# Part VI

# ZF without the Axiom of Choice

# Chapter 19

# Cardinality Revisited

With the axiom of choice out of hand, we cannot use the notion of the cardinality of a set $A$ as it was defined when we had the axiom of choice at hand. The reason if that if the least ordinal $\alpha$ such that there exists a bijection $A \xrightarrow{bij.} \alpha$ always exists when $A$ can be well-ordered — because the order type of this well-ordering yields at least one ordinal which satisfies $A \xrightarrow{bij.} \alpha$, so the class of all ordinals that are equipotent to $A$ being non-empty admits a minimal element. This may not be the case when deprived of the axiom of choice. For instance, as we will see in Section 22.2 Theorem 364, one can force the set of reals to lack any well-ordering at all.

## 19.1 Injections and Surjections Revisited

We first introduce some notations for the existence of an injection or a surjection.

**Notation 341** (**ZF**)**.** *Given any sets $A, B$, we write*

- $A \overset{1-1}{\lesssim} B$ *whenever there exists some mapping* $f : A \xrightarrow{1-1} B$*;*
- $A \overset{1-1}{\not\lesssim} B$ *whenever* $A \overset{1-1}{\lesssim} B$ *does not hold;*
- $A \overset{onto}{\lesssim} B$ *whenever there exists some mapping* $f : B \xrightarrow{onto} A$*;*
- $A \overset{onto}{\not\lesssim} B$ *whenever* $A \overset{onto}{\lesssim} B$ *does not hold.*

The following definition of being infinite is known as "Dedkind-infinite".

**Definition 342.** *Let $A$ be any set.*

$$A \text{ is Dedekind-infinite if } \omega \overset{1-1}{\lesssim} A.$$

*($A$ is Dedekind-finite if $\omega \overset{1-1}{\not\lesssim} A$.)*

We will see later[1] that — unless inconsistent — **ZF** does not prove that every *Dedekind-finite*-set is finite. But of course **ZFC** proves that every set can be well-ordered, therefore $\omega \overset{1-1}{\not\lesssim} A$ only holds when $A$ is finite.

**Lemma 343.**

$$\mathbf{ZF} \vdash_c \Big(\mathbf{AC} \longrightarrow \forall A\, \forall B\ \big(A \overset{onto}{\lesssim} B \longrightarrow A \overset{1-1}{\lesssim} B\big)\Big)$$

*Proof of Lemma* 343:
The result is trivial when $A$ is empty, for $B$ must be empty as well. So, we assume $A$ and $B$ are non-empty. Since $A \overset{onto}{\lesssim} B$, take any $g : B \xrightarrow{onto} A$ and form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function $c$ which for each $a \in A$ provides a *unique* $c(a) \in B$ such that $c(a) \in g^{-1}(a)$. By construction, $c : A \xrightarrow{1-1} B$ witnesses that $A \overset{1-1}{\lesssim} B$.

□ 343

**Corollary 344.** *Given any sets* $A, B$

$$\mathbf{ZFC} \vdash_c \Big(\big(A \overset{1-1}{\lesssim} B \wedge B \overset{onto}{\lesssim} A\big) \longrightarrow A \simeq B\Big).$$

*Proof of Corollary* 344: Immediate from Lemma 343 and Cantor Schroeder Bernstein Theorem (page 57).

□ 344

However, as we will see later, $A \overset{onto}{\lesssim} B \Longrightarrow A \overset{1-1}{\lesssim} B$ may fail in the absence of the Axiom of Choice. Nonetheless, we have this equivalence between the Axiom of Choice and the existence of inverses of surjections.

**Lemma 345.**

$$\mathbf{ZF} \vdash_c \big(\mathbf{AC} \ \longleftrightarrow\ \forall A\, \forall B\ \forall g : B \xrightarrow{onto} A\ \ \exists f : A \xrightarrow{1-1} B \quad g \circ f = id\big)$$

*Proof of Lemma* 345:

($\Longrightarrow$) Given any family $(A_i)_{i \in I}$ of non-empty disjoint sets, we obtain a choice function $f : I \to \bigcup_{i \in I} A_i$ by letting $g : \bigcup_{i \in I} A_i \xrightarrow{onto} I$ be defined as $g(a) = i$ iff $a \in A_i$ and $f : I \xrightarrow{1-1} \bigcup_{i \in I} A_i$ be any function such that $g \circ f = id$ — which guarantees that $f(i) \in A_i$ holds for every $i \in I$.

[1]Such a result can be found in Claim 365 on page 301

($\Longleftarrow$) The result is trivial when $A$ is empty, for $B$ must be empty as well. So, we assume $A$ and $B$ are non-empty. Since $g : B \xrightarrow{onto} A$, form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function $f$ which for each $a \in A$ provides a unique $f(a) \in g^{-1}(a)$. By construction, $f : A \xrightarrow{1-1} B$ and $g \circ f = id$ both hold.

$\square$ 345

**Lemma 346 (ZF).** *Given any non-empty sets $A$ and $B$,*

*(1) if there exists $f : A \xrightarrow{1-1} B$, then there exists $g : B \xrightarrow{onto} A$,*

*(2) if there exists $f : A \xrightarrow{1-1} B$, then there exists $g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$.*

*Proof of Lemma* 346:

(1) Assume $f : A \xrightarrow{1-1} B$, then take any element $a' \in A$ and define $g : B \xrightarrow{onto} A$ by $g(x) = a'$ if $x \notin f[A]$, and $g(x) = a$ if $f(x) = a$. The fact that $f$ is 1-1 guarantees that $g$ is onto.

(2) Given $f : A \xrightarrow{1-1} B$, define $g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$ by $g(C) = f[C]$.

$\square$ 346

## 19.2 Hartogs' Lemma

Without the axiom of choice, there may be sets that do not inject into any ordinal number. But, for any set, there is always some ordinal which does not injects into that set.

**Hartogs' Lemma (ZF).** *Given any set $A$, there exists some ordinal $\alpha$ such that*

$$\alpha \overset{1-1}{\not\lesssim} A.$$

*Proof of Hartogs' Lemma*: We consider the following set:

$$\mathcal{W} = \{(B, <_B) \subseteq A \times \mathscr{P}(A \times A) \mid (B, <_B) \text{ is a well-ordering}\}.$$

Notice that this set is non-empty since the empty ordering $(\varnothing, \varnothing)$ belongs to $\mathcal{W}$. We then consider the functional $\mathbf{F} : \mathcal{W} \to \mathbf{On}$ defined by

$$\mathbf{F}\big((B, <_B)\big) = \text{ the unique ordinal } \beta \text{ s.t. } (\beta, \in_\beta) \simeq (B, <_B).$$

We set

$$\alpha = \sup\left\{\mathbf{F}\big((B,<_B)\big)+1 \mid (B,<_B) \in \mathcal{W}\right\}.$$

It turns out that $\alpha \overset{1-1}{\not\lesssim} A$ holds; for otherwise if we let $f : \alpha \xrightarrow{1-1} A$ and set

$$B = f[\alpha] \text{ and } <_B = \left\{\big(f(\gamma), f(\delta)\big) \mid \gamma < \delta < \alpha\right\},$$

we then obtain $(B,<_B) \in \mathcal{W}$, hence $\alpha \in \mathbf{F}[\mathcal{W}]$, contradicting $\alpha > \mathbf{F}\big((B,<_B)\big)$.

☐ Hartogs' Lemma

## 19.3 Cardinals without the Axiom of Choice

**Definition 348** (**ZF**)**.** *Given any set $A$, we define the cardinal of $A$ — denoted by $|A|$ — by*

$$|A| = \{B \in \mathbf{V}_{\alpha+1} \mid B \simeq A\}$$

*where $\alpha$ is the least ordinal such that there exists some $B \in \mathbf{V}_{\alpha+1}$ that satisfies $B \simeq A$.*

**Notation 349** (**ZF**)**.** *Given any set $A$, we by denote $\alpha_{|A|}$ the least ordinal such that there exists some $B \in \mathbf{V}_{\alpha+1}$ that satisfies $B \simeq A$.*

With this definition we notice that

**Lemma 350** (**ZF**)**.** *Given any non-empty sets $A$ and $A'$,*

*(1) $|A|$ is a set;*

*(2) $|A| = |A'| \Longleftrightarrow A \simeq A'$;*

*(3) $|A| = |A'| \Longleftrightarrow A \overset{1-1}{\lesssim} A'$ and $A' \overset{1-1}{\lesssim} A$.*

*Proof of Lemma 350:*

(1) Obvious.

(2) (⟹) If $|A| = |A'|$, then we have the following equality between non-empty sets

$$\left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A\right\} = \left\{B \in \mathbf{V}_{\alpha_{|A'|}+1} \mid B \simeq A'\right\}$$

which yields $\alpha_{|A|} = \alpha_{|A'|}$, hence

$$\left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A\right\} = \left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A'\right\}$$

which leads to $A \simeq A'$.

**($\Longleftarrow$)** If $A \simeq A'$, then $A' \simeq B$ holds for every $B$ such that $A \simeq B$. Therefore $A' \simeq B$ holds for all $B \in |A| = \left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A\right\}$ which yields $\alpha_{|A'|} \leqslant \alpha_{|A|}$. By symmetry, one also has $\alpha_{|A|} \leqslant \alpha_{|A'|}$, thus $\alpha_{|A|} = \alpha_{|A'|}$, which leads to $|A| = |A'|$.

(3) This is immediate via Cantor Schroeder Bernstein Theorem (page 57).

$\square$ 350

# Chapter 20

# About $\mathbb{R}$ without the Axiom of Choice

## 20.1 Variations on the Reals

In this chapter, we will not really be interested at the reals as an algebraic structure nor a topological structure. We will concentrate on the reals as a set which is equipotent to the power set of the integers. This is the reason why we first recall the following relations:

**Lemma 351 (ZF).**

$$\mathbb{R} \simeq {}^{\omega}\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2).$$

*Proof of Lemma 351:*

(1) $\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}2$;

We recall ${}^{\omega}\omega = \{f : \mathbb{N} \to \mathbb{N}\}$ and ${}^{\omega}2 = \{f : \mathbb{N} \to \{0,1\}\}$.

By Cantor Schroeder Bernstein Theorem (page 57), we only need to show $\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}\omega \overset{1-1}{\lesssim} {}^{\omega}2 \overset{1-1}{\lesssim} \mathbb{R}$

$\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}\omega$**:** assume every real $r$ is written in base 10 as

- in case $0 \leqslant r$:

$$r = +\, e_0\, e_1\, e_2 \ldots e_k,\ d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$$

- in case $r < 0$:

$$r = -\, e_0\, e_1\, e_2 \ldots e_k,\ d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$$

where

(a) $k$ is finite,
(b) for each $i \leqslant k$ and each $j \in \mathbb{N}$, $e_i, d_j \in \{0,1,2,3,4,5,6,7,8,9\}$
(c) $e_0 = 0 \Longrightarrow k = 0$,
(d) $\langle d_j / j \in \mathbb{N}\rangle$ satisfies $\forall j \, \exists j' > j \; d_j \neq 9$. i.e., it is not ultimately constant with value 9. This means for instance that the real $0,23999999999999\ldots$ is rather represented by $+0,24000000000000\ldots$ and the integer $-3$ by $-3,00000000000\ldots$.

We describe the following mapping $f : \mathbb{R} \xrightarrow{1-1} {}^{\omega}\omega$ by

- If $r = +\, e_0\, e_1\, e_2 \ldots e_k,\; d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$, then

$$f(r) = \langle 8, 1+e_0, 1+e_1, \ldots, 1+e_k, 0, 1+d_0, 1+d_1, \ldots, 1+d_n, 1+d_{n+1}, \ldots\rangle$$

- If $r = +, e_0, e_1, e_2 \ldots e_k, , d_0, d_1, d_2, d_3 \ldots d_n, d_{n+1}, d_{n+2} \ldots\ldots$, then

$$f(r) = \langle 9, 1+e_0, 1+e_1, \ldots, 1+e_k, 0, 1+d_0, 1+d_1, \ldots, 1+d_n, 1+d_{n+1}, \ldots\rangle$$

**${}^{\omega}\omega \overset{1-1}{\lesssim} {}^{\omega}2$:** we define $g : {}^{\omega}\omega \xrightarrow{1-1} {}^{\omega}2$ by

$$g\big(\langle a_i / i \in \omega\rangle\big) = 1\, \underbrace{0 \ldots 0}_{a_0}\, 1\, \underbrace{0 \ldots 0}_{a_1}\, 1\, \underbrace{0 \ldots 0}_{a_2}\, 1 \ldots$$

**${}^{\omega}2 \overset{1-1}{\lesssim} \mathbb{R}$:** we define $h : {}^{\omega}2 \xrightarrow{1-1} \mathbb{R}$ by

$$g\big(\langle a_i / i \in \omega\rangle\big) = 0, a_0\, a_1\, a_2\, \ldots\, a_n\, a_{n+1}\, \ldots\ldots$$

(2) ${}^{\omega}\mathbb{R} \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}({}^{\omega}2)$.

It is enough to show that whenever $A \overset{1-1}{\lesssim} B$ holds for non-empty sets $A$ and $B$, then ${}^{\omega}A \overset{1-1}{\lesssim} {}^{\omega}B$ holds as well. So, given any $f : A \xrightarrow{1-1} B$, define $h : {}^{\omega}A \xrightarrow{1-1} {}^{\omega}B$ by

$$h\big(\langle a_i / i \in \omega\rangle\big) = \langle f(a_i) / i \in \omega\rangle\,.$$

(3) ${}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2)$.

${}^{\omega}2 \overset{1-1}{\lesssim} {}^{\omega}({}^{\omega}2)$ is obvious. We show ${}^{\omega}({}^{\omega}2) \overset{1-1}{\lesssim} {}^{\omega}2$ by providing $f : {}^{\omega}({}^{\omega}2) \xrightarrow{1-1} {}^{\omega}2$ defined by

$$f\Big(\Big\langle\, \langle a_{i,j} / j < \omega\rangle\; i < \omega \Big\rangle\Big) = \langle b_k / k < \omega\rangle$$

where $b_k = a_{i,j}$ iff $k = \frac{(i+j)(i+j+1)}{2} + i$.

Notice that the mapping $(i,j) \mapsto \frac{(i+j)(i+j+1)}{2} + i$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

□ 351

**Lemma 352 (ZF).**

$$\omega_1 \overset{onto}{\lesssim} {}^{\omega}2.$$

*Proof of Lemma 352:* We construct $f : {}^{\omega}2 \xrightarrow{onto} \omega_1$.

(1) we define a mapping $\ulcorner \quad \urcorner : \mathbb{N} \times \mathbb{N} \xrightarrow{1-1} \mathbb{N}$ by $\ulcorner n, m \urcorner = 2^{n+1} \cdot 3^{m+1}$.

(2) For each $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ we set

- if $\exists i \, \forall j \geqslant i \; a_j = 0$, then $f(s) = i$ for the least such $i$;
- if $\forall i \, \exists j \geqslant i \; a_j = 1$, then
  - if $\exists i \, \forall n \, \forall m \; (a_i = 1 \wedge \ulcorner n, m \urcorner \neq i)$, then $f(s) = 0$
  - if $\forall i \, \exists n \, \exists m \; (a_i = 1 \longrightarrow \ulcorner n, m \urcorner = i)$, then
    - if $\left(\mathbb{N}, \left\{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\right\}\right)$ is not a well-ordering, then $f(s) = 0$;
    - if $\left(\mathbb{N}, \left\{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\right\}\right)$ is a well-ordering, then $f(s) = \alpha$ where $\alpha$ is the unique ordinal isomorphic to $\left(\mathbb{N}, \left\{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\right\}\right)$. Notice that $\alpha \in \omega_1$ since $\alpha$ is countable.

      To show that $f$ is onto, it is enough to show that for every infinite countable ordinal $\alpha$ there exists some $s \in {}^{\omega}2$ such that $f(s) = \alpha$. For this, notice that $\alpha$ being countable, any bijection $h : \mathbb{N} \xleftrightarrow{bij.} \alpha$ induces a well-ordering on $\mathbb{N}$ of type $\alpha$. Namely, $(\mathbb{N}, <_\alpha)$ where $<_\alpha = \{(n,m) h(n) < h(m)\}$.

      By construction, $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ defined by $a_i = 1$ iff there exists $(n,m) \in <_\alpha$ such that $\ulcorner n, m \urcorner = i$.

$\square$ 352

We will see later that it is consistent with **ZF** to have $\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2$. This means, if **ZF** is consistent, there exists a model of **ZF** in which there exists some surjection from ${}^{\omega}2$ to $\omega_1$, but no injection from from $\omega_1$ to ${}^{\omega}2$.

Nice examples of such models where $\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2$ holds are given by those where the set of reals is a countable union of countable sets (see Section 22.1).

**Notation 353.** *Given any sets $A$ and $B$, the disjoint union of $A$ and $B$ is*

$$A \sqcup B := \left(A \times \{0\}\right) \cup \left(B \times \{1\}\right).$$

**Lemma 354** (**ZF**).

$$^{\omega}2 \cup \omega_1 \overset{onto}{\lesssim} {}^{\omega}2.$$

*Proof of Lemma 354:* We construct $f : {}^{\omega}2 \xrightarrow{onto} {}^{\omega}2 \cup \omega_1$

From Lemma 352, we are granted with a mapping $f' : {}^{\omega}2 \xrightarrow{onto} \omega_1$. Given any $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ we define $f(s)$ as follows:

- if $a_0 = 0$, then $f(s) = \langle a_{i+1} / i \in \omega \rangle$;
- if $a_0 = 1$, then $f(s) = f'\big( \langle a_{i+1} / i \in \omega \rangle \big)$.

□ 354

## 20.2 Outcomes of $\mathbb{R}$ as a Countable Union of Countable Sets

**Proposition 355** (**ZF**). *If $\mathbb{R}$ is a countable union of countable sets, then*

$$\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2.$$

*Proof of Proposition 355:* Notice first that since $\mathbb{R} \simeq {}^{\omega}\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2)$ holds by Lemma 351, the assumption is equivalent to saying that any of these sets is is a countable union of countable sets. So, we assume that it is the case of ${}^{\omega}({}^{\omega}2)$. i.e., there exists $(G_n)_{n<\omega}$ where for each integer $n$, $G_n$ is non-empty, countable and

$${}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} G_n.$$

Towards a contradiction, we assume that $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$ holds, so that there exists some $f : \omega_1 \xrightarrow{1-1} {}^{\omega}2$. We set

$$H_n = \{ s \in {}^{\omega}2 \mid \exists S \in G_n \ \exists k < \omega \ S(k) = s \}.$$

We notice that, for each integer $n$, we have $H_n \overset{1-1}{\lesssim} \omega$. Indeed, we take any $g : G_n \xrightarrow{1-1} \omega$ and construct $\mathcal{I} : H_n \xrightarrow{1-1} \omega$ by $\mathcal{I}(s) = \frac{(i+j)(i+j+1)}{2} + i$ where $i$ is the least integer such that there exists $S \in G_n$ with $g(S) = i$ and there exists some $k < \omega \ S(k) = n$; and $j$ is the least such $k$.

We then define $h : \omega \longrightarrow {}^{\omega}2$ so that for each integer $n$ we have

$$h(n) = f(\alpha_n) \quad \text{where} \quad \alpha_n = \min \{ \alpha \in \omega_1 \mid f(\alpha) \notin H_n \}.$$

By definition, $h \in {}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} G_n$, hence for some integer $n$ we have $h \in G_n$. As usual with this kind of diagonal argument, it is enough to consider $h(n)$ to obtain a contradiction for we end up with

- $h(n) \in H_n$ because $h \in G_n$
- $h(n) \notin H_n$ because $h(n) = f(\alpha_n) \notin H_n$.

$\square$ 355

**Corollary 356 (ZF).** *If $\mathbb{R}$ is a countable union of countable sets, then there exists some partition $\mathcal{R}$ of $\mathbb{R}$ such that*

$$\mathbb{R} \lnsim \mathcal{R}.$$

*Proof of Corollary 356:* The statement $\mathbb{R} \lnsim \mathcal{R}$ stands for $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$. which sums up both

$$\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R} \quad \text{and} \quad \mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}.$$

It is equivalent to the existence of some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \lnsim \mathcal{C}$.

Indeed, if $\mathbb{R} \lnsim \mathcal{R}$ holds, then take any $f : \mathbb{R} \xleftrightarrow{bij.} {}^{\omega}2$ and define $\mathcal{C} = \{f[p] \mid p \in \mathcal{R}\}$. Clearly $\mathcal{C}$ is a partition of ${}^{\omega}2$ that satisfies $\mathcal{R} \simeq \mathcal{C}$, which yields ${}^{\omega}2 \lnsim \mathcal{C}$ since one has ${}^{\omega}2 \simeq \mathbb{R} \lnsim \mathcal{R} \simeq \mathcal{C}$.

Similarly, if ${}^{\omega}2 \lnsim \mathcal{C}$ holds, then take any $g : {}^{\omega}2 \xleftrightarrow{bij.} \mathbb{R}$ in order to obtain the partition $\mathcal{R} = \{f[p] \mid p \in \mathcal{C}\}$ that satisfies $\mathcal{C} \simeq \mathcal{R}$ which leads to $\mathbb{R} \lnsim \mathcal{R}$ since one has $\mathbb{R} \simeq {}^{\omega}2 \lnsim \mathcal{C} \simeq \mathcal{R}$.

So, we prove that there exists some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \lnsim \mathcal{C}$. For this, we come back to Lemma 354 which stated that ${}^{\omega}2 \cup \omega_1 \overset{onto}{\lesssim} {}^{\omega}2$ holds and take any $f : {}^{\omega}2 \xrightarrow{onto} {}^{\omega}2 \cup \omega_1$ to form the partition

$$\begin{aligned} \mathcal{C} &= \Big\{ \{s \in {}^{\omega}2 \mid f(s) = x\} \mid x \in {}^{\omega}2 \cup \omega_1 \Big\} \\ &= \Big\{ f^{-1}\big[\{x\}\big] \mid x \in {}^{\omega}2 \cup \omega_1 \Big\}. \end{aligned}$$

We obtain

**${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$:** The mapping $g : {}^{\omega}2 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned} g(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\{x\}\big] \end{aligned}$$

is obviously $1-1$, hence witnesses that ${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$ holds.

**$\mathcal{C} \overset{1-1}{\not\lesssim} {}^{\omega}2$:** Towards a contradiction, we assume $\mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$. We notice that $\omega_1 \overset{1-1}{\lesssim} \mathcal{C}$ holds for the following mapping is 1-1: $h : \omega_1 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned} h(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\,\{x\}\,\big] \end{aligned}$$

Therefore, we have both

$$\omega_1 \overset{1-1}{\lesssim} \mathcal{C} \quad \text{and} \quad \mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$$

which leads to $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$, contradicting Proposition 355.

$\square$ 356

# Chapter 21

# Symmetric Submodels of Generic Extensions

## 21.1 Symmetry Groups and Hereditarily Symmetric $\mathbb{P}$-names

We recall Definition 331 which states that given $\mathbf{M}$ any *c.t.m.* of "**ZFC**" and $(\mathbb{P}, \leqslant, \mathbb{1})$ any partial order over $\mathbf{M}$, an *automorphism* of $\mathbb{P}$ is a mapping $\pi : \mathbb{P} \xleftrightarrow{bij.} \mathbb{P}$ such that

$$\forall p \in \mathbb{P}\ \forall q \in \mathbb{P}\ \ \big(p \leqslant q \ \longleftrightarrow\ \pi(p) \leqslant \pi(q)\big).$$

We also recall that this definition implies $\pi(\mathbb{1}) = \mathbb{1}$ and any such automorphism $\pi$ induces an automorphism $\tilde{\pi}$ on the class of $\mathbb{P}$-names $\mathbf{M}^{\mathbb{P}}$ defined by transfinite recursion (see Definition 333):

$$\begin{aligned} \tilde{\pi} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}^{\mathbb{P}} \\ \tau &\longmapsto \{(\tilde{\pi}(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}. \end{aligned}$$

Notice that we have $\tilde{\pi}(\varnothing) = \varnothing$, and more generally, for every canonical $\mathbb{P}$-name $\check{x}$, we have $\tilde{\pi}(\check{x}) = \check{x}$.

