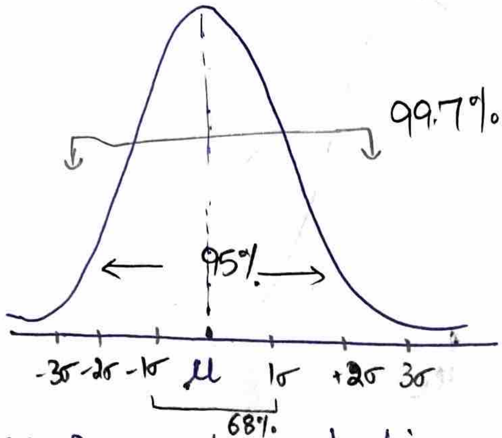
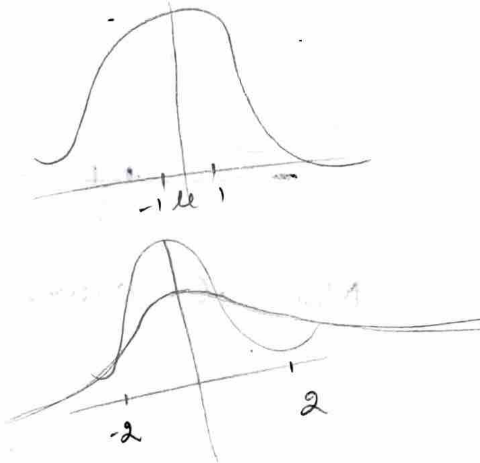
Normal Distribution

Normal Distribution is also known as Gaussian distribution or Bell-shaped Curve distribution. Its graph, called the normal curve, is the bell shaped curve.





Eg:- 1. Weight of newborn babies.
2. Height of 15 year old girls.
3. ~~Shoe size~~ Temparature
4. Blood pressure
5. ~~Students marks~~ Rainfall.

A continuous random variable X having the bell-shaped distribution is called a <u>normal random variable</u>.

Normal Distribution

The density function of the normal random variable X with mean μ and variance σ^2, is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

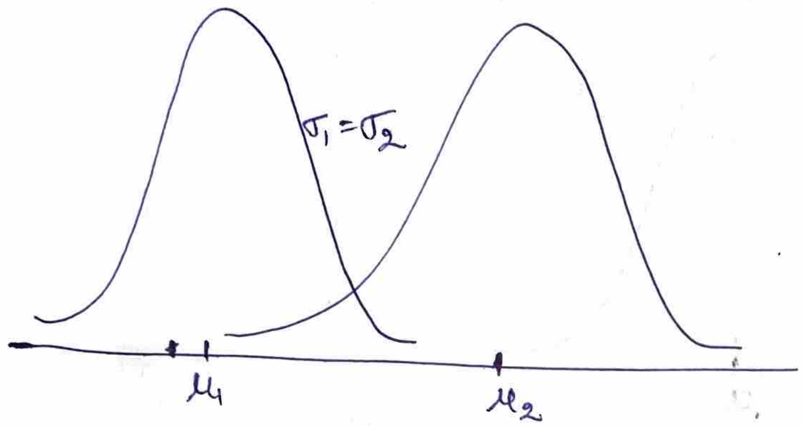
where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$

Properties of Normal Distribution

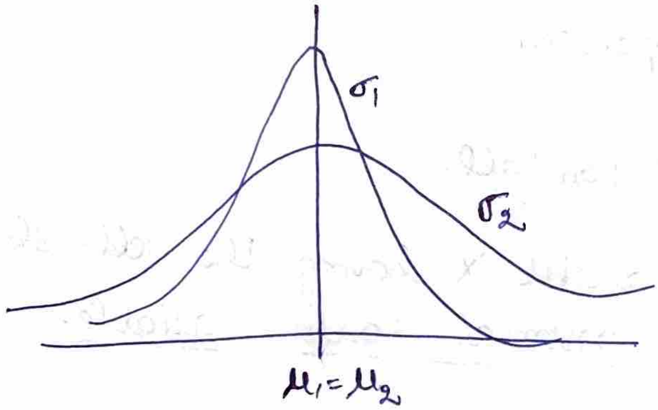
1. The mean, median and mode are all equal.
2. The curve is symmetric about the mean.
3. The total area under the curve and above the horizontal

