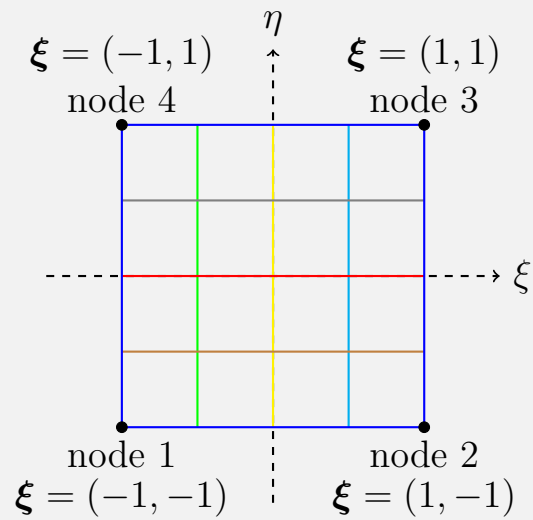


Quadrilateral Quality

C.B. Hovey

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(a) Parametric Space $\boldsymbol{\xi} = (\xi, \eta)$



(b) Physical Space $\boldsymbol{x} = (x, y)$

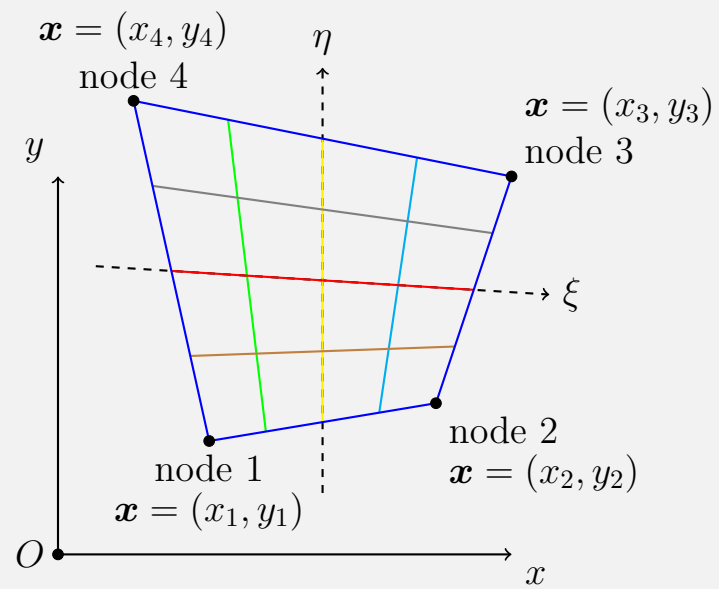


Figure 1: Parametric mapping $\boldsymbol{x} = f(\boldsymbol{\xi})$ from parametric space to physical space.

0.1 Isoparametric Mapping

Let the parametric mapping $f : \boldsymbol{\xi} \mapsto \boldsymbol{x}$ be defined as

$$x(\xi, \eta) = \sum_{a=1}^4 N_a(\xi, \eta) x_a, \quad (1)$$

$$y(\xi, \eta) = \sum_{a=1}^4 N_a(\xi, \eta) y_a, \quad (2)$$

where a nodal **shape function** is defined for each of the four nodes

$$N_1(\xi, \eta) \triangleq \frac{1}{4}(1 - \xi)(1 - \eta), \quad (3)$$

$$N_2(\xi, \eta) \triangleq \frac{1}{4}(1 + \xi)(1 - \eta), \quad (4)$$

$$N_3(\xi, \eta) \triangleq \frac{1}{4}(1 + \xi)(1 + \eta), \quad (5)$$

$$N_4(\xi, \eta) \triangleq \frac{1}{4}(1 - \xi)(1 + \eta). \quad (6)$$

0.2 Jacobian

For the quadrilateral element, the Jacobian \mathbf{J} is calculated as the partial matrix of derivatives of $\mathbf{x} = (x, y)$ with respect to $\boldsymbol{\xi} = (\xi, \eta)$,

$$\mathbf{J}(\xi, \eta) \triangleq \left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right] = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}. \quad (7)$$

Substituting $x(\xi, \eta)$ and $y(\xi, \eta)$ with shape function equations (1)-(2) and expanding terms, the Jacobian takes the form

$$\mathbf{J}(\xi, \eta) = \frac{1}{4} \begin{bmatrix} -1 + \eta & 1 - \eta & 1 + \eta & -1 - \eta \\ -1 + \xi & -1 - \xi & 1 + \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \quad (8)$$

