B-Spline Geometry

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Contents

Chapter 1

Introduction

1.1 Parameter Space

In Bézier geometry, parameter space t for curves[1] is a real number between zero and unity, inclusive,

$$t \in \mathbb{R} \subset [\,0,\ 1\,]. \tag{1.1}$$

For B-spline geometry, the parameter space is taken as a real number between zero and some number, t_κ, typically larger than unity as seen in the forthcoming discussion. For

[1] This is extended by u for s<u>u</u>rfaces and again by v for <u>v</u>olumes.

now, we say

$$t \in \mathbb{R} \subset [\ t_0,\ t_\kappa\ ]. \tag{1.2}$$

So, the parameter space for Bézier curves will be a special case of the parameter space for B-spline curves when $t_0 = 0$ and $t_\kappa = 1$.

1.2 Knots, Knot Spans, Knot Vectors

Next, we identify discrete, non-decreasing values along this interval $[\ 0,\ t_\kappa\ ]$, and define these values as **knots**. **Knots** decompose the parameter space into sequential intervals, called **knot spans**.[2] The set of $(\kappa + 1)$ knots compose a **knot vector** $\boldsymbol{T}$,

$$\boldsymbol{T} = \langle\ t_0, t_1, t_2, \ldots, t_\kappa\ \rangle. \tag{1.3}$$

Because knots mark the termination points, beginning and end, of knot spans, they impart a measure on the knot span, which is simply the difference between the values at sequential knots, and may be as small as zero, since knot sequence values are non-decreasing. For example, the value of the first knot span is equal to the value $(t_1 - t_0)$. A knot vector with $(\kappa + 1)$ knots has (κ) knot spans.

[2]For curves, a knot in parameter space will get mapped to a point in physical space. For surfaces, a knot will get mapped to a curve. For volumes, a knot will get mapped to a surface. For now, consider only curves with knots.

Remark 1.2.1. Recastability of the Parameter Space
Since the B-Spline domain $[\,0,\ t_\kappa\,]$ is a *parameter* space, it can be recast. Two examples of such as recast are

- **Normalization**: The entire interval can be divided by t_κ, making the new parameter space be $[\,0,\ 1\,]$, which is a recovery of the Bézier parameter space.
- **Offset**: The interval may be shifted up or down by some constant value. Thus, t_0 is not necessarily always zero.

Remark 1.2.2. Unit Knot Span Convention
It is a convention to denote knot values as non-negative *integer* values starting from zero, though they actually have non-negative *real* values. For example, the knot vector $\boldsymbol{T} = \langle\ 0.0,\ 0.5,\ 1.0\ \rangle$ can be equally-well reprented as $\boldsymbol{T} = \langle\ 0,\ 1,\ 2\ \rangle$. Both have three knots but only the latter has a unit knot span. The unit knot span convention is used because it is often convienient to count knots, one by one.

1.3 Uniform Knot Vectors

When all the knot spans of a given knot vector are equal, the knot vector is **uniform**. Otherwise, the knot vector is **non-uniform**. We will begin the discussion with uniform knot vectors because they are the easier of the two variants to develop.

Example 1.
A *uniform* knot vector containing 10 knots might be written as

$$\boldsymbol{T} = \langle\, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \,\rangle, \text{ with } t \in \mathbb{R} \subset [\, 0,\ 9\,]. \tag{1.4}$$

□

Example 2.
A *non-uniform* knot vector containing 10 knots might be written as

$$\boldsymbol{T} = \langle\, 0, 0, 2, 3, 4, 5, 6, 7, 8, 9 \,\rangle, \text{ with } t \in \mathbb{R} \subset [\, 0,\ 9\,]. \tag{1.5}$$

The first two knot spans have a value of zero and two, respectively. The remaining knot spans have a value of one. Thus, the knot vector is non-uniform. Notice also the repeated knot value of zero at the beginning of the knot vector. This is allowed since knots are a non-decreasing sequence. Repeated knots will have a particular significance, as shown later. □

Remark 1.3.3. Connection to Finite Element Analysis (FEA)
Knot spans will also be known as **elements** because we perform numerical quadrature over a knot span in isogeometric analysis (IGA) [Cottrell et al., 2009]. In IGA, the parent (or local or parameterized) element is the knot span. All of the knot spans described by a single knot vector are defined as a **patch**. A patch spans the B-spline parameter space.

In contrast, isoparametric analysis used for FEA has two notions of element: the parent (or local or parameterized) element and the physical (global) element.

1.4 Basis Functions

Let the B-spline normalized basis function of degree p be written N^p. Here, p denotes degree; it is not an exponent. After developing the basis functions, we then use them to construct B-spline curves in Chapter 2. The "B" in B-spline stands for **basis**.

The first normalized basis function is the unit piecewise constant, defined as

$$\text{for } p = 0: \quad N_i^0(t) \triangleq \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Notice that the non-zero interval $(t_i \leq t < t_{i+1})$

- left-hand-side uses $\leq$, but the
- right-hand-side uses $<$ (and not $\leq$).

Example 3.

B-spline constant. Figure 1.1 shows $N_i^0(t)$ from (1.6), the unit piecewise constant basis function (degree $p = 0$), in parametric space, $t \in [\, t_0,\ t_\kappa \,]$, for the uniform knot vector

$$\boldsymbol{T} = \langle\, t_0, t_1, t_2, t_3, t_4, t_5, t_6 \,\rangle, \tag{1.7}$$

$$= \langle\, 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6 \,\rangle. \tag{1.8}$$

□

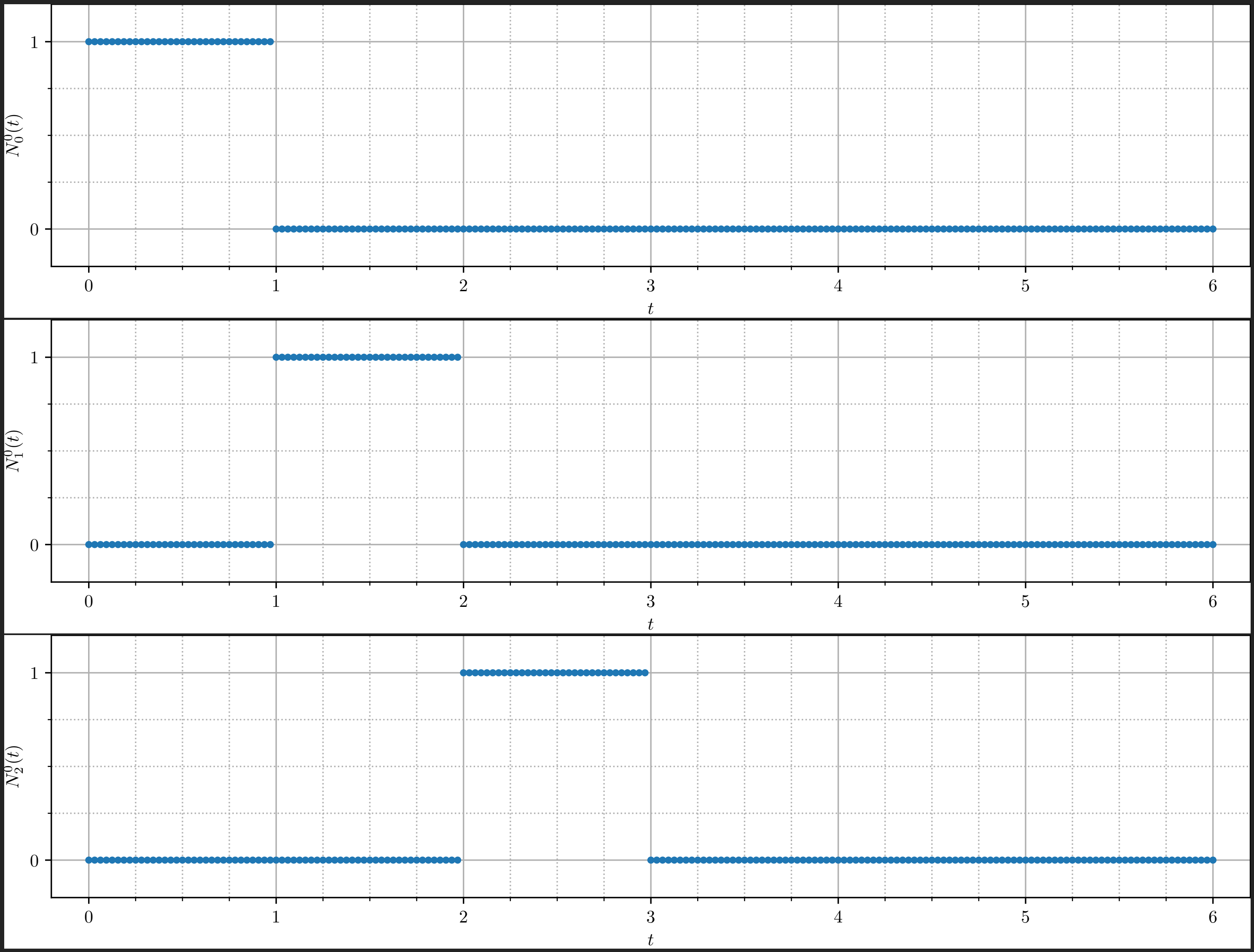

Figure 1.1: B-spline constant ($p = 0$) basis function. Source code on GitHub.