axis is equal to 1. $\left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$

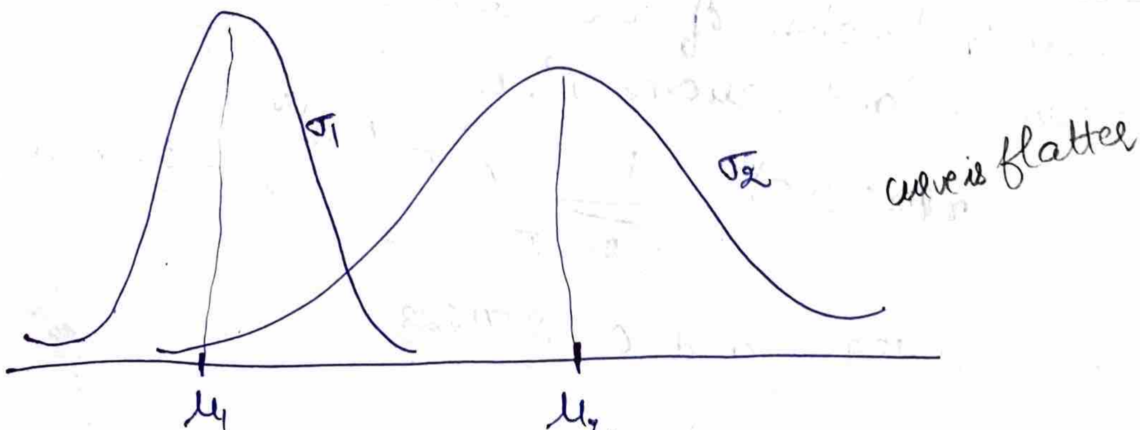
4. Area to the left and area to the right about the mean are same, i.e., 0.5.



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.

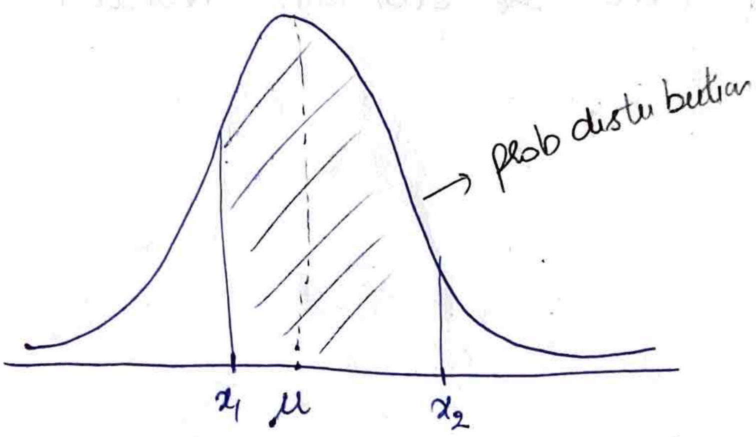


Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

Areas under the Normal Curve



$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)\, dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

For normal distributed variable X, the variable

$$Z = \frac{X-\mu}{\sigma}$$

is called standard normal variate.

$$\therefore P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}\, dz$$

$$= \int_{z_1}^{z_2} n(z; 0, 1)\, dz$$

$$= P(z_1 < Z < z_2),$$

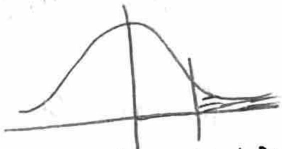
where Z is seen to be a normal random variable with mean 0 and variance 1.

Def : The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution.

Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.