By Lemma 335, for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$, and all automorphism $\pi$,

$$\mathbf{M}\big[\pi[G]\big] = \mathbf{M}[G].$$

Moreover, by Lemma 336, for all $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$, and $p \in \mathbb{P}$,

$$p \Vdash \varphi\left(\tau_1, \ldots, \tau_n\right) \Longleftrightarrow \pi(p) \Vdash \varphi\big(\tilde{\pi}(\tau_1), \ldots, \tilde{\pi}(\tau_n)\big).$$

**Definition 357** (Symmetry Group)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*.*

*For each $\mathbb{P}$-name $\tau$ the symmetry group of $\tau$ is*

$$sym_{\mathscr{G}}(\tau) = \{\pi \in \mathcal{G} \mid \tilde{\pi}(\tau) = \tau\}.$$

We notice:

(1) For any canonical $\mathbb{P}$-name $\check{x}$,

$$sym_{\mathscr{G}}(\check{x}) = \{\pi \in \mathcal{G} \mid \pi(\check{x}) = \check{x}\} = \mathcal{G}.$$

(2) For all $\mathbb{P}$-name $\tau$, and all automorphism $\pi \in \mathcal{G}$, one has

$$sym_{\mathscr{G}}\big(\tilde{\pi}(\tau)\big) = \pi \circ sym_{\mathscr{G}}(\tau) \circ \pi^{-1}$$

since, given any automorphism $\mu \in sym_{\mathscr{G}}(\tau)$, we have

$$(\tilde{\pi} \circ \tilde{\mu} \circ \tilde{\pi}^{-1})(\tilde{\pi}(\tau)) = \tilde{\pi} \circ \tilde{\mu} \circ (\tilde{\pi}^{-1} \circ \tilde{\pi})(\tau) = \tilde{\pi} \circ \tilde{\mu}(\tau) = \tilde{\pi}(\tau).$$

**Definition 358** (Normal Filter)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*.*

$\mathscr{F}$ *is a normal filter on* $\mathcal{G}$ *if*

$\mathscr{F}$ *is a set of subgroups of* $\mathcal{G}$ *such that for all subgroups* $\mathcal{H}$, $\mathcal{K}$ *of* $\mathcal{G}$ *and all* $\pi \in \mathcal{G}$*:*

*(1)* $\mathcal{G} \in \mathscr{F}$

*(2) if* $\mathcal{H} \in \mathscr{F}$ *and* $\mathcal{H} \subseteq \mathcal{K}$*, then* $\mathcal{K} \in \mathscr{F}$

*(3) if* $\mathcal{H} \in \mathscr{F}$ *and* $\mathcal{K} \in \mathscr{F}$*, then* $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$

*(4) if* $\mathcal{H} \in \mathscr{F}$*, then* $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$

Granted with some normal filters, the idea is then to consider the $\mathbb{P}$-names whose symmetry group belongs to the filter, and this iterated hereditarily the following way:

**Definition 359** (Hereditarily Symmetric $\mathbb{P}$-names)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*.*
*The set of all hereditarily symmetric* $\mathbb{P}$*-names* $\mathbf{HS}_{\mathscr{F}} \subseteq \mathbf{M}^{\mathbb{P}}$ *is defined by transfinite recursion:*

$$\tau \in \mathbf{HS}_{\mathscr{F}} \iff sym_{\mathscr{G}}(\tau) \in \mathscr{F} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathscr{F}}.$$

So we have

$$\mathbf{HS}_{\mathscr{F}} = \left\{\tau \in \mathbf{M}^{\mathbb{P}} \mid sym_{\mathscr{G}}(\tau) \in \mathscr{F} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathscr{F}}\right\}.$$

Notice that

- every canonical $\mathbb{P}$-name $\check{x} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathscr{G}}(\check{x}) = \mathcal{G} \in \mathscr{F}$.
- If $\tau \in \mathbf{HS}_{\mathscr{F}}$, then $\tilde{\pi}(\tau) \in \mathbf{HS}_{\mathscr{F}}$ holds for any $\pi \in \mathcal{G}$, hence for all $\mathbb{P}$-name $\tau$

$$\tau \in \mathbf{HS}_{\mathscr{F}} \iff \tilde{\pi}(\tau) \in \mathbf{HS}_{\mathscr{F}}.$$

## 21.2 The Symmetric Submodel

We now define the symmetric submodel of a generic extension as its restriction to the only elements that admit an hereditarily symmetric $\mathbb{P}$-name.

**Definition 360** (Symmetric Submodel)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*. The symmetric submodel of the generic extension* $\mathbf{M}[G]$ *is*

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} = \{(\tau)_G \in \mathbf{M}[G] \mid \tau \in \mathbf{HS}_{\mathscr{F}}\}.$$

So, a symmetric submodel is a structure in between $\mathbf{M}$ (since every canonical $\mathbb{P}$-name is hereditarily symmetric) and the generic extension $\mathbf{M}[G]$ (since hereditarily symmetric $\mathbb{P}$-name is a $\mathbb{P}$-name). The properties that make the symmetric submodel very interesting is that it is transitive and satisfies all the axioms of $\mathbf{ZF}$, but contrary to the generic extension, it does not necessarily satisfy the axiom of choice.

**Proposition 361.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*.*

*(1)* $\mathbf{M} \subseteq \widehat{\mathbf{M}[G]}^{\mathscr{F}} \subseteq \mathbf{M}[G]$*;*

*(2)* $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ *is a transitive set;*

*(3)* $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ *satisfies "*$\mathbf{ZF}$*".*

*Proof of Proposition* 361:

(1) is immediate.

(2) Assume $x \in (\tau)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ with $\tau \in \mathbf{HS}_{\mathscr{F}}$. Then, there exists $p \in \mathbf{G}$ and $\sigma \in \mathbf{M}^{\mathbb{P}}$ such that $(\sigma, p) \in \tau$ and $x = (\sigma)_G$. Since $\tau \in \mathbf{HS}_{\mathscr{F}}$, it follows that $\sigma \in \mathbf{HS}_{\mathscr{F}}$, hence $x = (\sigma)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

(3) $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ satisfies "**ZF**":

**Extensionality** holds in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ is transitive.

**Comprehension Schema** We want to show that for all $\sigma, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n \in \mathbf{HS}_{\mathscr{F}}$ and $\varphi(x, y_1, \ldots, y_n)$:

$$u = \left\{ z \in (\sigma)_G \mid \left(\varphi(z, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \right\} \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}.$$

We must find some $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $u = (\tau)_G$. For this purpose, we first modify $\sigma$ and consider $\overline{\sigma}$ instead:

$$\overline{\sigma} = \{(\underset{\sim}{z}, p) \mid \exists q \in \mathbb{P}\ p \leqslant q \text{ and } (\underset{\sim}{z}, q) \in \sigma\}$$

Clearly, $sym_{\mathscr{G}}(\overline{\sigma}) \supseteq sym_{\mathscr{G}}(\sigma) \in \mathscr{F}$, hence $sym_{\mathscr{G}}(\overline{\sigma}) \in \mathscr{F}$. Moreover, since $dom(\overline{\sigma}) = dom(\sigma) \subseteq \mathbf{HS}_{\mathscr{F}}$, it follows $\overline{\sigma} \in \mathbf{HS}_{\mathscr{F}}$.
We set

$$\underset{\sim}{u} = \left\{ (\underset{\sim}{z}, p) \in \overline{\sigma} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \left(\varphi(\underset{\sim}{z}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n)\right)^{\mathbf{HS}_{\mathscr{F}}} \right\}.$$

We show $\underset{\sim}{u} \in \mathbf{HS}_{\mathscr{F}}$. Since $\overline{\sigma} \in \mathbf{HS}_{\mathscr{F}}$, it only remains to show that $sym_{\mathscr{G}}(\underset{\sim}{u}) \in \mathscr{F}$. For this, we consider

$$\mathscr{G}' = sym_{\mathscr{G}}(\overline{\sigma}) \cap sym_{\mathscr{G}}(\underset{\sim}{z}_1) \cap \ldots \cap sym_{\mathscr{G}}(\underset{\sim}{z}_n)$$

$\mathscr{G}' \in \mathscr{F}$ holds since $\mathscr{F}$ is a filter, and for each $\pi \in \mathscr{G}'$, one has

$$\tilde{\pi}(\overline{\sigma}) = \overline{\sigma}, \quad \tilde{\pi}(\tau_1) = \tau_1, \quad \ldots, \tilde{\pi}(\underset{\sim}{z}_n) = \underset{\sim}{z}_n.$$

and also

$$\begin{aligned}
\tilde{\pi}(\underset{\sim}{u}) &= \left\{ \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \mid (\underset{\sim}{z}, p) \in \underset{\sim}{u} \right\} \\
&= \left\{ \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \mid (\underset{\sim}{z}, p) \in \overline{\sigma} \wedge p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n) \right\} \\
&= \left\{ \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \mid \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \in \tilde{\pi}(\overline{\sigma}) \wedge \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\left(\tilde{\pi}(\underset{\sim}{z}), \tilde{\pi}(\underset{\sim}{z}_1), \ldots, \tilde{\pi}(\underset{\sim}{z}_n)\right) \right\} \\
&= \left\{ \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \mid \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \in \overline{\sigma} \wedge \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\left(\tilde{\pi}(\underset{\sim}{z}), \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n\right) \right\} \\
&= \left\{ \left(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\right) \in \overline{\sigma} \mid \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\left(\tilde{\pi}(\underset{\sim}{z}), \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n\right) \right\} \\
&= \left\{ (\underset{\sim}{z}, p) \in \overline{\sigma} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}), \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n) \right\} \\
&= \underset{\sim}{u}
\end{aligned}$$

We have just shown $\mathscr{G}' \subseteq sym_{\mathscr{G}}(\underset{\sim}{u})$, which proves that $sym_{\mathscr{G}}(\underset{\sim}{u}) \in \mathscr{F}$ and completes the proof that $\underset{\sim}{u} \in \mathbf{HS}_{\mathscr{F}}$.

It remains to show that $(\underset{\sim}{u})_G = u$.

$(\underset{\sim}{\mathbf{u}})_{\mathbf{G}} \subseteq \mathbf{u}$**:** if $(\underset{\sim}{z})_G \in (\underset{\sim}{u})_G$, then there exists $p \in \mathbf{G}$ such that both $(\underset{\sim}{z}, p) \in \overline{\sigma}$ and $p \Vdash_{\mathbb{P},\mathbf{M}} \Big(\varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n})\Big)^{\mathbf{HS}_{\mathscr{F}}}$ hold, which yields $\Big(\varphi((\underset{\sim}{z})_G, (\underset{\sim}{z_1})_G, \ldots, (\underset{\sim}{z_n})_G)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. By construction, there exists $q \geqslant p$ such that $(\underset{\sim}{z}, q) \in \sigma$. Thus, since $q \geqslant p \in G$, holds, $q \in G$ which gives $(\underset{\sim}{z})_G \in (\sigma)_G$. Putting together

$$(\underset{\sim}{z})_G \in (\sigma)_G \text{ and } \Big(\varphi((\underset{\sim}{z})_G, (\underset{\sim}{z_1})_G, \ldots, (\underset{\sim}{z_n})_G)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$$

we obtain $(\underset{\sim}{z})_G \in u$.

$\mathbf{u} \subseteq (\underset{\sim}{\mathbf{u}})_{\mathbf{G}}$**:** if $z \in u$, then $z \in (\sigma)_G$ and $\Big(\varphi\big(z, (\underset{\sim}{z_1})_G, \ldots, (\underset{\sim}{z_n})_G\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. So, there exists some hereditarily symmetric $\mathbb{P}$,-name $\underset{\sim}{z}$ for $z$. Namely, some $\underset{\sim}{z} \in \sigma$ which satisfies both $\underset{\sim}{z} \in \mathbf{HS}_{\mathscr{F}}$ and $(\underset{\sim}{z})_G = z$. Moreover, there exists $p \in G$ such that $(\underset{\sim}{z}, p) \in \sigma$. Hence, $(\underset{\sim}{z}, p) \in \overline{\sigma}$, and since $\Big(\varphi\big((\underset{\sim}{z})_G, (\underset{\sim}{z_1})_G, \ldots, (\underset{\sim}{z_n})_G\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ it follows $p \Vdash_{\mathbb{P},\mathbf{M}} \Big(\varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n})\Big)^{\mathbf{HS}_{\mathscr{F}}}$, which shows that $(\underset{\sim}{z}, p) \in \underset{\sim}{u}$, and finally $z = (\underset{\sim}{z})_G \in (\underset{\sim}{u})_G$.

**Pairing** If $(\tau)_G, (\sigma)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ — i.e., with $\tau, \sigma \in \mathbf{HS}_{\mathscr{F}}$ — then $\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathscr{G}}(\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}) \supseteq sym_{\mathscr{G}}(\sigma) \cap sym_{\mathscr{G}}(\tau)$. We obtain

$$\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}_G = \{(\tau)_G, (\sigma)_G\} \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}.$$

**Union** Let $\sigma \in \mathbf{HS}_{\mathscr{F}}$, to prove that $\bigcup(\sigma)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$, it is enough to show that there exists $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $\bigcup(\sigma)_G \subseteq (\tau)_G$. We recall that

$$dom(\sigma) = \{\delta \in \mathbf{HS}_{\mathscr{F}} \mid \exists p \in \mathbb{P}\ (\delta, p) \in \sigma\}$$

We set

$$X = \bigcup \big\{ dom(\delta) \mid \delta \in dom(\sigma) \big\}.$$

Notice that, for every $\pi \in sym_{\mathscr{G}}(\sigma)$, $\tilde{\pi}(X) = X$ holds since $\sigma \in \mathbf{HS}_{\mathscr{F}}$. Thus, $X \times \{\mathbb{1}\}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and $(X \times \{\mathbb{1}\})_G \supseteq \bigcup \sigma$.

**Infinity** Since $\omega \in \mathbf{M}$, one has $\check{\omega} \in \mathbf{HS}_{\mathscr{F}}$, hence $\omega = (\check{\omega})_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

**Power Set** Let $\sigma \in \mathbf{HS}_{\mathscr{F}}$, it is enough to show there exists $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $\mathscr{P}\big((\sigma)_G\big) \cap \widehat{\mathbf{M}[G]}^{\mathscr{F}} \subseteq (\tau)_G$.

Notice first that for every subset $X \subseteq dom(\sigma)$, the $\mathbb{P}$-name $\sigma_X = \big\{(\underset{\sim}{x}, \mathbb{1}) \mid \underset{\sim}{x} \in X\big\}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ since every $\underset{\sim}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and $sym_{\mathscr{G}}(\sigma_X) \supseteq sym_{\mathscr{G}}(\sigma) \in \mathscr{F}$.

We consider :

$$\tau = \big\{(\sigma_X, \mathbb{1}) \mid X \subseteq dom(\sigma)\big\}$$

Notice that $\tau \in \mathbf{HS}_{\mathscr{F}}$ since every $\sigma_X$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and

$$sym_{\mathscr{G}}(\tau) \supseteq \bigcap_{X \subseteq dom(\sigma)} sym_{\mathscr{G}}(\sigma_X) \supseteq sym_{\mathscr{G}}(\sigma) \in \mathscr{F}.$$

Given any $Y \subseteq (\sigma)_G$, it appears that $Y \in (\tau)_G$, since $Y = \Big(\sigma_{\{\underset{\sim}{y} \mid (\underset{\sim}{y})_G \in Y\}}\Big)_G$ and by construction $Y \in (\tau)_G$.

**Foundation** holds in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ is transitive and **Foundation** holds in the ground model. Indeed, the $\in$-well-foundedness of each element of $\mathbf{M}^{\mathbb{P}}$ yields the $\in$-well-foundedness of each element of $\mathbf{HS}_{\mathscr{F}}$, hence each element of $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

**Replacement Schema** for each formula $\varphi(x, y, z_1, \ldots, z_n)$, we want to prove that: $\forall z_1 \in \mathbf{M}[G] \ldots \forall z_n \in \mathbf{M}[G]$

$$\left(\begin{array}{c} \forall x \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}\ \exists! y \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}\ \ \big(\varphi(x, y, z_1, \ldots, z_n)\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \\ \longrightarrow \\ \forall u \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}\ \exists v \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}\ \forall x \in u\ \exists y \in v\ \ \big(\varphi(x, y, z_1, \ldots, z_n)\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \end{array}\right).$$

We fix $z_1 = (\underset{\sim}{z}_1)_G, \ldots, z_n = (\underset{\sim}{z}_n)_G$, and $u = (\underset{\sim}{u})_G$. <u>Inside $\mathbf{M}$</u> we define:

$$\begin{aligned} \mathbf{F} : dom(\underset{\sim}{u}) \times \mathbb{P} &\to \mathbf{On} \\ (\underset{\sim}{x}, p) &\to \begin{cases} \text{least } \alpha \text{ s.t. } \exists \underset{\sim}{y} \in \mathbf{HS}_{\mathscr{F}} \cap \mathbf{V}_\alpha\ \ p \Vdash \varphi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n)^{\mathbf{HS}_{\mathscr{F}}} \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Since $\mathbf{M}$ satisfies the instances of the various replacement schema that we need to complete the proof, there exists $\beta \in \big(\mathbf{On}\big)^{\mathbf{M}}$ such that $\mathbf{F}\left[dom(\underset{\sim}{u}) \times \mathbb{P}\right] \subseteq \beta$. We claim that $v = \mathbf{V}_\beta$ works:
Let $(\underset{\sim}{x})_G \in (\underset{\sim}{u})_G$, by hypothesis there exists — a unique — $(\underset{\sim}{y})_G$ such that

$$\Big(\varphi\big((\underset{\sim}{x})_G, (\underset{\sim}{y})_G, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

Therefore there exists $p \in G$ such that

$$p \Vdash \varphi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n)^{\mathbf{HS}_{\mathscr{F}}}$$

It follows that there exists $\hat{\underset{\sim}{y}} \in \mathbf{V}_\beta$ such that

$$p \Vdash \varphi(\underset{\sim}{x}, \hat{\underset{\sim}{y}}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n).$$

The Truth Lemma yields

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \varphi\big(\,(\underset{\sim}{x})_G\,, \big(\hat{\underset{\sim}{y}}\big)_G\,, \big(\underset{\sim}{z}_1\big)_G\,, \ldots, \big(\underset{\sim}{z}_n\big)_G\,\big)^{\mathbf{HS}_{\mathscr{F}}}$$

so, by unicity, we obtain $\big(\underset{\sim}{y}\big)_G = \big(\hat{\underset{\sim}{y}}\big)_G$. Therefore $\mathbf{V}_\beta$ is (one of) the set we were looking for, since it satisfies

$$\left\{ \big(\underset{\sim}{y}\big)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \mid \exists\, (\underset{\sim}{x})_G \in (\underset{\sim}{u})_G \ \Big(\varphi\big(\,(\underset{\sim}{x})_G\,, \big(\underset{\sim}{y}\big)_G\,, \big(\underset{\sim}{z}_1\big)_G\,, \ldots, \big(\underset{\sim}{z}_n\big)_G\,\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \right\} \subseteq \mathbf{V}_\beta = \big(\check{\mathbf{V}}_\beta\big)_G.$$

$\square$ 361

# Chapter 22

# Some Applications of the Symmetric Submodel Technique

## 22.1 Forcing $\mathbb{R}$ as a Countable Union of Countable Sets

The following blatantly contradicts the countable axiom of choice (**CC**) which is precisely what was used — for the special case of a countable family of countable sets — to prove Lemma 104 which established that any countable union of countable sets is countable, although with the help of Cantor's Theorem (see page 61), we proved that the set of reals is uncountable.

**Theorem 362** (Feferman & Lévy)**.**

$$cons(\mathbf{ZF}) \implies cons(\mathbf{ZF} + \text{“}\,\mathbb{R} \textit{ is a countable union of countable sets”}).$$

*Proof of Theorem* 362: Instead of showing that there exists some model that satisfies

$$\text{“}\,\mathbb{R} \text{ is a countable union of countable sets”}$$

we will show that there exists some model that satisfies the equivalent statement

$$\text{“}\mathcal{P}(\omega) \text{ is a countable union of countable sets”}.$$

We start with $\mathbf{M}$ any *c.t.m.* of "$\mathbf{ZFC}{+}\forall n < \omega \;\; 2^{\aleph_n} = \aleph_{n+1}$" and force with the poset: $(\mathbb{P}_{\text{Levy}}, \leqslant, \mathbb{1})$ defined <u>inside $\mathbf{M}$</u> by

$$\mathbb{P}_{\text{Levy}} = \Big\{ f \;:\; \omega \times \omega \to \aleph_\omega \mid dom(f) \text{ is finite} \wedge \forall n \in \omega \; \forall m \in \omega \;\; f(n,m) \in \aleph_n \Big\}$$

and $f \leqslant g \iff f \supseteq g$, so that $\mathbb{1} = \varnothing$.

We let $G$ be $\mathbb{P}_{\text{Levy}}$-generic over $\mathbf{M}$ and construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ of $\mathbf{M}[G]$. For this, we consider the group $\mathscr{G}_{per.}$ of the permutations of $\omega \times \omega$ which do not move the first projection. i.e.,

$$\mathscr{G}_{per.} = \left\{\pi : \omega \times \omega \xleftrightarrow{bij.} \omega \times \omega \mid \forall i, j, k, n < \omega \ \big(\pi(k, i) = (n, j) \longrightarrow k = n\big)\right\}.$$

For each integer $n$, any such permutation $\pi$ induces an permutation $\pi_n$ of $\omega$ defined by

$$\forall i \quad < \omega \ \pi(n, i) = \big(n, \pi_n(i)\big).$$

Every permutation $\pi \in \mathscr{G}_{per.}$ induces an automorphism $\pi : \mathbb{P} \xrightarrow{aut.} \mathbb{P}$ by having for each forcing condition $p \in \mathbb{P}$:

$$dom\,(\pi(p)) = \pi\,[dom\,(\pi)] = \big\{(n, \pi_n(i)) \mid (n, i) \in dom\,(p)\,\big\}$$

and

$$\pi(p)\big(n, \pi_n(i)\big) = p(n, i).$$

Or, to say it directly,

$$\pi(p) = \Big\{\big(n, \pi_n(i), \alpha\big) \mid (n, i, \alpha) \in p\Big\}.$$

We let $\mathscr{G}_{aut.}$ be the group of automorphisms of $\mathbb{P}$ induced by the group of permutations $\mathscr{G}_{per.}$:

$$\mathscr{G}_{aut.} = \big\{\pi \mid \pi \in \mathscr{G}_{per.}\big\}.$$

We let $\mathscr{H}_k$ be the subgroup of $\mathscr{G}_{aut.}$ formed of all permutations $\pi$ such that for every integer $n < k$, the permutation $\pi_n$ is the identity. i.e.,

$$\mathscr{H}_k = \Big\{\pi \in \mathscr{G}_{aut.} \mid \forall n < k \ \ \forall i < \omega \ \ \pi(n, i) = (n, i)\Big\}.$$

We also let $\mathscr{F}$ be the filter:

$$\mathscr{F} = \Big\{\mathscr{S} \subseteq \mathscr{G}_{aut.} \mid \text{“}\mathscr{S} \text{ is a subgroup”} \ \wedge \ \exists n < \omega \ \ \mathscr{H}_n \subseteq \mathscr{S}\Big\}.$$

$\mathscr{F}$ is a normal filter on $\mathscr{G}_{aut.}$ since $\mathscr{F}$ is a set of subgroups of $\mathscr{G}_{aut.}$ such that for all subgroups $\mathscr{S}$, $\mathscr{T}$ of $\mathscr{G}_{aut.}$:

(1) $\mathscr{G}_{aut.} \in \mathscr{F}$ holds since $\mathscr{G}_{aut.} = \mathscr{H}_0$;

(2) if $\mathscr{S} \in \mathscr{F}$ and $\mathscr{S} \subseteq \mathscr{T}$ for $\mathscr{T}$ some subgroup of $\mathscr{G}_{aut.}$, then there exists $n < \omega$ such that $\mathscr{H}_n \subseteq \mathscr{S} \subseteq \mathscr{T}$, hence $\mathscr{T} \in \mathscr{F}$;

(3) if $\mathscr{S} \in \mathscr{F}$ and $\mathscr{T} \in \mathscr{F}$, then there exist $n, k < \omega$ $\mathscr{H}_n \subseteq \mathscr{S}$ and $\mathscr{H}_k \subseteq \mathscr{T}$, hence $\mathscr{H}_{\sup\{n,k\}} \subseteq \mathscr{S} \cap \mathscr{T}$ which gives $\mathscr{S} \cap \mathscr{T} \in \mathscr{F}$;

(4) if $\mathscr{S} \in \mathscr{F}$, then there exists $n < \omega$ such that $\mathscr{H}_n \subseteq \mathscr{S}$ and for all $\pi \in \mathscr{G}_{aut.}$, one has $\pi \circ \mathscr{H}_n \circ \pi^{-1} = \mathscr{H}_n$, so that $\mathscr{H}_n = \pi \circ \mathscr{H}_n \circ \pi^{-1} \subseteq \pi \circ \mathscr{S} \circ \pi^{-1}$ which shows $\pi \circ \mathscr{S} \circ \pi^{-1} \in \mathscr{F}$.

We then construct canonical symmetric $\mathbb{P}$-names for the reals[1].

Let $\underset{\sim}{x}$ be any name in $\mathbf{HS}_{\mathscr{F}}$ for a subset $x$ of the integers — i.e., $\big(x \in \mathcal{P}(\omega)\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. We have

$$\underset{\sim}{x} \subseteq \Big\{(\underset{\sim}{k}, p) \mid \underset{\sim}{k} \in \mathbf{HS}_{\mathscr{F}} \wedge p \in \mathbb{P} \wedge p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}\Big\}.$$

Since $\underset{\sim}{x} \in \mathbf{HS}_{\mathscr{F}}$, there exists some integer $n_{\underset{\sim}{x}}$ such that $\mathscr{H}_{n_{\underset{\sim}{x}}} \subseteq sym_{\mathscr{G}_{aut.}}(\underset{\sim}{x})$, which yields for every $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$:

$$\tilde{\pi}(\underset{\sim}{x}) = \Big\{\big(\tilde{\pi}(\underset{\sim}{k}), \pi(p)\big) \mid (\underset{\sim}{k}, p) \in \underset{\sim}{x}\Big\} = \underset{\sim}{x}$$

We define a canonical $\mathbb{P}$-name $\underset{\dot\sim}{x}$ for $x$ by:

$$\underset{\dot\sim}{x} = \Big\{(\check{n}, q) \mid \exists (\underset{\sim}{k}, p) \in \underset{\sim}{x} \;\; \exists r \in \mathbb{P} \;\; \big(r \leqslant p \wedge q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}\big)\Big\}.$$

We first check that $\underset{\dot\sim}{x} \in \mathbf{HS}_{\mathscr{F}}$. Notice first that given any $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$ and $(\check{n}, q) \in \underset{\dot\sim}{x}$, we have

$$\tilde{\pi}(\check{n}, q) = \big(\check{n}, \pi(q)\big).$$

Since $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$, we also have $dom(q) \subseteq n_{\underset{\sim}{x}} \times \omega$, which gives $\pi(q) = q$ so that we obtain

$$\tilde{\pi}(\check{n}, q) = \big(\check{n}, \pi(q)\big) = (\check{n}, q),$$

which yields $\tilde{\pi}(\underset{\dot\sim}{x}) = \underset{\dot\sim}{x}$. We have shown $\underset{\dot\sim}{x}$ is symmetric and since every $\mathbb{P}$-name $\sigma$ such that $(\sigma, p) \in \underset{\dot\sim}{x}$ holds for some $p \in \mathbb{P}$ is of the form $\sigma = \check{n}$, it follows $\sigma \in \mathbf{HS}_{\mathscr{F}}$, hence we have shown that $\underset{\dot\sim}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$.

