

**Part V**

**Forcing**



## Chapter 13

# Forcing Conditions and Generic Filters

### 13.1 Introduction

Assuming that **ZFC** is consistent, the aim of this chapter is to prove that there exists a model of **ZFC** that does not satisfies the Continuum Hypothesis. In other words, we are going to prove that **ZFC** +  $\neg\mathbf{CH}$  is not inconsistent assuming **ZFC** is consistent. To do so, we proceed by contraposition and prove:

If **ZFC** +  $\neg\mathbf{CH}$  is inconsistent, then **ZFC** is already inconsistent.

i.e.,

$$\mathbf{ZFC} + \neg\mathbf{CH} \vdash_c \perp \implies \mathbf{ZFC} \vdash_c \perp.$$

Now, if  $\mathbf{ZFC} + \neg\mathbf{CH} \vdash_c \perp$  is satisfied, then such a proof of its inconsistency involves only finitely many formulas. Therefore, there exist  $\varphi_1, \dots, \varphi_n$  in  $\mathbf{ZFC} + \neg\mathbf{CH}$  such that for any closed formula  $\varphi$ , we have

$$\varphi_1, \dots, \varphi_n \vdash_c (\varphi \wedge \neg\varphi).$$

i.e.,

$$\vdash_c \left( (\varphi_1 \wedge \dots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi) \right).$$

From Lemma [167](#), we have that from any non-empty class **M** be any non-empty class.

$$\vdash_c \left( (\varphi_1 \wedge \dots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi) \right) \implies \vdash_c \left( (\varphi_1 \wedge \dots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi) \right)^{\mathbf{M}}.$$

Notice that

$$\begin{aligned} \left( (\varphi_1 \wedge \dots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi) \right)^{\mathbf{M}} &= \left( (\varphi_1 \wedge \dots \wedge \varphi_n) \right)^{\mathbf{M}} \longrightarrow \left( (\varphi \wedge \neg\varphi) \right)^{\mathbf{M}} \\ &= \left( (\varphi_1)^{\mathbf{M}} \wedge \dots \wedge (\varphi_n)^{\mathbf{M}} \right) \longrightarrow \left( (\varphi)^{\mathbf{M}} \wedge (\neg\varphi)^{\mathbf{M}} \right) \\ &= \left( (\varphi_1)^{\mathbf{M}} \wedge \dots \wedge (\varphi_n)^{\mathbf{M}} \right) \longrightarrow \left( (\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}} \right) \end{aligned}$$

By using *forcing* methods, one can prove that there exists some  $\mathbf{N}$  such that

$$\mathbf{ZFC} \vdash_c \left( (\varphi_1)^{\mathbf{N}} \wedge \dots \wedge (\varphi_n)^{\mathbf{N}} \right).$$

Since we also have

$$\vdash_c \left( (\varphi_1)^{\mathbf{N}} \wedge \dots \wedge (\varphi_n)^{\mathbf{N}} \right) \longrightarrow \left( (\varphi)^{\mathbf{N}} \wedge \neg(\varphi)^{\mathbf{N}} \right)$$

we obtain

$$\mathbf{ZFC} \vdash_c \left( (\varphi)^{\mathbf{N}} \wedge \neg(\varphi)^{\mathbf{N}} \right)$$

thus

$$\mathbf{ZFC} \vdash \perp.$$

As for the proof of

$$\mathbf{ZFC} \vdash_c \exists \mathbf{N} \left( (\varphi_1)^{\mathbf{N}} \wedge \dots \wedge (\varphi_n)^{\mathbf{N}} \right).$$

there exist only a finite number of formulas  $\psi_1, \dots, \psi_k$  from  $\mathbf{ZFC}$  that are really needed to conduct the proof. So, it really is

$$\psi_1, \dots, \psi_k \vdash_c \exists \mathbf{N} \left( (\varphi_1)^{\mathbf{N}} \wedge \dots \wedge (\varphi_n)^{\mathbf{N}} \right).$$

So, what we will do is consider any transitive countable model<sup>1</sup>  $\mathbf{M}$  (given by the Montague's Reflection Principle (see page 204) such that

$$\mathbf{M} \models (\psi_1 \wedge \dots \wedge \psi_k).$$

By forcing, we will obtain a transitive model

$$\mathbf{N} = \mathbf{M}[G] \models (\varphi_1 \wedge \dots \wedge \varphi_n).$$

## 13.2 Montague's Reflection Principle

**Montague's Reflection Principle** (Lévy & Montague). *Let  $\varphi_0, \dots, \varphi_n$  be any  $\mathcal{L}_{\text{ST}}$ -formulas.*

$$\mathbf{ZF} \vdash_c \forall \alpha \in \mathbf{On} \exists \beta > \alpha \quad \text{"}\varphi_0, \dots, \varphi_n \text{ are absolute for } \mathbf{V}(\beta), \mathbf{V} \text{"}$$

<sup>1</sup>Notice that both  $\mathbf{M}$  and  $\mathbf{N} = \mathbf{M}[G]$  will be sets.

*Proof of Montague's Reflection Principle:* The proof is similar to the proof of Theorem 266. First, without loss of generality we may assume that the set of formulas  $\{\varphi_0, \dots, \varphi_n\}$  is closed under sub-formulas and only contains formulas using  $\neg, \wedge$  as connectors and  $\exists$  as quantifiers. For each integer  $i \leq n$  such that  $\varphi_i$  is of the form  $\exists x \varphi_j(x, y_1, \dots, y_{k_i})$ , we define a functional  $\mathbf{G}_i : \underbrace{\mathbf{V} \times \dots \times \mathbf{V}}_{k_i} \rightarrow \mathbf{On}$  by

$$\begin{aligned} \mathbf{G}_i(y_1, \dots, y_{k_i}) &= 0 \text{ if } \left( \neg \exists x \varphi_j(x, y_1, \dots, y_{k_i}) \right)^{\mathbf{V}} \\ &= \text{least } \theta \text{ s.t. } \exists x \in \mathbf{V}(\theta) \left( \varphi_j(x, y_1, \dots, y_{k_i}) \right)^{\mathbf{V}} \end{aligned}$$

Then, for each integer  $i \leq n$  we define a functional  $\mathbf{F}_i : \mathbf{On} \rightarrow \mathbf{On}$  by

$$\begin{aligned} \mathbf{F}_i(\xi) &= \sup \{ \mathbf{G}_i(y_1, \dots, y_{k_i}) \mid y_1, \dots, y_{k_i} \in \mathbf{V}(\xi) \} \text{ if } \mathbf{G}_i \text{ is defined} \\ \mathbf{F}_i(\xi) &= 0 \text{ otherwise.} \end{aligned}$$

Given any ordinal  $\alpha$ , one defines the strictly increasing sequence  $(\beta_k)_{k \in \omega}$  and a limit ordinal  $\beta$  by:

- $\beta_0 = \alpha$
- $\beta_{k+1} = \sup \{ \beta_k + 1, \mathbf{F}_1(\beta_k), \dots, \mathbf{F}_n(\beta_k) \}$
- $\beta = \sup_{k \in \omega} \beta_k$

We show — by induction on the height of the formula — that for each integer  $i \leq n$ , one has

$$\forall y_1 \in \mathbf{V}(\beta) \dots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \varphi_i(y_1, \dots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \varphi_i(y_1, \dots, y_{k_i})^{\mathbf{V}} \right)$$

It turns out that the only interesting case is when  $\varphi_i$  is of the form  $\exists x \varphi_j(x, y_1, \dots, y_{k_i})$ . So, we have to check that

$$\forall y_1 \in \mathbf{V}(\beta) \dots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \left( \exists x \varphi_j(x, y_1, \dots, y_{k_i}) \right)^{\mathbf{V}(\beta)} \longleftrightarrow \left( \exists x \varphi_j(x, y_1, \dots, y_{k_i}) \right)^{\mathbf{V}} \right)$$

i.e.,

$$\forall y_1 \in \mathbf{V}(\beta) \dots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V}(\beta) \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \exists x \in \mathbf{V} \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}} \right)$$

Clearly, the direction

$$\forall y_1 \in \mathbf{V}(\beta) \dots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V}(\beta) \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}(\beta)} \longrightarrow \exists x \in \mathbf{V} \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}} \right)$$

is taken care of by the induction hypothesis. So, we show

$$\forall y_1 \in \mathbf{V}(\beta) \dots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V} \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}} \longrightarrow \exists x \in \mathbf{V}(\beta) \varphi_j(x, y_1, \dots, y_{k_i})^{\mathbf{V}(\beta)} \right)$$

We fix  $y_1 \in \mathbf{V}(\beta), \dots, y_{k_i} \in \mathbf{V}(\beta)$ . For some large enough integer  $p$ , one has

$$\{y_1, \dots, y_{k_i}\} \subseteq \mathbf{V}(\beta_p).$$

By construction, there exists  $x \in \mathbf{V}(\mathbf{G}_i(y_1, \dots, y_{k_i}))$  such that  $(\varphi_j(x, y_1, \dots, y_{k_i}))^{\mathbf{V}}$ . Since  $\mathbf{G}_i(y_1, \dots, y_{k_i}) \leq \mathbf{F}_i(\beta_p) \leq \beta_{p+1}$ , it follows that there exists  $x \in \mathbf{V}(\beta_{p+1}) \subseteq \mathbf{V}(\beta)$  such that  $(\varphi_j(x, y_1, \dots, y_{k_i}))^{\mathbf{V}}$ . Finally, by induction hypothesis, there exists  $x \in \mathbf{V}(\beta)$  such that  $(\varphi_j(x, y_1, \dots, y_{k_i}))^{\mathbf{V}(\beta)}$ .

□ Montague's Reflection Principle

**Corollary 278.** *For every finite set of formulas  $\{\varphi_0, \dots, \varphi_n\} \subseteq \mathbf{ZFC}$ ,*

$$\mathbf{ZFC} \vdash_c \exists \mathbf{M} \left( |\mathbf{M}| = \aleph_0 \wedge \text{"}\mathbf{M} \text{ is transitive"} \wedge \left( \bigwedge_{0 \leq i \leq n} \varphi_i \right)^{\mathbf{M}} \right).$$

*Proof of Corollary 278:* By Montague's Reflection Principle, since  $\bigwedge_{0 \leq i \leq n} \varphi_i$  holds in  $\mathbf{V}$ , there exists some ordinal  $\beta$  such that  $\bigwedge_{0 \leq i \leq n} \varphi_i$  holds in  $\mathbf{V}(\beta)$ . Then, since the language of set theory is finite, by Löwenheim-Skolem Theorem (see [2, 3, 4, 5, 6, 33]), there exists some countable model  $\mathbf{N}$  such that

$$\mathbf{N} \models \bigwedge_{0 \leq i \leq n} \varphi_i.$$

Notice that, although  $\mathbf{V}(\beta)$  is transitive, this may not be the case with  $\mathbf{N}$ . But by the Mostowski Collapsing Theorem (page 113) there exists both some transitive class  $\mathbf{M}$ , and an isomorphism  $\iota : (\mathbf{N}, \in) \xrightarrow{\text{isom.}} (\mathbf{M}, \in)$ . Finally, being isomorphic,  $\mathbf{N}$  and  $\mathbf{M}$  are elementary equivalent, i.e., they satisfy the same closed formulas, which yields  $\mathbf{M}$  is a transitive countable set that satisfies

$$\mathbf{M} \models \bigwedge_{0 \leq i \leq n} \varphi_i.$$

□ 278

### 13.3 Posets and Generic Filters

A *countable transitive model (c.t.m.)* of “**ZFC**” is a countable transitive model of a “sufficiently large number of axioms of **ZFC**”.

**Definition 279** (Notion of Forcing).

- A notion of forcing is a partial order  $(\mathbb{P}, \leq)$ .  
It is often abbreviated as  $\mathbb{P}$ , and referred to as a poset.  
We also use the notation  $(\mathbb{P}, \leq, \mathbb{1})$  when the poset admits a maximum element  $\mathbb{1}$ .
- The elements of  $\mathbb{P}$  are called conditions.
- Given two conditions  $p, q \in \mathbb{P}$ , we say that  $p$  is stronger than  $q$  if  $p \leq q$ .

**Definition 280** (Poset). Let  $(\mathbb{P}, \leq, \mathbb{1})$  be a poset with maximal element  $\mathbb{1}$ , and let  $p, q \in \mathbb{P}$ . We say that

- $p$  and  $q$  are comparable if either  $p \leq q$  or  $q \leq p$  holds;
- $p$  and  $q$  are compatible if there exists  $r \in \mathbb{P}$  such that  $r \leq p$  and  $r \leq q$ ;
- we write  $p \perp q$  when  $p$  and  $q$  incompatible. i.e., when they are not compatible;
- a subset  $A \subseteq \mathbb{P}$  is an (strong) antichain if for all  $p, q \in A$ ,  $p \perp q$ ;
- a subset  $D \subseteq \mathbb{P}$  is dense in  $\mathbb{P}$  if for all  $p \in \mathbb{P}$  there exists  $q \in D$  such that  $q \leq p$ .

**Example 281.** Let  $\mathbb{P} = \mathcal{P}(X) \setminus \{\emptyset\}$ , with  $p \leq q$  if and only if  $p \subseteq q$ . In this case, one has

- $\mathbb{1} = X$
- if  $p \cap q \neq \emptyset$ , then  $p \cap q \leq p, q$
- $p \perp q$  if and only if  $p \cap q = \emptyset$
- $\left\{ \{p\} \mid n \in X \right\}$  is both an antichain and dense in  $\mathbb{P}$ .

**Definition 282** (Filter). Let  $(\mathbb{P}, \leq, \mathbb{1})$  be a notion of forcing.

$$G \subseteq \mathbb{P} \text{ is a filter on } \mathbb{P} \iff \left\{ \begin{array}{l} \forall p \in G \ \forall q \in G \ \exists r \in G \ (r \leq p \wedge r \leq q) \\ \text{and} \\ \forall p \in G \ \forall q \in \mathbb{P} \ (p \leq q \longrightarrow q \in G). \end{array} \right.$$

All the filters we will consider will be non-constructive. We will essentially claim that “there exists some filter  $G...$ ” by mean of a proof by contradiction. i.e., assuming that such a filter does not exist, leads to some contradiction. Therefore, there exists such a filter... So, asking for examples of such filters is useless.

**Definition 283** (Genericity). *Let  $(\mathbb{P}, \leq, \mathbb{1})$  be a notion of forcing and  $\mathbf{M}$  be any set (or class).*

$$G \subseteq \mathbb{P} \text{ is } \mathbb{P}\text{-generic over } \mathbf{M} \iff \begin{cases} G \text{ is a filter on } \mathbb{P} \\ \text{and} \\ \forall D \subseteq \mathbb{P} \quad \left( (D \text{ dense in } \mathbb{P} \wedge D \in \mathbf{M}) \longrightarrow D \cap G \neq \emptyset \right). \end{cases}$$

**Example 284.** *Let  $\mathbb{P}$  be the set of functions such that the domain is finite and included in  $\omega$  and the image is included in  $\omega_1$ .*

$$\mathbb{P} = \left\{ p \subseteq (\omega \times \omega_1) \mid |p| < \omega \text{ and } \forall a, b, c \in \omega \quad \left( ((a, b) \in p \wedge (a, c) \in p) \longrightarrow b = c \right) \right\}$$

*For  $p, q \in \mathbb{P}$ , let  $p \leq q$  if and only if  $p \supseteq q$  ( $p$  extends  $q$ ). We thus have  $\mathbb{1} = \emptyset$ .*

*Towards a contradiction, assume  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{V}$ . Set  $f = \bigcup G$  and notice that*

- *$f$  is a binary relation.*
- *$f$  is a function: given any integer  $n$ , if both  $(n, x)$  and  $(n, y)$  belong to  $f = \bigcup G$ , then consider any  $p, q \in G$  such that  $(n, x) \in p$  and  $(n, y) \in q$ . Notice that  $G$  being a filter, there exists  $r \in G$  such that  $r \leq p, q$  (i.e.,  $r$  extends both  $p$  and  $q$ ). Therefore,  $r(n) = p(n) = q(n)$  holds which shows that  $x = y$ .*
- *$\text{dom}(f) = \omega$ , since for all  $n \in \omega$  the set*

$$D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$$

*is dense in  $\mathbb{P}$  and its intersection with  $G$  is thus nonempty since  $D_n \in \mathbf{V}$  and  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{V}$ .*

- *$\text{ran}(f) = \omega_1$  since for all ordinals  $\alpha < \omega_1$ , the set:*

$$E_\alpha = \{p \in \mathbb{P} \mid \alpha \in \text{ran}(p)\}$$

*is dense in  $\mathbb{P}$  and belongs to  $\mathbf{V}$ .*

So, we have obtained  $f = \bigcup G : \omega \xrightarrow{\text{onto}} \omega_1$ , which is a contradiction.  
 This shows that our assumption fails. i.e., there is no  $\mathbb{P}$ -generic filter over  $\mathbf{V}$ .

That may seem a problem at first glance, but since our aim it to consider countable transitive models of “**ZFC**”, the latter result is not in our scope. To, the contrary, when  $\mathbf{M}$  is a countable set as opposed to the whole universe  $\mathbf{V}$ , we have a positive result.

**Lemma 285.** *Let  $\mathbf{M}$  be any countable set,  $\mathbb{P}$  any poset in  $\mathbf{M}$ , and  $p \in \mathbb{P}$ . There exists  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  such that  $p \in G$ .*

Notice that we do not claim that  $G$  belongs to  $\mathbf{M}$ . In fact, most of the time we will have  $G \notin \mathbf{M}$ .  
*Proof of Lemma 285:* Let us consider (in  $\mathbf{V}$ ) an enumeration  $(D_n)_{n \in \omega}$  of the sets of the form  $D$  that satisfy both

- (1)  $D \subseteq \mathbb{P}$  is dense in  $\mathbb{P}$ .
- (2)  $D \in \mathbf{M}$ .

Let  $p_0 \in D_0$  be such that  $p_0 \leq p$ ; we define by induction on  $\omega$  a sequence  $(p_n)_{n \in \omega}$  such that:

$$p_{n+1} \leq p_n \quad \text{and} \quad p_{n+1} \in D_{n+1}.$$

Let us consider  $G$ , the filter generated by  $(p_n)_{n \in \omega}$ :

$$G = \{q \in \mathbb{P} \mid \exists n \in \omega, p_n \leq q\}.$$

Since the formula “ $D$  is dense in  $\mathbb{P}$ ” is absolute — i.e., “ $D$  is dense in  $\mathbb{P}$ ”  $\longleftrightarrow$  (“ $D$  is dense in  $\mathbb{P}$ ”) $^{\mathbf{M}}$  —  $G$  is a filter whose intersection with the dense sets of  $\mathbf{M}$  is nonempty.  $G$  is thus  $\mathbb{P}$ -generic over  $\mathbf{M}$  and  $p \in G$ .

□ 285

We said that in most cases, the generic filter will not belong to the ground model  $\mathbf{M}$ . We present now a condition on  $\mathbb{P}$  which guarantees  $G \notin \mathbf{M}$ .

**Lemma 286.** *Let  $\mathbf{M}$  be any transitive model of “**ZFC**”, and  $\mathbb{P} \in \mathbf{M}$  a notion of forcing that satisfies  $\forall p \in \mathbb{P} \exists r, q \in \mathbb{P} (q \leq p \wedge r \leq p \wedge q \perp r)$ .*

$$G \text{ is } \mathbb{P}\text{-generic over } \mathbf{M} \implies G \notin \mathbf{M}.$$

Notice that in this lemma, we do not simply consider any transitive set  $\mathbf{M}$ , but rather a transitive model of “**ZFC**”. The reason for this, is that we need  $\mathbf{M}$  to satisfy some very basic properties. For instance, we need that given  $\mathbb{P}$  and  $G$  that belong to  $\mathbf{M}$ , the set  $\mathbb{P} \setminus G$  also belongs to  $\mathbf{M}$ .

*Proof of Lemma 286.* Suppose, for the sake of contradiction, that  $G \in \mathbf{M}$ .

- We let  $D = \mathbb{P} \setminus G$  and  $\mathbf{M}$  satisfy enough axioms from **ZFC** such that  $D \in \mathbf{M}$  holds.
- We then show that  $D$  is dense in  $\mathbb{P}$ : take any  $p \in \mathbb{P}$ , there exist  $r, q \in \mathbb{P}$  such that  $q \leq p$ ,  $r \leq p$  and  $q \perp r$ . But  $r$  and  $q$  cannot both be elements of  $G$ , for otherwise, since  $G$  is a filter on  $\mathbb{P}$ , there would exist  $s \in \mathbb{P}$  such that  $s \leq r$  and  $s \leq q$ , and this would contradict the fact that  $p$  and  $r$  are incompatible. It follows that either  $r$  or  $q$  is a member of  $D$ , therefore  $D$  is dense in  $\mathbb{P}$ .

Finally, we have the following contradiction:

$$(1) \ D \in \mathbf{M} \qquad (2) \ G \text{ is } \mathbb{P}\text{-generic over } \mathbf{M} \qquad (3) \ \begin{aligned} D \cap G &= (\mathbb{P} \setminus G) \cap G \\ &= \emptyset. \end{aligned}$$

□ 286

We need a last result which seems technical at first glance but will prove extremely useful later on.

**Definition 287.** Let  $\mathbb{P}$  be any poset,  $E \subseteq \mathbb{P}$ , and  $p \in \mathbb{P}$ .

$$E \text{ is } \textbf{dense below } p \iff \forall q \leq p \ \exists r \in E \ r \leq q.$$

So, being dense below  $p$  is really what it says it is: being dense but only with regards to the sub-poset formed of all forcing conditions that lies below  $p$ .

**Lemma 288.** Let  $\mathbf{M}$  be a transitive model of “**ZFC**”,  $\mathbb{P}$  a notion of forcing such that  $\mathbb{P} \in \mathbf{M}$ , and  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ . Let also  $p \in \mathbb{P}$  and  $E \subseteq \mathbb{P}$  be such that  $E \in \mathbf{M}$ . Then,

- either  $G \cap E \neq \emptyset$ , or
- there exists  $q \in G$  such that for all  $r \in E$ ,  $r \perp q$ .

Furthermore, if  $E$  is dense below  $p \in G$ , then  $G \cap E \neq \emptyset$ .

We will use many times the latter statement: every set which is dense below some element which belongs to the generic filter  $G$  also intersects this filter  $G$ .

*Proof of Lemma 288:* To prove the first part of the lemma, let

$$D = \{s \in \mathbb{P} \mid \exists r \in E \ s \leq r\} \cup \{s \in \mathbb{P} \mid \forall r \in E \ s \perp r\}.$$

First, we notice that  $D$  is dense. Indeed, let  $s \in \mathbb{P}$ , then either there exists  $r \in E$  such that  $r$  and  $s$  are compatible, and so there exists  $q \in \mathbb{P}$  with  $q \leq s$  and  $q \leq r$ , which implies that  $q \in D$ ; or, for all  $r \in E$ , we have  $r \perp s$  and thus  $s \in D$ . Since  $s \leq s$ ,  $D$  is dense in  $\mathbb{P}$ .

Moreover,  $D \in \mathbf{M}$  because  $E \in \mathbf{M}$ . As a result of  $G$  being  $\mathbb{P}$ -generic over  $\mathbf{M}$ , its intersection with  $D$  is non-empty. Take any  $s \in D \cap G$ . Since  $s \in D$ , either there exists  $r \in E$  such that  $s \leq r$ , and so, since  $G$  is a filter,  $r \in G$  and  $G \cap E \neq \emptyset$ ; or for all  $r \in E$ ,  $s \perp r$ .

For the second part of the lemma, we assume  $p \in G$  and  $E$  is dense below  $p$ . Towards a contradiction we also assume  $G \cap E = \emptyset$ . Then, the previous result provides some  $q \in G$  such that for all  $r \in E$ ,  $r \perp q$ . Since  $G$  is a filter, there exists  $s \in G$  such that  $s \leq p$  and  $s \leq q$ . But  $E$  is dense below  $p$ , so there exists  $r \in E$  such that  $r \leq s$ . We have obtained  $r \in E$  such that  $r \leq q$ . This contradicts the property that  $q$  satisfies:  $\forall r' \in E \ r' \perp q$ .

