). So, we have

$$\begin{aligned}\forall \xi < \alpha\ \exists x \in \mathbf{L}(\alpha) \quad x = \mathbf{L}(\xi) &\longleftrightarrow \forall \xi \in \mathbf{On}\ \exists x \in \mathbf{L}(\alpha)\ \ \exists y \in \mathbf{L}(\alpha)\ \ \psi(x, y, \xi) \\ &\longleftrightarrow \big(\forall \xi \in \mathbf{On}\ \exists x\ \ \exists y\ \psi(x, y, \xi)\big)^{\mathbf{L}(\alpha)} \\ &\longleftrightarrow \big(\forall \xi \in \mathbf{On}\ \exists x\ \ \exists y\ \psi(x, y, \xi)\big)^{M} \\ &\longleftrightarrow \forall \xi \in \mathbf{On} \cap M\ \exists x \in M\ \ \exists y \in M\ \big(\psi(x, y, \xi)\big)^M \\ &\longleftrightarrow \forall \xi \in \mathbf{On} \cap M\ \exists x \in M\ \ \underbrace{\exists y \in M\ \ \psi(x, y, \xi)}_{x = \mathbf{L}(\xi)}\end{aligned}$$

Set $\beta = M \cap \mathbf{On}$ (notice that β is an ordinal by transitivity of M). By the result above we have

$$\forall \xi < \beta \ \ \mathbf{L}(\xi) \in M.$$

We now use the fact that α is a limit ordinal, to show that β is also a limit ordinal. For this, we make use of the fact that "$z \in \mathbf{On}$" is Δ_0^{0-rud}, hence absolute for transitive classes.

$$\begin{aligned}\alpha \text{ is a limit ordinal} &\longrightarrow \forall \xi \in \alpha\ \exists \zeta \in \alpha\ \ \xi \in \zeta \\ &\longrightarrow \big(\forall \xi \in \mathbf{On}\ \exists \zeta \in \mathbf{On}\ \ \xi \in \zeta\big)^{\mathbf{L}(\alpha)} \\ &\longrightarrow \big(\forall \xi \in \mathbf{On}\ \exists \zeta \in \mathbf{On}\ \ \xi \in \zeta\big)^{M}.\end{aligned}$$

[3] As it is the Mostowski collapse of an extensional class (set!) X.

So we have

$$\left.\begin{array}{r}\beta \text{ is a limit ordinal} \\ \forall \xi < \beta \quad \mathbf{L}(\xi) \in M\end{array}\right\} \Longrightarrow \mathbf{L}(\beta) = \bigcup_{\xi<\beta} \mathbf{L}(\xi) \subseteq M.$$

To show $M \subseteq \mathbf{L}(\beta) = \bigcup_{\xi<\beta} \mathbf{L}(\xi)$, it is enough to notice that

$$\begin{aligned}
& \forall x \in \mathbf{L}(\alpha)\ \exists v \in \mathbf{L}(\alpha)\ \exists \xi \in \mathbf{On} \cap \mathbf{L}(\alpha)\ \Big(\overbrace{v = \mathbf{L}(\xi)}^{\exists z \in \mathbf{L}(\alpha)\ \overbrace{\psi(z,\xi,v)}^{\Delta_0^{0-rud}}} \ \wedge\ x \in v \Big) \\
\longleftrightarrow\ & \Big(\forall x\ \exists v\ \exists \xi \in \mathbf{On}\ \ \big(\exists z\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big)\Big)^{\mathbf{L}(\alpha)} \\
\longleftrightarrow\ & \Big(\forall x\ \exists v\ \exists \xi \in \mathbf{On}\ \ \big(\exists z\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big)\Big)^{M} \\
\longleftrightarrow\ & \forall x \in M\ \exists v \in M\ \exists \xi \in \mathbf{On} \cap M\ \ \big(\exists z \in M\ \ \psi(z,\xi,v)^{M}\ \wedge\ x \in v\big) \\
\longleftrightarrow\ & \forall x \in M\ \exists v \in M\ \exists \xi \in \mathbf{On} \cap M\ \ \big(\exists z \in M\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big) \\
\longleftrightarrow\ & \forall x \in M\ \exists v \in M\ \exists \xi \in \mathbf{On} \cap M\ \ \big(v = \mathbf{L}(\xi)\ \wedge\ x \in v\big)
\end{aligned}$$

which shows that $M \subseteq \bigcup_{\xi<\beta} \mathbf{L}(\xi) = \mathbf{L}(\beta)$.

□ 274

We saw that any elementary submodel of some $\mathbf{L}(\alpha)$ for some limit ordinal α is in fact isomorphic to some $\mathbf{L}(\beta)$ for some limit ordinal $\beta \leqslant \alpha$. We now show that every subset of $\mathbf{L}(\alpha)$ can be extended in an elementary submodel of $\mathbf{L}(\alpha)$ whose cardinality does not exceed the one of X, provided that X is infinite.

