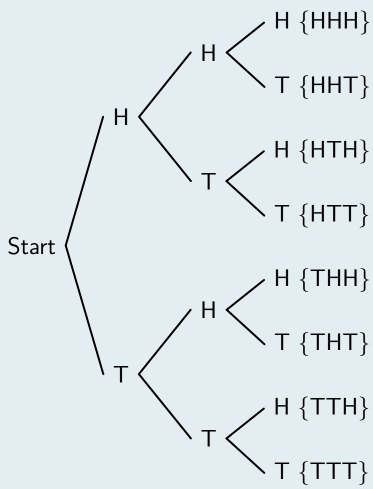
Module 18: Introduction to Discrete Probability

1 Three fair coins are tossed. Find each of the following

(a) The sample space, S

Solution:



(b) The event, E_1, that exactly two are tails.

Solution: $E_1 = \{HTT, THT, TTH\}$

(c) The event, E_2, that at least two are tails.

Solution: $E_2 = \{HTT, THT, TTH, TTT\}$

(d) The probability of E_1

Solution:

$$p(E_1) = \frac{|E_1|}{|S|} = \frac{3}{8}$$

(e) The probability of E_2

Solution:

$$p(E_2) = \frac{|E_2|}{|S|} = \frac{4}{8} = \frac{1}{2}$$

2 A card is dealt from a well-shuffled, standard deck of 52 playing cards. Find the probability of being dealt each of the following.

(a) A red card

Solution: $\frac{|\text{red cards}|}{|\text{cards}|} = \frac{26}{52} = \frac{1}{2}$

(b) A queen

Solution: $\frac{|\text{queens}|}{|\text{cards}|} = \frac{4}{52} = \frac{1}{13}$

(c) A club

Solution: $\frac{|\text{clubs}|}{|\text{cards}|} = \frac{13}{52} = \frac{1}{4}$

(d) The queen of clubs

Solution: $\frac{|\text{queen of clubs}|}{|\text{cards}|} = \frac{1}{52}$

(e) A queen or a club

Solution: $\frac{|\text{queens or clubs}|}{|\text{cards}|} = \frac{4+13-1}{52} = \frac{16}{52} = \frac{4}{13}$

(f) Not a queen

Solution: $1 - P(\text{queen}) = 1 - \frac{4}{52} = \frac{48}{52} = \frac{12}{13}$

3 Five cards are dealt (without replacement) from a well-shuffled 52 card standard deck. Find the probability of the following:

(a) being dealt 5 spades.

Solution:

$$\frac{\binom{13}{5}}{\binom{52}{5}} \approx 0.0004951$$

(b) being dealt 4 spades and 1 heart.

Solution:

$$\frac{\binom{13}{4}\binom{13}{1}}{\binom{52}{5}} \approx 0.003576$$

(c) being dealt all the same suit.

Solution:

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} \approx 0.00198$$

(d) not being dealt all the same suit

Solution: This is just 1 minus the answer above:

$$1 - \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} \approx 0.99801$$

4 A drawer has 7 socks. 4 socks are black and 3 are white socks. John randomly pulls out 4 socks. Find the probability of the following:

(a) all 4 socks are black.

Solution:

$$\frac{\binom{4}{4}}{\binom{7}{4}} = \frac{1}{35}$$

(b) exactly 2 are white.

Solution: Exactly 2 socks are white so 2 must be black.

$$\frac{\binom{3}{2}\binom{4}{2}}{\binom{7}{4}} = \frac{3 \cdot 6}{35} = \frac{18}{35}$$

(c) at least 3 are white.

Solution: In this situation, at least 3 white socks is equivalent to exactly 3 white socks (and 1 black) OR exactly 4 white socks (and 0 black). Note that choosing 4 white socks from a set of 3 white socks is impossible yielding a 0 below.

$$\frac{\binom{3}{3}\binom{4}{1}+\binom{3}{4}\binom{4}{0}}{\binom{7}{4}}=\frac{1\cdot 4+0\cdot 1}{35}=\frac{4}{35}$$

(d) at most 2 are black.

