Part III

The Consistency of ZF

Chapter 9

Arithmetic and Recursivity

9.1 Recursive Sets of Integers and Functions $\mathbb{N}^p \longrightarrow \mathbb{N}$

Definition 203.

(1) (dom_f, f) *is a* partial function $\mathbb{N}^p \longrightarrow \mathbb{N}$ *if* f *is a mapping* $dom_f \longrightarrow \mathbb{N}$ *where* $dom_f \subseteq \mathbb{N}^p$.

(2) (dom_f, f) *is a* total function $\mathbb{N}^p \longrightarrow \mathbb{N}$ *if* $dom_f = \mathbb{N}^p$ *holds.*

- *We say that* f *is undefined on* x *— or* $f(x)$ *is undefined — if* $x \notin dom_f$.
- *We use the notation* $f \in \mathbb{N}^{(dom \subseteq \mathbb{N}^p)}$ *to signify that* (dom_f, f) *is a* partial function $\mathbb{N}^p \longrightarrow \mathbb{N}$ *whose domain is* dom_f.

Definition 204 (Partial Recursive Functions)**.** *The set of* partial recursive *($\mathcal{P}art.\ \mathcal{R}ec.$) functions is the least that*

(1) *contains:*

  (a) *All constants* $\mathbb{N}^p \longrightarrow \mathbb{N}$ *(all* $\bar{i} \in \mathbb{N}^{(\mathbb{N}^p)}$ *s.t.* $\bar{i}(\overrightarrow{x}) = i$ *— any* $i, p \in \mathbb{N}$*).*

  (b) *All projections* π_i^p *(any* $p \in \mathbb{N}$, *any* $1 \leqslant i \leqslant p$*)*

  (c) *The successor function* $S \in \mathbb{N}^{\mathbb{N}}$.

(2) *and is closed under*

  (a) *composition*

  (b) *recursion*

(c) *minimisation.*

Definition 205 (Composition)**.** *Given* $f_1, \dots, f_n \in \mathbb{N}^{(dom \subseteq \mathbb{N}^p)}$ *and* $g \in \mathbb{N}^{(dom \subseteq \mathbb{N}^n)}$, *the composition* $h = g(f_1, \dots, f_n) \in \mathbb{N}^{(\mathbb{N}^p)}$ *is defined by*

$$\begin{cases} h(\overrightarrow{x}) \text{ is undefined iff } \begin{cases} \overrightarrow{x} \notin \bigcap\limits_{1 \leqslant i \leqslant n} dom_{f_i} \\ \\ \text{or otherwise} \\ \\ \big(f_1(\overrightarrow{x}), \dots, f_n(\overrightarrow{x})\big) \notin dom_g. \end{cases} \\ \\ h(\overrightarrow{x}) \text{ is defined otherwise and } h(\overrightarrow{x}) = g\big(f_1(\overrightarrow{x}), \dots, f_n(\overrightarrow{x})\big). \end{cases}$$

$$= g\big(f_1(x_1, \dots, x_p), \dots, f_n(x_1, \dots, x_p)\big)$$

Definition 206 (Recursion)**.** *Given* $g \in \mathbb{N}^{(dom \subseteq \mathbb{N}^p)}$ *and* $h \in \mathbb{N}^{(dom \subseteq \mathbb{N}^{p+2})}$, *there exists a unique* $f \in \mathbb{N}^{(dom \subseteq \mathbb{N}^{p+1})}$ *such that for all* $\overrightarrow{x} \in \mathbb{N}^p$ *and* $y \in \mathbb{N}$:

(1)

$$\begin{cases} f(\overrightarrow{x}, 0) \text{ is undefined if } \overrightarrow{x} \notin dom_g \\ \text{and} \\ f(\overrightarrow{x}, 0) \text{ is defined otherwise with } f(\overrightarrow{x}, 0) = g(\overrightarrow{x}). \end{cases}$$

(2)

$$\begin{cases} f(\overrightarrow{x}, y+1) \text{ is undefined if } \begin{cases} (\overrightarrow{x}, y) \notin dom_f \\ \\ \text{or} \\ \\ \big(\overrightarrow{x}, y, f(\overrightarrow{x}, y)\big) \notin dom_h. \end{cases} \\ \text{and} \\ \text{otherwise } f(\overrightarrow{x}, y+1) \text{ is defined and } f(\overrightarrow{x}, y+1) = h\big(\overrightarrow{x}, y, f(\overrightarrow{x}, y)\big). \end{cases}$$

Definition 207 (Minimization)**.** *Given $f \in \mathbb{N}^{(dom \subseteq \mathbb{N}^{p+1})}$, we define $g \in \mathbb{N}^{(dom \subseteq \mathbb{N}^{p})}$ by:*

$$g(\overrightarrow{x}) = \mu y \quad f(\overrightarrow{x}, y) = 0.$$

Notice that

$$g(\overrightarrow{x}) = y \iff \begin{cases} \forall z < y \begin{cases} f(\overrightarrow{x}, z) \text{ is defined!} \\ \text{and} \\ f(\overrightarrow{x}, z) > 0 \end{cases} \\ \\ \text{and} \\ f(\overrightarrow{x}, y) = 0. \end{cases}$$

Theorem 208. *For every $k > 0$ and every $f \in \mathbb{N}^{(dom \subseteq \mathbb{N}^{k})}$ the following are equivalent*

- *f is $\mathcal{P}art.\ \mathcal{R}ec.$,*
- *f is Turing computable.*

Definition 209 (Primitive Recursive Functions)**.** *The set of* primitive recursive *($\mathcal{P}rim.\ \mathcal{R}ec.$) functions is the least that*

(1) contains:

(a) All constants $\mathbb{N}^p \longrightarrow \mathbb{N}$ (all $\bar{i} \in \mathbb{N}^{(\mathbb{N}^p)}$ s.t. $\bar{i}(\overrightarrow{x}) = i$ — any $i, p \in \mathbb{N}$).

(b) All projections π_i^p (any $p \in \mathbb{N}$, any $1 \leqslant i \leqslant p$)

(c) The successor function $S \in \mathbb{N}^{\mathbb{N}}$.

(2) and is closed under

(a) composition

(b) recursion

Remark 210. The set $\mathcal{P}rim.\,\mathcal{R}ec.$ of *primitive recursive* functions only contains **total** recursive ones. Moreover, it is a *strict subset* of the set of all $\mathcal{P}art.\,\mathcal{R}ec.$-functions that are total.

9.2 Robinson Arithmetic

Definition 211 (Robinson Arithmetic)**.** *The language we consider is* $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ *where*

(1) 0 *is a constant symbol,*

(2) S *is a unary function symbol*[1]*,*

(3) $+$ *and* $\cdot$ *are binary function symbols*[2]*.*

Robinson Arithmetic *is formed of the following 7 axioms:*

axiom 1. $\forall x \; Sx \neq 0$

axiom 2. $\forall x \; \exists y \; (x \neq 0 \rightarrow Sy = x)$

axiom 3. $\forall x \; \forall y \; (Sx = Sy \rightarrow x = y)$

axiom 4. $\forall x \; x+0 = x$

axiom 5. $\forall x \; \forall y \; \big(x+Sy = S(x+y)\big)$

axiom 6. $\forall x \; x\cdot 0 = 0$

axiom 7. $\forall x \; \forall y \; \big(x\cdot Sy = (x\cdot y)+x\big)$

Notation 212. *We introduce "* $x \leqslant z$ *" to abbreviate the formula "* $\exists y \; y+x = z$ *"; and "* $x < z$ *" for the formula "* $\exists y \; \big(y+x = z \;\wedge\; x \neq z\big)$ *".*

Definition 213. *Let* $f \in \mathbb{N}^{(\mathbb{N}^n)}$ *and* $\varphi(x_0, x_1, \ldots, x_n)$ *be any* $\mathcal{L}_{\mathcal{A}}$*-formula whose free variables are among* $\{x_0, x_1, \ldots, x_n\}$*.*

[1] for any terms of $\mathcal{L}_{\mathcal{A}}$ t, we use the notation St instead of $S(t)$.

[2] for any terms of $\mathcal{L}_{\mathcal{A}}$ t_0, t_1, we use the notation t_0+t_1 (respectively $t_0\cdot t_1$) instead of $+(t_0, t_1)$ (respectively $\cdot(t_0, t_1)$).

$\varphi(x_0, x_1, \ldots, x_n)$ represents the function f if for all $i_1, \ldots, i_n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f(i_1, \ldots, i_n) = x_0 \longleftrightarrow \varphi(x_0, i_1, \ldots, i_n) \Big).$$

Definition 214. *Let $A \subseteq \mathbb{N}^n$ and $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathcal{A}}$-formula whose free variables are among $\{x_1, \ldots, x_n\}$.*
$\varphi(x_1, \ldots, x_n)$ represents the set A if for all $i_1, \ldots, i_n \in \mathbb{N}$ we have:

- *if $(i_1, \ldots, i_n) \in A$, then $\mathcal{R}ob. \vdash_c \varphi(i_1, \ldots, i_n)$;*
- *if $(i_1, \ldots, i_n) \notin A$, then $\mathcal{R}ob. \vdash_c \neg\varphi(i_1, \ldots, i_n)$.*

Proposition 215. *For any $A \subseteq \mathbb{N}^n$,*

A is representable if and only if χ_A is representable.

Theorem 216. *All total recursive functions are representable.*

9.3 Coding Sequences of Integers

We define $\mathcal{P}rim.\ \mathcal{R}ec.$ functions that allow to treat finite sequences of integers as integers. Every sequence $\langle x_1, \ldots, x_p \rangle$ will be *"coded"* by a single integer $\alpha_p(x_1, \ldots, x_p)$. And from this single integer $\alpha_p(x_1, \ldots, x_p)$ one will be able to recover the elements of the original sequence by having $\mathcal{P}rim.\ \mathcal{R}ec.$ functions β_p^i that satisfy

$$\beta_p^i\Big(\alpha_p(x_1, \ldots, x_p)\Big) = x_i.$$

Proposition 217. *For every non-zero $p \in \mathbb{N}$ there exists $\mathcal{P}rim.\ \mathcal{R}ec.$ functions $\beta_p^1, \beta_p^2, \ldots, \beta_p^p \in$*

$\mathbb{N}^{\mathbb{N}}$ and $\alpha_p \in \mathbb{N}^{(\mathbb{N}^p)}$ such that

$$\begin{cases} \alpha_p : \mathbb{N}^p \xleftrightarrow{1-1 \text{ and onto}} \mathbb{N} \\ \\ \qquad\qquad \textit{and} \\ \\ \alpha_p^{-1}(x) = \big(\beta_p^1(x), \ldots, \beta_p^p(x)\big). \end{cases}$$

Proof of Proposition 217: We start by defining $\alpha_1 = \beta_1^1 = id$. Then we move on to