Lemma 275 (ZFC). *If α is any ordinal, and X is any set such that $X \subseteq \mathbf{L}(\alpha)$. Then,*

there exists M such that $|M| = \sup\{|X|, \aleph_0\}$, $X \subseteq M$, and $M \prec \mathbf{L}(\alpha)$.

Proof of Lemma 275: We make use of the Tarski-Vaught criterion [2, 3, 4, 5, 6, 33]. This criterion states that

$$M \prec \mathbf{L}(\alpha)$$

$$\Longleftrightarrow$$

$$\text{for each } \mathcal{L}_{\text{ST}}\text{-formula } \varphi(x_0, x_1, \ldots, x_n) \text{ and } a_1 \in M, \ldots, a_n \in M$$

$$\big(\exists x_0\ \varphi(x_0, x_1, \ldots, x_n)\big)^{\mathbf{L}(\alpha)} \longrightarrow \big(\exists x_0\ \varphi(x_0, x_1, \ldots, x_n)\big)^{M}.$$

We construct M that satisfies the Tarski-Vaught criterion by recursion on the integers:

- $M_0 = X \cup \omega$

- $M_{n+1} = M_n \cup \left\{ x \in \mathbf{L}(\alpha) \;\middle|\; \begin{array}{c} \text{there exist } \varphi(x, x_1, \ldots, x_k), a_1 \in M_n, \ldots, a_k \in M_n \\ \left(\varphi(x, a_1, \ldots, a_k)\right)^{\mathbf{L}(\alpha)} \\ \wedge \\ \forall y \in \mathbf{L}(\alpha) \quad \left(\left(\varphi(y, a_1, \ldots, a_k)\right)^{\mathbf{L}(\alpha)} \longrightarrow y \nless_{\mathbf{L}} x\right) \end{array} \right\}$
- $M = \bigcup_{n \in \omega} M_n$.

It is then easy to check that M satisfies both

(1) $X \cup \omega \subseteq M \prec \mathbf{L}(\alpha)$.

(2) $|X \cup \omega| = \sup\{|X|, \aleph_0\} = |M|$.

$\square$ 275

Everything is now ready for proving the main Lemma 272 which says: if $\mathbf{V} = \mathbf{L}$, then for every infinite cardinal κ, one has $\mathscr{P}(\kappa) \subseteq \mathbf{L}(\kappa^+)$. In other words, one does not have to look farther than $\mathbf{L}(\kappa^+)$ in order to find all constructible subsets of κ.

Proof of Lemma 272: Let $Y \in \mathscr{P}(\kappa)$ and α be the least limit ordinal such that $Y \in \mathbf{L}(\alpha)$. We set

$$X = \kappa \cup \{Y\}$$

By Lemma 275 there exists M such that

- $|M| = \sup\{|X|, \aleph_0\} = \kappa$
- $X \subseteq M$
- $M \prec \mathbf{L}(\alpha)$.

By Lemma 274 there exist

- an ordinal $\beta \leqslant \alpha$
- $\pi : \left(M, \in\right) \approx \left(\mathbf{L}(\beta), \in\right)$.

The Mostowski collapsing function is the identity on every transitive set. So, κ being an ordinal is transitive, hence

$$\pi \restriction \kappa = id.$$

Thus,

$$\begin{aligned} \pi(Y) &= \{\pi(\xi) \mid \xi \in Y\} \\ &= \{\xi \mid \xi \in Y\} \\ &= Y. \end{aligned}$$

So, it follows

$$\pi(Y) = Y \in \mathbf{L}(\beta).$$

Since $|M| = \kappa$ and $(M, \in) \approx (\mathbf{L}(\beta), \in)$ we obtain

$$|\mathbf{L}(\beta)| = \kappa.$$

Finally, by Lemma 263, we get

$$|\mathbf{L}(\beta)| = |\beta| = \kappa.$$

Thus

$$\beta < \kappa^+$$

and

$$Y \in \mathbf{L}(\kappa^+)$$

which shows that

$$\mathscr{P}(\kappa) \subseteq \mathbf{L}(\kappa^+).$$

$\square$ 272

12.3 Inner Models

Definition 276 (Inner Model)**.** *A class* $\mathbf{M}$ *is an* inner model *of* **ZFC** *if*

(1) $\mathbf{M}$ *is transitive* *(2)* $\mathbf{On} \cap \mathbf{M} = \mathbf{On} \cap \mathbf{V}$ *(3)* $(\mathbf{ZFC})^{\mathbf{M}}$.

Clearly, the Gödel's Constructible Universe is an inner model. Moreover, it turns out that $\mathbf{L}$ is the $\subseteq$-least inner model in the following sense: if $\mathbf{I}$ is an inner model, then $\mathbf{L} \subseteq \mathbf{I}$.