Sol: (a) $P(Z > 1.84)$

$= 1 - 0.9671$

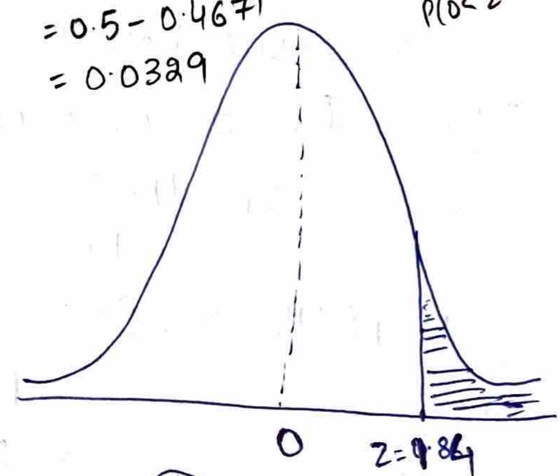
$= 0.0329.$

$= 0.5 - P(0 < z < 1.84)$

$= 0.5 - 0.4671$

$= 0.0329$

$P(0 < z < z_1)$



(b) $P(-1.97 < Z < 0.86)$

$= P(Z < 0.86) - P(Z < -1.97)$

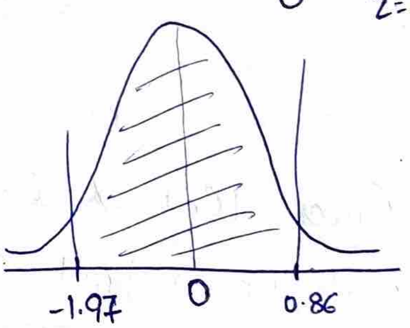
$= 0.8051 - 0.0244$

$= 0.7807.$

$P(0 < z < 0.86) = 0.3051$

$P(0 < z < 1.97) = 0.4756$

$\xrightarrow{+} 0.7807$



Q: Given a standard normal distribution, find the value of K such that

(a) $P(Z > K) = 0.3015$ and

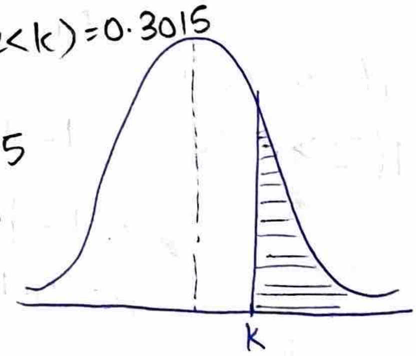
(b) $P(K < Z < -0.18) = 0.4197.$

$0.5 - P(0 < z < k) = 0.3015$

$\Rightarrow P(0 < z < k)$

$= 0.1985$

$K = 0.52$



Sol: (a) $P(z > k) = 1 - P(Z < k) = 0.3015$

$\Rightarrow P(z < k) = 1 - 0.3015 = 0.6985$

$\therefore k = 0.52.$

(b) $P(k < Z < -0.18) = 0.4197$

$\Rightarrow P(z < -0.18) - P(z < k) = 0.4197$

$\Rightarrow 0.4286 - P(z < k) = 0.4197$

$\Rightarrow P(z < k) = 0.4286 - 0.4197 = 0.0089$

$P(0 < z < -k) -$

$P(0 < z < 0.18)$

$= 0.4197$

$\Rightarrow P(0 < z < -k)$

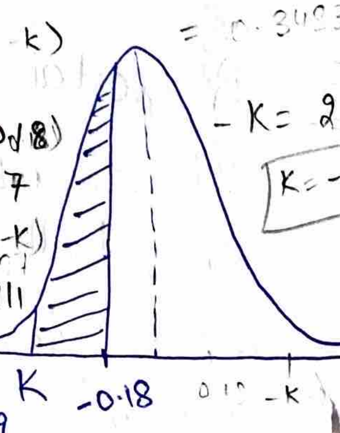
$= 0.4911$

$\Rightarrow P(0 < z < -k)$

$= 0.3493$

$-K = 2.37$

$\boxed{K = -2.37}$



$\Rightarrow P(Z<K) = 0.0089$

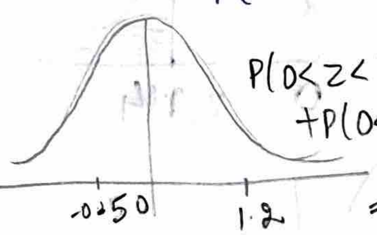
$K = -2.37.$

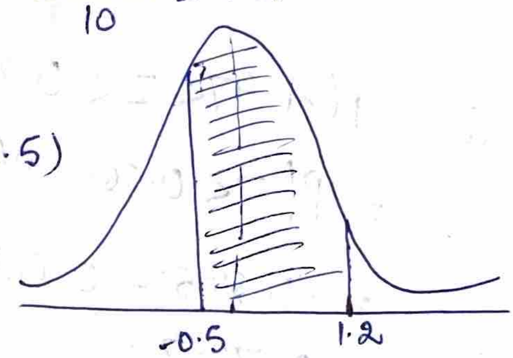
Ex:- Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Sol:- The z value corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$Z_1 = \frac{45-50}{10} = -0.5 \text{ and } Z_2 = \frac{62-50}{10} = 1.2.$$

$$\therefore P(45 < X < 62) = P(-0.5 < Z < 1.2)$$





$$= P(Z < 1.2) - P(Z < -0.5)$$

$$= 0.8849 - 0.3085$$

$$= 0.5764.$$

$P(0<Z<1.2) + P(0<Z<0.5) = 0.3489 + 0.1915 =$

Ex:- Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Sol:- $$Z = \frac{362 - 300}{50} = 1.24.$$

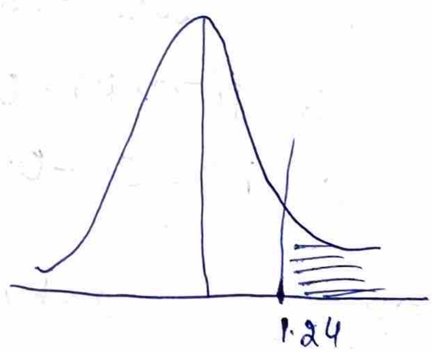
$$\left[Z = \frac{X - \mu}{\sigma} \right]$$

