Some Discrete Probability Distributions

The Bernoulli Process

An experiment often consists of repeated trials, each with two possible outcomes that may be labeled success or failure.

Eg: Tossing a coin ten times and finding the probability of number of heads.

Head — Success
Tail — Failure.

The process is referred to as a Bernoulli process. Each trial is called a Bernoulli trial.

The Bernoulli ~~trial~~ process must possess the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Note: In drawing cards from a deck, the probabilities for repeated trials change if the card is not replaced.
The probability of selecting a heart on first draw is $\frac{13}{52}$ and on second draw it is conditional probability, $\frac{13}{51}$ or $\frac{12}{51}$ depending on whether a heart appeared on first draw.
$\therefore$ This would not be considered as a bernoulli trial.

Binomial Distribution

The number X of successes in n Bernoulli trials is called a binomial random variable. The probability distribution of this discrete random variable is called the binomial distribution and its value is denoted by $b(x; n, p)$.

Def: A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x; n, p) = {}^nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

$$\boxed{{}^nC_x = \binom{n}{x}}$$

Ex: Three items are selected at random from a manufacturing process, inspected and classified as defective or nondefective. Find the probability distribution for number of defectives assuming that 25% items are defective.

Sol. Let X be a random variable representing number of defectives. $P(S) = p = \frac{1}{4}$, $q = \frac{3}{4}$

$S = \{NNN, NDN, NND, DNN, NDD, DND, DDN, DDD\}$

X	0	1	2	3
$f(x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

$$f(0) = P(NNN) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{27}{64}$$

$$f(1) = 3\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{27}{64}$$

$$f(2) = 3\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{9}{64}$$

$$f(3) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$$

$$b\left(x; 3, \frac{1}{4}\right) = {}^3C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x},$$

$n = 0, 1, 2, 3.$

Ex: Ten ~~Three~~ coins are thrown simultaneously. Find the probability of getting at least seven heads.

Sol: p = Probability of getting a head $= \frac{1}{2}$

q = Probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$b\left(x; 10, \tfrac{1}{2}\right) = {}^{10}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{10-x}, \quad x = 0, 1, 2, --, 10$$

Probability of getting at least 7 heads

$$= P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10).$$

$$= {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + {}^{10}C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}C_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right)$$

$$+ {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^0$$

$$= \left(\tfrac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \frac{1}{1024} \left[\frac{10!}{7!\,3!} + \frac{10!}{8!\,2!} + \frac{10!}{9!\,1!} + \frac{10!}{0!\,10!} \right]$$

$$= \frac{1}{1024} \left[\frac{10 \cdot 9 \cdot 8}{3 \cdot 2} + \frac{10 \cdot 9}{2} + 10 + 1 \right]$$

$$= \frac{1}{1024} [120 + 45 + 11]$$

$$= \frac{176}{1024}.$$

Ex The probability that a certain kind of component will survive a shock test is $\frac{3}{4}$. Find the probability that exactly 2 of the next 4 components tested survive.

Sol:

$p = \frac{3}{4}, \; q = \frac{1}{4}$

$$b\left(2; 4, \frac{3}{4}\right) = {}^{4}C_{2}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)^{2} = 6 \cdot \frac{9}{16} \cdot \frac{1}{16} = \frac{27}{128}.$$

Ex The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

(a) at least 10 survive

(b) from 3 to 8 survive

(c) exactly 5 survive?

Moment Generating Function (m.g.f.)

The m.g.f. of a RV X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\,dx, & \text{for continuous probability distribution} \\ \sum e^{tx} f(x), & \text{for discrete probability distribution} \end{cases}$$

$$M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

where
$$\mu_r' = E(X^r) = \begin{cases} \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{for continuous distribution} \\ \sum_x x^r f(x), & \text{for discrete distribution,} \end{cases}$$

is the rth moment of X about origin.

Thus, μ_r' (about origin) = Coefficient of $\frac{t^r}{r!}$ in $M_X(t)$.