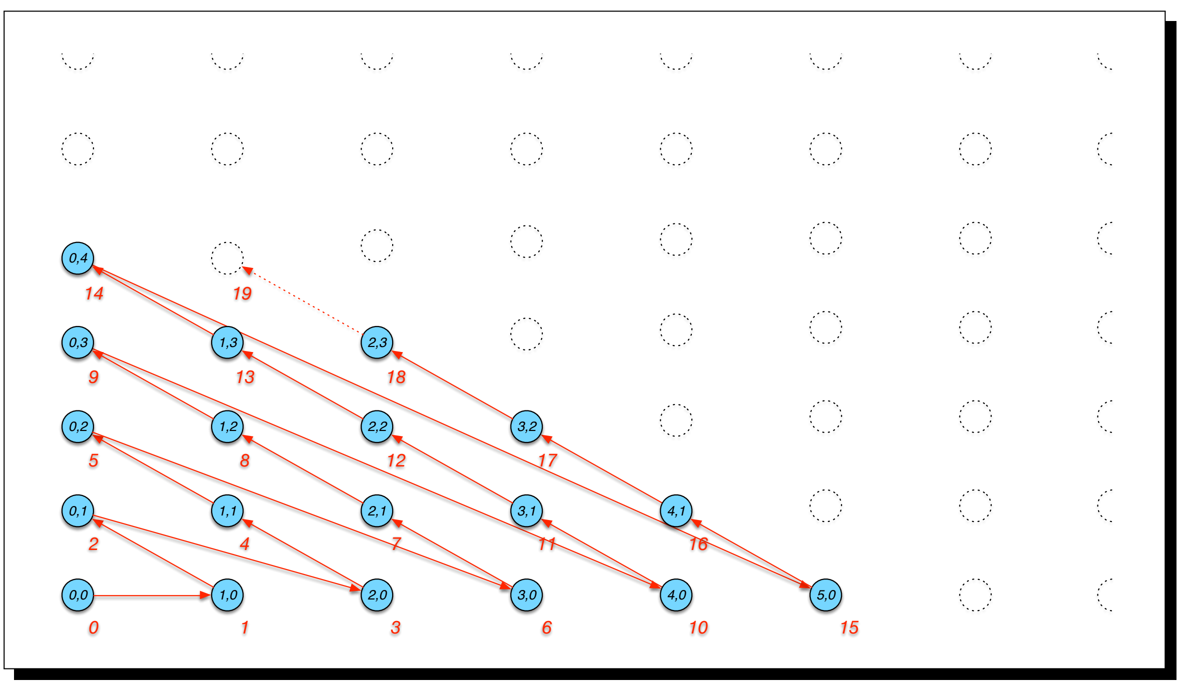
$$\alpha_2(x, y) = \frac{1}{2}(x + y + 1)(x + y) + y$$

$$\alpha_2(x, y) = \frac{(x + y) \cdot (x + y + 1)}{2} + y.$$

This is obtained by looking at the following picture and noticing that

(1) $\alpha_2(x, y) = \alpha_2(x + y, 0) + y$, and

(2) $$\begin{aligned} \alpha_2(x + y, 0) &= 1 + 2 + \cdots + (x + y) \\ &= \frac{1}{2}\left(\begin{matrix} 1 \\ + \\ x+y \end{matrix} \; + \; \begin{matrix} 2 \\ + \\ x+y-1 \end{matrix} \; + \cdots + \; \begin{matrix} x+y \\ + \\ 1 \end{matrix} \right). \end{aligned}$$



We have

(1) $\beta_2^1(n) = \mu x \leqslant n \quad \exists t \leqslant n \ \ \alpha_2(x,t) = n$

(2) $\beta_2^2(n) = \mu y \leqslant n \quad \exists t \leqslant n \ \ \alpha_2(t,y) = n.$

Then we define $\alpha_{p+1},\ \beta_{p+1}^1,\ \beta_{p+1}^2, \ldots\ldots,\ \beta_{p+1}^{p-1},\ \beta_{p+1}^p$ and β_{p+1}^{p+1} by induction on $p \in \mathbb{N}$:

- $\alpha_{p+1}(x_1,\ldots,x_p,x_{p+1}) = \alpha_p\big(x_1,\ldots,x_{p-1},\alpha_2(x_p,x_{p+1})\big)$
- $\beta_{p+1}^1 = \beta_p^1$;
- $\beta_{p+1}^2 = \beta_p^2$;

$$\vdots$$

- $\beta_{p+1}^{p-1} = \beta_p^{p-1}$;
- $\beta_{p+1}^p = \beta_2^1 \circ \beta_p^p$;
- $\beta_{p+1}^{p+1} = \beta_2^2 \circ \beta_p^p$.

$\square$ 217

Example 218. *A different way of coding sequences of integers:*

$$\begin{cases} c(\varepsilon) & = & 1 \\ c(x_0,\ldots,x_p) & = & \Pi\,(0)^{x_0+1}\cdot\Pi\,(1)^{x_1+1}\cdots\Pi\,(p)^{x_p+1}. \end{cases}$$

From $n \in \mathbb{N}\smallsetminus\{0\}$ we recover the sequence $\langle x_0,\ldots,x_p\rangle$ such that $c(x_0,\ldots,x_p) = n$ by considering the $\mathcal{P}rim.\ \mathcal{R}ec.$ function $d \in \mathbb{N}^{(\mathbb{N}^2)}$ which yields the exponents of the prime numbers:

$$d(i,n) = \mu x \leqslant n \quad \Pi\,(i)^{x+1} \text{ does not divide } n.$$

9.4 Gödel Numbers

The idea is the following: intuitively, formulas from arithmetic talk about integers — no matter whether these are standard or not — we ca turn them into formulas that talk about the arithmetic itself by encoding formulas, proofs, etc. by integers. This way a formula $\varphi(x)$ may say something like "*x is the code of a closed formula from our language* $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$" or $\psi(x,y)$ may eventually say "*x is the code of a closed formula θ from our language and* y *is the code of a proof of θ in Robinson arithmetic*".
As always we start with the terms: given term t we write $\ulcorner t \urcorner$ for its code.

Definition 219 (Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-terms)**.** *The Gödel numbering of the terms from the language* $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ *is*

$$\begin{array}{rclcrcl}
t & = & 0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(0,0,0) \\
t & = & x_n & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(n+1,0,0) \\
t & = & St_0 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, 0, 1) \\
t & = & t_0 + t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 2) \\
t & = & t_0 \cdot t_1 & \rightsquigarrow & \ulcorner t \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 3)
\end{array}$$

Lemma 220. *The set* $\mathcal{T}$ *of all codes of terms from* $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{T} = \left\{ \ulcorner t \urcorner \mid t \text{ is a term from } \mathcal{L}_{\mathcal{A}} \right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$

Definition 221 (Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-formulas)**.** *The Gödel numbering of the* $\mathcal{L}_{\mathcal{A}}$*-formulas is*

$$\begin{array}{rclcrcl}
\varphi & = & t_0 = t_1 & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 4) \\
\varphi & = & \neg\psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, 0, 5) \\
\varphi & = & (\varphi_0 \wedge \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 6) \\
\varphi & = & (\varphi_0 \vee \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 7) \\
\varphi & = & (\varphi_0 \longrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 8) \\
\varphi & = & (\varphi_0 \longleftrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 9) \\
\varphi & = & \forall x_n\ \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, n, 10) \\
\varphi & = & \exists x_n\ \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = & \alpha_3(\ulcorner \psi \urcorner, n, 11).
\end{array}$$

Notice that for every formula φ, we have $\ulcorner\varphi\urcorner > 0$.

Lemma 222. *The following sets are $\mathcal{P}rim.\ \mathcal{R}ec.$:*

- *The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$*

$$\{\ulcorner\varphi\urcorner \mid \varphi \text{ is a formula from } \mathcal{L}_\mathcal{A}\}$$

- *The set of all codes of terms from $\mathcal{L}_\mathcal{A}$ that contain the variable x_n*

$$\mathcal{T}_{\checkmark x} = \{(\ulcorner t\urcorner, n) \mid t \text{ is a term from } \mathcal{L}_\mathcal{A} \text{ and } t \text{ contains } x_n\}$$

- *The set of all codes of terms from $\mathcal{L}_\mathcal{A}$ that do not contain the variable x_n*

$$\mathcal{T}_{\times x} = \{(\ulcorner t\urcorner, n) \mid t \text{ is a term from } \mathcal{L}_\mathcal{A} \text{ and } t \text{ does not contain } x_n\}$$

- *The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$ that contain the variable x_n*

$$\mathcal{F}_{\checkmark x} = \{(\ulcorner\varphi\urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_\mathcal{A} \text{ and } \varphi \text{ contains } x_n\}$$

- *The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$ that do not contain the variable x_n*

$$\mathcal{F}_{\times x} = \{(\ulcorner\varphi\urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_\mathcal{A} \text{ and } \varphi \text{ does not contain } x_n\}$$

- *The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$ that contain x_n as a free variable*

$$\mathcal{F}_{\checkmark x\ free} = \{(\ulcorner\varphi\urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_\mathcal{A} \text{ and } x_n \text{ is free in } \varphi\}$$

- *The set of all codes of formulas from $\mathcal{L}_\mathcal{A}$ that contain x_n as a bound variable*

$$\mathcal{F}_{\checkmark x\ bound} = \{(\ulcorner\varphi\urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_\mathcal{A} \text{ and } x_n \text{ is bound in } \varphi\}$$

- *The set of all codes of closed formulas from $\mathcal{L}_\mathcal{A}$*

$$\mathcal{F}_{\checkmark closed} = \{\ulcorner\varphi\urcorner \mid \varphi \text{ is a closed formula from } \mathcal{L}_\mathcal{A}\}$$

Lemma 223. *The functions defined below are Prim. Rec.:*

- $\mathcal{S}^{\mathcal{T}}_{ub.} \in \mathbb{N}^{(\mathbb{N}^3)}$

$$\mathcal{S}^{\mathcal{T}}_{ub.}(n_u, n_t, n) \begin{cases} \ulcorner u_{[t/x_n]} \urcorner & \text{if } n_u \in \mathcal{T},\ n_t \in \mathcal{T} \text{ and } n_u = \ulcorner u \urcorner, n_t = \ulcorner t \urcorner \\ 0 & \text{otherwise .} \end{cases}$$

- $\mathcal{S}^{\mathcal{F}}_{ub.} \in \mathbb{N}^{(\mathbb{N}^3)}$

$$\mathcal{S}^{\mathcal{F}}_{ub.}(n_\varphi, n_t, n) = \begin{cases} \ulcorner \varphi[t/x_n] \urcorner & \text{if } n_\varphi = \ulcorner \varphi \urcorner \in \mathcal{F},\ n_t = \ulcorner t \urcorner \in \mathcal{T} \\ 0 & \text{otherwise .} \end{cases}$$

We now define a way of coding (finite) sets of formulas. We do not really encode the set, but some finite sequence of formulas, because we do not care about the ordering of such a sequence. (So, even if what we really encode is the sequence, we do handle it as if it were a set.)

Definition 224 (coding and decoding sequences)**.** *We define both* $\ulcorner\ \urcorner : \mathbb{N}^{<\omega} \longrightarrow \mathbb{N}$ *and* $\llcorner\ \urcorner : \mathbb{N}^2 \longrightarrow \mathbb{N}$ *by*

$$\begin{cases} \ulcorner \varepsilon \urcorner & = 0 \\ \ulcorner x_0, \dots, x_p \urcorner & = \Pi(0)^{x_0} \cdot \Pi(1)^{x_1} \cdots \Pi(p)^{x_p}. \end{cases}$$

Where $\Pi(i)$ *enumerates the prime numbers*[1].
And

$$\llcorner n \urcorner^i = \mu x \leqslant n \quad \Pi(i)^{x+1} \textit{ does not divide } n.$$

Notice that for all $i \leqslant p$ we have $\llcorner \ulcorner x_0, \dots, x_p \urcorner \urcorner^i = x_i$. Furthermore, for every formula φ, the integer $\ulcorner \varphi \urcorner$ is strictly positive. Therefore, given any sequence $\langle x_0, \dots, x_p \rangle \in \mathbb{N}^{<\omega}$ if $\llcorner \ulcorner x_0, \dots, x_p \urcorner \urcorner^i = 0$ then we know for sure that x_i does not code a formula.