$P(X > 362)$

$= P(Z > 1.24) = 1 - P(Z < 1.24)$

$= 1 - [0.8925]$

$= 0.1075.$

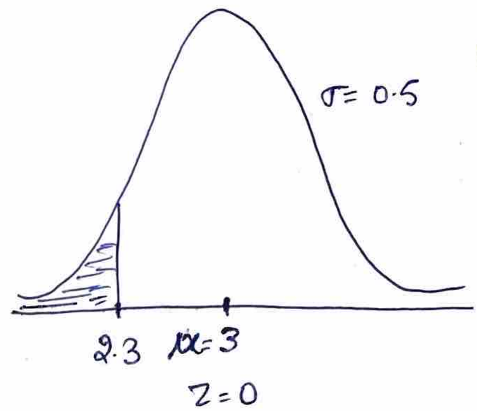


Ex A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

$$Z = \frac{x-\mu}{\sigma} = \frac{2.3-3}{0.5} = -1.4.$$

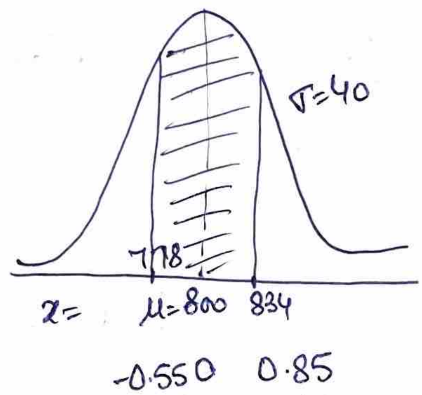
$P(x < 2.3) = P(Z < -1.4)$

$= 0.0808.$



Ex An elictrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

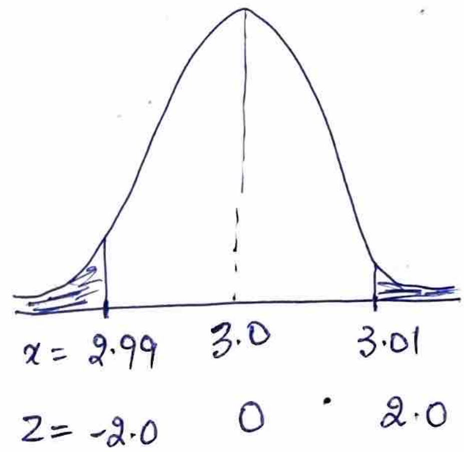
Sol:
$$Z = \frac{X-\mu}{\sigma} \Rightarrow Z_1 = \frac{X_1-\mu}{\sigma}$$
$$= \frac{778-800}{40}$$
$$= -0.55$$



$$Z_2 = \frac{x_2-\mu}{\sigma} = \frac{834-800}{40} = 0.85$$

$$P(778 < X < 834) = P(-0.55 < Z < 0.85)$$
$$= P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912$$
$$= 0.5111.$$

Ex In an industrial process, the diameter of a ball bearing is an important measurement. The buyer sets specifications for the diameter to be 3.0 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean $\mu = 3.0$ and standard deviation $\sigma = 0.005$. On average, how many manufactured ball bearings will be scrapped?



<u>Sol</u>:

$$Z_1 = \frac{2.99 - 3}{0.005} = -2.0$$

$$Z_2 = \frac{3.01 - 3}{0.005} = 2.0$$

$\therefore$ ~~P(x > 2~~

$$\therefore P(X < 2.99) = P(Z < -2)$$
$$= 0.0228$$

and $P(X > 3.01) = P(Z > 2.0) = 1 - P(Z < 2)$
$$= 1 - 0.9772$$
$$= 0.0228$$

$\therefore$ The ball bearing will be scrapped if

$$P(X < 2.99) + P(X > 3.01) = P(Z < -2) + P(Z > 2)$$
$$= 2(0.0228)$$
$$= 0.0456.$$

On average,

$\therefore$ ~~45.6%~~ 4.56% of the manufactured ball bearings will be scrapped.

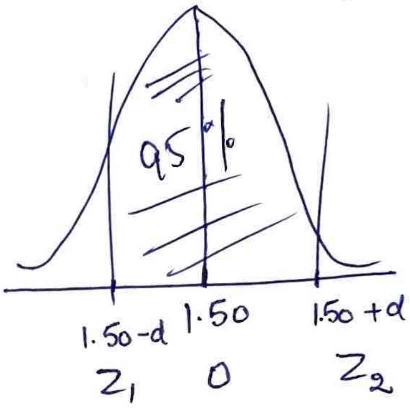
<u>Or</u> by symmetry, $P(Z < -2) + P(Z > 2) = 2[P(Z < -2)]$
$$= 2 \times 0.228$$
$$= 0.0456.$$

$$2[0.5 - P(z < 2)] = [0.5 - 0.4772]2$$
$$= 0.0228 \times 2$$
$$= 0.0456$$

<u>Ex</u> Scales / Indicators
Gauges are used to reject all components for which a certain dimension is not within the specification $1.50 \pm d$. It is known that this measurement is normally

distributed with mean 1.50 and standard deviation 0.2. Determine the value d such that the specifications cover 95% of the measurements.