Now that we know both $\underset{\sim}{x}$ and$\underset{\dot\sim}{x}$ belong to $\mathbf{HS}_{\mathscr{F}}$, we need to show that they both give birth to the same set $x$. Namely,

**Claim 363.**

$$\big(\underset{\dot\sim}{x}\big)_G = (\underset{\sim}{x})_G\,.$$

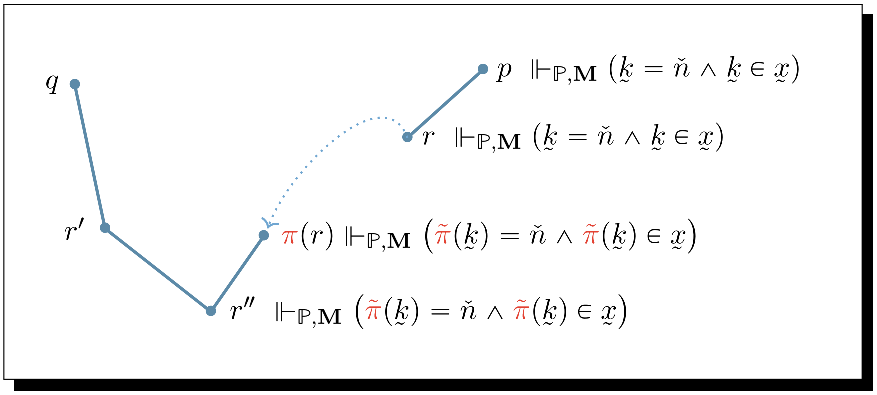
*Proof of Claim 363:*

$\big(\underset{\dot\sim}{x}\big)_G \subseteq (\underset{\sim}{x})_G$ : Let $n \in \big(\underset{\dot\sim}{x}\big)_G$, and consider any $q \in G$ such that $(\check{n}, q) \in \underset{\dot\sim}{x}$. So, there exists $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ such that

$$r \leqslant p \text{ and } r \restriction_{n_{\underset{\sim}{x}} \times \omega} = q \text{ and } r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}.$$

Now, for every condition $r' \leqslant q$, we may find a condition $r''$ and an automorphism $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$ such that the picture below holds.

[1] Here the word real stands for any subset of integers.




i.e., both $r'' \leqslant r'$ and $r'' \leqslant \pi(r)$ hold. Notice first the following:

- $p \Vdash_{\mathbb{P},\mathbf{M}} (\underset{\sim}{k} = \check{n} \wedge \underset{\sim}{k} \in \underset{\sim}{x})$ holds for
  - $p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}$ holds since $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$.
  - $p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \underset{\sim}{x}$ holds, since for every $G$ $\mathbb{P}$-generic over $\mathbf{M}$ such that $p \in G$, the statement $(\underset{\sim}{k})_G \in (\underset{\sim}{x})_G$ holds.
- $r \Vdash_{\mathbb{P},\mathbf{M}} (\underset{\sim}{k} = \check{n} \wedge \underset{\sim}{k} \in \underset{\sim}{x})$ since $r \leqslant p$.
- $\pi(r) \Vdash_{\mathbb{P},\mathbf{M}} \big(\tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \underset{\sim}{x}\big)$ since $\pi(r) \Vdash_{\mathbb{P},\mathbf{M}} \big(\tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \tilde{\pi}(\underset{\sim}{x})\big)$ and $\tilde{\pi}(\underset{\sim}{x}) = \underset{\sim}{x}$.
- $q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} = \pi(r) \restriction_{n_{\underset{\sim}{x}} \times \omega}$

So, in particular since $q = \pi(r) \restriction_{n_{\underset{\sim}{x}} \times \omega}$ and $r' \supseteq q$, $\pi(r)$ and $r'$ agree on the domain $n_{\underset{\sim}{x}} \times \omega$. i.e., since $q \subseteq r'$, $r \restriction_{n_{\underset{\sim}{x}} \times \omega} = q$ and $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$, we have

$$\pi(r) \restriction_{n_{\underset{\sim}{x}} \times \omega} = r \restriction_{n_{\underset{\sim}{x}} \times \omega} = q = r' \restriction_{dom(q)} .$$

So, consider

$$U = \big\{(n,i) \in dom(r) \cap dom\,(r') \mid r(n,i) \neq r'(n,i)\big\}$$

(Notice that $n_{\underset{\sim}{x}} < n$ holds for each $(n,i) \in U$.) Let $\pi \in \mathscr{G}_{aut.}$ satisfy that for each $s \in \mathbb{P}$ and $(n,i) \in dom\,(s)$:

- If $(n,i) \notin U$, then $\pi(s)(n,i) = s(n,i)$;
- If $(n,i) \in U$ but both $s(n,i) \neq r(n,i)$ and $s(n,i) \neq r'(n,i)$ hold, then $\pi(s)(n,i) = s(n,i)$;
- If $(n,i) \in U$ and $s(n,i) = r(n,i)$, then $\pi(s)(n,i) = r'(n,i)$;
- If $(n,i) \in U$ and $s(n,i) = r'(n,i)$, then $\pi(s)(n,i) = r(n,i)$.

Notice that $\pi$ belongs to $\mathscr{H}_{n_{\underset{\sim}{x}}}$ and also $\pi(r)$ agrees with $r'$ on their common domain. Therefore, there exists some $r'' \leqslant r', \pi(r)$ which necessarily satisfies

$$r'' \Vdash_{\mathbb{P},\mathbf{M}} \left(\tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \underset{\sim}{x}\right).$$

It follows that the set $\mathscr{D} = \left\{r'' \in \mathbb{P} \mid r'' \Vdash_{\mathbb{P},\mathbf{M}} \check{n} \in \underset{\sim}{x}\right\}$ is dense below $q$. So, $\mathscr{D}$ intersects [2] the generic filter $G$, which finally yields $n \in (\underset{\sim}{x})_G$.

$(\underset{\sim}{x})_G \subseteq (\underset{\dot\sim}{x})_G$ : If $k \in (\underset{\sim}{x})_G$, then there exists $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ such that both $p \in G$ and $p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}$. So, it follows that there exist $n \in \omega$ and some $r \leqslant p$ with $r \in G$ such that

$$r \Vdash_{\mathbb{P},\mathbf{M}} \left(\underset{\sim}{k} \in \check{\omega} \wedge \underset{\sim}{k} \in \underset{\sim}{x} \wedge \check{n} = \underset{\sim}{k}\right);$$

which finally yields $(\check{n}, r \restriction_{n_{\underset{\sim}{x}} \times \omega}) \in \underset{\dot\sim}{x}$.

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For each integer $n$, set

$$\underset{\sim}{R_n} = \left\{(\underset{\dot\sim}{x}, \mathbb{1}) \mid \text{“}\underset{\dot\sim}{x} \text{ is a canonical } \mathbb{P}\text{-name for a real” and } \underset{\dot\sim}{x} \in D_n\right\}$$

where

$$\underset{\dot\sim}{x} \in D_n \iff \forall m \in \omega\ \forall p \in \mathbb{P}\ \ \forall (i,j) \in dom\,(p)\ \left((\check{m}, p) \in \underset{\dot\sim}{x} \longrightarrow i < n\right).$$

Now, for every $\pi \in \mathscr{H}_n$, we have $\tilde{\pi}(\underset{\sim}{R_n}) = \underset{\sim}{R_n}$, hence $\mathscr{H}_n \subseteq sym_{\mathscr{G}_{aut.}}\left(\left(\underset{\sim}{R_n}\right)\right) \in \mathscr{F}$. Since every $\underset{\dot\sim}{x} \in dom\left(\underset{\sim}{R_n}\right)$ belongs to $\mathbf{HS}_{\mathscr{F}}$, it follows both that $\underset{\sim}{R_n} \in \mathbf{HS}_{\mathscr{F}}$ and $\left(\underset{\sim}{R_n}\right)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

We recall that *couple* : $\mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \rightarrow \mathbf{M}^{\mathbb{P}}$ was introduced in Example 300 so that given any $\tau, \sigma \in \mathbf{M}^{\mathbb{P}}$, and any $G$ $\mathbb{P}$-generic over $\mathbf{M}$ one has $\left(couple(\tau,\sigma)\right)_G = \left((\tau)_G, (\sigma)_G\right)$. This is

$$couple(\tau, \sigma) = \left\{\left(\{(\tau, \mathbb{1})\}, \mathbb{1}\right), \left(\{(\tau, \mathbb{1}), (\sigma, \mathbb{1})\}\mathbb{1}\right)\right\}$$

We set $R_n = \left(\underset{\sim}{R_n}\right)_G$ and define a $\mathbb{P}$-name for the function that maps $n$ to $R_n$:

$$\underset{\sim}{F} = \left\{\left(couple(\check{n}, \underset{\sim}{R_n}), \mathbb{1}\right) \mid n \in \omega\right\}$$

By construction, $\underset{\sim}{F} \in \mathbf{HS}_{\mathscr{F}}$, therefore, the function $(\underset{\sim}{F})_G$ belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

$$\begin{aligned} (\underset{\sim}{F})_G : \omega &\rightarrow \widehat{\mathbf{M}[G]}^{\mathscr{F}} \\ n &\mapsto \quad R_n \end{aligned}$$

[2] i.e., there exists some $r'' \in \mathscr{D} \cap G$.

Notice that every real that belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ belongs to some $R_n$, so that

$$\left(\mathcal{P}(\omega) = \bigcup \{R_n \mid n \in \omega\}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

So, it just remains to prove that, for each $n$, $\left(\underset{\sim}{R_n} \text{ is countable}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$.

(1) We first compute the size of $\underset{\sim}{R_n}$ inside the ground model. Since

$$\underset{\sim}{R_n} = \left\{(\underset{\approx}{x}, \mathbb{1}) \mid \text{“}\underset{\approx}{x} \text{ is a canonical } \mathbb{P}\text{-name for a real” and } \underset{\approx}{x} \in D_n\right\}$$

it is enough to count how many canonical $\mathbb{P}$-names of the form $\underset{\approx}{x}$ there are.

$$\underset{\approx}{x} = \left\{(\check{n}, q) \mid \exists(\underset{\sim}{k}, p) \in \underset{\approx}{x}\, \exists r \in \mathbb{P}\, \left(r \leqslant p \wedge q = r \upharpoonright_{n_{\underset{\approx}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}\right)\right\}.$$

By construction, if $(\check{n}, q) \in \underset{\approx}{x}$ and $(\underset{\approx}{x}, \mathbb{1}) \in \underset{\sim}{R_n}$, then $dom(q)$ is finite and $q : n \times \omega \to \aleph_n$. So, inside $\mathbf{M}$ there are at most $\aleph_n$ many such forcing conditions $q$ and $\aleph_0$ many canonical $\mathbb{P}_{\text{Levy}}$-names of the form $\check{n}$, which yields $2^{\aleph_n}$ many $\underset{\approx}{x}$. Since $\mathbf{M}$ satisfies $\forall k < \omega \;\; 2^{\aleph_k} = \aleph_{k+1}$, we obtain

$$\left(|\underset{\sim}{R_n}| \leqslant 2^{\aleph_n} = \aleph_{n+1}\right)^{\mathbf{M}}.$$

(2) We define, for each integer $n$, a $\mathbb{P}$-name for a function from $\omega$ onto $\left(\underset{\sim}{R_n}\right)_G$:

$$\underset{\sim}{f_n} = \left\{\left(couple(\check{k}, \check{\alpha}), p\right) \mid p \in \mathbb{P} \wedge dom(p) \subseteq (n+1) \times \omega \wedge p(n,k) = \alpha\right\}.$$

(a) By construction, $sym_{\mathscr{G}_{aut.}}\left(\underset{\sim}{f_n}\right) \supseteq \mathscr{H}_{n+1}$ and $couple(\check{k}, \check{\alpha}) \in \mathbf{HS}_{\mathscr{F}}$. Therefore, $\underset{\sim}{f_n} \in \mathbf{HS}_{\mathscr{F}}$ and $f_n = \left(\underset{\sim}{f_n}\right)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

(b) We show now that, inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, we have $\left(\underset{\sim}{f_n}\right)_G = f_n : \omega \xrightarrow{onto} (\aleph_n)^{\mathbf{M}}$:

$f_n$ **is a function from** $\boldsymbol{\omega}$ **to** $(\boldsymbol{\aleph_n})^{\mathbf{M}}$**:**

- $f_n \subseteq \omega \times (\aleph_n)^{\mathbf{M}}$ holds by construction.
- If both $(k, \alpha)$ and $(k, \beta)$ belong to $f_n$, then there exist $p_\alpha, p_\beta \in G$ with $p_\alpha(n,k) = \alpha$ and $p_\beta(n,k) = \beta$. Since both $p_\alpha$ and $p_\beta$ belong to $G$, they agree on their common domain, hence $p_\alpha(n,k) = p_\beta(n,k)$. i.e., $\alpha = \beta$.

$f_n$ **is onto:** Given any $\alpha \in (\aleph_n)^{\mathbf{M}}$, the set

$$\left\{p \in \mathbb{P} \mid dom(p) \subseteq (n+1) \times \omega \wedge \exists k \in \omega\, \left((n,k) \in dom(p) \wedge p(n,k) = \alpha\right)\right\}.$$

is dense which shows that there exists some integer $k$ such that $f_n(k) = \alpha$.

Inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, since $\omega \subseteq (\aleph_n)^{\mathbf{M}}$, we have $\omega \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}}$. Now, define $g_n : (\aleph_n)^{\mathbf{M}} \xrightarrow{1-1} \omega$ by

$$g_n(\alpha) = \bigcap \{k \in \omega \mid f_n(k) = \alpha\},$$

or, in other words, $g_n(\alpha)$ is the least integer $k$ such that $f_n(k) = \alpha$ (notice that such a $k$ exists because $f_n$ is onto). So, we have shown $(\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$, and by Cantor-Schröder-Bernstein Theorem (see page 57), $\omega \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$ yields $\omega \simeq (\aleph_n)^{\mathbf{M}}$.

It remains to show that $\left(R_n \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. For this purpose, we notice that since

$$\left(|\underset{\sim}{R_n}| \leqslant 2^{\aleph_n} = \aleph_{n+1}\right)^{\mathbf{M}},$$

there exists inside $\mathbf{M}$ some mapping

$$f'_n : dom\left(\underset{\sim}{R_n}\right) \xrightarrow{1-1} (\aleph_{n+1})^{\mathbf{M}}$$

which maps each $\underset{\approx}{x} \in dom\left(\underset{\sim}{R_n}\right)$ to some ordinal $\alpha \in (\aleph_{n+1})^{\mathbf{M}}$. We consider

$$\underset{\sim}{B} = \left\{\left(couple(\underset{\approx}{x}, \check{\alpha}), \mathbb{1}\right) \mid \alpha = f'_n(\underset{\approx}{x}).\right\}$$

We set $(\underset{\sim}{B})_G = B$. Since every $\underset{\approx}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$, we have $\underset{\sim}{B}$ also belongs to $\mathbf{HS}_{\mathscr{F}}$, hence $B \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ and

$$\begin{aligned} B &= \left\{\left(couple(\underset{\approx}{x}, \check{\alpha})\right)_G \mid \left(\left(\underset{\approx}{x}, \mathbb{1}\right) \in \underset{\sim}{R_n} \wedge f'_n(\underset{\approx}{x}) = \alpha\right)\right\} \\ &= \left\{\left(\left(\underset{\approx}{x}\right)_G, \alpha\right) \mid \left(\left(\underset{\approx}{x}, \mathbb{1}\right) \in \underset{\sim}{R_n} \wedge f'_n(\underset{\approx}{x}) = \alpha\right)\right\}. \end{aligned}$$

Inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, we define $h_n : R_n \xrightarrow{1-1} (\aleph_n)^{\mathbf{M}}$ by

$$h_n(x) = \bigcap \{\alpha \in (\aleph_n)^{\mathbf{M}} \mid (x, \alpha) \in B\}.$$

Since inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ we already have $(\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$, we have proved

$$R_n \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$$

hence $R_n \overset{1-1}{\lesssim} \omega$, which means

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{“}R_n \text{ is countable”}.$$

All in all, we have shown

$$\left(\mathcal{P}(\omega) = \bigcup\{R_n \mid n \in \omega\} \wedge \forall n \in \omega \left(R_n \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}} \wedge (\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \aleph_0\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$$

i.e.,

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \mathcal{P}(\omega) = \bigcup\{R_n \mid n \in \omega\} \wedge \forall n \in \omega \text{ “} R_n \text{ is countable”}.$$

which proves

$$\left(\text{“}\mathcal{P}(\omega) \text{ is a countable union of countable sets”}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

and since $\mathbb{R} \simeq \mathcal{P}(\omega)$ holds in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ because $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ satisfies “**ZF**” and **ZF** $\vdash_c \mathbb{R} \simeq \mathcal{P}(\omega)$, we have also proved

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{ “}\mathbb{R} \text{ is a countable union of countable sets”}.$$

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Among the major consequences of this result is Proposition 355 which states that if $\mathbb{R}$ is a countable union of countable sets, then

$$\omega_1 \overset{1-1}{\not\lesssim} \mathbb{R}.$$

Notice that this result holds in a model which satisfies “**ZF**”, hence in which there is no bijection between $\omega$ and $\mathbb{R}$. In other words, the real numbers are uncountable and the model knows it, but there is no injection from the least uncountable ordinal to the real numbers.

Another disturbing result which is a consequence of the real numbers being a countable union of countable sets is Corollary 356 which states that if $\mathbb{R}$ is a countable union of countable sets, then there exists some partition $\mathcal{R}$ of $\mathbb{R}$ together with an injection from $\mathbb{R}$ to $\mathcal{R}$ (showing that this partition is extremely fine) but somehow, no injection from the partition to the real numbers:

$$\mathbb{R} \lnsim \mathcal{R}.$$

Such a result of course, highly contradicts the axiom of choice since

(1) inside a world where **AC** holds, one could precisely make use of this axiom to pick from every element of the partition which is non-empty, some element to then form a 1-1 mapping from $\mathcal{R}$ to $\mathbb{R}$.

(2) Also, getting used of working with the axiom of choice at hand, our initial reaction is to understand $\mathbb{R} \lnsim \mathcal{R}$ as saying the the set of all the real numbers is strictly smaller than it partition, which seems extremely bizarre.

## 22.2 Forcing the Well-Orderings of the Reals Out

In this section we show that it is consistent with **ZF** that there exists no well-ordering of the reals. Notice that this implies that there is no bijection between any ordinal and the set of all reals, and even that there is no injection of the reals into the class of the ordinals.

**Theorem 364** (Cohen)**.**

$$cons\big(\mathbf{ZF}\big) \Longrightarrow cons\big(\mathbf{ZF} + \text{“}\textit{there is no well-ordering of } \mathbb{R}\text{”}\big).$$

*Proof of Theorem* 364: To do so, we prove that given $\mathbf{M}$ any *c.t.m.* of "**ZFC**" with $\mathbb{P}_{\aleph_0} \in \mathbf{M}$, if $G$ is $\mathbb{P}_{\aleph_0}$-generic over $\mathbf{M}$, then there exists some countable set of reals[3] $A \in \mathbf{M}[G]$ and a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, such that $\underline{\text{inside } \widehat{\mathbf{M}[G]}^{\mathscr{F}}}$: $A$ still exists, remains infinite, but contains no proper subset $B \subsetneq A$ such that $B \simeq A$.
This will clearly give the result because if There exists a well-ordering of the reals, then every subset of the reals can also be well-ordered. So, in particular there would exists a well ordering $(A, <_A)$ whose order-type would be some infinite ordinal. From there, it would be an easy exercise to design a proper subset $B \subsetneq A$ such that $B \simeq A$.

We force with $\mathbb{P} = \big(\mathbb{P}_{\aleph_0}, \leqslant, \mathbb{1}\big)$ where

$$\mathbb{P}_{\aleph_0} = \Big\{f : \omega \times \omega \longrightarrow \{0,1\} \mid dom(f) \ \textit{ is finite}\Big\}; \quad f \leqslant g \iff f \supseteq g; \quad \mathbb{1} = \varnothing.$$

Given any $G$ $\mathbb{P}$-generic over $\mathbf{M}$, we have $\bigcup G = \mathcal{F} \in \mathbf{M}[G]$ satisfies

$$\mathcal{F} : \omega \times \omega \to \{0,1\}.$$

For each integer $k$, we set

$$\underset{\sim}{a_k} = \big\{(\check{n}, p) \in dom(\check{\omega}) \times \mathbb{P} \mid p(k,n) = 1\big\} \qquad \text{and} \qquad \underset{\sim}{A} = \big\{(\underset{\sim}{a_k}, \mathbb{1}) \mid k \in \omega\big\}.$$

We let $\big(\underset{\sim}{a_k}\big)_G = a_k$ and $(\underset{\sim}{A})_G = A$, so that we have

$$a_k = \big\{n < \omega \mid \mathcal{F}(k,n) = 1\big\} \qquad \text{and} \qquad A = \big\{a_k \mid k \in \omega\big\}.$$

Since for all integers $l, m, n$ the following sets $D_{n,l}$ and $E_{n,m}$ are dense in $\mathbb{P}$:

$$D_{n,l} = \big\{p \in \mathbb{P} \mid \exists k > l \ \ p(n,k) = 1\big\}$$

and

$$E_{n,m} = \big\{p \in \mathbb{P} \mid \exists k \leqslant \omega \ \ p(n,k) \neq p(m,k)\big\}.$$

Using the notation $[\omega]^\omega$ for the set of infinite subsets of $\omega$, it follows that

[3] Here, reals stand for subsets of integers.

- $\left(a_n \in [\omega]^\omega\right)^{\mathbf{M}[G]}$,
- $A \in \mathbf{M}[G] \setminus \mathbf{M}$, and
- $\Big(\forall n \in \omega \left(a_n \in A \wedge \forall m \in \omega \; (n \neq m \longleftrightarrow a_n \neq a_m)\right)\Big)^{\mathbf{M}[G]}$.

Therefore, $A$ is infinite.

We then construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ which still contains the infinite set $A$ but no injection from $\omega$ to $A$.

Every permutation of the integers $\rho : \omega \overset{bij.}{\longleftrightarrow} \omega$ induces an automorphism $\pi_\rho : \mathbb{P} \overset{aut.}{\longleftrightarrow} \mathbb{P}$ defined by

$$\pi_\rho(p) = \Big\{ \Big(\big(\rho(n), m\big), i\Big) \subseteq (\omega \times \omega) \times \omega \mid \big((n,m), i\big) \in p \Big\}.$$

We consider the group of such automorphisms

$$\mathscr{G} = \left\{\pi_\rho \mid \rho : \omega \overset{bij.}{\longleftrightarrow} \omega\right\}$$

and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ the filter generated by

$$\left\{\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{\mathit{fin}}(\omega)\right\}$$

where

$$\mathit{fix}_{\mathscr{G}}(F) = \left\{\pi_\rho \in \mathscr{G} \mid \forall n \in F \;\; \rho(n) = n\right\}.$$

$\mathscr{F}$ is a normal filter on $\mathcal{G}$ since $\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$:

(1) $\mathscr{G} \in \mathscr{F}$ because $\mathscr{G} = \mathit{fix}_{\mathscr{G}}(\varnothing)$.

(2) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H} \subseteq \mathcal{K}$ holds for some finite $F \subseteq \omega$, which shows $\mathcal{K} \in \mathscr{F}$.

(3) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then both $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H}$ and $\mathit{fix}_{\mathscr{G}}(E) \subseteq \mathcal{K}$ hold for finite $E, F \subseteq \omega$. Thus, $\mathit{fix}_{\mathscr{G}}(E \cup F) \subseteq \mathcal{H} \cap \mathcal{K} \in \mathscr{F}$ holds which shows that $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.

(4) If $\mathcal{H} \in \mathscr{F}$, then given any finite $F \subseteq \omega$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H}$, one has $\pi_\rho \circ \mathit{fix}_{\mathscr{G}}(F) \circ \pi_\rho^{-1} = \mathit{fix}_{\mathscr{G}}(\rho[F])$; so that $\mathit{fix}_{\mathscr{G}}(\rho[F]) \subseteq \pi_\rho \circ \mathcal{H} \circ \pi_\rho^{-1}$. Thus, $\pi_\rho \circ \mathcal{H} \circ \pi_\rho^{-1} \in \mathscr{F}$.

So, we can define $\mathbf{HS}_{\mathscr{F}}$ as the class of all hereditarily symmetric $\mathbb{P}$-names, and $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ as the symmetric submodel of the generic extension $\mathbf{M}[G]$ induced by $\mathbf{HS}_{\mathscr{F}}$.