□ 288

Our goal is now, starting from any *c.t.m.* of “**ZFC**”  $\mathbf{M}$  and any  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ , to construct a *c.t.m.* of “**ZFC**”  $\mathbf{M}[G]$  — called a generic extension — which satisfies

$$\begin{array}{lll} \circ \mathbf{M} \subseteq \mathbf{M}[G]; & \circ (\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}; & \circ G \in \mathbf{M}[G]. \end{array}$$



# Chapter 14

## $\mathbb{P}$ -names and Generic Extensions

### 14.1 $\mathbb{P}$ -names

Some of the elements in  $\mathbf{M}[G]$  will be new sets in the sense that they do not exist inside  $\mathbf{M}$ . However, each and every one of these elements will have a *name* in  $\mathbf{M}$ . To say it differently and see things the other way round, every set that belongs to  $\mathbf{M}[G]$  already pre-exists in  $\mathbf{M}$  in that it already has a name, even though a key that is required to decode and identify it is missing in  $\mathbf{M}$  (this key is the filter  $G$ ).

**Definition 289** ( $\mathbb{P}$ -name).  $\tau$  is a  $\mathbb{P}$ -name if and only if  $\tau$  is a binary relation and for all  $(\sigma, p) \in \tau$ ,  $\sigma$  is a  $\mathbb{P}$ -name and  $p \in \mathbb{P}$ .

Notice that  $\emptyset$  satisfies this definition, hence  $\emptyset$  is a  $\mathbb{P}$ -name.

Formally,  $\mathbb{P}$ -names are defined recursively. First consider the following binary relation  $E$  on  $\mathbb{P}$ -names:

$$\sigma E \tau \iff \exists p \in \mathbb{P} (\sigma, p) \in \tau.$$

$E$  is well-founded since:

$$\sigma E \tau \implies rk(\sigma) < rk(\tau).$$

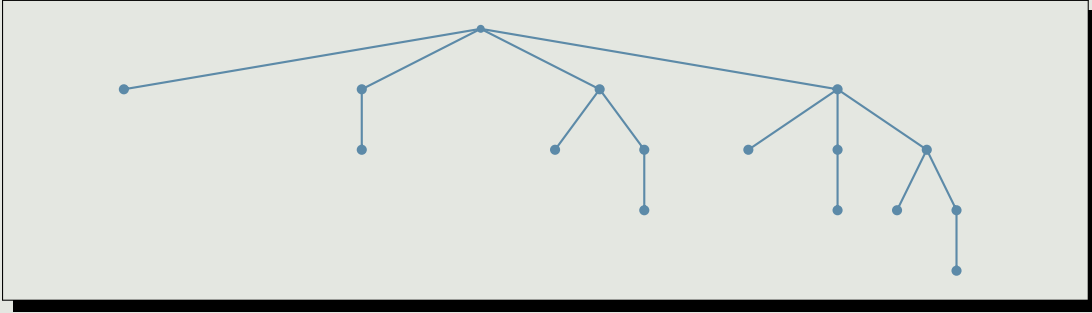
We set:

$$\mathbf{F}(\tau) = \begin{cases} 1, & \text{if } \forall x \in \tau ("x \text{ is a couple } (x_1, x_2)" \wedge x_2 \in \mathbb{P} \wedge \mathbf{F}(x_1) = 1); \\ 0, & \text{otherwise.} \end{cases}$$

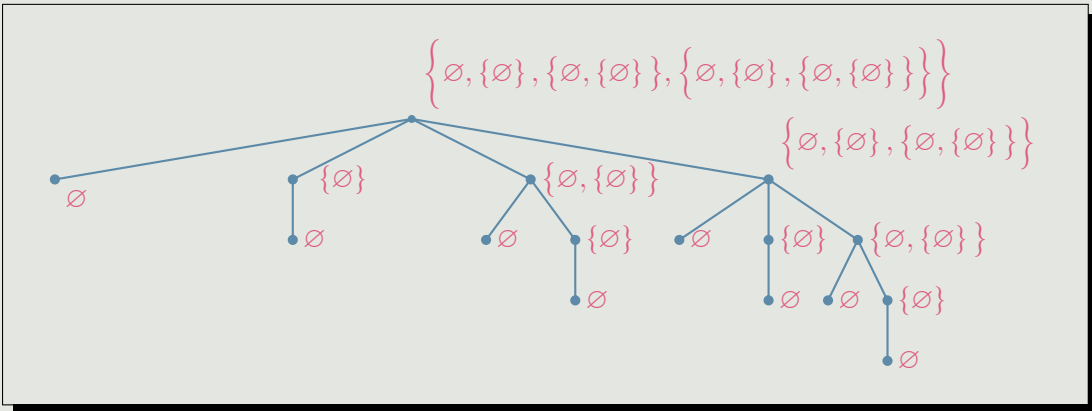
We may define  $\mathbf{F}$  as  $\mathbf{F}(\tau) = \mathbf{H}(\mathbf{F} \upharpoonright_{\text{pred}_E(\tau)}, \tau, \mathbb{P})$  where all notions used to define  $\mathbf{H}$  are  $\Delta_0^{rud}$ , hence  $\mathbf{H}$  is absolute for transitive models of “**ZFC**”. Then, the class of all  $\mathbb{P}$ -names is the set

$$\{\tau \mid \mathbf{F}(\tau) = 1\}.$$

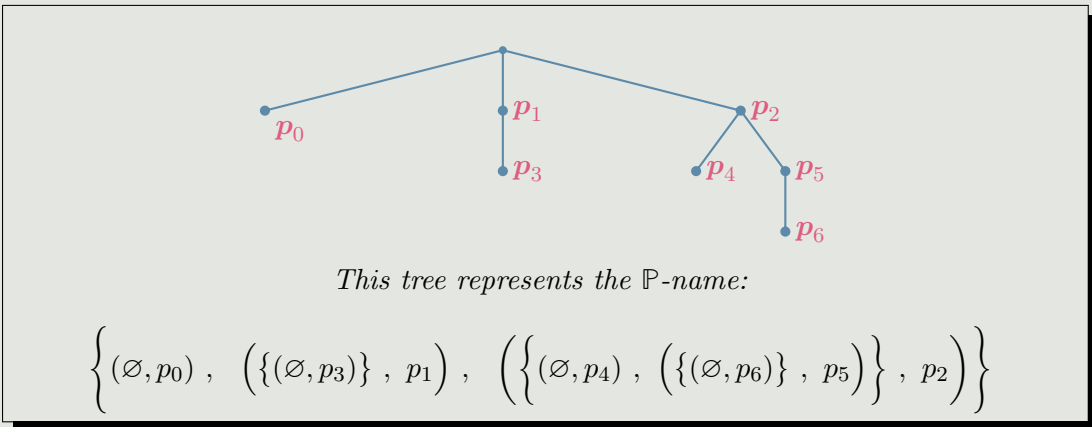
**Example 290.** *In order to get the right intuition about  $\mathbb{P}$ -names, it is fruitful to go back to the way we represented well-founded sets by well-founded trees. For instance, in Example 152 where we presented a tree*



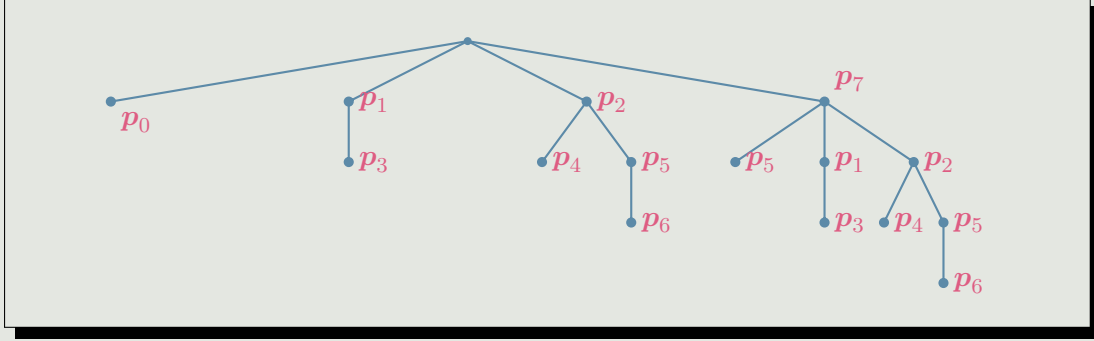
that represents the ordinal 4 when we associate to each node  $\mathbf{n}$  the set  $\hat{\mathbf{n}} = \{\hat{\mathbf{c}} \mid \mathbf{c} \text{ is a child of } \mathbf{n}\}$ :



When represented by trees,  $\mathbb{P}$ -names are not just well-founded trees, but rather some particular **colored** well-founded trees: those whose nodes — except for the root — are “colored” by forcing conditions. For instance:



Some  $\mathbb{P}$ -name which is a coloring of the set 4:



This tree represents the  $\mathbb{P}$ -name:

$$\left\{ \begin{array}{l} (\emptyset, p_0), \\ \left( \left\{ (\emptyset, p_3) \right\}, p_1 \right), \left( \left\{ (\emptyset, p_4), \left( \left\{ (\emptyset, p_6) \right\}, p_5) \right\}, p_2 \right), \\ \left( \left\{ (\emptyset, p_5), \left( \left\{ (\emptyset, p_3) \right\}, p_1 \right), \left( \left\{ (\emptyset, p_4), \left( \left\{ (\emptyset, p_6) \right\}, p_5) \right\}, p_2 \right) \right\}, p_7 \right) \end{array} \right\}$$

**Definition 291.** Let  $\mathbf{V}^{\mathbb{P}} = \{\mathbb{P}\text{-names}\}$ . If  $\mathbf{M}$  is a transitive model of “ZFC”, then

$$\mathbf{M}^{\mathbb{P}} = \mathbf{M} \cap \mathbf{V}^{\mathbb{P}}.$$

By absoluteness,

$$\mathbf{M}^{\mathbb{P}} = \left\{ \tau \in \mathbf{M} \mid (\tau \text{ is a } \mathbb{P}\text{-name})^{\mathbf{M}} \right\}.$$

## 14.2 Generic Extensions

Starting from  $\mathbf{M}$  any transitive model of “ZFC”, we create another model — known as a generic extension — by considering all the  $\mathbb{P}$ -names that belong to  $\mathbf{M}$  ( $\mathbf{M}^{\mathbb{P}} = \mathbf{M} \cap \mathbf{V}^{\mathbb{P}}$ ) and

“unscrambling” them with the use a filter  $G$  — that is generic over  $\mathbf{M}$  — which plays the role of a decryption key.

**Definition 292** (Generic Extension). *Let  $(\mathbb{P}, \leq, \mathbb{1})$  be a notion of forcing*

(1) *Given any  $\tau \in \mathbf{V}^{\mathbb{P}}$ , and  $G \subseteq \mathbb{P}$  a filter, we recursively define*

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G (\sigma, p) \in \tau\}.$$

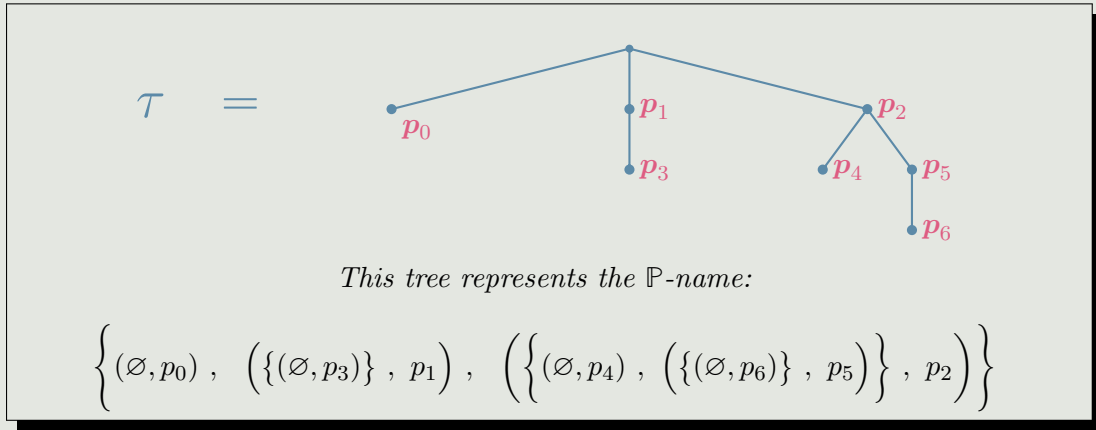
(2) *Given  $\mathbf{M}$  any transitive model of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  and  $G \subseteq \mathbb{P}$  a filter, we define*

$$\mathbf{M}[G] = \{(\tau)_G \mid \tau \in \mathbf{M}^{\mathbb{P}}\}.$$

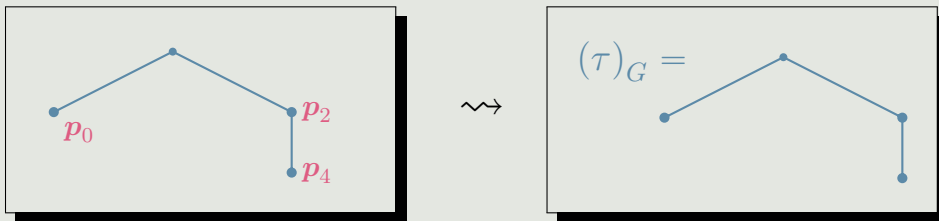
**Notation 293.** *Given  $\mathbf{M}$  any transitive model of “ZFC”,  $\mathbb{P} \in \mathbf{M}$ ,  $G \subseteq \mathbb{P}$  any filter  $\mathbb{P}$ -generic over  $\mathbf{M}$ , and  $x \in \mathbf{M}[G]$ , we write  $\dot{x}$  for any  $\mathbb{P}$ -name for  $x$ . i.e.,*

$$\dot{x} \in \mathbf{M}^{\mathbb{P}} \text{ and } (\dot{x})_G = x.$$

**Example 294.** *Consider the following  $\mathbb{P}$ -name  $\tau$  that was introduced in Example 290:*



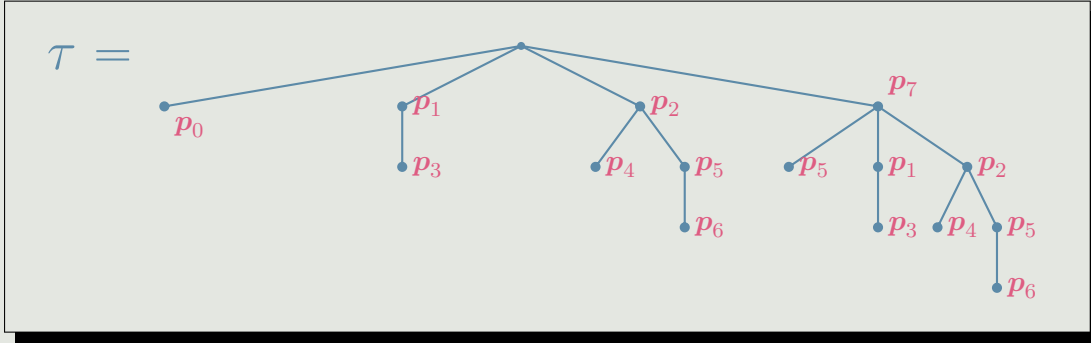
If for each integer  $n$ ,  $p_n \in G \iff n$  is even. Then  $(\tau)_G$  is obtained by removing the nodes colored by forcing conditions not in the filter, then getting rid of the coloring:



So, we obtain

$$(\tau)_G = \{\emptyset, \{\emptyset\}\}.$$

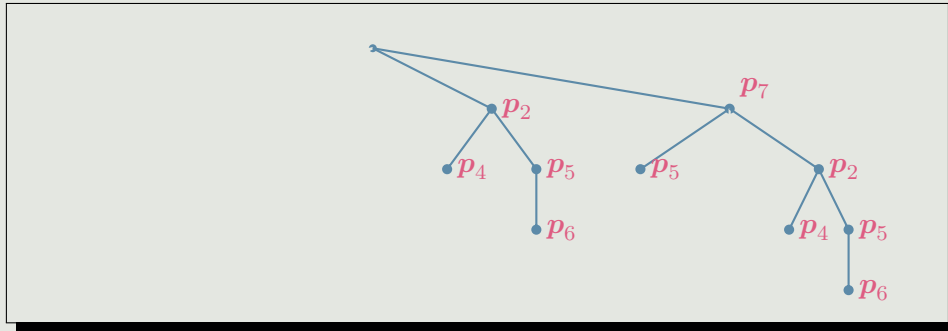
**Example 295.** Consider the following  $\mathbb{P}$ -name  $\tau$  that was introduced in Example 290:



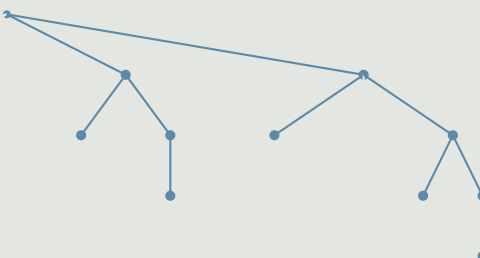
This tree represents the  $\mathbb{P}$ -name:

$$\tau = \left\{ \begin{array}{l} (\emptyset, p_0), \\ \left( \{(\emptyset, p_3)\}, p_1 \right), \left( \left\{ (\emptyset, p_4), \left( \{(\emptyset, p_6)\}, p_5) \right\}, p_2 \right\}, p_7 \right) \\ \left( \left\{ (\emptyset, p_5), \left( \{(\emptyset, p_3)\}, p_1 \right), \left( \left\{ (\emptyset, p_4), \left( \{(\emptyset, p_6)\}, p_5) \right\}, p_2 \right) \right\}, p_7 \right) \right\} \end{array} \right.$$

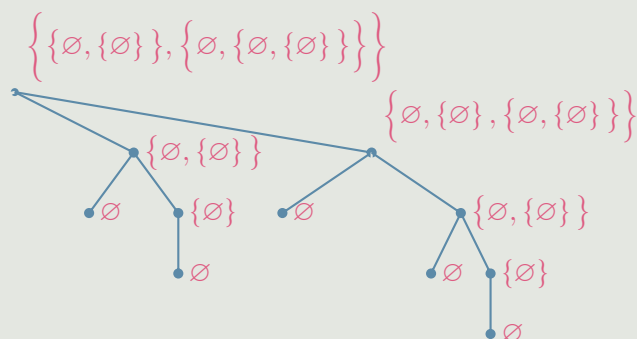
If for each integer  $n$ ,  $p_n \in G \iff n \notin \{0, 1, 3\}$ . Then  $(\tau)_G$  is obtained by removing the nodes colored by forcing conditions not in the filter, then getting rid of the coloring:



$$(\tau)_G =$$



$$(\tau)_G =$$

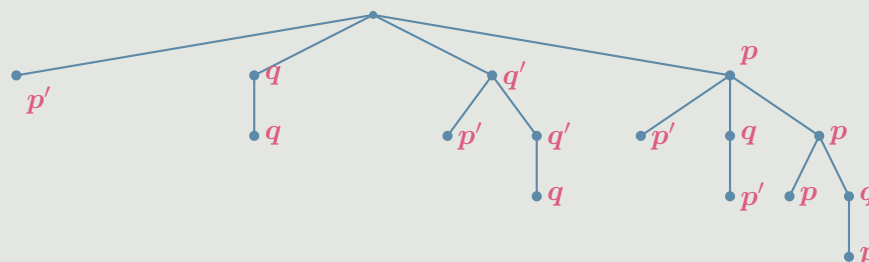


So, we obtain

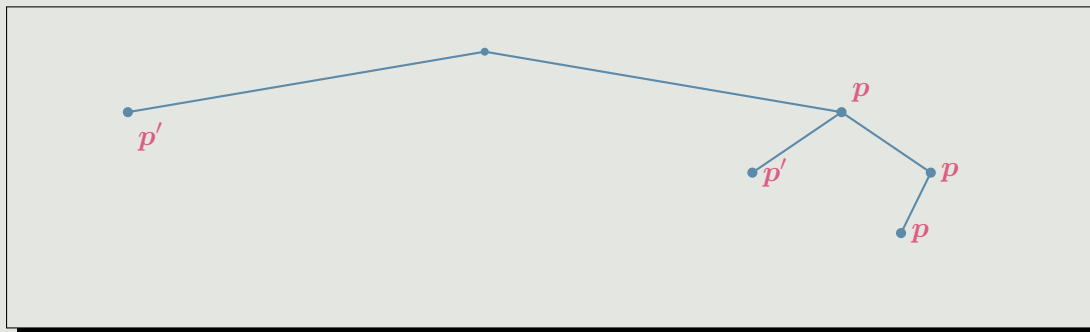
$$(\tau)_G = \left\{ \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \right\}.$$

**Example 296.** Consider

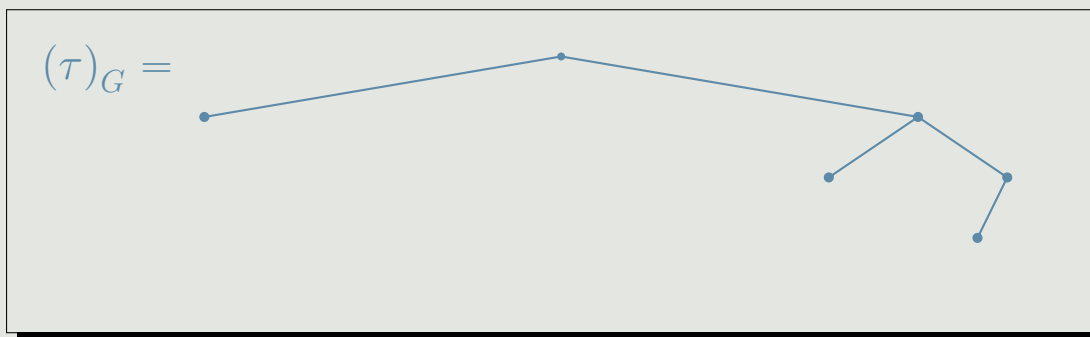
$$\tau =$$



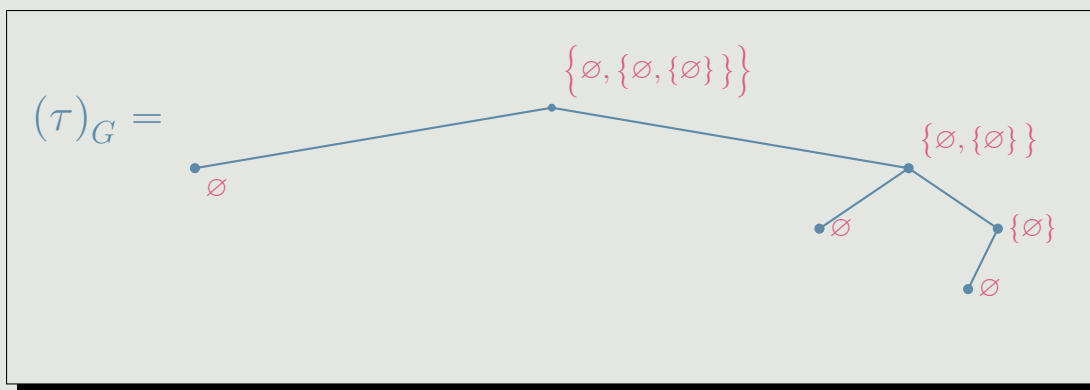
with  $p, p' \in G$ , but  $q, q' \notin G$ . This yields the following tree:



and by dropping the forcing conditions:



which yields



hence,

$$(\tau)_G = \left\{ \emptyset, \left\{ \emptyset, \left\{ \emptyset \right\} \right\} \right\}.$$

**Definition 297.** By  $\in$ -induction, we define for any  $x \in \mathbf{M}$ ,

$$\check{x} = \{(\check{y}, \mathbb{1}) \mid y \in x\}.$$

We will also consider

$$\Gamma = \{(\check{p}, p) \mid p \in \mathbb{P}\}.$$

These names are called *canonical names* for any set that belongs to  $\mathbf{M}$  and the filter, respectively.