Solution: At most 2 black socks is equivalent to exactly 0 black OR exactly 1 black OR exactly 2 black socks.

$$\frac{\binom{4}{0}\binom{3}{4}+\binom{4}{1}\binom{3}{3}+\binom{4}{2}\binom{3}{2}}{\binom{7}{4}}=\frac{1\cdot 0+4\cdot 1+6\cdot 3}{35}=\frac{22}{35}$$

5 A fair six-sided dice is rolled 5 times. Find the probability of the following:

(a) All 2's are rolled.

Solution: $p(\text{five } 2's) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{6^5}$

(b) Four 2's are rolled and the 5^{th} roll is a 1.

Solution: $p(\text{four } 2's \cap \text{one } 1) = (\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}) \cdot \frac{1}{6} = \frac{1}{6^5}$

(c) Not all 2's are rolled.

Solution: $1 - \frac{1}{6^5}$

Module 19: Conditional Probability and Bayes' Theorem

Solution: Recall **Bayes' Theorem:** For events X and F from a common sample space where $P(X), P(F) \neq 0$,

$$P(F|X) = \frac{P(X|F)P(F)}{P(X)} \qquad \text{(version 1)}$$

and equivalently,

$$P(F|X) = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\overline{F})P(\overline{F})} \qquad \text{(version 2)}$$

While the first version does not appear in the textbook or pre-assessment practice problems, this equivalent definition is listed in the lesson summary.

Note that $P(F \cap X) = P(X|F)P(F)$. So version 1 is often shown as:

$$P(F|X) = \frac{P(X \cap F)}{P(X)} \qquad \text{(version 3)}$$

6 A car dealership has created an app that will randomly select one vehicle from their inventory to show a potential customer. The current state of their inventory is shown in the following table.

Color\Type	Sedan	Sports	SUV
Black	12	3	7
Blue	10	4	6
Red	9	9	4
White	9	5	2

(a) What is the probability that a red sports car is shown?

Solution: $\frac{9}{80}$

(b) What is the probability that the car shown is blue or red?

Solution: $\frac{21}{40}$

Since the events B = blue car and R = red car are disjoint

$$P(B \cup R) = P(B) + P(R) = \frac{20}{80} + \frac{22}{80} = \frac{42}{80} = \frac{21}{40}.$$

(c) What is the probability that the car shown is black or a sedan?

Solution: $\frac{5}{8}$

Since events F = black car and G = sedan are not disjoint, we use the inclusion/exclusion rule (Lesson 5.5) to get

$$P(F \cup G) = P(F) + P(G) - P(F \cap G) = \frac{22}{80} + \frac{40}{80} - \frac{12}{80} = \frac{50}{80} = \frac{5}{8}$$

(d) The app has filters and white is chosen. What is the probability that the car shown is an SUV?

Solution:

The question can be reworded as "What is the probability of an SUV given that the car is white?" This is a conditional probability. Define events W = white car and V = SUV.

$$P(V|W) = \frac{P(V \cap W)}{P(W)} = \frac{|V \cap W|}{|W|} = \frac{2}{16} = \frac{1}{8}$$

(e) Let A be the event of showing a blue car and D be the event of showing a sedan. Are A and D independent events?

Solution: Yes.

Looking at lesson 5.9, we show $P(A) = P(A|D)$. There are 20 blue cars out of 80 total so $P(A) = \frac{20}{80} = \frac{1}{4}$. There are 40 sedans and 10 blue sedans so $P(A|D) = \frac{|A \cap D|}{|D|} = \frac{10}{40} = \frac{1}{4}$. Since $P(A) = P(A|D)$, A and D are independent.

Alternatively, it could be shown that $P(A \cap D) = P(A) \cdot P(D)$ or that $P(D) = P(D|A)$.

(f) Let C be the event of showing a black car and E be the event of showing a sports car. Are C and E independent events?