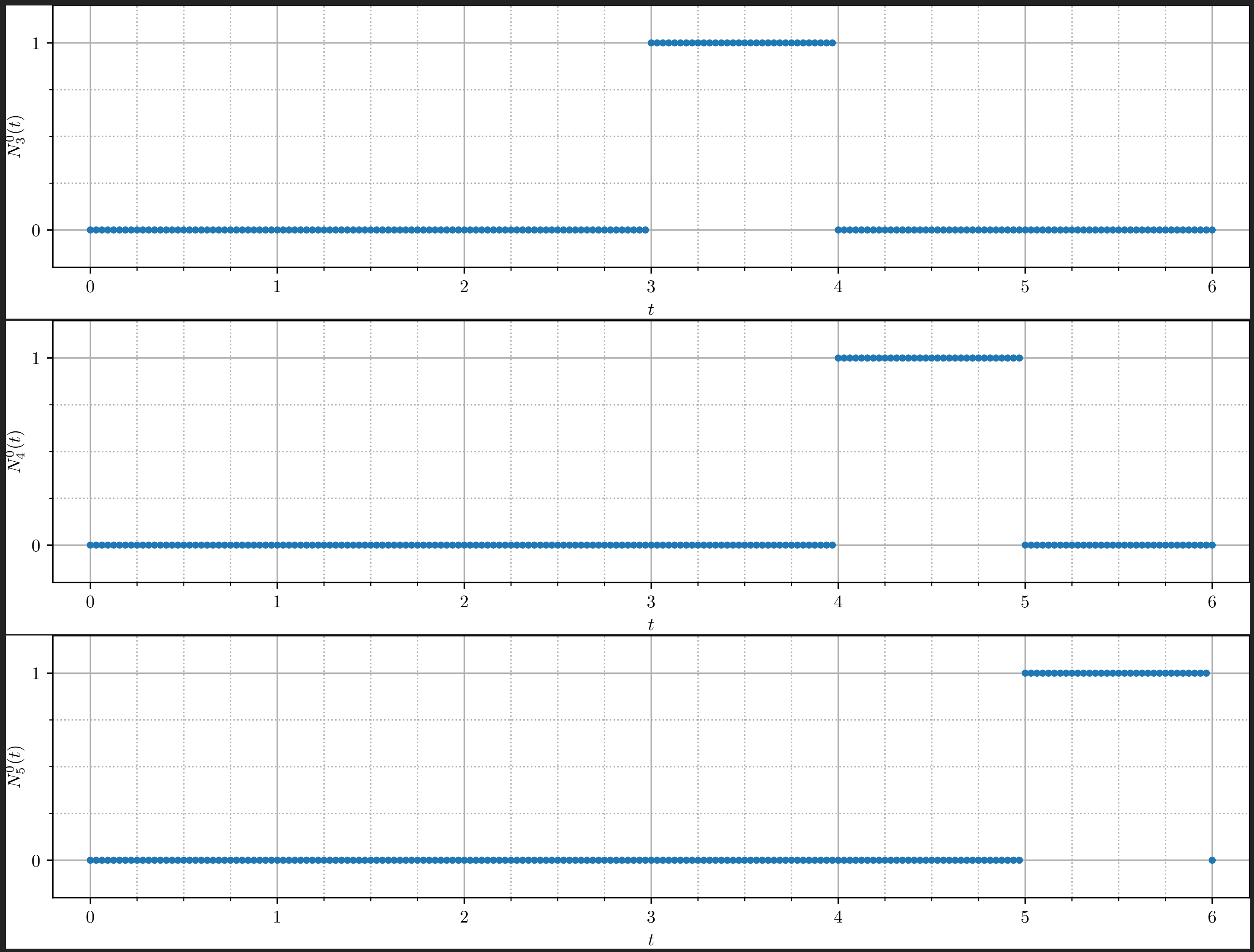

Figure 1.2: Continued from previous figure.

For basis function of degree $p > 0$, *e.g.*, $p = 1, 2, 3, \ldots$, the normalized basis functions are defined by the **Cox-de Boor recursion formula**:

$$\text{for } p \geq 1: \quad N_i^p(t) \triangleq \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1}^{p-1}(t). \tag{1.9}$$

Example 4.

B-spline linear. Using (1.9), the first $(i = 0)$ normalized basis function of degree $(p = 1)$ is

$$N_0^1(t) = \frac{t - t_0}{t_{0+1} - t_0} N_0^{1-1}(t) + \frac{t_{0+1+1} - t}{t_{0+1+1} - t_{0+1}} N_{0+1}^{1-1}(t), \tag{1.10}$$

$$= \frac{t - t_0}{t_1 - t_0} N_0^0(t) + \frac{t_2 - t}{t_2 - t_1} N_1^0(t). \tag{1.11}$$

Review of Figure 1.1 shows $N_0^0(t)$ and $N_1^0(t)$ act as "on" and "off" switches, since

$$N_0^0(t) = \begin{cases} 1 & \text{if } t_0 \leq t < t_1, \\ 0 & \text{otherwise;} \end{cases} \qquad \text{and,} \qquad N_1^0(t) = \begin{cases} 1 & \text{if } t_1 \leq t < t_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

Thus,

$$N_0^1(t) = \begin{cases} (t - t_0) \,/\, (t_1 - t_0) & \text{if } t_0 \leq t < t_1, \\ (t_2 - t) \,/\, (t_2 - t_1) & \text{if } t_1 \leq t < t_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

Figure 1.3 shows the B-spline linear basis functions over the same knot vector used for Figure 1.1. Note, for the given knot vector $\boldsymbol{T} = \langle\, 0, 1, 2, 3, 4, 5, 6 \,\rangle$, there is one fewer complete *linear* basis function than there is complete *constant* basis function. Explained below, this is due to local support, which increases with increasing degree and thus decreases the number of complete basis functions that can exist in the extents of the knot vector. □

This example illustrates the pattern of **local support**. This pattern can be stated as follows:

> A B-spline basis function of degree p
> will have local support over $(p + 1)$ knot spans.

This example shows the a pattern of **periodicity** in the basis functions for $N_i^1(t)$. In general, the unit piecewise linear (degree $p = 1$) basis function at knot t_i can be written as

$$N_i^1(t) = \begin{cases} (t - t_i) \;/\; (t_{i+1} - t_i) & \text{if } t_i \le t < t_{i+1}, \\ (t_{i+2} - t) \;/\; (t_{i+2} - t_{i+1}) & \text{if } t_{i+1} \le t < t_{i+2}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

The periodicity pattern exists for all B-spline basis functions $N_i^p(t)$ of any degree $p \ge 0$.

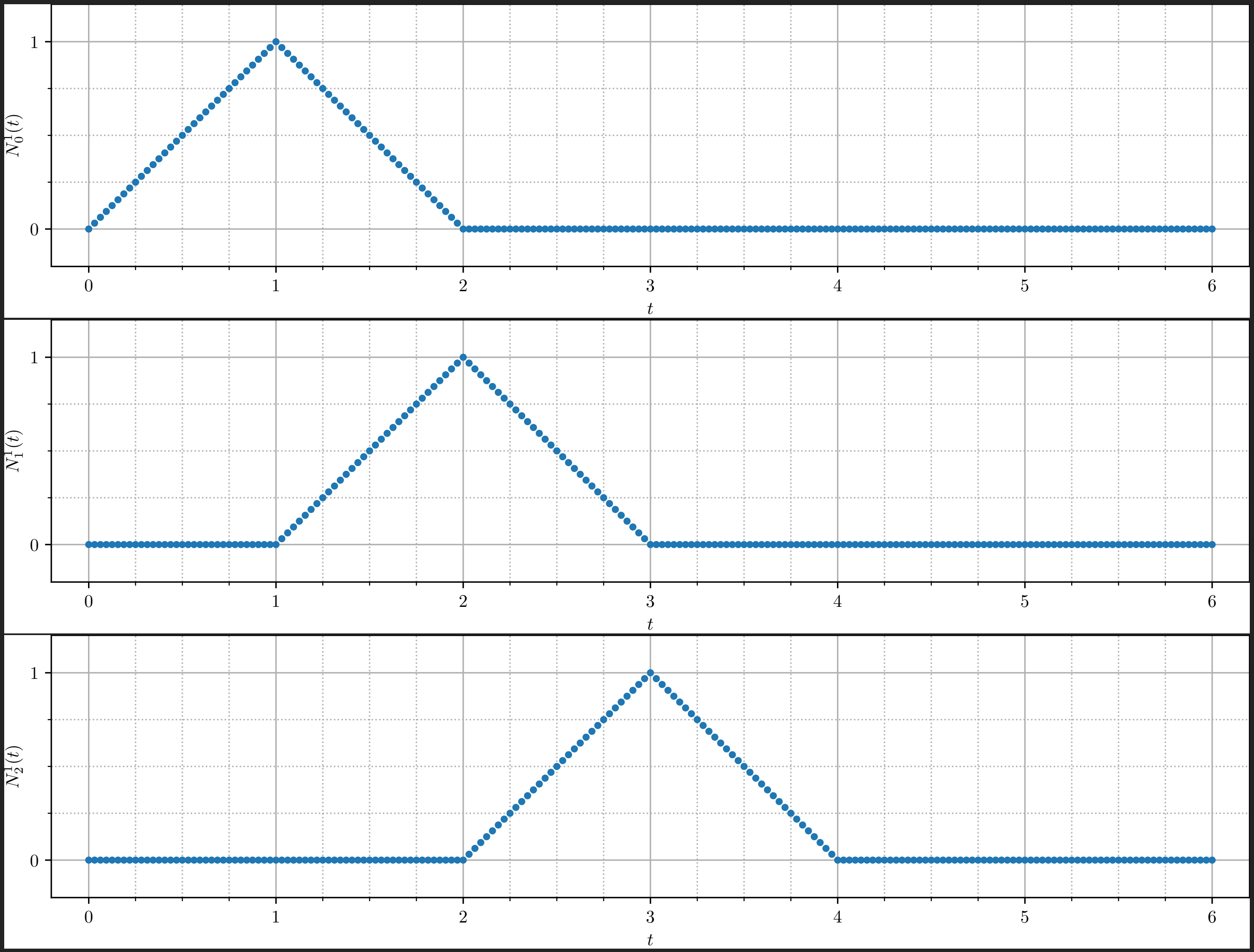

Figure 1.3: B-spline linear ($p = 1$) basis function. Source code on GitHub.