We say that the integer 1 codes the empty set — which is also an empty set of formulas — and another integer codes the set $\Delta = \{\varphi_0, \varphi_1, \dots, \varphi_p\}$ if this integer is of the form $\Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \cdots \Pi(i_p)^{\ulcorner \varphi_p \urcorner}$.

Definition 225 (Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-finite sets of formulas)**.** *The Gödel numbering of any set $\Delta = \{\varphi_0, \varphi_1, \ldots, \varphi_p\}$ of $\mathcal{L}_{\mathcal{A}}$-formulas is any integer of the form*

$$\begin{cases} \ulcorner \varnothing \urcorner & = & 1 \\ \ulcorner \Delta \urcorner & = & \Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \cdots \Pi(i_p)^{\ulcorner \varphi_p \urcorner}. \end{cases}$$

We denote $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ the set of codes of finite sets of formulas:

$$\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} = \{\ulcorner \Delta \urcorner \mid \Delta \textit{ is some finite set of } \mathcal{L}_{\mathcal{A}} \textit{ formulas}\}.$$

Lemma 226.

- *There exist two $\mathcal{P}rim.\ \mathcal{R}ec.$ functions $\mathcal{R}_{em.} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ and $\mathcal{A}_{dd.} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that*

$$\mathcal{A}_{dd.}(n,m) = \begin{cases} \ulcorner \Delta \cup \{\varphi\} \urcorner & \textit{if} \quad n = \ulcorner \varphi \urcorner \in \mathcal{F} \quad \textit{and} \quad m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \end{cases}$$

$$\mathcal{R}_{em.}(n,m) = \begin{cases} \ulcorner \Delta \setminus \{\varphi\} \urcorner & \textit{if} \quad n = \ulcorner \varphi \urcorner \in \mathcal{F} \quad \textit{and} \quad m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \textit{if} \quad n \notin \mathcal{F} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \end{cases}$$

- *There exists a $\mathcal{P}rim.\ \mathcal{R}ec.$ function $\mathcal{U}_{nion} \in \mathbb{N}^{(\mathbb{N}^2)}$ such that*

$$\mathcal{U}_{nion}(n,m) = \begin{cases} 0 & \textit{if} \quad n \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{or} \quad m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ \ulcorner \Gamma' \cup \Delta' \urcorner & \textit{if} \quad n = \ulcorner \Gamma \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad \begin{cases} m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ \textit{and} \\ \ulcorner \Gamma' \cup \Delta' \urcorner \ \mathcal{E}_{qu.} \ \ulcorner \Gamma \cup \Delta \urcorner. \end{cases} \end{cases}$$

The following sets are $\mathcal{P}rim.\ \mathcal{R}ec.$

- *The set $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ of codes of finite sets of formulas is $\mathcal{P}rim.\ \mathcal{R}ec.$*

- $$\mathcal{I}_{ns.} = \left\{(\ulcorner\varphi\urcorner, \ulcorner\Delta\urcorner) \in \mathbb{N}^2 \mid \ulcorner\varphi\urcorner \in \mathcal{F},\ \ulcorner\Delta\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ \textit{and} \ \ \varphi \in \Delta\right\}.$$
- $$\mathcal{E}_{qu.} = \left\{(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner) \in \mathbb{N}^2 \mid \ulcorner\Gamma\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})},\ \ulcorner\Delta\urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \ \ \textit{and} \ \ \Gamma = \Delta\right\}.$$

Definition 227 (Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$-sequents from *sequent calculus*)**.** *The Gödel numbering of any sequent* $\Gamma \vdash \Delta$ *is*

$$\ulcorner\Gamma \vdash \Delta\urcorner = \alpha_2(\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner)$$

We denote $\mathcal{SQ}$ *the set of* $\mathcal{L}_{\mathcal{A}}$*-sequents:*

$$\mathcal{SQ} = \{\ulcorner\Gamma \vdash \Delta\urcorner \mid \Gamma,\ \Delta \textit{ finite sets of } \mathcal{L}_{\mathcal{A}} \textit{ formulas}\}.$$

Given any integer n we use the notation ${}^{l}n$ for $\beta_2^1(n)$ and ${}^{r}n$ for $\beta_2^2(n)$. This way,

$$\text{if } n = \ulcorner\Gamma \vdash \Delta\urcorner, \text{ then } {}^{l}n = \ulcorner\Gamma\urcorner \text{ and } {}^{r}n = \ulcorner\Delta\urcorner.$$

Lemma 228. *The set* $\mathcal{SQ}$ *of codes of sequents of* sequent calculus *is* $\mathcal{P}rim.\ \mathcal{R}ec.$

Proof of Lemma 228:

$$\chi_{\mathcal{SQ}}(n) = \begin{cases} 1 & \textit{if} \quad {}^{l}n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \textit{and} \quad {}^{r}n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \textit{else} \end{cases}$$

□ 228

We will now denote $\mathcal{AX}$ the set of codes of axioms of sequent calculus which are not to be mistaken for the axiom of Robinson arithmetic.

Definition 229 (Gödel numbering of the axioms of *sequent calculus*)**.**

$$\mathcal{AX} = \left\{\alpha_2\left(2^{\ulcorner\varphi\urcorner}, 2^{\ulcorner\varphi\urcorner}\right) \mid \ulcorner\varphi\urcorner \in \mathcal{F}\right\}.$$

Lemma 230. *The set $\mathcal{AX}$ of codes of axioms of* sequent calculus *is $\mathcal{Prim.\ Rec}$.*

Proof of Lemma 230:

$$\chi_{\mathcal{AX}}(n) = \begin{cases} 1 & \textit{if} \quad \beta_2^1(n) = \beta_2^2(n) \quad \textit{and} \quad \llcorner\beta_2^1(n)\urcorner^0 \in \mathcal{F} \\ 0 & \textit{else} \end{cases}$$

9.5 Coding the Proofs

We recall that a proof in Sequent Calculus is a tree of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\ (\varphi \to \psi) \vdash \forall x\ (\varphi \to \psi)}{\scriptstyle ax}}{\forall x\ (\varphi \longrightarrow \psi) \vdash \varphi_{[y/x]} \to \psi_{[y/x]}}{\scriptstyle \forall e} \qquad \dfrac{\dfrac{}{\forall x\ \varphi \vdash \forall x\ \varphi}{\scriptstyle ax}}{\forall x\ \varphi \vdash \varphi_{[y/x]}}{\scriptstyle \forall e}}{\forall x\ (\varphi \to \psi), \forall x\ \varphi \vdash \psi_{[y/x]}}{\scriptstyle \to e}}{\forall x\ (\varphi \to \psi), \forall x\ \varphi \vdash \forall x\ \psi}{\scriptstyle \forall i}}{\forall x\ (\varphi \to \psi) \vdash \forall x\ \varphi \to \forall x\ \psi}{\scriptstyle \to i}$$

where the shape of the tree is controlled by the rules of Sequent Calculus.

We are now ready to define for each rule of the Sequent Calculus, a set of tuples of codes of sequents that satisfy the property that the rule defines.
We will successively define

(1) $\circ\ \mathcal{R}_{ax} \subseteq \mathbb{N}$

(2)

- $\circ\ \mathcal{R}_{\wedge_{l1}} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\wedge_{l2}} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\neg_l} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\forall_l} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\exists_l} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\vee_{r1}} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\vee_{r2}} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\neg_r} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\forall_r} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{\exists_r} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{wkn_l} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{wkn_r} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{ctr\ l\&r} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{cut} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{Rep} \subseteq \mathbb{N}^2$
- $\circ\ \mathcal{R}_{Ref} \subseteq \mathbb{N}^2$

(3)
- $\circ\ \mathcal{R}_{\vee_l} \subseteq \mathbb{N}^3$
- $\circ\ \mathcal{R}_{\rightarrow_l} \subseteq \mathbb{N}^3$
- $\circ\ \mathcal{R}_{\wedge_r} \subseteq \mathbb{N}^3$

and for each of them, the fact that it is $\mathcal{P}rim.\ \mathcal{R}ec.$ will derive from its definition. We first recall what the rules are.

Sequent Calculus

Axioms

$$\dfrac{}{\varphi \vdash \varphi}\ {\scriptstyle ax}$$

Logical Rules

$$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \land \psi \vdash \Delta}\ {\scriptstyle \land_{l1}} \qquad \dfrac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \land \psi \vdash \Delta}\ {\scriptstyle \land_{l2}} \qquad \dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \land \psi, \Delta}\ {\scriptstyle \land_{r}}$$

$$\dfrac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \lor \psi \vdash \Delta}\ {\scriptstyle \lor_{l}} \qquad \dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \lor \psi, \Delta}\ {\scriptstyle \lor_{r1}} \qquad \dfrac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \lor \psi, \Delta}\ {\scriptstyle \lor_{r2}}$$

$$\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ {\scriptstyle \to_{l}} \qquad \dfrac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ {\scriptstyle \to_{r}}$$

$$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ {\scriptstyle \neg_{l}} \qquad \dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ {\scriptstyle \neg_{r}}$$

$$\dfrac{\Gamma, \varphi_{[t/x]} \vdash \Delta\ ^{1}}{\Gamma, \forall x\ \varphi \vdash \Delta}\ {\scriptstyle \forall_{l}} \qquad \dfrac{\Gamma \vdash \varphi_{[y/x]}, \Delta}{\Gamma \vdash \forall x\ \varphi, \Delta\ ^{2}}\ {\scriptstyle \forall_{r}}$$

$$\dfrac{\Gamma, \varphi_{[y/x]} \vdash \Delta}{\Gamma, \exists x\ \varphi \vdash \Delta\ ^{2}}\ {\scriptstyle \exists_{l}} \qquad \dfrac{\Gamma \vdash \varphi_{[t/x]}, \Delta\ ^{1}}{\Gamma \vdash \exists x\ \varphi, \Delta}\ {\scriptstyle \exists_{r}}$$

$$\dfrac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta}\ {\scriptstyle Ref} \qquad \dfrac{\Gamma, t = s, \varphi_{[s/x]}, \varphi_{[t/x]} \vdash \Delta}{\Gamma, s = t, \varphi_{[t/x]} \vdash \Delta}\ {\scriptstyle Rep}$$

Structural Rules

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ {\scriptstyle wkn_{l}} \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\ {\scriptstyle wkn_{r}}$$

$$\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ {\scriptstyle ctr_{l}} \qquad \dfrac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\ {\scriptstyle ctr_{r}}$$