Notice that for each integer $k$ and each $\pi_\rho \in \mathscr{G}$, we have

$$\begin{aligned}\tilde{\pi}_\rho(\underset{\sim}{a}_k) &= \Big\{\big(\tilde{\pi}_\rho(\check{n}), \pi_\rho(p)\big) \in dom(\check{\omega}) \times \mathbb{P} \mid p(k,n) = 1\Big\} \\ &= \Big\{\big(\check{n}, \pi_\rho(p)\big) \in dom(\check{\omega}) \times \mathbb{P} \mid \pi_\rho(p)\big(\rho(k), n\big) = 1\Big\} \\ &= \Big\{\big(\check{n}, q\big) \in dom(\check{\omega}) \times \mathbb{P} \mid q\big(\rho(k), n\big) = 1\Big\} \\ &= \underset{\sim}{a}_{\rho(k)}.\end{aligned}$$

We see that for every permutation $\rho$ such that $\rho(k) = k$, we have $\tilde{\pi}_\rho(\underset{\sim}{a}_k) = \underset{\sim}{a}_{\rho(k)} = \underset{\sim}{a}_k$. Therefore, $fix_{\mathscr{G}}(\{k\}) \subseteq sym_{\mathscr{G}}(\underset{\sim}{a}_k)$. And since each element of $dom(\underset{\sim}{a}_k)$ which is of the form $\check{n}$ is in $\mathbf{HS}_{\mathscr{F}}$, we have $\underset{\sim}{a}_k \in \mathbf{HS}_{\mathscr{F}}$, hence $a_k \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$. Moreover, for each $\pi_\rho \in \mathscr{G}$, we have

$$\begin{aligned}\tilde{\pi}_\rho(\underset{\sim}{A}) &= \Big\{\big(\tilde{\pi}_\rho(\underset{\sim}{a}_k), \pi_\rho(\mathbb{1})\big) \mid k \in \omega\Big\} \\ &= \Big\{\big(\underset{\sim}{a}_{\rho(k)}, \mathbb{1}\big) \mid k \in \omega\Big\} \\ &= \underset{\sim}{A}.\end{aligned}$$

Which shows that $\mathscr{G} \subseteq sym_{\mathscr{G}}(\underset{\sim}{A})$, hence $\underset{\sim}{A} \in \mathbf{HS}_{\mathscr{F}}$ and $A \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

We already have $\big(a_n \in [\omega]^\omega\big)^{\mathbf{M}[G]}$ and

$$\Big(\forall n \in \omega\ \big(a_n \in A \wedge \forall m \in \omega\ (n \neq m \longleftrightarrow a_n \neq a_m)\big)\Big)^{\mathbf{M}[G]}.$$

The set $[\omega]^\omega$ belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since it belongs to $M$. So, we have for each integer $n$, $\big(a_n \in [\omega]^\omega\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ and $\big(\text{“}A \text{ is infinite”}\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. We show that $\big(\text{“}A \text{ is Dedekind-finite”}\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ — see Definition 342.

**Claim 365.**

$$\big(\textit{“There is no 1-1 mapping from } \omega \textit{ to } A\textit{”}\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

*Proof of Claim* 365: Towards a contradiction, we assume that there exists in $\mathbf{M}$ an hereditarily symmetric name $\underset{\sim}{f} \in \mathbf{HS}_{\mathscr{F}}$ for some mapping that exists inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, namely,

$$f = \big(\underset{\sim}{f}\big)_G : \omega \xrightarrow{1-1} A.$$

So, there exists $p \in G$ such that

$$p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f} : \check{\omega} \xrightarrow{1-1} \underset{\sim}{A}.$$

Since $\underset{\sim}{f} \in \mathbf{HS}_{\mathscr{F}}$, we have $sym_{\mathscr{G}}\left(\underset{\sim}{f}\right) \in \mathscr{F}$, hence there exists some finite set $F_{\underset{\sim}{f}} \subsetneq \omega$ such that $fix_{\mathscr{G}}(F_{\underset{\sim}{f}}) \subseteq sym_{\mathscr{G}}\left(\underset{\sim}{f}\right)$. Then, $f$ being injective, there exist $n_f \in \omega \setminus F_{\underset{\sim}{f}}$ and $k \in \omega$ such that $f(k) = a_{n_f}$. So, by the Truth Lemma, there also exists $p_f \in G$ with $p_f \leqslant p$ and

$$p_f \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{k}) = \underset{\sim}{a}_{n_f}.$$

We consider any permutation $\rho : \omega \overset{bij.}{\longleftrightarrow} \omega$ such that $\pi_\rho \in fix_{\mathscr{G}}(F_{\underset{\sim}{f}})$, $\rho(n_f) \neq n_f$ and there exists $q \leqslant \pi_\rho(p_f), p_f$ — i.e., $\pi_\rho(p_f)$ and $p_f$ are compatible.

From $\pi_\rho \in fix_{\mathscr{G}}(F_{\underset{\sim}{f}})$, we obtain $\pi_\rho \in sym_{\mathscr{G}}\left(\underset{\sim}{f}\right)$, hence $\tilde{\pi}_\rho(\underset{\sim}{f}) = \underset{\sim}{f}$. So, we have

$$\pi_\rho(p_f) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_\rho(\underset{\sim}{f})\big(\tilde{\pi}_\rho(\check{k})\big) = \tilde{\pi}_\rho(\underset{\sim}{a}_{n_f})$$

i.e.,

$$\pi_\rho(p_f) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\left(\check{k}\right) = \underset{\sim}{a}_{\rho(n_f)}.$$

Any $q \in \mathbb{P}$ which satisfies both $q \leqslant \pi_\rho(p_f)$ and $q \leqslant p_f$ yields both

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{k}) = \underset{\sim}{a}_{n_f}$$

and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\left(\check{k}\right) = \underset{\sim}{a}_{\rho(n_f)},$$

hence

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a}_{n_f} = \underset{\sim}{a}_{\rho(n_f)}.$$

Now, for every filter $H$ $\mathbb{P}$-generic over $\boldsymbol{\mathcal{M}}$ that contains $q$ we would get

$$\left(a_{n_f} = a_{\rho(n_f)}\right)^{\mathbf{M}[H]}$$

but since $n_f \neq \rho(n_f)$, this would contradict

$$\Big(\forall n \in \omega\, \forall m \in \omega\ (n \neq m \longrightarrow a_n \neq a_m)\Big)^{\mathbf{M}[H]}.$$

This proves that there is no hereditarily symmetric $\mathbb{P}$-name for an injection from $\omega$ to $A$.

$\square$ 365

So, we have shown that there is no hereditarily symmetric name for an injection from $\omega$ to the infinite set $A$. This result implies that there is no well-ordering of $\mathbb{R}$, since any well-ordering of the reals would yield some bijection $f : \alpha \overset{\text{bij.}}{\longleftrightarrow} \mathbb{R}$ which would yield an injection $g : \omega \xrightarrow{1-1} A$ defined by recursion on the integers by

$$\begin{aligned} g(n) = f(\beta) \text{ where } \beta &= \min\left\{\xi \in \alpha \mid f(\xi) \in A \setminus \{g(i) \mid i < n\}\right\} \\ &= \min\Big\{\xi \in \alpha \mid \big(f(\xi) \in A \wedge \forall i < n\ f(\xi) \neq g(i)\big)\Big\}. \end{aligned}$$

$\square$ 364

## 22.3 Forcing Every Ultrafilter on $\omega$ is Principal

**Definition 366.** *Let $X$ be any non-empty set.*

- *An ultrafilter $\mathscr{U}$ on $X$ is any non-empty set $\mathscr{U} \subseteq \mathscr{P}(X)$ which satisfies*

$$\left.\begin{array}{l} (1)\ \varnothing \notin \mathscr{U} \\ (2)\ \text{if } A, B \in \mathscr{U}, \text{ then } A \cap B \in \mathscr{U} \\ (3)\ \text{if } A \in \mathscr{U} \text{ and } A \subseteq B, \text{ then } B \in \mathscr{U} \end{array}\right\} \text{Filter}$$

$$(4)\ \text{for all } A \subseteq X,\ A \in \mathscr{U} \text{ or } A^{\complement} \in \mathscr{U} \quad \} \text{ Ultra}$$

- *An ultrafilter $\mathscr{U}$ on $X$ is principal (or trivial) if there exists some $A \subseteq X$ such that*

$$\mathscr{U} = \{B \subseteq X \mid A \subseteq B\}.$$

- *An ultrafilter $\mathscr{U}$ on $X$ is free if it is non-principal*

An ultrafilter is trivial if and only if it contains some $\subseteq$-least element. Every filter $\mathscr{F} \subseteq \mathscr{P}(X)$ which contains some $\subseteq$-least element $A$ can trivially be extended into an ultrafilter, namely

$$\begin{aligned} \mathscr{U} &= \{B \subseteq X \mid \exists C \in \mathscr{F}\ C \subseteq B\} \\ &= \{B \subseteq X \mid A \subseteq B\} \end{aligned}$$

This question is far more involved with non-trivial filters. With the axiom of choice, of course, every filter can be extended into an ultrafilter. But the converse is not necessarily true. Even for the Fréchet Filter — $\mathscr{F}_{réchet} = \{A \subseteq \omega \mid \omega \smallsetminus A \text{ is finite}\}$ — as shown by the next result, it is consistent with **ZF** that it cannot be extended by any ultrafilter.

**Theorem 367** (Feferman)**.**

$$cons(\mathbf{ZF}) \Longrightarrow cons(\mathbf{ZF} + \text{“}every\ ultrafilter\ on\ \omega\ is\ trivial\text{”}).$$

*Proof of Theorem 367*: As with the proof of Theorem 364, we start with **M** any *c.t.m.* of "**ZFC**" and force with $\mathbb{P} = (\mathbb{P}_{\aleph_0}, \leqslant, \mathbb{1})$ where

$$\mathbb{P}_{\aleph_0} = \Big\{f : \omega \times \omega \longrightarrow \{0,1\} \mid dom(f) \text{ is finite}\Big\}; \quad f \leqslant g \iff f \supseteq g; \quad \mathbb{1} = \varnothing.$$

Given any $G$ which is $\mathbb{P}$-generic over $\mathbf{M}$, we have $\bigcup G = \mathcal{F} \in \mathbf{M}[G]$ satisfies

$$\mathcal{F} : \omega \times \omega \to \{0, 1\}.$$

For each integer $k$, we set

$$\underset{\sim}{a_k} = \big\{(\check{n}, p) \in dom\,(\check{\omega}) \times \mathbb{P} \mid p(k,n) = 1\big\}.$$

We let $\Big(\underset{\sim}{a_k}\Big)_G = a_n$, so that we have

$$a_k = \big\{n < \omega \mid \mathcal{F}(k,n) = 1\big\}.$$

Since for all integers $l, m, n$ the sets $D_{n,l}$ and $E_{n,m}$ below are dense in $\mathbb{P}$:

$$D_{n,l} = \big\{p \in \mathbb{P} \mid \exists k > l \;\; p(n,k) = 1\big\}$$

and

$$E_{n,m} = \big\{p \in \mathbb{P} \mid \exists k \leqslant \omega \;\; p(n,k) \neq p(m,k)\big\}$$

it follows that

$$\Big(a_n \in [\omega]^\omega \;\wedge\; \forall n \in \omega\, \forall m \in \omega \;(n \neq m \longleftrightarrow a_n \neq a_m)\Big)^{\mathbf{M}[G]}.$$

We construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ by considering, for each $S \subseteq \omega\times\omega$, an automorphism $\pi_S : \mathbb{P} \xrightarrow{aut.} \mathbb{P}$ defined for each $p \in \mathbb{P}$ by:

$$\begin{aligned} \pi_S(p) : dom\,(p) &\longrightarrow 2 \\ (n,m) &\mapsto \begin{cases} 1 - p(n,m) & \text{if} \;\; (n,m) \in S \\ p(n,m) & \text{if} \;\; (n,m) \notin S. \end{cases} \end{aligned}$$

We let $\mathscr{G}$ be the group of all such automorphisms and given any $F \in \mathcal{P}_{fin}(\omega)$,

$$fix_{\mathscr{G}}(F \times \omega) = \big\{\pi_S \in \mathscr{G} \mid S \cap (F \times \omega) = \varnothing\big\},$$

and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the filter generated by

$$\big\{fix_{\mathscr{G}}(F \times \omega) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\omega)\big\}.$$

We verify that $\mathscr{F}$ is a normal filter on $\mathcal{G}$.
$\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$:

(1) $\mathscr{G} \in \mathscr{F}$ because $\mathscr{G} = fix_{\mathscr{G}}(\varnothing) = fix_{\mathscr{G}}(\varnothing \times \omega)$

(2) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $fix_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H} \subseteq \mathcal{K}$ holds for some finite $F \subseteq \omega$, which shows $\mathcal{K} \in \mathscr{F}$

(3) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then both $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H}$ and $\mathit{fix}_{\mathscr{G}}(E \times \omega) \subseteq \mathcal{K}$ hold for finite $E, F \subseteq \omega$. Thus, $\mathit{fix}_{\mathscr{G}}\big((E \cup F) \times \omega\big) \subseteq \mathcal{H} \cap \mathcal{K} \in \mathscr{F}$ holds which shows that $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.

(4) if $\mathcal{H} \in \mathscr{F}$, then given any finite $F \subseteq \omega$ such that $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H}$, one has

$$\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \pi_S \circ \mathit{fix}_{\mathscr{G}}(F \times \omega) \circ \pi_S^{-1},$$

thus, $\pi_S \circ \mathcal{H} \circ \pi_S^{-1} \in \mathscr{F}$.

So, we can define $\mathbf{HS}_{\mathscr{F}}$ as the class of all hereditarily symmetric $\mathbb{P}$-names, and $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ as the symmetric submodel of the generic extension $\mathbf{M}[G]$ induced by $\mathbf{HS}_{\mathscr{F}}$.

We let $\mathscr{U}$ be any ultrafilter in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, together with $\underset{\sim}{\mathscr{U}} \in \mathbf{HS}_{\mathscr{F}}$ any $\mathbb{P}$-name for $\mathscr{U}$, and any $p \in G$ with

$$p \Vdash_{\mathbb{P},\mathbf{M}} \text{“}\underset{\sim}{\mathscr{U}} \text{ is an ultrafilter over } \check{\omega}\text{”}.$$

We take any finite $F \in \mathcal{P}_{\mathit{fin}}(\omega)$ such that $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{\mathscr{U}})$ as well as any integer $k \notin F$. We distinguish between $a_k \in \mathscr{U}$ and $a_k \notin \mathscr{U}$.

If $a_k \in \mathscr{U}$: we pick any $q \in G$ such that $q \leqslant p$ and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \in \underset{\sim}{\mathscr{U}}.$$

we consider any $k' \in \omega$ large enough such that

$$\big\{(k,n) \in \omega \times \omega \mid n \geqslant k'\big\} \cap \mathit{dom}(q) = \varnothing$$

We notice that $S = \big\{(k,n) \in \omega \times \omega \mid n \geqslant k'\big\}$ satisfies $S \cap (F \times \omega) = \varnothing$ and form $\pi_S$ and consider $\underset{\sim}{b_k} = \tilde{\pi}_S(\underset{\sim}{a_k})$ and write $b_k$ for $(\underset{\sim}{b_k})_G$. By construction, we see that for each integer $n \geqslant k'$ we have

$$n \in a_k \iff n \notin b_k$$

which yields

$$a_k \cap b_k \subseteq k'$$

which shows that this set is finite. Building on $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \in \underset{\sim}{\mathscr{U}}$, we reach

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_S(\underset{\sim}{a_k}) \in \tilde{\pi}_S(\underset{\sim}{\mathscr{U}}).$$

Since $\mathit{fix}_{\mathscr{G}}(F \times \omega) = \big\{\pi_{S'} \in \mathscr{G} \mid S' \cap (F \times \omega) = \varnothing\big\}$ and $S \cap (F \times \omega) = \varnothing$, we have

$$\pi_S \in \mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{\mathscr{U}}),$$

which gives

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b_k} \in \underset{\sim}{\mathscr{U}};$$

and since $S \cap dom(q) = \varnothing$, we have $\pi_S(q) = q$, so that we finally obtain

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b_k} \in \underset{\sim}{\mathscr{U}}.$$

So, we end up with both $a_k \in \mathscr{U}$ and $b_k \in \mathscr{U}$, hence $a_k \cap b_k \in \mathscr{U}$. Since $a_k \cap b_k \subseteq k'$, we obtain

$$\mathscr{U} = \{X \subseteq \omega \mid c \subseteq X\}$$

where $c$ is the finite set defined by

$$c = \bigcap \{Y \in \mathscr{U} \mid Y \subseteq a_k \cap b_k\}.$$

Thus, $\mathscr{U}$ is principal.

If $a_k \notin \mathscr{U}$: we pick any $q \in G$ such that $q \leqslant p$ and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \notin \underset{\sim}{\mathscr{U}}.$$

we consider any $k' \in \omega$ large enough such that

$$\{(k,n) \in \omega \times \omega \mid n \geqslant k'\} \cap dom(q) = \varnothing$$

We notice that $S = \{(k,n) \in \omega \times \omega \mid n \geqslant k'\}$ satisfies $S \cap (F \times \omega) = \varnothing$ and form $\pi_S$ and consider $\underset{\sim}{b_k} = \tilde{\pi}_S(\underset{\sim}{a_k})$ and write $b_k$ for $(\underset{\sim}{b_k})_G$. By construction, we see that for each integer $n \geqslant k'$ we have

$$n \in a_k \iff n \notin b_k$$

which yields

$$(\omega \smallsetminus a_k) \cap (\omega \smallsetminus b_k) = \{n \in \omega \mid n \notin a_k \wedge n \notin b_k\} \subseteq k',$$

hence this set is finite. From $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \notin \underset{\sim}{\mathscr{U}}$, we get

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_S(\underset{\sim}{a_k}) \notin \tilde{\pi}_S(\underset{\sim}{\mathscr{U}}).$$

Since $fix_{\mathscr{G}}(F \times \omega) = \{\pi_{S'} \notin \mathscr{G} \mid S' \cap (F \times \omega) = \varnothing\}$ and $S \cap (F \times \omega) = \varnothing$, we have

$$\pi_S \in fix_{\mathscr{G}}(F \times \omega) \subseteq sym_{\mathscr{G}}(\underset{\sim}{\mathscr{U}}),$$

which gives

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b_k} \notin \underset{\sim}{\mathscr{U}};$$

and since $S \cap dom(q) = \varnothing$, we have $\pi_S(q) = q$, so that we finally obtain

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b_k} \notin \underset{\sim}{\mathscr{U}}.$$

So, we end up with both $a_k \notin \mathscr{U}$ and $b_k \notin \mathscr{U}$, which gives $(\omega \setminus a_k) \in \mathscr{U}$ and $(\omega \setminus b_k) \in \mathscr{U}$ and finally $(\omega \setminus a_k) \cap (\omega \setminus b_k) \in \mathscr{U}$. Now, since $(\omega \setminus a_k) \cap (\omega \setminus b_k) \subseteq k'$, we obtain

$$\mathscr{U} = \{X \subseteq \omega \mid c \subseteq X\}$$

where $c$ is the finite set defined by

$$c = \bigcap \{Y \in \mathscr{U} \mid Y \subseteq (\omega \setminus a_k) \cap (\omega \setminus b_k)\}.$$

Thus, $\mathscr{U}$ is principal.

$\square$ 367

We have constructed a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ in which there is no free ultrafilter on $\omega$ because every ultrafilter on $\omega$ is principal. So, in particular, the Fréchet filter — which belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ because it belongs to $\mathbf{M}$ and $\mathbf{M} \subseteq \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ — cannot be extended into any ultrafilter.

# Chapter 23

# Set Theory with Atoms

Set theory with atoms — denoted **ZFA** — is slightly different from **ZF** in that it not only contains the empty set as a set that does not contain any other set, but also some other sets that bear the same property. These other basic elements are known as **atoms** or as **Urelements** (usually written **urelements**).

## 23.1 Zermelo-Fraenkel with Atoms (ZFA)

The language of **ZFA** is the same as the language of **ZF** augmented with a constant symbol $\mathcal{A}$ whose interpretation is a "set of atoms" denoted by $\mathbb{A}$. Since atoms do not contain any element, the axiom of extensionality is modified and a few other axioms are added to **ZF**.

**Empty Set Existence for ZFA**

$$\exists x(\forall y\ y \notin x \wedge x \notin \mathcal{A}).$$

This axiom claims that the empty set exists and is different from any atom.

**Extensionality for ZFA**

$$\forall x\, \forall y\Big((x \notin \mathcal{A} \wedge y \notin \mathcal{A}) \longrightarrow \big(\forall z\ (z \in x \longleftrightarrow z \in y) \longrightarrow x = y\big)\Big).$$

This axiom claims that all sets that are not atoms are the same if and only if they contain the same elements. Notice that the Axiom of **Extensionality for ZFA** implies that the empty set is unique, which guarantees the use of the usual constant symbol $\varnothing$.

**Axiom of Atoms**

$$\forall x\ \big(x \in \mathcal{A}\ \longleftrightarrow\ (x \neq \varnothing \wedge \forall z\ z \notin x)\big).$$

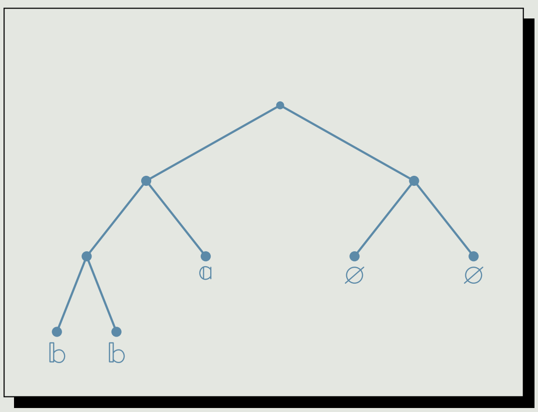
This axiom claims that, apart from the empty set — which is not an atom — the atoms are the only sets which do not contain any other set. By convention, we assume also that the

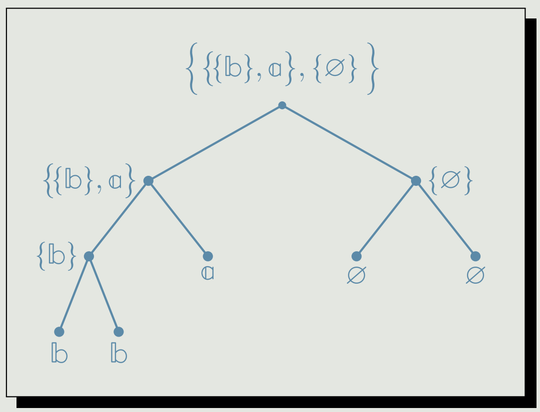
**Comprehension Schema** is strengthened in order to have the empty subset of any set to be the empty set as opposed to some atom. For instance,

$$\forall w_1 ... \forall w_n \, \forall x \, \forall z \, \Big( \big( z \in x \longrightarrow \neg \varphi_{(x,z,w_1,\ldots,w_n)} \big) \longrightarrow \big\{ z \in x \mid \varphi_{(x,z,w_1,\ldots,w_n)} \big\} = \varnothing \Big)$$

The whole theory of **ZFA** is developed the same way the theory of **ZF** is — in particular ordinals are constructed from the empty set and not from atoms. The tree representation of a set — as a well-founded tree since we work with **Foundation** — still holds, except that with **ZF** the leaves correspond to the empty set, whereas with **ZFA**, these leaves can also represent atoms and not just the empty set.

**Example 368.** *We assume* $\mathbb{a}, \mathbb{b} \in \mathbb{A}$. *Below is a tree that represents the set* $\Big\{ \{\{\mathbb{b}\}, \mathbb{a}\}, \{\varnothing\} \Big\}$.




**Definition 369** (**ZFA**)**.** *Given any set $S$, we define $\boldsymbol{\mathcal{P}}^{\infty}(S)$ by transfinite recursion.*

- $\boldsymbol{\mathcal{P}}^{0}(S) = S$
- $\boldsymbol{\mathcal{P}}^{\alpha+1}(S) = \boldsymbol{\mathcal{P}}^{\alpha}(S) \cup \mathscr{P}\big(\boldsymbol{\mathcal{P}}^{\alpha}(S)\big)$
- $\boldsymbol{\mathcal{P}}^{\alpha}(S) = \bigcup_{\xi<\alpha} \boldsymbol{\mathcal{P}}^{\xi}(S)$ *(when $\alpha$ is limit).*

$$\boldsymbol{\mathcal{P}}^{\infty}(S) = \bigcup_{\alpha \in \mathbf{On}} \boldsymbol{\mathcal{P}}^{\xi}(S) \, .$$

If $\mathbb{A}$ is any set of atoms, and $\boldsymbol{\mathcal{M}}$ is any model of **ZFA**, then we have

$$\forall x \ \big(x \in \boldsymbol{\mathcal{M}} \longleftrightarrow x \in \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})\big).$$

Whereas the subclass $\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)$ — known as the **kernel** — is the domain of a model of **ZF**.

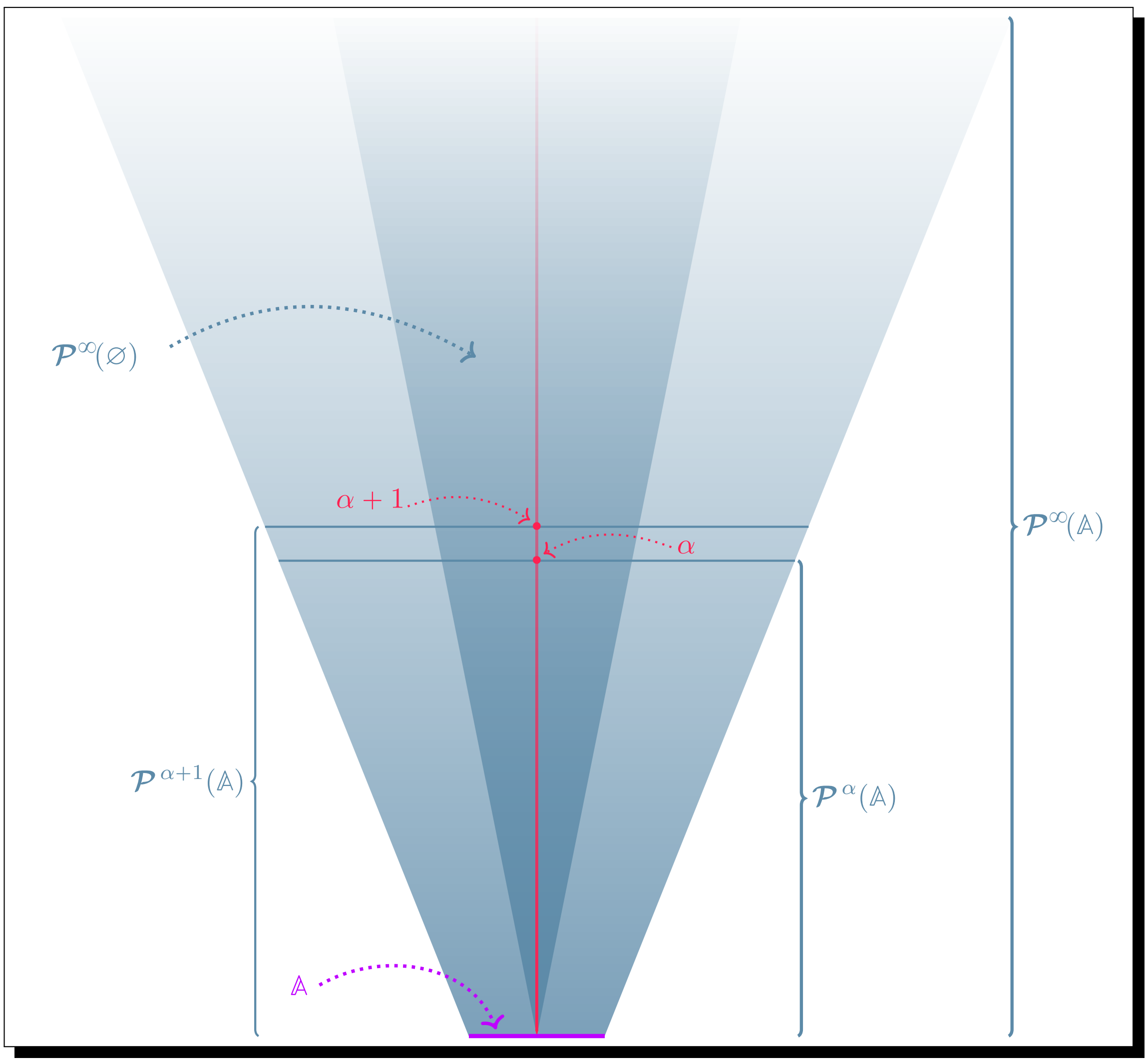



Figure 23.1: The Classes $\boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$ and $\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)$.