Sol:



$P(Z_1 < Z < Z_2) = 0.95$

$2[0.5 - P(Z < Z_1)] = 0.95$ $\quad 2 \times P(0 < Z < Z_1) = 0.95$

$\Rightarrow 1 - 2P(Z < Z_1) = 0.95$ $\quad \Rightarrow P(0 < Z < Z_1) = 0.475$

$\Rightarrow 2P(Z < Z_1) = 0.05$

$\Rightarrow P(Z < Z_1) = 0.025$ $\quad Z_1 = 1.96$

$Z_1 = -1.96$

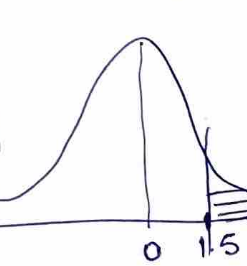
$Z_2 = 1.96$

$Z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow -1.96 = \frac{1.50 - d - 1.50}{0.2}$

$\Rightarrow -1.96 = \frac{-d}{0.2}$

$\Rightarrow d = 0.392$

Ex A certain machine makes electrical resistors having a mean resistance of 40 ohms (measurement of resistence) and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distibution and can be measured to any degree of accuracy, what % of resistors will have a resistence exceeding 43 ohms?



Sol: $\mu = 40, \sigma = 2$

~~$X = Z - \mu$~~ $\quad Z = \frac{X - \mu}{\sigma} = \frac{43 - 40}{2} = 1.5$

$P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332$

$= 0.0668.$

Hence, 6.68% of the resistors will have a resistance exceeding 43 ohms.

Normal Approximation to the Binomial Distribution

Th^m If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Normal distribution is a limiting case of binomial distribution under the following conditions:-

(i) n, the number of trials is indefinitely large, i.e., $n \to \infty$

(ii) neither p nor q is very small.

and the approximation will be good if np and ~~np(1-q~~ $np(1-p)$ are greater than or equal to 5.

Normal Approximation to the Binomial Distribution

Let X be a binomial random variable with parameters n and p. For large n, X has approximately a normal distribution with $\mu = np$ and $\sigma^2 = npq = np(1-p)$ and

$$P(X \le x) = \sum_{k=0}^{x} b(k; n, p)$$

$\approx$ area under normal curve to the left of $x + 0.5$

$$= P\left(Z \le \frac{x + 0.5 - np}{\sqrt{npq}}\right),$$

and the approximation will be good if np and $np(1-p)$ are greater than or equal to 5.

Q:- The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

Sol:- $n = 100$, $\mu = np = 40$, $\sigma^2 = npq = 24$, $p = 0.4$, $q = 0.6$

$\sigma = 4.899$

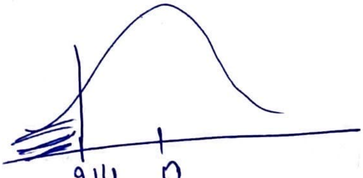
$$P(X < 30) = \cancel{P\left(\frac{29.5-40}{4.899}\right.} \quad P\left(Z < \frac{29.5-40}{4.899}\right) = P(Z < -2.14)$$

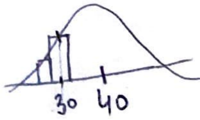
$= 0.0162$

or $0.5 - P(Z > 2.14)$

$= 0.5 - 0.4838$

$= 0.0162.$





Mgf of Normal distribution

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\mu+\sigma z)t} e^{\frac{-z^2}{2}} dz$$

Let $z = \frac{x-\mu}{\sigma}$

$x = \mu + \sigma z$

$dx = \sigma dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} e^{\sigma z t} e^{-z^2/2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z t)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z t + (\sigma t)^2 - (\sigma t)^2)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-\sigma t)^2}{2}} e^{\frac{(\sigma t)^2}{2}} dz$$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\, du$$

let $z - \sigma t = u$

$dz = du$

$$= \frac{2e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2}\, du$$

$\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

if $f(x)$ is even.

$$= \frac{2e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-v} \frac{dv}{\sqrt{2v}}$$

let $\frac{u^2}{2} = v$

$2u\, du = 2dv$

$\Rightarrow du = \frac{dv}{u}$

$= \frac{dv}{\sqrt{2v}}$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{\pi}} \int_{0}^{\infty} v^{-1/2} e^{-v}\, dv$$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}\right)$$

$\Gamma\alpha = \int_{0}^{\infty} x^{\alpha-1} e^{-x}\, dx, \ \alpha > 0$

$$= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{\pi}} \sqrt{\pi} = e^{\mu t + \sigma^2 t^2/2}$$

$$\boxed{M_X(t) = e^{\mu t + \sigma^2 t^2/2}}$$

$$\frac{d}{dt}(M_X(t)) = e^{\mu t + \sigma^2 t^2/2}\left(\mu + \frac{\sigma^2}{2} \cdot 2t\right)$$

$$= (\mu + \sigma^2 t)\, e^{\mu t + \sigma^2 t^2/2}$$

$$\frac{d^2}{dt^2}(M_x(t)) = (\mu+\sigma^2 t)\, e^{\mu t+\sigma^2 t^2/2} \cdot \left(\mu + \frac{\sigma^2}{2}\, 2t\right)$$
$$+ e^{\mu t+\sigma^2 t^2/2} \cdot \sigma^2$$

$E(X) = \mu$

$E(X^2) = \mu^2 + \sigma^2$

Variance $= E(X^2) - [E(X)]^2 = \sigma^2$.

