Module 13: Counting by Bijections and Products of Sets

1 A certain model of pickup truck is available in five exterior colors, three interior colors, and three interior styles. In addition, the transmission can be either manual or automatic, and the truck can have either two-wheel or four-wheel drive. How many different versions of the pickup truck can be ordered?

Solution: We have five categories, __ × __ × __ × __ × __. Filling in the number of choices for each category we have,

$$\underline{5} \times \underline{3} \times \underline{3} \times \underline{2} \times \underline{2} = 180$$

2 A local charity organization must elect a president, vice president, and treasurer from its 30 members. How many ways can this be done?

Solution: There are two good approaches for this solution. First, you can use the generalized product rule. Pick the president, then the vice president, then the treasurer in order. There are 30 options for president, then 29 for vice president, and then 28 for treasurer giving us $30 \cdot 29 \cdot 28 = 24,360$ ways.

This can also be viewed as a permutation. From the 30 members, we are creating a list of 3: first is president, second is vice president, and third treasurer. Order matters because of their roles.

$$P(30,3) = \frac{30!}{(30-3)!} = \frac{30!}{27!} = 30 \cdot 29 \cdot 28 = 24,360$$

3 Which of the following functions maps an $11-$to-1 correspondence from the set $\{1,2,3,\ldots,44\}$ to the set $\{1,2,3,4\}$? Note that option B uses the floor function and option C uses the ceiling function.

A. $f(x) = \binom{x}{3}$ B. $f(x) = \lfloor \frac{x}{11} \rfloor$ C. $f(x) = \lceil \frac{x}{11} \rceil$ D. $f(x) = \frac{x}{11}$

Solution: C. Review the k-to-1 correspondence in Lesson 4.4.

Using $x = 1$, we see that options A, B, and D have outputs that are not in the target set and therefore not correct. Our only real option is C. It is good to check that 11 elements from the domain map to each element in the target.

The notation $\lceil x \rceil$ is called a *ceiling* function. It is the smallest integer greater than or equal to x. For example, $\lceil \frac{1}{11} \rceil = \lceil .0\overline{909} \rceil = 1$. So then,

$$\lceil \frac{1}{11} \rceil = \lceil \frac{2}{11} \rceil = \lceil \frac{3}{11} \rceil = \cdots \lceil \frac{11}{11} \rceil = 1$$

Similar to this, show that 11 inputs map to outputs 2, 3, and 4.

4 Which of the following functions maps an $8-$to-1 correspondence from $\{0, 1, 2, 3, \ldots, 55\}$ to $\{0, 1, \ldots, 6\}$?

A. $f(x) = \sin(\pi x)$ B. $f(x) = x \bmod 7$ C. $f(x) = x \bmod 8$ D. $f(x) = \lceil \frac{x}{7} \rceil$

Solution: B. Review the k-to-1 correspondence in Lesson 4.4.

The outputs of $x \bmod 7$ fall within the range 0 through 6, as required. We just need to check that 8 inputs map to each output. The following table shows this to be true.

x	$x \bmod 7$
0, 7, 14, 21, 28, 35, 42, 49	0
1, 8, 15, 22, 29, 36, 43, 50	1
2, 9, 16, 23, 30, 37, 44, 51	2
3, 10, 17, 24, 31, 38, 45, 52	3
4, 11, 18, 25, 32, 39, 46, 53	4
5, 12, 19, 26, 33, 40, 47, 54	5
6, 13, 20, 27, 34, 41, 48, 55	6

You can use trial-and-error to eliminate the other options. For option A, $\sin(\pi x)$ maps every value in the domain to 0. For option C, $x \bmod 8$ produces outputs from 0 through 7 (outside of range). For option D, $\lceil \frac{x}{7} \rceil$ also produces outputs outside the range, for example $\lceil \frac{50}{7} \rceil = 8$

5 Suppose that $B = \{0, 1, 2, \ldots 8\}$. If $f : A \to B$ is 3-to-1 mapping of A onto B, what is $|A|$?

Solution: Recall that $|A|$ notates the number of elements in the set A. Review the k-to-1 rule in Lesson 4.4: If there is a k-to-1 correspondence from A to B, then

$$|B| = \frac{|A|}{k}$$

Since $|B| = 9$ and f is a 3-to-1 correspondence, we get

$$9 = \frac{|A|}{3} \quad \to \quad |A| = 27$$

Module 14: Counting with Permutations and Combinations

6 The local branch of a national business chain has 27 employees. If they all shake each others hand at the company picnic, at least how many handshakes occurred?