Cut Rule

$$\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ {\scriptstyle cut}$$

[1] t a term
[2] y with no free occurrence the sequent concluding the rule (not in $\Gamma, \exists x\ \varphi$ nor $\forall x\ \varphi$, nor Δ)

$$\boxed{\dfrac{}{\varphi \vdash \varphi}\,{}^{ax}}$$

$$U \in \mathcal{R}_{ax} \iff U \in \mathcal{AX}$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\,{}^{\wedge_{l1}}}$$

$$(U, D) \in \mathcal{R}_{\wedge_{l1}}$$

$$\iff$$

$$\begin{cases} U \in \mathcal{SQ} \\ \textit{and} \\ D \in \mathcal{SQ} \\ \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U \ \exists \ulcorner\psi\urcorner \leqslant {}^{l}D \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \ulcorner\varphi \wedge \psi\urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.} \left(\ulcorner\varphi\urcorner, {}^{l}U\right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.} \left(\ulcorner\varphi \wedge \psi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

where

- “ $\ulcorner\varphi \wedge \psi\urcorner$” stands for “ $\alpha_3(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner, 6)$”.
- “ $\exists \ulcorner\varphi\urcorner \leqslant k \quad \theta_{[\ulcorner\varphi\urcorner/y]}$]” stands for “ $\exists n \leqslant k \quad \big(n \in \mathcal{F} \ \wedge \ \theta_{[n/y]}\big)$” and more generally
- “ $\exists \ulcorner\varphi_1\urcorner \leqslant k_1 \ \ldots\ldots \exists \ulcorner\varphi_n\urcorner \leqslant k_n \ \ \theta_{[\ulcorner\varphi_1\urcorner/y_1, \ldots, \ulcorner\varphi_n\urcorner/y_n]}$” stands for “ $\exists p \leqslant \alpha_n(k_1, \ldots, k_n) \quad \big(\bigwedge_{i \leqslant n} \big(\beta_n^i(p) \in \mathcal{F} \ \wedge \ \beta_n^i(p) \leqslant k_i\big) \quad \wedge \quad \theta_{[\beta_n^1(p)/y_1, \ldots, \beta_n^n(p)/y_n]}\big)$”.

....

$$\boxed{\dfrac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ {\scriptstyle \wedge_{l2}}}$$

$$(U, D) \in \mathcal{R}_{\wedge_{l2}}$$

$$\Longleftrightarrow$$

$$\begin{cases} U \in \mathcal{SQ} \\ \quad \textit{and} \\ D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner \psi \urcorner \leqslant {}^{l}U \ \exists \ulcorner \varphi \urcorner \leqslant {}^{l}D \quad \left(\begin{array}{c} \ulcorner \psi \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \ulcorner \varphi \wedge \psi \urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.} \left(\ulcorner \psi \urcorner, {}^{l}U \right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.} \left(\ulcorner \varphi \wedge \psi \urcorner, {}^{l}D \right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ {\scriptstyle \vee_{l}}}$$

$$(U_l, U_r, d) \in \mathcal{R}_{\vee_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U_l \ \mathcal{E}_{qu.} \ {}^{r}U_r \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner \varphi \urcorner \leqslant {}^{l}U_l \ \exists \ulcorner \psi \urcorner \leqslant {}^{l}U_r \quad \left(\begin{array}{c} \ulcorner \varphi \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U_l \quad \textit{and} \quad \ulcorner \psi \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U_r \quad \textit{and} \quad \ulcorner \varphi \wedge \psi \urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.} \left(\ulcorner \varphi \urcorner, {}^{l}U_l \right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.} \left(\ulcorner \psi \urcorner, {}^{l}U_r \right) \ \mathcal{E}_{qu.} \ \mathcal{R}_{em.} \left(\ulcorner \varphi \wedge \psi \urcorner, {}^{l}D \right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ {\scriptstyle \to_l}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\to_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \qquad \textit{and} \\ {}^{r}U_r\ \mathcal{E}_{qu.}\ {}^{r}D \\ \qquad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U_l\ \exists \ulcorner\psi\urcorner \leqslant {}^{l}U_r \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U_l \quad \textit{and} \quad \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}U_r \quad \textit{and} \quad \ulcorner\varphi \wedge \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}U_l\right)\ \mathcal{E}_{qu.}\ {}^{r}U_r \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^{l}U_r\right)\ \mathcal{E}_{qu.}\ {}^{l}U_l\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\varphi \to \psi\urcorner, {}^{l}D\right) \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\psi\urcorner, {}^{l}U_r\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\varphi \to \psi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

- where " $A\ \ \mathcal{E}_{qu.}\ \ B\ \ \mathcal{E}_{qu.}\ \ C$ " stands for " $A\ \ \mathcal{E}_{qu.}\ \ B$ *and* $B\ \ \mathcal{E}_{qu.}\ \ C$ "

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ {\scriptstyle \neg_l}}$$

$$(U, D) \in \mathcal{R}_{\neg_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \qquad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \qquad \textit{and} \qquad \ulcorner\neg\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}U\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\neg\varphi\urcorner, {}^{l}D\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi_{[t/x_n]} \vdash \Delta}{\Gamma, \forall x_n\ \varphi \vdash \Delta}\ {\scriptstyle \forall_l}}$$

$$(U, D) \in \mathcal{R}_{\forall_l}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{cc}
U, D \in \mathcal{SQ} & \\
\textit{and} & \\
{}^rU\ \mathcal{E}_{qu.}\ {}^rD & \\
\textit{and} & \\
\exists n \leqslant {}^lD\ \ \exists \ulcorner\forall x_n\varphi\urcorner \leqslant {}^lD\ \ \exists \ulcorner t\urcorner \leqslant {}^lU &
\left(\begin{array}{c}
\left(\begin{array}{c}
\ulcorner\forall x_n\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lD \qquad \textit{and} \qquad \left(\ulcorner\varphi\urcorner, n\right) \in \mathcal{F}_{\checkmark x\ free} \\
\textit{and} \\
\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^lU \\
\textit{and} \\
\mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right), {}^lU\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\forall x_n\varphi\urcorner, {}^lD\right)
\end{array}\right) \\
\\
\textit{or} \\
\\
\left(\begin{array}{c}
\ulcorner\forall x_n\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lD \\
\textit{and} \\
\left(\ulcorner\varphi\urcorner, n\right) \in \left(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\right) \\
\textit{and} \\
\ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lU \\
\textit{and} \\
\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner\varphi\urcorner, \ulcorner t\urcorner, n\right)\ \mathcal{I}_{ns.}\ {}^lU \\
\textit{and} \\
\mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^lU\right)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\left(\ulcorner\forall x_n\varphi\urcorner, {}^lD\right)
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

- "$\exists n \leqslant {}^rU\ \ \exists \ulcorner\forall x_n\varphi\urcorner \leqslant {}^rU\ \ \ldots$" stands for "$\exists n \leqslant {}^rU\ \ \exists m \leqslant {}^rU\ \ \left(m \in \mathcal{F}\ \wedge\ \beta^3_3(m) = 10\ \wedge\ \beta^2_3(m) = n\ \wedge\ \beta^1_3(m) = \ulcorner\varphi\urcorner\ \wedge\ \ldots\right)$"
- "$\ulcorner\varphi\urcorner$" stands for "$\beta^1_3\left(\ulcorner\forall x_n\varphi\urcorner\right)$"
- "$\exists \ulcorner t\urcorner \leqslant {}^lU\ \ldots$" stands for "$\exists v \leqslant {}^lU\left(v \in \mathcal{T}\ \wedge\ \ldots\right)$"

....

$$\boxed{\dfrac{\Gamma, \varphi_{[x_k/x_n]} \vdash \Delta}{\Gamma, \exists x_n\ \varphi \vdash \Delta}\ {}^{\exists_l}}$$ [2]

$$(U, D) \in \mathcal{R}_{\exists_l}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l}
U, D \in \mathcal{SQ} \\
\textit{and} \\
{}^{r}U\ \mathcal{E}_{qu.}\ {}^{r}D \\
\textit{and} \\
\exists \ulcorner x_n \urcorner \leqslant {}^{l}D\ \ \exists \ulcorner x_k \urcorner \leqslant {}^{l}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{l}D\ \ \exists \ulcorner \psi \urcorner \leqslant {}^{l}U \left(\begin{array}{c}
\left(\begin{array}{c}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}U \\
\textit{and} \\
(\ulcorner \psi \urcorner, k) \in \mathcal{F}_{\checkmark x\ free} \ \ \textit{and} \ \ \Big(k \neq n \to (\ulcorner \psi \urcorner, n) \in (\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound})\Big) \\
\textit{and} \\
\alpha_3(\mathcal{S}^{\mathcal{F}}_{ub.}(\ulcorner \psi \urcorner, \ulcorner x_n \urcorner, k), n, 11) = \ulcorner \exists x_n \varphi \urcorner \\
\textit{and} \\
\mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{l}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^{l}D) \\
\textit{and} \\
\forall \ulcorner \theta \urcorner \leqslant {}^{l}D\ \ \Big(\ulcorner \theta \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \to (\ulcorner \theta \urcorner, k) \in (\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound})\Big) \\
\textit{and} \\
\forall \ulcorner \delta \urcorner \leqslant {}^{r}D\ \ \Big(\ulcorner \delta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \to (\ulcorner \delta \urcorner, k) \in (\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound})\Big)
\end{array}\right) \\
\\
\textit{or} \\
\\
\left(\begin{array}{c}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{l}U \quad \textit{and} \\
(\ulcorner \psi \urcorner, k) \in (\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}) \quad \textit{and} \quad (\ulcorner \psi \urcorner, n) \in (\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}) \\
\textit{and} \quad \alpha_3(\ulcorner \psi \urcorner, n, 11) = \ulcorner \exists x_n \varphi \urcorner \quad \textit{and} \\
\mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{l}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^{l}D)
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

where

- " $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \forall x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$" stands for " $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta_3^3(m) = 11\ \wedge\ \beta_3^2(m) = n\ \wedge\ \beta_3^1(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\Big)$"
- " $\ulcorner \varphi \urcorner$ stands for " $\beta_3^1(\ulcorner \forall x_n \varphi \urcorner)$"
- " $\exists \ulcorner x_k \urcorner \leqslant {}^{l}U\ \ldots$" stands for " $\exists k \leqslant {}^{l}U(\ulcorner x_k \urcorner = \alpha_3(k+1, 0, 0)\ \wedge\ \ldots)$"

....