$\therefore$ If X is a normal variate, $E(X) = \mu$, $Var(X) = \sigma^2$.

Let Z be a standard normal variate.

$\therefore$ $E(Z) = 0$, $Var(Z) = 1$. $\quad \frac{d}{dt}[M_Z(t)] = e^{t^2/2} \cdot \frac{1}{2}(2t) = te^{t^2/2}$

$E(Z) = 0$

$M_Z(t) = e^{t^2/2}$. $\quad \frac{d^2}{dt^2}[M_Z(t)] = te^{t^2/2} \cdot t + e^{t^2/2}$

$E(Z^2) = 1 \Rightarrow Var(Z) = 1$.

Q:- A multiple choice quiz has 200 questions, each with 4 possible answers of which only 1 is correct. What is the probability that sheer guesswork yields from 25 to 30 correct answers for the 80 of the 200 problems about which the student has no knowledge.

Sol:- $p = \frac{1}{4}$.

Let X represents the number of correct answers resulting from guesswork, then

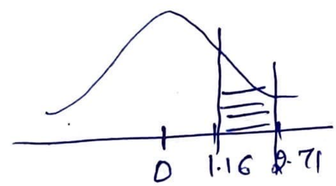
$$P(25 \le X \le 30) = \sum_{x=25}^{30} b(x;\, 80,\, \tfrac{1}{4})$$

$$\mu = np = 80 \cdot \frac{1}{4} = 20$$

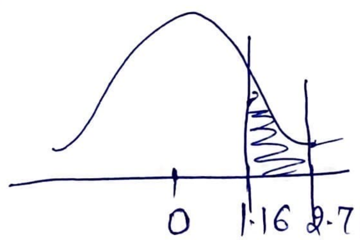
$$\sigma = \sqrt{npq} = \sqrt{80 \cdot \frac{1}{4} \cdot \frac{3}{4}} = \sqrt{15} = 3.873$$

$$x_1 = 24.5,\ x_2 = 30.5,\quad Z_1 = \frac{24.5-20}{3.873} = 1.16,\ Z_2 = \frac{30.5-20}{3.873} = 2.71$$

$$P(25 \le X \le 30) = P(1.16 < Z < 2.71)$$
$$= P(Z < 2.71) - P(Z < 1.16)$$
$$= 0.9966 - 0.8770$$
$$= 0.1196$$



or $$P(25 \le X \le 30) = P(1.16 < Z < 2.71)$$
$$= P(0 < Z < 2.71) - P(0 < Z < 1.16)$$
$$= 0.4966 - 0.3770$$
$$= 0.1196.$$



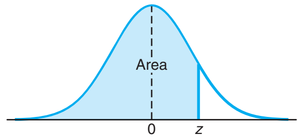


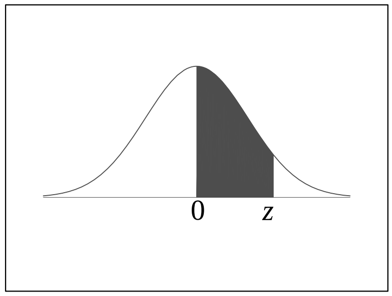
Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

Gamma and Exponential Distributions

Gamma Distribution is used to find the time until k events occur. When we are interested in the time until rth event occurs.

Ex: 1. The time until k customers arrive.
2. The time until you've been invited to k parties.

(Poisson: The prob that k customers will arrive in a fixed interval).

Gamma Function

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \text{ for } \alpha > 0.$$

Ex: $\int_0^\infty x^2 e^{-x} dx = \Gamma(3)$.

Properties:
1. $\Gamma n = (n-1)(n-2) \ldots 1\,\Gamma 1$, for positive integer n.
2. $\Gamma n = (n-1)!$ for a positive integer n.
3. $\Gamma 1 = 1$.
4. $\Gamma \frac{1}{2} = \sqrt{\pi}$.

Gamma Distribution

The continuous random variable X has a gamma distribution with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$, $\beta > 0$.

The special gamma distribution for which $\alpha=1$ is called the exponential distribution.

Exponential Distribution

The continuous random variable X has an exponential distribution, with parameter β, if its density function is given by

$$f(x;\beta)=\begin{cases} \frac{1}{\beta}e^{-x/\beta}, & x>0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\beta>0$.