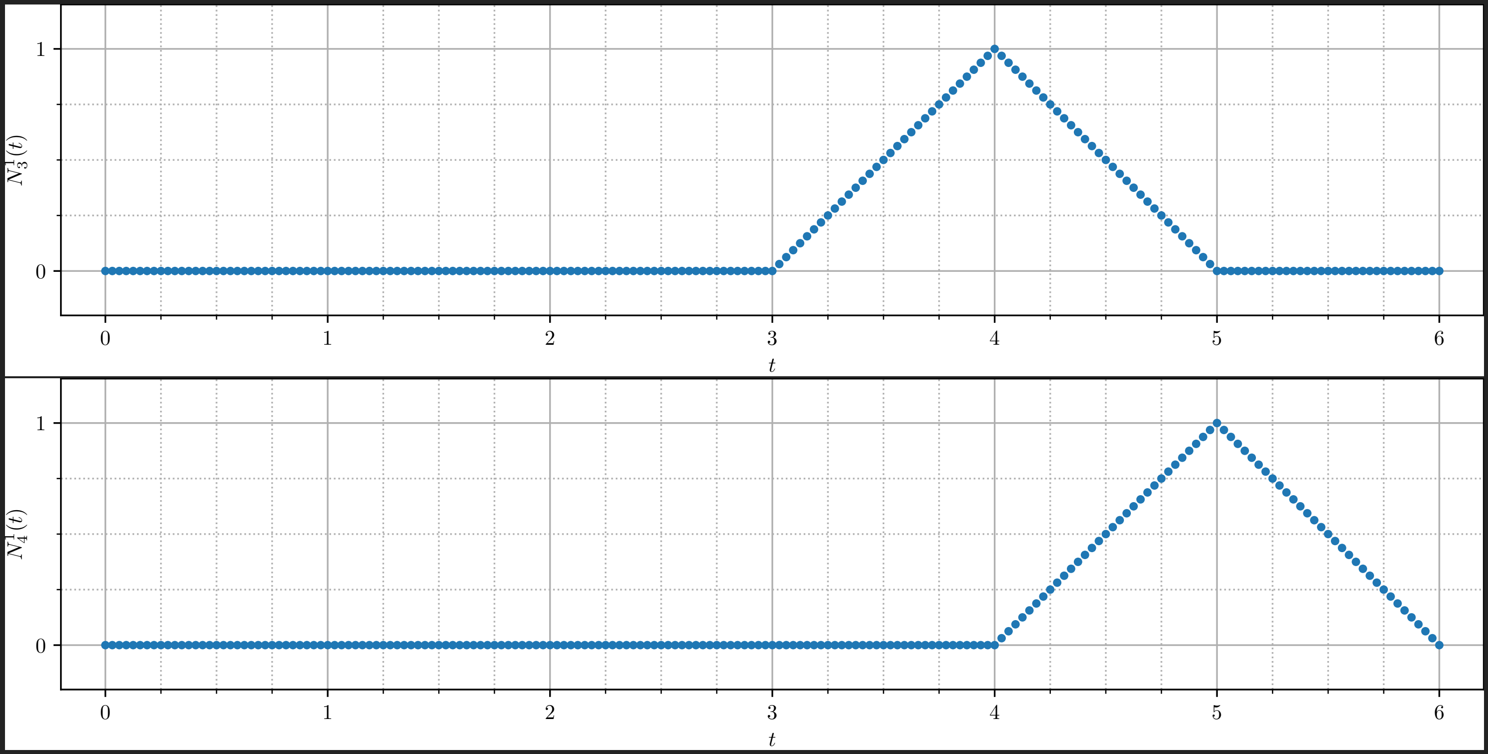

Figure 1.4: Continued from previous figure.

The local support property means that B-spline basis functions of increasing degree require an increasing number of knots to be defined. Increasing the degree of the B-spline basis tends to both increase the duration and decrease the amplitude of the non-zero values of the function. A basis function of degree p also depends on the basis functions of decreasing order, e.g., $(p-1)$, $(p-2)$, and so on. This dependence is defined through the Cox-de Boor relationships.

Figure 1.5 illustrates the Cox-de Boor recursion algorithm, with local support over knot intervals.

$$
\begin{array}{ccccccccc}
t_i & & t_{i+1} & & t_{i+2} & & t_{i+3} & & t_{i+4} \\
\vdots & \iddots & \vdots & \iddots & \vdots & \iddots & \vdots & \iddots & \\
N_i^0 & & N_{i+1}^0 & & N_{i+2}^0 & & N_{i+3}^0 & & \\
\vdots & \iddots & \vdots & \iddots & \vdots & \iddots & & & \\
N_i^1 & & N_{i+1}^1 & & N_{i+2}^1 & & & & \\
\vdots & \iddots & \vdots & \iddots & & & & & \\
N_i^2 & & N_{i+1}^2 & & & & & & \\
\vdots & \iddots & & & & & & & \\
N_i^3 & & & & & & & &
\end{array}
$$

Figure 1.5: Graphical illustration of Cox-de Boor recursion algorithm up to the degree of cubic ($p = 3$).

Figure 1.6 illustrates the Cox-de Boor recursion algorithm, with local support over knot intervals, reimagined with a gridded shape.

$$
\begin{array}{ccccccccccccc}
 & & & t_i & & & & & & & & & \\
\underline{\text{degree}} & \underline{\text{spans}} & \underline{\text{knots}} & \uparrow & & & & & & & & & \\
p=0 & 1 & 2 & N_i^0 & \longrightarrow & t_{i+1} & & & & & & & \\
 & & & \uparrow & & \uparrow & & & & & & & \\
p=1 & 2 & 3 & N_i^1 & \longrightarrow & N_{i+1}^0 & \longrightarrow & t_{i+2} & & & & & \\
 & & & \uparrow & & \uparrow & & \uparrow & & & & & \\
p=2 & 3 & 4 & N_i^2 & \longrightarrow & N_{i+1}^1 & \longrightarrow & N_{i+2}^0 & \longrightarrow & t_{i+3} & & & \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
p=3 & 4 & 5 & N_i^3 & \longrightarrow & N_{i+1}^2 & \longrightarrow & N_{i+2}^1 & \longrightarrow & N_{i+3}^0 & \longrightarrow & t_{i+4} &
\end{array}
$$

Figure 1.6: Graphical illustration of Cox-de Boor recursion algorithm up to the degree of cubic ($p = 3$), with gridded arrangement.

Example 5.

B-spline quadratic. Using (1.9), the i^{th} normalized basis function of degree $(p = 2)$ is

$$N_i^2(t) = \frac{t - t_i}{t_{i+2} - t_i} N_i^1(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} N_{i+1}^1(t), \tag{1.15}$$

$$= \frac{t - t_i}{t_{i+2} - t_i} \left\{ \frac{t - t_i}{t_{i+1} - t_i} N_i^0(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} N_{i+1}^0(t) \right\} +$$

$$\frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} \left\{ \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}} N_{i+1}^0(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+2}} N_{i+2}^0(t). \right\} \tag{1.16}$$

$$N_i^2(t) = \begin{cases} \dfrac{t - t_i}{t_{i+2} - t_i} \cdot \dfrac{t - t_i}{t_{i+1} - t_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots & \text{if } t_i \le t < t_{i+1}, \\[2ex] \dfrac{t - t_i}{t_{i+2} - t_i} \cdot \dfrac{t_{i+2} - t}{t_{i+2} - t_{i+1}} + \dfrac{t_{i+3} - t}{t_{i+3} - t_{i+1}} \cdot \dfrac{t - t_{i+1}}{t_{i+2} - t_{i+1}} & \text{if } t_{i+1} \le t < t_{i+2}, \\[2ex] \dfrac{t_{i+3} - t}{t_{i+3} - t_{i+1}} \cdot \dfrac{t_{i+3} - t}{t_{i+3} - t_{i+2}} \quad \ldots\ldots\ldots\ldots\ldots & \text{if } t_{i+2} \le t < t_{i+3}, \\[2ex] 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{otherwise.} \end{cases} \tag{1.17}$$