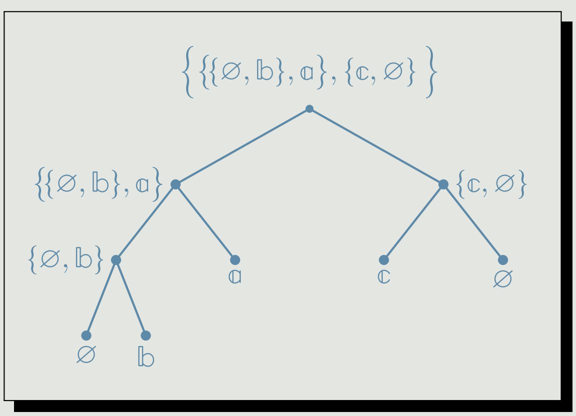
## 23.2 Permutation Models

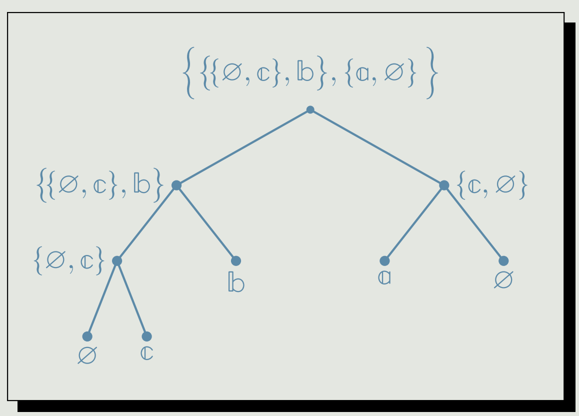
**Definition 370 (ZFA).** *Given any permutation* $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$*, the functional* $\breve{\pi} : \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) \to \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$ *is defined for every set* $x$ *by*

- *if* $x = \varnothing$*, then* $\breve{\pi}(x) = \breve{\pi}(\varnothing) = \varnothing$
- *if* $x \in \mathbb{A}$*, then* $\breve{\pi}(x) = \pi(x)$
- *if* $x \notin \mathbb{A} \cup \{\varnothing\}$*, then* $\breve{\pi}(x) = \{\breve{\pi}(y) \mid y \in x\}$.

**Example 371.** *We assume* $\mathbb{A} = \{\mathbb{a}, \mathbb{b}, \mathbb{c}\}$*. From the tree that represents the set* $x = \Big\{\{\{\varnothing, \mathbb{b}\}, \mathbb{a}\}, \{\mathbb{c}, \varnothing\}\Big\}$ *(left picture) we obtain* $\breve{\pi}(x) = \Big\{\{\{\varnothing, \mathbb{c}\}, \mathbb{b}\}, \{\mathbb{a}, \varnothing\}\Big\}$ *(right picture) by applying the following permutation:*

$$\begin{array}{rccc} \pi : & \mathbb{A} & \xleftrightarrow{bij.} & \mathbb{A} \\ & \mathbb{a} & \longmapsto & \mathbb{b} \\ & \mathbb{b} & \longmapsto & \mathbb{c} \\ & \mathbb{c} & \longmapsto & \mathbb{a} \end{array}$$




**Definition 372.** *Given any model* $\boldsymbol{\mathcal{M}}$*, a functional* $\boldsymbol{\mathcal{F}} : \boldsymbol{\mathcal{M}} \xrightarrow{bij.} \boldsymbol{\mathcal{M}}$ *is an* $\in$*-automorphism if it satisfies*

$$\forall x \in \boldsymbol{\mathcal{M}}\, \forall y \in \boldsymbol{\mathcal{M}}\, \Big(x \in y \longleftrightarrow \boldsymbol{\mathcal{F}}(x) \in \boldsymbol{\mathcal{F}}(y)\Big).$$

**Remark 373.** Given any set of atoms $\mathbb{A}$, any permutation $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$ yields a functional $\breve{\pi} : \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) \to \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$ which satisfies

(1) $\breve{\pi}$ is 1-1 and onto, and $\breve{\pi}^{-1} = \breve{\pi^{-1}}$;

(2) $\forall x \in \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})\ \forall y \in \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})\ \Big(x \in y \longleftrightarrow \breve{\pi}(x) \in \breve{\pi}(y)\Big)$.

*Proof of Remark 373*:

(1) (a) $\breve{\pi}$ is 1-1: if $x \neq y$, then by symmetry, there exists $z \in x \smallsetminus y$, thence

$$\left.\begin{array}{c} z \in x \Longrightarrow \breve{\pi}(z) \in \breve{\pi}(x) \\ \text{and} \\ z \notin y \Longrightarrow \breve{\pi}(z) \notin \breve{\pi}(y) \end{array}\right\} \Longrightarrow \breve{\pi}(z) \in \breve{\pi}(x) \smallsetminus \breve{\pi}(y)$$

(b) $\breve{\pi}$ is onto: towards a contradiction assume that for some minimal ordinal $\alpha$ there exists some $y \in \boldsymbol{\mathcal{P}}^{\alpha+1}(\mathbb{A})$ such that $\breve{\pi}(x) \neq y$ holds for all $x \in \boldsymbol{\mathcal{M}}$. By minimality of $\alpha$, every element of $y$ is in the range or $\breve{\pi}$, hence there exists $S \in \boldsymbol{\mathcal{M}}$ such that $y = \big\{\breve{\pi}(x) \mid x \in S\big\}$, which yields $y = \breve{\pi}(S)$, a contradiction.

(c) is immediate by induction on $rk_{\boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})}(x) =$ least ordinal $\alpha$ such that $x \in \boldsymbol{\mathcal{P}}^{\alpha+1}(\mathbb{A})$.

(2) is immediate.

$\square$ 373

So, any permutation $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$ yields an $\in$-automorphism $\breve{\pi} : \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) \to \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$. We recall that a group $\mathscr{G}$ of permutations of $\mathbb{A}$ is some subgroup of $\big\{\pi : \mathbb{A} \to \mathbb{A} \mid \pi$ is 1-1 and onto$\big\}$. equipped with $\circ : (g, f) \mapsto g \circ f$.

**Definition 374** (Permutation Normal Filter)**.** *Let $\boldsymbol{\mathcal{M}}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, and $\mathscr{G}$ be a group of permutations of $\mathbb{A}$. $\mathscr{F}$ is a **normal filter** on $\mathcal{G}$ if $\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$:*

*(1) $\mathcal{G} \in \mathscr{F}$,*

*(2) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} \in \mathscr{F}$,*

*(3) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$,*

*(4) if $\mathcal{H} \in \mathscr{F}$, then $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$,*

*(5) for each atom $a \in \mathbb{A}$, $\big\{\pi \in \mathcal{G} \mid \pi(a) = a\big\} \in \mathscr{F}$.*

**Definition 375** (Symmetry Group and Symmetric Set)**.** *Let $\boldsymbol{\mathcal{M}}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. For each set $x \in \boldsymbol{\mathcal{M}}$,*

- *the* ***symmetry group*** *of $x$ is*

$$sym_{\mathscr{G}}(x) = \{\pi \in \mathcal{G} \mid \breve{\pi}(x) = x\};$$

- *we write "$x$ is* ***symmetric****" for*

$$sym_{\mathscr{G}}(x) \in \mathscr{F}.$$

**Definition 376** (Hereditarily Symmetric Sets)**.** *Let $\boldsymbol{\mathcal{M}}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. The set of all hereditarily symmetric sets $\mathbf{HS}_{\mathscr{F}} \subseteq \boldsymbol{\mathcal{M}}$ is defined by transfinite recursion:*

$$x \in \mathbf{HS}_{\mathscr{F}} \iff \begin{cases} sym_{\mathscr{G}}(x) \in \mathscr{F} \\ \quad\text{and} \\ x \subseteq \mathbf{HS}_{\mathscr{F}}. \end{cases}$$

**Remark 377.** Let $\boldsymbol{\mathcal{M}}$ be any model of **ZFA** with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. For all sets $x, y \in \boldsymbol{\mathcal{M}}$ and all permutations $\pi \in \mathcal{G}$,

(1) $x = y \iff \breve{\pi}(x) = \breve{\pi}(y)$;

(2) $x \in \mathbf{HS}_{\mathscr{F}} \iff \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$.

*Proof of Remark* 377:

(1) $x \in y \iff \breve{\pi}(x) \in \breve{\pi}(y)$ yields
$$\begin{aligned} x \subseteq y &\iff \forall z \in x \;\; z \in y \\ &\iff \forall z \in x \;\; \breve{\pi}(z) \in \breve{\pi}(y) \\ &\iff \forall \breve{\pi}(z) \in \breve{\pi}(x) \;\; \breve{\pi}(z) \in \breve{\pi}(y) \\ &\iff \breve{\pi}(x) \subseteq \breve{\pi}(y). \end{aligned}$$

Thus,

$$x = y \iff x \subseteq y \text{ and } y \subseteq x \iff \breve{\pi}(x) \subseteq \breve{\pi}(y) \text{ and } \breve{\pi}(y) \subseteq \breve{\pi}(x) \iff \breve{\pi}(x) = \breve{\pi}(y).$$

(2) (a) We show that $sym_{\mathscr{G}}(x) \in \mathscr{F} \implies sym_{\mathscr{G}}\big(\breve{\pi}(x)\big) \in \mathscr{F}$.
For every permutation $\rho$ we have

$$\begin{aligned} \breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi}(x) = x \quad &\implies \quad \breve{\pi} \circ \breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi}(x) = \breve{\pi}(x) \\ &\implies \quad \breve{\rho}\big(\breve{\pi}(x)\big) = \breve{\pi}(x) \\ &\implies \quad \rho \in sym_{\mathscr{G}}\big(\breve{\pi}(x)\big); \end{aligned}$$

which shows

$$\breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi} \in sym_{\mathscr{G}}(x) \implies \rho \in sym_{\mathscr{G}}\big(\breve{\pi}(x)\big)$$

or, equivalenty,

$$\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1} \subseteq sym_{\mathscr{G}}\big(\breve{\pi}(x)\big).$$

ISince $\mathscr{F}$ being a normal filter, it follows

$$\underbrace{sym_{\mathscr{G}}(x)}_{\in\mathscr{F}} \implies \underbrace{\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1}}_{\in\mathscr{F}} \implies \underbrace{\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1}}_{\in\mathscr{F}} \subseteq \underbrace{sym_{\mathscr{G}}\big(\breve{\pi}(x)\big)}_{\in\mathscr{F}}.$$

(b) We show that $x \in \mathbf{HS}_{\mathscr{F}} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$ by induction on $rk_{\boldsymbol{\mathcal{P}}^{\alpha}(\mathbb{A})}(x)$ = least ordinal $\alpha$ such that $x \in \boldsymbol{\mathcal{P}}^{\alpha+1}(\mathbb{A})$.

- If $rk_{\boldsymbol{\mathcal{P}}^{\alpha}(\mathbb{A})}(x) = 0$, then $x$ is a set — possibly empty — of atoms.

$$x \in \mathbf{HS}_{\mathscr{F}} \implies \left\{ \begin{array}{ccc} sym_{\mathscr{G}}(x) \in \mathscr{F} & \overset{by\ (a)}{\implies} & sym_{\mathscr{G}}(\breve{\pi}(x)) \in \mathscr{F} \\ \text{and} & & \text{and} \\ \underbrace{\forall a \in x \;\; a \in \mathbf{HS}_{\mathscr{F}}}_{\forall a\in\mathbb{A}\ sym_{\mathscr{G}}(a)\in\mathscr{F}} & \implies & \underbrace{\forall a \in x \;\; \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}}_{\forall a\in\mathbb{A}\ sym_{\mathscr{G}}(a)\in\mathscr{F}} \end{array} \right\} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}.$$

- If $rk_{\boldsymbol{\mathcal{P}}^{\alpha}(\mathbb{A})}(x) > 0$

$$x \in \mathbf{HS}_{\mathscr{F}} \implies \left\{ \begin{array}{ccc} sym_{\mathscr{G}}(x) \in \mathscr{F} & \overset{by\ (a)}{\implies} & sym_{\mathscr{G}}(\breve{\pi}(x)) \in \mathscr{F} \\ \text{and} & & \text{and} \\ \forall y \in x \;\; y \in \mathbf{HS}_{\mathscr{F}} & \implies & \underbrace{\forall y \in x \;\; \breve{\pi}(y) \in \mathbf{HS}_{\mathscr{F}}}_{by\ induction\ hypothesis} \end{array} \right\} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}.$$

We have shown $x \in \mathbf{HS}_{\mathscr{F}} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$ holds for all $x$ and all $\breve{\pi}$. So, in particular for $x := \breve{\pi}(x)$ and $\breve{\pi} := \breve{\pi}^{-1}$ we have

$$\begin{aligned} \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}} \quad &\implies \quad \breve{\pi}^{-1} \circ \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}} \\ &\implies \quad x \in \mathbf{HS}_{\mathscr{F}}. \end{aligned}$$

$\square$ 377

**Lemma 378.** *Let $\mathcal{M}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. Let $\varphi(z_1, \ldots, z_n)$ be any $\mathcal{L}_{\mathrm{ST}}$-formula whose free variables are among $z_1, \ldots, z_n$. If $\pi \in \mathcal{G}$, then for all $x_1, \ldots, x_n \in \mathcal{M}$,*

$$\mathcal{M} \models \varphi(x_1, \ldots, x_n) \Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big).$$

*Proof of Lemma 378:* As always, the proof is by induction on the height of $\varphi$. Without loss of generality, we may assume that $\varphi$ only contains $\neg$ and $\wedge$ as connectors and $\exists$ as sole quantifier.

(1) If $\varphi$ is an atomic formula, then we already saw that

- $\mathcal{M} \models x_1 = x_2 \iff \mathcal{M} \models \breve{\pi}(x_1) = \breve{\pi}(x_2)$
- $\mathcal{M} \models x_1 \in x_2 \iff \mathcal{M} \models \breve{\pi}(x_1) \in \breve{\pi}(x_2)$.

(2) If $\varphi = \neg\psi$, then

$$\begin{aligned}\mathcal{M} \models \varphi(x_1, \ldots, x_n) &\Longleftrightarrow \mathcal{M} \not\models \psi(x_1, \ldots, x_n)\\ &\Longleftrightarrow \mathcal{M} \not\models \psi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big)\\ &\Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big).\end{aligned}$$

(3) If $\varphi = (\psi \wedge \theta)$, then

$$\begin{aligned}\mathcal{M} \models \varphi(x_1, \ldots, x_n) &\Longleftrightarrow \mathcal{M} \models \psi(x_1, \ldots, x_n) \text{ and } \mathcal{M} \models \theta(x_1, \ldots, x_n)\\ &\Longleftrightarrow \mathcal{M} \models \psi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big) \text{ and } \mathcal{M} \models \theta\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big)\\ &\Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big).\end{aligned}$$

(4) If $\varphi = \exists x \psi$, then

$$\begin{aligned}\mathcal{M} \models \varphi(x_1, \ldots, x_n) &\Longleftrightarrow \text{there exists } y \in \mathcal{M},\quad \mathcal{M} \models \psi(y/x, x_1, \ldots, x_n)\\ &\Longleftrightarrow \text{there exists } \breve{\pi}(y) \in \mathcal{M},\quad \mathcal{M} \models \psi\big(\breve{\pi}(y)/x, \breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big)\\ &\Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(x_1), \ldots, \breve{\pi}(x_n)\big).\end{aligned}$$

$\square$ 378

We now define the symmetric submodel of $\mathcal{M}$ — denoted by $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ —- as the restriction of $\mathcal{M}$ to $\mathbf{HS}_{\mathscr{F}}$.

**Definition 379** (Permutation Model)**.** *Let $\boldsymbol{\mathcal{M}}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. The submodel of $\boldsymbol{\mathcal{M}}$ formed of all the symmetric sets of $\boldsymbol{\mathcal{M}}$ is called the permutation model and denoted by:*

$$\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}} = \boldsymbol{\mathcal{M}} \cap \mathbf{HS}_{\mathscr{F}}.$$

We show that every permutation model satisfies **ZFA**.

**Proposition 380.** *Let $\boldsymbol{\mathcal{M}}$ be any transitive model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$.*

*(1) $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ is transitive;*

*(2) $\boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \subseteq \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$;*

*(3) $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ satisfies* **ZFA***.*

*Proof of Proposition* 380:

(1) If $x \in y \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$, then $x \in \boldsymbol{\mathcal{M}}$ since $\boldsymbol{\mathcal{M}}$ is transitive and $x \in \mathbf{HS}_{\mathscr{F}}$ since $y \subseteq \mathbf{HS}_{\mathscr{F}}$, hence $x \in \boldsymbol{\mathcal{M}} \cap \mathbf{HS}_{\mathscr{F}} = \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$

(2) For all $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing)$, and for all $\pi \in \mathcal{G}$,

$$\left.\begin{array}{l} \circ\ \breve{\pi}(x) = x, \text{ hence } sym_{\mathcal{G}}(x) = \mathcal{G} \in \mathscr{F}; \\ \circ\ tc(x) \subseteq \boldsymbol{\mathcal{P}}^{\infty}(\varnothing), \text{ hence } x \subseteq \mathbf{HS}_{\mathscr{F}} \end{array}\right\} x \in \mathbf{HS}_{\mathscr{F}}$$

(3) $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ satisfies **ZFA**:

**Empty Set Existence for ZFA** comes from $\boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \subseteq \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$.

**Extensionality for ZFA** is from $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ being transitive.

**Comprehension Schema** We want to show that for all $w, w_1, \dots, w_n \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ and formula $\varphi(x, y, y_1, \dots, y_n)$:

$$u = \left\{ v \in w \mid \Big(\varphi\big(v/x, w/y, w_1/y_1, \dots, w_n/y_n\big)\Big)^{\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}} \right\} \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}.$$

For this it is enough to consider the following subgroup of $\mathcal{S} \in \mathscr{F}$:

$$\mathcal{S} = sym_{\mathscr{G}}(w) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n).$$

Notice that, for any $\pi \in \mathcal{S}$ and $v \in w$, we have

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(v/x, w/y, w_1/y_1, \ldots, w_n/y_n\big) &\iff \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\Big(\breve{\pi}(v)/x, \breve{\pi}(w)/y, \breve{\pi}(w_1)/y_1, \ldots, \breve{\pi}(w_n)/y_n\Big)\\ &\iff \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(\breve{\pi}(v)/x, w/y, w_1/y_1, \ldots, w_n/y_n\big)\end{aligned}$$

So, for every $v \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ and every $\pi \in \mathcal{S}$, we have $v \in u \iff \breve{\pi}(v) \in u$. Since $\breve{\pi}(u) = \big\{\breve{\pi}(v) \mid v \in u\big\}$, we have shown that shows that $\breve{\pi}(u) = u$ holds for every $v \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$. Hence,

$$\mathcal{S} = sym_{\mathscr{G}}(w) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n) \subseteq sym_{\mathscr{G}}(u) \in \mathscr{F}.$$

which shows that $u \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$.

**Pairing** If $x, y \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$, then $\{x, y\} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathscr{G}}\big(\{x, y\}\big) \supseteq sym_{\mathscr{G}}(x) \cap sym_{\mathscr{G}}(y)$ and $x, y \in \mathbf{HS}_{\mathscr{F}}$. We obtain

**Union** Let $x \in \mathbf{HS}_{\mathscr{F}}$, to prove that $\bigcup x \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$, it is enough to show that there exists $u \in \mathbf{HS}_{\mathscr{F}}$ such that $\bigcup x \subseteq u$.

$$u = \Big\{\breve{\pi}(z) \in \breve{\pi}\Big[\bigcup x\Big] \mid \exists y \in x \;\; z \in y \;\wedge\; \pi \in \mathcal{G}\Big\}$$

One has $\bigcup x \subseteq u$, $sym_{\mathscr{G}}(u) = \mathcal{G}$ and every $\breve{\pi}(z) \in u$ satisfies $\breve{\pi}(z) \in \mathbf{HS}_{\mathscr{F}}$ since $z \in \mathbf{HS}_{\mathscr{F}}$ holds.

**Infinity** Since $\omega$ belongs to the kernel, it belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$.

**Power Set** Let $x \in \mathbf{HS}_{\mathscr{F}}$, it is enough to show there exists $u \in \mathbf{HS}_{\mathscr{F}}$ such that $\mathscr{P}\big((x)\big) \cap \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}} \subseteq u$.

$$\begin{aligned} u &= \Big\{\breve{\pi}(y) \mid y \in \mathcal{P}(x) \;\wedge\; \pi \in \mathcal{G}\Big\}\\ &= \bigcup\Big\{\breve{\pi}\Big[\mathcal{P}(x)\Big] \mid \pi \in \mathcal{G}\Big\}.\end{aligned}$$

One has $\mathcal{P}(x) \subseteq u$, $sym_{\mathscr{G}}(u) = \mathcal{G}$ and every $\breve{\pi}(y) \in u$ satisfies $\breve{\pi}(y) \in \mathbf{HS}_{\mathscr{F}}$ since $y \in \mathbf{HS}_{\mathscr{F}}$ holds.

**Foundation** holds in $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ since $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}$ is transitive and **Foundation** holds in $\boldsymbol{\mathcal{M}}$.

**Replacement Schema** for each formula $\varphi(x, y, z_1, \ldots, z_n)$, we want to prove that given any $w_1 \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}, \ldots, w_n \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ :

$$\left(\begin{array}{c} \forall x \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \exists! y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \ \big(\varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \\ \longrightarrow \\ \forall u \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \exists v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \forall x \in u\ \exists y \in v\ \ \big(\varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \end{array}\right).$$

We fix $w_1 \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}, \ldots, w_n \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and $u \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and consider (inside $\mathcal{M}$ which satisfies the **Replacement Schema** since it satisfies **ZFA**) the following set

$$v = \left\{y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \mid \big(\exists x \in u\ \ \varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}}\right\}$$

We consider the subgroup

$$\mathcal{S} = sym_{\mathscr{G}}(u) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n).$$

Notice that, for any $\pi \in \mathcal{S}$, any $x \in u$ and any $y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$, we have $\breve{\pi}(x) \in \breve{\pi}(u) = u$ and

$$\begin{aligned} \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(\varphi(x, y, w_1, \ldots, w_n)\big) &\Longleftrightarrow \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\Big(\breve{\pi}(x), \breve{\pi}(y), \breve{\pi}(w_1), \ldots, \breve{\pi}(w_n)\Big) \\ &\Longleftrightarrow \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(\breve{\pi}(x), \breve{\pi}(y), w_1, \ldots, w_n\big). \end{aligned}$$

Since we have $x \in u \iff \breve{\pi}(x) \in u$, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \exists x \in u\ \ \varphi\big(x, y, w_1, \ldots, w_n\big) \iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \exists \breve{\pi}(x) \in u\ \ \varphi\big(\breve{\pi}(x), \breve{\pi}(y), w_1, \ldots, w_n\big).$$

Therefore, we have $y \in v \iff \breve{\pi}(y) \in v$, which shows that $\breve{\pi}(v) = v$, hence

$$\mathcal{S} = sym_{\mathscr{G}}(u) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n) \subseteq sym_{\mathscr{G}}(v) \in \mathscr{F},$$

which shows that $v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

$\square$ 380

## 23.3 The Basic Fraenkel Model

**Definition 381** (Basic Fraenkel Model)**.** *Let $\mathcal{M}$ be any transitive model of* **ZFA** *with any countable infinite set of atoms $\mathbb{A}$, $\mathscr{G}$ be the group of permutations of $\mathbb{A}$, and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the normal filter generated by*

$$\left\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\right\}$$

*where*

$$fix_{\mathscr{G}}(F) = \left\{\pi \in \mathscr{G} \mid \forall x \in F\ \ \breve{\pi}(x) = x\right\}.$$

*The submodel of $\mathcal{M}$ formed of all its symmetric sets is the permutation model known as the basic Fraenkel Model:*

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}} = \mathcal{M} \cap \mathbf{HS}_{\mathscr{F}}.$$

For any set $y$, we call **support** of $y$ any $F_x \in \mathcal{P}_{fin}(\mathbb{A})$ which satisfies $fix_{\mathscr{G}}(F_x) \subseteq sym_{\mathscr{G}}(y)$. Notice that if $F_x$ is a support of $y$ and $F_x \subseteq F \in \mathcal{P}_{fin}(\mathbb{A})$ holds, then $fix_{\mathscr{G}}(F) \subseteq fix_{\mathscr{G}}(F_x) \subseteq sym_{\mathscr{G}}(y)$ holds as well, so that $F$ is also a support of $y$.