[2] x_k has no free occurrence in Γ, $\exists x_n\ \varphi$ and Δ

$$\boxed{\dfrac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta}\ {\scriptstyle Ref}}$$

$$(U, D) \in \mathcal{R}_{\scriptscriptstyle Ref}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner t \urcorner \leqslant {}^{l}U \quad \left(\begin{array}{c} \ulcorner t = t \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner t = t \urcorner, {}^{l}U\right) \quad \mathcal{E}_{qu.} \ {}^{l}D \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, t = s, \varphi_{[s/x_n]}, \varphi_{[t/x_n]} \vdash \Delta}{\Gamma, s = t, \varphi_{[t/x_n]} \vdash \Delta}\ {\scriptstyle Rep}}$$

$$(U, D) \in \mathcal{R}_{\scriptscriptstyle Rep}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^{r}U \ \mathcal{E}_{qu.} \ {}^{r}D \\ \quad \textit{and} \\ \exists \ulcorner t \urcorner \leqslant {}^{l}U \ \exists \ulcorner s \urcorner \leqslant {}^{l}U \ \exists n \leqslant U \ \exists \ulcorner \varphi \urcorner \leqslant U^{U} \quad \left(\begin{array}{c} \ulcorner t = s \urcorner \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner s \urcorner, n\right) \ \mathcal{I}_{ns.} \ {}^{l}U \quad \textit{and} \quad \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right) \ \mathcal{I}_{ns.} \ {}^{l}U \\ \textit{and} \\ \ulcorner s = t \urcorner \ \mathcal{I}_{ns.} \ {}^{l}D \quad \textit{and} \quad \mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right) \ \mathcal{I}_{ns.} \ {}^{l}D \\ \textit{and} \\ \mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right), \mathcal{R}_{em.}\left(\ulcorner s = t \urcorner, {}^{l}D\right)\right) \\ \mathcal{E}_{qu.} \\ \mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner s \urcorner, n\right), \mathcal{R}_{em.}\left(\mathcal{S}^{\mathcal{F}}_{ub.}\left(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n\right), \mathcal{R}_{em.}\left(\ulcorner t = s \urcorner, {}^{l}U\right)\right)\right) \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ {\scriptstyle \wedge_r}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\wedge_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \textit{and} \\ {}^l U_l\ \mathcal{E}_{qu.}\ {}^l U_r\ \mathcal{E}_{qu.}\ {}^l D \\ \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^r U_l\ \exists \ulcorner\psi\urcorner \leqslant {}^r U_r \quad \begin{pmatrix} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^r U_l \quad \textit{and} \quad \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^r U_r \quad \textit{and} \quad \ulcorner\varphi \wedge \psi\urcorner\ \mathcal{I}_{ns.}\ {}^r D \\ \textit{and} \\ \mathcal{R}_{em.}(\ulcorner\varphi\urcorner, {}^r U_l)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner\psi\urcorner, {}^r U_r)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner\varphi \wedge \psi\urcorner, {}^r D) \end{pmatrix} \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ {\scriptstyle \vee_{r1}}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r1}}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \textit{and} \\ {}^l U\ \mathcal{E}_{qu.}\ {}^l D \\ \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^r U\ \exists \ulcorner\psi\urcorner \leqslant {}^r D \quad \begin{pmatrix} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^r U \quad \textit{and} \quad \ulcorner\varphi \vee \psi\urcorner\ \mathcal{I}_{ns.}\ {}^r D \\ \textit{and} \\ \mathcal{R}_{em.}(\ulcorner\varphi\urcorner, {}^r U)\ \ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner\varphi \vee \psi\urcorner, {}^r D) \end{pmatrix} \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\,{\scriptstyle \vee_{r2}}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r2}}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l} \quad U, D \in \mathcal{SQ} \\ \qquad \textit{and} \\ \quad {}^{l}U \ \mathcal{E}_{qu.}\ {}^{l}D \\ \qquad \textit{and} \\ \exists \ulcorner\psi\urcorner \leqslant {}^{r}U \ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}D \quad \left(\begin{array}{c} \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \quad \textit{and} \quad \ulcorner\varphi \vee \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \mathcal{R}_{em.}\,(\ulcorner\psi\urcorner, {}^{r}U) \quad \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner\varphi \vee \psi\urcorner, {}^{r}D) \end{array}\right) \end{array}\right.$$

....

$$\boxed{\dfrac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\,{\scriptstyle \to_{r}}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\to_r}$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l} \quad U_l, U_r, D \in \mathcal{SQ} \\ \qquad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U_l \ \ \exists \ulcorner\psi\urcorner \leqslant {}^{r}U_r \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^{l}U_l \quad \textit{and} \quad \ulcorner\psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}U_r \quad \textit{and} \quad \ulcorner\varphi \to \psi\urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\ \textit{and} \\ \mathcal{R}_{em.}\,(\ulcorner\psi\urcorner, {}^{r}U_r) \quad \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner\varphi \to \psi\urcorner, {}^{r}D) \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U_l\right) \quad \mathcal{E}_{qu.}\ {}^{l}D \end{array}\right) \end{array}\right.$$

....

$$\boxed{\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\,\neg_r}$$

$$(U, D) \in \mathcal{R}_{\neg_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{l}U \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner \ \mathcal{I}_{ns.}\ {}^{l}U \qquad and \qquad \ulcorner\neg\varphi\urcorner \ \mathcal{I}_{ns.}\ {}^{r}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U\right)\ \mathcal{E}_{qu.}\ {}^{l}D \\ and \\ \mathcal{R}_{em.}\left(\ulcorner\neg\varphi\urcorner, {}^{r}D\right)\ \mathcal{E}_{qu.}\ {}^{r}U \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi_{[x_k/x_n]}, \Delta}{\Gamma \vdash \forall x_n\ \varphi, \Delta^{\,2}}\ {\scriptstyle \forall_r}}$$

$$(U, D) \in \mathcal{R}_{\forall_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \textit{and} \\ {}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\ \textit{and} \\ \exists \ulcorner x_n \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner x_k \urcorner \leqslant {}^{r}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner \psi \urcorner \leqslant {}^{r}U \left(\begin{array}{c} \left(\begin{array}{c} \ulcorner \forall x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \\ \textit{and} \\ (\ulcorner \psi \urcorner, k) \in \mathcal{F}_{\checkmark x\ free} \ \ \textit{and}\ \ \Big(k \neq n \to (\ulcorner \psi \urcorner, n) \in \big(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\big)\Big) \\ \textit{and} \\ \alpha_3(\mathcal{S}^{\mathcal{F}}_{ub.}(\ulcorner \psi \urcorner, \ulcorner x_n \urcorner, k), n, 11) = \ulcorner \forall x_n \varphi \urcorner \\ \textit{and} \\ \mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{r}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \forall x_n \varphi \urcorner, {}^{r}D) \\ \textit{and} \\ \forall \ulcorner \theta \urcorner \leqslant {}^{r}D\ \Big(\ulcorner \theta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \to (\ulcorner \theta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big) \\ \textit{and} \\ \forall \ulcorner \delta \urcorner \leqslant {}^{r}D\ \Big(\ulcorner \delta \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \to (\ulcorner \delta \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big)\Big) \end{array} \right) \\ \\ \textit{or} \\ \\ \left(\begin{array}{c} \ulcorner \forall x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \textit{and} \quad \ulcorner \psi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \quad \textit{and} \\ (\ulcorner \psi \urcorner, k) \in \big(\mathcal{F}_{\times x_k} \cup \mathcal{F}_{\checkmark x_k\ bound}\big) \quad \textit{and} \quad (\ulcorner \psi \urcorner, n) \in \big(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\big) \\ \textit{and} \quad \alpha_3(\ulcorner \psi \urcorner, n, 11) = \ulcorner \forall x_n \varphi \urcorner \quad \textit{and} \\ \mathcal{R}_{em.}(\ulcorner \psi \urcorner, {}^{r}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}(\ulcorner \forall x_n \varphi \urcorner, {}^{r}D) \end{array} \right) \end{array} \right) \end{cases}$$

where

- “ $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \forall x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$” stands for
  “ $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta_3^3(m) = 10\ \wedge\ \beta_3^2(m) = n\ \wedge\ \beta_3^1(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\Big)$”
- “ $\ulcorner \varphi \urcorner$ stands for “ $\beta_3^1\big(\ulcorner \forall x_n \varphi \urcorner\big)$”
- “ $\exists \ulcorner x_k \urcorner \leqslant {}^{r}U\ \ldots$” stands for “ $\exists k \leqslant {}^{r}U\big(\ulcorner x_k \urcorner = \alpha_3(k+1, 0, 0)\ \wedge\ \ldots\big)$”

....

[2] x_k has no free occurrence in Γ, $\forall x_n\ \varphi$ and Δ

$$\boxed{\dfrac{\Gamma \vdash \varphi_{[t/x_n]}, \Delta}{\Gamma \vdash \exists x_n\ \varphi, \Delta}\ {\scriptstyle \exists_r}}$$

$$(U, D) \in \mathcal{R}_{\exists_r}$$

$$\Longleftrightarrow$$

$$\begin{cases}
U, D \in \mathcal{SQ} \\
\textit{and} \\
{}^{l}U\ \mathcal{E}_{qu.}\ {}^{l}D \\
\textit{and} \\
\exists n \leqslant {}^{r}D\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}D\ \ \exists \ulcorner t \urcorner \leqslant {}^{r}U \left(\begin{array}{c}
\begin{pmatrix}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \quad \textit{and} \quad \big(\ulcorner \varphi \urcorner, n\big) \in \mathcal{F}_{\checkmark x\ free} \\
\textit{and} \\
\mathcal{S}^{\mathcal{F}}_{ub.}\,(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n)\ \mathcal{I}_{ns.}\ {}^{r}U \\
\textit{and} \\
\mathcal{R}_{em.}\,\big(\mathcal{S}^{\mathcal{F}}_{ub.}\,(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n), {}^{r}U\big)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner \exists x_n \varphi \urcorner, {}^{r}D)
\end{pmatrix} \\
\\
\textit{or} \\
\\
\begin{pmatrix}
\ulcorner \exists x_n \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}D \\
\textit{and} \\
\big(\ulcorner \varphi \urcorner, n\big) \in \big(\mathcal{F}_{\times x} \cup \mathcal{F}_{\checkmark x\ bound}\big) \\
\textit{and} \\
\ulcorner \varphi \urcorner\ \mathcal{I}_{ns.}\ {}^{r}U \\
\textit{and} \\
\mathcal{S}^{\mathcal{F}}_{ub.}\,(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n)\ \mathcal{I}_{ns.}\ {}^{r}U \\
\textit{and} \\
\mathcal{R}_{em.}\,(\ulcorner \varphi \urcorner, {}^{r}U)\ \mathcal{E}_{qu.}\ \mathcal{R}_{em.}\,(\ulcorner \exists x_n \varphi \urcorner, {}^{r}D)
\end{pmatrix}
\end{array} \right)
\end{cases}$$

- " $\exists n \leqslant {}^{r}U\ \ \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^{r}U\ \ \ldots$" stands for
  " $\exists n \leqslant {}^{r}U\ \ \exists m \leqslant {}^{r}U\ \ \Big(m \in \mathcal{F}\ \wedge\ \beta_3^3(m) = 11\ \wedge\ \beta_3^2(m) = n\ \wedge\ \beta_3^1(m) = \ulcorner \varphi \urcorner\ \wedge\ \ldots\Big)$"
- " $\ulcorner \varphi \urcorner$ stands for " $\beta_3^1\big(\ulcorner \exists x_n \varphi \urcorner\big)$"
- " $\exists \ulcorner t \urcorner \leqslant {}^{r}U\ \ldots$" stands for " $\exists v \leqslant {}^{r}U\big(v \in \mathcal{T}\ \wedge\ \ldots\big)$"

....