Since, $M_X(t)$ generates moments, it is known as m.g.f.

or $\mu_r' = \left| \frac{d^r}{dt^r} \{ M_X(t) \} \right|_{t=0}$

In general, the m.g.f. of X about the point $X = a$ is defined as:

$$M_X(t) \text{ (about } a) = E\left[e^{t(X-a)} \right]$$

$$= E\left[1 + t(X-a) + \frac{t^2 (X-a)^2}{2!} + \cdots + \frac{t^r}{r!}(X-a)^r + \cdots \right]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

where $\mu_r' = E[(X-a)^r]$, is the rth moment about the point $X = a$.

Find the Moment Generating Function of Binomial Distribution and use it to find the μ and σ^2

Let X be a binomial random variable.

$\therefore \; b(x; n, p) = {}^nC_x \, p^x q^{n-x}, \; x = 0, 1, 2, \ldots, n.$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q + pe^t)^n$$

$$\left[\text{Binomial Expansion}: (a+x)^n = {}^nC_0 a^0 x^n + {}^nC_1 a^1 x^{n-1} + \cdots + {}^nC_n a^n x^0 \right]$$

... and Variance of Binomial Distribution

$\mu = E(x) = \mu_1'$, the first moment about origin.

$\sigma^2 = E(x^2) - [E(x)]^2 = \mu_2' - (\mu_1')^2$,

where μ_2' is the second moment about origin.

Now, $E(x^2) = \frac{d^2}{dt^2}\left[M_x(t)\right]_{t=0}$

$\therefore\ E(x) = \frac{d}{dt}\left[M_x(t)\right|_{t=0} = \left| n(q+pe^t)^{n-1} pe^t \right|_{t=0}$

$= np(q+p)^{n-1}$

$= np \qquad [\because q+p=1]$

$\therefore\ \boxed{\mu = E(x) = np}$

$E(x^2) = \left| \frac{d^2}{dt^2}(M_x(t)) \right|_{t=0}$

$= \left| \frac{d}{dt}\left(npe^t (q+pe^t)^{n-1} \right) \right|_{t=0}$

$= \left| np\left\{ e^t (n-1)(q+pe^t)^{n-2} pe^t + e^t (q+pe^t)^{n-1} \right\} \right|_{t=0}$

$= np\left[(n-1)(q+p)p + (q+p)^{n-1}\right]$

$= np[np - p + 1] = n^2p^2 - np^2 + np.$

$$\sigma^2 = n^2p^2 - np^2 + np - n^2p^2 \cancel{= np(np}$$

$$= np(1-p) = npq$$

$$\boxed{\sigma^2 = npq}$$

Negative Binomial and Geometric Distributions

Negative Binomial Experiments

Consider an experiment where the properties are the same as those listed for a binomial experiment, with the exception that the trials will be replaced until a fixed number of successes occur.

Therefore, instead of the probability of x successes in n trials, where n is fixed, we are now interested in the probability that the kth success occurs on the xth trial. Experiments of this kind are called negative binomial experiments.

Negative Binomial Random Variable

The number X of trials required to produce k successes in a negative binomial experiment is called a negative binomial random variable and its probability distribution is called the negative binomial distribution.

Negative Binomial distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1-p$, then the probability distribution of the random variable X, the number of the trial on which the kth success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, \ldots$$

1\. Q:- Find the probability that a person flipping a coin gets

(a) the third head on the seventh flip

(b) the first head on the fourth flip.

Sol: (a) $b^*(7, 3, \frac{1}{2}) = {}^6C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^4$

$$= \frac{15}{128} = 0.1172$$

$\cancel{b^*(4, 1, \frac{1}{2})}$ (b) $b^*(4, 1, \frac{1}{2}) = {}^3C_0 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = \frac{1}{16} = 0.0625$

Ex In an NBA (National Basketball Association) Championship series, the team that wins four games out of seven is the winner. Suppose that the teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team ~~B~~ A will win the series? ~~in 6 games?~~

(c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

Sol: (a) $b^*(6; 4, 0.55) = {}^5C_3 (0.55)^4 (0.45)^2$

$$= 0.1853$$

(b) P(A will win the series)

$= P(X \geq 4)$

$= b^*(4; 4, 0.55) + b^*(5; 4, 0.55) + b^*(6; 4, 0.55) + b^*(7; 4, 0.55)$

$= {}^3C_3 (0.55)^4 (0.45)^0 + {}^4C_3 (0.55)^4 (0.45)^1 + {}^5C_3 (0.55)^4 (0.45)^2 + {}^6C_3 (0.55)^4 (0.45)^3$

$= 0.0915 + 0.1647 + 0.1853 + 0.1668$

$= 0.6083$.