The determinant of the Jacobian, $\det(\mathbf{J})$, can be found to be

$$\det(\mathbf{J}(\xi, \eta)) = c_0 + c_1\xi + c_2\eta, \quad (9)$$

where

$$c_0 = \frac{1}{8} [(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)], \quad (10)$$

$$c_1 = \frac{1}{8} [(x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)], \quad (11)$$

$$c_2 = \frac{1}{8} [(x_2 - x_3)(y_1 - y_4) - (x_1 - x_4)(y_2 - y_3)]. \quad (12)$$

0.3 Quality

We follow *The Verdict Geometry Quality Library* documentation¹ and implementation² for the definitions of quality metrics. The SNL Cubit help manual is also helpful.³

0.3.1 Preliminaries

Let the four edge vectors and their respective lengths be defined as

$$\mathbf{e}_1 \triangleq \mathbf{x}_2 - \mathbf{x}_1, \quad \ell_1 \triangleq \|\mathbf{e}_1\|, \quad (13)$$

$$\mathbf{e}_2 \triangleq \mathbf{x}_3 - \mathbf{x}_2, \quad \ell_2 \triangleq \|\mathbf{e}_2\|, \quad (14)$$

$$\mathbf{e}_3 \triangleq \mathbf{x}_4 - \mathbf{x}_3, \quad \ell_3 \triangleq \|\mathbf{e}_3\|, \quad (15)$$

$$\mathbf{e}_4 \triangleq \mathbf{x}_1 - \mathbf{x}_4, \quad \ell_4 \triangleq \|\mathbf{e}_4\|. \quad (16)$$

The two (non-normalized) *principal axes* are defined through vector addition of the two opposing side lengths

$$\mathbf{X} \triangleq \mathbf{e}_1 - \mathbf{e}_3 = (\mathbf{x}_2 - \mathbf{x}_1) - (\mathbf{x}_4 - \mathbf{x}_3), \quad (17)$$

$$\mathbf{Y} \triangleq \mathbf{e}_2 - \mathbf{e}_4 = (\mathbf{x}_3 - \mathbf{x}_2) - (\mathbf{x}_1 - \mathbf{x}_4). \quad (18)$$

¹Knupp PM, Ernst CD, Thompson DC, Stimpson CJ, Pebay PP. The verdict geometric quality library. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA (United States); 2006 Mar 1. OSTI <https://www.osti.gov/servlets/purl/901967>.

²See <https://github.com/Kitware/VTK/blob/master/ThirdParty/verdict/vtkverdict/> and in particular, the `quad_scaled_jacobian` function in the `V_QuadMetric.cpp` implementation.

³See https://cubit.sandia.gov/files/cubit/16.04/help_manual/WebHelp/cubithelp.htm

At each vertex, there is a normal vector and its respective normalized unit vector

$$\mathbf{N}_1 \triangleq \mathbf{e}_4 \times \mathbf{e}_1, \quad \hat{\mathbf{n}}_1 \triangleq \mathbf{N}_1 / \|\mathbf{N}_1\|, \quad (19)$$

$$\mathbf{N}_2 \triangleq \mathbf{e}_1 \times \mathbf{e}_2, \quad \hat{\mathbf{n}}_2 \triangleq \mathbf{N}_2 / \|\mathbf{N}_2\|, \quad (20)$$

$$\mathbf{N}_3 \triangleq \mathbf{e}_2 \times \mathbf{e}_3, \quad \hat{\mathbf{n}}_3 \triangleq \mathbf{N}_3 / \|\mathbf{N}_3\|, \quad (21)$$

$$\mathbf{N}_4 \triangleq \mathbf{e}_3 \times \mathbf{e}_4, \quad \hat{\mathbf{n}}_4 \triangleq \mathbf{N}_4 / \|\mathbf{N}_4\|. \quad (22)$$

At the center of the element, there is principal axis normal as well

$$\mathbf{N}_c \triangleq \mathbf{X} \times \mathbf{Y}, \quad \hat{\mathbf{n}}_c \triangleq \mathbf{N}_c / \|\mathbf{N}_c\|. \quad (23)$$