$$\boxed{\dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ {\scriptstyle wkn_l}}$$

$$(U, D) \in \mathcal{R}_{wkn_l}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^rU\ \mathcal{E}_{qu.}\ {}^rD \\ \quad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^lD \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^lD \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^lD\right)\ \mathcal{E}_{qu.}\ {}^lU \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\ {\scriptstyle wkn_r}}$$

$$(U, D) \in \mathcal{R}_{wkn_r}$$

$$\Longleftrightarrow$$

$$\begin{cases} U, D \in \mathcal{SQ} \\ \quad \textit{and} \\ {}^lU\ \mathcal{E}_{qu.}\ {}^lD \\ \quad \textit{and} \\ \exists \ulcorner\varphi\urcorner \leqslant {}^rD \quad \left(\begin{array}{c} \ulcorner\varphi\urcorner\ \mathcal{I}_{ns.}\ {}^rD \\ \textit{and} \\ \mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^rD\right)\ \mathcal{E}_{qu.}\ {}^rU \end{array} \right) \end{cases}$$

....

$$\boxed{\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,{\scriptstyle ctr_l}} \qquad \boxed{\dfrac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\,{\scriptstyle ctr_r}}$$

$$(U, D) \in \mathcal{R}_{ctr\;l\&r} \iff \begin{cases} U, D \in \mathcal{SQ} \\ \quad\; and \\ {}^{l}U \;\; \mathcal{E}_{qu.} \; {}^{l}D \\ \quad\; and \\ {}^{r}U \;\; \mathcal{E}_{qu.} \; {}^{r}D \end{cases}$$

....

$$\boxed{\dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\scriptstyle cut}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{cut}$$

$$\iff$$

$$\begin{cases} U_l, U_r, D \in \mathcal{SQ} \\ \qquad and \\ \exists \ulcorner\varphi\urcorner \leqslant {}^{r}U_l \qquad \begin{pmatrix} \ulcorner\varphi\urcorner \; \mathcal{I}_{ns.} \; {}^{r}U_l \;\; and \;\; \ulcorner\varphi\urcorner \; \mathcal{I}_{ns.} \; {}^{l}U_r \\ and \\ \mathcal{U}_{nion}\left(\mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{l}U_r\right), {}^{l}U_l\right) \;\; \mathcal{E}_{qu.} \; {}^{l}D \\ and \\ \mathcal{U}_{nion}\left(\mathcal{R}_{em.}\left(\ulcorner\varphi\urcorner, {}^{r}U_l\right), {}^{r}U_r\right) \;\; \mathcal{E}_{qu.} \; {}^{r}D \end{pmatrix} \end{cases}$$

....

We write

$$\begin{cases} \mathcal{R}^0 = \mathcal{R}_{ax} \\ \\ \mathcal{R}^1 = \begin{cases} \mathcal{R}_{\wedge l1} \cup \mathcal{R}_{\wedge l2} \cup \mathcal{R}_{\neg_l} \cup \mathcal{R}_{\forall_l} \cup \mathcal{R}_{\exists_l} \cup \mathcal{R}_{\vee r1} \\ \cup \mathcal{R}_{\vee r2} \cup \mathcal{R}_{\neg_r} \cup \mathcal{R}_{\forall_r} \cup \mathcal{R}_{\exists_r} \cup \mathcal{R}_{wkn_l} \cup \\ \mathcal{R}_{wkn_r} \cup \mathcal{R}_{ctr\,l\&r} \cup \mathcal{R}_{Rep} \cup \mathcal{R}_{Ref} \cup \mathcal{R}_{cut} \end{cases} \\ \\ \mathcal{R}^2 = \mathcal{R}_{\vee_l} \cup \mathcal{R}_{\rightarrow_l} \cup \mathcal{R}_{\wedge r} \end{cases}$$

We say an integer codes a proof if it is of the form

$$\alpha_4(\texttt{node}, \texttt{left proof-tree}, \texttt{right proof-tree}, \texttt{arity of the rule}).$$

Definition 231. *The set $\mathcal{P}_{roofs}$ of the codes of all possible proofs is defined by*

$$k = \alpha_4(n_1, n_2, n_3, n_4) \in \mathcal{P}_{roofs}$$

$$\Longleftrightarrow$$

$$\begin{cases} n_4 = 0 & and & n_3 = 0 & and & n_2 = 0 & and & n_1 \in \mathcal{R}^0 \\ & & & or & & & \\ n_4 = 1 & and & n_3 = 0 & and & n_2 \in \mathcal{P}_{roofs} & and & \big(\beta_4^1(n_2), n_1\big) \in \mathcal{R}^1 \\ & & & or & & & \\ n_4 = 2 & and & n_3 \in \mathcal{P}_{roofs} & and & n_2 \in \mathcal{P}_{roofs} & and & \big(\beta_4^1(n_3), \beta_4^1(n_2), n_1\big) \in \mathcal{R}^2. \end{cases}$$

Notation 232. *Given any proof P we write $\ulcorner P \urcorner$ for the integer described above that codes this proof.*

Lemma 233. *The set $\mathcal{P}_{roofs}$ is $\mathcal{P}rim.\ \mathcal{R}ec.$.*

Chapter 10

Undecidability Results

10.1 Undecidability of Robinson Arithmetic

Definition 234.

(1) A theory T is recursive if the following set is recursive:

$$\left\{ \ulcorner \varphi \urcorner \mid \varphi \in T \right\}.$$

(2) A theory T is decidable if the following set is recursive:

$$thms(T) = \left\{ \ulcorner \varphi \urcorner \mid T \vdash_c \varphi \right\}.$$

Informally, this means that a theory is decidable if one has an algorithm which on any input that represents a formula φ stops and accepts if T proves φ, and stops and rejects if T does not prove φ.

Theorem 235. *Given any theory T, the set*

$$\left\{ \left(\ulcorner P \urcorner, \ulcorner \varphi \urcorner \right) \in \mathbb{N}^2 \mid P \text{ is a proof of } T \vdash_c \varphi \right\}$$

is

- *primitive recursive if T is primitive recursive,*
- *recursive if T is recursive.*

Proposition 236. *Given any theory T,*

$$\left\{\ulcorner\psi\urcorner \mid \psi \in T\right\} \text{ is recursive } \Longrightarrow \left\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\right\} \text{ is recursively enumerable.}$$

We recall that a theory is complete if it is both consistent and satisfies for each formula φ either

$$T \vdash_c \varphi \quad \text{or} \quad T \vdash_c \neg\varphi.$$

Corollary 237. *Let T be any recursive theory.*

If T is complete, then T is decidable.

Theorem 238. *Let $T \supseteq \mathcal{R}ob.$ be any theory,*

$$T \text{ is consistent } \Longleftrightarrow T \text{ is undecidable.}$$

Proof of Theorem 238:

($\Leftarrow$) T inconsistent $\Rightarrow$ T decidable is straightforward since in this case we have

$$\left\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\right\} = \mathcal{F}.$$

($\Rightarrow$) Towards a contradiction, we assume that T is decidable. We then consider

$$\mathcal{F}_{\checkmark x_0\ !free} = \left\{\ulcorner\varphi\urcorner \mid \varphi \text{ is a formula whose only free variable is } x_0\right\}.$$

Since we already know that the set

$$\mathcal{F}_{\checkmark x\ free} = \left\{\left(\ulcorner\varphi\urcorner, n\right) \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathcal{A}} \text{ and } x_n \text{ is free in } \varphi\right\}$$

is $\mathcal{P}rim.\ \mathcal{R}ec.$ and we have

$$\ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \Longleftrightarrow \left(\ulcorner\varphi\urcorner, 0\right) \in \mathcal{F}_{\checkmark x\ free} \quad \text{and} \quad \forall n \leqslant \ulcorner\varphi\urcorner \ \Big(n \neq 0 \rightarrow \left(\ulcorner\varphi\urcorner, n\right) \notin \mathcal{F}_{\checkmark x\ free}\Big).$$

an immediate consequence is that $\mathcal{F}_{\checkmark x_0\ !free}$ is also $\mathcal{P}rim.\ \mathcal{R}ec.$

Then the set[1]

$$\Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \ \ \textit{and} \ \ T \vdash_c \varphi[n/x_0]\Big\}$$

$$=$$

$$\Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \ \ \textit{and} \ \ \mathcal{S}^{\mathcal{F}}_{ub.}\,(\ulcorner\varphi\urcorner, \ulcorner n\urcorner, 0) \in \{\ulcorner\psi\urcorner \mid T \vdash_c \psi\}\Big\}$$

is recursive.

We then consider the following set

$$\overset{\ddot\smile}{\mathcal{D}}_{iag.} = \Big\{k \in \mathbb{N} \mid (k,k) \notin \Big\{\big(\ulcorner\varphi\urcorner, n\big) \mid \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free} \ \ \textit{and} \ \ T \vdash_c \varphi_{[n/x_0]}\Big\}\Big\}.$$

$\overset{\ddot\smile}{\mathcal{D}}_{iag.}$ is clearly recursive, therefore there exists some formula $\varphi_{\ddot\smile}(x_0)$ that represents $\overset{\ddot\smile}{\mathcal{D}}_{iag.}$. This means that for all $k \in \mathbb{N}$ we have:

- $k \in \overset{\ddot\smile}{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \varphi_{\ddot\smile[k/x_0]}$
- $k \notin \overset{\ddot\smile}{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \neg\varphi_{\ddot\smile[k/x_0]}$.

It is enough to consider the closed formula $\varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]}$ where $[\ulcorner\varphi_{\ddot\smile}\urcorner]$ stands for the term $\overbrace{S\ldots S}^{\ulcorner\varphi_{\ddot\smile}\urcorner}0$.

We ask the question whether or not T proves $\varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]}$. This depends on whether $\ulcorner\varphi_{\ddot\smile}\urcorner$ belongs to $\overset{\ddot\smile}{\mathcal{D}}_{iag.}$ or not.

- $\ulcorner\varphi_{\ddot\smile}\urcorner \in \overset{\ddot\smile}{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]} \implies T \vdash_c \varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]} \implies \ulcorner\varphi_{\ddot\smile}\urcorner \notin \overset{\ddot\smile}{\mathcal{D}}_{iag.}$
- $\ulcorner\varphi_{\ddot\smile}\urcorner \notin \overset{\ddot\smile}{\mathcal{D}}_{iag.} \implies \mathcal{R}ob. \vdash_c \neg\varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]} \implies T \vdash_c \neg\varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]}$.

  Since T is consistent we cannot have both

$$T \vdash_c \neg\varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]} \qquad \textit{and} \qquad T \vdash_c \varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]}.$$

  Therefore, we have $T \nvdash_c \varphi_{\ddot\smile[[\ulcorner\varphi_{\ddot\smile}\urcorner]/x_0]}$ which immediately implies $\ulcorner\varphi_{\ddot\smile}\urcorner \in \overset{\ddot\smile}{\mathcal{D}}_{iag.}$.

---

[1] we recall that we defined $\mathcal{S}^{\mathcal{F}}_{ub.}\,(n_u, n_t, n) = \begin{cases} \ulcorner\varphi[t/x_n]\urcorner & \textit{if} \ \ n_\varphi = \ulcorner\varphi\urcorner \in \mathcal{F},\ n_t = \ulcorner t\urcorner \in \mathcal{T} \\ 0 & \textit{otherwise}\ . \end{cases}$

We obtain

$$\ulcorner \varphi_{\ddot\smile} \urcorner \in \ddot{\mathcal{D}}_{iag.} \iff \ulcorner \varphi_{\ddot\smile} \urcorner \notin \ddot{\mathcal{D}}_{iag.}.$$

This contradiction finishes the proof that T is undecidable.