	μ	σ^2
Gamma distribution	$\alpha\beta$	$\alpha\beta^2$
Exponential distribution	β	β^2.

Gamma Distribution

- Model (represent) the time between independent events that occur at constant average rate.

Ex: Modeling the time until 3rd or 4th accident occurs.

- Model the elapsed time between various number of events.

$\beta \to$ Mean time between events.

If $\beta=2$ and measuring the time between the vehicles passing through a traffic signal in [illegible] minutes.

$\Rightarrow$ There are ~~4 vehicles~~ 2 minutes between vehicles passing on average.

Exponential Distribution :- Model the time till next event occurs.

Note: $\lambda = \frac{1}{\beta}$,

where λ represents the average number of events per unit time and β is mean time between events.

Ex Suppose that telephone calls arriving at a call center follow a poisson process with an average of 5 calls per minute. What is the probability that upto a minute will elapse by the time 2 calls have come in to the call center?

Sol: The poisson process applies, with time until 2 Poisson events occur following gamma distribution.

$\therefore \beta = \frac{1}{5}, \ \alpha = 2.$

$$P(X \leq 1) = \int_0^1 25 \frac{1}{\Gamma 2} x e^{-5x} dx$$

$$= 25 \int_0^1 x e^{-5x} dx \qquad \left[\Gamma n = (n-1)!\right]$$

$$= 25 \left[x \frac{e^{-5x}}{-5} - 1 \cdot \frac{e^{-5x}}{25} \right]_0^1$$

$$= \cancel{\frac{25}{5}\left\{-e^{-5} + e^{-5} + 0 + 1\right\}}$$

$$= -\frac{25}{25} \left[5x e^{-5x} + e^{-5x} \right]_0^1$$

$$= -1 \left[5e^{-5} + e^{-5} - 1 \right]$$

$$= -6e^{-5} + 1 = 1 - 6(0.0067)$$

$$= 0.96.$$

Ex In a biomedical study with rats, a dose response investigation is used to determine the effect of the dose of a toxicant on their survival time. For a certain dose of the toxicant, the study determines that the survival time, in weeks, has a gamma distribution with $\alpha=5$ and $\beta=10$. What is the probability that a rat survives no longer than 60 weeks?

Sol: Let the random variable X be the survival time.

The reqd probability is

$$P(X \le 60) = \frac{1}{\beta^5} \int_0^{60} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma 5} dx$$

$\alpha = 5, \ \beta = 10$

$$\Rightarrow P(X \le 60) = \frac{1}{4!\,10^5} \int_0^{60} x^4 e^{-x/10} dx$$

$$= \frac{1}{4!\,10^5}\left[x^4 \frac{e^{-x/10}}{-1/10} - 4x^3 \frac{e^{-x/10}}{1/10^2} + 12x^2 \frac{e^{-x/10}}{-1/10^3} - \frac{24x\, e^{-x/10}}{1/10^4} + \frac{24 e^{-x/10}}{-1/10^5} \right]_0^{60}$$

$$= \frac{1}{4!\,10^5}\left[\left(-10x^4 - 400x^3 - 12000x^2 - 240000x - 2400000\right)e^{-x/10}\Big|_{x=60} + 2400000 \right]$$

$$= \frac{1}{4!\,10^5}\left[0.0025[(-12960000) - 8640000 - 43200000 - 2400000) + 2400000 \right]$$

$$= \frac{1}{4!\,10^5}\left[-654000 + 2400000 \right] = \frac{1746000}{2400000} = 0.72.$$

Ex Suppose that a system contains a certain type of Component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these Components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Sol: The probability that a given component is still functioning after 8 years is

$$P(T > 8) = \frac{1}{5}\int_8^{\infty} e^{-t/5}\, dt = \frac{1}{5}\left[\frac{e^{-t/5}}{-1/5}\right]_8^{\infty}$$

$$= -\left[0 - e^{-8/5}\right]$$

$$= e^{-8/5} = 0.2$$

Let X represent the number of components functioning after 8 years.

$\therefore$ The probability that x components will work after 8 years is

$$b(x; 5, 0.2) = {}^5C_x (0.2)^x (0.8)^{5-x}, \quad x = 0, 1, 2, \ldots, 5.$$

Reqd probability is

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \left[P(X = 0) + P(X = 1)\right]$$

$$= 1 - \left[(0.8)^5 + 5(0.2)(0.8)^4 \right]$$

$$= 1 - [0.3277 + 0.4096]$$

$$= 0.2627.$$

<u>Ex</u> Based on extensive testing, it is determined that the time Y in years before a major repair is required for a certain washing machine is characterized by the density function

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4}, & y \geq 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Note that Y is an exponential random variable with $\mu = 4$ years. The machine is considered a ~~good~~ bargain if it is unlikely to require a major repair before the sixth year. What is the probability $P(Y > 6)$?