Solution: No.

Looking at lesson 5.9, we show $P(C) \neq P(C|E)$. There are 22 black cars so $P(C) = \frac{22}{80} = \frac{11}{40}$. There are 21 sports cars and 3 of those are black so $P(C|E) = \frac{|C \cap E|}{|E|} = \frac{3}{21} = \frac{1}{7}$. Since $P(C) \neq P(C|E)$, C and E are not independent.

7 Bag #1 contains 10 blue marbles. Bag #2 appears identical but contains 5 red marbles and 5 blue marbles. Suppose you randomly choose a bag and then randomly pick a blue marble. What is the probability that you choose Bag #2?

Solution: We know that a blue marble was chosen. So we are looking for the probability of selecting Bag #2 *given* that a blue marble was picked, e.g., $P(\#2|B)$. The conditional statement indicates we should use Bayes' Theorem. We'll use first version (seen above):

$$P(\#2|B) = \frac{P(B|\#2)P(\#2)}{P(B)}$$

So we need to find the following:

- The probability of selecting Bag #2: $P(\#2) = \frac{1}{2}$
- The probability of selecting Bag a blue marble given that Bag #2 was chosen: $P(B|\#2) = \frac{5}{10}$
- The probability of selecting a blue marble. We'd have to select Bag #1 or Bag #2 and then select a blue marble: $P(B) = \frac{1}{2} \cdot \frac{5}{10} + \frac{1}{2} \cdot 1 = \frac{3}{4}$. When the case arises that you can't find this, use the other version of Bayes' Theorem.

Now we plug and play:

$$P(\#2|B) = \frac{P(B|\#2)P(\#2)}{P(B)} = \frac{\frac{5}{10} \cdot (\frac{1}{2})}{\frac{3}{4}} = \frac{1}{3}$$

Note that we could have used the other version of Bayes' Theorem:

$$P(\#2|B) = \frac{P(B|\#2)P(\#2)}{P(B|\#2)P(\#2) + P(B|\overline{\#2})P(\overline{\#2})} = \frac{\frac{5}{10} \cdot (\frac{1}{2})}{\frac{5}{10} \cdot (\frac{1}{2}) + 1 \cdot (\frac{1}{2})} = \frac{1}{3}$$

8 The zombie apocalypse has come. However, it's not as bad as the movies. Only 1% of the population will get infected. 90% of the people who are infected with this disease will test positive, but 5% of the people tested will have false positives (meaning the test shows positive but they are **not** infected). What is the probability that a person is infected if the test shows positive? Approximate your answer to the 4^{th} decimal place.

Solution: We are looking for the probability that someone is infected given that they have tested positive, e.g., $P(I|T)$. As this is a conditional probability questions, we should immediately consider using Bayes' theorem:

$$P(I|T) = \frac{P(T|I)P(I)}{P(T|I)P(I) + P(T|\bar{I})P(\bar{I})}$$

So we need the following:

- The probability that a person is infected: $P(I) = .01$
- The probability that an infected person tests positive (true positive): $P(T|I) = .90$

- The probability that a person is not infected: $P(\bar{I}) = .99$
- The probability that a non-infected person tests positive (false positive): $P(T|\bar{I}) = .05$

Now we plug these values into the formula:

$$P(I|T) = \frac{P(T|I)P(I)}{P(T|I)P(I) + P(T|\bar{I})P(\bar{I})} = \frac{(.90)(.01)}{(.90)(.01) + (.05)(.99)} \approx 0.1538$$

Note that using this version of Bayes' theorem makes sense because we were given the probabilities of a false positive and a true positive. To use the other version,

$$P(I|T) = \frac{P(T|I)P(I)}{P(T)}$$

we would need to find $P(T)$, but this is the probability that someone who is infected tests positive OR someone who is not infected tests positive, i.e.,

$$P(T) = P(T \cap I) + P(T \cap \bar{I}) = P(T|I)P(I) + P(T|\bar{I})P(\bar{I})$$

The denominator of the other version. Though more complicated to write, it also applies to more general situations.