Solution: Each handshake is a group of 2 different people. Order does NOT matter (Joe shaking hands with Betty is the same as Betty shaking hands with Joe). No repeats and order does not matter means we'll use a combination: How many ways can we select a group of 2 people from the 27?

$$C(27, 2) = \binom{27}{2} = \frac{27!}{(27-2)!2!} = \frac{27!}{25!2!} = \frac{27 \cdot 26}{2} = 351$$

7 Consider a standard deck of 52 playing cards. If you are unfamiliar with playing cards, check out this explanation of playing cards.

a. How many ways can five cards be drawn if their order is important?

Solution: Here order matters, e.g., $\{7\heartsuit, K\spadesuit, 2\spadesuit, 3\clubsuit, 3\diamondsuit\}$ is a different way to draw the cards then $\{2\spadesuit, 3\clubsuit, 3\diamondsuit, 7\heartsuit, K\spadesuit,\}$ since the order is different. So we use a permutation:

$$P(52, 5) = \frac{52!}{(52-5)!} = \frac{52!}{47!} = 311,875,200$$

b. How many five card hands exist where the order doesn't matter?

Solution: Here order does not matter, e.g., $\{7\heartsuit, K\spadesuit, 2\spadesuit, 3\clubsuit, 3\diamondsuit\}$ is the same hand as
$\{2\spadesuit, 3\clubsuit, 3\diamondsuit, 7\heartsuit, K\spadesuit\}$. So we use a combination:

$$C(52, 5) = \binom{52}{5} = \frac{52!}{(52-5)!5!} = 2,598,960$$

8 At a university there are 10 faculty members in the mathematics department and 12 faculty members in the computer science department.

(a) How many ways are there to form a committee from three members of the mathematics department.

Solution: No repeats and order doesn't matter⇒combination.

$$C(10,3) = \binom{10}{3} = \frac{10!}{(10-3)!3!} = \frac{10 \cdot 9 \cdot 8}{3!} = \frac{720}{6} = 120$$

(b) How many ways are there to select a department head, a graduate chair, and undergraduate chair for the computer science faculty. One person can not occupy multiple positions.

Solution: No repeats and order does matter⇒permutation.

$$P(12,3) = \frac{12!}{(12-3)!} = 12 \cdot 11 \cdot 10 = 1320$$

(c) How many ways are there to form a committee from one member of the computer science department and 2 members of the mathematics department.

Solution: From the 12 CS faulty, 1 is chosen. There are $C(12,1) = \binom{12}{1} = \frac{12!}{(12-1)!1!} = 12$ ways to select the CS Faculty member.

From the 10 math faculty, 2 are chosen, There are $C(10,2) = \binom{10}{2} = \frac{10!}{(10-2)!2!} = \frac{10 \cdot 9}{2} = 45$ ways to select the math faculty members.

So there are a total of $12 \times 45 = 540$ possibilities.

(d) How many ways are there to form a committee from one faculty member of the mathematics department and two faculty members of the computer science department if one of these faculty members is selected as the spokesperson of the committee.

Solution: Similar to above, there are $C(10,1) = \binom{10}{1} = \frac{10!}{(10-1)!1!} = 10$ ways to select the math faculty and $C(12,2) = \binom{12}{2} = \frac{12!}{(12-2)!2!} = 66$ ways to select the computer science faculty. Once the committee is chosen, there are 3 ways to select the spokesperson, so we have $10 \times 66 \times 3 = 1980$ total possibilities.

9 A coin is flipped 20 times and the results are recorded. How many outcomes have exactly 9 heads?

Solution: $C(20, 9) = \binom{20}{9} = 167,960$

To see why this is a combination, label the flips 1 through 20. We want to select a subset of 9 values to represent which ones are heads. Heads in flips labelled $\{3, 5, 6, 7, 11, 13, 15, 16, 19\}$ is the same outcome as heads in flips $\{13, 11, 16, 6, 7, 19, 5, 3, 15\}$, so order doesn't matter. So we get 20 choose 9.

10 Each student at State University has a student ID number consisting of five digits (the first digit is nonzero, and the digits can be repeated) followed by two of the letters A, B, C, and D (letters cannot be repeated). How many different student numbers are possible?

Solution: We have five numbers $0 - 9$ which can be repeated, but the the first is nonzero: $9 \times 10 \times 10 \times 10 \times 10 = 9 \cdot 10^4 = 90,000$ possibilities, and then two of four letters which cannot be repeated. It is implied that the order of the letters does matter since $10000AB$ and $10000BA$ would be two different IDs. So for the letters we have, $P(4, 2) = \frac{4!}{(4-2)!} = \frac{24}{2} = 12$ possibilities. For a total of $9 \times 10^4 \times 12 = 1,080,000$ possible ID numbers.

Module 15: Counting with Multisets

11 There are 250 pieces of candy in a jar and each one has a color, either red, orange, yellow, green, blue, or purple. The jar contains 37 red, 48 blue, and 55 yellow. How many pieces of candy are either blue, green, purple, or orange?