**Lemma 298.** Let  $(\mathbb{P}, \leq, \mathbb{1})$  be a notion of forcing, and  $G \subseteq \mathbb{P}$  a filter.

$$(1) \quad (\check{x})_G = x.$$

$$(2) \quad (\Gamma)_G = G.$$

An immediate consequence of this lemma is that as long as  $\mathbf{M}$  is a model which is closed under the “check” operation<sup>1</sup> — which comes to asking that  $\mathbf{M}$  be a model of “ZFC”, where “ZFC” contains the axioms that are necessary to prove that  $\mathbf{V}$  is closed under the “check” operation — then both  $\mathbf{M} \subseteq \mathbf{M}[G]$  and  $G \in \mathbf{M}[G]$  hold.

*Proof of Lemma 298:*

(1) By  $\in$ -induction, since  $\check{\emptyset} = \emptyset$  and  $(\emptyset)_G = \emptyset$ :

$$\begin{aligned} (\check{x})_G &= \{(\check{y})_G \mid \exists p \in G \ (\check{y}, p) \in \check{x}\} \\ &= \{(\check{y})_G \mid (\check{y}, \mathbb{1}) \in \check{x}\} \\ &= \{y \mid y \in x\} \\ &= x. \end{aligned}$$

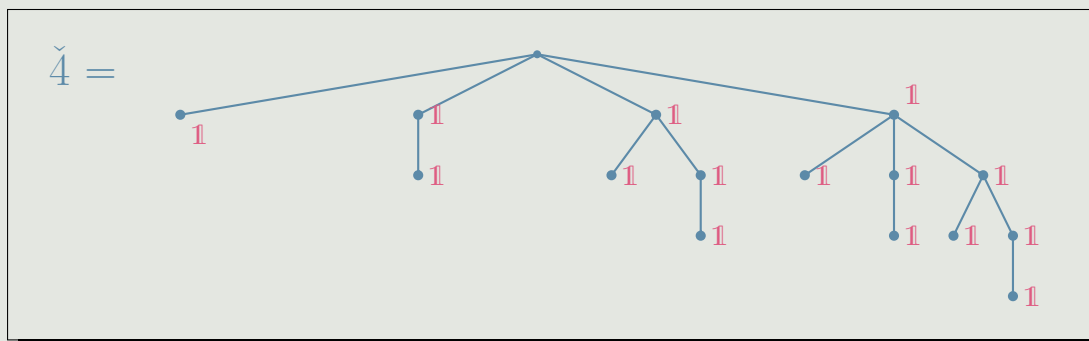
(2)

$$\begin{aligned} (\Gamma)_G &= \{(\check{p})_G \mid \exists p \in G \ (\check{p}, p) \in \Gamma\} \\ &= \{p \mid p \in G\} \\ &= G. \end{aligned}$$

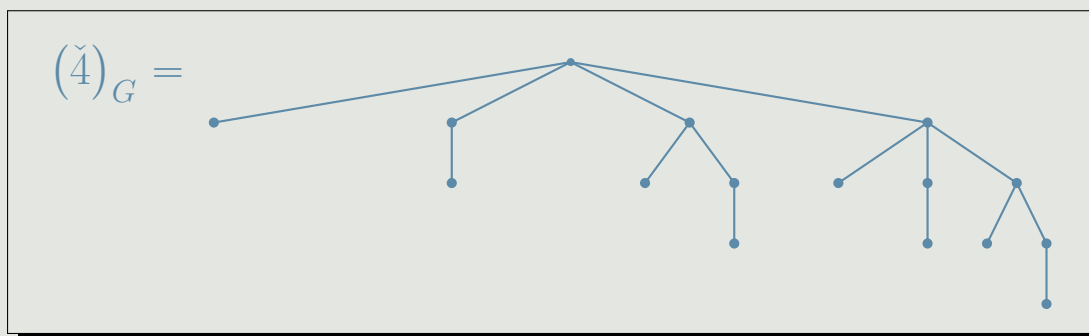
□ 298

<sup>1</sup>This means that  $\mathbf{M}$  satisfies  $\check{x} \in \mathbf{M}$  holds for every  $x \in \mathbf{M}$

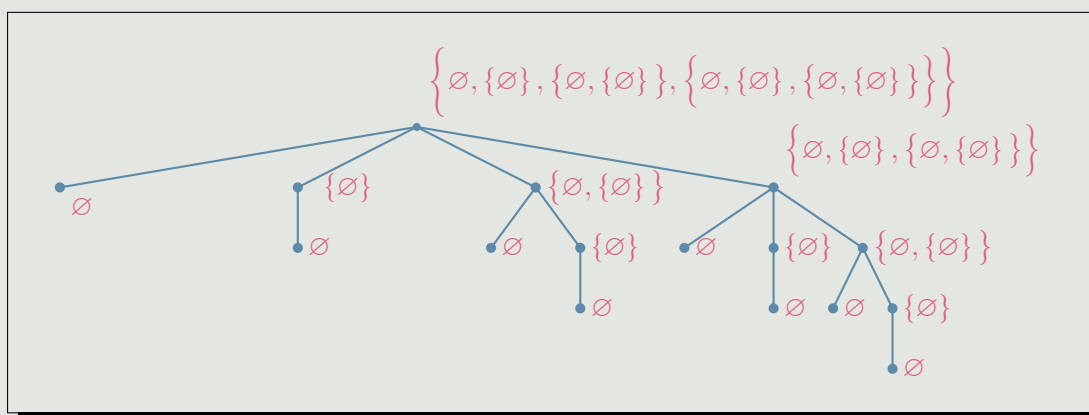
**Example 299.** For instance,  $\check{4}$  corresponds to:



which yields



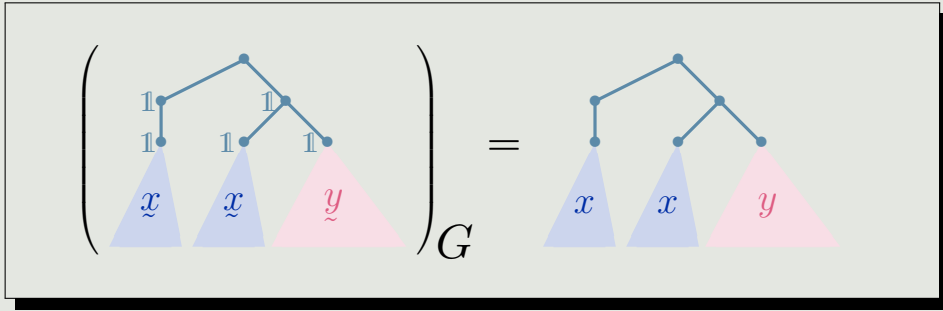
which is nothing but the ordinal 4:



**Example 300.** We define  $\text{couple} : \mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \rightarrow \mathbf{M}^{\mathbb{P}}$  so that given any  $\underline{x}, \underline{y} \in \mathbf{M}^{\mathbb{P}}$ , and any  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ ,  $\text{couple}(\underline{x}, \underline{y}) = \tau$  with  $(\tau)_G = ((\underline{x})_G, (\underline{y})_G)$ . This is the canonical name

$$\tau = \overline{\{\{\underline{x}\}, \{\underline{x}, \underline{y}\}\}} = \left\{ \left( \{(\underline{x}, \mathbb{1})\}, \mathbb{1} \right), \left( \{(\underline{x}, \mathbb{1}), (\underline{y}, \mathbb{1})\}, \mathbb{1} \right) \right\},$$

as shown in the picture below.



**Lemma 301.** If  $\mathbf{M}$  is a transitive model of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  is a notion of forcing, and  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ , then

- (1)  $\mathbf{M} \subseteq \mathbf{M}[G]$
- (2)  $G \in \mathbf{M}[G]$ .

*Proof of Lemma 301:* Both statements are consequence of previous Lemma 298. Indeed, since for all  $x \in \mathbf{M}$ , one has  $\check{x} \in \mathbf{M}^{\mathbb{P}}$  and  $x = (\check{x})_G \in \mathbf{M}[G]$ , it follows that  $\mathbf{M} \subseteq \mathbf{M}[G]$ . Moreover,  $\Gamma \in \mathbf{M}^{\mathbb{P}}$ , so  $G = (\Gamma)_G \in \mathbf{M}[G]$ .

□ 301

**Lemma 302.** Let  $\mathbf{M}$  be a transitive model of “ZFC”,  $\mathbb{P}$  a notion of forcing, and  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ . Then

- (1)  $\mathbf{M}[G]$  is transitive,
- (2) if  $\mathbf{N}$  is a transitive model of “ZFC” with  $\mathbf{M} \subseteq \mathbf{N}$  such that  $G \in \mathbf{N}$ , then  $\mathbf{M}[G] \subseteq \mathbf{N}$ .

*Proof of Lemma 302:*

- (1) Given any  $x \in (\tau)_G \in \mathbf{M}[G]$ , there exists  $\sigma \in \mathbf{M}^{\mathbb{P}}$  and  $p \in \mathbb{P}$  such that  $(\sigma, p) \in \tau$  and  $x = (\sigma)_G$ . Thus,  $x = (\sigma)_G \in \mathbf{M}[G]$ .
- (2) Recall that

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G (\sigma, p) \in \tau\},$$

and that we defined the functional  $F$  as:

$$\begin{aligned} \mathbf{F} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}[G] \\ \tau &\longmapsto (\tau)_G \end{aligned}$$

Therefore:

$$\mathbf{F}(\tau) = \mathbf{H}(\mathbf{F} \upharpoonright_{\text{pred}_E(\tau)}, G, \tau).$$

Since  $\mathbf{H}$  is absolute, we have  $((\tau)_G)^{\mathbf{N}} = (\tau)_G$  and therefore  $\mathbf{M}[G] \subseteq \mathbf{N}$ .

□ 302

If  $\mathbf{M}[G]$  is a transitive model of “**ZFC**”, the second part of this Lemma states that  $\mathbf{M}[G]$  is the smallest transitive model of “**ZFC**” such that both  $\mathbf{M} \subseteq \mathbf{M}[G]$  and  $G \in \mathbf{M}[G]$  hold.

**Lemma 303.** *Let  $\mathbf{M}$  be a transitive model of “**ZFC**”,  $\mathbb{P}$  a notion of forcing with  $\mathbb{P} \in \mathbf{M}$ , and  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ .*

$$(\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}$$

*Proof of Lemma 303:* We prove by induction on the rank that for all  $\tau \in \mathbf{M}^{\mathbb{P}}$ ,  $rk((\tau)_G) \leq rk(\tau)$ . Indeed

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G (\sigma, p) \in \tau\},$$

it follows that

$$rk((\tau)_G) = \sup \{rk((\sigma)_G) + 1 \mid \exists p \in G (\sigma, p) \in \tau\}.$$

By inductive hypothesis, one has

$$\begin{aligned} rk((\tau)_G) &\leq \sup \{rk(\sigma) + 1 \mid \exists p \in G (\sigma, p) \in \tau\} \\ &\leq \sup \{rk(\sigma) + 1 \mid \exists p \in \mathbb{P} (\sigma, p) \in \tau\} \\ &\leq \sup \{rk(z) + 1 \mid z \in \tau\} \\ &\leq rk(\tau). \end{aligned}$$

This implies  $(\mathbf{On})^{\mathbf{M}[G]} \subseteq (\mathbf{On})^{\mathbf{M}}$  which combined with  $\mathbf{M} \subseteq \mathbf{M}[G]$  yields  $(\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}$ .

□ 303



## Chapter 15

# The Truth Lemma

The Truth Lemma is about connecting the truth inside the generic extension to the truth inside  $\mathbf{V}$  and the truth inside  $\mathbf{M}$ .

### 15.1 Forcing from inside $\mathbf{V}$

**Definition 304.** Let  $\mathbf{M}$  be a c.t.m. of “ZFC”,  $\mathbb{P}$  a notion of forcing with  $\mathbb{P} \in \mathbf{M}$ . Let also  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula whose free variables are among  $x_1, \dots, x_n$ ,  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$  and  $p \in \mathbb{P}$ . We say that  $p$  forces  $\varphi(\tau_1, \dots, \tau_n)$  and write

$$p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$$

$$\Longleftrightarrow$$

for all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  such that  $p \in G$ , one has

$$\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G).$$

Notice that this definition is not made in  $\mathbf{M}$ , but rather in  $\mathbf{V}$ .

**Lemma 305.** Let  $\mathbf{M}$  be a c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  a notion of forcing,  $\varphi(x_1, \dots, x_n), \psi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula whose free variables are among  $x_1, \dots, x_n$ , and  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ , and  $p \in \mathbb{P}$ .

- (1) If  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$  and  $q \leq p$ , then  $q \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$ .
- (2) If  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$  and  $p \Vdash_{\mathbb{P}, \mathbf{M}} \psi(\tau_1, \dots, \tau_n)$ , then  $p \Vdash_{\mathbb{P}, \mathbf{M}} (\varphi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n))$ .

*Proof of Lemma 305:*

- (1) Suppose that  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$  and  $q \leq p$ . Let  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$  and such that  $q \in G$ . Since  $G$  is a filter, one has  $p \in G$ , which yields  $\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G)$ . So, we have shown that for all  $G$ ,  $\mathbb{P}$ -generic over  $\mathbf{M}$ , such that  $q \in G$ ,  $\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G)$  holds which means — by definition —  $q \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$ .
- (2) Immediate.

□ 305

## 15.2 Forcing from inside $\mathbf{M}$

The idea now, is to define another notion of forcing, not inside  $\mathbf{V}$  but inside  $\mathbf{M}$  so that the two coincide. i.e., we want to define  $\Vdash_*$  in  $\mathbf{V}$  such that for all  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$  and  $p \in \mathbb{P}$ :

$$p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n) \iff (p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

As we will see, the whole construction is relatively tedious and cumbersome but the results such a construction will provide are worth the effort.

**Definition 306.** Let  $\tau_1, \dots, \tau_n \in \mathbf{V}^{\mathbb{P}}$ . We let:

- $p \Vdash_* \tau_1 = \tau_2$  if and only if both

- (1) for all  $(\pi_1, s_1) \in \tau_1$ , the following set is dense below  $p$ :

$$D_\alpha(\pi_1, s_1, \tau_2) = \left\{ q \in \mathbb{P} \mid q \leq s_1 \longrightarrow \exists (\pi_2, s_2) \in \tau_2 \ (q \leq s_2 \wedge q \Vdash_* \pi_1 = \pi_2) \right\}$$

- (2) for all  $(\pi_2, s_2) \in \tau_2$ , the following set is dense below  $p$ :

$$D_\beta(\pi_2, s_2, \tau_1) = \left\{ q \in \mathbb{P} \mid q \leq s_2 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \ (q \leq s_1 \wedge q \Vdash_* \pi_2 = \pi_1) \right\}$$

- $p \Vdash_* \tau_1 \in \tau_2$  if and only if the following set is dense below  $p$ :

$$\left\{ q \in \mathbb{P} \mid \exists (\pi, s) \in \tau_2 \ (q \leq s \wedge q \Vdash_* \pi = \tau_1) \right\}$$

- $p \Vdash_* (\varphi \wedge \psi)$  if and only if  $p \Vdash_* \varphi$  and  $p \Vdash_* \psi$ ;
- $p \Vdash_* \neg \varphi$  if and only if for all  $q \leq p$ ,  $q \nVdash_* \varphi$ ;
- $p \Vdash_* \exists x \varphi(x, \tau_1, \dots, \tau_n)$  if and only if the following set is dense below  $p$ :

$$\left\{ q \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \ q \Vdash_* \varphi(\sigma, \tau_1, \dots, \tau_n) \right\}.$$

**Example 307.** Notice that the empty set is a  $\mathbb{P}$ -name, it is even a  $\mathbb{P}$ -name for the empty set and the canonical  $\mathbb{P}$ -name for the empty set is nothing but the empty set:

$$\check{\emptyset} = \{(\check{\sigma}, \mathbf{1}) \mid \sigma \in \emptyset\} = \emptyset \quad \text{and} \quad (\emptyset)_G = \emptyset.$$

For every forcing condition  $p$ , and any  $\mathbb{P}$ -name  $\tau$  one has

(1)  $p \Vdash_* \tau \in \emptyset$  because the following set being empty, is definitely not dense below  $p$ :

$$\left\{ q \in \mathbb{P} \mid \exists (\pi, s) \in \emptyset \quad (q \leq s \wedge q \Vdash_* \pi = \tau) \right\} = \emptyset.$$

(2)  $p \Vdash_* \emptyset = \emptyset$  because

(a) “for all  $(\pi_1, s_1) \in \tau_1$ , the following set is dense below  $p$ :

$$D_\alpha(\pi_1, s_1, \emptyset) = \left\{ q \in \mathbb{P} \mid q \leq s_1 \longrightarrow \exists (\pi_2, s_2) \in \emptyset \quad (q \leq s_2 \wedge q \Vdash_* \pi_1 = \pi_2) \right\}”$$

is a true statement because it is of the form

$$\forall (\pi_1, s_1) \quad ((\pi_1, s_1) \in \tau_1 \longrightarrow D_\alpha(\pi_1, s_1, \emptyset) \text{ “is dense below } p”)$$

(b) “for all  $(\pi_2, s_2) \in \emptyset$ , the set  $D_\beta(\pi_2, s_2, \emptyset)$  is dense below  $p$ ” holds also for the same reason.

(3)  $p \Vdash_* \emptyset \in \{\check{\emptyset}\}$  because we have

$$\widetilde{\{\check{\emptyset}\}} = \{(\check{\sigma}, \mathbf{1}) \mid \sigma \in \{\emptyset\}\} = \{(\check{\emptyset}, \mathbf{1})\} = \{(\emptyset, \mathbf{1})\}$$

and the following set is dense below  $p$ :

$$\begin{aligned} & \left\{ q \in \mathbb{P} \mid \exists (\pi, s) \in \widetilde{\{\check{\emptyset}\}} \quad (q \leq s \wedge q \Vdash_* \pi = \emptyset) \right\} \\ &= \left\{ q \in \mathbb{P} \mid (q \leq \mathbf{1} \wedge q \Vdash_* \emptyset = \emptyset) \right\} \\ &= \mathbb{P}. \end{aligned}$$

**Lemma 308.** Let  $\mathbb{P}$  be a notion of forcing, and  $p \in \mathbb{P}$ . Let also  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula whose free variables are among  $x_1, \dots, x_n$ , and  $\tau_1, \dots, \tau_n \in \mathbf{V}^{\mathbb{P}}$ .

The following are equivalent:

- (1)  $p \Vdash_* \varphi(\tau_1, \dots, \tau_n)$ ;
- (2) for all  $r \leq p$ ,  $r \Vdash_* \varphi(\tau_1, \dots, \tau_n)$ ;
- (3) the set  $\{r \in \mathbb{P} \mid r \Vdash_* \varphi(\tau_1, \dots, \tau_n)\}$  is dense below  $p$ .

*Proof of Lemma 308:*

(1)  $\Rightarrow$  (2) By induction on the height of  $\varphi$ .

$\varphi : x_1 = x_2$

Take any  $r \leq p$ , and suppose that  $p \Vdash_* \tau_1 = \tau_2$ , which means that

for all  $(\pi_1, s_1) \in \tau_1$ ,  $D_\alpha(\pi_1, s_1, \tau_2)$  is dense below  $p$ .

Since  $r \leq p$ ,  $D_\alpha(\pi_1, s_1, \tau_2)$  is also dense below  $r$ . Analogously, for all  $(\pi_2, s_2) \in \tau_2$ ,  $D_\beta(\pi_2, s_2, \tau_1)$  is dense below  $r$ . Therefore  $r \Vdash_* \tau_1 = \tau_2$ .

$\varphi : x_1 \in x_2$

Take any  $r \leq p$ , and suppose that  $p \Vdash_* \tau_1 \in \tau_2$ , which means that the set

$$\left\{ q \in \mathbb{P} \mid \exists (\pi, s) \in \tau_2 \left( q \leq s \wedge q \Vdash_* \pi = \tau_1 \right) \right\}$$

is dense below  $p$ . It follows that the same set is dense below  $r$ .

$\varphi : \exists x \psi(x, x_1, \dots, x_n)$

Take any  $r \leq p$ , and suppose that  $p \Vdash_* \exists x \psi(x, \tau_1, \dots, \tau_n)$ , which means that the set

$$\left\{ q \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \quad q \Vdash_* \psi(\sigma, \tau_1, \dots, \tau_n) \right\}$$

is dense below  $p$ . It follows that the same set is dense below  $r$ .

$\varphi : (\theta \wedge \psi)$

Take any  $r \leq p$ , and suppose that  $p \Vdash_* \theta \wedge \psi$ , which means that both  $p \Vdash_* \theta$  and  $p \Vdash_* \psi$ . So by induction hypothesis, one has  $r \Vdash_* \theta$  and  $r \Vdash_* \psi$ , which comes down to  $r \Vdash_* \theta \wedge \psi$ .

$\varphi : \neg\psi$

Take any  $r \leq p$ , and suppose that  $p \Vdash_* \neg\psi$ , which means that for all  $q \leq p$ ,  $q \nVdash_* \psi$ . So in particular, for all  $q \leq r$ ,  $q \nVdash_* \psi$ ; which means  $r \nVdash_* \psi$ .

(2)  $\Rightarrow$  (3) is immediate.

(3)  $\Rightarrow$  (1) By induction on the height of  $\varphi$ .

$\varphi : x_1 = x_2$

We suppose the set  $D = \{r \in \mathbb{P} \mid r \Vdash_* \tau_1 = \tau_2\}$  is dense below  $p$ . So, for all  $(\pi_1, s_1) \in \tau_1$ ,  $D_\alpha(\pi_1, s_1, \tau_2)$  is dense below  $r$  for all  $r \in D$ . But since  $D$  is dense below  $p$ ,  $D_\alpha(\pi_1, s_1, \tau_2)$  is dense below  $p$  as well, and the same holds for  $D_\beta(\pi_2, s_2, \tau_1)$ . So,  $p \Vdash_* \tau_1 = \tau_2$ .

$\varphi : x_1 \in x_2$

We suppose the set  $D = \{r \in \mathbb{P} \mid r \Vdash_* \tau_1 \in \tau_2\}$  is dense below  $p$ . So, the set

$$\left\{q \in \mathbb{P} \mid \exists (\pi, s) \in \tau_2 \left( q \leq s \wedge q \Vdash_* \pi = \tau_1 \right)\right\}$$

is dense below  $r$  for all  $r \in D$ . Hence it is also dense below  $p$ , which yields  $p \Vdash_* \tau_1 \in \tau_2$ .

$\varphi : \exists x \psi(x, x_1, \dots, x_n)$

We suppose once again that the set  $D = \{r \in \mathbb{P} \mid r \Vdash_* \exists x \varphi(x, \tau_1, \dots, \tau_n)\}$  is dense below  $p$ . So, the set

$$\left\{q \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \quad q \Vdash_* \psi(\sigma, \tau_1, \dots, \tau_n)\right\}.$$

is dense below each  $r \in D$ , which implies that it is dense below  $p$ . Henceforth,  $p \Vdash_* \exists x \varphi(x, \tau_1, \dots, \tau_n)$ .

$\varphi : (\theta \wedge \psi)$

We assume that the set  $D = \{r \in \mathbb{P} \mid r \Vdash_* (\theta \wedge \psi)\}$  is dense below  $p$ . So, both sets

$$\{r \in \mathbb{P} \mid r \Vdash_* \theta\} \quad \text{and} \quad \{r \in \mathbb{P} \mid r \Vdash_* \psi\}$$

are dense below  $p$ . By induction hypothesis, this leads to  $p \Vdash_* \theta$  and  $p \Vdash_* \psi$ , and finally to  $p \Vdash_* (\theta \wedge \psi)$ .