(c) P(team A wins the playoff)

$= P(X \geq 3)$

$= b^*(3; 3, 0.55) + b^*(4; 3, 0.55) + b^*(5; 3, 0.55)$

$= {}^2C_2 (0.55)^3 (0.45)^0 + {}^3C_2 (0.55)^3 (0.45)^1 + {}^4C_2 (0.55)^3 (0.45)^2$

$= 0.1664 + 0.2246 + 0.2021$

$= 0.5931$

If we consider the special case of the negative binomial distribution where $k=1$, we have a probability distribution for the number of trials required for a single success.

For eg: Tossing a coin until head occurs.

We might be interested in the probability that the first head occurs on the fourth toss.

The negative binomial reduces to

$$b^*(x; 1, p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots$$

Geometric Distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots$$

1 Q (b) $g(4; \frac{1}{2}) = (\frac{1}{2})(\frac{1}{2})^3 = \frac{1}{2^4} = \frac{1}{16} = 0.0625$

Ex For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Sol: $g(5; 0.01) = (0.01)(0.99)^4$ $\qquad p = \frac{1}{100} = 0.01$

$= 0.0096$ $\qquad q = 0.99$

Ex: At a busy time, a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. Suppose that we let $p = 0.05$ be the probability of a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call.

Sol: $g(5; 0.05) = (0.05)(0.95)^4 = 0.0407.$

Mean and Variance of a random variable following the Geometric Distribution:

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{q}{p^2}.$$

Q: In the above example, find the expected number of calls necessary to make a connection?

Sol: $\mu = \frac{1}{0.05} = 20.$

Poisson Process and Poisson Distribution

Some experiments results in Counting the number of particular events occur in given time interval or in a specified region, known as Poisson Experiments.

The time interval may be of any length, such as a minute, a day, a week, a month or even a year.

Eg:- 1. Number of telephone calls received per hour by an office.

2. How many vehicles pass through a traffic signal in a day.

3. How many people arrive at a railway station from 9 am to 11 am.

4. How many people enter in the door of a shopping mall in January.

The specified region can be a line segment, an area, a volume or perhaps a piece of material.

Eg:- 1. Number of field mice per acre.

2. Number of typing errors per page.

Poisson Process

Poisson Process represents observations | occurrences | happenings over time | area.

Properties of Poisson Process

1. The number of outcomes | occurrences during disjoint time intervals are independent.

Eg:- No. of earthquakes (accidents) recorded in 2021-22 is independent of the no. of earthquakes (accidents) recorded in 2001-02.

2. The probability of a single occurrence during a small time interval is proportional to the length of the interval.

$P_1(h) = P(X(h) = 1) = \lambda h$ (Rate of occurrence of an event)

3. The probability of more than one occurrence during a small time interval in negligible.

<u>Eg</u>: If there a train accident at 9.00 am at a particular place, then it is highly ~~that~~ unlikely that there will be a train accident at 9.03 am.

Poisson Random Variable and Poisson Distribution

The number X of outcomes occurring during a Poisson experiment is called a Poisson Random Variable and its probability distribution is called the Poisson distribution.

Poisson Distribution

<u>Def</u>: The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is the average number of outcomes per unit time, distance, area or volume and $e = 2.71828\ldots$

<u>Ex</u> During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 milisecond is 4. What is the probability that 6 particles enter the counter in a given milisecond?