□

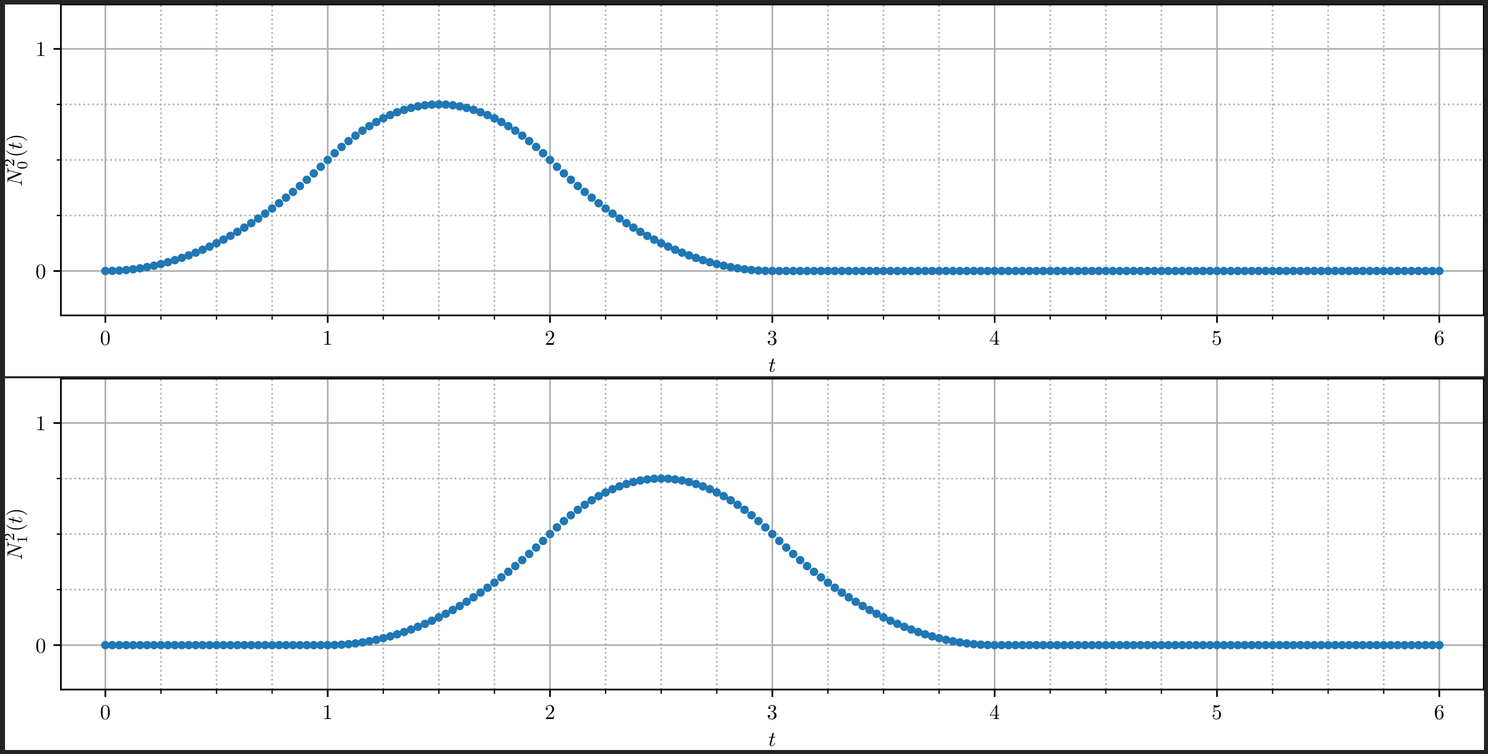

Figure 1.7: B-spline quadratic ($p = 2$) basis function. Source code on GitHub.

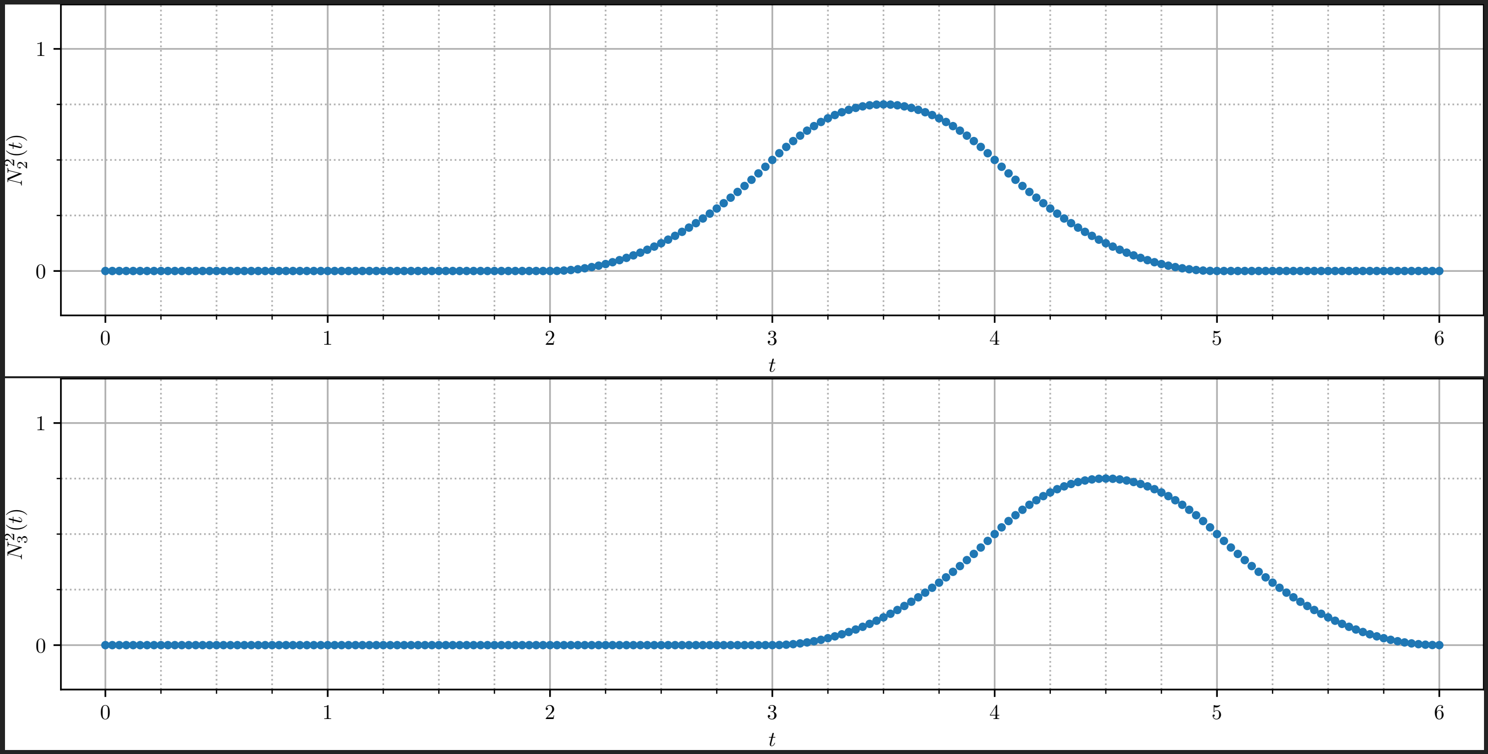

Figure 1.8: Continued from previous figure.

Example 6.

Using the patterns generated in Figure 1.5, construct an expression for $N_0^3(t)$, the first normalized cubic basis function (degree $p = 3$).