$\varphi : \neg\psi$

We assume that the set  $D = \{r \in \mathbb{P} \mid r \Vdash_* \neg\psi\}$  is dense below  $p$  and proceed by contradiction. So, we suppose  $p \nVdash_* \neg\psi$ , which means that there exists  $q \leq p$  such that  $q \Vdash_* \psi$ . By (1)  $\Rightarrow$  (2) we see that for all  $r \leq q$ ,  $r \Vdash_* \psi$ . Since  $D$  is dense below  $p$ , it is also dense below  $q$ . Now, any  $r \in D \cap \{s \in \mathbb{P} \mid s \leq q\}$  satisfies both  $r \Vdash_* \neg\psi$  and  $r \Vdash_* \psi$ , a contradiction.

□ 308

### 15.3 Connecting the Truth in $\mathbf{M}[G]$ to the Truth in $\mathbf{M}$

Providing we have access to the filter  $G$ , we show that we can go back and forth between the truth in  $\mathbf{M}$  and the truth in  $\mathbf{M}[G]$ .

**Lemma 309.** *Let  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula,  $\mathbf{M}$  any c.t.m. of “ZFC”,  $\mathbb{P}$  any notion of forcing on  $\mathbf{M}$ ,  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ , and  $G$  any filter  $\mathbb{P}$ -generic over  $\mathbf{M}$ .*

(1) *If  $p \in G$  and  $(p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}$ , then*

$$\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G).$$

(2) *If  $\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G)$ , then there exists  $p \in G$  such that*

$$(p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

Viewed from the perspective of the generic extension — in the sense that we start from picking elements in  $\mathbf{M}[G]$  and find a name for them later on, as opposed to firstly starting with  $\mathbb{P}$ -names and secondly decoding them — this Lemma states that for all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ , and all sets  $a_1, \dots, a_n$  in  $\mathbf{M}[G]$ , we have

(1) *If  $p \in G$  and  $(p \Vdash_* \varphi(q_1, \dots, q_n))^{\mathbf{M}}$ , then*

$$\mathbf{M}[G] \models \varphi(a_1, \dots, a_n).$$

(2) *If  $\mathbf{M}[G] \models \varphi(a_1, \dots, a_n)$ , then there exists  $p \in G$  such that*

$$(p \Vdash_* \varphi(q_1, \dots, q_n))^{\mathbf{M}}.$$

**Definition 310.** *Given  $\mathbb{P}$  a notion of forcing, and  $\pi_1, \pi_2, \tau_1, \tau_2 \in \mathbf{V}^{\mathbb{P}}$ , we define*

$$(\pi_1, \pi_2) < (\tau_1, \tau_2) \iff \pi_1 \in \text{dom}(\tau_1) \text{ and } \pi_2 \in \text{dom}(\tau_2).$$

Notice that this definition yields  $<$  is well-founded since  $rk(\pi_1) < rk(\tau_1)$  and  $rk(\pi_2) < rk(\tau_2)$  both hold, therefore,  $\min \{rk(\pi_1), rk(\pi_2)\} < \min \{rk(\tau_1), rk(\tau_2)\}$ .

*Proof of Lemma 309:* We prove (1) and (2) simultaneously by induction on the height of  $\varphi$ .

$\varphi : \mathbf{x}_1 = \mathbf{x}_2$  We prove (1) and (2) by  $<$ -induction.

(1) Let  $p \in G$  be such that  $(p \Vdash_* \tau_1 = \tau_2)^{\mathbf{M}}$ , we want to show both

$$\mathbf{M}[G] \models (\tau_1)_G \subseteq (\tau_2)_G \quad \text{and} \quad \mathbf{M}[G] \models (\tau_2)_G \subseteq (\tau_1)_G.$$

We recall that

$$(\tau_1)_G = \{(\pi_1)_G \mid \exists s_1 \in G \ (\pi_1, s_1) \in \tau_1\}.$$

Let  $(\pi_1, s_1) \in \tau_1$ , and let us show that  $(\pi_1)_G \in (\tau_2)_G$ . To do so, we are reaching for some  $s_2 \in G$  such that  $(\pi_1, s_2) \in \tau_2$ .

Since  $p$  and  $s_1$  are elements of the filter  $G$ , there exists  $q \in G$  such that  $q \leq p$  and  $q \leq s_1$ . The set  $D_\alpha(\pi_1, s_1, \tau_2)$  is dense below  $p$  and thus under  $q$ . By Lemma 288, one has  $G \cap D_\alpha(\pi_1, s_1, \tau_2) \neq \emptyset$ . Then take any  $r \in G \cap D_\alpha(\pi_1, s_1, \tau_2) \neq \emptyset$ . There thus exists  $(\pi_2, s_2) \in \tau_2$  such that  $r \leq s_2$  and  $r \Vdash_* \pi_1 = \pi_2$ . Moreover, since  $r \in G$  and  $r \leq s_2$ , one obtains  $s_2 \in G$ . It follows that  $\mathbf{M}[G] \models (\pi_2)_G \in (\tau_2)_G$ .

We have  $(\pi_1, \pi_2) < (\tau_1, \tau_2)$ ,  $r \Vdash_* \pi_1 = \pi_2$ , and  $r \in G$ . So, the induction hypothesis, gives  $\mathbf{M}[G] \models (\pi_1)_G = (\pi_2)_G$ .

Therefore, we have shown that  $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_2)_G$  holds for every  $(\pi_1)_G \in (\tau_1)_G$ , and so  $\mathbf{M}[G] \models (\tau_1)_G \subseteq (\tau_2)_G$ . The opposite inclusion is achieved in a similar fashion.

- (2) Suppose that  $\mathbf{M}[G] \models (\tau_1)_G = (\tau_2)_G$ . Let

$$D = \{r \in \mathbb{P} \mid r \Vdash_* \tau_1 = \tau_2 \vee \psi_1(r) \vee \psi_2(r)\},$$

where

$$\psi_1(x) : \exists(\pi_1, s_1) \in \tau_1 \ (x \leq s_1 \wedge \forall(\pi_2, s_2) \in \tau_2 \ \forall q \leq s_2 \ (q \Vdash_* \pi_1 = \pi_2 \longrightarrow q \perp x))$$

and

$$\psi_2(x) : \exists(\pi_2, s_2) \in \tau_2 \ (x \leq s_2 \wedge \forall(\pi_1, s_1) \in \tau_1 \ \forall q \leq s_1 \ (q \Vdash_* \pi_2 = \pi_1 \longrightarrow q \perp x)).$$

Let us show that  $D$  is dense in  $\mathbb{P}$ . Let  $p \in \mathbb{P}$ , if  $p \Vdash_* \tau_1 = \tau_2$ , then  $p \in D$ . Otherwise, there exists  $(\pi_1, s_1) \in \tau_1$  such that  $D_\alpha(\pi_1, s_1, \tau_2)$  is not dense below  $p$ , so there exists  $(\pi_2, s_2) \in \tau_2$  such that  $D_\beta(\pi_2, s_2, \tau_1)$  is not dense below  $p$ .

Suppose that there exists  $(\pi_1, s_1) \in \tau_1$  such that  $D_\alpha(\pi_1, s_1, \tau_2)$  is not dense below  $p$ , which means that there exists  $r \leq p$  such that for all  $q \leq r$ ,  $q \notin D_\alpha(\pi_1, s_1, \tau_2)$ . We show that  $r$  satisfies  $\psi_1$ .

Let  $q \leq r$ ,  $q \notin D_\alpha(\pi_1, s_1, \tau_2)$ , so  $q \leq s_1$ . Furthermore, for all  $(\pi_2, s_2) \in \tau_2$ ,  $q \not\leq s_2$  on  $q \not\Vdash_* \pi_1 = \pi_2$ .

For all  $t \in \mathbb{P}$  and for all  $(\pi_2, s_2) \in \tau_2$ , if  $t \leq s_2$  and  $t \Vdash_* \pi_1 = \pi_2$  then  $t \perp r$ . Indeed, if this is not the case, there would exist  $t' \leq r$  such that  $t' \leq s_2$ ,  $t' \Vdash_* \pi_1 = \pi_2$ , but the last two properties assure us that  $t' \in D_\alpha(\pi_1, s_1, \tau_2)$  which contradicts the definition of  $r$ . Therefore  $r$  satisfies  $\psi_1$ . We reason in a similar manner if there exists  $(\pi_2, s_2) \in \tau_2$  such that  $D_\beta(\pi_2, s_2, \tau_1)$  is not dense below  $p$ . Hence,  $D$  is dense in  $\mathbb{P}$ .

Let us now show that if  $p \in G$ , then  $p$  does not satisfy neither  $\psi_1$ , nor  $\psi_2$ . Suppose towards contradiction that  $p \in G$  and that  $p$  satisfies  $\psi_1$ . Fix  $(\pi_1, s_1) \in \tau_1$  such that:

$$p \leq s_1 \wedge \forall(\pi_2, s_2) \in \tau_2 \ \forall q \leq s_2 \ (q \Vdash_* \tau_1 = \tau_2 \rightarrow q \perp p).$$

We have  $p \leq s_1$  and  $p \in G$ , so  $s_1 \in G$ . Hence  $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_1)_G$ . Now,  $\mathbf{M}[G] \models (\tau_1)_G = (\tau_2)_G$ , so  $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_2)_G$ . There thus exists  $(\pi_2, s_2) \in \tau_2$  such that  $(\pi_1)_G = (\pi_2)_G$ .

By induction hypothesis, there exists  $r \in G$  such that  $r \Vdash_* \pi_1 = \pi_2$ . It follows that there exists  $q \leq r, s_2, p$  such that  $q \Vdash_* \pi_1 = \pi_2$ . But since  $p$  satisfies  $\psi_1$ , from  $q \leq s_2$  and  $q \Vdash_* \pi_1 = \pi_2$  we deduce that  $q \perp p$ , but this contradicts the fact that  $q \leq p$ . The case of  $\psi_2$  is analogous.

We can conclude by remarking that since  $D$  is dense, there exists  $p \in G \cap D$  such that  $p \Vdash_* \tau_1 = \tau_2$ .

$\varphi : x_1 \in x_2$

- (1) Suppose that there exists  $p \in G$  such that  $p \Vdash_* \tau_1 \in \tau_2$ . The set

$$D = \{q \in \mathbb{P} \mid \exists (\pi_2, s_2) \in \tau_2 \ (q \leq s_2 \wedge q \Vdash_* \tau_1 = \pi_2)\}$$

is thus dense below  $p$ . Since  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ ,  $G \cap D \neq \emptyset$ . Let  $q \in G \cap D$  and  $(\pi_2, s_2) \in \tau_2$  be such that  $q \leq s_2$  and  $q \Vdash_* \tau_1 = \pi_2$ .  $G$  is a filter, so  $s_2 \in G$ , which in turn implies

$$\mathbf{M}[G] \models (\pi_2)_G \in (\tau_2)_G.$$

Furthermore,  $q \in G$  and  $q \Vdash_* \tau_1 = \pi_2$ , so

$$\mathbf{M}[G] \models (\tau_1)_G = (\pi_2)_G.$$

Hence,  $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$ .

- (2) Suppose that  $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$ . There thus exists  $s_2 \in G$  such that  $(\pi_2, s_2) \in \tau_2$  and  $(\pi_2)_G = (\tau_1)_G$ . Hence, by [2](#) for equality, there exists  $q \in G$  such that  $q \Vdash_* \pi_2 = \tau_1$ . Since  $G$  is a filter, there exists  $p \in G$  such that  $p \leq q$  and  $p \leq s_2$ . Since  $p \leq q$  and  $q \Vdash_* \pi_2 = \tau_1$ ,  $p$  moreover verifies  $p \Vdash_* \pi_2 = \tau_1$ . The set

$$D = \{q \in \mathbb{P} \mid \exists (\pi_2, s_2) \in \tau_2 \ (q \leq s_2 \wedge q \Vdash_* \tau_1 = \pi_2)\}$$

is then dense below  $p$  since all  $q' \leq p$  verify  $q' \leq s_2$  and  $q' \Vdash_* \tau_1 = \pi_2$ . Hence  $p \in G$  verifies  $p \Vdash_* \tau_1 \in \tau_2$ .

$\varphi : (\varphi \wedge \psi)$

- (1) Suppose that there exists  $p \in G$  such that  $(p \Vdash_* (\varphi \wedge \psi))^{\mathbf{M}}$ . In particular, this means there exists  $p \in G$  such that  $(p \Vdash_* \varphi)^{\mathbf{M}}$  and  $(p \Vdash_* \psi)^{\mathbf{M}}$  and by induction hypothesis, that  $\mathbf{M}[G] \models \varphi$  and  $\mathbf{M}[G] \models \psi$  both hold. Thus,  $\mathbf{M}[G] \models (\varphi \wedge \psi)$  holds as well.

- (2) Suppose that  $\mathbf{M}[G] \models (\varphi \wedge \psi)$ , so  $\mathbf{M}[G] \models \varphi$  and  $\mathbf{M}[G] \models \psi$ . There thus exist  $p, q \in G$  such that  $(p \Vdash_* \varphi)^{\mathbf{M}}$  and  $(q \Vdash_* \psi)^{\mathbf{M}}$ . But since  $G$  is a filter, there exists  $r \in G$  such that  $r \leq p$  and  $r \leq q$ , moreover such that  $r$  verifies  $(r \Vdash_* \varphi)^{\mathbf{M}}$  and  $(r \Vdash_* \psi)^{\mathbf{M}}$ . Hence,  $(p \Vdash_* (\varphi \wedge \psi))^{\mathbf{M}}$ .

$\varphi : \neg\varphi$

- (1) Suppose that there exists  $p \in G$  such that  $(p \Vdash_* \neg\varphi)^{\mathbf{M}}$ . For the sake of contradiction, also suppose that  $\mathbf{M}[G] \not\models \neg\varphi$ . Then  $\mathbf{M}[G] \models \varphi$ , and so there exists  $q \in G$  such that  $(q \Vdash_* \varphi)^{\mathbf{M}}$ . Since  $G$  is a filter, there exists  $r \in G$  such that  $r \leq p$  and  $r \leq q$ . From  $r \leq q$  and  $(q \Vdash_* \varphi)^{\mathbf{M}}$ , it follows that  $(r \Vdash_* \varphi)^{\mathbf{M}}$ . But  $r \leq p$ , so  $(p \not\Vdash_* \neg\varphi)^{\mathbf{M}}$ , which contradicts the assumptions we made on  $p$ .
- (2) Suppose that  $\mathbf{M}[G] \models \neg\varphi$ . Let

$$D = \left\{ q \in \mathbb{P} \mid (q \Vdash_* \varphi)^{\mathbf{M}} \vee (q \Vdash_* \neg\varphi)^{\mathbf{M}} \right\}.$$

The set  $D$  is dense below  $p \in \mathbb{P}$  in any case. Indeed, let  $p \in \mathbb{P}$  and  $q \leq p$ , then we have two possible cases: either  $(q \Vdash_* \neg\varphi)^{\mathbf{M}}$ , and therefore  $q \in D$ , or there exists  $r \leq q$  such that  $(r \Vdash_* \varphi)^{\mathbf{M}}$  and  $r \in D$ .

Since for all  $p \in \mathbb{P}$ ,  $D$  is dense below  $p$ ,  $D \cap G \neq \emptyset$ . Let  $q \in D \cap G$ , then either  $(q \Vdash_* \neg\varphi)^{\mathbf{M}}$ , and the conclusion follows, or  $(q \Vdash_* \varphi)^{\mathbf{M}}$ . But the latter case is to exclude since it would imply that  $\mathbf{M}[G] \models \varphi$ .

$\exists x \ \varphi(x, \mathbf{a}_1, \dots, \mathbf{a}_n)$  Let  $\tau = (\tau_1, \dots, \tau_n)$ .

- (1) Suppose that there exists  $p \in G$  such that  $(p \Vdash_* \exists x \ \varphi(x, \tau))^{\mathbf{M}}$ . The set

$$D = \left\{ r \in \mathbb{P} \mid \exists \sigma \in \mathbf{M}^{\mathbb{P}} (r \Vdash_* \varphi(\sigma, \tau))^{\mathbf{M}} \right\}$$

is thus dense below  $p$  and  $D \cap G \neq \emptyset$ . Let  $q \in D \cap G$ , there exists  $\sigma \in \mathbf{M}^{\mathbb{P}}$  such that  $(q \Vdash_* \varphi(\sigma, \tau))^{\mathbf{M}}$ . Hence  $\mathbf{M}[G] \models \varphi((\sigma)_G, (\tau)_G)$ . Therefore,  $\mathbf{M}[G] \models \exists x \ \varphi(x, (\tau)_G)$ .

- (2) Suppose that  $\mathbf{M}[G] \models \exists x \ \varphi(x, (\tau)_G)$ . Let  $(\sigma)_G$  be such that  $\mathbf{M}[G] \models \varphi((\sigma)_G, (\tau)_G)$ . By induction, there exists  $p \in G$  such that  $(p \Vdash_* \varphi(\sigma, \tau))^{\mathbf{M}}$ , so for all  $r \leq p$ ,  $(r \Vdash_* \varphi(\sigma, \tau))^{\mathbf{M}}$ . Thus

$$D = \left\{ r \in \mathbb{P} \mid \exists \sigma \in \mathbf{M}^{\mathbb{P}} (r \Vdash_* \varphi(\sigma, \tau))^{\mathbf{M}} \right\}$$

is dense below  $p$  and it follows that  $(p \Vdash_* \exists x \ \varphi(x, \tau))^{\mathbf{M}}$ .

□ 309

At last, we are now able to prove the main result that connects the truth in  $\mathbf{V}$  to the truth inside  $\mathbf{M}$ .

**Lemma 311.** *Let  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula,  $\mathbf{M}$  any c.t.m. of “ZFC”,  $\mathbb{P}$  any notion of forcing on  $\mathbf{M}$ , and  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ .*

*For all  $p \in \mathbb{P}$ ,*

$$p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n) \iff (p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

*Proof of Lemma 311:*

( $\Leftarrow$ ) Consider any  $p \in \mathbb{P}$  such that  $(p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}$ . By Lemma 309 (1) for any<sup>1</sup>  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  such that  $p \in G$  one has  $\mathbf{M}[G] \models \varphi(\tau_1, \dots, \tau_n)$ , which, by definition, is equivalent to  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$ .

( $\Rightarrow$ ) Fix  $p \in \mathbb{P}$  such that  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$ , and let

$$D = \left\{ r \in \mathbb{P} \mid (r \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}} \right\}.$$

$D$  is dense below  $p$ . Indeed, if this were not the case, there would exist  $q \leq p$  such that for all  $r \leq q$ ,  $r \notin D$ , i.e.,

$$(\forall r \leq q \ r \not\Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

It would follow that  $(q \Vdash_* \neg \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}$ . By the reverse implication proved above, it would follow that  $q \Vdash \neg \varphi(\tau_1, \dots, \tau_n)$  and thus, for  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  with  $q \in G$ ,

$$\mathbf{M}[G] \models \neg \varphi((\tau_1)_G, \dots, (\tau_n)_G)$$

would hold. But if  $q \in G$ , then  $p \in G$  and having

$$\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G)$$

would yield the desired contradiction. □ 311

The next result is the main result of this chapter. It is really a theorem which builds on the lemmas that were proved before. nevertheless, following the tradition of the “founding fathers”, we do not call it a theorem, but a lemma. However, its title — the “Truth Lemma” — indicates that it is of major importance.

**The Truth Lemma.** *Let  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula,  $\mathbf{M}$  any c.t.m. of “ZFC”,  $\mathbb{P}$  any notion of forcing on  $\mathbf{M}$ , and  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ .*

*For all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ ,*

$$\mathbf{M}[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G) \iff \exists p \in G (p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

<sup>1</sup>Since  $\mathbf{M}$  is a c.t.m. of “ZFC”, such a  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  exists by Lemma 285.

Viewed from the perspective of the generic extension — in the sense that we start from picking elements in  $\mathbf{M}[G]$  and find a name for them later on, as opposed to beginning with  $\mathbb{P}$ -names — the Truth Lemma states that for all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ , and all sets  $a_1, \dots, a_n$  in  $\mathbf{M}[G]$ , we have

$$\mathbf{M}[G] \models \varphi(a_1, \dots, a_n) \iff \exists p \in G (p \Vdash_* \varphi(\underline{a}_1, \dots, \underline{a}_n))^{\mathbf{M}}.$$

*Proof of the Truth Lemma:* This is an immediate consequence of Lemmas 309 and 311.

□ Truth Lemma

Combining the Truth Lemma with Lemma 311 we obtain the following picture:

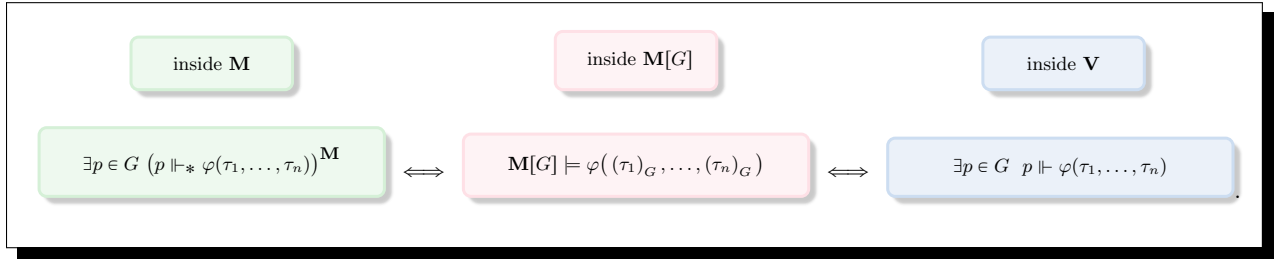


Figure 15.1: The connections between forcing and the generic extension.

Assuming the ground model is a countable transitive model of “ZFC”, the forcing relation preserves all logical consequences.

**Proposition 313.** Let  $\varphi(x_1, \dots, x_n)$ ,  $\psi(x_1, \dots, x_n)$ , be any  $\mathcal{L}_{\text{ST}}$ -formula,  $\mathbf{M}$  any c.t.m. of “ZFC”,  $\mathbb{P}$  any notion of forcing on  $\mathbf{M}$ , and  $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ .

For all  $p \in \mathbb{P}$ ,

$$\left. \begin{array}{l} (p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}} \\ \text{and} \\ \vdash_c \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \longrightarrow \psi(x_1, \dots, x_n)) \end{array} \right\} \implies (p \Vdash_* \psi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

*Proof of Proposition 313:* Consider in  $\mathbf{M}$  the following set:

$$Q = \{q \in \mathbb{P} \mid q \Vdash_* \psi(\tau_1, \dots, \tau_n)\}.$$

We show that  $Q$  is dense below  $p$ . Towards a contradiction, let us assume that there exists some  $s \leq p$  such that for all  $t \leq s$

$$t \not\Vdash_* \psi(\tau_1, \dots, \tau_n).$$

This implies

$$s \Vdash_* \neg\psi(\tau_1, \dots, \tau_n).$$

Since by Lemma 305 we have  $s \leq p$  gives

$$s \Vdash_* \varphi(\tau_1, \dots, \tau_n)$$

we end up with

$$s \Vdash_* (\varphi(\tau_1, \dots, \tau_n) \wedge \neg\psi(\tau_1, \dots, \tau_n)).$$

By Lemma 285 there exists some filter  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$  such that  $s \in G$ . By the Truth Lemma, we have

$$(s \in G \wedge (s \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}) \implies \mathbf{M}[G] \models (\varphi(\tau_1, \dots, \tau_n) \wedge \neg\psi(\tau_1, \dots, \tau_n))$$

which yields

$$\mathbf{M}[G] \models (\varphi(\tau_1, \dots, \tau_n) \wedge \neg\psi(\tau_1, \dots, \tau_n)).$$

Now, since

$$\vdash_c \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \longrightarrow \psi(x_1, \dots, x_n))$$

we have

$$\models \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \longrightarrow \psi(x_1, \dots, x_n))$$

and in particular

$$\mathbf{M}[G] \models \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \longrightarrow \psi(x_1, \dots, x_n))$$

which yields

$$\mathbf{M}[G] \models (\varphi(\tau_1, \dots, \tau_n) \longrightarrow \psi(\tau_1, \dots, \tau_n))$$

By *modus ponens* this gives

$$\mathbf{M}[G] \models \psi(\tau_1, \dots, \tau_n)$$

which yields the following contradiction

$$\mathbf{M}[G] \models (\psi(\tau_1, \dots, \tau_n) \wedge \neg\psi(\tau_1, \dots, \tau_n)).$$

So, we have shown that  $Q$  is dense below  $p$ , and by Lemma 308 we obtain

$$(p \Vdash_* \psi(\tau_1, \dots, \tau_n))^{\mathbf{M}}.$$

□ 313

From now on, if  $\mathbf{M}$  is fixed, we will identify  $p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\tau_1, \dots, \tau_n)$  and  $(p \Vdash_* \varphi(\tau_1, \dots, \tau_n))^{\mathbf{M}}$  and in case both  $\mathbb{P}$  and  $\mathbf{M}$  are clear from the context, we will simply write  $p \Vdash \varphi(\tau_1, \dots, \tau_n)$ .