□ 238

We propose a picture that illustrates this diagonal argument:

If $(\varphi_i)_{i \in \mathbb{N}}$ is a enumeration of all the formulas with x_0 as one and only free variable, we make sure to define a formula which satisfies this requirement although it is none of them.

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5		φ_n	
$\ulcorner \varphi_0 \urcorner$	0	1	1	0	1	0	...	0	...
$\ulcorner \varphi_1 \urcorner$	1	1	1	0	0	0	...	0	...
$\ulcorner \varphi_2 \urcorner$	1	0	1	0	0	0	...	1	...
$\ulcorner \varphi_3 \urcorner$	0	0	1	0	1	0	...	0	...
$\ulcorner \varphi_4 \urcorner$	0	1	0	1	1	1	...	0	...
$\ulcorner \varphi_5 \urcorner$	1	1	0	0	0	0	...	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$\ulcorner \varphi_n \urcorner$	1	0	0	0	1	1	...	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

There is a 1 on — for instance row 3 and column 2 — if $T \vdash_c \varphi_2(\lceil \ulcorner \varphi_3 \urcorner \rceil)$, and there is a 0 — for instance on row 2 and column 5 — if $T \nvdash_c \varphi_5(\lceil \ulcorner \varphi_2 \urcorner \rceil)$.

Now if T is decidable, the whole array is decidable. This means there is a Decider that on any input (n, m) accepts if there is a 1 on position (n, m), and rejects if there is a 0. Furthermore, for the whole array is decidable, its diagonal is also decidable. Hence the complement of the diagonal is decidable as well. Finally, since all recursive sets are representable, the complement of the diagonal is represented by some formula among the enumeration — say φ_n — which inevitably stumbles on $[\ulcorner\varphi_n\urcorner]$).

Theorem 239 (undecidability of first order logic)**.** *The set*

$$\left\{\ulcorner\varphi\urcorner \mid \quad \vdash_c \varphi\right\}.$$

is not recursive

Proof of Theorem 239: Since $\mathcal{R}ob.$ is a finite theory, we let $\phi_{\mathcal{R}ob.}$ be the conjunction of the seven axioms from $\mathcal{R}ob.$. For any formula ψ we have

$$\mathcal{R}ob. \vdash_c \psi \iff \phi_{\mathcal{R}ob.} \vdash_c \psi \iff \quad \vdash_c \phi_{\mathcal{R}ob.} \to \psi.$$

$$\ulcorner\psi\urcorner \in \left\{\ulcorner\varphi\urcorner \mid \mathcal{R}ob. \vdash_c \varphi\right\} \iff \ulcorner\phi_{\mathcal{R}ob.} \to \psi\urcorner \in \left\{\ulcorner\varphi\urcorner \mid \quad \vdash_c \varphi\right\}.$$

Therefore, if the set of codes of universally valid formulas were decidable, then Robinson arithmetic would also be decidable.

$\square$ 239

Theorem 240 (Gödel's first incompleteness theorem)**.** *Let $T \supseteq \mathcal{R}ob.$ be any theory both consistent and recursive.*

T is incomplete.

Proof of Theorem 240: By corollary 237 every recursive complete theory is decidable. By Theorem 238 the theory T is undecidable.

$\square$ 240

10.2 Peano Arithmetic and $I\Sigma^0_1$

To prove Gödel's second incompleteness theorem we need to work in a theory slightly more expressive than $\mathcal{R}ob$. For this reason we introduce the theory of *Peano arithmetic*.
$\mathcal{P}eano$ is a theory based on the same language as Robinson arithmetic : $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$

$\mathcal{P}eano$ has infinitely many axioms:

axiom 1. $\forall x\ Sx \neq 0$

axiom 2. $\forall x\ \exists y\ (x \neq 0 \rightarrow Sy = x)$

axiom 3. $\forall x\ \forall y\ (Sx = Sy \rightarrow x = y)$

axiom 4. $\forall x\ x+0 = x$

axiom 5. $\forall x\ \forall y\ \big(x+Sy = S(x+y)\big)$

axiom 6. $\forall x\ x \cdot 0 = 0$

axiom 7. $\forall x\ \forall y\ \big(x \cdot Sy = (x \cdot y)+x\big)$

axiom schema (induction) $\forall x_0\ \forall x_1\ \ldots\ \forall x_n\ \Big(\Big(\varphi_{[0/x_0]} \wedge \forall x_0\ \big(\varphi \rightarrow \varphi_{[Sx_0/x_0]}\big)\Big) \rightarrow \forall x_0\ \varphi\Big)$ (for any formula $\varphi_{[x_0,x_1,\ldots,x_n]}$)[2].

So we see that $\mathcal{P}eano$ is nothing but $\mathcal{R}ob.$ plus the induction schema for all formulas constructed on the language of arithmetic. In fact we will not need to work within $\mathcal{P}eano$ but only a part of it formed of $\mathcal{R}ob.$ plus the induction schema restricted to the sole Σ^0_1-formulas (see next section). This theory is called $\mathcal{R}ob. + I\Sigma^0_1$

10.3 The Arithmetical Hierarchy

For the purpose of defining the arithmetical hierarchy we add a binary symbol " $<$" to our language but essentially for the purpose of denoting bounded formulas such as $\exists y \leqslant t\ \ \varphi$ and $\forall y \leqslant t\ \varphi$. In a sense, this differs from the use of this same symbol inside Robinson arithmetic (see page 134) where it was an abbreviation for " $\exists y\ \big(y+x = z\ \wedge\ x \neq z\big)$". For the reason that in what follows we will have

- " $\exists y\ \big(y+x = z\ \wedge\ x \neq z\big)$" is a Σ^0_1-formula, and
- " $\exists x \leqslant z\ \ x \neq z$" is a Δ^0_0-formula.

We will be working with $\mathcal{R}ob. + I\Sigma^0_1$ so for every x and y we will have both

$$x \leqslant y\ \vee\ y \leqslant x$$

and

$$\exists z\ \ z+x = y \iff \exists z\ \ x+z = y.$$

[2] the notation $\varphi_{[x_0,x_1,\ldots,x_n]}$ means that the free variable of φ are all among $x_0, x_1, \ldots, x_n$.

Definition 241 (Δ^0_0-formulas)**.** *The set of Δ^0_0-formulas is the least that*

(1) contains all atomic formulas: $t_0 = t_1$

(2) is closed under conjunctions, disjunctions and negations

(3) is closed under bounded quantifications

$$\text{if } \varphi \in \Delta^0_0 \text{ and } t \text{ is a term, then } \forall x < t\ \varphi \text{ and } \exists x < t\ \ \varphi \text{ both belong to } \Delta^0_0.$$

Definition 242 (arithmetical hierarchy)**.** *The hierarchy of formulas from arithmetic is defined by induction on $n \in \mathbb{N}$:*

(1) $\Sigma^0_0 = \Pi^0_0 = \Delta^0_0$

(2) Σ^0_{n+1} *is the set of all formulas of the form* $\exists x_1 \ldots \exists x_k\ \varphi$ *where* $\varphi \in \Pi^0_n$.

(3) Π^0_{n+1} *is the set of all formulas of the form* $\forall x_1 \ldots \forall x_k \varphi$ *where* $\varphi \in \Sigma^0_n$.

(4) $\Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}$

Theorem 243. *Every total recursive function is representable by some Σ^0_1-formula.*

10.4 Gödel's Second Incompleteness Theorem

We first recall that by Theorem 235 the set below is

$$\left\{ \left(\ulcorner P \urcorner, \ulcorner \varphi \urcorner \right) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T\ \vdash_c\ \varphi \right\}$$

- primitive recursive if T is primitive recursive,
- recursive if T is recursive.

We consider any recursive theory $T \supseteq \mathcal{R}ob.$ and consider some Σ^0_1-formula $\phi_{proof_T}(x_1, x_2)$ which represents the set above. This means that for all $i_1, i_2 \in \mathbb{N}$ we have:

- if $(i_1, i_2) \in \left\{(\ulcorner P \urcorner, \ulcorner \varphi \urcorner) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T \vdash_c \varphi\right\}$, then $\mathcal{R}ob. \vdash_c \phi_{proof_T}(i_1, i_2)$;
- if $(i_1, i_2) \notin \left\{(\ulcorner P \urcorner, \ulcorner \varphi \urcorner) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T \vdash_c \varphi\right\}$, then $\mathcal{R}ob. \vdash_c \neg\phi_{proof_T}(i_1, i_2)$.

so in particular if T is consistent, we have

$$P \text{ is a proof of } T \vdash_c \varphi \iff \mathcal{R}ob. \vdash_c \phi_{proof_T}(\lceil \ulcorner P \urcorner \rceil, \lceil \ulcorner \varphi \urcorner \rceil).$$

We consider the following primitive recursive function $diag : \mathbb{N} \longrightarrow \mathbb{N}$.

$$diag(n) = \begin{cases} \ulcorner \varphi_{[\lceil \ulcorner \varphi \urcorner \rceil / x_0]} \urcorner & \textit{if} \quad n = \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \ !free} \\ 0 & \textit{otherwise} \end{cases}$$

together with any Σ^0_1-formula $\varphi_{diag}(x_0, x_1)$ that represents $diag$. This means we have for all $n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(diag(n) = x_0 \longleftrightarrow \varphi_{diag}(x_0, n)\right).$$

We define the Σ^0_1-formula $\Xi(x_0)$ by

$$\Xi(x_0) := \exists x_1 \, \exists x_2 \quad (\phi_{proof_T}(x_1, x_2) \quad \wedge \quad \varphi_{diag}(x_2, x_0))$$

Proposition 244. *For every integer n we have*

$$\mathbb{N} \models \Xi(n) \quad \iff \quad \mathcal{R}ob. \vdash_c \Xi(n).$$

Proof of Proposition 244:

(1) if $n = \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \ !free}$ and there is a proof P of $T \vdash_c \varphi_{[\lceil \ulcorner \varphi \urcorner \rceil / x_0]}$ we have both

$$\mathcal{R}ob. \vdash_c \varphi_{diag}(\lceil \ulcorner \varphi_{[\lceil \ulcorner \varphi \urcorner \rceil / x_0]} \urcorner \rceil, n)$$

and

$$\mathcal{R}ob. \vdash_c \phi_{proof_T}(\lceil \ulcorner P \urcorner \rceil, \lceil \ulcorner \varphi_{[\lceil \ulcorner \varphi \urcorner \rceil / x_0]} \urcorner \rceil)$$

therefore

$$\mathcal{R}ob. \vdash_c \exists x_1 \, \exists x_2 \quad (\phi_{proof_T}(x_1, x_2) \quad \wedge \quad \varphi_{diag}(x_2, n))$$

which is

$$\mathcal{R}ob. \vdash_c \Xi(n).$$

(2) if $n = \ulcorner\varphi\urcorner \in \mathcal{F}_{\checkmark x_0\ !free}$ and there is no proof P of $T\ \vdash_c\ \varphi_{[[\ulcorner\varphi\urcorner]/x_0]}$ we have for all proofs P

$$\mathcal{R}ob.\ \vdash_c\ \forall x_2\Big(\varphi_{diag}(x_2, n) \longleftrightarrow x_2 = \lceil\ulcorner\varphi_{[[\ulcorner\varphi\urcorner]/x_0]}\urcorner\rceil\Big)$$

and

$$\mathcal{R}ob.\ \vdash_c\ \neg\phi_{proof_T}(\lceil\ulcorner P\urcorner\rceil, \lceil\ulcorner\varphi_{[[\ulcorner\varphi\urcorner]/x_0]}\urcorner\rceil)$$

and furthermore for every integer i

$$\mathcal{R}ob.\ \vdash_c\ \neg\phi_{proof_T}(i, \lceil\ulcorner\varphi_{[[\ulcorner\varphi\urcorner]/x_0]}\urcorner\rceil)$$

therefore, since $\mathbb{N} \models \phi_{\mathcal{R}ob.}$, by the soundness theorem we have

$$\mathcal{R}ob. \nvdash_c \Xi\,(n)\,.$$

(3) if $n \notin \mathcal{F}_{\checkmark x_0\ !free}$, then for every integer i_1,

$$\mathcal{R}ob.\ \vdash_c\ \neg\phi_{proof_T}(i_1, diag(n))$$

for the reason that for all integer i_1

$$(i_1, 0) \notin \Big\{(\ulcorner P\urcorner, \ulcorner\varphi\urcorner) \in \mathbb{N}^2 \mid P\ \textit{is a proof of}\ T\ \vdash_c\ \varphi\Big\}$$

because 0 is never the code of a formula. Hence, by application of the soundness theorem we have

$$\mathcal{R}ob. \nvdash_c \Xi\,(n)\,.$$

$\square$ 244

So to speak, $\mathbb{N} \models \Xi\,(n)$ asserts that there exists a proof that there is a 1 on position (i_n, i_n) in the array on page 164, where n is the integer that codes the formula φ_{i_n}.