$\square$

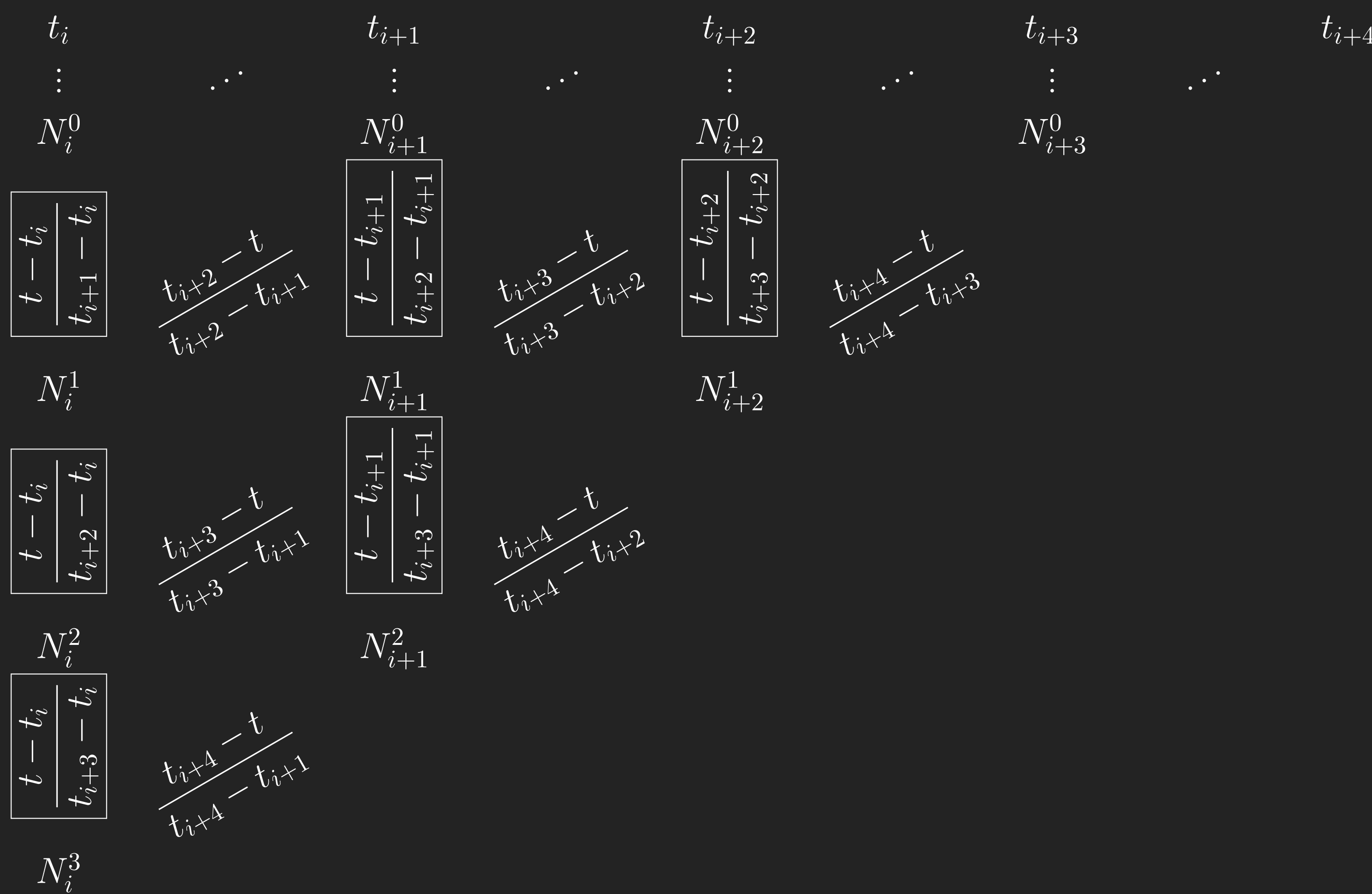

Figure 1.9: Cox-de Boor recursion algorithm up to the degree of cubic ($p = 3$), with factors composed of knot values and parameter t.

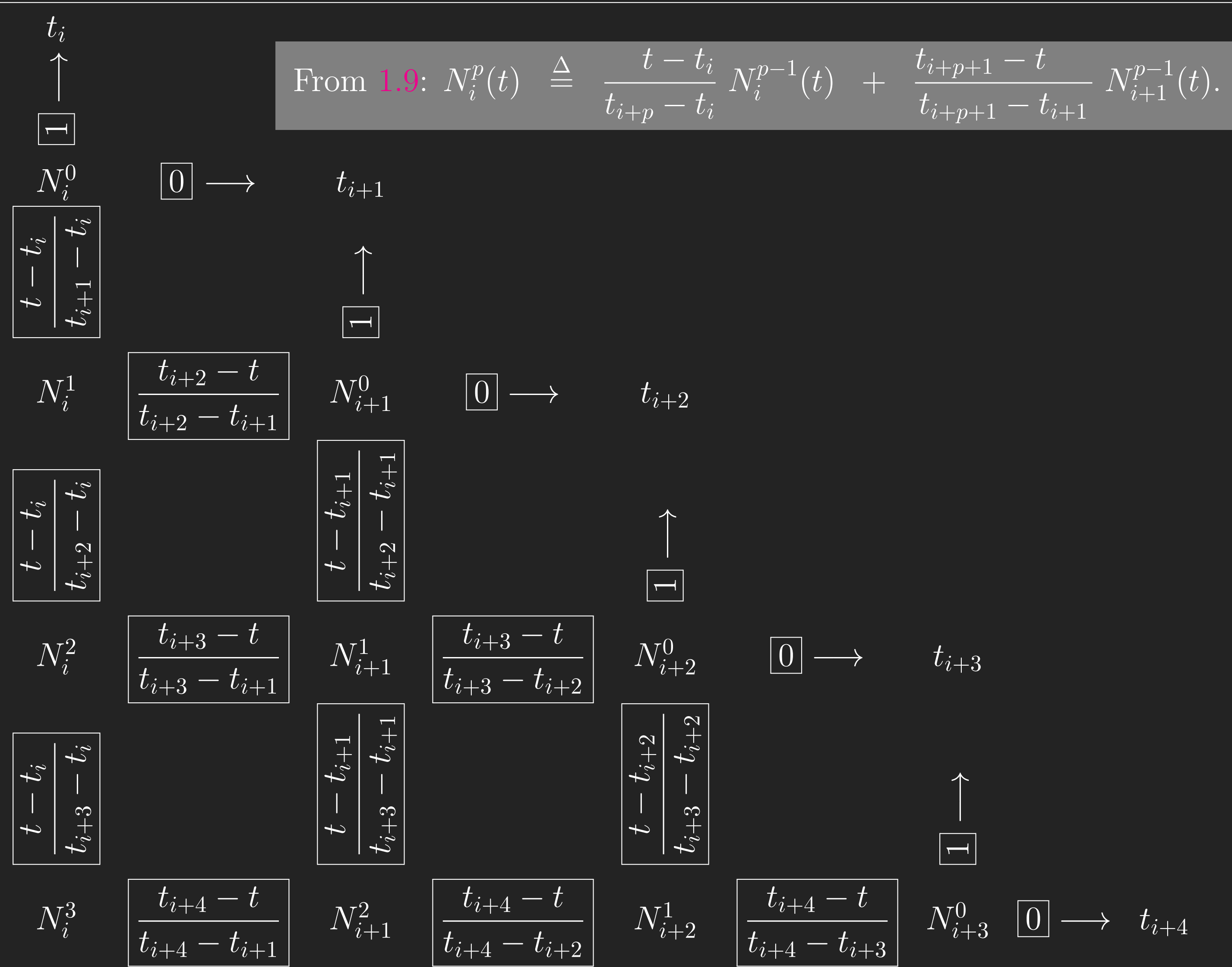

Figure 1.10: Explicit Cox-de Boor illustration up to cubic ($p = 3$).

1.5 Non-Uniform Knot Vectors

The repetition of a knot value in the knot vector causes a knot span to go to zero, which is one way to cause knot vector to change from uniform to non-uninform.[3]

- In Section 1.5.1, we introduce non-uniform knot vectors by reviewing cases where the first and last knots are repeated one or more times.
- In Section 1.5.2, we see how results in the preceeding section can give rise to the Bézier basis functions as a special case of the B-spline basis functions.
- In Section 1.5.3, we examine repeated knots that are repeated in general throughout the knot vector (both at the knot vector endpoints as well as within the knot vector).
- In Section 1.5.4, we generalized the B-spline basis functions further, by allowing for non-uniform (and non-zero) knot spans within the knot vector.

[3] The other way to cause a uniform knot vector to become non-uniform without repeated knot values is to have two or more knot spans with non-equal (and non-zero because repeated knot values are absent) knot interval distance.

1.5.1 Repeated Knot Values at Knot Vector Endpoints

Example 7.

The nine B-spline linear basis functions ($p = 1$) for the knot vector composed of 11 knots $\boldsymbol{T} = \langle\, t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10}\,\rangle = \langle\, 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 8\,\rangle$ produce eight elements (eight non-zero knot spans) as shown in Fig. 1.11. □

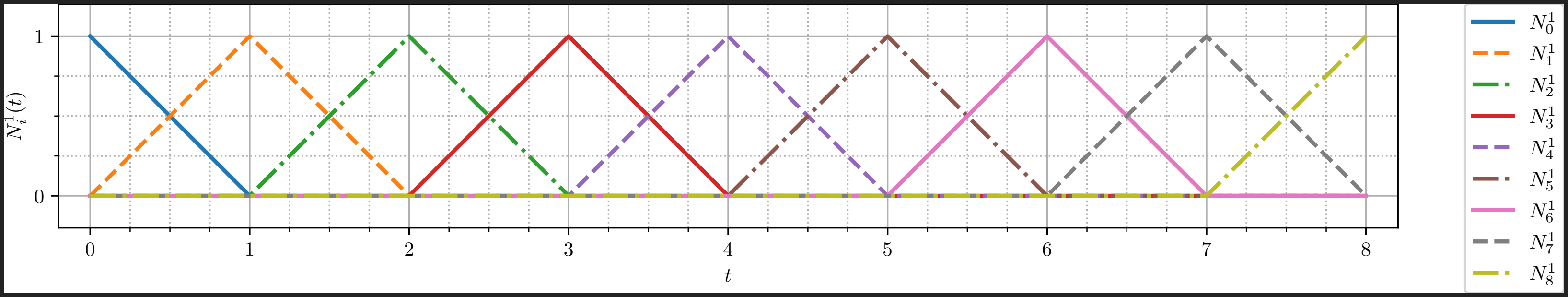

Figure 1.11: Nine B-spline linear basis functions. Source code on GitHub.

Example 8.