## Chapter 16

# ZFC within the Generic Extension and Cardinal Preservation

### 16.1 ZFC within the Generic Extension

This whole section is dedicated to proving, providing one starts with a ground model  $\mathbf{M}$  that satisfies “ZFC”, that the generic extension  $\mathbf{M}[G]$  also satisfies “ZFC”. Of, course, this statement should be understood backwards: whatever finite set of axioms  $\Delta$  from **ZFC** we consider, we will end up with some generic extension  $\mathbf{M}[G]$  that satisfies  $\Delta$ , providing we start from a ground model  $\mathbf{M}$  that satisfies some (other) finite set  $\Gamma$  of axioms from **ZFC**, where the relation between  $\Gamma$  and  $\Delta$  could be made explicit (but will never be).

**Theorem 314.** *Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $(\mathbb{P}, \leq, 1) \in \mathbf{M}$  be any partial order and  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ .*

$\mathbf{M}[G]$  satisfies “ZFC”.

This theorem really states that given any finite sub-theory  $\Delta \subsetneq \mathbf{ZFC}$ , there exists some finite sub-theory  $\Gamma \subsetneq \mathbf{ZFC}$  such that in order to have  $\mathbf{M}[G] \models \Delta$ , it is enough to start from any c.t.m.  $\mathbf{M}$  which satisfies  $\mathbf{M} \models \Gamma$ .

*Proof of Lemma [314](#):*

**Extensionality:** holds in  $\mathbf{M}[G]$  since  $\mathbf{M}[G]$  is transitive.

**Comprehension Schema:** We want to show that for all  $\sigma, \pi_1, \dots, \pi_n \in \mathbf{M}^{\mathbb{P}}$  and  $\varphi(x, y_1, \dots, y_n)$ :

$$u = \left\{ z \in (\sigma)_G \mid \left( \varphi(z, (\pi_1)_G, \dots, (\pi_n)_G) \right)^{\mathbf{M}[G]} \right\} \in \mathbf{M}[G].$$

We must find some  $\tau \in \mathbf{M}^{\mathbb{P}}$  such that  $u = (\tau)_G$ . So, we set

$$\tau = \{(\theta, p) \in \text{dom}(\sigma) \times \mathbb{P} \mid p \Vdash_{\mathbb{P}, \mathbf{M}} (\theta \in \sigma \wedge \varphi(\theta, \pi_1, \dots, \pi_n))\}.$$

We show that  $(\tau)_G = u$ .

$$\begin{aligned} (\tau)_G &= \{(\theta)_G \mid \exists p \in G \ (\theta, p) \in \tau\} \\ &= \{(\theta)_G \mid \theta \in \text{dom}(\sigma) \wedge \exists p \in G \ p \Vdash_{\mathbb{P}, \mathbf{M}} (\theta \in \sigma \wedge \varphi(\theta, \pi_1, \dots, \pi_n))\} \\ &= \left\{(\theta)_G \mid \theta \in \text{dom}(\sigma) \wedge \left((\theta)_G \in (\sigma)_G \wedge \varphi((\theta)_G, (\pi_1)_G, \dots, (\pi_n)_G)\right)^{\mathbf{M}[G]}\right\} \\ &= \left\{(\theta)_G \mid \theta \in \text{dom}(\sigma) \wedge (\theta)_G \in (\sigma)_G \wedge \left(\varphi((\theta)_G, (\pi_1)_G, \dots, (\pi_n)_G)\right)^{\mathbf{M}[G]}\right\} \\ &= u. \end{aligned}$$

**Pairing:** We assume  $\mathbf{M}$  is a *c.t.m.* of sufficiently enough finitely many formulas from “**ZFC**”, so that given any  $\mathbb{P}$ -names  $\tau, \sigma \in \mathbf{M}^{\mathbb{P}}$ , we have<sup>1</sup>  $\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\} \in \mathbf{M}^{\mathbb{P}}$ . Then we make use of the fact  $\mathbb{1}$  belongs to  $G$  to obtain:

$$\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}_G = \{(\tau)_G, (\sigma)_G\} \in \mathbf{M}[G].$$

**Union:** Let  $\sigma \in \mathbf{M}^{\mathbb{P}}$ , to prove that  $\bigcup (\sigma)_G \in \mathbf{M}[G]$ , it is enough to show that there exists  $\tau \in \mathbf{M}^{\mathbb{P}}$  such that  $\bigcup (\sigma)_G \subseteq (\tau)_G$ . We recall that

$$\text{dom}(\sigma) = \{\pi \in \mathbf{M}^{\mathbb{P}} \mid \exists p \in \mathbb{P} \ (\pi, p) \in \sigma\}.$$

We set

$$\tau = \bigcup \text{dom}(\sigma).$$

Since  $\mathbf{M}$  is a *c.t.m.* of “a sufficiently large enough amount of axioms from **ZFC**,” we have  $\tau \in \mathbf{M}^{\mathbb{P}}$ . Let  $\pi \in \text{dom}(\sigma)$ , then  $\pi \subseteq \bigcup \text{dom}(\sigma) = \tau$ , and thus  $(\pi)_G \subseteq (\tau)_G$ , which yields

$$\begin{aligned} \bigcup (\sigma)_G &= \bigcup \{(\pi)_G \mid \exists p \in G \ (\pi, p) \in \sigma\} \\ &\subseteq \bigcup \{(\pi)_G \mid (\pi, p) \in \sigma\} \\ &= \left(\bigcup \text{dom}(\sigma)\right)_G \\ &= (\tau)_G. \end{aligned}$$

<sup>1</sup> It is enough to guarantee, for instance, that  $\mathbf{M}$  is closed under the functional  $(x, y) \mapsto \{(x, \mathbb{1}), (y, \mathbb{1})\}$ .

**Infinity:** This axiom holds in  $\mathbf{M}[G]$  since both  $\omega \in \mathbf{M}$  and  $\check{\omega} \in \mathbf{M}^{\mathbb{P}}$  are satisfied, and  $\omega = (\check{\omega})_G \in \mathbf{M}[G]$ .

**Power Set:** Let  $\sigma \in \mathbf{M}^{\mathbb{P}}$ , we must show that the set  $\mathcal{P}((\sigma)_G) \cap \mathbf{M}[G]$  belongs to  $\mathbf{M}[G]$ . For this, it is enough to show that there exists  $\tau \in \mathbf{M}^{\mathbb{P}}$  such that  $\mathcal{P}((\sigma)_G) \cap \mathbf{M}[G] \subseteq (\tau)_G$ , we then get the result by making use of both an instance of the comprehension schema and the axiom of extensionality.

We consider :

$$\begin{aligned} S &= \{ \mu \in \mathbf{M}^{\mathbb{P}} \mid \text{dom}(\mu) \subseteq \text{dom}(\sigma) \} \\ &= \{ \mu \in \mathbf{M}^{\mathbb{P}} \mid \mu \subseteq (\text{dom}(\sigma) \times \mathbb{P}) \} \\ &= \left( \mathcal{P}(\text{dom}(\sigma) \times \mathbb{P}) \right) \cap \mathbf{M}^{\mathbb{P}} \\ &= \left( \mathcal{P}(\text{dom}(\sigma) \times \mathbb{P}) \right)^{\mathbf{M}}. \end{aligned}$$

Notice that given any  $b \in \mathcal{P}((\sigma)_G) \cap \mathbf{M}[G]$  and any  $\mathbb{P}$ -name  $\check{b}$  for  $b$ , we have both both

- (1) the set  $\check{b}' = \{ (\theta, p) \in \text{dom}(\sigma) \times \mathbb{P} \mid p \Vdash_{\mathbb{P}, \mathbf{M}} \theta \in \check{b} \}$  belongs to  $S$ ;
- (2) and

$$\begin{aligned} (\check{b}')_G &= \{ (\theta)_G \in \mathbf{M}[G] \mid \theta \in \text{dom}(\sigma) \wedge \exists p \in G \ p \Vdash \theta \in \check{b} \} \\ &= \{ (\theta)_G \in \mathbf{M}[G] \mid \theta \in \text{dom}(\sigma) \wedge (\theta)_G \in (\check{b})_G \} \\ &= \{ (\theta)_G \in \mathbf{M}[G] \mid \theta \in \text{dom}(\sigma) \wedge (\theta)_G \in b \} \\ &= \{ (\theta)_G \in \mathbf{M}[G] \mid (\theta)_G \in b \} \\ &= \{ a \in \mathbf{M}[G] \mid a \in b \} \\ &= b. \end{aligned}$$

Assuming “**ZFC**,” contains enough axioms to guarantee that  $S \times \{1\}$  is some  $\mathbb{P}$ -name that

belongs to  $\mathbf{M}$ , we set  $\tau = S \times \{\mathbf{1}\}$  and we obtain:

$$\begin{aligned}
 (\tau)_G &= (S \times \{\mathbf{1}\})_G \\
 &= \{(\mu)_G \mid \mu \in S\} \\
 &= \{(\mu)_G \mid \mu \in \mathbf{M}^{\mathbb{P}} \wedge \mu \subseteq (\text{dom}(\sigma) \times \mathbb{P})\} \\
 &\supseteq \{(b')_G \mid (b)_G \in \mathcal{P}((\sigma)_G) \cap \mathbf{M}[G]\} \\
 &= \{(b)_G \mid (b)_G \in \mathcal{P}((\sigma)_G) \cap \mathbf{M}[G]\} \\
 &= \mathcal{P}((\sigma)_G) \cap \mathbf{M}[G].
 \end{aligned}$$

**Foundation:** This axiom holds in  $\mathbf{M}[G]$  because  $\mathbf{M}[G]$  is transitive and  $\mathbf{M}$  satisfies the axiom of **Foundation** — which simply means that the axiom of **Foundation** belongs to “**ZFC**”. To show this, we simply show that any infinite  $\ni$ -descending chain in  $\mathbf{M}[G]$ , would yield some other infinite  $\ni$ -descending chain in  $\mathbf{M}$ .

Notice that for all sets  $a, b \in \mathbf{M}[G]$  with  $\dot{b}$  any  $\mathbb{P}$ -name such that  $(\dot{b})_G = b$ , we have

$$\begin{aligned}
 a \in b \in \mathbf{M}[G] &\implies \exists \dot{a} \in \text{dom}(\dot{b}) \quad (\dot{a})_G = a \\
 &\implies \exists \dot{a} \text{ rk}(\dot{a}) < \text{rk}(\dot{b});
 \end{aligned}$$

which induces

$$\left( \exists (a_i)_{i \in \omega} \forall i \in \omega \quad a_{i+1} \in a_i \right)^{\mathbf{M}[G]} \implies \left( \exists (\dot{a}_i)_{i \in \omega} \forall i \in \omega \quad \dot{a}_{i+1} \in \dot{a}_i \right)^{\mathbf{M}}$$

and equivalently

$$\begin{aligned}
 \neg \left( \exists (\dot{a}_i)_{i \in \omega} \forall i \in \omega \quad \dot{a}_{i+1} \in \dot{a}_i \right)^{\mathbf{M}} &\implies \neg \left( \exists (a_i)_{i \in \omega} \forall i \in \omega \quad a_{i+1} \in a_i \right)^{\mathbf{M}[G]} \\
 \parallel &\qquad \qquad \qquad \parallel \\
 \left( \neg \exists (\dot{a}_i)_{i \in \omega} \forall i \in \omega \quad \dot{a}_{i+1} \in \dot{a}_i \right)^{\mathbf{M}} &\qquad \left( \neg \exists (a_i)_{i \in \omega} \forall i \in \omega \quad a_{i+1} \in a_i \right)^{\mathbf{M}[G]}
 \end{aligned}$$

Since

$$\left( \neg \exists (\dot{a}_i)_{i \in \omega} \forall i \in \omega \quad \dot{a}_{i+1} \in \dot{a}_i \right)^{\mathbf{M}}$$

holds, it follows that

$$\left( \neg \exists (a_i)_{i \in \omega} \forall i \in \omega \quad a_{i+1} \in a_i \right)^{\mathbf{M}[G]}$$

holds as well.

**Replacement Schema:** for each formula  $\varphi(x, y, z_1, \dots, z_n)$ , we want to prove that:

$$\forall z_1, \dots, \forall z_n \in \mathbf{M}[G] \left( \begin{array}{c} \forall x \in \mathbf{M}[G] \exists! y \in \mathbf{M}[G] \quad (\varphi(x, y, z_1, \dots, z_n))^{\mathbf{M}[G]} \\ \longrightarrow \\ \forall u \in \mathbf{M}[G] \exists v \in \mathbf{M}[G] \forall x \in u \exists y \in v \quad (\varphi(x, y, z_1, \dots, z_n))^{\mathbf{M}[G]} \end{array} \right).$$

We fix  $a_1 = (a_1)_G, \dots, a_n = (a_n)_G$ , and  $u = (u)_G$ . Inside  $\mathbf{M}$  we define:

$$\begin{aligned} \mathbf{F} : \text{dom}(u) \times \mathbb{P} &\rightarrow \mathbf{On} \\ (a, p) &\rightarrow \begin{cases} \text{least } \alpha \in \mathbf{On} \text{ s.t. } \exists \underline{b} \in \mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\alpha \quad p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(a, \underline{b}, a_1, \dots, a_n) \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Since  $\mathbf{M}$  satisfies the finitely many instances of the replacement schema our proof requires, there exists  $\beta \in (\mathbf{On})^{\mathbf{M}}$  such that  $\mathbf{F}[\text{dom}(u) \times \mathbb{P}] \subseteq \beta$ . We set:

$$\eta = (\mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\beta) \times \{\mathbf{1}\} \in \mathbf{M}.$$

We assume

$$\forall x \in \mathbf{M}[G] \exists! y \in \mathbf{M}[G] \quad (\varphi(x, y, a_1, \dots, a_n))^{\mathbf{M}[G]}$$

and let  $a \in u$ . It follows that there exists some — unique —  $b \in \mathbf{M}[G]$  such that

$$(\varphi(a, b, a_1, \dots, a_n))^{\mathbf{M}[G]}.$$

Therefore there exists  $p \in G$  such that given any  $\mathbb{P}$ -names  $\underline{a} \in \text{dom}(u)$  and  $\underline{b} \in \mathbf{M}^{\mathbb{P}}$  which satisfy  $(\underline{a})_G = a$  and  $(\underline{b})_G = b$ , respectively, we have

$$p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\underline{a}, \underline{b}, a_1, \dots, a_n)$$

It follows that there exists  $\underline{b}' \in \eta = (\mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\beta) \times \{\mathbf{1}\}$  such that

$$p \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\underline{a}, \underline{b}', a_1, \dots, a_n)$$

The Truth Lemma yields

$$\mathbf{M}[G] \models \varphi(a, (\underline{b}')_G, a_1, \dots, a_n).$$

Finally, by unicity, we obtain  $b = (\underline{b}')_G$ , which shows that  $b \in (\eta)_G$ . Therefore  $(\eta)_G$  satisfies

$$\left\{ b \in \mathbf{M}[G] \mid \exists a \in u \quad (\varphi(a, b, a_1, \dots, a_n))^{\mathbf{M}[G]} \right\} \subseteq (\eta)_G.$$

**Choice:** In order to establish that  $(\mathbf{AC})^{\mathbf{M}[G]}$  it is enough to show that given any  $A \in \mathbf{M}[G]$ , there exists an ordinal  $\alpha$  and a mapping  $f : \alpha \xrightarrow{\text{onto}} A$ .

Since we assume  $\mathbf{M}$  is a *c.t.m.* of finitely many axioms from **ZFC**, we assume in particular that **AC** is one of those. Thus, **AC** holds in  $\mathbf{M}$ .

Given any  $A \in \mathbf{M}[G]$ , we let  $\check{A} \in \mathbf{M}^{\mathbb{P}}$  satisfy  $(\check{A})_G = A$ . Inside  $\mathbf{M}$ , there exist some ordinal  $\alpha$  and some mapping

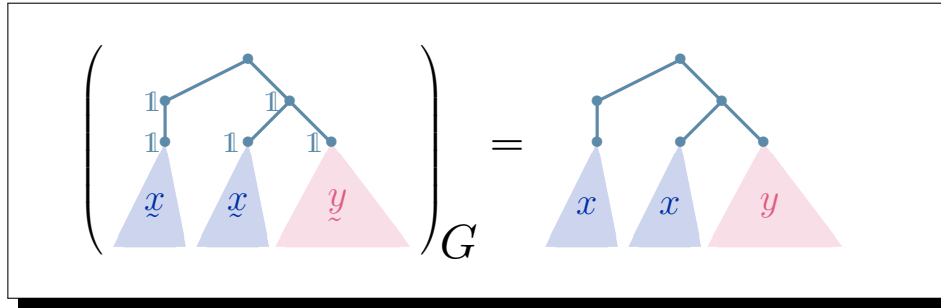
$$g : \alpha \xrightarrow{\text{onto}} \text{dom}(\check{A}).$$

We make use of the functional  $\text{couple} : \mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \rightarrow \mathbf{M}^{\mathbb{P}}$  that was defined in Example 300. We recall that

$$\text{couple}(\check{x}, \check{y}) = \left\{ \left( \{(\check{x}, \mathbb{1})\}, \mathbb{1} \right), \left( \{(\check{x}, \mathbb{1}), (\check{y}, \mathbb{1})\} \right) \right\}$$

as shown in the picture below, providing  $(\check{x})_G = x$  and  $(\check{y})_G = y$ , we have

$$\left( \text{couple}(\check{x}, \check{y}) \right)_G = (x, y).$$



We consider the  $\mathbb{P}$ -name  $\check{f} \in \mathbf{M}^{\mathbb{P}}$  defined by

$$\check{f} = \left\{ \left( \text{couple}(\check{\beta}, g(\beta)), \mathbb{1} \right) \in \mathbf{M}^{\mathbb{P}} \times \{\mathbb{1}\} \mid \beta \in \alpha \right\}$$

We then have

$$\begin{aligned} f &= (\check{f})_G = \left\{ \left( \text{couple}(\check{\beta}, g(\beta)) \right)_G \in \mathbf{M}[G] \mid (\beta \in \alpha)^{\mathbf{M}[G]} \right\} \\ &= \left\{ \left( (\check{\beta})_G, (g(\beta))_G \right) \in \mathbf{M}[G] \mid \beta \in \alpha \right\} \\ &= \left\{ \left( \beta, (g(\beta))_G \right) \in \mathbf{M}[G] \mid \beta \in \alpha \right\}. \end{aligned}$$

It remains to show

$$f : \alpha \xrightarrow{\text{onto}} (A)_G = \left\{ (\mu)_G \mid \mu \in \text{dom}(\check{A}) \right\}$$

To show that  $f$  is a function from  $\alpha$  to  $A = (A)_G = \left\{ (\mu)_G \mid \mu \in \text{dom}(\underline{A}) \right\}$  is immediate.

To show that  $f$  is onto, it suffices to notice that for every  $b \in A$ , there exists some  $\underline{b} \in \text{dom}(\underline{A})$  such that  $b = (\underline{b})_G$  and use the fact that  $g : \alpha \xrightarrow{\text{onto}} \text{dom}(\underline{A})$  is surjective to get some  $\beta < \alpha$  such that  $g(\beta) = \underline{b}$ . Thus we finally obtain

$$b = (\underline{b})_G = (g(\beta))_G = f(\beta)$$

which shows that

$$f : \alpha \xrightarrow{\text{onto}} A.$$

□ 314

## 16.2 A First Attempt to Deny CH

We try to apply our knowledge of the generic extensions and propose a notion of forcing  $\mathbb{P} \in \mathbf{M}$ , where  $\mathbf{M}$  is a *c.t.m.* of “**ZFC**”, such that for every  $G$  that is  $\mathbb{P}$ -generic over  $\mathbf{M}$  one has

$$\mathbf{M}[G] \models \neg \text{CH}.$$

As we will later see, this first attempt will hit the target, but in order for us to realize that, we will need to discuss the notion of “cardinal preservation”.

**Example 315.** We let  $\mathbb{P}$  be the following notion of forcing, and  $\mathbf{M}$  be a *c.t.m.* of “**ZFC**” with  $\mathbb{P} \in \mathbf{M}$ :

$$\mathbb{P} = \left\{ f : (\omega_2)^{\mathbf{M}} \times \omega \rightarrow \{0, 1\} \mid \text{dom}(f) \text{ is finite} \right\}$$

with

$$f \leq g \iff f \supseteq g$$

and

$$1 = \emptyset.$$

Let  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$  and  $\mathcal{F} = \bigcup G$ . From Exercise 284 we already know that

(1)  $\mathcal{F}$  is a function (see Exercise 284):

$$\mathcal{F} : (\omega_2)^{\mathbf{M}} \times \omega \longrightarrow \{0, 1\}.$$

(2) Given any  $p \in \mathbb{P}$  and any ordinal  $\alpha < (\omega_2)^{\mathbf{M}}$  and any integer  $n$  such that  $(\alpha, n) \notin \text{dom}(p)$ , one has  $r = p \cup \{((\alpha, n), 0)\}$  and  $q = p \cup \{((\alpha, n), 1)\}$  satisfy

$$q \leq p \wedge r \leq p \wedge q \perp r$$

hence, by Lemma 286,  $G \notin \mathbf{M}$ .

(3) By Lemma 298,  $G \in \mathbf{M}[G]$ , hence  $\mathcal{F} \in \mathbf{M}[G]$ .

For  $\alpha < \beta < (\omega_2)^{\mathbf{M}}$ , we consider:

$$D_{\alpha,\beta} = \left\{ p \in \mathbb{P} \mid \exists n < \omega \left( (\alpha, n) \in \text{dom}(p) \wedge (\beta, n) \in \text{dom}(p) \wedge p(\alpha, n) \neq p(\beta, n) \right) \right\}.$$

We show that  $D_{\alpha,\beta}$  is dense in  $\mathbb{P}$ . Indeed, let  $q \in \mathbb{P}$ , since  $\text{dom}(q)$  is finite, there exists  $n \in \omega$  such that  $(\alpha, n)$  and  $(\beta, n)$  do not belong to  $\text{dom}(q)$ . Set

$$p = q \cup \left\{ ((\alpha, n), 0), ((\beta, n), 1) \right\},$$

to obtain  $p \leq q$  and  $p \in D_{\alpha,\beta}$ , which shows that  $D_{\alpha,\beta}$  is dense in  $\mathbb{P}$ .