We now consider the formula $\neg\Xi\,(x_0)$ — that we write $\neg\Xi$ — together with the term that represents its code $\ulcorner\neg\Xi\urcorner$ and the term that represents the code of the formula $\neg\Xi\,(x_0)$ which "eats up" its own code $\ulcorner\neg\Xi_{[[\ulcorner\neg\Xi\urcorner]/x_0]}\urcorner$

Claim 245.

$$\mathcal{R}ob.\ \vdash_c\ \Xi_{[[\ulcorner\neg\Xi\urcorner]/x_0]} \longleftrightarrow \exists x_1\phi_{proof_T}(x_1, \lceil\ulcorner\neg\Xi_{[[\ulcorner\neg\Xi\urcorner]/x_0]}\urcorner\rceil)$$

which is precisely

$$\mathcal{R}ob.\ \vdash_c\ \exists x_1\ \exists x_2\ \ (\phi_{proof_T}(x_1, x_2)\ \ \wedge\ \ \varphi_{diag}(x_2, \lceil\ulcorner\neg\Xi\urcorner\rceil)) \longleftrightarrow \exists x_1\phi_{proof_T}(x_1, \lceil\ulcorner\neg\Xi_{[[\ulcorner\neg\Xi\urcorner]/x_0]}\urcorner\rceil)\,.$$

Proof of Claim 245:

($\Leftarrow$) By the very definition of the function *diag* and that φ_{diag} represents that function we have

$$\mathcal{R}ob. \vdash_c \varphi_{diag}(\ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)$$

thus

$$\mathcal{R}ob. \vdash_c \exists x_1 \phi_{proof_T}(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner) \longrightarrow \exists x_1 \exists x_2 \ \left(\phi_{proof_T}(x_1, x_2) \ \wedge \ \varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)\right).$$

($\Rightarrow$) Since φ_{diag} represents the function *diag* we have

$$\mathcal{R}ob. \vdash_c \forall x_2 \left(\varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)\right) \longleftrightarrow x_2 = \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\Big)$$

hence

$$\mathcal{R}ob. \vdash_c \left(\exists x_1 \exists x_2 \ \left(\phi_{proof_T}(x_1, x_2) \ \wedge \ \varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)\right) \longrightarrow x_2 = \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)$$

therefore

$$\mathcal{R}ob. \vdash_c \exists x_1 \exists x_2 \ \left(\phi_{proof_T}(x_1, x_2) \ \wedge \ \varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)\right) \longrightarrow \exists x_1 \phi_{proof_T}(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner).$$

$\square$ 245

Claim 246.

$$T \nvdash_c \neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}.$$

Proof of Claim 246: Towards a contradiction, we assume that

$$T \vdash_c \neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}.$$

It follows that there exists an integer $\ulcorner P\urcorner$ such that

$$\left(\ulcorner P\urcorner \ , \ \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right) \in \left\{\left(\ulcorner Q\urcorner, \ulcorner\varphi\urcorner\right) \in \mathbb{N}^2 \mid Q \textit{ is a proof of } T \ \vdash_c \ \varphi\right\}.$$

Therefore, since ϕ_{proof_T} represents the set above, we have

$$\mathcal{R}ob. \vdash_c \phi_{proof_T}(\ulcorner\ulcorner P\urcorner\urcorner, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner)$$

and by Claim 245 we obtain

$$\mathcal{R}ob. \vdash_c \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}.$$

Since $\mathcal{R}ob. \subseteq T$ we obtain

$$T \vdash_c \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}$$

which contradicts the fact that T is consistent for we obtain both

$$T \vdash_c \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \quad \text{and} \quad T \vdash_c \neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}.$$

$\square$ 246

Claim 247.

$$\left.\begin{array}{c}\mathcal{R}ob. \\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\end{array}\right] \vdash_c \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \neg cons(T).$$

Where $\neg cons(T)$ stands for the formula[3]

$$\exists\ulcorner\varphi\urcorner \left(\exists x_0\ \phi_{proof_T}(x_0, \ulcorner\varphi\urcorner)\ \wedge\ \exists x_0\ \phi_{proof_T}(x_0, \ulcorner\neg\varphi\urcorner)\right)$$

Proof of Claim 247: From Claim 245 we obtain

$$\mathcal{R}ob.\ \vdash_c\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_T}\left(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right).$$

Thus we have both

- $\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\ \vdash_c\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)$
- $\mathcal{R}ob.\ \vdash_c\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_T}\left(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)$

which leads to

$$\left.\begin{array}{c}\mathcal{R}\text{ob.} \\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\end{array}\right\} \vdash_c\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \begin{pmatrix}\exists x_1\ \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right) \\ \bigwedge \\ \exists x_1\ \phi_{proof_T}\left(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\end{pmatrix}$$

By the very definition[4] of ϕ_{proof_T} and $\phi_{proof_{\mathcal{R}ob.}}$ we have

- $\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \forall x_1\ \left(\phi_{proof_{\mathcal{R}ob.}}(x_0, x_1) \longrightarrow \phi_{proof_T}(x_0, x_1)\right).$

Therefore we obtain

$$\left.\begin{array}{c}\mathcal{R}\text{ob.} \\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \exists x_1 \phi_{proof_{\mathcal{R}ob.}}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\end{array}\right\} \vdash_c\ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \to \begin{pmatrix}\exists x_1\ \phi_{proof_T}\left(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right) \\ \bigwedge \\ \exists x_1\ \phi_{proof_T}\left(x_1, \ulcorner\ulcorner\neg\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner\right)\end{pmatrix}$$

---

[3] We recall that we write $\exists\ulcorner\varphi\urcorner \ldots$ for $\exists x\ (\varphi_{\mathcal{F}}(x)\ \wedge\ \ldots$.

[4] this means "if we choose wisely the Σ^0_1-formulas ϕ_{proof_T} and $\phi_{proof_{\mathcal{R}ob.}}$ that represent the two recursive sets

$$\left\{(\ulcorner P\urcorner, \ulcorner\varphi\urcorner) \in \mathbb{N}^2 \mid P \textit{ is a proof of } T\ \vdash_c\ \varphi\right\} \quad \text{and} \quad \left\{(\ulcorner P\urcorner, \ulcorner\varphi\urcorner) \in \mathbb{N}^2 \mid P \textit{ is a proof of } \mathcal{R}ob.\ \vdash_c\ \varphi\right\}.$$

which yields the result.

□ 247

Lemma 248. *Let $T \supseteq \mathcal{R}ob.$ be any consistent recursive theory.*
If

$$T \ \vdash_c \ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner)\,,$$

then

$$T \nvdash_c cons(T).$$

Proof of Lemma 248: Follows immediately from Claims 246 and 247.

□ 248

So we are left with the problem of characterising the consistent theories that both extend Robinson arithmetic and prove this very strange formula: $\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner)$. Ultimately we will show that $\mathcal{R}ob. + I\Sigma^0_1$ is a good candidate. That is we will have

$$\mathcal{R}ob. + I\Sigma^0_1 \ \vdash_c \ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner)\,.$$

It can easily be seen that this formula is Σ^0_1. We will prove a more general result.

Lemma 249. *Let φ be any closed Σ^0_1-formula.*

$$\mathcal{R}ob. + I\Sigma^0_1 \ \vdash_c \ \varphi \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}(x_1, \ulcorner\varphi\urcorner)\,.$$

Theorem 250 (Gödel's second incompleteness theorem)**.** *Let $T \supseteq \mathcal{R}ob. + I\Sigma^0_1$ be any consistent recursive theory.*

$$T \nvdash_c cons(T).$$

Proof of Theorem 250: Follows immediately from Lemmas 248 and 249. For the condition required by Lemma 248 on a theory $T \supseteq \mathcal{R}ob.$ to satisfy the conditions of the second incompleteness theorem was that

$$T \ \vdash_c \ \Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]} \longrightarrow \exists x_1 \phi_{proof_{\mathcal{R}ob.}}(x_1, \ulcorner\ulcorner\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}\urcorner\urcorner)\,.$$

And it turns out that the formula $\Xi_{[\ulcorner\ulcorner\neg\Xi\urcorner\urcorner/x_0]}$ is both closed and Σ^0_1 for it is the formula

$$\exists x_1\ \exists x_2\ \left(\phi_{proof_T}(x_1, x_2)\ \wedge\ \varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)\right)$$

where both $\phi_{proof_T}(x_1, x_2)$ and $\varphi_{diag}(x_2, \ulcorner\ulcorner\neg\Xi\urcorner\urcorner)$ are formulas that represent primitive recursive relations, hence Σ^0_1.

$\square$ 250

Of course, the different versions of set theory (**ZF**, **ZFC**, etc.) are not built on the language of arithmetic $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ but on a rudimentary one: $\mathcal{L}_{\mathrm{ST}}$. However, one can not only introduce these new symbols $0, S, +, \cdot$ and work in an extension of the original theory by definition, but one can also very easily simulate arithmetic within the different versions of set theory. Indeed, one has

Lemma 251.

$$\mathbf{ZF} \vdash_c (\mathcal{P}eano)^{\omega}.$$

Proof of Lemma 251: An easy exercise.

$\square$ 251

Clearly, the different versions of set theory that we consider (**ZF**, **ZFC**, etc.) are all recursive (even primitive recursive). As a consequence we obtain

Corollary 252.

- *If* **ZF** *is consistent, then it does not prove its own consistency:*

$$\textit{if } \mathbf{ZF} \nvdash_c \bot, \textit{ then } \mathbf{ZF} \nvdash_c cons(\mathbf{ZF}).$$

- $$\textit{If } \mathbf{ZFC} \nvdash_c \bot, \textit{ then } \mathbf{ZFC} \nvdash_c cons(\mathbf{ZFC}).$$

Proof of Corollary 252: An easy exercise. $\square$ 252