**Lemma 382.** *With same notation[1] as in the definition of $\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}}$. Let $F \in \mathcal{P}_{fin}(\mathbb{A})$ and $S \subseteq \mathbb{A}$. If $fix_{\mathscr{G}}(F) \subseteq sym_{\mathscr{G}}(S)$, then $S$ is either finite or co-finite and*

- *if $S$ is finite, then $S \subseteq F$;*
- *if $S$ is co-finite, then $(\mathbb{A} \smallsetminus S) \subseteq F$.*

*Proof of Lemma 382:*

- If $S \cap (\mathbb{A} \smallsetminus F) = \varnothing$, then $S \subseteq F$.
- If $S \cap (\mathbb{A} \smallsetminus F) \neq \varnothing$, we show that $S \supseteq (\mathbb{A} \smallsetminus F) \neq \varnothing$. We fix some $a \in S \cap (\mathbb{A} \smallsetminus F)$ and consider any $b \in S \cap (\mathbb{A} \smallsetminus F)$ such that $b \neq a$. The permutation $\pi_{a\leftrightarrow b}$ which exchanges $a$ and $b$, and is the identity everywhere else, belongs to $fix_{\mathscr{G}}(F)$. Now, $\breve{\pi}_{a\leftrightarrow b}(S) = S$ implies $a \in S \iff \breve{\pi}_{a\leftrightarrow b}(a) \in S$, which shows that $b$ belongs to $S$. Thus $(\mathbb{A} \smallsetminus S) \subseteq F$.

$\square$ 382

We now show that inside the basic Fraenkel model, there exists some set which is both infinite and Dedekind-finite.

**Proposition 383.** *Let $\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}}$ be the basic Fraenkel model with $\mathbb{A}$ as set of atoms.*

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}} \models \aleph_0 \overset{1-1}{\not\leq} \mathbb{A}.$$

So, although the basic Fraenkel model is built from a set of atoms which is infinite and countable, the model itself cannot recognize this fact, for there is no injection from the integers to the set of atoms.

*Proof of Proposition 383:* Towards a contradiction, we assume that inside $\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathbb{A}$. Then the set

$$S = \{f(2n) \in \mathbb{A} \mid n \in \omega\}$$

[1] See Definition 381.

belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$. So, we have $sym_{\mathscr{G}}(S) \in \mathscr{F}$, hence there exists some finite $F \subseteq \mathbb{A}$ such that $fix_{\mathscr{G}}(F) \subseteq sym_{\mathscr{G}}(S)$. By Lemma 382, $S$ is either finite or co-finite, a contradiction.

□ 383

**Proposition 384.** *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ be the basic Fraenkel model with $\mathbb{A}$ as set of atoms.*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\lesssim} \mathcal{P}(\mathbb{A}).$$

*Proof of Proposition 384:* Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathcal{P}(\mathbb{A})$. Since $f$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$, there exists some finite $F_f \subseteq \mathbb{A}$ such that

$$fix_{\mathscr{G}}(F_f) \subseteq sym_{\mathscr{G}}(f).$$

By Lemma 382, any $S \subseteq \mathbb{A}$ that satisfies $fix_{\mathscr{G}}(F_f) \subseteq sym_{\mathscr{G}}(S)$ satisfies also either $S \subseteq F_f$ or $(\mathbb{A} \smallsetminus S) \subseteq F_f$. Therefore, there exist only finitely many such sets $S$. So, take any $n \in \omega$ such that $f(n) \subseteq \mathbb{A}$ satisfies

$$fix_{\mathscr{G}}(F_f) \nsubseteq sym_{\mathscr{G}}\big(f(n)\big).$$

Take any $\pi \in fix_{\mathscr{G}}(F_f) \smallsetminus sym_{\mathscr{G}}\big(f(n)\big)$ in order to have both

$$\breve{\pi}(f) = f \quad\text{and}\quad \breve{\pi}\big(f(n)\big) \neq f(n).$$

Since $n$ belongs to the kernel, $\breve{\pi}(n) = n$ holds, which leads to $f\big(\breve{\pi}(n)\big) = f(n)$.
By construction,

$$\begin{aligned}\breve{\pi}(f) &= \breve{\pi}\Big(\big\{\big(k, f(k)\big) \mid k \in \omega\big\}\Big)\\ &= \Big\{\Big(\breve{\pi}(k), \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}\\ &= \Big\{\Big(k, \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}.\end{aligned}$$

So that, in particular, we have

$$\breve{\pi}(f)(n) = \breve{\pi}\big(f(n)\big).$$

But, since $\pi \in fix_{\mathscr{G}}(F_f)$, we also have $\breve{\pi}(f) = f$, hence $\breve{\pi}(f)(n) = f(n)$ which contradicts $\breve{\pi}\big(f(n)\big) \neq f(n)$.

□ 384

## 23.4 The Second Fraenkel Model

**Definition 385** (Second Fraenkel Model)**.** *Let $\mathcal{M}$ be any transitive model of* **ZFA** *whose set of atoms is*

$$\mathbb{A} = \bigcup_{n \in \omega} P_n, \text{ where } P_n = \{a_n, b_n\};$$

*group of permutation is*

$$\mathscr{G} = \{\pi : \mathbb{A} \xrightarrow{bij.} \mathbb{A} \mid \forall n \in \omega \ \breve{\pi}(P_n) = P_n\};$$

*(i.e., $\mathscr{G}$ is the group of permutations of $\mathbb{A}$ which preserves the pairs), and normal filter is $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ generated by*

$$\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\}$$

*where*

$$fix_{\mathscr{G}}(F) = \{\pi \in \mathscr{G} \mid \forall x \in F \ \breve{\pi}(x) = x\}.$$

*The submodel of $\mathcal{M}$ formed of all its symmetric sets is the permutation model known as the second Fraenkel Model:*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} = \mathcal{M} \cap \mathbf{HS}_{\mathscr{F}}.$$

Notice that the set of atoms of the second Fraenkel model is made up of the elements of countably many disjoints pairs.

**Lemma 386.** *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model whose set of atoms is*

$$\mathbb{A} = \bigcup_{n \in \omega} P_n, \text{ where } P_n = \{a_n, b_n\},$$

*(1) Each set $P_n$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$;*

*(2) the mapping $f = \{(n, P_n) \mid n \in \omega\}$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.*

*Proof of Lemma 386:*

(1) By construction, every $\pi \in \mathcal{G}$ satisfies $\breve{\pi}(P_n) = P_n$, hence $sym_{\mathscr{G}}(P_n) = \mathcal{G} \in \mathscr{F}$. So, each set $P_n$ is symmetric, hence hereditarily symmetric, so it belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.

(2) For each $\pi \in \mathcal{G}$, and each $n \in \omega$, we have $\breve{\pi}(n) = n$ because $n$ belongs to the kernel. Therefore,

$$\breve{\pi}(f) = \breve{\pi}\Big(\{(n, P_n) \mid n \in \omega\}\Big) = \Big\{\big(\breve{\pi}(n), \breve{\pi}(P_n)\big) \mid n \in \omega\Big\} = \{(n, P_n) \mid n \in \omega\}.$$

So, $sym_{\mathscr{G}}(f) = \mathcal{G} \in \mathscr{F}$ which shows that $f$ is symmetric. Since all elements of $f$ are hereditarily symmetric, $f$ is hereditarily symmetric as well, hence $f$ belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.

$\square$ 386

**Theorem 387.** *Let $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model with $\mathbb{A} = \bigcup_{n\in\omega} P_n$ and $P_n = \{a_n, b_n\}$ as set of atoms.*

$$\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“}\big\{P_n \mid n \in \omega\big\} \textit{ does not admit any choice function”}.$$

*i.e.,*

$$\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \not\models \exists \mathsf{c} : \omega \longrightarrow \mathbb{A} \ \forall n \in \omega \ \ \mathsf{c}(n) \in P_n.$$

This theorem says that there is no function $\mathsf{c} : \omega \longrightarrow \mathbb{A}$ which satisfies that for each integer $n$, $\mathsf{c}(n)$ belongs to $P_n$. Since each set $P_n$ contains exactly two elements, one may think of them as socks which the model considers so indistinguishable that it cannot pick exactly one of them in each pair.

*Proof of Theorem* 387: Towards a contradiction, we assume that inside $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ there exists such a choice function $\mathsf{c} : \omega \xrightarrow{1-1} \bigcup\big\{P_n \mid n \in \omega\big\}$. Since $\mathsf{c}$ belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$, there exists some finite $F_{\mathsf{c}} \subseteq \mathbb{A}$ such that

$$fix_{\mathscr{G}}(F_{\mathsf{c}}) \subseteq sym_{\mathscr{G}}(\mathsf{c}).$$

Pick $n$ large enough such that $F_{\mathsf{c}} \cap P_n = \varnothing$ as well as $\pi \in fix_{\mathscr{G}}(F_{\mathsf{c}})$ that satisfies both $\breve{\pi}(a_n) = b_n$ and $\breve{\pi}(b_n) = a_n$. We then have the following contradiction:

- $\breve{\pi}(\mathsf{c}) = \mathsf{c}$ (because $\pi \in fix_{\mathscr{G}}(F_{\mathsf{c}})$),
- $\breve{\pi}\big(\mathsf{c}(n)\big) \neq \mathsf{c}(n)$ (by construction), and
- $$\begin{aligned}\breve{\pi}(\mathsf{c}) &= \Big\{\breve{\pi}\big(n, \mathsf{c}(n)\big) \mid n \in \omega\Big\} \\ &= \Big\{\Big(\breve{\pi}(n), \breve{\pi}\big(\mathsf{c}(n)\big)\Big) \mid n \in \omega\Big\} \\ &= \Big\{\Big(n, \breve{\pi}\big(\mathsf{c}(n)\big)\Big) \mid n \in \omega\Big\} \neq \Big\{\big(n, \mathsf{c}(n)\big) \mid n \in \omega\Big\} = \mathsf{c}.\end{aligned}$$

$\square$ 387

**Theorem 388.** *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model whose set of atoms is made up of $\mathbb{A} = \bigcup_{n\in\omega} P_n$, where $P_n = \{a_n, b_n\}$,*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{"there is an infinite binary tree without any infinite branch".}$$

*Proof of Theorem 388*: We set

$$T = \bigcup_{n\in\omega} \Big\{ s \in {}^{n}\mathbb{A} \mid \forall k \in n \ s(k) \in P_k \Big\}.$$

Any infinite branch would yield a choice function contradicting Theorem 387.

□ 388

Theorem 388 typically contradicts König Lemma which is the following well-known result:

**König Lemma** ($\mathbf{AC}_\omega$)**.** *Every infinite finitely branching tree admits an infinite branch.*

*Proof of König Lemma*: Let $T \subseteq E^{<\omega}$ be infinite. Since $T$ is finitely branching, for every integer $n$, $T \cap {}^{n}E$ is finite, hence $T$ is countable[2]. With the help of $\mathbf{AC}_\omega$ we can equip $T$ with a well-ordering $<_T$. By recursion on the integers we define $b : \omega \to T$ such that for each $n$ we have $b(n) \in T \cap {}^{n}E$ by

- $b(0) = \varnothing$
- $b(n+1) =<_T$ -least in $\Big\{ s \in T \cap {}^{n+1}E \mid s \upharpoonright n = b(n) \wedge \big\{ s' \in T \mid s \upharpoonright (n+1) = s' \upharpoonright (n+1) \big\}$ is infinite$\Big\}$.

□ König Lemma

## 23.5 The Ordered Mostowski Model

**Definition 390** (Ordered Mostowski Model)**.** *Let $\mathcal{M}$ be any transitive model of* **ZFA** *whose set of atoms is a countable set $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which is a dense ordering without least nor greatest element. i.e., $(\mathbb{A}, <_{\mathbf{M}})$ is isomorphic to $(\mathbb{Q}, <)$.*

[2] Notice that since we assume $\mathbf{AC}_\omega$, we have that a countable union of countable sets is countable.

*We let $\mathscr{G}$ be the group of all order preserving permutations of $\mathbb{A}$. i.e.,*

$$\mathscr{G} = \Big\{\pi : \mathbb{A} \xrightarrow{bij.} \mathbb{A} \mid \forall \mathbb{a} \in \mathbb{A}\, \forall \mathbb{b} \in \mathbb{A}\, \big(\mathbb{a} <_{\mathbf{M}} \mathbb{b} \longleftrightarrow \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b})\big)\Big\}.$$

*Let $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the normal filter generated by $\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\}$, which can be proved to be normal. The ordered Mostowski model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ is the corresponding permutation model.*

For any set $y$, we call **support** of $y$ any $F_y \in \mathcal{P}_{fin}(\mathbb{A})$ which satisfies $fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)$. Notice that if $F_y$ is a support of $y$ and $F_y \subseteq F \in \mathcal{P}_{fin}(\mathbb{A})$ holds, then $fix_{\mathscr{G}}(F) \subseteq fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)$ holds as well, so that $F$ is also a support of $y$.

**Lemma 391.** *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ be the ordered Mostowski model.*

*(1) The set $<_{\mathbf{M}} = \{(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A} \mid \mathbb{a} <_{\mathbf{M}} \mathbb{b}\}$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$.*

*(2) (a) If $F$ and $F'$ are two supports of $y$, then $F \cap F'$ is also a support of $y$.*

*(b) For each set $x \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$, there exists some $\subseteq$-least support of $x$.*

*(c) The following class is symmetric:*

$$\{(x, E) \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \times \mathcal{P}_{fin}(\mathbb{A}) \mid E \text{ is the } \subseteq \text{-least support of } x\}.$$

*(3) For all $F \in \mathcal{P}_{fin}(\mathbb{A})$, if $F$ has $n$ elements, then there exist exactly $2^{2n+1}$ sets of the form $S \subseteq \mathbb{A}$ such that $F$ is a support of $S$.*

*Proof of Lemma* 391:

(1) For every permutation $\pi \in \mathscr{G}$ and every $(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A}$ we have

$$\begin{aligned} (\mathbb{a}, \mathbb{b}) \in <_{\mathbf{M}} &\iff \mathbb{a} <_{\mathbf{M}} \mathbb{b} \\ &\iff \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b}) \\ &\iff \big(\pi(\mathbb{a}), \pi(\mathbb{b})\big) \in <_{\mathbf{M}}) \\ &\iff \breve{\pi}\big(\mathbb{a}, \mathbb{b}\big) \in <_{\mathbf{M}} . \end{aligned}$$

Hence, $\breve{\pi}\big( <_{\mathbf{M}} \big) = <_{\mathbf{M}}$ yields $sym_{\mathscr{G}}\big( <_{\mathbf{M}} \big) = \mathscr{G} \in \mathscr{F}$, thus $<_{\mathbf{M}} \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$.

(2) (a) Notice that given any permutation $\pi \in fix_{\mathscr{G}}(F \cap F')$, there exists permutations $\rho_1, \ldots, \rho_k \in fix_{\mathscr{G}}(F)$ and $\rho'_1, \ldots, \rho'_k \in fix_{\mathscr{G}}(F')$ — for some $k$ large enough — such

that $\rho_1 \circ \rho'_1 \circ \rho_2 \circ \rho'_2 \circ \ldots \rho_k \circ \rho'_k = \pi$.This is better seen on an example: assume $F = \{\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{a}_4\}$ and $F' = \{\mathbb{a}_1, \mathbb{b}_2, \mathbb{a}_4\}$ with $F \cap F' = \{\mathbb{a}_1, \mathbb{a}_4\}$ and

$$\mathbb{a}_1 <_{\mathbf{M}} \mathbb{a}_2 <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \mathbb{a}_3 <_{\mathbf{M}} \mathbb{a}_4$$

Assume $\pi$ satisfies $\mathbb{a}_2 <_{\mathbf{M}} \pi(\mathbb{a}_2) <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \pi(\mathbb{b}_2) <_{\mathbf{M}} \pi(\mathbb{a}_3) <_{\mathbf{M}} \mathbb{a}_3$, then take:

(A) $\rho'$ defined by
- on $]-\infty, \mathbb{a}_2]$, $\rho' = \pi$
- on $]\mathbb{a}_2, \mathbb{b}_2[$, $\rho' = \theta$ for some (any) order isomorphism between $]\mathbb{a}_2, \mathbb{b}_2[$ and $]\pi(\mathbb{a}_2), \mathbb{b}_2[$
- $\rho'(\mathbb{b}_2) = \mathbb{b}_2$
- on $]\mathbb{b}_2, \mathbb{a}_3[$, $\rho' = \delta$ for some (any) order isomorphism between $]\mathbb{b}_2, \mathbb{a}_3[$ and $]\mathbb{b}_2, \pi(\mathbb{a}_3)[$
- $\rho'(\mathbb{a}_3) = \pi(\mathbb{a}_3)$
- on $]\mathbb{a}_3, +\infty]$, $\rho' = \pi$

(B) $\rho$ defined by
- on $]-\infty, \pi(\mathbb{a}_2)]$, $\rho = id$
- on $]\pi(\mathbb{a}_2), \mathbb{b}_2[$, $\rho$ satisfies $\theta \circ \rho = \pi$
- $\rho(\mathbb{b}_2) = \pi(\mathbb{b}_2)$
- on $]\mathbb{b}_2, \mathbb{a}_3[$, $\rho$ satisfies $\delta \circ \rho = \pi$
- $\rho(\mathbb{a}_3) = \mathbb{a}_3$
- on $]\mathbb{a}_3, +\infty]$, $\rho = id$

Notice that $\rho' \in \mathit{fix}_{\mathscr{G}}(F')$ and $\rho \in \mathit{fix}_{\mathscr{G}}(F)$ and $\rho \circ \rho' = \pi$.

(b) Take any $F \in \mathcal{P}_{fin}(\mathbb{A})$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and consider

$$E = \bigcap \{F' \subseteq F \mid \mathit{fix}_{\mathscr{G}}(F') \subseteq \mathit{sym}_{\mathscr{G}}(x)\}.$$

Clearly $\mathit{fix}_{\mathscr{G}}(E) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and $E$ is $\subseteq$-minimal.

(c) For any $\pi \in \mathscr{G}$ we have $\breve{\pi}(x, E) = (\breve{\pi}(x), \breve{\pi}(E))$. Moreover, $\mathit{fix}_{\mathscr{G}}(\breve{\pi}(E)) = \pi \circ \mathit{fix}_{\mathscr{G}}(E) \circ \pi^{-1}$ and $\mathit{sym}_{\mathscr{G}}(\breve{\pi}(x)) = \pi \circ \mathit{sym}_{\mathscr{G}}(x) \circ \pi^{-1}$. So, if $E$ is the $\subseteq$-least support of $x$, then $\breve{\pi}(E)$ is the $\subseteq$-least support of $\breve{\pi}(x)$. Therefore, we have shown that for all $\pi \in \mathscr{G}$,

$$\mathit{sym}_{\mathscr{G}}\Big(\big\{(x, E) \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \times \mathcal{P}_{fin}(\mathbb{A}) \mid E \text{ is least support of } x\big\}\Big) = \mathscr{G} \in \mathscr{F}.$$

(3) Assume $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$ and $F$ is a support of $S$. We have for every $\mathbb{b} \in S$:

(a) if $\mathbb{b} <_{\mathbf{M}} \mathbb{a}_1$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{c} <_{\mathbf{M}} \mathbb{a}_1\} \subseteq S$ holds since for any $\mathbb{c} <_{\mathbf{M}} \mathbb{a}_1$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

(b) if $\mathbb{a}_n <_{\mathbf{M}} \mathbb{b}$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_n <_{\mathbf{M}} \mathbb{c}\} \subseteq S$ since for any $\mathbb{a}_n <_{\mathbf{M}} \mathbb{c}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

(c) if $\mathbb{a}_i <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{i+1}$ then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}\} \subseteq S$ since for any $\mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

So, there are exactly $n + 1$ such intervals, each of them either entirely belongs to $S$ or is disjoint from $S$. There are also $n$ atoms in $F$, each of which may or may not belong to $S$. So, there are as many sets of the form $S$ as there are mappings from $n + 1 + n$ into $\{0, 1\}$ which makes a total of $2^{2n+1}$ different subsets of $\mathbb{A}$.

$\square$ 391

**Theorem 392.** *Let $\mathcal{M}^{\mathrm{HS}_{\mathscr{F}}}_{ost.}$ be the ordered Mostowski model*

$$\mathcal{M}^{\mathrm{HS}_{\mathscr{F}}}_{ost.} \models \text{“there exists some mapping } f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{\text{onto}} \mathcal{P}(\mathbb{A})\text{”}.$$

*Proof of Theorem* 392:

(1) For all support $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$, define a mapping

$$\begin{array}{ccc} {}^{2n+1}2 & \longrightarrow & \mathcal{P}(\mathbb{A}) \\ \chi & \mapsto & S_\chi \end{array}$$

so that $\{S_\chi \mid \chi \in {}^{2n+1}2\}$ is the set of all subsets of $\mathbb{A}$ which have $F$ as support.

We write

- $I_0$ for $]-\infty, \mathbb{a}_1[ = \{\mathbb{b} \in \mathbb{A} \mid \mathbb{b} <_{\mathbf{M}} \mathbb{a}_1\}$
- $I_k$ for $]\mathbb{a}_k, \mathbb{a}_{k+1}[ = \{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_k <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{k+1}\}$ (any $1 \leqslant k < n$)
- $I_n$ for $]\mathbb{a}_n, +\infty[ = \{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_1 <_{\mathbf{M}} \mathbb{b}\}$

We map every sequence $\chi \in {}^{2n+1}2$ to $S_\chi \subseteq \mathbb{A}$ defined by

$$S_\chi = \bigcup \Big(\{I_k \subseteq \mathbb{A} \mid 0 \leqslant k \leqslant n \ \wedge\ \chi(2k) = 1\} \cup \{a_k \in \mathbb{A} \mid 1 \leqslant k \leqslant n \ \wedge\ \chi(2k-1) = 1\}\Big)$$

so that $\{S_\chi \mid \chi \in {}^{2n+1}2\}$ is the set of all subsets of $\mathbb{A}$ which have $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ as support.

(2) We are now able to show that in the *Mostowski model* $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{I}}}_{ost.}$ there exists some mapping

$$f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}).$$

We equip $2^{<\omega}$ with the lexicographic ordering $<_{lex.}$ defined by

$$\chi <_{lex.} \chi' \iff \exists i \left(\chi(i) = 0 \;\wedge\; \chi(i) = 1 \;\wedge\; \forall j < i \; \chi(j) = \chi'(j)\right).$$

For every sequence $\chi \in 2^{<\omega}$ we write $\overleftrightsquigarrow{\chi}$ for the sequence of same length as $\chi$ that satisfies $\chi(n) = 1 - \overleftrightsquigarrow{\chi}(n)$ (any integer $n < lh(\chi)$). We define a mapping $g : 2^{<\omega} \to 2^{<\omega}$ by $g(\varnothing) = \varnothing$ and for $\chi$ a non-empty sequence,

$$\begin{array}{rcccl} g(\chi) & = & \chi & \text{if} & \chi(0) = 0 \\ & = & \overleftrightsquigarrow{\chi} & \text{if} & \chi(0) = 1 \end{array}$$

So, $g(\chi)$ is the one among $\chi$ and its dual $\overleftrightsquigarrow{\chi}$ which starts with a 0.

For every integer $n$ and every $\chi \in 2^n$ we write $\chi 0$ for the sequence in $2^{n+1}$ which satisfies $\chi 0 \restriction n = \chi$ and $\chi 0(n) = 0$.

We define an ordering $\prec_n$ on ${}^{2n+1}2$ by

$$\chi \prec_n \chi' \iff g(\chi 0) <_{lex.} g(\chi' 0).$$

and denote by

$$\begin{array}{rcl} h : 2^{2n+1} & \xrightarrow{onto} & {}^{2n+1}2 \\ i & \mapsto & \chi_{(i,n)} \end{array}$$

the enumeration of ${}^{2n+1}2$ along $\prec_n$. i.e., we have

$$\chi_{(0,n)} \prec_n \chi_{(1,n)} \prec_n \ldots \prec_n \chi_{(2^{2n+1}-1,n)}.$$

We finally define the surjection by

$$\begin{array}{rcl} f : \mathcal{P}_{fin}(\mathbb{A}) & \xrightarrow{onto} & \mathcal{P}(\mathbb{A}) \\ F \neq \varnothing & \mapsto & S_{\chi_{(|F|,|F|)}} \\ \varnothing & \mapsto & \varnothing. \end{array}$$

So, if the cardinality of $F$ is $n$, then $\chi_{(|F|,|F|)}$ is the $n^{\text{th}}$ mapping — with regard to the ordering $\prec_n$ — of the form $\chi : 2n+1 \to \{0,1\}$.

This mapping belongs to the *Mostowski model* $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{I}}}_{ost.}$, essentially because, as a permutation model, it satisfies **ZFA** .

It remains to show that $f$ is onto. For this purpose, take any $S \in \mathcal{P}(\mathbb{A}) \setminus \varnothing$ with $E$ the $\subseteq$-least support of $S$ and $|E| = n$. By construction, there exists some integer $i < 2^{2n+1}$ such that $S_{\chi_{(i,n)}} = S$. The way the ordering $\prec_n$ is defined guarantees $i \geqslant n$: this is because $E$ being the $\subseteq$-least support of $S$, there are at least $n$ many 1's in the sequence $\chi_{(i,n)}$ (by construction, $\chi_{(i,n)}(2j-1) = 1$ holds for all $1 \leqslant i \leqslant n$). So, if $i = n$, then we are done. Otherwise, it is tedious but straightforward to see that $E$ can be extended into a set $F \supseteq E$ which satisfies $|F| = i$ and $S_{\chi_{(i,i)}} = S_{\chi_{(i,n)}}$, which gives the result.