Sol: $\lambda t = 4,\ x = 6$

$$p(6;4) = \frac{e^{-4}(4)^6}{6!} = \frac{(0.0183)(4096)}{720} = \frac{74.9568}{720} = 0.1041$$

Mean and Variance of Poisson Distribution $p(x;\lambda t)$

$$\boxed{\mu = \lambda t,\ \sigma^2 = \lambda t}$$

$$\mu = E(X) = \sum_{x=0}^{\infty} x\frac{e^{-\lambda t}(\lambda t)^x}{x!} = \sum_{x=1}^{\infty} \frac{x e^{-\lambda t}(\lambda t)^x}{x!}$$

$$= e^{-\lambda t}(\lambda t)\sum_{x=1}^{\infty}\frac{(\lambda t)^{x-1}}{(x-1)!}$$

$$= e^{-\lambda t}(\lambda t)\left[1 + \frac{\lambda t}{1} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots\right]$$

$$= \lambda t\, e^{-\lambda t} e^{\lambda t}$$

$$\boxed{\mu = \lambda t}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2\frac{e^{-\lambda t}(\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=1}^{\infty}(x^2 - x + x)\frac{e^{-\lambda t}(\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{e^{-\lambda t} (\lambda t)^x}{\cancel{x(x-1)}(x-2)!} + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{(x-2)!} + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots \right] + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 e^{\lambda t} + \lambda t - \lambda^2 t^2$$

$$= \lambda^2 t^2 + \lambda t - \lambda^2 t^2$$

$$= \lambda t$$

$$\boxed{\sigma^2 = \lambda t}$$

Approximation of Binomial Distribution by a Poisson Distribution

Poisson distribution is a limiting case of the binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large, i.e., $n \to \infty$

(ii) p, the constant probability of success for each trial is indefinitely small, i.e., $p \to 0$.

(iii) ~~$np = \lambda$~~, $np = \mu$, is finite.

Thm Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$ and $np \xrightarrow{n\to\infty} \mu$ remains constant,

$$b(x; n, p) \xrightarrow{n \to \infty} p(x; \mu).$$

$$p(x; \mu) = \frac{e^{-\mu} \mu^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Ex In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

Sol: (a) What is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with accidents?

Sol: Let X be a binomial random variable with $n = 400$ and $p = 0.005$.

Thus, $np = 400 \times 0.005 = 2$

Using Poisson Process,

(a) $P(X=1) = \dfrac{e^{-2}2^1}{1!} = (0.1353)(2) = 0.2706$

(b) $P(X \le 3) = P(X=1) + P(X=2) + P(X=3) + P(X=0)$

$= 0.2706 + \dfrac{e^{-2}2^2}{2!} + \dfrac{e^{-2}2^3}{3!} + \dfrac{e^{-2}2^0}{0!}$

$= 0.2706 + \dfrac{(0.1353)(4)}{2} + \dfrac{(0.1353)(8)}{6} + 0.1353$

$= 0.2706 + 0.2706 + 0.1804 + 0.1353$

$= 0.7216 + 0.1353$

$= 0.8569$

Ex In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Sol: It is a binomial experiment with $n = 8000$ and $p = 0.001$

Since p is very close to 0 and n is quite large, we will use Poisson distribution.

$$\mu = 8000 \times 0.001 = 8.$$

Let X represent the number of bubbles.

$$P(X<7) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$= e^{-8}\left[\frac{8^0}{0!}+\frac{8^1}{1!}+\frac{8^2}{2!}+\frac{8^3}{3!}+\frac{8^4}{4!}+\frac{8^5}{5!}+\frac{8^6}{6!}\right]$$

$$= e^{-8}\left[1+8+32+85.3333+170.6667+273.0667+364.0889\right]$$

$$= \cancel{0.00033}[934.1556] \times 0.0003355$$

$$= 0.3134$$

Ex A manufacturer, who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain:
(i) no defective (ii) at least two defectives.