Module 20: Random Variables

9 A hand of three cards is dealt from a well-shuffled standard deck of 52 playing cards. Let C denote the number of clubs in the hand.

(a) What is the range of C?

Solution: $0, 1, 2, 3$

(b) What is the distribution over C?

Solution:

$$\left(0, \frac{\binom{39}{3}}{\binom{52}{3}}\right), \left(1, \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}}\right), \left(2, \frac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}}\right), \left(3, \frac{\binom{13}{3}}{\binom{52}{3}}\right)$$

When rounded to the nearest hundredth:

$$(0, 0.41), (1, 0.44), (2, 0.14), (3, 0.01)$$

(c) What is the expected value of C and interpret this value in context?

Solution: On average, a hand of 3 cards will have 0.75 clubs.

$$E[C] = 0 \cdot 0.41 + 1 \cdot 0.44 + 2 \cdot 0.14 + 3 \cdot 0.01 = 0.75$$

10 In a certain game, a player pays \$5 to play. There is a 25% chance that the player wins and receives \$7, a 25% chance the player wins and receives \$15, and a 50% chance the player wins nothing. What is the expected outcome of this game? Round to the nearest cent.

Solution: \$0.50

Recall the definition from Lesson 5.13 of *expected value* of a discrete random variable X is

$$E[X] = \sum_{s \in S} X(s)p(s)$$

where S is the set of possible outcomes of X, $p(s)$ is the probability of outcome s, and $X(s)$ is the value of X for s. In other words, multiply the value of each outcome to its probability and add them all up. If the player won the \$7 prize, their net outcome is \$2 (\$7 won minus \$5 to play).

$$E[X] = 2(.25) + 10(.25) + (-5)(.50) = 0.50$$

11 WGU is holding a raffle to raise money for a new internet service. The tickets cost $\$10$ each. The value and number of each prize to be given away is listed in the provided table. Find the expected value of a ticket under the given circumstances.

Prize	Value	#
Vacation	$\$3,000$	1
Stereo	$\$500$	5
Books	$\$100$	10
T-shirt	$\$20$	100

Solution: Recall the definition from Lesson 5.13 of *expected value* of a discrete random variable X is

$$E[X] = \sum_{s \in S} X(s)p(s)$$

where S is the set of possible outcomes of X, $p(s)$ is the probability of outcome s, and $X(s)$ is the value of X for s.

(a) 200 tickets

Solution: $\$32.50$

There are $200 - 116 = 84$ losing tickets. Taking into account the cost to play, we get:

Outcome, s	**Value,** $X(s)$	**Probability,** $p(s)$
Vacation	$\$2,990$	$1/200$
Stereo	$\$490$	$5/200$
Books	$\$90$	$10/200$
T-shirt	$\$10$	$100/200$
Nothing	$-\$10$	$84/200$

$$E[X] = 2990\left(\frac{1}{200}\right) + 490\left(\frac{5}{200}\right) + 90\left(\frac{10}{200}\right) + 10\left(\frac{100}{200}\right) - 10\left(\frac{84}{200}\right) = \$32.50$$

(b) 1000 tickets

Solution: $-\$1.50$

There are $1000-116 = 884$ losing tickets. Taking into account the cost to play, we get:

Outcome, s	**Value,** $X(s)$	**Probability,** $p(s)$
Vacation	$\$2,990$	$1/1000$
Stereo	$\$490$	$5/1000$
Books	$\$90$	$10/1000$
T-shirt	$\$10$	$100/1000$
Nothing	$-\$10$	$884/1000$

$$E[X] = 2990\left(\frac{1}{1000}\right) + 490\left(\frac{5}{1000}\right) + 90\left(\frac{10}{1000}\right) + 10\left(\frac{100}{1000}\right) - 10\left(\frac{884}{1000}\right) = -\$1.50$$