Solution: 158

We can use the complement rule described in 4.12: If S is the set of all outcomes and $P \subseteq S$, then $|P| = |S| - |\overline{P}|$. Let P be the set of blue, green, purple, or orange candy.

|blue, green, purple, or orange| = |total| - |NOT(blue, green, purple, or orange)|

= |total| - |red or yellow|

= 250 - (37 + 55) = 158

12 Twenty coins are flipped. How many outcomes have at least 4 heads facing up?

Solution: 1,047,225

The quickest approach is to use the complement rule as described in Lesson 4.12: If S is the set of all outcomes and $P \subseteq S$, then $|P| = |S| - |\overline{P}|$. We use P = set of outcomes with at least 4 heads. Then $\overline{P}$ = set of outcomes with less than 4 heads, which is easier to think about as zero, one, two, or three heads.

$$|\overline{P}| = |\text{no heads}| + |\text{one head}| + |\text{two heads}| + |\text{three heads}|$$

$$= \binom{20}{0} + \binom{20}{1} + \binom{20}{2} + \binom{20}{3}$$
$$= 1 + 20 + 190 + 1140 = 1351$$

So we end up with
$$|P| = |S| - |\overline{P}| = 2^{20} - 1351 = 1,047,225$$

13 How many ways are there to place 6 identical objects into 3 different bins?

Solution: 28

In general, the number of ways to place n indistinguishable objects into m groups is given below. The formula on the right is on the formula sheet.

$$\binom{n+m-1}{n} = \binom{n+m-1}{m-1}$$

This is often called a "star and bar" approach. We can line up our identical objects (stars) and place dividers between them (bars) to indicate which bin they will go into. For example,

$$\star\star|\star\star|\star\star$$

places 2 objects in each bin, and
$$\star\star\star|\star\star\star|$$

places 3 into the first two bins and none in the last. Note that we need to count the ways to place $3-1=2$ bars in $6+2$ spots. Order does not matter:

$$\binom{6+(3-1)}{(3-1)} = \binom{8}{2} = 28$$

Similarly, we can count the number of ways to place 6 stars in the eight places:

$$\binom{8}{6} = 28$$

14 There are 15 varieties of donuts sold at a bakery. How many ways are there to select a dozen donuts?

Solution: 9,657,700

We need to distribute the 12 doughnuts over the 15 varieties, making this a multiset question: $n = 12$ and $m = 15$. The number of ways to select the 12 doughnuts is

$$\binom{12+15-1}{15-1} = \binom{26}{14} = 9,657,700.$$

15 A program is asked to ping a server exactly 100 times in the course of a seven day week. How many different schedules are there for the program to ping the server if it must do at least 20 on each weekend day (Saturday and Sunday)? A schedule consists of the number of pings the program does on each of the seven days of the week, for example, Mon: 15, Tue: 0, Wed: 15, Thu: 15, Fri: 5, Sat: 25, Sun: 25.

Solution: 90,858,768

Start by giving both Saturday and Sunday 20 pings. These days could get more so they are still included in the next step.

We now count how many ways there are to distribute the remaining 60 pings over the seven days. This is a multiset with $n = 60$ and $m = 7$.

$$\binom{60+7-1}{7-1} = \binom{66}{6} = 90,858,768$$

16 How many ways can we add three non-negative integers such that they sum to 9? (order matters)

Solution: 55

The problem is asking us to count the number of ordered sets of non-negative integers, (a, b, c), such that

$$a + b + c = 9$$

We need to describe the situation in terms of indistinguishable objects (as above). So we'll use a "star and bar" approach. Only we'll use ones and zeros instead (using bars for 0's would get confusing!).

Think of a, b, c as bins and the ones as indistinguishable objects. So we will need 9 ones and 2 zeros. For example

$$11101101111$$

corresponds to $3+2+4=9$ and

$$11111001111$$

corresponds to $5+0+4$. So now we have a bijection between the solutions to $a+b+c=9$ and these sequences of ones and zeros. And the problem is equivalent to counting the number of ways to choose 2 out 11 objects

$$\binom{11}{2} = 55$$

Module 16: Generating Permutations and Combinations

17 Write the following 5-tuples in lexicographic order.

- $(5, 1, 4, 3, 2)$
- $(4, 3, 2, 2, 2)$
- $(5, 2, 1, 2, 1)$
- $(4, 3, 3, 5, 1)$

Solution: $(4, 3, 2, 2, 2)$, $(4, 3, 3, 5, 1)$, $(5, 1, 4, 3, 2)$, $(5, 2, 1, 2, 1)$

Review Lesson 4.15 for comparing n-tuples lexicographically.

18 Write the next 5 permutations in lexicographic order immediately following $(2, 3, 4, 1)$.

Solution: $(2, 4, 1, 3)$, $(2, 4, 3, 1)$, $(3, 1, 2, 4)$, $(3, 1, 4, 2)$, $(3, 2, 1, 4)$

This concept is discussed in Lesson 4.15.