Sol: $n = 500$, $p = 0.001$, $np = 0.5$

Let X be a random variable denote the number of defective bottles in a box of 500. The prob of x defective bottle in a box is

$$\therefore P(X=x) = \frac{e^{-0.5}(0.5)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$\therefore$ The number of boxes containing x defective bottles in a consignment of 100 boxes is

$$100 \times P(X=x) = 100 \times \frac{e^{-0.5} \times (0.5)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) Number of boxes containing no defective bottles is

$$100 \times P(X=0) = 100 \times \frac{e^{-0.5} \times (0.5)^0}{0!}$$

$$= 100 \times 0.6065$$

$$= 6065$$

$$\simeq 61.$$

(ii) Number of boxes containing at least two defective bottles is

$$100 \times P(X \geq 2) = 100 \left[1 - P(X<2)\right]$$

$$= 100\left[1 - P(X=0) - P(X=1)\right]$$

$$= 100\left[1 - 0.6065 \times 1 - \frac{0.6065.(0.5)}{1!}\right]$$

$$= 100\left[1 - 0.6065 - 0.3033\right]$$

$$= 100\left[0.0902\right]$$

$$= 9.02$$

$$\simeq 9.$$

Moment Generating Function (m.g.f.)

The m.g.f. of a RV X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\,dx, & \text{for Continuous probability distribution} \\ \sum e^{tx} f(x), & \text{for discrete probability distribution} \end{cases}$$

$$M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

where $\mu_r' = E(X^r) = \begin{cases} \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{for Continuous distribution} \\ \sum_x x^r f(x), & \text{for discrete distribution,} \end{cases}$

is the rth moment of X about origin.

Thus, μ_r' (about origin) = Coefficient of $\frac{t^r}{r!}$ in $M_X(t)$.

Since, $M_X(t)$ generates moments, it is known as m.g.f.

or $\mu_r' = \left| \frac{d^r}{dt^r} \{M_X(t)\} \right|_{t=0}$

In general, the m.g.f. of X about the point X = a is defin as:

$$M_X(t) \text{ (about } a) = E\left[e^{t(X-a)}\right]$$

$$= E\left[1 + t(X-a) + \frac{t^2(X-a)^2}{2!} + \cdots + \frac{t^r}{r!}(X-a)^r + \cdots\right]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

where $\mu_r' = E[(X-a)^r]$, is the rth moment about the point X = a.

<u>Find the Moment Generating Function of Binomial Distribution and use it to find the μ and σ^2</u>

Let X be a binomial random variable.

$$\therefore \; b(x; n, p) = {}^nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q + pe^t)^n$$

$$\left[\text{Binomial Expansion: } (a+x)^n = {}^nC_0 a^0 x^n + {}^nC_1 a^1 x^{n-1} + \cdots + {}^nC_n a^n x^0\right]$$

... and Variance of Binomial Distribution

$\mu = E(X) = \mu_1'$, the first moment about origin.

$\sigma^2 = E(X^2) - [E(X)]^2 = \mu_2' - (\mu_1')^2$,

where μ_2' is the second moment about origin.

Now, $E(X^2) = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$

$\therefore \; E(X) = \frac{d}{dt}\left[M_X(t)\right|_{t=0} = \left| n(q+pe^t)^{n-1} pe^t \right|_{t=0}$

$= np(q+p)^{n-1}$

$= np \qquad \left[\because q+p=1\right]$

$\therefore \; \boxed{\mu = E(X) = np}$

$\Rightarrow E(X^2) = \left|\frac{d^2}{dt^2}(M_X(t))\right|_{t=0}$

$= \left|\frac{d}{dt}\left(npe^t(q+pe^t)^{n-1}\right)\right|_{t=0}$

$= \left| np\left\{ e^t(n-1)(q+pe^t)^{n-2} pe^t + e^t(q+pe^t)^{n-1} \right\}\right|_{t=0}$

$= np\left[(n-1)(q+p)p + (q+p)^{n-1}\right]$

$= np\left[np - p + 1\right] = n^2p^2 - np^2 + np$.