What is the probability that a major repair is required in the first year?

<u>Sol</u>:

$$F(y) = \frac{1}{\beta} \int_0^y e^{-t/\beta} \, dt = \frac{1}{\beta} \left[\frac{-e^{-t/\beta}}{1/\beta} \right]_0^y$$

$$= \frac{-\beta}{\beta} \left[e^{-y/\beta} - 1 \right]$$

$$= 1 - e^{-y/\beta}.$$

$$\therefore P(Y > 6) = 1 - P(Y \leq 6)$$

$$= 1 - F(6)$$

$$= 1 - 1 + e^{-6/4}$$

$$= 0.2231.$$

The probability that a major repair is required in the first year is

$$P(Y < 1) = 1 - e^{-1/4} = 1 - 0.7788 = 0.2212.$$

In this case, machine is not a good bargain.

mgf of Gamma Distribution

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{tx - \frac{x}{\beta}} x^{\alpha-1} dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{-\frac{x}{\beta}(1-t\beta)} x^{\alpha-1} dx$$

Let $\frac{x}{\beta}(1-t\beta) = u$

$\frac{1}{\beta}(1-t\beta)dx = du$

$dx = \frac{du}{\frac{1}{\beta}(1-t\beta)}$

$= \frac{\beta . du}{(1-t\beta)}$

when $x=0$, $u=0$

when $x=\infty$, $u=\infty$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{-u} \frac{(\beta u)^{\alpha-1}}{(1-t\beta)^{\alpha-1}} \frac{\beta du}{1-t\beta}$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \frac{\beta^\alpha}{(1-t\beta)^\alpha} \int_0^\infty e^{-u} u^{\alpha-1} du$$

$$= \frac{1}{\Gamma\alpha} (1-t\beta)^{-\alpha} \Gamma\alpha$$

$$\boxed{M_X(t) = (1-t\beta)^{-\alpha}}$$

$$E(X) = \left| \frac{d}{dt}[M_X(t)] \right|_{t=0}$$

$$E(X^2) = \left| \frac{d^2}{dt^2}(M_X(t)) \right|_{t=0}$$

~~E(X)~~ =

$$\frac{d}{dt}(M_X(t)) = (-\alpha)(1-t\beta)^{-\alpha-1}(-\beta) = \alpha\beta(1-t\beta)^{-\alpha-1}$$

$$\frac{d^2}{dt^2}[M_X(t)] = (\alpha\beta)(-\alpha-1)(1-t\beta)^{-\alpha-1-1}(-\beta)$$
$$= \alpha\beta^2(\alpha+1)(1-t\beta)^{-\alpha-2}$$

$$E(X) = \alpha\beta$$

$$E(X^2) = \alpha\beta^2(\alpha+1) = \alpha^2\beta^2 + \alpha\beta^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2.$$

$$\boxed{\begin{array}{l} \text{Mean} = \alpha\beta \\ \text{Variance} = \alpha\beta^2 \end{array}}$$

Remark: Standard Gamma Distribution | One parameter Gamma Distribution

A RV X is said to have a gamma distribution with parameter ~~f(x)=~~ $\alpha > 0$, if its pdf is

$$f(x) = \begin{cases} \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x}, & \alpha > 0,\ x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\boxed{\begin{array}{l} M_X(t) = (1-t)^{-\alpha} \\ \text{Mean} = \alpha \\ \text{Variance} = \alpha \end{array}}$$

mgf of Exponential Distribution

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}\,\frac{1}{\beta}e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\int_0^\infty e^{tx - x/\beta}\,dx = \frac{1}{\beta}\int_0^\infty e^{-\frac{x}{\beta}(1-\beta t)}\,dx$$

$$\cancel{= \frac{1}{\beta}\int_0^\infty e^{\left(t-\frac{1}{\beta}\right)x}\,dx}$$

$$= \frac{1}{\beta}\left[\frac{e^{-\frac{x}{\beta}(1-\beta t)}}{-\frac{1}{\beta}(1-\beta t)}\right]_0^\infty$$

$$= \frac{-\beta}{\beta(1-\beta t)}\left[0-1\right]$$

$$= (1-\beta t)^{-1}$$

$$\boxed{M_X(t) = (1-\beta t)^{-1}}$$

$$\frac{d}{dt}\left(M_X(t)\right) = -1(1-\beta t)^{-2}(-\beta) = \beta(1-\beta t)^{-2}$$

$$\frac{d^2}{dt^2}\left[M_X(t)\right] = -2\beta(1-\beta t)^{-3}(-\beta) = 2\beta^2(1-\beta t)^{-3}$$

$E(X) = \beta$

$E(X^2) = 2\beta^2$

Variance $= 2\beta^2 - \beta^2 = \beta^2.$

$$\boxed{\begin{array}{l}\text{Mean} = \beta \\ \text{Variance} = \beta^2\end{array}}$$