The nine B-spline quadratic basis functions ($p = 2$) for the knot vector composed of 12 knots $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10},\ t_{11}\ \rangle = \langle\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 7,\ 7\ \rangle$ produce seven elements (seven non-zero knot spans) as shown in Fig. 1.12. □

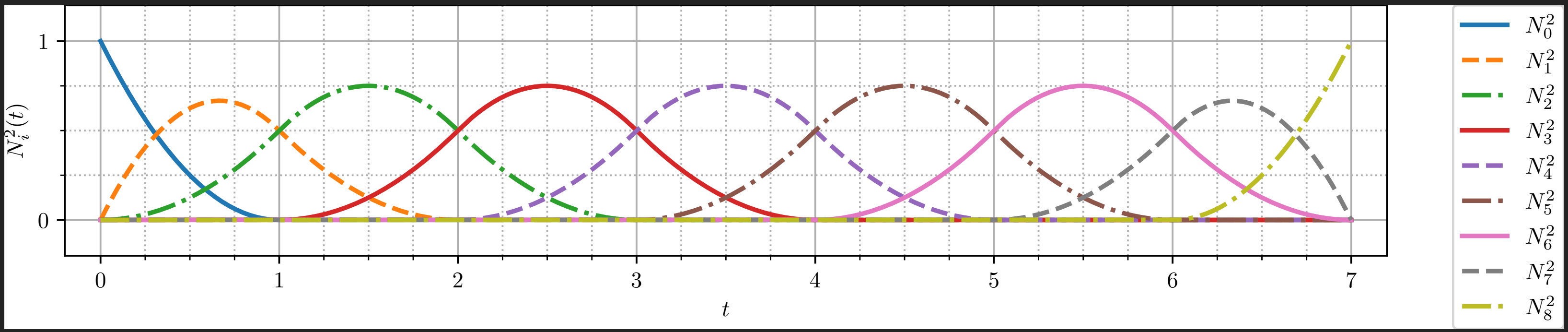

Figure 1.12: Nine B-spline quadratic basis functions. Source code on GitHub.

Example 9.
The nine B-spline cubic basis functions ($p = 3$) for the knot vector composed of 13 knots $\boldsymbol{T} = \langle\, t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10},\ t_{11},\ t_{12}\, \rangle =$ $\langle\, 0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 6,\ 6,\ 6\, \rangle$ produce six elements (six non-zero knot spans) as shown in Fig. 1.13. $\square$

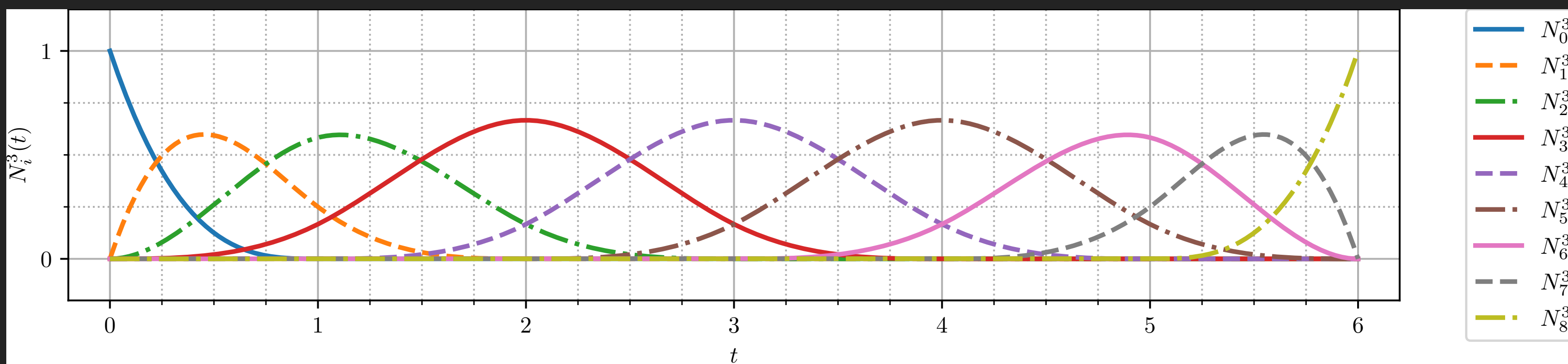

Figure 1.13: Nine B-spline cubic basis functions. Source code on GitHub.

Example 10.
The nine B-spline quartic basis functions ($p = 4$) for the knot vector composed of 14 knots $\boldsymbol{T} = \langle\, t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10},\ t_{11},\ t_{12},\ t_{13}\,\rangle =$ $\langle\, 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5\,\rangle$ produce five elements (five non-zero knot spans) as shown in Fig. 1.14. $\square$

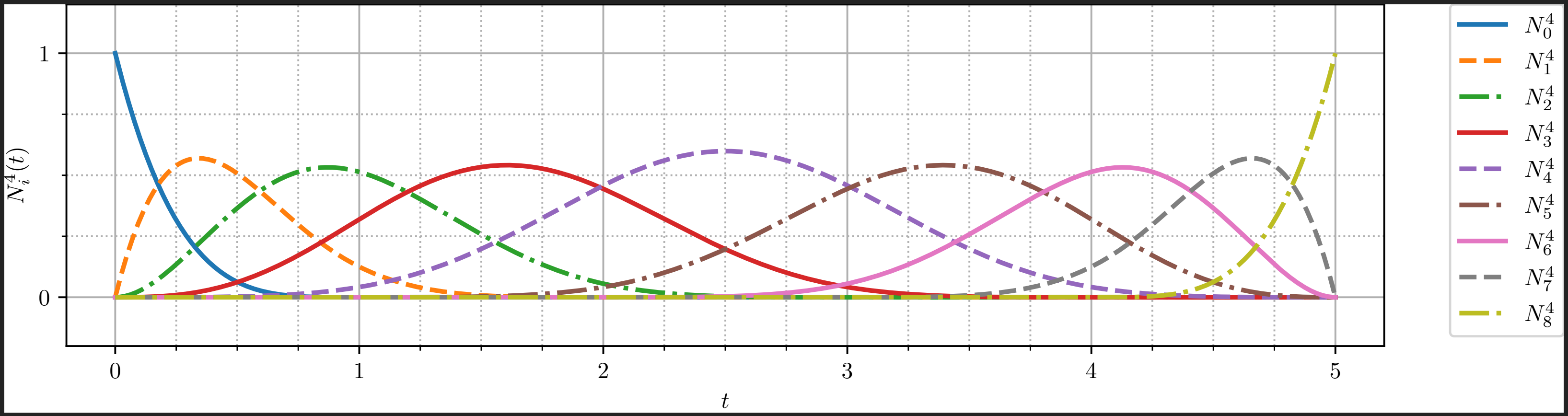

Figure 1.14: Nine B-spline quartic basis functions. Source code on GitHub.

1.5.2 Recovery of Bézier Basis Functions

Example 11.

Recovery of **Bézier linear** basis as a special case (Fig. 1.15).

The two Bézier linear basis functions B_i^1, $i \in [\ 0,\ 1\ ]$, are obtained as a special case of the B-spline linear basis functions N_i^1, $i \in [\ 0,\ 1\ ]$ when the knot vector $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3\ \rangle = \langle\ 0,\ 0,\ 1,\ 1\ \rangle$. $\square$

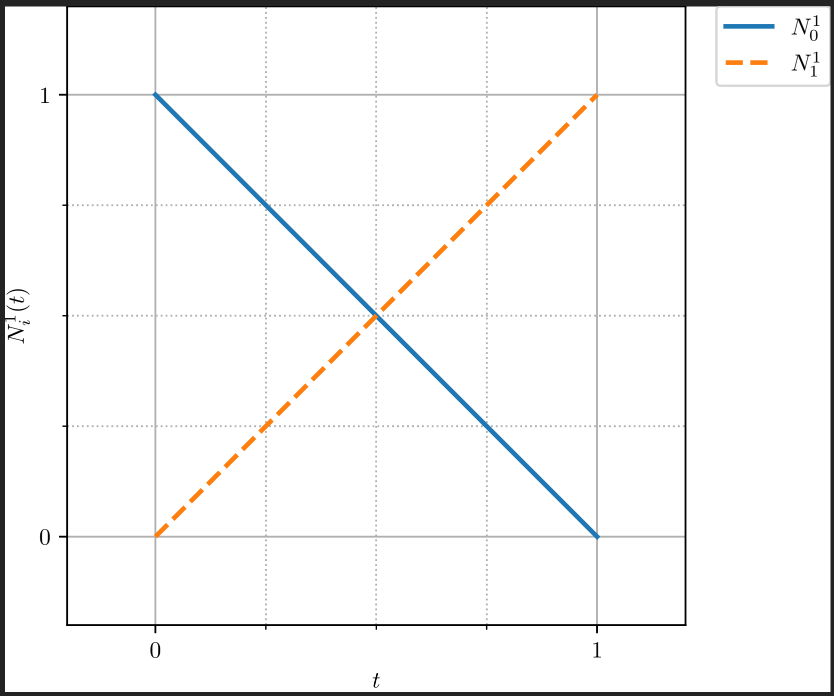

Figure 1.15: Recovery of Bézier linear basis functions from B-spline linear basis functions. Source code on GitHub.