$$\sigma^2 = n^2p^2 - np^2 + np - n^2p^2 \cancel{- np(np}$$

$$= np(1-p) = npq$$

$$\boxed{\sigma^2 = npq}$$

M.g.f. of Negative Binomial Distribution

$$M_X(t) = E(e^{tX}) = \sum_{x=k}^{\infty} e^{tx} \binom{x-1}{k-1} q^{x-k} p^k$$

$$= p^k e^{tk} \sum_{x=k}^{\infty} e^{t(x-k)} \binom{x-1}{k-1} q^{x-k}$$

$$= p^k e^{tk} \sum_{x=k}^{\infty} \binom{x-1}{k-1} (qe^t)^{x-k}$$

Let $r = x-k$
$\Rightarrow x = k+r$

$$= p^k e^{tk} \sum_{r=0}^{\infty} \binom{k+r-1}{k-1} (qe^t)^r$$

$$= (pe^t)^k \left[1 + k(qe^t) + \frac{k(k+1)}{2!}(qe^t)^2 + \frac{k(k+1)(k+2)}{3!}(qe^t)^3 + \cdots \right]$$

$$= (pe^t)^k (1-qe^t)^{-k}$$

$$\left[(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \text{ provided } |x|<1 \right]$$

$$\therefore M_X(t) = \frac{(pe^t)^k}{(1-qe^t)^k}, \text{ provided } |qe^t| < 1.$$

At $t=0$,
Mean $=E(X)$
Variance $=E(X^2)-\mu^2$.

$$E(X)=\frac{d}{dt}M_X(t)$$

$$=\frac{(1-qe^t)^k\,k(pe^t)^{k-1}pe^t-(pe^t)^k\,k(1-qe^t)^{k-1}.(-qe^t)}{(1-qe^t)^{2k}}$$

$$=\frac{k(1-qe^t)^{k-1}(pe^t)^k\left[1-qe^t+qe^t\right]}{(1-qe^t)^{2k}}$$

$$=\frac{k(1-qe^t)^{k-1}(pe^t)^k}{(1-qe^t)^{2k}}$$

At $t=0$, $\mu=\frac{k(1-q)^{k-1}p^k}{(1-q)^{2k}}=\frac{kp^{k-1}p^k}{p^{2k}}=\frac{kp^{2k-1}}{p^{2k}}$

$$\boxed{\mu=\frac{k}{p}}$$

$$E(X^2)=\frac{d^2}{dt^2}M_X(t)=\frac{d}{dt}\left[\frac{k(1-qe^t)^{k-1}(pe^t)^k}{(1-qe^t)^{2k}}\right]$$

$$=k\left[\frac{(1-qe^t)^{2k}\frac{d}{dt}\left[(1-qe^t)^{k-1}(pe^t)^k\right]-(1-qe^t)^{k-1}(pe^t)^k(-qe^t)2k(1-qe^t)^{2k-1}}{(1-qe^t)^{4k}}\right.$$

$$= k\left[\frac{(1-qe^t)^{2k}(1-qe^t)^{k-1}k(pe^t)^{k-1}pe^t + (pe^t)^k(k-1)(1-qe^t)^{k-2}(-qe^t) + 2k(1-qe^t)^{3k-2}(pe^t)^k qe^t}{(1-qe^t)^{4k}}\right]$$

At $t=0$,

$$E(x^2) = \frac{k\left[(1-q)^{2k}(1-q)^{k-1}k(p)^{k-1}p + (k-1)(p)^k(1-q)^{k-2}(-q) + 2kqp^k(1-q)^{3k-2}\right]}{(1-q)^{4k}}$$

$$= \frac{k\left[kp^{2k+k-1+k-1+1} + (k-1)(-q)p^{k+k-2} + 2kqp^{k+3k-2}\right]}{p^{4k}}$$

$$= \frac{k\left[kp^{4k-1} - q(k-1)p^{4k-2} + 2kqp^{4k-2}\right]}{p^{4k}}$$

$$= \frac{k\left[kp^{4k-1} - qkp^{4k-2} + qp^{4k-2} + 2kqp^{4k-2}\right]}{p^{4k}}$$

$$= \frac{k\left[kp^{4k-1} + qp^{4k-2} + qkp^{4k-2}\right]}{p^{4k}}$$

$$= \frac{k\cdot p^{4k}}{p^{4k}}\left[\frac{k}{p} + \frac{q}{p^2} + \frac{qk}{p^2}\right] = k\left[\frac{k}{p} + \frac{q}{p^2} + \frac{qk}{p^2}\right]$$