We also have  $D_{\alpha,\beta} \in \mathbf{M}$  holds for each every  $\alpha < \beta < (\omega_2)^{\mathbf{M}}$  and, since  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ , for every  $\alpha < \beta < (\omega_2)^{\mathbf{M}}$ :

$$D_{\alpha,\beta} \cap G \neq \emptyset.$$

Thus there exist  $p \in G$  and  $n \in \omega$  such that  $p(\alpha, n) \neq p(\beta, n)$ . It follows that for all  $\alpha < \beta < (\omega_2)^{\mathbf{M}}$ , there exists an integer  $n$  such that

$$\mathcal{F}(\alpha, n) \neq \mathcal{F}(\beta, n).$$

For each ordinal  $\alpha < (\omega_2)^{\mathbf{M}}$ , we consider

$$a_\alpha = \{ n < \omega \mid \mathcal{F}(\alpha, n) = 1 \}.$$

If  $\alpha < \beta < (\omega_2)^{\mathbf{M}}$ , since there exists  $n \in \omega$  such that  $\mathcal{F}(\alpha, n) \neq \mathcal{F}(\beta, n)$ , we have

$$a_\alpha \neq a_\beta.$$

It follows that there exist at least  $(\omega_2)^{\mathbf{M}}$ -many different subsets of  $\omega$  in  $\mathbf{M}[G]$ .

In the Example above, a question remains:

what is the cardinality of  $(\omega_2)^{\mathbf{M}}$  inside  $\mathbf{M}[G]$  ?

i.e.,

$$\text{what is } \left| (\omega_2)^{\mathbf{M}} \right|^{\mathbf{M}[G]} ?$$

In order to succeed in our attempt, we would like two things:

(1) to claim  $\mathbf{M}[G] \models 2^{\aleph_0} \geq \aleph_2$ , and

(2) to carefully be able to determine whether or not  $(\omega_2)^{\mathbf{M}} = (\omega_2)^{\mathbf{M}[G]}$  holds.

In order to answer these questions, we need to investigate the collapse of cardinal numbers that may occur during the move from  $\mathbf{M}$  to  $\mathbf{M}[G]$ . In particular, since the ordinals of  $\mathbf{M}$  and  $\mathbf{M}[G]$  are the same, we would like to know of some conditions which guarantee that the ordinals that are cardinals in  $\mathbf{M}$  still remain cardinals in  $\mathbf{M}[G]$ .

### 16.3 Cardinal Preservation

We recall from Definition 280, that given  $\mathbf{M}$  any c.t.m. of “ZFC”,  $(\mathbb{P} \leq, \mathbb{1}) \in \mathbf{M}$  any notion of forcing on  $\mathbf{M}$ ,

$$\mathcal{A} \subseteq \mathbb{P} \text{ is a (strong) antichain} \iff \forall p \in \mathcal{A} \forall q \in \mathcal{A} \ (p \neq q \longrightarrow p \perp q).$$

**Definition 316.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  any notion of forcing on  $\mathbf{M}$ , and  $(\lambda \text{ is a cardinal})^{\mathbf{M}}$ , we say

$\mathbb{P}$  has the  $\lambda$ -chain condition — or  $\mathbb{P}$  is  $\lambda$ -c.c. —

$$\iff$$

in  $\mathbf{M}$ , every antichain of  $\mathbb{P}$  has cardinality strictly less than  $\lambda$ .

We say that  $\mathbb{P}$  is c.c.c. if  $\mathbb{P}$  is  $\aleph_1$ -c.c..

$\lambda$ -chain condition is the wording commonly adopted. However, the correct formulation should rather be  $\lambda$ -antichain condition, or even  $\lambda$ -strong antichain condition.

**Definition 317.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  any notion of forcing on  $\mathbf{M}$ , and  $(\lambda \text{ is a cardinal})^{\mathbf{M}}$ .

$\mathbb{P}$  preserves cardinals  $\geq \lambda$  (respectively  $\leq \lambda$ )

$$\iff$$

For all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ ,  $\mathbf{M}$  and  $\mathbf{M}[G]$  have the same cardinals  $\geq \lambda$  (respectively  $\leq \lambda$ ).

The following theorem gives an explicit condition on the poset which guarantees that the cardinals above some threshold are preserved.

**Theorem 318.** *Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  any notion of forcing on  $\mathbf{M}$ .*

$$\left. \begin{array}{l} (\lambda \text{ is a regular cardinal})^{\mathbf{M}} \\ \text{and} \\ (\mathbb{P} \text{ is } \lambda\text{-c.c.})^{\mathbf{M}} \end{array} \right\} \implies \mathbb{P} \text{ preserves cardinals } \geq \lambda.$$

In particular, if  $\mathbf{M} \models \lambda = \aleph_1$ , we have  $\mathbf{M} \models \text{“}\aleph_1 \text{ is regular”}$  because we assume **AC** is part of the finitely many axioms that  $\mathbf{M}$  satisfies. Therefore, given any  $\mathbb{P} \in \mathbf{M}$  which satisfies  $\mathbf{M} \models \mathbb{P} \text{ is } \lambda\text{-c.c.}$  ( $\mathbb{P} \text{ is } \aleph_1\text{-c.c.}$ ), we have  $\mathbb{P}$  preserves all cardinals  $\geq \aleph_1$ . Since  $\aleph_0$  and all finite cardinals are all absolute for transitive classes, we have that  $\mathbf{M}$  and  $\mathbf{M}[G]$  have exactly the same cardinals.

*Proof of Theorem 318:* Let  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ . Towards a contradiction, we suppose there exists  $\lambda' \geq \lambda$  a cardinal such that  $\lambda'$  is collapsed. We would have then some ordinal  $\mu < \lambda'$  and  $f \in \mathbf{M}[G]$  a mapping from  $\mu$  onto  $\lambda'$ . We let  $\tilde{f} \in \mathbf{M}^{\mathbb{P}}$  satisfy  $f = (\tilde{f})_G$ . By the Truth Lemma, there exists  $p_0 \in G$  such that:

$$p_0 \Vdash \tilde{f} : \check{\mu} \xrightarrow{\text{onto}} \check{\lambda}'.$$

We define inside  $\mathbf{M}$ :

$$\begin{aligned} \mathbf{F} : \mu &\longrightarrow \mathcal{P}(\lambda') \\ \alpha &\longmapsto \{\beta < \lambda' \mid \exists q \leq p_0 \quad q \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}\}. \end{aligned}$$

We show that, in  $\mathbf{M}[G]$ ,  $f(\alpha) \in \mathbf{F}(\alpha)$  holds for all  $\alpha < \mu$ . For this, we let  $\beta = f(\alpha)$ , i.e.,  $\mathbf{M}[G] \models f(\alpha) = \beta$ . By the Truth Lemma, there exists  $q_0 \in G$  such that :

$$q_0 \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}.$$

Therefore, there exists  $r \in G$  such that  $r \leq q_0$ ,  $r \leq p_0$  and  $r \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}$ , hence  $\beta \in \mathbf{F}(\alpha)$  and thus for all  $\alpha < \mu$ ,  $f(\alpha) \in \mathbf{F}(\alpha)$ .

We now show that for all  $\alpha < \mu$ ,  $(|\mathbf{F}(\alpha)| < \lambda)^{\mathbf{M}}$ . Let  $\alpha < \mu$  and  $\beta$  in  $\mathbf{F}(\alpha)$ . With the help of **AC** — which holds in  $\mathbf{M}$  — we map  $\beta$  to some  $q_\beta \in \mathbb{P}$  which satisfies both

$$(1) \quad q_\beta \leq p_0 \qquad (2) \quad q_\beta \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}.$$

We notice that the underlying mapping:  $\mathbf{H} : \mathbf{F}(\alpha) \xrightarrow{1-1} \mathbb{P}$  is 1-1.

This relies on the fact that not only do we have  $\beta \neq \beta' \implies q_\beta \neq q_{\beta'}$ , but we even have  $\beta \neq \beta' \implies q_\beta \perp q_{\beta'}$ .

To show this, let us assume towards a contradiction, that there exist  $\beta \neq \beta'$  with  $q_\beta$  and  $q_{\beta'}$  compatible. Then there would also exist some  $q \in \mathbb{P}$  such that  $q \leq q_\beta$  and  $q \leq q_{\beta'}$ , hence:

$$(1) \quad q \Vdash_{\mathbb{P}, \mathbf{M}} f : \check{\mu} \xrightarrow{\text{onto}} \check{\lambda}' \quad (2) \quad q \Vdash_{\mathbb{P}, \mathbf{M}} f(\check{\alpha}) = \check{\beta} \quad (3) \quad q \Vdash_{\mathbb{P}, \mathbf{M}} f(\check{\alpha}) = \check{\beta}'.$$

Then, in  $\mathbf{V}$ , we could get some filter  $J$  which contains  $q$  and is  $\mathbb{P}$ -generic over  $\mathbf{M}$ . By the Truth Lemma, this would lead to some generic extension  $\mathbf{M}[J]$  that would satisfy:

$$(1) \quad \mathbf{M}[J] \models f : \mu \xrightarrow{\text{onto}} \lambda' \quad (2) \quad \mathbf{M}[J] \models f(\alpha) = \beta \quad (3) \quad \mathbf{M}[J] \models f(\alpha) = \beta',$$

which leads to  $\mathbf{M}[J] \models \beta = \beta'$ , contradicting our hypotheses about  $\beta$  and  $\beta'$ .

So, not only have we shown that  $\mathbf{H}$  is injective, but we have also shown that  $\mathbf{H}[\mathbf{F}(\alpha)]$  is an antichain. Now, since  $\mathbb{P}$  is  $\lambda$ -c.c., we obtain

$$\left( |\mathbf{H}[\mathbf{F}(\alpha)]| < \lambda \right)^{\mathbf{M}}.$$

and since  $\mathbf{H}$  is 1-1,

$$\left( |\mathbf{F}(\alpha)| < \lambda \right)^{\mathbf{M}}.$$

Inside  $\mathbf{M}$ , we define  $S \subseteq \lambda'$  by

$$S = \bigcup \{ \mathbf{F}(\alpha) \mid \alpha < \mu \}.$$

For each  $\alpha < \mu$ , we have  $f(\alpha) \in S$  and  $f : \mu \xrightarrow{\text{onto}} \lambda'$ , therefore we have

$$f[\mu] = \lambda' \subseteq S.$$

Since both  $S \subseteq \lambda'$  and  $\lambda' \subseteq S$  hold, we obtain

$$S = \lambda'.$$

For each  $\alpha < \mu$ , we set

$$|\mathbf{F}(\alpha)| = \lambda_\alpha.$$

We distinguish between  $\mu < \lambda$  and  $\mu \geq \lambda$  in order to show that in any case,  $|S| < \lambda'$  holds which contradicts  $S = \lambda'$ :

- If  $\mu < \lambda$ , then since  $\lambda$  is regular,  $|S| = \sup \{ \lambda_\alpha \mid \alpha < \mu \} < \lambda < \lambda'$  holds.
- If  $\mu \geq \lambda$ , then  $|S| \leq \mu \cdot \lambda = \max \mu, \lambda = \mu < \lambda'$ .

□ **318**

We already have a condition on posets — being  $\lambda$ -c.c. for some  $\lambda$  regular — which guarantees that cardinals above a certain threshold are preserved. We now propose another condition on posets which guarantees the same type of preservation not above but below the same kind of threshold. The first condition relied on sizes of antichains, this new one deals with sizes of chains.

**Definition 319.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  any notion of forcing, and  $(\lambda \text{ is a cardinal})^{\mathbf{M}}$ .

$\mathbb{P}$  is  $\lambda$ -closed

$\Longleftrightarrow$

for all  $\gamma < \lambda$  and decreasing sequence  $(p_\xi)_{\xi < \gamma}$  from  $\mathbb{P}$ , there exists  $p \in \mathbb{P}$  s.t.  $p \leq p_\xi$  (any  $\xi < \gamma$ ).

**Theorem 320.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  be any notion of forcing.

If  $(\lambda \text{ is a cardinal})^{\mathbf{M}}$  and  $(\mathbb{P} \text{ is } \lambda\text{-closed})^{\mathbf{M}}$ , then  $\mathbb{P}$  preserves all cardinals  $\leq \lambda$ .

Before proving this theorem, we need some easy preliminary result.

**Lemma 321.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  be any notion of forcing, and  $p \in \mathbb{P}$  be any forcing condition. Let also  $\varphi(x, x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula, and  $\underline{b}, \underline{a}_1, \dots, \underline{a}_n \in \mathbf{M}^{\mathbb{P}}$ .

If  $p \Vdash_{\mathbb{P}, \mathbf{M}} \exists x (x \in \underline{b} \wedge \varphi(x, \underline{a}_1, \dots, \underline{a}_n))$ , then there exists  $q \leq p$  and  $\underline{c} \in \text{dom}(\underline{b})$  s.t.

$$q \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\underline{c}, \underline{a}_1, \dots, \underline{a}_n).$$

*Proof of Lemma 321:* Let  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$  such that  $p \in \mathbf{M}[G]$  and set  $b = (b)_G, a_1 = (\underline{a}_1)_G, \dots, a_n = (\underline{a}_n)_G$ . We have:

$$\mathbf{M}[G] \models \exists x (x \in b \wedge \varphi(x, a_1, \dots, a_n));$$

therefore there exists — by the very definition of  $b$  — some  $c = (\underline{c})_G$  with  $\underline{c} \in \text{dom}(\underline{b})$  such that

$$\mathbf{M}[G] \models (c \in b \wedge \varphi(c, a_1, \dots, a_n)).$$

By the Truth Lemma, there exists  $p' \in G$  such that

$$p' \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\underline{c}, \underline{a}_1, \dots, \underline{a}_n).$$

Since both  $p$  and  $p'$  belong to  $G$ , there exists  $q \in G$  such that both  $q \leq p$  and  $q \leq p'$  hold. This yields

$$q \Vdash_{\mathbb{P}, \mathbf{M}} \varphi(\underline{c}, \underline{a}_1, \dots, \underline{a}_n).$$

□ 321

*Proof of Theorem 320:* We simply show the following which will immediately give the result:  
For all  $\mu < \lambda$ , for all  $\xi \in \mathbf{On}$ , for all  $f : \mu \xrightarrow{\text{onto}} \xi$ ,

$$\text{if } f \in \mathbf{M}[G], \text{ then } f \in \mathbf{M}.$$

We assume  $f = (f)_G \in \mathbf{M}[G]$ , and let  $p_0 \in G$  such that  $p_0 \Vdash_{\mathbb{P}, \mathbf{M}} \check{f} : \check{\mu} \xrightarrow{\text{onto}} \check{\xi}$ . We set

$$D = \left\{ p \in \mathbb{P} \mid \left( \exists \check{g} \forall \alpha < \mu \exists ! x \left( \text{couple}(\check{\alpha}, x) \in \check{g} \wedge p \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = x \right) \right)^{\mathbf{M}} \right\}.$$

which we summarize as

$$D = \{ p \in \mathbb{P} \mid \exists \check{g} \in \mathbf{M} \forall \alpha < \mu \ p \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{g}(\check{\alpha}) \}.$$

We show that  $D$  is dense below  $p_0$ . For this purpose, we define both a  $\leq_{\mathbb{P}}$ -decreasing sequence  $(p_\alpha)_{\alpha < \mu}$  and a sequence of ordinals  $(\xi_\alpha)_{\alpha < \mu}$  such that for all  $\alpha < \beta < \mu$ :

$$p_\beta \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{\xi}_\alpha.$$

The definition is by recursion on  $\alpha < \mu$ . At each step  $\alpha$ , both  $(p_\zeta)_{\zeta \leq \alpha}$  and  $(\xi_\zeta)_{\zeta < \alpha}$  are defined. In particular, all  $\xi_\zeta$  are defined at successor level (even for  $\zeta$  limit).

**$\alpha := 0$ :** Nothing needs to be defined at this stage, since only  $p_0$  is required and it is already defined.

**$\alpha := \alpha + 1$ :** we define  $p_{\alpha+1} \leq p_\alpha$  and  $\xi_\alpha$ . By Construction, we have  $p_\alpha \leq p_0$ , hence

$$p_\alpha \Vdash_{\mathbb{P}, \mathbf{M}} \check{f} : \check{\mu} \xrightarrow{\text{onto}} \check{\xi}.$$

Since  $\alpha < \mu$ , it follows that

$$p_\alpha \Vdash_{\mathbb{P}, \mathbf{M}} \exists x \in \check{\xi} \ \check{f}(\check{\alpha}) = x.$$

By definition,  $\check{\xi} = \{(\check{\eta}, \mathbb{1}) \mid \eta < \xi\}$  and  $\text{dom}(\check{\xi}) = \{\check{\eta} \mid \eta < \xi\}$ . From Lemma 321, there exists  $p_{\alpha+1} \leq p_\alpha$  and  $\xi_\alpha < \xi$  such that

$$p_{\alpha+1} \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{\xi}_\alpha.$$

**$\alpha$  limit:** assuming the decreasing sequence  $(p_\zeta)_{\zeta < \alpha}$  has been constructed, since  $\mathbb{P}$  is  $\lambda$ -closed, there exists  $p_\alpha$  which is below every  $p_\zeta$ . Since  $\alpha$  is limit, there is no other condition on  $p_\alpha$  to satisfy and there is no ordinal of the form  $\xi_\zeta$  to define.

Since  $\alpha < \mu < \lambda$  and  $\mathbb{P}$  is  $\lambda$ -closed, there exists some  $p_\mu \in \mathbb{P}$  such that for all  $\alpha < \mu$  we have

$$p_\mu \leq p_\alpha \quad \text{and} \quad p_\mu \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{\xi}_\alpha.$$

Inside  $\mathbf{M}$ , we set  $g(\alpha) = \xi_\alpha$  so that we have

$$p_\mu \leq p_0 \quad \text{and} \quad \forall \alpha < \mu \quad p_\mu \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{g}(\check{\alpha})$$

which shows that  $p_\mu$  belongs to  $D$  and completes the proof that  $D$  is dense below  $p_0$ .

Now, since  $p_0 \in G$ , we have  $D \cap G \neq \emptyset$ . For any  $q \in D \cap G$ , by the very definition of  $D$  there exists  $g \in \mathbf{M}$  such that

$$\forall \alpha < \mu \quad q \Vdash_{\mathbb{P}, \mathbf{M}} \check{f}(\check{\alpha}) = \check{g}(\check{\alpha}).$$

Thus, we have for all  $\alpha < \mu$ ,

$$\mathbf{M}[G] \models (f(\alpha))_G = (g(\alpha))_G$$

i.e.,

$$\mathbf{M}[G] \models f(\alpha) = g(\alpha).$$

which shows that  $f = g \restriction \mu$  which belongs to  $\mathbf{M}$ . So, finally we obtain  $f \in \mathbf{M}$ .

□ 320

## Chapter 17

# Independence of CH

In this chapter, we will prove that if the theory **ZF** is consistent, so is the theory **ZF** +  $2^{\aleph_0} = \aleph_2$ . Since we already know that if **ZF** is consistent, so is **ZF** +  $2^{\aleph_0} = \aleph_1$  (see Theorem 271), this new result will show that  $2^{\aleph_0} = \aleph_1$  is independent from **ZF**. i.e., if **ZF** is consistent, then

- **ZF**  $\not\models_c 2^{\aleph_0} = \aleph_1$
- **ZF**  $\not\models_c 2^{\aleph_0} \neq \aleph_1$ .

The same result holds for **ZFC** as well. i.e., if **ZFC** is consistent, then

- **ZFC**  $\not\models_c 2^{\aleph_0} = \aleph_1$
- **ZFC**  $\not\models_c 2^{\aleph_0} \neq \aleph_1$ .

In the next chapter, we will also have similar results for **AC** instead of **CH**. i.e., if **ZF** is consistent, then

- **ZF**  $\not\models_c \mathbf{AC}$
- **ZF**  $\not\models_c \neg \mathbf{AC}$ .

Moreover, many more independence results can be obtained by applying forcing techniques. We only illustrate the by a few samples, but many more can be found in the literature.

### 17.1 Forcing $2^{\aleph_0} = \aleph_2$

We go back to the poset that was introduced in Example 315:

$$\mathbb{P}_{\omega_2} = \{f : (\omega_2)^{\mathbf{M}} \times \omega \longrightarrow 2 \mid |dom(f)| < \omega\}.$$

In order to conclude that, when we forced with this notion of forcing, the generic extension satisfied  $2^{\aleph_0} = \aleph_2$  we needed to make sure that cardinals were preserved. This is precisely what this section will establish: simply by proving that  $\mathbb{P}_{\omega_2}$  has the c.c.c..

As a preliminary, we need to prove some purely combinatorial result.

**Lemma 322.** *Let  $F$  be any family of finite sets such that  $|F| = \aleph_1$ . There exist  $F' \subseteq F$  and  $r$  finite such that:*

- $|F'| = \aleph_1$ ;
- for all  $a, b \in F'$ ,  $a \cap b = r$ .

$F'$  is called a  $\Delta$ -system.

*Proof of Lemma 322.* Since  $F$  is some family of finite sets, and  $|F| = \aleph_1$ , there exists an integer  $n$  and a subset of  $F$  with cardinality  $\aleph_1$  only containing sets of cardinality  $n$ . So without loss of generality we may assume that for all  $a \in F$ ,  $|a| = n$ . The proof then goes by induction on  $n$ .

If  $n = 1$ , then  $F' = F$  and  $r = \emptyset$  works.

We now suppose that the property holds for  $n$  and show that it also holds for  $n + 1$ . So we let  $F$  be such that for all  $a \in F$ ,  $|a| = n + 1$ . We then distinguish between two cases.

- (1) There exists  $x$  such that  $|F_x| = \aleph_1$  where  $F_x = \{a \in F \mid x \in a\}$ . We then set  $F_0 = \{a \setminus \{x\} \mid a \in F_x\}$ . We obtain  $|F_0| = \aleph_1$  and for all  $a \in F_0$ ,  $|a| = n$ . By induction hypothesis, there exists  $F'_0 \subseteq F_0$  and  $r_0$  such that  $|F'_0| = \aleph_1$  and for all  $a, b \in F'_0$ ,  $a \cap b = r_0$ . We then set

$$F' = \{a_0 \cup \{x\} \mid a \in F'_0\} \text{ and } r = r_0 \cup \{x\}.$$

Notice that we have  $|F'| = \aleph_1$  and for all  $a, b \in F'$ ,  $a \cap b = r$ .

- (2) For every  $x$ ,  $|F_x| < \aleph_1$ . We then define by induction on  $\xi$  a sequence  $(a_\xi)_{\xi < \omega_1}$  of two by two disjoint elements of  $F$ . We start with  $a_0$  being any element, and for each  $\beta < \omega_1$ , we choose  $a_\beta$  such that for all  $\xi < \beta$ ,  $a_\xi \cap a_\beta = \emptyset$ . We can do so, for otherwise there would exist some (least)  $\beta < \omega_1$  such that no  $a \in F$  satisfies that for all  $\xi < \beta$ ,  $a_\xi \cap a = \emptyset$ . We then define the following mapping:

$$\begin{aligned} f : (F \setminus \{a_\xi \mid \xi < \beta\}) &\longrightarrow \beta \\ a &\longmapsto \xi \text{ such that } a \cap a_\xi \neq \emptyset. \end{aligned}$$

Since  $|F \setminus \{a_\xi \mid \xi < \beta\}| = \aleph_1$  and  $|\beta| < \aleph_1$ , there exists  $\xi < \beta$  such that  $|f^{-1}[\xi]| = \aleph_1$ . But  $a_\xi$  is finite and there are  $\aleph_1$ -many elements in  $F$  that have one element in common with  $a_\xi$ , therefore there exists  $x \in a_\xi$  such that  $|F_x| = \aleph_1$ , which contradicts the hypothesis. This guarantees the existence of the sequence  $(a_\xi)_{\xi < \omega_1}$ . We then finally set  $F' = \{a_\xi \mid \xi < \omega_1\}$  and  $r = \emptyset$ .

□ 322

The following notion of forcing was introduced in Example 315.

**Definition 323.** The notion of forcing  $(\mathbb{P}_{\omega_2}, \leq, \mathbb{1})$  is defined by

$$(1) \mathbb{P}_{\omega_2} = \left\{ f : (\omega_2)^{\mathbf{M}} \times \omega \rightarrow \{0, 1\} \mid \text{dom}(f) \text{ is finite} \right\}$$

$$(2) f \leq f' \iff f \supseteq f'$$

$$(3) \mathbb{1} = \emptyset$$

**Lemma 324.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC” and  $(\mathbb{P}_{\omega_2}, \leq, \mathbb{1}) \in \mathbf{M}$ .