Example 12.
Recovery of **Bézier quadratic** basis as a special case (Fig. 1.16).
The three Bézier quadratic basis functions B_i^2, $i \in [\ 0,\ 1,\ 2\ ]$, are obtained as a special case of the B-spline quadratic basis functions N_i^2, $i \in [\ 0,\ 1,\ 2\ ]$ when the knot vector $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5\ \rangle = \langle\ 0,\ 0,\ 0,\ 1,\ 1,\ 1\ \rangle$. $\square$

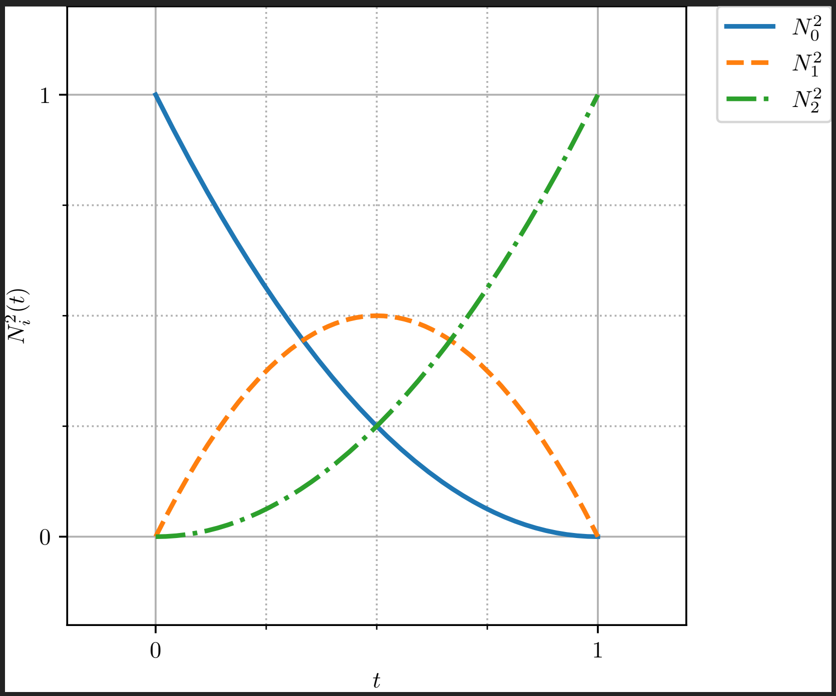

Figure 1.16: Recovery of Bézier quadratic basis functions from B-spline quadratic basis functions. Source code on GitHub.

Example 13.

Recovery of **Bézier cubic** basis as a special case (Fig. 1.17).

The four Bézier cubic basis functions B_i^3, $i \in [\ 0,\ 1,\ 2,\ 3\ ]$, are obtained as a special case of the B-spline cubic basis functions N_i^3, $i \in [\ 0,\ 1,\ 2,\ 3\ ]$ when the knot vector $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7\ \rangle = \langle\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1\ \rangle$. $\square$

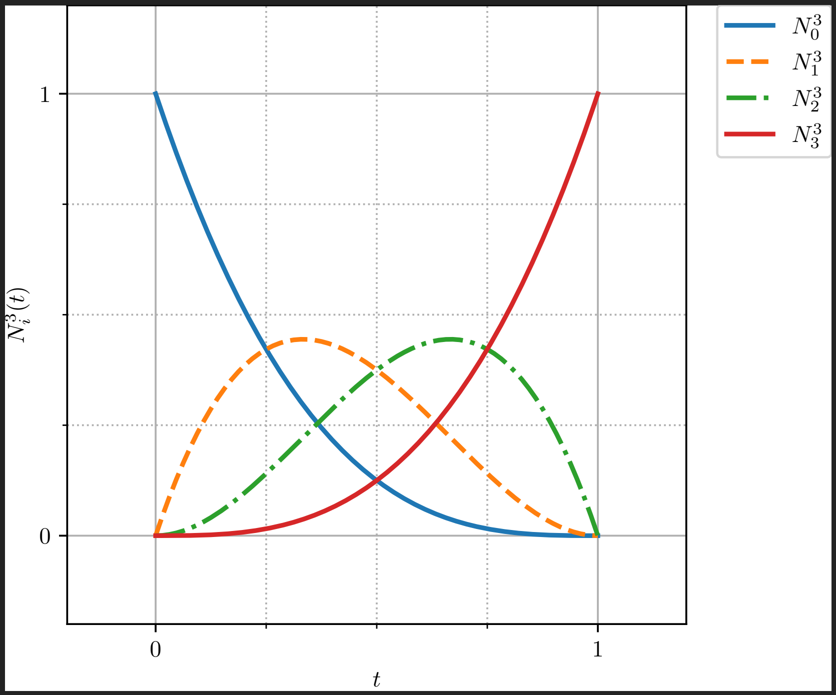

Figure 1.17: Recovery of Bézier cubic basis functions from B-spline cubic basis functions. Source code on GitHub.

Example 14.

Recovery of **Bézier quartic** basis as a special case (Fig. 1.18).

The five Bézier quartic basis functions B_i^4, $i \in [\ 0,\ 1,\ 2,\ 3,\ 4\ ]$, are obtained as a special case of the B-spline quartic basis functions N_i^4, $i \in [\ 0,\ 1,\ 2,\ 3,\ 4\ ]$ when the knot vector $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9\ \rangle = \langle\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1\ \rangle$. $\square$

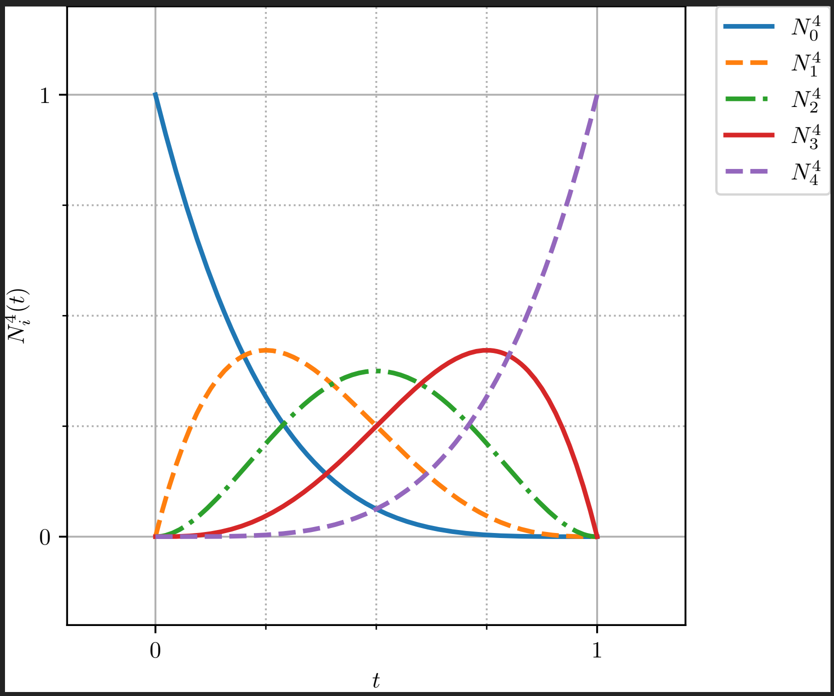

Figure 1.18: Recovery of Bézier quartic basis functions from B-spline quartic basis functions. Source code on GitHub.

Remark 1.5.4. Local Support
One important distinction: normalized basis functions of B-splines have local support; whereas, Bernstein polynomial basis of Béziers do *not* have local support.

For the B-spline basis, a single basis function is zero except for the spans over which it is defined as non-zero. Moving a knot, accomplished by changing its value, will modify only the bases that use that particular knot in a non-zero sense; all other bases remain unchanged. This is easy to conceptualize through study of Figure 1.1, were a single knot value increased or decreased.

In contrast, for the Bernstein polynomials, contributions from each basis function span the entire parameter domain. Bernstein polynomials provide global support, not local support.

Local support will be shown to be advantageous because a local modification to the curves, surfaces, and volumes created by B-splines will not alter the entire geometry; it only causes changes locally.

1.5.3 Repeated Knot Values In General

Example 15.
Reproduction of [Cottrell et al., 2009] Figure 2.5 (and [Piegl and Tiller, 1997] Figure 2.6): The eight B-spline quadratic basis functions ($p = 2$) for the knot vector composed of 11 knots $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10}\ \rangle = \langle\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 4,\ 5,\ 5,\ 5\ \rangle$ produce five elements (five non-zero knot spans) as shown in Fig. 1.19. $\square$

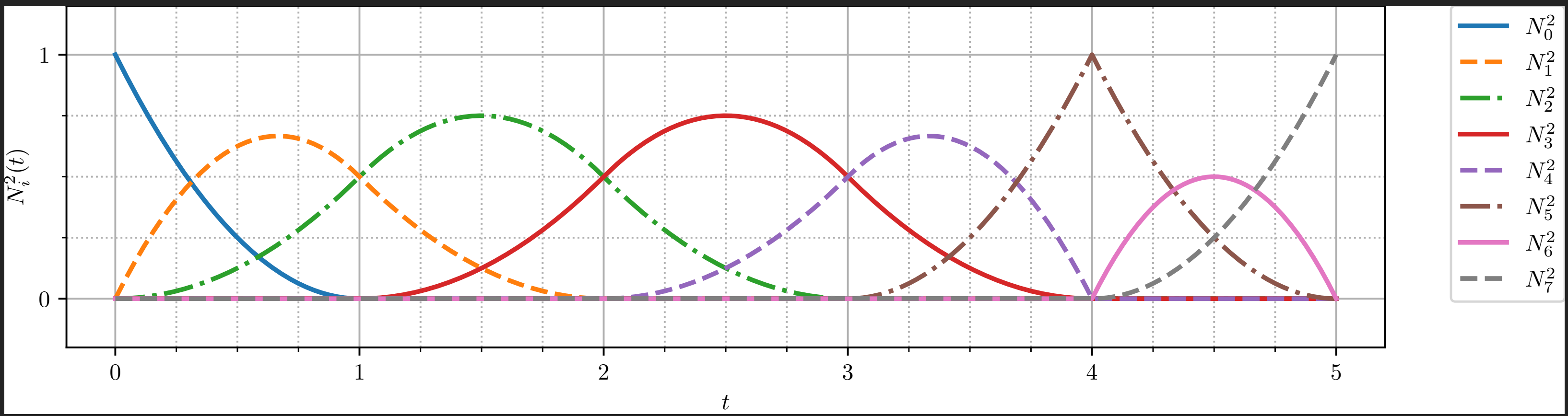

Figure 1.19: Reproduction of [Cottrell et al., 2009] Figure 2.5. Source code on GitHub.