$$= k\left[\frac{pk+q+qk}{p^2}\right]$$

$$= k\left[\frac{k+q}{p^2}\right]$$

$$= \frac{k}{p^2}\left[k+q\right]$$

$$\text{Variance} = \frac{k}{p^2}(k+q) - \frac{k^2}{p^2} = \frac{k^2}{p^2} + \frac{kq}{p^2} - \frac{k^2}{p^2}$$

$$= \frac{kq}{p^2}.$$

$$\boxed{\text{Variance} = \frac{kq}{p^2}}$$

m.g.f. of Geometric Distribution

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} e^{tx} q^{x}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} [q e^{t}]^{x}$$

Infinite G.P.

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-r},$$

provided common ratio $r < 1$

$$= \frac{p}{q} \frac{qe^t}{1-qe^t}, \text{ provided } qe^t < 1.$$

$$= \frac{pe^t}{1-qe^t} \text{ provided } qe^t < 1.$$

$$\boxed{M_X(t) = \frac{pe^t}{1-qe^t} \text{ provided } qe^t < 1}$$

Mean and Variance

At $t=0$,

$$E(X) = \frac{(1-qe^t)\, pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= pe^t\left[1-qe^t + qe^t\right] \Big/ (1-qe^t)^2$$

$$E(X) = \frac{pe^t}{(1-qe^t)^2}$$

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \Rightarrow \boxed{\mu = \frac{1}{p}}$$

$$E(X^2) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

At $t=0$,

$$E(X^2) = \frac{(1-q)^2 p + 2pq(1-q)}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{p^4}$$

$$= \frac{1}{p} + \frac{2q}{p^2}$$

$$\text{Variance} = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2} = \frac{p+2q-1}{p^2} = \frac{1+q-1}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

M.g.f. of Poisson Distribution

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t \lambda} \qquad \left[e^x = 1 + x + \frac{x^2}{2!} + \cdots \right]$$

$$\cancel{= e^{-\lambda(1+}}$$

$$M_X(t) = e^{-\lambda(1-e^t)}$$

Mean and Variance of Poisson Distribution

$$E(X) = \frac{d}{dt}\left[M_X(t) \right] = e^{-\lambda(1-e^t)} (-\lambda)(-e^t)$$

$$= (\lambda e^t) e^{-\lambda(1-e^t)}$$

$$= \lambda e^{t - \lambda + \lambda e^t}$$

At $t=0$, $E(X) = \lambda e^{-\lambda+\lambda} = \lambda e^0 = \lambda$

$$\boxed{\therefore \ E(X) = \lambda}$$

$$E(X^2) = \lambda\left[e^t\{-\lambda(-e^t)\, e^{-\lambda(1-e^t)}\} + e^t e^{-\lambda(1-e^t)} \right]$$

At $t=0$, $E(X^2) = \lambda\left[1 + \lambda e^{-\lambda(1-1)} + e^{-\lambda(1-1)}\right]$

$$= \lambda[1 + \lambda + 1] = 2\lambda + \lambda^2$$

Variance $= 2\lambda + \lambda^2 - \lambda^2 =$

$$E(x^2) = \frac{d^2}{dt^2}\left[M_x(t)\right]$$

$$= \frac{d}{dt}\left[(\lambda e^t)\, e^{-\lambda(1-e^t)}\right]$$

$$= \lambda\left[e^t e^{-\lambda(1-e^t)} + e^t e^{-\lambda(1-e^t)} (+\lambda e^t)\right]$$

At $t=0$

$$E(x^2) = \lambda\left[e^{-\lambda(1-1)} + e^0 e^{-\lambda(1-1)} \lambda\right]$$

$$= \lambda[1+\lambda] = \lambda + \lambda^2$$

Variance $= \lambda + \lambda^2 - \lambda^2 = \lambda$

$$\boxed{\text{Variance} = \lambda}$$