$\square$ 392

# Chapter 24

# Simulating Permutation Models by Symmetric Models

The main result in this chapter is that one can simulate arbitrary large fragments of permutation models by symmetric submodels of generic extensions. Indeed, we may have $\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A})$, for $\gamma$ as large as needed, embed into some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$. This way, most of the results obtained in the context of permutation models, henceforth in the realm of **ZFA** as opposed to **ZF**, may now be transferred to proper models of **ZF**. This is the case, for instance, of Proposition 383 which states that the basic Fraenkel model cannot recognize that it is built from some infinite and countable set of atoms. This is also the case of Theorem 387 which says that there is a countable family of pairs for which no choice function succeeds in picking exactly one element in each pair.

## 24.1 The Jech-Sochor Embedding Theorem

**The Jech-Sochor Embedding Theorem.** *Let $\mathcal{Z}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}_{\mathbb{A}}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathcal{F}_{\mathbb{A}}$ any normal filter on $\mathcal{G}_{\mathbb{A}}$. Let $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ be the permutation model induced by $\mathcal{Z}$ and $\mathcal{F}_{\mathbb{A}}$. Let also $\gamma$ be any ordinal and*

$$\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \mathbf{ZFA} + \big(\mathbf{AC}\big)^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}.$$

*There exist*

- *a symmetric model*

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$$

- *an embedding:*

$$\begin{array}{rccl} (\underset{\sim}{\ })_G : & \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} & \xrightarrow{1-1} & \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \\ & x & \longmapsto & (\underset{\sim}{x})_G \end{array}$$

○ *whose restriction to $\mathcal{P}^{\gamma}(\mathbb{A}) \cap \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ is an $\in$-isomorphism:*

$$\begin{aligned} (\underset{\sim}{\ })_G : \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A}) &\xleftrightarrow{\in-isom.} \left(\mathcal{P}^{\gamma}\left((\underset{\sim}{\mathbb{A}})_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathbb{A}}}} \\ x &\mapsto (\underset{\sim}{x})_G. \end{aligned}$$

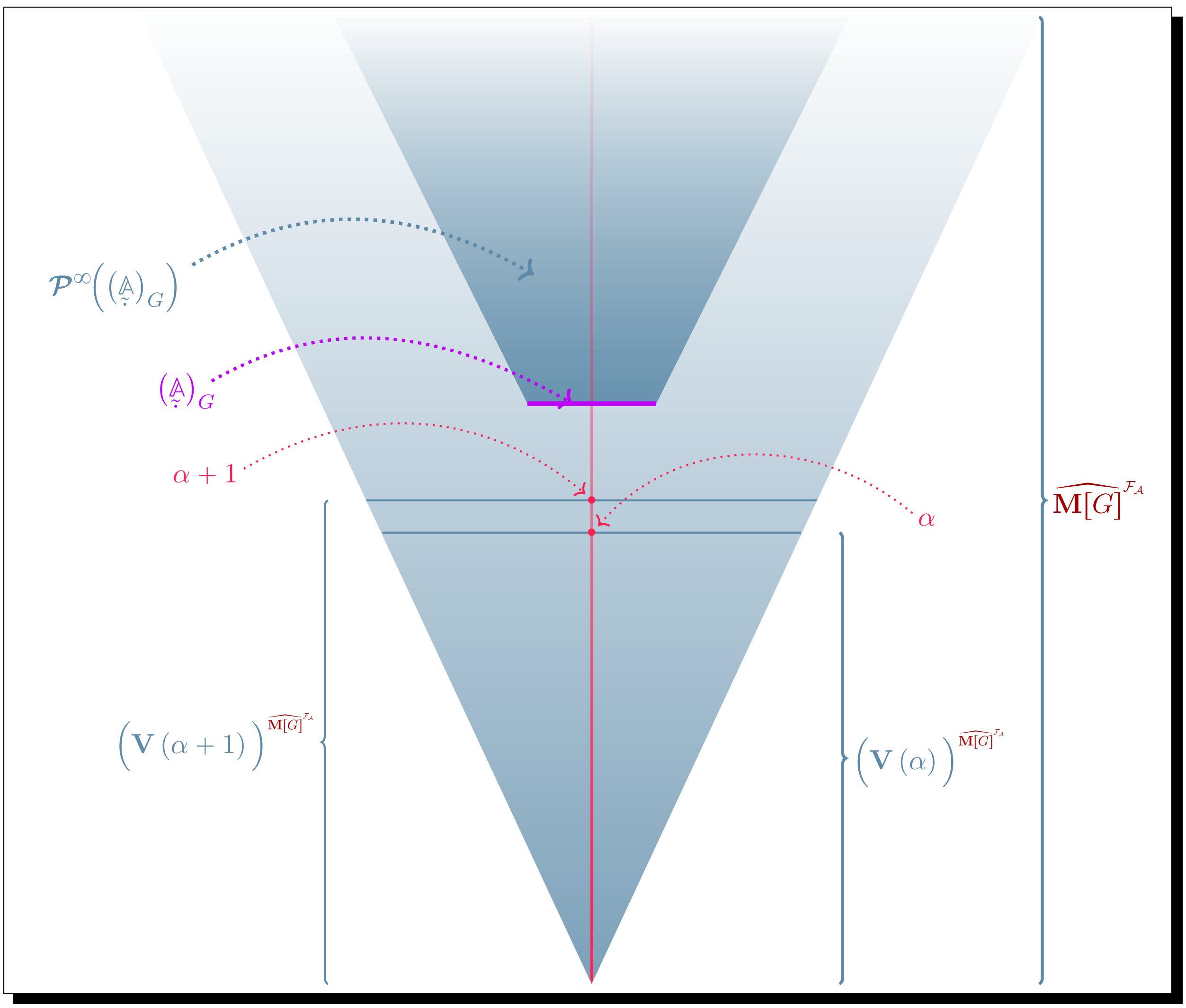



Figure 24.1: The Class $\mathcal{P}^{\infty}(\mathbb{A})$ embedded inside a symmetric submodel of $\mathbf{M}[G]$.

*Proof of the Jech-Sochor Embedding Theorem*: We let $\mathbf{M}$ be the kernel of the model of **ZFA**:

$$\mathbf{M} = \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}.$$

Inside $\boldsymbol{\mathcal{Z}}$, we pick any set $\mathcal{A}$ that belongs to the kernel ($\mathcal{A} \in \mathbf{M}$) and satisfies $|\mathcal{A}| = |\mathbb{A}|$, together with any bijection $\iota : \mathbb{A} \overset{\text{bij.}}{\longleftrightarrow} \mathcal{A}$ that witnesses that $\mathcal{A}$ and $\mathbb{A}$ have same cardinalities.
Inside $\mathbf{M}$ we choose a cardinal $\kappa$ large enough so that the following holds:

$$\mathbf{M} \models \text{“}\,\kappa \text{ is a regular cardinal}\,\text{”} \quad \text{and} \quad |\boldsymbol{\mathcal{P}}^{\,\gamma}(\mathcal{A})\,| < \kappa.$$

We force with $(\mathbb{P}, \leqslant, \mathbb{1})$ defined by

$$\mathbb{P} = \left\{ p : \big(\mathcal{A} \times \kappa \times \kappa\big) \to \{0,1\} \,\Big|\, |\,dom\,(p)\,| < \kappa \right\}$$

$$p \leqslant q \iff p \supseteq q$$

$$\mathbb{1} = \varnothing$$

.

By choice of $\kappa$, $\mathbb{P}$ is a $\kappa$-closed notion of forcing. i.e., if $\langle p_\xi / \xi < \delta\rangle$ is a $\leqslant$-decreasing sequence for some $\delta < \kappa$, then $\Big|\bigcup_{\xi<\delta} dom\,(p_\xi)\Big| < \kappa$, hence $p = \bigcup_{\xi<\delta} p_\xi \in \mathbb{P}$.

We then define, for each element $z$ of $\boldsymbol{\mathcal{Z}}$, some canonical $\mathbb{P}$-name $\underset{\sim}{z}$ which belong to $\mathbf{M}$ as follows. For each $\mathbb{a} \in \mathbb{A}$ and $\xi < \kappa$, we set

(1)
$$\underset{\sim}{x}_{\mathbb{a},\xi} = \left\{ (\check{\zeta}, p) \mid p \in \mathbb{P} \wedge p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \right\}$$

(2)
$$\underset{\sim}{\mathbb{a}} = \left\{ (\underset{\sim}{x}_{\mathbb{a},\xi}, \mathbb{1}) \mid \xi \in \kappa \right\}$$

(3)
$$\underset{\sim}{\mathbb{A}} = \left\{ \underset{\sim}{\mathbb{a}} \mid \mathbb{a} \in \mathbb{A} \right\}.$$

Finally, for each set $z \in \boldsymbol{\mathcal{Z}}$, by recursion, we define

$$\begin{array}{rcll} \underset{\sim}{z} & = & \left\{ (\underset{\sim}{y}, \mathbb{1}) \mid y \in z \right\} & \text{if} \quad z \notin \mathbb{A}. \\ & = & \underset{\sim}{\mathbb{a}} & \text{if} \quad z = \mathbb{a} \in \mathbb{A}. \end{array}$$

**Claim 394.** *Let $G$ which is $\mathbb{P}$-generic over $\mathbf{M}$. For all $x \in \boldsymbol{\mathcal{Z}}$, for all $\mathbb{a}, \mathbb{a}' \in \mathbb{A}$, and for all $\xi < \kappa$, we have*

*(1)* $\mathbb{a} \neq \mathbb{a}' \Rightarrow \left(\underset{\sim}{\mathbb{a}}\right)_G \neq \left(\underset{\sim}{\mathbb{a}'}\right)_G$

*(2)* $\left(\underset{\sim}{x}\right)_G \notin \left(\underset{\sim}{\mathbb{a}}\right)_G$

*(3)* $\left(\underset{\sim}{x}\right)_G \neq \left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$.

*Proof of Claim 394:*

(1) $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G = \left(\underset{\sim}{x}_{\mathbb{a}',\xi'}\right)_G \iff \mathbb{a} = \mathbb{a}'$ and $\xi = \xi'$.

($\Longrightarrow$) We show ($\mathbb{a} \neq \mathbb{a}'$ or $\xi \neq \xi'$) $\Longrightarrow \left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \neq \left(\underset{\sim}{x}_{\mathbb{a}',\xi'}\right)_G$. So, we assume that either $\mathbb{a} \neq \mathbb{a}'$ or $\xi \neq \xi'$ holds and first show that the following set is dense:

$$S = \left\{p \in \mathbb{P} \mid \exists \zeta < \kappa \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \wedge p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0\right\}$$

Indeed, take any $q \in \mathbb{P}$. Since $|dom\,(q)| < \kappa$ and $cof(\kappa) = \kappa$, we have

$$\{\zeta < \kappa \mid (\iota(\mathbb{a}), \xi, \zeta) \notin dom\,(q)\} \cap \{\zeta < \kappa \mid (\iota(\mathbb{a}'), \xi', \zeta) \notin dom\,(q)\} \neq \varnothing$$

Take any ordinal $\zeta$ in this set and form $p$ such that

$$dom\,(p) = dom\,(q) \cup \{(\iota(\mathbb{a}), \xi, \zeta), (\iota(\mathbb{a}'), \xi', \zeta)\}$$

with $p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1$ and $p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0$.
Since $S \in \mathbf{M}$ and $S$ is dense, there exists some $p \in S \cap G$. Therefore, for some ordinal $\zeta < \kappa$, one has $(\check{\zeta}, p) \in \underset{\sim}{x}_{\mathbb{a},\xi}$ and $(\check{\zeta}, p) \notin \underset{\sim}{x}_{\mathbb{a}',\xi'}$. Henceforth, $\underset{\sim}{x}_{\mathbb{a},\xi} \neq \underset{\sim}{x}_{\mathbb{a}',\xi'}$.

($\Longleftarrow$) is immediate.

(2) $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \neq \left(\underset{\sim}{x}\right)_G$ holds for all $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}$.
Notice first that for any $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}$, by construction we precisely have $\underset{\sim}{x} = \check{x}$, so that $(\check{x})_G = x$. So, it is enough to show that $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \notin \mathbf{M}$. Towards a contradiction, we assume $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \in \mathbf{M}$. Then, the following set also belongs to $\mathbf{M}$:

$$D = \left\{p \in \mathbb{P} \mid \exists \zeta < \kappa \;\; \left(\big(\iota(\mathbb{a}), \xi, \zeta\big) \in dom\,(p) \wedge \left(p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G\right)\right)\right\}$$

We show that $D$ is dense. Indeed, given any $p \in \mathbb{P}$, there exists some $\zeta < \kappa$, with $\big(\iota(\mathbb{a}), \xi, \zeta\big) \notin dom\,(p)$ so that we can extend $p$ by $q$ and $r$ the following way:

- $dom\,(q) = dom\,(r) = dom\,(p) \cup \{\, (\iota(\mathbb{a}), \xi, \zeta) \,\}$,
- $q \restriction dom\,(p) = r \restriction dom\,(p) = p$,
- $q\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1$ and $r\big(\iota(\mathbb{a}), \xi, \zeta\big) = 0$.

Since we have $q \perp r$, we have either $q \in D$ or $r \in D$, which shows $D$ is dense.

Now, since $D$ is dense, we take any $p \in D \cap G$, and any $\zeta$ which satisfy

$$p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G .$$

Then, the definition of $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G$ leads to the following contradiction.

$$\zeta \in \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \longleftrightarrow p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G .$$

So, we have shown $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \notin \mathbf{M}$.

(3) For all $x \in \boldsymbol{\mathcal{Z}}$ and all $\mathbb{a} \in \mathbb{A}$, $\big(\underset{\sim}{x}\big)_G \notin \big(\underset{\sim}{\mathbb{a}}\big)_G$.

We recall that

$$\big(\underset{\sim}{\mathbb{a}}\big)_G = \Big(\Big\{(\underset{\sim}{x}_{\mathbb{a},\xi}, \mathbb{1}) \mid \xi \in \kappa\Big\}\Big)_G = \Big\{(\underset{\sim}{x}_{\mathbb{a},\xi})_G \mid \xi \in \kappa.\Big\}$$

Towards a contradiction, we assume that there exists some $\xi \in \kappa$ such that $\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G$. i.e.,

$$\begin{aligned}
\big(\underset{\sim}{x}\big)_G &= \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \\
&= \Big(\Big\{(\check{\zeta}, p) \mid p \in \mathbb{P} \wedge p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\}\Big)_G \\
&= \Big\{\, \big(\check{\zeta}\big)_G \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\} \\
&= \Big\{\zeta < \kappa \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\}.
\end{aligned}$$

(a) If $x = \mathbb{a}' \in \mathbb{A}$, then $\underset{\sim}{x} = \underset{\sim}{\mathbb{a}}' = \Big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'}, \mathbb{1}) \mid \xi' \in \kappa\Big\}$ and

$$\begin{aligned}
\big(\underset{\sim}{x}\big)_G &= \big(\underset{\sim}{\mathbb{a}}'\big)_G \\
&= \Big(\big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'}, \mathbb{1}) \mid \xi' \in \kappa\big\}\Big)_G \\
&= \Big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'})_G \mid \xi' \in \kappa\Big\} \\
&= \Big\{\Big\{\, \big(\check{\zeta}\big)_G \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\Big\} \mid \xi' \in \kappa\Big\} \\
&= \Big\{\, \underbrace{\Big\{\zeta < \kappa \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\Big\}}_{S_{\xi'}} \mid \xi' \in \kappa\Big\}
\end{aligned}$$

Now, it is enough to show that no set $S_{\xi'} = \left\{\zeta < \kappa \mid \exists p \in G \; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\right\}$ is an ordinal to get a contradiction. To see this, simply notice that the following set is trivially dense in $\mathbb{P}$:

$$\left\{ p \in \mathbb{P} \;\middle|\; \exists \zeta < \zeta' < \kappa \begin{pmatrix} \big(\iota(\mathbb{a}'), \xi', \zeta\big) \in dom\,(p) \quad \wedge \quad p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0 \\ \text{and} \\ \big(\iota(\mathbb{a}'), \xi', \zeta'\big) \in dom\,(p) \quad \wedge \quad p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1 \end{pmatrix} \right\}$$

which implies that $S_{\xi'}$ is a non-empty set of ordinals which is not an initial segment of the ordinals, henceforth it is not an ordinal.

(b) If $x \notin \mathbb{A}$, then $\underset{\sim}{x} = \{(\underset{\sim}{y}, \mathbb{1}) \mid y \in x\}$ and $\big(\underset{\sim}{x}\big)_G = \big\{\big(\underset{\sim}{y}\big)_G \mid y \in x\big\}$.

- If $tc\,(x)$ does not contain any atom ($x$ belongs to the kernel), then by construction $\underset{\sim}{x} = \check{x}$ and by Claim 394 (2) we have $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \neq (\check{x})_G$.
- If $tc\,(x)$ contains an atom $\mathbb{a}'$, then $tc\,((x)_G)$ contains $(\mathbb{a}')_G$ which is impossible by case (3)(a).

$\square$ 394

**Claim 395.** *Let $G$ be $\mathbb{P}$-generic over $\mathbf{M}$. For all $x, y \in \boldsymbol{\mathcal{Z}}$, we have*

*(1)* $(x \in y)^{\boldsymbol{\mathcal{Z}}} \iff \Big(\big(\underset{\sim}{x}\big)_G \in \big(\underset{\sim}{y}\big)_G\Big)^{\mathbf{M}[G]}$

*(2)* $(x = y)^{\boldsymbol{\mathcal{Z}}} \iff \Big(\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{y}\big)_G\Big)^{\mathbf{M}[G]}$.

*Proof of Claim 395:* The proof is by induction on $rk_{\mathcal{P}^\infty(\mathbb{A})}\,(y)$. We prove simultaneously (1) and (2).

$\boldsymbol{rk_{\mathcal{P}^\infty(\mathbb{A})}\,(y) = 0}$**:** corresponds to $y$ being an atom of the form $\mathbb{a} \in \mathbb{A}$.

(1) $(\Longrightarrow)$ $(x \in \mathbb{a})^{\boldsymbol{\mathcal{Z}}}$ never holds, so the result is immediate.

$(\Longleftarrow)$ $\Big(\big(\underset{\sim}{x}\big)_G \in \big(\underset{\sim}{\mathbb{a}}\big)_G\Big)^{\mathbf{M}[G]}$ never holds, as we saw above, so the result is immediate.

(2) $(\Longrightarrow)$ is trivial.

$(\Longleftarrow)$
- If $x = \mathbb{a}' \in \mathbb{A}$, then $\big(\underset{\sim}{\mathbb{a}}\big)_G = \big(\underset{\sim}{\mathbb{a}}'\big)_G \Longrightarrow \mathbb{a} = \mathbb{a}'$.
- If $x \notin \mathbb{A}$, then $\underset{\sim}{x} = \big\{(\underset{\sim}{z}, \mathbb{1}) \mid z \in x\big\}$ and $\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{\mathbb{a}}\big)_G$ implies $\big(\underset{\sim}{z}\big)_G \in \big(\underset{\sim}{\mathbb{a}}\big)_G$ holds for some $z \in x$, which contradicts Claim 394 (2).

$\boldsymbol{rk_{\mathcal{P}^\infty(\mathbb{A})}\,(y) > 0}$**:** corresponds to $y$ being an atom of the form $\mathbb{a} \in \mathbb{A}$.

(1) $(\Longrightarrow)$ follows by definition of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

$(\Longleftarrow)$ by construction of $\underset{\sim}{y} = \big\{(\underset{\sim}{z}, \mathbb{1}) \mid z \in y\big\}$, there exists some $z \in y$ such that $\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{z}\big)_G$. By induction hypothesis, we obtain $\big(x = z\big)^{\boldsymbol{\mathcal{Z}}}$, hence $\big(x \in y\big)^{\boldsymbol{\mathcal{Z}}}$.

(2) $(\Longrightarrow)$ is trivial.

$(\Longleftarrow)$ If $\big(x \neq y\big)^{\boldsymbol{\mathcal{Z}}}$, then by symmetry, there exists $\big(z \in y \wedge z \notin x\big)^{\boldsymbol{\mathcal{Z}}}$ and by (24.1) we obtain $\big(\underset{\sim}{z}\big)_G \in \big(\underset{\sim}{y}\big)_G$ and $\big(\underset{\sim}{z}\big)_G \notin \big(\underset{\sim}{x}\big)_G$ which yields $\big(\underset{\sim}{x}\big)_G \neq \big(\underset{\sim}{y}\big)_G$.

$\square$ 395

We then associate to every permutation $\rho \in \mathcal{G}_{\mathbb{A}}$ (where $\rho : \mathbb{A} \xrightarrow{bij.} \mathbb{A}$ and $\mathcal{G}_{\mathbb{A}}$ is the subgroup of the group of permutations of $\mathbb{A}$) the following set $\Pi_\rho$ of permutations $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$:

$$\begin{aligned}
\Pi_\rho &= \Big\{\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa \mid \forall \mathbb{a} \in \mathbb{A}\ \forall \xi < \kappa\ \exists \zeta < \kappa\ \ \pi\big(\iota(\mathbb{a}), \xi\big) = \Big(\iota\big(\rho(\mathbb{a})\big), \zeta\Big)\Big\} \\
&= \left\{\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa \mid \forall \mathbb{a} \in \mathbb{A}\ \ \pi\Big[\big\{\underbrace{\iota(\mathbb{a})}_{\mathtt{a}}\big\} \times \kappa\Big] = \Big\{\underbrace{\iota\big(\overbrace{\rho(\mathbb{a})}^{\mathbb{b}}\big)}_{\mathtt{b}}\Big\} \times \kappa\right\}.
\end{aligned}$$

The intuition behind all this is that $\mathcal{A} \times \kappa$ should be regarded as as many disjoint copies of $\kappa$ as there are atoms ($\mathbb{A}$-many or equivalently $\mathcal{A}$-many). Then, every permutation $\rho_{\mathbb{A}} : \mathbb{A} \xrightarrow{bij.} \mathbb{A}$ induces a permutation $\rho_{\mathcal{A}} : \mathcal{A} \xrightarrow{bij.} \mathcal{A}$ via the bijection $\iota : \mathbb{A} \xleftrightarrow{\text{bij.}} \mathcal{A}$. Now, we only consider the permutations $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$ which, for every $\mathtt{a} \in \mathcal{A}$, map $\{\mathtt{a}\} \times \kappa$ to $\{\mathtt{b}\} \times \kappa$ — where the relation between $\mathtt{a}$ and $\mathtt{b}$ is given by $\iota(\mathbb{a}) = \mathtt{a}$ and $\iota \circ \rho(\mathbb{a}) = \mathtt{b}$.
So, as shown in Figure 24.2, each permutation in $\Pi_\rho$ can be regarded as as many permutations of $\kappa$ as there are atoms, since for every $\mathtt{a} \in \mathcal{A}$:

$$\pi_{\restriction\{\mathtt{a}\}\times\kappa} : \{\mathtt{a}\} \times \kappa \xrightarrow{bij.} \{\mathtt{b}\} \times \kappa.$$

We set

$$\mathcal{G}_{\mathcal{A}} = \bigcup\{\Pi_\rho \mid \rho \in \mathcal{G}_{\mathbb{A}}\}.$$

For every subgroup $\mathcal{H}_{\mathbb{A}} \subseteq \mathcal{G}_{\mathbb{A}}$, we set

$$\mathcal{H}_{\mathcal{A}} = \bigcup\{\Pi_\rho \mid \rho \in \mathcal{H}_{\mathbb{A}}\}.$$

Now, every permutation $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$ induces an automorphism $\pi_{\mathbb{P}} : \mathbb{P} \xrightarrow{bij.} \mathbb{P}$ defined by

$$\pi_{\mathbb{P}}\big(\pi\,(\mathtt{a}, \xi)\,, \zeta\big) = \big(\mathtt{a}, \xi, \zeta\big)$$

or, to say it differently:

$$\pi_{\mathbb{P}}\big(\mathtt{a}, \xi, \zeta\big) = \big(\pi^{-1}\,(\mathtt{a}, \xi)\,, \zeta\big)$$

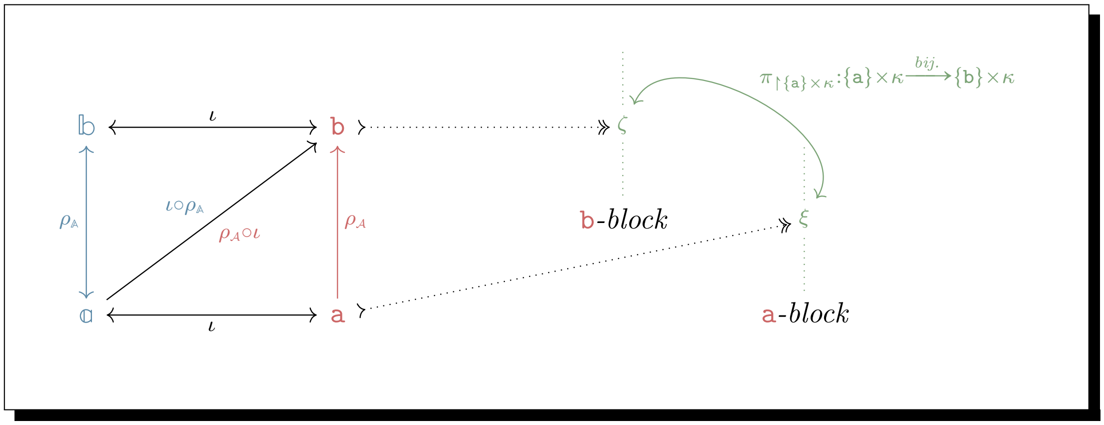



Figure 24.2: The permutation of blocks induced by $\pi \in \Pi_\rho$.