$\mathbb{P}_{\omega_2}$  has the c.c.c..

*Proof of Lemma 324:* Towards a contradiction, we suppose that  $\mathcal{A}$  is an antichain in  $\mathbb{P}_{\omega_2}$  with cardinality  $\aleph_1$ . We set  $F = \{\text{dom}(p) \mid p \in \mathcal{A}\}$ . Since in  $\mathbb{P}_{\omega_2}$  there exist only finitely many different functions over any finite fixed domain, we necessarily have that  $|F| = \aleph_1$ . By Lemma 322, there exists some  $\Delta$ -system  $F' \subseteq F$  such that  $|F'| = \aleph_1$  and  $r$  finite such that for any two different  $a, b \in F'$ ,  $a \cap b = r$ . We let  $(p_\alpha)_{\alpha < \omega_1}$  be a sequence of elements of  $\mathcal{A}$  such that for any  $\alpha < \omega_1$ ,  $\text{dom}(p_\alpha) \in F'$ . For  $\alpha < \alpha' < \omega_1$ , we have  $p_\alpha \perp p_{\alpha'}$ , hence

$$p_\alpha \restriction r \neq p_{\alpha'} \restriction r.$$

It follows that the mapping

$$\begin{aligned} \aleph_1 &\xrightarrow{1-1} 2^r \\ \alpha &\longmapsto p_\alpha \restriction r \end{aligned}$$

is injective which is impossible because  $2^r$  is finite (recall  $r$  is finite).

□ 324

**Corollary 325.** Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P}_{\omega_2} \in \mathbf{M}$ , and  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ .

$\mathbb{P}_{\omega_2}$  preserves all cardinals.

i.e., for all  $\alpha \in \mathbf{On}$ ,

$$\aleph_\alpha^{\mathbf{M}[G]} = \aleph_\alpha^{\mathbf{M}}.$$

*Proof of Corollary 325:* By Lemma 324,  $\mathbb{P}_{\omega_2}$  has the countable chain condition (c.c.c.). By Theorem 318, it preserves all cardinals  $\geq \aleph_1$ . Moreover, by absoluteness,  $\aleph_0^{\mathbf{M}} = \aleph_0^{\mathbf{M}[G]}$  and also for each integer  $n$ ,  $n^{\mathbf{M}} = n^{\mathbf{M}[G]}$ . So,  $\mathbb{P}_{\omega_2}$  preserves all cardinals.

□ 325

This Corollary guarantees that forcing with  $\mathbb{P}_{\omega_2}$  yields at least  $(\omega_2)^{\mathbf{M}}$ -many different subsets of  $\omega$  in  $\mathbf{M}[G]$ . Now, we know from Corollary 325 that in  $\mathbf{M}[G]$  there are at least  $(\omega_2)^{\mathbf{M}[G]}$ -many different subsets of  $\omega$ . Therefore,  $\mathbf{M}[G] \models 2^{\aleph_0} \geq \aleph_2$ . Some more work is still required to show that  $\mathbf{M}[G] \models 2^{\aleph_0} = \aleph_2$ . Namely, to show that  $\mathbf{M}[G] \models 2^{\aleph_0} \leq \aleph_2$ . This is what the next result is about: obtaining a bound on the size of  $\mathcal{P}(\lambda)$  in  $\mathbf{M}[G]$  that depends on some properties of the notion of forcing  $\mathbb{P}$ .

**Lemma 326.** *Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  any notion of forcing.*

*If  $(\mathbb{P}$  is c.c.c.) $^{\mathbf{M}}$ ,  $(\lambda$  cardinal) $^{\mathbf{M}}$ , and  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ . Then*

$$|\mathcal{P}(\lambda)|^{\mathbf{M}[G]} \leq \left(|\mathbb{P}|^\lambda\right)^{\mathbf{M}}.$$

*Proof of Lemma 326:* Let  $X \in \mathcal{P}(\lambda)^{\mathbf{M}[G]}$ , we choose a  $\mathbb{P}$ -name  $\underline{X}$  in  $\mathbf{M}[G]$  such that  $(\underline{X})_G = X$ . Inside  $\mathbf{M}$  we define:

$$\begin{aligned} f_{\underline{X}} : \lambda &\longrightarrow \mathcal{P}(\mathbb{P}) \\ \alpha &\longmapsto \mathcal{A}_\alpha \text{ some maximal antichain in } \{p \in \mathbb{P} \mid p \Vdash \check{\alpha} \in \underline{X}\}. \end{aligned}$$

We define inside  $\mathbf{M}[G]$ :

$$\begin{aligned} \mathbf{F} : \mathcal{P}(\lambda) &\longrightarrow \left({}^\lambda \{\text{antichains of } \mathbb{P}\}\right)^{\mathbf{M}} \\ X &\longmapsto f_{\underline{X}}. \end{aligned}$$

We show that  $\mathbf{F}$  is 1-1. For this we let  $X, X' \in \mathcal{P}(\lambda)^{\mathbf{M}[G]}$  be distinct and  $\underline{X}, \underline{X}'$  such that  $(\underline{X})_G = X$  and  $(\underline{X}')_G = X'$ . Pick  $\alpha \in (X \setminus X') \cup (X' \setminus X)$ . By symmetry, we assume  $\alpha \in X \setminus X'$ . By the Truth Lemma, there exists  $p_0 \in G$  such that

$$p_0 \Vdash (\check{\alpha} \in \underline{X} \wedge \check{\alpha} \notin \underline{X}').$$

Since  $f_{\underline{X}}(\alpha) = \mathcal{A}_\alpha$  is some maximal antichain, there exists  $p \in \mathcal{A}_\alpha$  such that  $p_0$  and  $p$  are compatible. Moreover,  $p_0$  is not compatible with any element from  $\mathcal{A}'_\alpha$ . Indeed, if  $p_0$  were compatible with some  $p' \in \mathcal{A}'_\alpha$  there would exist  $r$  such that  $r \leq p_0$ ,  $r \leq p'$  and

$$r \Vdash \check{\alpha} \notin \underline{X}' \quad \text{and} \quad r \Vdash \check{\alpha} \in \underline{X}'.$$

A contradiction since

$$r \Vdash \check{\alpha} \notin \underline{X}' \iff r \Vdash \neg \check{\alpha} \in \underline{X}' \iff \forall t \leq r \ t \nVdash \check{\alpha} \in \underline{X}' \implies r \nVdash \check{\alpha} \in \underline{X}'.$$

Since  $p_0$  is compatible with some element from  $f_X(\alpha) = \mathcal{A}_\alpha$  but no element from  $f_{X'}(\alpha) = \mathcal{A}'_\alpha$ , we conclude that  $\mathcal{A}_\alpha \neq \mathcal{A}'_\alpha$  and  $f_X \neq f_{X'}$ , which shows that  $\mathbf{F}$  is 1-1; which in turn gives

$$|\mathcal{P}(\lambda)|^{\mathbf{M}[G]} \leq \left| {}^\lambda \{\text{antichains of } \mathbb{P}\} \right|^{\mathbf{M}}.$$

Since  $\mathbb{P}$  has the c.c.c., every antichain is countable. So, we have

$$|\{\text{antichains of } \mathbb{P}\}|^{\mathbf{M}} \leq (|\mathbb{P}|^{\aleph_0})^{\mathbf{M}}.$$

Finally, an easy computation gives the result:

$$|\mathcal{P}(\lambda)|^{\mathbf{M}[G]} \leq \left| {}^\lambda \{\text{antichains of } \mathbb{P}\} \right|^{\mathbf{M}} \leq \left| \left( |\mathbb{P}|^{\aleph_0} \right)^\lambda \right|^{\mathbf{M}} = \left| |\mathbb{P}|^{\aleph_0 \cdot \lambda} \right|^{\mathbf{M}} \leq \left( |\mathbb{P}|^\lambda \right)^{\mathbf{M}}.$$

□ 326

**Corollary 327** (Cohen). *Let  $\mathbf{M}$  be any c.t.m. of “ZFC + CH”,  $\mathbb{P} = (\mathbb{P}_{\omega_2})^{\mathbf{M}}$ , and  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ .*

$$\left( 2^{\aleph_0} = \aleph_2 \right)^{\mathbf{M}[G]}.$$

Notice that we start with a ground model  $\mathbf{M}$  that satisfies **CH**. i.e.,  $(2^{\aleph_0} = \aleph_1)^{\mathbf{M}}$ .

*Proof of Corollary 327:* By Lemma 324,  $\mathbb{P}_{\omega_2}$  has the c.c.c., so by Lemma 326, we have:

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} \leq \left( |\mathbb{P}_{\omega_2}|^{\aleph_0} \right)^{\mathbf{M}}.$$

Notice that

$$f \in \mathbb{P}_{\omega_2} \iff f \text{ is some finite function } : \omega_2 \times \omega \rightarrow \{0, 1\}.$$

So formally,

$$f \in \mathbb{P}_{\omega_2} \iff \left( f \subseteq ((\omega_2 \times \omega) \times \{0, 1\}) \text{ and } f \text{ is finite} \right),$$

hence,

$$|\mathbb{P}_{\omega_2}|^{\mathbf{M}} = \aleph_2^{\mathbf{M}}.$$

In  $\mathbf{M}$ ,  $\omega_2$  is regular since **AC** is satisfied. So, any function from  $\omega$  into  $\omega_2$  is indeed some mapping from  $\omega$  into some  $\alpha < \omega_2$ , and certainly  $|\alpha| \leq \aleph_1$ . So, in  $\mathbf{M}$ :

$$|\omega \alpha| \leq \aleph_1^{\aleph_0}.$$

Since  $\mathbf{M}$  is a c.t.m. of “ZFC + CH” we have  $(\aleph_1 = 2^{\aleph_0})^{\mathbf{M}}$ . Thus, in  $\mathbf{M}$ , for every  $\alpha < \omega_2$  we also have:

$$|\omega \alpha| \leq \aleph_1^{\aleph_0} = \left( 2^{\aleph_0} \right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \aleph_1.$$

When  $\alpha$  varies over  $\omega_2$ , we obtain:

$$\aleph_2 \leq \aleph_2^{\aleph_0} = \left| \aleph_0 \aleph_2 \right| = \left| \bigcup_{\alpha < \omega_2} {}^\omega \alpha \right| \leq \left| \left\{ (\alpha, f) \mid \alpha < \omega_2 \wedge f \in {}^\omega \alpha \right\} \right| \leq \aleph_2 \cdot \aleph_1 = \aleph_2.$$

So, we have shown

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} \leq \left( |\mathbb{P}_{\omega_2}|^{\aleph_0} \right)^{\mathbf{M}} = \aleph_2^{\mathbf{M}}.$$

In Example [315](#) we obtained

$$\aleph_2^{\mathbf{M}} \leq |\mathcal{P}(\omega)|^{\mathbf{M}[G]}.$$

Altogether, this gives

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} = \aleph_2^{\mathbf{M}}.$$

i.e.,

$$\left( 2^{\aleph_0} \right)^{\mathbf{M}[G]} = \aleph_2^{\mathbf{M}}.$$

In Corollary [325](#) we proved that  $\mathbb{P}_{\omega_2}$  preserves all cardinals, so in particular

$$\aleph_2^{\mathbf{M}} = \aleph_2^{\mathbf{M}[G]}$$

which finally leads to

$$\left( 2^{\aleph_0} = \aleph_2 \right)^{\mathbf{M}[G]}.$$

□ [327](#)

## 17.2 Reflecting back on forcing $2^{\aleph_0} = \aleph_2$

Formally, in order to show

$$\text{cons}(\mathbf{ZFC}) \longrightarrow \text{cons}(\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2)$$

we proceeded by contraposition and proved:

$$\neg \text{cons}(\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2) \longrightarrow \neg \text{cons}(\mathbf{ZFC}).$$

For this purpose, we supposed there exist axioms  $\varphi_1, \dots, \varphi_n$  in  $\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2$  such that:

$$\varphi_1, \dots, \varphi_n \vdash_c \perp.$$

One can then determine within “**ZFC**” — in advance and *independently of  $\mathbf{M}$*  — some other formulas  $\psi_1, \dots, \psi_k$  in “**ZFC**” such that if  $\mathbf{M}$  is a model of  $\psi_1, \dots, \psi_k$  and  $G$  is  $\mathbb{P}_{\omega_2}$ -generic over  $\mathbf{M}$ , then  $\mathbf{M}[G]$  is a model of  $\varphi_1, \dots, \varphi_n$ . We add to  $\psi_1, \dots, \psi_k$  other formulas  $\psi_{k+1}, \dots, \psi_l$  which enable us to prove other results such as the ones on cardinal preservations, on  $\Delta$ -systems, or on absoluteness, etc.

Then, we work in **ZFC**:

$$\mathbf{ZFC} \vdash_c \left( \begin{array}{c} \text{“ } \exists \mathbf{M} \text{ a c.t.m. s.t. } \left( \{\psi_1, \dots, \psi_l\} \right)^{\mathbf{M}} \text{”} \\ \wedge \\ \text{“ } \exists G \text{ } (\mathbb{P}_{\omega_2})^{\mathbf{M}}\text{-generic over } \mathbf{M} \text{”} \end{array} \right) \Rightarrow \exists \mathbf{M}[G] \left( \{\varphi_1, \dots, \varphi_n\} \right)^{\mathbf{M}[G]}$$

Since,

$$\mathbf{ZFC} \vdash_c \text{“ } \exists \mathbf{M} \text{ a c.t.m. s.t. } \left( \{\psi_1, \dots, \psi_l\} \right)^{\mathbf{M}} \text{”} \wedge \text{“ } \exists G \text{ } (\mathbb{P}_{\omega_2})^{\mathbf{M}}\text{-generic over } \mathbf{M} \text{”}.$$

by *modus ponens* follows,

$$\mathbf{ZFC} \vdash_c \exists \mathbf{M}[G] \left( \{\varphi_1, \dots, \varphi_n\} \right)^{\mathbf{M}[G]}.$$

or more generally,

$$\mathbf{ZFC} \vdash_c \underbrace{\exists N \left( \{\varphi_1, \dots, \varphi_n\} \right)^N}_{\psi}.$$

Since  $\varphi_1, \dots, \varphi_n \vdash_c \perp$ ,

$$\mathbf{ZFC} \vdash_c \underbrace{\neg \exists N \left( \{\varphi_1, \dots, \varphi_n\} \right)^N}_{\neg \psi}.$$

Therefore,

$$\mathbf{ZFC} \vdash_c \perp.$$

As put by Kenneth Kunen: “*The inelegant part of this argument is that the procedure of finding  $\psi_1, \dots, \psi_l$ , although straightforward, completely effective, and finitistically valid, is also very tedious*” [20, p. 233]

### 17.3 Forcing $2^{\aleph_0} = \aleph_{\alpha+1}$

The same argument, *mutatis mutandis* yields, for any ordinal  $\alpha$ , the equiconsistency of **ZFC** and **ZFC** +  $2^{\aleph_0} = \aleph_{\alpha+1}$ .

**Definition 328.** Let  $\mathbf{M}$  be any c.t.m. of “**ZFC**”. Given any  $\alpha \in \mathbf{On}$ , we let  $(\mathbb{P}_{\aleph_\alpha}, \leq, \mathbb{1})$  be

- (1)  $\mathbb{P}_{\aleph_\alpha} = \left\{ f : (\aleph_\alpha)^{\mathbf{M}} \times \omega \rightarrow \{0, 1\} \mid \text{dom}(f) \text{ is finite} \right\}$
- (2)  $f \leq f' \iff f \supseteq f'$
- (3)  $\mathbb{1} = \emptyset$

We first need to show that  $\mathbb{P}_{\aleph_\alpha}$  has the c.c.c. which will guarantee that all cardinals are preserved.

**Lemma 329.** *Let  $\mathbf{M}$  be any c.t.m. of “ZFC”,  $0 < \alpha \in \mathbf{On}$  and  $\mathbb{P}_{\aleph_\alpha} \in \mathbf{M}$ .*

*$\mathbb{P}_{\aleph_\alpha}$  has the c.c.c..*

*Proof of Lemma 329:* *Mutatis mutandis*, identical to the proof of Lemma 324. Towards a contradiction, we suppose that  $\mathcal{A}$  is an antichain in  $\mathbb{P}_{\aleph_\alpha}$  with cardinality  $\aleph_1$ . We set

$$F = \{ \text{dom}(p) \mid p \in \mathcal{A} \}.$$

Since in  $\mathbb{P}_{\aleph_\alpha}$  there exist only finitely many different functions over any finite fixed domain, we necessarily have that  $|F| = \aleph_1$ . By Lemma 322, there exists some  $\Delta$ -system  $F' \subseteq F$  such that  $|F'| = \aleph_1$  and  $r$  finite such that for any two different  $a, b \in F'$ ,  $a \cap b = r$ . We let  $(p_\alpha)_{\alpha < \omega_1}$  be a sequence of elements of  $\mathcal{A}$  such that for any  $\alpha < \omega_1$ ,  $\text{dom}(p_\alpha) \in F'$ . For  $\alpha < \alpha' < \omega_1$ , we have  $p_\alpha \perp p_{\alpha'}$ , hence  $p_\alpha \restriction r \neq p_{\alpha'} \restriction r$ . It follows that the mapping

$$\begin{aligned} \aleph_1 &\xrightarrow{1-1} 2^r \\ \alpha &\longmapsto p_\alpha \restriction r \end{aligned}$$

is injective which is impossible because  $2^r$  is finite (recall  $r$  is finite).

□ 329

**Theorem 330** (Cohen). *Let  $\mathbf{M}$  be any c.t.m. of “ZFC+GCH” and  $0 < \alpha \in \mathbf{On}$ .*

*If  $\mathbb{P} = (\mathbb{P}_{\aleph_{\alpha+1}})^{\mathbf{M}}$  and  $(\mathbf{GCH})^{\mathbf{M}}$ , then for all  $G$   $\mathbb{P}$ -generic over  $\mathbf{M}$ ,*

$$\left( 2^{\aleph_0} = \aleph_{\alpha+1} \right)^{\mathbf{M}[G]}.$$

*Proof of Lemma 330:* By Lemma 117  $\aleph_{\alpha+1}$  is regular and by Lemma 329  $\mathbb{P}_{\aleph_{\alpha+1}}$  has the c.c.c., thus  $\mathbb{P}_{\aleph_{\alpha+1}}$  preserves all cardinals. By Lemma 326, we have

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} \leq \left( |\mathbb{P}_{\aleph_{\alpha+1}}|^{\aleph_0} \right)^{\mathbf{M}}.$$

So, we need to compute  $\left( |\mathbb{P}_{\aleph_{\alpha+1}}|^{\aleph_0} \right)^{\mathbf{M}}$ . i.e.,  $|\aleph_0 \mathbb{P}_{\aleph_{\alpha+1}}|^{\mathbf{M}}$ .

Inside  $\mathbf{M}$ , one has

$$\circ \quad |\mathbb{P}_{\aleph_{\alpha+1}}| = \aleph_{\alpha+1};$$

- since  $\mathbf{M}$  is a *c.t.m.* of “**ZFC** + **GCH**” we have  $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ ;
- $\aleph_{\alpha+1}$  is regular since **AC** is satisfied. So, any function from  $\omega$  into  $\aleph_{\alpha+1}$  is indeed some mapping from  $\omega$  into some  $\xi < \omega_{\alpha+1}$ , and certainly  $|\xi| \leq \aleph_\alpha$ . So,

$$|\omega \xi| \leq \aleph_\alpha^{\aleph_0}.$$

- The mapping  $\chi$  that associates to each function  $f$  from  $\aleph_0$  into  $\aleph_\alpha$ , its characteristic function  $\chi_f : \aleph_0 \times \aleph_\alpha \longrightarrow \{0, 1\}$  is 1-1. Thus

$$\aleph_\alpha^{\aleph_0} \leq 2^{\aleph_0 \cdot \aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

- So we obtain:

$$\begin{aligned} \aleph_{\alpha+1} &\leq \aleph_{\alpha+1}^{\aleph_0} \\ &= |\aleph_0 \aleph_{\alpha+1}| \\ &= \left| \bigcup_{\xi < \aleph_{\alpha+1}} \xi^{\aleph_0} \right| \\ &\leq \left| \left\{ (\xi, f) \mid \xi < \aleph_{\alpha+1} \wedge f \in {}^\omega \xi \right\} \right| \\ &\leq \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_0} \\ &\leq \aleph_{\alpha+1} \cdot \aleph_{\alpha+1} \\ &= \aleph_{\alpha+1} \end{aligned}$$

which yields

$$|\mathbb{P}_{\aleph_{\alpha+1}}|^{\aleph_0} = \aleph_{\alpha+1}^{\aleph_0} = \aleph_{\alpha+1}.$$

Thus, by applying Lemma 326 we obtain:

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} \leq (|\mathbb{P}_{\aleph_{\alpha+1}}|^{\aleph_0})^{\mathbf{M}} = (\aleph_{\alpha+1})^{\mathbf{M}} = (\aleph_{\alpha+1})^{\mathbf{M}[G]}.$$

i.e.,

$$\left( |\mathcal{P}(\omega)| \leq \aleph_{\alpha+1} \right)^{\mathbf{M}[G]}.$$

For the other inequality, we set  $\mathcal{F} = \bigcup G$  and notice that

- (1)  $\mathcal{F}$  is a function (see Exercise 284):

$$\mathcal{F} : (\aleph_{\alpha+1})^{\mathbf{M}} \times \omega \longrightarrow \{0, 1\}.$$

- (2)  $G \notin \mathbf{M}$  since by Lemma 286, given any  $p \in \mathbb{P}_{\aleph_{\alpha+1}}$  and any integer  $n \notin \text{dom}(p)$ , one has  $r = p \cup \{(n, 0)\}$  and  $q = p \cup \{(n, 1)\}$  satisfy

$$q \leq p \wedge r \leq p \wedge q \perp r.$$

(3)  $G \in \mathbf{M}[G]$  (see Lemma 298), hence  $\mathcal{F} \in \mathbf{M}[G]$ .