19 Write the following 4-subsets of $\{$a, b, c, d, e, f, g$\}$ in lexicographical order.

- $\{$c, e, f, g$\}$
- $\{$a, d, g, e$\}$
- $\{$c, f, e, a$\}$
- $\{$g, f, e, b$\}$

Solution: When comparing **subsets** (see Lesson 4.16), list the elements of each set in increasing order:

- $\{$c, e, f, g$\}$ = $\{$c, e, f, g$\}$
- $\{$a, d, g, e$\}$ = $\{$a, d, e, g$\}$
- $\{$c, f, e, a$\}$ = $\{$a, c, e, f$\}$
- $\{$g, f, e, b$\}$ = $\{$b, e, f, g$\}$

Then we compare the subsets starting with the left-most elements to get:

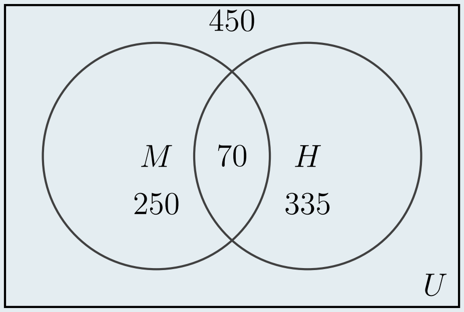
1. $\{$c, f, e, a$\}$ = $\{$a, c, e, f$\}$
2. $\{$a, d, g, e$\}$ = $\{$a, d, e, g$\}$
3. $\{$g, f, e, b$\}$ = $\{$b, e, f, g$\}$
4. $\{$c, e, f, g$\}$ = $\{$c, e, f, g$\}$

Module 17: Advanced Counting Techniques

20 A college has 1105 students. This semester 320 students are taking a math course, 405 of the students are taking a history class, and 250 of the students taking a math class are not enrolled in a history class.

(a) Construct a Venn diagram, and write the appropriate numbers in the four regions

Solution:



$|M \cap H| = 320 - 250$=70, $|H - M| = 405 - 70 = 335$, and $|\overline{M \cup H}| = 1105 - 250 - 70 - 335 = 450$.

(b) How many of the students are taking math and history?

Solution: $|M \cap H| = 70$

(c) How many students are taking history but not math?

Solution: $|H - M| = 335$

(d) How many students are taking math or history?

Solution: $|M \cup H| = 250 + 70 + 335 = 655$

(e) How many students are taking neither math nor history?

Solution: $|\overline{M \cup H}| = |U| - |M \cup H| = 1105 - 655 = 450$

21 Computer science students were surveyed about which operating systems they have used recently. Results showed that 70 students have used Windows, 62 have used macOS, and 16 have used Linux recently. Of these, 24 used both Windows and macOS, 7 used Windows and Linux, and 10 used macOS and Linux recently. Furthermore, 3 students report using all three operating systems recently. How many students reported using at least one of the operating systems recently?

Solution: 110

Define W to be the set of students who have used Windows recently, M the set the set of students who have used macOS recently, and L the set of students who have used Linux recently. The the inclusion/exclusion principle with three sets (Lesson 4.18) gives

$$|W \cup M \cup L| = |W| + |M| + |L| - |W \cap M| - |W \cap L| - |M \cap L| + |W \cap M \cap L|$$

$$= 70 + 62 + 16 - 24 - 7 - 10 + 3$$

$$= 110$$

22 What is the coefficient of x^6y^4 in $(x-2y)^{10}$?

Solution: 3360

This formula for the Binomial Theorem is the same as what's on the Formula Sheet and recommend practicing with it. Note that the book provides an equivalent, yet slightly different formula.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Matching with $(x-2y)^{10}$ we'll use $n=10$, $a=x$, and $b=-2y$. Be careful with the sign!

$$(x-2y)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^k (-2y)^{10-k}$$

Looking at the provided variables x^6y^4 we only need $k=6$ instead of the whole sum. Now we just plug this in:

$$\binom{10}{6} x^6 (-2y)^{10-6} = 210x^6(-2)^4y^4 = 210x^6 16y^4 = 3360x^6y^4$$

23 How many cards must be drawn from a standard deck of 52 playing cards to guarantee:

a. at least two have the same suite?

Solution: 5

By the pigeonhole principle (Lesson 4.20), we need one more card than total number of suits.

b. at least 5 have the same suit?

Solution: 17

This time, use the "Contrapositive of the generalized pigeonhole principle" in Lesson 4.20 where $k=4$ (the four suits), and $b=5$ (at least five of the same suit). So n must be at least $4(5-1)+1=17$. That is we need at least 17 cards to guarantee 5 have the same suit.