Following this, we regard every permutation $\pi \in \mathcal{G}_\mathcal{A}$ as the automorphism $\pi_\mathbb{P}$ of $\mathbb{P}$ that it induces. For every $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$ we set

$$fix_{\mathcal{G}_\mathcal{A}}(F) = \left\{\pi \in \mathcal{G}_\mathcal{A} \mid \forall (\mathsf{a}, \xi) \in F \ \ \pi(\mathsf{a}, \xi) = (\mathsf{a}, \xi)\right\}$$

Finally, we set $\mathcal{F}_\mathcal{A}$ is the filter generated by

$$\left\{\mathcal{H}_\mathcal{A} \mid \mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A}\right\} \cup \left\{fix_{\mathcal{G}_\mathcal{A}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}.$$

i.e.,

$$\mathcal{H} \in \mathcal{F}_\mathcal{A} \iff \begin{cases} \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_n \subseteq \mathcal{H} \subseteq \mathcal{G}_\mathcal{A} \text{ and } \text{ for some } \mathcal{H}_1, \ldots, \mathcal{H}_n \in \\ \quad \left\{\mathcal{H}_\mathcal{A} \mid \mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A}\right\} \cup \left\{fix_{\mathcal{G}_\mathcal{A}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}. \end{cases}$$

Clearly, $\left\{\mathcal{H}_\mathcal{A} \mid \mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A}\right\}$ is closed under finite intersections because $\mathcal{F}_\mathbb{A}$ is closed under finite intersections. Also, $\left\{fix_{\mathcal{G}_\mathcal{A}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}$ is closed under finite intersections because for $F_1, \ldots, F_n \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$, we have

$$fix_{\mathcal{G}_\mathcal{A}}(F_1) \cap \ldots \cap fix_{\mathcal{G}_\mathcal{A}}(F_n) = fix_{\mathcal{G}_\mathcal{A}}(F_1 \cup \ldots \cup F_n).$$

Therefore,

$$\mathcal{J} \in \mathcal{F}_\mathcal{A} \iff \mathcal{H}_\mathcal{A} \cap fix_{\mathcal{G}_\mathcal{A}}(F) \subseteq \mathcal{J} \subseteq \mathcal{G}_\mathcal{A} \text{ for some } \mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A} \text{ and } F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa).$$

We check that $\mathcal{F}_\mathcal{A}$ is a normal filter on $\mathcal{G}_\mathcal{A}$. i.e., $\mathcal{F}_\mathcal{A}$ is a set of subgroups of $\mathcal{G}_\mathcal{A}$ such that for all subgroups $\mathcal{H}_\mathcal{A}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}_\mathcal{A}$:

(1) $\mathcal{G}_\mathcal{A} \in \mathcal{F}_\mathcal{A}$: we have $\mathcal{G}_\mathcal{A} = \bigcup\left\{\Pi_\rho \mid \rho \in \mathcal{G}_\mathbb{A}\right\}$ and $\mathcal{G}_\mathbb{A} \in \mathcal{F}_\mathbb{A}$ so, $\mathcal{G}_\mathcal{A} \in \left\{\mathcal{H}_\mathcal{A} \mid \mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A}\right\} \subseteq \mathcal{F}_\mathcal{A}$.

(2) if $\mathcal{H} \in \mathcal{F}_\mathcal{A}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} \in \mathcal{F}_\mathcal{A}$: This is by the very definition of $\mathcal{F}_\mathcal{A}$.

(3) if $\mathcal{H} \in \mathcal{F}_{\mathcal{A}}$ and $\mathcal{K} \in \mathcal{F}_{\mathcal{A}}$, then $\mathcal{H} \cap \mathcal{K} = \mathcal{J} \in \mathcal{F}_{\mathcal{A}}$: assume for some $\mathcal{H}_{\mathbb{A}}, \mathcal{K}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $H, K \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$

$$\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H} \text{ and } \mathcal{K}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(K) \subseteq \mathcal{K}.$$

Then

$$\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(H) \cap \mathcal{K}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(K) \subseteq \mathcal{H} \cap \mathcal{K}$$

i.e.,

$$\underbrace{\mathcal{H}_{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}}_{\substack{\mathcal{J}_{\mathcal{A}} \\ with \\ \mathcal{J}_{\mathbb{A}} = \mathcal{H}_{\mathbb{A}} \cap \mathcal{K}_{\mathbb{A}}}} \cap \underbrace{fix_{\mathcal{G}_{\mathcal{A}}}(H) \cap fix_{\mathcal{G}_{\mathcal{A}}}(K)}_{fix_{\mathcal{G}_{\mathcal{A}}}(H \cup K)} \subseteq \mathcal{H} \cap \mathcal{K}$$

(4) if $\mathcal{H} \in \mathcal{F}_{\mathcal{A}}$, then $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathcal{F}_{\mathcal{A}}$: assume for some $\mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $H \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$

$$\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H}.$$

Then, since $\pi \in \mathcal{G}_{\mathcal{A}}$, there exists some $\rho \in \mathcal{G}_{\mathbb{A}}$ such that $\pi \in \Pi_{\rho}$. Now,

$$\pi \circ \mathcal{H}_{\mathcal{A}} \circ \pi^{-1} = \mathcal{H}_{\mathcal{A}}{}' \text{ where } \mathcal{H}_{\mathbb{A}}{}' = \rho \circ \mathcal{H}_{\mathbb{A}} \circ \rho^{-1} \in \mathcal{F}_{\mathbb{A}}$$

and

$$\begin{aligned}\pi \circ fix_{\mathcal{G}_{\mathcal{A}}}(H) \circ \pi^{-1} &= \left\{\pi \circ \tau \circ \pi^{-1} \mid \tau \in \mathcal{G}_{\mathcal{A}} \textit{ and } \forall (\mathtt{a}, \xi) \in H \;\; \tau(\mathtt{a}, \xi) = (\mathtt{a}, \xi)\right\} \\ &= \left\{\tau \in \mathcal{G}_{\mathcal{A}} \mid \forall (\mathtt{b}, \xi) \in \pi[H] \;\; \tau(\mathtt{b}, \xi) = (\mathtt{b}, \xi)\right\} \in \mathcal{F}_{\mathcal{A}}\end{aligned}$$

Now, since $\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H}$, we have

$$\pi \circ \mathcal{H}_{\mathcal{A}} \circ \pi^{-1} \cap \pi \circ fix_{\mathcal{G}_{\mathcal{A}}}(H) \circ \pi^{-1} \subseteq \pi \circ \mathcal{H} \circ \pi^{-1} \in \mathcal{F}_{\mathcal{A}}.$$

We let the set of all hereditarily symmetric $\mathbb{P}$-names $\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \subseteq \mathbf{M}^{\mathbb{P}}$

$$\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} = \left\{\tau \in \mathbf{M}^{\mathbb{P}} \mid sym_{\mathcal{G}_{\mathcal{A}}}(\tau) \in \mathcal{F}_{\mathcal{A}} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}\right\}.$$

i.e.,

$$\tau \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \iff sym_{\mathcal{G}_{\mathcal{A}}}(\tau) \in \mathcal{F}_{\mathcal{A}} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}.$$

We denote the symmetric submodel of the generic extension $\mathbf{M}[G]$ by

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} = \left\{ (\tau)_G \in \mathbf{M}[G] \mid \tau \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \right\}.$$

We notice that the following sets belong to the symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$:

- $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$ (for all $\mathbb{a} \in \mathbb{A}$ and all $\xi < \kappa$) because

$$sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right) = fix_{\mathcal{G}_{\mathcal{A}}}\left(\left\{ (\iota(\mathbb{a}), \xi) \right\}\right).$$

- $\left(\underset{\sim}{\mathbb{a}}\right)_G$ (for all $\mathbb{a} \in \mathbb{A}$) because

$$sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{\mathbb{a}}\right) = \bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})\right\}.$$

- $\left(\underset{\sim}{\mathbb{A}}\right)_G$ because

$$sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{\mathbb{A}}\right) = \mathcal{G}_{\mathcal{A}}.$$

So, we already have $\left(\underset{\sim}{\mathbb{A}}\right)_G$, each $\left(\underset{\sim}{\mathbb{a}}\right)_G$, and each $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$ all belong to the symmetric submodel.

We now show that $x$ belongs to the permutation model $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ if and only if $\left(\underset{\sim}{x}\right)_G$ belongs to the symmetric submodel of the generic extension $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$.

**Claim 396.** *For all $x \in \mathcal{Z}$,*

$$x \in \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \iff \underset{\sim}{x} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}.$$

*Proof of Claim 396:* This comes down to proving

$$sym_{\mathcal{G}_{\mathbb{A}}}(x) \in \mathcal{F}_{\mathbb{A}} \iff sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_{\mathcal{A}}.$$

($\Longrightarrow$) We have $sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}\right) = \bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_{\mathbb{A}}}(x)\right\}$. So, if $sym_{\mathcal{G}_{\mathbb{A}}}(x) \in \mathcal{F}_{\mathbb{A}}$, then, by definition,

$$\bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_{\mathbb{A}}}(x)\right\} = sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_{\mathcal{A}}.$$

($\Longleftarrow$) If $sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_{\mathcal{A}}$, then, for some $\mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$, we have

$$\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(F) \subseteq sym_{\mathcal{G}_{\mathcal{A}}}\left(\underset{\sim}{x}\right) \subseteq \mathcal{G}_{\mathcal{A}}$$

where $\mathcal{H}_{\mathcal{A}} = \bigcup\left\{\Pi_\rho \mid \rho \in \mathcal{H}_{\mathbb{A}}\right\}$. We set

$$\mathbb{F} = \left\{\mathbb{a} \mid \exists \xi < \kappa \quad \left(\iota(\mathbb{a}), \xi\right) \in F\right\}.$$

We have

$$\underbrace{\underbrace{\mathcal{H}_{\mathbb{A}}}_{\in \mathcal{F}_{\mathbb{A}}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} \underbrace{fix_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})}_{\in \mathcal{F}_{\mathbb{A}}}}_{\in \mathcal{F}_{\mathbb{A}}} \subseteq sym_{\mathcal{G}_{\mathbb{A}}}(x) \subseteq \mathcal{G}_{\mathbb{A}}$$

since $\mathcal{F}_{\mathbb{A}}$ is finite, this shows that $sym_{\mathcal{G}_{\mathbb{A}}}(x) \in \mathcal{F}_{\mathbb{A}}$.

$\square$ 396

**Claim 397.** *For all $x \in \boldsymbol{\mathcal{Z}}$,*

$$x \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \iff (\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$$

*Proof of Claim 397*:

($\Longrightarrow$) This is a consequence of Claim 396 since $x \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \Longrightarrow \underset{\sim}{x} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \Longrightarrow (\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$.

($\Longleftarrow$) We proceed by contradiction, assuming there exists $x \in \boldsymbol{\mathcal{Z}}$ such that $(\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ but $x \notin \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$. We assume $x$ be the $\in$-least such set in the sense that $y \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ holds for all $\left(\underset{\sim}{y}\right)_G \in (\underset{\sim}{x})_G$.

Since $(\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$, there exists some $\mathbb{P}$-name $\underset{\sim}{z} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}$ and some forcing condition $p_z \in G$ such that

$$p_z \Vdash \underset{\sim}{z} = \underset{\sim}{x}.$$

Since $\underset{\sim}{z} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}$, there exist both $\mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$ such that

$$\mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(F) \subseteq \mathit{sym}_{\mathcal{G}_{\mathcal{A}}}(\underset{\sim}{z}) \subseteq \mathcal{G}_{\mathcal{A}}$$

Since, $\mathit{sym}_{\mathcal{G}_{\mathbb{A}}}(x) \notin \mathcal{F}_{\mathbb{A}}$, we have, for $\mathbb{F} = \left\{\mathbb{a} \in \mathbb{A} \mid \exists \xi < \kappa \quad \left(\iota(\mathbb{a}), \xi\right) \in F\right\}$,

$$\underbrace{\underbrace{\mathcal{H}_{\mathbb{A}}}_{\in \mathcal{F}_{\mathbb{A}}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} \underbrace{\mathit{fix}_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})}_{\in \mathcal{F}_{\mathbb{A}}}}_{\in \mathcal{F}_{\mathbb{A}}} \nsubseteq \mathit{sym}_{\mathcal{G}_{\mathbb{A}}}(x) \subseteq \mathcal{G}_{\mathbb{A}}.$$

Therefore, there exists

$$\rho \in \left(\mathcal{H}_{\mathbb{A}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} \mathit{fix}_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})\right) \setminus \mathit{sym}_{\mathcal{G}_{\mathbb{A}}}(x).$$

In particular, we have $\rho(x) \neq x$.

Since $|\mathit{dom}(p_z)| < \kappa$, there exists some $\delta < \kappa$ such that

$$\left\{(a, \xi) \in \mathcal{A} \times \kappa \mid \delta < \xi\right\} \cap \left(F \cup \mathit{dom}(p_z) \upharpoonright \mathcal{A} \times \kappa\right) = \varnothing.$$

So, in order to have some $\pi \in \Pi_\rho$ satisfy both

(1) $\pi \in \mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(F)$ (2) $\pi(p_z)$ and $p_z$ are compatible.

we can define $\pi$ as follows:

$$\begin{array}{lll}
\text{for } \mathbb{a} \in \mathbb{F} \text{ and } \xi < \kappa: & \pi\big(\iota(\mathbb{a}), \xi\big) & = \big(\iota(\mathbb{a}), \xi\big) \\
\text{for } \mathbb{a} \notin \mathbb{F} \text{ and } \xi < \delta: & \pi\big(\iota(\mathbb{a}), \xi\big) & = \big(\iota \circ \rho(\mathbb{a}), \delta + \xi\big) \\
 & \pi\big(\iota(\mathbb{a}), \delta + \xi\big) & = \big(\iota \circ \rho(\mathbb{a}), \xi\big) \\
\text{for } \mathbb{a} \notin \mathbb{F} \text{ and } \delta < \xi + 1 < \kappa: & \pi\big(\iota(\mathbb{a}), \delta + \xi\big) & = \big(\iota \circ \rho(\mathbb{a}), \delta + \xi\big).
\end{array}$$

We then have

- $\tilde{\pi}(\underset{\sim}{z}) = \underset{\sim}{z}$ (because $\pi \in \mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(F)$);
- $p_z \Vdash \tilde{\pi}(\underset{\sim}{x}) \neq \underset{\sim}{x}$ (because $\boldsymbol{\mathcal{Z}} \models \rho(x) \neq x$ and by Claim 395 (2)

$$\boldsymbol{\mathcal{Z}} \models \rho(x) \neq x \iff \mathbf{M}[G] \models \Big(\underset{\sim}{\rho(x)}\Big)_G \neq \big(\underset{\sim}{x}\big)_G$$

$$\iff \mathbf{M}[G] \models \big(\tilde{\pi}(\underset{\sim}{x})\big)_G \neq \big(\underset{\sim}{x}\big)_G .)$$

- there exists $q \in \mathbb{P}$ such that $q \leqslant p_z$ and $q \leqslant \pi(p_z)$ which leads to the following contradiction:

$$q \Vdash \underset{\sim}{z} = \underset{\sim}{x} \quad \text{and} \quad q \Vdash \tilde{\pi}(\underset{\sim}{x}) \neq \underset{\sim}{x} \quad \text{and} \quad q \Vdash \tilde{\pi}(\underset{\sim}{x}) = \underset{\sim}{z}.$$

□ 397

**Claim 398.** *For all $x \in \boldsymbol{\mathcal{Z}}$, and all ordinal $\gamma$,*

$$\big\{ \big(\underset{\sim}{x}\big)_G \mid x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}} \big\} = \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big(\big(\underset{\sim}{\mathbb{A}}\big)_G\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}.$$

*Proof of Claim 398:*

($\subseteq$) is immediate.

($\supseteq$) The proof is by $\in$-induction. We let $x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}}$ with $\big(\underset{\sim}{x}\big)_G \in \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big(\big(\underset{\sim}{\mathbb{A}}\big)_G\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$ and $y \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ be such that $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models y \in \big(\underset{\sim}{x}\big)_G$. We consider $\underset{\sim}{y}$ any $\mathbb{P}$-name for $y$. Now, for each $u \in x$, the following set is dense:

$$D_u = \big\{q \in \mathbb{P} \mid q \Vdash \underset{\sim}{u} \in \underset{\sim}{y} \quad \text{or} \quad q \Vdash \underset{\sim}{u} \notin \underset{\sim}{y}\big\}.$$

Since $|\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}| < \kappa$, we have $|x| < \kappa$, hence $|\{D_u \mid u \in x\}| < \kappa$ and, since $\mathbb{P}$ is $\kappa$-closed, there exists some forcing condition $p \in \mathbb{P}$ which "decides" for each $u \in x$, whether $\underset{\cdot}{u} \in \underset{\sim}{y}$ or $\underset{\cdot}{u} \notin \underset{\sim}{y}$ holds. Namely,

$$p \in G \cap \bigcap_{u \in x} \{q \in \mathbb{P} \mid q \Vdash \underset{\cdot}{u} \in \underset{\sim}{y} \ \text{ or } \ q \Vdash \underset{\cdot}{u} \notin \underset{\sim}{y}\}.$$

We take $z = \{u \in x \mid p \Vdash \underset{\cdot}{u} \in \underset{\sim}{y}\}$ so that we have $(\underset{\cdot}{z})_G = y$ and since $(\underset{\cdot}{z})_G$ belongs to $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$, we also have $z \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ by Claim 397.

$\square$ 398

Claim 398 yields that the embedding $\begin{array}{ccc} \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} & \to & \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \\ x & \mapsto & (\underset{\cdot}{x})_G \end{array}$ satisfies

$$\{ (\underset{\cdot}{x})_G \mid x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \} = \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big( (\underset{\cdot}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$$

and for all $x, y \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ we have

$$\boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models y \in x \iff \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models (\underset{\cdot}{y})_G \in (\underset{\cdot}{x})_G .$$

So, it follows that the mapping $x \mapsto (\underset{\cdot}{x})_G$ is an $\in$-isomorphism between $\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ and $\Big(\boldsymbol{\mathcal{P}}^{\gamma}\big( (\underset{\cdot}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$.

$\square$ Jech-Sochor Embedding Theorem

## 24.2 Some applications of the Jech-Sochor Embedding Theorem

**Corollary 399.** *Let $\boldsymbol{\mathcal{Z}}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}_{\mathbb{A}}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathcal{F}_{\mathbb{A}}$ any normal filter on $\mathcal{G}_{\mathbb{A}}$ satisfy $\boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \mathbf{ZFA} + (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$. Let $\boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ be the permutation model induced by $\boldsymbol{\mathcal{Z}}$ and $\mathcal{F}_{\mathbb{A}}$. Let also $\alpha$ be any ordinal and $\varphi$ be any formula of the form $\varphi := \exists x \ \ \psi(x)$ where $\psi$ is some $\Delta_0^{0-rud}$-formula whose quantifiers are all bounded by $\boldsymbol{\mathcal{P}}^{\alpha}(x)$.*

$$\textit{If } \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \varphi, \textit{ then there exists } \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \textit{ such that } \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \varphi.$$

*Proof of Corollary 399*: Take any $B \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ such that $\boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \psi(B)$ and any $\gamma$ large enough so that $\boldsymbol{\mathcal{P}}^{\alpha}(B) \subseteq \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A})$. By the Jech-Sochor Embedding Theorem (on page 333) there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A})$ and $\left(\mathcal{P}^{\gamma}\left(\left(\underset{\sim}{\mathbb{A}}\right)_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$ are $\in$-isomorphic, hence

$$\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \varphi \implies \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A}) \models \varphi \implies \left(\mathcal{P}^{\gamma}\left(\left(\underset{\sim}{\mathbb{A}}\right)_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}} \models \varphi \implies \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \varphi.$$

$\square$ 399

**Corollary 400.** *If* **ZF** *is consistent, so are the following theories:*

*(1)* **ZF**+ *"there exist some infinite Dedekind-finite set".*

*(2)* **ZF**+ *"there exist some infinite set $\mathcal{A}$ such that $\mathcal{P}(\mathcal{A})$ is Dedekind-finite".*

*(3)* **ZF**+ *"there exist some countable family of pairs which does not admit any choice function".*

*(4)* **ZF**+ *"there is an infinite binary tree without any infinite branch".*

*(5)* **ZF**+ *"there exist an infinite set $\mathbb{A}$ and a mapping $f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})$".*

*Proof of Corollary 400*: First, notice that **ZF** and **ZFC** are equiconsistent. Then,

(1) By Proposition 383, the basic Fraenkel model $\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}}$ contains an infinite set of atoms $\mathbb{A}$ which is Dedekind-finite:

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \models \aleph_0 \overset{1-1}{\not\leq} \mathbb{A}.$$

For any integer $n$ and any atom $\mathbb{a}$, we have

$$(n, \mathbb{a}) \in \mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \cap \mathcal{P}^{\omega}(\mathbb{A}).$$

Hence,

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \models \{f \subseteq \omega \times \mathbb{A} \mid f : \omega \to \mathbb{A}\} \subseteq \mathcal{P}^{\omega+1}(\mathbb{A}).$$

Moreover,

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \models (\mathbf{AC})^{\mathcal{P}^{\infty}(\varnothing)}.$$

Now,

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \models \text{"there is no } f : \omega \xrightarrow{1-1} \mathbb{A}\text{"} \iff \mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathcal{F}}} \models \left(\text{"there is no } f : \omega \xrightarrow{1-1} \mathbb{A}\text{"}\right)^{\mathcal{P}^{\omega+1}(\mathbb{A})}.$$

By Corollary 399, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \text{"there is no } f : \omega \xrightarrow{1-1} \left(\underset{\sim}{\mathbb{A}}\right)_G\text{"}$$

(2) *mutatis mutandis*, the proof is the same as for (1): By Proposition 384, the basic Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ contains an infinite set of atoms $\mathbb{A}$ which satisfies:

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\leq} \mathcal{P}(\mathbb{A}).$$

For any integer $n$ and any set of atom $\mathbb{B} \subseteq \mathbb{A}$, we have

$$(n, \mathbb{B}) \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \quad \Longrightarrow \quad (n, \mathbb{B}) \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \cap \boldsymbol{\mathcal{P}}^{\omega+3}(\mathbb{A}).$$

Hence,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models {}^{\omega}\mathcal{P}(\mathbb{A}) \subseteq \boldsymbol{\mathcal{P}}^{\omega+4}(\mathbb{A})\,.$$

we have,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \text{“there is no } f : \omega \xrightarrow{1-1} \mathcal{P}(\mathbb{A})\text{”} \iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \left(\text{“there is no } f : \omega \xrightarrow{1-1} \mathcal{P}(\mathbb{A})\text{”}\right)^{\boldsymbol{\mathcal{P}}^{\omega+4}(\mathbb{A})}.$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$, by Corollary 399, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_A}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_A} \models \text{“there is no } f : \omega \xrightarrow{1-1} \mathcal{P}\left(\left(\underset{\sim}{\mathbb{A}}\right)_G\right)\text{”}.$$

(3) By Theorem 387, the second Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ with $\mathbb{A} = \bigcup_{n \in \omega} P_n$ and $P_n = \{a_n, b_n\}$ as set of atoms satisfies

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“}\left\{P_n \mid n \in \omega\right\} \text{ does not admit any choice function”}.$$

Now, a choice function is an element $f \in {}^{\omega}\mathbb{A}$ which satisfies $\forall n \in \omega\, f(n) \in P_n$ and we have,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models {}^{\omega}\mathbb{A} \subseteq \boldsymbol{\mathcal{P}}^{\omega+1}(\mathbb{A})\,.$$

Therefore we obtain

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“there is no choice function for } \left\{P_n \mid n \in \omega\right\}\text{”}$$

$$\Longleftrightarrow$$

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \left(\text{“there is no choice function for } \left\{P_n \mid n \in \omega\right\}\text{”}\right)^{\boldsymbol{\mathcal{P}}^{\omega+1}(\mathbb{A})}.$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$, by Corollary 399, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_A}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_A} \models \text{“there is no choice function for } \left\{P_n \mid n \in \omega\right\}\text{”}.$$

(4) By Theorem 388, Weak König Lemma fails inside the second Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ with $\mathbb{A} = \bigcup_{n\in\omega} P_n$ and $P_n = \{a_n, b_n\}$ as set of atoms. Because the infinite binary tree

$$T = \bigcup_{n\in\omega} \Big\{ s \in {}^n\mathbb{A} \mid \forall k \in n \; s(k) \in P_k \Big\}.$$

does not have any infinite branch (for the reason such an infinite branch would yield a choice function that would contradict Theorem 387).

Now, every element of this tree belongs to some $\boldsymbol{\mathcal{P}}^k(\mathbb{A})$ for some integer $k$ large enough. Hence, $T \subseteq \boldsymbol{\mathcal{P}}^\omega(\mathbb{A})$ which yields

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models T \in \boldsymbol{\mathcal{P}}^{\omega+1}(\mathbb{A})\,.$$

Therefore we obtain

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \Big(\text{“there exists an infinite binary tree with no infinite branch”}\Big)^{\boldsymbol{\mathcal{P}}^{\omega+1}(\mathbb{A})}.$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$, by Corollary 399, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \text{“there exists an infinite binary tree with no infinite branch”}.$$

(5) By Theorem 392, the ordered Mostowski model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ whose set of atoms is a countable set $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which is a dense ordering without least nor greatest element satisfies

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \text{“there exists a mapping } f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})\text{”}$$

Now, every element of this mapping is of the form $(\mathbb{F}, \mathbb{B})$ for some $\mathbb{F} \subseteq \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ and $\mathbb{B} \subseteq \mathcal{P}(\mathbb{A})$. Since both $\mathbb{F}$ and $\mathbb{B}$ belong to $\boldsymbol{\mathcal{P}}^1(\mathbb{A})$ we have $(\mathbb{F}, \mathbb{B}) = \big\{ \{\mathbb{F}\}, (\mathbb{F}, \mathbb{B}) \big\}$ belongs to $\boldsymbol{\mathcal{P}}^3(\mathbb{A})$. This shows that the mapping $f$ belongs to $\boldsymbol{\mathcal{P}}^4(\mathbb{A})$.

Therefore we obtain

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \Big(\text{“there exist an infinite set } \mathbb{A} \text{ and a mapping } f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})\text{”}\Big)^{\boldsymbol{\mathcal{P}}^4(\mathbb{A})}.$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$, by Corollary 399, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \text{“there exist an infinite set } \mathbb{A} \text{ and a mapping } f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})\text{”}.$$

$\square$ 400

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