For  $\alpha < \beta < (\aleph_{\alpha+1})^{\mathbf{M}}$ , we consider:

$$D_{\alpha,\beta} = \left\{ p \in \mathbb{P} \mid \exists n < \omega \ ((\alpha, n) \in \text{dom}(p) \wedge (\beta, n) \in \text{dom}(p) \wedge p(\alpha, n) \neq p(\beta, n)) \right\}.$$

$D_{\alpha,\beta}$  is dense in  $\mathbb{P}$  because given any  $q \in \mathbb{P}$ , since  $\text{dom}(q)$  is finite, there exists  $n \in \omega$  such that  $(\alpha, n)$  and  $(\beta, n)$  do not belong to  $\text{dom}(q)$ , thus the following forcing condition  $p \leq q$  belongs to  $D_{\alpha,\beta}$ .

$$p = q \cup \left\{ ((\alpha, n), 0), ((\beta, n), 1) \right\}.$$

Since each  $D_{\alpha,\beta}$  is dense and belongs to  $\mathbf{M}$ , and  $G$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ ,

$$D_{\alpha,\beta} \cap G \neq \emptyset.$$

Thus there exists  $p \in G$  and  $n \in \omega$  such that  $p(\alpha, n) \neq p(\beta, n)$ . It follows that for all  $\alpha < \beta < (\aleph_{\alpha+1})^{\mathbf{M}}$ , there exists an integer  $n$  such that

$$\mathcal{F}(\alpha, n) \neq \mathcal{F}(\beta, n).$$

For each ordinal  $\alpha < (\aleph_{\alpha+1})^{\mathbf{M}}$ , we consider

$$X_\alpha = \{ n < \omega \mid \mathcal{F}(\alpha, n) = 1 \}.$$

If  $\alpha < \beta < (\aleph_{\alpha+1})^{\mathbf{M}}$ , since there exists  $n \in \omega$  such that  $\mathcal{F}(\alpha, n) \neq \mathcal{F}(\beta, n)$ , we have

$$X_\alpha \neq X_\beta.$$

It follows that there exist at least  $(\aleph_{\alpha+1})^{\mathbf{M}}$ -many subsets of  $\omega$  in  $\mathbf{M}[G]$ . Thus,

$$|\mathcal{P}(\omega)|^{\mathbf{M}[G]} \geq (\aleph_{\alpha+1})^{\mathbf{M}} = (\aleph_{\alpha+1})^{\mathbf{M}[G]}.$$

i.e.,

$$\left( |\mathcal{P}(\omega)| \geq \aleph_{\alpha+1} \right)^{\mathbf{M}[G]}.$$

Finally, we have shown

$$\left( \aleph_{\alpha+1} \leq |\mathcal{P}(\omega)| \leq \aleph_{\alpha+1} \right)^{\mathbf{M}[G]}$$

which yields

$$\left( 2^{\aleph_0} = \aleph_{\alpha+1} \right)^{\mathbf{M}[G]}.$$

## Chapter 18

# Independence of AC

### 18.1 Notion of Forcing and Automorphisms

We shift our attention to the Axiom of Choice and intend to prove:

$$\mathbf{ZFC} \vdash_c \text{cons}(\mathbf{ZFC}) \longrightarrow \text{cons}(\mathbf{ZF} + \neg \mathbf{AC}).$$

We will do this by first forcing from a ground model  $\mathbf{M}$  which satisfies “**ZFC**”. This will provide us with a generic extension  $\mathbf{M}[G]$  which will also satisfy “**ZFC**” as shown by Theorem 314. So, there is no chance we get a model in which the axiom of choice fails this way. However, we will consider a submodel of the generic extension for which we will be able to prove that it denies the axiom of choice.

**Definition 331.** Let  $\mathbf{M}$  be a c.t.m. of “**ZFC**” and  $(\mathbb{P}, \leq, \mathbb{1})$  a partial order over  $\mathbf{M}$ .

Any mapping  $\pi : \mathbb{P} \longrightarrow \mathbb{P}$  is an automorphism of  $\mathbb{P}$  if

- $\pi$  is a bijection;
- $\forall p \in \mathbb{P} \ \forall q \in \mathbb{P} \ (p \leq q \iff \pi(p) \leq \pi(q))$ ;
- $\pi(\mathbb{1}) = \mathbb{1}$ .

**Lemma 332.** Let  $\mathbf{M}$  be a c.t.m. of “**ZFC**” and  $(\mathbb{P}, \leq, \mathbb{1})$  a partial order over  $\mathbf{M}$ . If  $\pi \in \mathbf{M}$  is an automorphism of  $\mathbb{P}$ , then

$$G \text{ is } \mathbb{P}\text{-generic over } \mathbf{M} \iff \pi[G] \text{ is } \mathbb{P}\text{-generic over } \mathbf{M}.$$

*Proof of Lemma 332:*

( $\implies$ ) In order to prove that  $\pi[G]$  is  $\mathbb{P}$ -generic over  $\mathbf{M}$ , we first show that  $\pi[G]$  is a filter over  $\mathbb{P}$ . We have

- (1) Given any  $\pi(p), \pi(q) \in \pi[G]$ , since  $G$  is a filter, there exists  $r \in G$  such that  $r \leq p$  and  $r \leq q$ . Since  $\pi$  is an automorphism, we have  $\pi(r) \in \pi[G]$ , together with

$$\pi(r) \leq \pi(p) \text{ and } \pi(r) \leq \pi(q).$$

- (2) If  $\pi(p) \in \pi[G]$  and  $\pi(p) \leq \pi(q)$ , then

$$p \leq q \iff \pi(p) \leq \pi(q)$$

holds, which yields  $q \in G$  (since  $G$  is a filter), hence  $\pi(q) \in \pi[G]$ .

- (3)  $1 = \pi(1)$ , thus  $1 \in \pi[G]$ .

We now check that  $\pi[G]$  satisfies the density clause:

For every  $D \in \mathbf{M}$  which is dense in  $\mathbb{P}$ ,  $\pi[G] \cap D \neq \emptyset$ .

It suffices to show that  $\pi^{-1}[D]$  is dense in  $\mathbb{P}$ , since

$$\pi[G] \cap D = \pi[G \cap \pi^{-1}[D]].$$

Let  $p \in \mathbb{P}$ ,  $D$  is dense, so there exists  $r \leq \pi(p)$  such that  $r \in D$ ; hence  $\pi^{-1}(r) \leq p$  and  $\pi^{-1}(r) \in \pi^{-1}[D]$ . Therefore,  $\pi^{-1}[D] \cap G \neq \emptyset$  and thus  $D \cap \pi[G] \neq \emptyset$ .  $\pi[G]$  is thus  $\mathbb{P}$ -generic over  $\mathbf{M}$ .

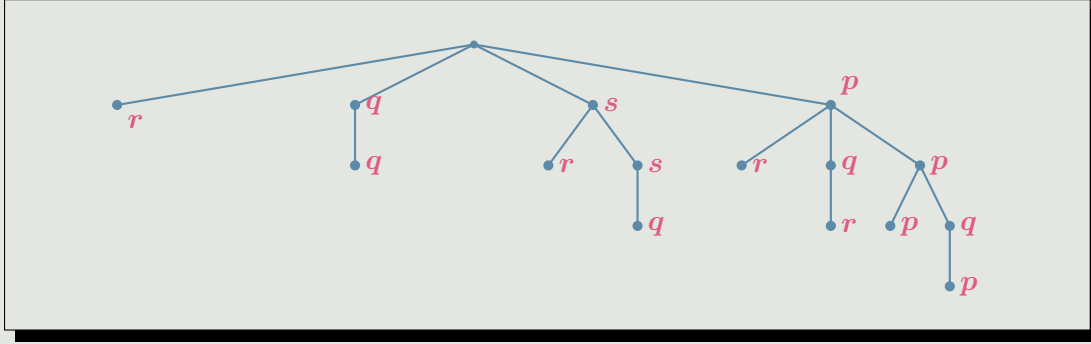
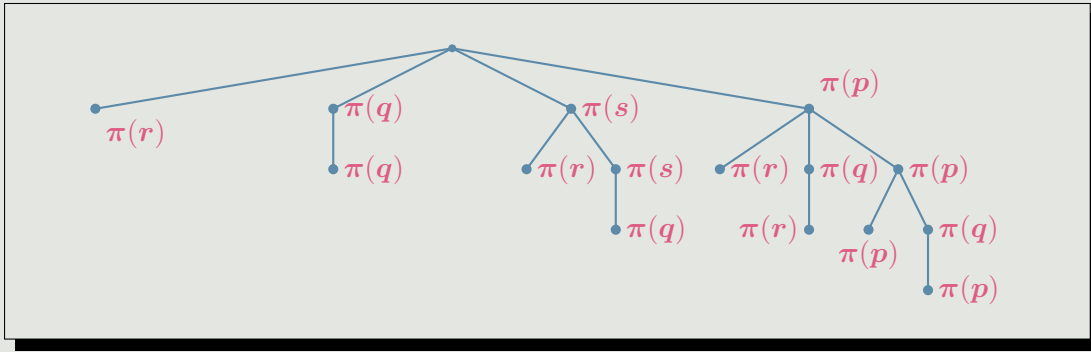
( $\impliedby$ ) The proof of the reverse implication is simply addressed by replacing  $\pi$  by  $\pi^{-1}$ .

□ **332**

**Definition 333.** Let  $\mathbf{M}$  be a c.t.m. of “ZFC”,  $(\mathbb{P}, \leq, 1)$  a partial order over  $\mathbf{M}$  and  $\pi : \mathbb{P} \longrightarrow \mathbb{P}$ . By transfinite recursion, we define

$$\begin{aligned} \tilde{\pi} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}^{\mathbb{P}} \\ \tau &\longmapsto \{(\tilde{\pi}(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}. \end{aligned}$$

**Example 334.**

The  $\mathbb{P}$ -name  $\tau$ :The  $\mathbb{P}$ -name  $\tilde{\pi}(\tau)$ :

We show that the image of a  $\mathbb{P}$ -generic filter over  $\mathbf{M}$  by an automorphism of  $\mathbb{P}$  yields exactly the same generic extension as the original filter.

**Lemma 335.** *Let  $\mathbf{M}$  be a c.t.m. of “ZFC”,  $\mathbb{P} \in \mathbf{M}$  be a notion of forcing,  $G$  be  $\mathbb{P}$ -generic over  $\mathbf{M}$ , and  $\pi \in \mathbf{M}$  be an automorphism of  $\mathbb{P}$ .*

$$\mathbf{M}[\pi[G]] = \mathbf{M}[G].$$

*Proof of Lemma 335:* Notice first that, for all  $\tau \in \mathbf{M}^{\mathbb{P}}$ , we have

$$(\tau)_G = (\tilde{\pi}(\tau))_{\pi[G]}.$$

(Indeed, given any  $b \in \mathbf{M}[\pi[G]]$ , and  $\tau \in \mathbf{M}^{\mathbb{P}}$  such that  $b = (\tau)_{\pi[G]}$  we have  $b = (\tau)_{\pi[G]} = (\tilde{\pi}^{-1}(\tau))_G$ .)

This yields

$$\mathbf{M}[\pi[G]] \subseteq \mathbf{M}[G].$$

For the reverse inclusion, we make use of Lemma 302 which stated that if  $\mathbf{N}$  is a transitive model of “ZFC” with  $\mathbf{M} \subseteq \mathbf{N}$  such that  $G \in \mathbf{N}$ , then  $\mathbf{M}[G] \subseteq \mathbf{N}$ . We notice that

- (1)  $\mathbf{M}[\pi[G]]$  is transitive.
- (2)  $\mathbf{M} \subseteq \mathbf{M}[\pi[G]]$ ;
- (3)  $G = \pi^{-1}[\pi[G]] \in \mathbf{M}[\pi[G]]$ .

This gives

$$\mathbf{M}[G] \subseteq \mathbf{M}[\pi[G]]$$

which yields

$$\mathbf{M}[\pi[G]] = \mathbf{M}[G].$$

□ 335

**Lemma 336.** *Let  $\mathbf{M}$  be a c.t.m. of “ZFC” and  $\mathbb{P}$  a notion of forcing over  $\mathbf{M}$ . Let also  $\varphi(x_1, \dots, x_n)$  be any  $\mathcal{L}_{\text{ST}}$ -formula. If  $\pi \in \mathbf{M}$  is an automorphism of  $\mathbb{P}$ , then*

- (1) *for all  $x \in \mathbf{M}$ ,  $\tilde{\pi}(\check{x}) = \check{x}$ ;*
- (2) *for all  $\check{a}_1, \dots, \check{a}_n \in \mathbf{M}^{\mathbb{P}}$ , and  $p \in \mathbb{P}$ ,*

$$p \Vdash \varphi(\check{a}_1, \dots, \check{a}_n) \iff \pi(p) \Vdash \varphi(\tilde{\pi}(\check{a}_1), \dots, \tilde{\pi}(\check{a}_n)).$$

*Proof of Lemma 336:* The proof of (1) is immediate. For (2), we write  $a_1, \dots, a_n$  for  $(\check{a}_1)_G, \dots, (\check{a}_n)_G$ ,

repsectively. By using Lemmas 332 and 335, we have

$$\begin{aligned}
 p \Vdash \varphi(\underline{a}_1, \dots, \underline{a}_n) &\iff \text{for all } G \text{ } \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
 &\quad \mathbf{M}[G] \models \varphi(a_1, \dots, a_n) \\
 &\iff \text{for all } G \text{ } \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
 &\quad \mathbf{M}[G] \models \varphi\left((\tilde{\pi}(\underline{a}_1))_{\pi[G]}, \dots, (\tilde{\pi}(\underline{a}_n))_{\pi[G]}\right) \\
 &\iff \text{for all } G \text{ } \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
 &\quad \mathbf{M}[\pi[G]] \models \varphi\left((\tilde{\pi}(\underline{a}_1))_{\pi[G]}, \dots, (\tilde{\pi}(\underline{a}_n))_{\pi[G]}\right) \\
 &\iff \text{for all } \pi[G] \text{ } \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } \pi(p) \in \pi[G] \\
 &\quad \mathbf{M}[\pi[G]] \models \varphi\left((\tilde{\pi}(\underline{a}_1))_{\pi[G]}, \dots, (\tilde{\pi}(\underline{a}_n))_{\pi[G]}\right) \\
 &\iff \pi(p) \Vdash \varphi(\tilde{\pi}(\underline{a}_1), \dots, \tilde{\pi}(\underline{a}_n)).
 \end{aligned}$$

□ 336

## 18.2 Hereditarily Ordinal Definable Sets

**Definition 337.** *Given any set  $A$ ,*

(1) **OD**( $A$ ) *is defined by*

$$b \in \mathbf{OD}(A)$$

$$\iff$$

*for some  $\mathcal{L}_{\text{ST}}$ -formula  $\varphi(x, x_1, \dots, x_n, y_1, \dots, y_k)$ , ordinals  $\alpha, \alpha_1, \dots, \alpha_n$  and  $a_1, \dots, a_k \in A$ .*

$$b = \left\{ z \in \mathbf{V}_\alpha \mid \left( \varphi(z, \alpha_1, \dots, \alpha_n, a_1, \dots, a_k, A) \right)^{\mathbf{V}_\alpha} \right\}.$$

(2) **HOD**( $A$ ) *is defined by*

$$b \in \mathbf{HOD}(A)$$

$$\iff$$

*$b \in \mathbf{OD}(A)$  and the transitive closure of  $b$  is included in  $\mathbf{OD}(A)$ .*

**Theorem 338.** *Let  $A$  be an arbitrary set.*

$$(\mathbf{ZF})^{\mathbf{HOD}(A)}.$$

*Proof of Lemma 338:* The proof is identical to the proof of  $(\mathbf{ZF})^{\mathbf{HOD}}$  — see exercises sheet.

□ 338

### 18.3 Forcing $\neg\mathbf{AC}$

This section is entirely dedicated to “constructing” a model of  $\mathbf{ZF}$  in which the axiom of choice fails. Namely,

**Theorem 339.**

$$\mathbf{ZFC} \vdash_c \text{cons}(\mathbf{ZFC}) \longrightarrow \text{cons}(\mathbf{ZF} + \neg\mathbf{AC}).$$

*Proof of Theorem 339:* To do so, we prove that given  $\mathbf{M}$  any *c.t.m.* of “ $\mathbf{ZFC}$ ” with  $\mathbb{P}_{\aleph_0} \in \mathbf{M}$ , if  $G$  is  $\mathbb{P}_{\aleph_0}$ -generic over  $\mathbf{M}$ , then there exists a set  $A \in \mathbf{M}[G]$  such that:

$$\mathbf{M}[G] \models (\neg\mathbf{AC})^{\mathbf{HOD}(A)}.$$

Or, to say it differently,

$$\left( (\neg\mathbf{AC})^{\mathbf{HOD}(A)} \right)^{\mathbf{M}[G]}.$$

We start by forcing with

$$\mathbb{P}_{\aleph_0} = \{f : \omega \times \omega \longrightarrow \{0, 1\} \mid \text{dom}(f) \text{ finite}\}.$$

Given any  $G$   $\mathbb{P}_{\aleph_0}$ -generic over  $\mathbf{M}$ , we have  $\mathcal{F} = \bigcup G$  satisfies

$$\mathcal{F} : \omega \times \omega \rightarrow \{0, 1\}.$$

Let

$$a_k = \{n < \omega \mid \mathcal{F}(k, n) = 1\} \quad \text{and} \quad A = \{a_k \mid k < \omega\}.$$

We have  $A \in \mathbf{M}[G]$ , and  $A \notin \mathbf{M}$  for otherwise, one could from recover from  $A$  some filter  $\pi[G]$  for some automorphism  $\pi$  of  $\mathbb{P}$ . This would yield  $\pi[G] \in \mathbf{M}$ , henceforth  $\mathbf{M}[\pi[G]] = \mathbf{M}[G] \in A$  which would contradict Lemma 286.

Also, since

$$D_{n,m} = \{p \in \mathbb{P}_{\aleph_0} \mid \exists k \leq \omega \ p(n, k) \neq p(m, k)\}$$

is dense in  $\mathbb{P}_{\aleph_0}$ , for all integers  $n \neq m$ , we have  $a_n \neq a_m$ , which shows that  $A$  is infinite.

Inside  $\mathbf{M}[G]$ , we verify that  $A$  is an element of  $\mathbf{HOD}(A)$ :

- (1)  $A \in \mathbf{OD}(A)$  since it is definable from itself.
- (2) If  $x \in A$ , then  $x$  is definable from itself, so  $x \in \mathbf{OD}(A)$ . If  $y \in x \in A$ , then  $y \in \omega$ , hence  $y \in \mathbf{OD} \subseteq \mathbf{OD}(A)$ . So, the transitive closure of  $A$  is included in  $\omega \cup A \subseteq \mathbf{OD}(A)$ .

Now, for each  $n \in \omega$ , we define *canonical*  $\mathbb{P}_{\aleph_0}$ -names  $\underline{a}_n$  and  $\underline{A}$  for, respectively,  $a_n$  and  $A$ :

$$\underline{a}_n = \{(\check{m}, p) \mid p(n, m) = 1\}.$$

and

$$\underline{A} = \{(\underline{a}_n, \mathbb{1}) \mid n < \omega\},$$

so that we have

$$(\underline{a}_n)_G = a_n \quad \text{and} \quad (\underline{A})_G = A.$$

We let

$$\mathbf{N} = (\mathbf{HOD}(A))^{\mathbf{M}[G]}.$$

Towards a contradiction, we suppose  $(\mathbf{AC})^{\mathbf{N}}$ .

So, inside  $\mathbf{N}$   $A$  can be well-ordered. In particular, there exists some mapping  $f : A \xrightarrow{1-1} \mathbf{On}$ . Since  $f \in \mathbf{N}$ , we have

$$(f \in \mathbf{OD}(A))^{\mathbf{M}[G]}$$

and therefore  $f$  is definable in  $\mathbf{M}[G]$  with parameters  $\alpha_1, \dots, \alpha_n \in \mathbf{On}$ ,  $a_1, \dots, a_k \in A$  and  $A$ . Let  $a_{k+1} \in A$  and  $\alpha \in \mathbf{On}$  such that  $f(a_{k+1}) = \alpha$ .  $a_{k+1}$  is definable in  $\mathbf{M}[G]$  with parameters  $\alpha, \alpha_1, \dots, \alpha_n \in \mathbf{On}$ ,  $a_1, \dots, a_k \in A$  and  $A$  and some  $\mathcal{L}_{\text{ST}}$ -formula  $\varphi$ :

$$\left( "a_{k+1} \text{ is the only } x \text{ such that } \varphi(x, \alpha, \alpha_1, \dots, \alpha_n, a_1, \dots, a_k, A)" \right)^{\mathbf{M}[G]}.$$

So, by the Truth Lemma, there exists  $r \in G$  such that

$$(r \Vdash " \underline{a}_{k+1} \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, \underline{a}_1, \dots, \underline{a}_k, \underline{A}) " )^{\mathbf{M}}.$$

We then consider the set:

$$D = \left\{ q \in \mathbb{P}_{\aleph_0} \mid \exists l > k+1 \quad (q \Vdash " \underline{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, \underline{a}_1, \dots, \underline{a}_k, \underline{A}) " )^{\mathbf{M}} \right\}.$$

We have  $D \in \mathbf{M}$  and we still need to show

**Claim 340.**

*$D$  is dense below  $r$ .*

*Proof of Claim 340:* Given any  $q \leq r$ , since  $\text{dom}(r)$  is finite,

$$\exists l > k+1 \quad \forall i < \omega \quad (l, i) \notin \text{dom}(q).$$

Then, we consider the permutation  $\rho : \omega \times \omega \longrightarrow \omega \times \omega$  defined for all  $i < \omega$  by

- $\rho(k+1, i) = (l, i)$ ;
- $\rho(l, i) = (k+1, i)$ ;
- $\rho(n, i) = (n, i)$  (any  $n \notin \{l, k+1\}$ ).

This permutation induces the automorphism  $\pi : \mathbb{P}_{\aleph_0} \longrightarrow \mathbb{P}_{\aleph_0}$  defined for all  $p \in \mathbb{P}_{\aleph_0}$  by

- $\text{dom}(\pi(p)) = \rho[\text{dom}(p)]$
- $\pi(p)(\rho(n, m)) = p(n, m)$ .

We denote by  $\tilde{\pi}$  its extension to  $\mathbf{M}^{\mathbb{P}}$ . For  $i \notin \{k+1, l\}$ , we have:

$$\begin{aligned} \tilde{\pi}(a_i) &= \{(\tilde{\pi}(\check{m}), \pi(p)) \mid p(i, m) = 1\} \\ &= \{(\check{m}, \pi(p)) \mid \pi(p)(i, m) = 1\} \\ &= \{(\check{m}, p) \mid p(i, m) = 1\} \\ &= a_i; \end{aligned}$$

$$\begin{aligned} \tilde{\pi}(a_{k+1}) &= \{(\tilde{\pi}(\check{m}), \pi(p)) \mid p(k+1, m) = 1\} \\ &= \{(\check{m}, \pi(p)) \mid \pi(p)(l, m) = 1\} \\ &= \{(\check{m}, p) \mid p(l, m) = 1\} \\ &= a_l; \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}(a_l) &= \{(\tilde{\pi}(\check{m}), \pi(p)) \mid p(l, m) = 1\} \\ &= \{(\check{m}, \pi(p)) \mid \pi(p)(k+1, m) = 1\} \\ &= \{(\check{m}, p) \mid p(k+1, m) = 1\} \\ &= a_{k+1}. \end{aligned}$$

In  $\mathbf{M}$ ,

$$q \Vdash "a_{k+1} \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, a_1, \dots, a_k, \check{A})"$$

yields

$$\pi(q) \Vdash "\tilde{\pi}(a_{k+1}) \text{ is the only } x \text{ such that } \varphi(x, \tilde{\pi}(\check{\alpha}), \tilde{\pi}(\check{\alpha}_1), \dots, \tilde{\pi}(\check{\alpha}_n), \tilde{\pi}(a_1), \dots, \tilde{\pi}(a_k), \tilde{\pi}(\check{A}))"$$

We thus have:

$$\pi(q) \Vdash "a_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, a_1, \dots, a_k, \check{A})".$$

But  $q$  is not defined over  $l$ , so  $\pi(q)$  is not defined over  $k+1$  and for all integers  $i \notin \{k+1, l\}$  and  $m < \omega$ , we have  $q(i, m) = \pi(q)(i, m)$ . Therefore,  $q$  and  $\pi(q)$  are compatible and  $s = q \cup \pi(q)$  satisfies both  $s \leq q$  and  $s \leq \pi(q)$ , and also

$$s \Vdash "a_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, a_1, \dots, a_k, \check{A})".$$

So, at last, we have found  $s \in D$ , which shows that  $D$  is dense below  $r$ .

□ 340

Finally, since  $G$  is  $\mathbb{P}_{\aleph_0}$ -generic over  $\mathbf{M}$ , one has  $D \cap G \neq \emptyset$ , but any  $q \in D \cap G$  yields that there exists  $l > k + 1$  such that

$$(q \Vdash \text{“} \dot{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \dots, \check{\alpha}_n, \dot{a}_1, \dots, \dot{a}_k, \dot{A}) \text{”})^{\mathbf{M}}$$

By the Truth Lemma, this gives

$$\left( \text{“} \dot{a}_l \text{ is the only } x \text{ such that } \varphi(x, \alpha, \alpha_1, \dots, \alpha_n, a_1, \dots, a_k, A) \text{”} \right)^{\mathbf{M}[G]}.$$

which contradicts the uniqueness of  $a_{k+1}$  in  $\mathbf{M}[G]$ .

□ 339