There are four contributions to the quadrilateral area from each of the four nodal areas⁴

$$\alpha_1 \triangleq \mathbf{N}_1 \cdot \hat{\mathbf{n}}_c, \quad (24)$$

$$\alpha_2 \triangleq \mathbf{N}_2 \cdot \hat{\mathbf{n}}_c, \quad (25)$$

$$\alpha_3 \triangleq \mathbf{N}_3 \cdot \hat{\mathbf{n}}_c, \quad (26)$$

$$\alpha_4 \triangleq \mathbf{N}_4 \cdot \hat{\mathbf{n}}_c. \quad (27)$$

⁴It may be tempting to (erroneously) write $\hat{\mathbf{n}}_1 \xrightarrow{2D} \hat{\mathbf{n}}_2 \xrightarrow{2D} \hat{\mathbf{n}}_3 \xrightarrow{2D} \hat{\mathbf{n}}_4 \xrightarrow{2D} \hat{\mathbf{n}}_c$ and $\alpha_1 \xrightarrow{2D} \|\mathbf{N}_1\|, \alpha_2 \xrightarrow{2D} \|\mathbf{N}_2\|, \alpha_3 \xrightarrow{2D} \|\mathbf{N}_3\|, \alpha_4 \xrightarrow{2D} \|\mathbf{N}_4\|$ given that for the 2D quadrilateral case, all unit norms are in the same plane. The problem with such a construction is that it destroys the sign information carried by each normal vector. For the non-degenerate case, the foregoing simplification is true, since all give normals will carry the same sign. However, for degenerate cases, such as when a quadrilateral folds over onto itself, the sign information is no longer homogenous, and the sign information *must* be retained to accurately calculate Jacobian metrics that go negative.

0.3.2 Signed Area

The **signed area** SA is defined as the average of all nodal area contributions:

$$\text{SA} \triangleq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}. \quad (28)$$

The metric dimension is L^2 and the idea (unit square) value is 1.0.

0.3.3 Aspect Ratio

The **aspect ratio** AR is defined as the maximum edge length ratios taken at the quadrilateral center.⁵ This can be expressed in terms of the norms of the principal axes as

$$\text{AR} = \max \left(\frac{\| \mathbf{X} \|}{\| \mathbf{Y} \|}, \frac{\| \mathbf{Y} \|}{\| \mathbf{X} \|} \right) \quad (29)$$

Alternatively, the perimeter length multiplied by the maximum side length, divided by four times the area to define a triangle aspect ratio that is meaningful for quadrilaterals,⁶ with dimension L^0 and acceptable range $[1.0, 1.3]$.

⁵Robinson J. CRE method of element testing and the Jacobian shape parameters. Engineering Computations. 1987 Feb 1.

⁶Knupp 2006, *op. cit.* at 38.

0.3.4 Minimum Jacobian

The **Minimum Jacobian** J_{\min} is defined as the minimum pointwise area of local map at the four corners and center of quadrilateral⁷

$$J_{\min} \triangleq \min(\alpha_1, \alpha_2, \alpha_3, \alpha_4). \quad (30)$$

0.3.5 Minimum Scaled Jacobian

The **Minimum Scaled Jacobian** \hat{J}_{\min} is the minimum nodal area divided by the lengths of the two edge vector connecting that point⁸

$$\hat{J}_{\min} \triangleq \min\left(\frac{\alpha_1}{\ell_4 \ell_1}, \frac{\alpha_2}{\ell_1 \ell_2}, \frac{\alpha_3}{\ell_2 \ell_3}, \frac{\alpha_4}{\ell_3 \ell_4}\right), \quad (31)$$

When all normals are in the same plane, the Minimum Scaled Jacobian reduces to

$$\hat{J}_{\min} \xrightarrow{2D} \min(\sin \theta_1, \sin \theta_2, \sin \theta_3, \sin \theta_4), \quad (32)$$

where θ_1 is the angle between \mathbf{e}_4 and \mathbf{e}_1 , θ_2 with \mathbf{e}_1 and \mathbf{e}_2 , θ_3 with \mathbf{e}_2 and \mathbf{e}_3 , and θ_4 with \mathbf{e}_3 and \mathbf{e}_4 .

The dimension is L^0 . The full range is $[-1.0, 1.0]$. The acceptable range is typically taken as $[0.3, 1.0]$ in the generous case and $[0.5, 1.0]$ in the more restricted case.

⁷Knupp 2006, *op. cit.* at 42.

⁸Knupp 2006, *op. cit.* at 51.