Example 16.
Reproduction of [Cottrell et al., 2009] Figure 2.6:
The 15 B-spline quartic basis functions ($p = 4$) for the knot vector composed of 20 knots $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10},\ t_{11},\ t_{12},\ t_{13},\ t_{14},\ t_{15},\ t_{16},\ t_{17},\ t_{18},\ t_{19}\ \rangle$ $= \langle\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5\ \rangle$ produce five elements (five non-zero knot spans) as shown in Fig. 1.20. $\square$

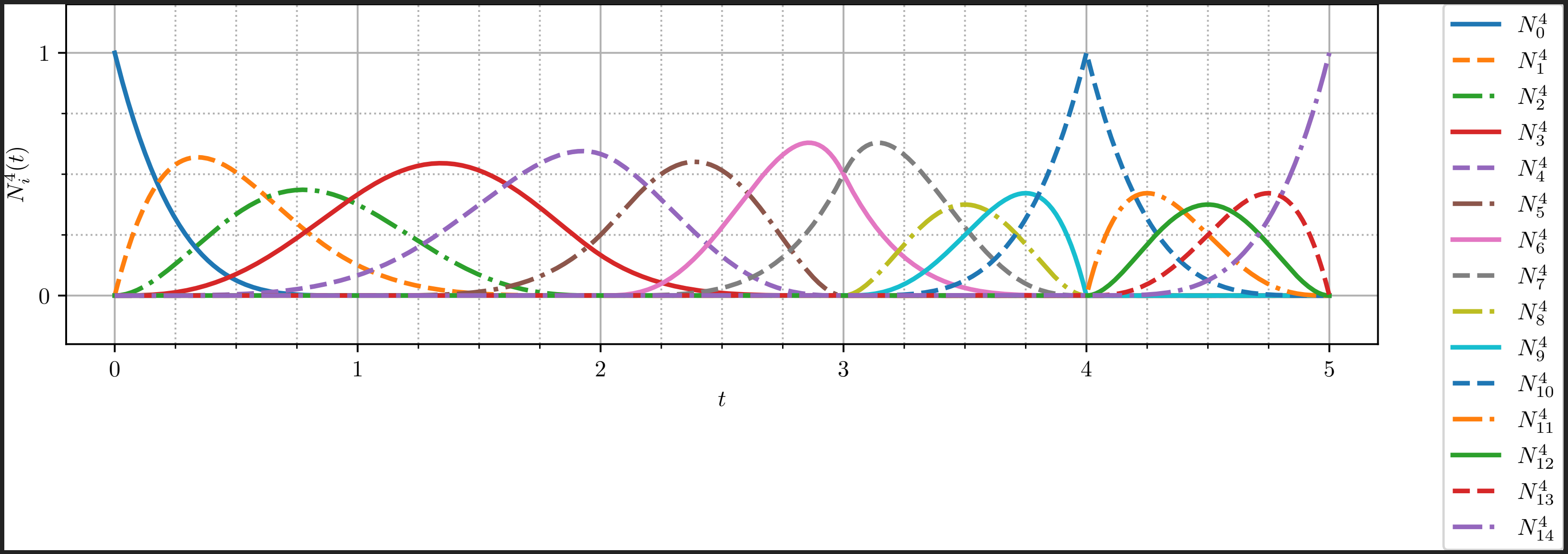

Figure 1.20: Reproduction of [Cottrell et al., 2009] Figure 2.6. Source code on GitHub.

1.5.4 Repeated Knot Values and Non-Zero, Non-Uniform Knot Spans

Example 17.

Reproduction of [Piegl and Tiller, 1997] Figure 2.12:

The seven B-spline cubic basis functions ($p = 3$) for the knot vector composed of 11 knots $\boldsymbol{T} = \langle\, t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10}\,\rangle = \langle\, 0,\ 0,\ 0,\ 0,\ 1,\ 5,\ 6,\ 8,\ 8,\ 8,\ 8\,\rangle$ produce four elements (four non-zero knot spans) as shown in Fig. 1.21. $\square$

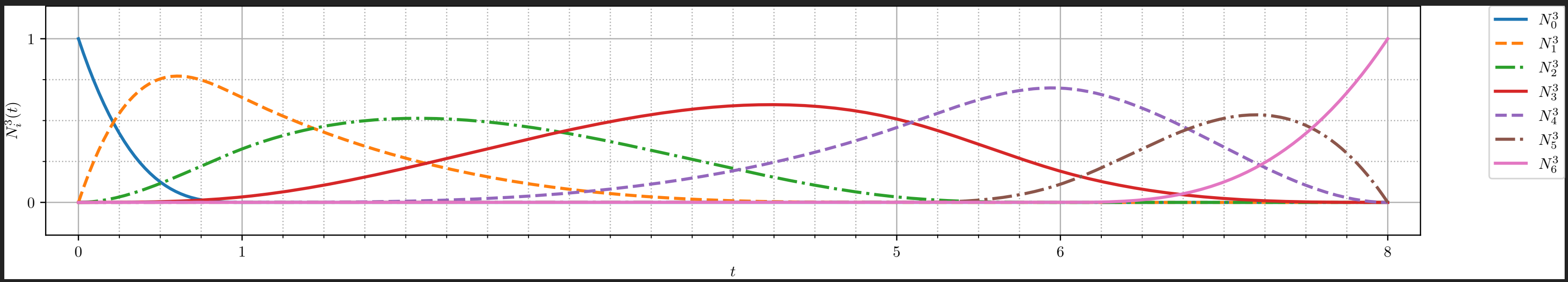

Figure 1.21: Reproduction of [Piegl and Tiller, 1997] Figure 2.12. Source code on GitHub.

Chapter 2

B-Spline Curves

The general form of a degree p ($p \geq 0$) B-spline curve $\mathbb{C}^p(t)$ defined by $(n+1)$ control points $\boldsymbol{P}_i$, $i = 0, 1, \ldots, n$, is given by

$$\mathbb{C}^p(t) \triangleq \sum_{i=0}^{n} N_i^p(t)\boldsymbol{P}_i, \qquad \text{for } t \in \mathbb{R} \subset [\, t_0,\ t_\kappa \,], \tag{2.1}$$

where $N_i^p(t)$ is a **B-spline basis function** of degree p, defined in (1.6) and (1.9), and t is a real number parameter bounded by the endpoints of the knot vector (see (1.2) and (1.3)).

A B-spline basis function of degree p
with $(n+1)$ control points will require
(κ) knot spans and thus $(\kappa+1)$ knots, where
$\kappa = p + n + 1.$

Equivalently,

$$\underbrace{(\ \kappa+1\ )}_{\#\text{ knots}} = \underbrace{(\ p+1\ )}_{\text{degree }+\ 1} + \underbrace{(\ n+1\ )}_{\#\text{ control points}}$$

Table 2.1: Requirements for number of knot spans, given a B-spline of degree p, up to cubic ($p = 3$), and number of control points $(n+1)$.

degree	**control points**	$(n+1)$	**knot spans** κ	**knot vector**	**parameter span**
$p=0$	$\boldsymbol{P}_0$	1	1	$\langle\ t_0, t_1\ \rangle$	$t \in [t_0, t_1)$
$p=1$	$\boldsymbol{P}_0, \boldsymbol{P}_1$	2	3	$\langle\ t_0, t_1, t_2, t_3\ \rangle$	$t \in [t_0, t_3)$
$p=2$	$\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2$	3	5	$\langle\ t_0, t_1, t_2, t_3, t_4, t_5\ \rangle$	$t \in [t_0, t_5)$
$p=3$	$\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3$	4	7	$\langle\ t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\ \rangle$	$t \in [t_0, t_7)$

Example 18.

Reproduction of [Piegl and Tiller, 1997] Figure 3.1:

A cubic $(p = 3)$ Bézier curve captured as a special case of a cubic B-spline curve (see basis functions and knot vector 1.17) with knot vector composed of eight knots $\boldsymbol{T} = \langle\, t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7 \,\rangle = \langle\, 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1 \,\rangle$, a single element (one non-zero knot span), and four control points $\{\boldsymbol{P}_i\}_{i=0}^{n} = [[0,0],[3,8],[10.5,9.5],[15,0]]$ to create the 2D B-spline curve in Fig. 2.1. $\square$

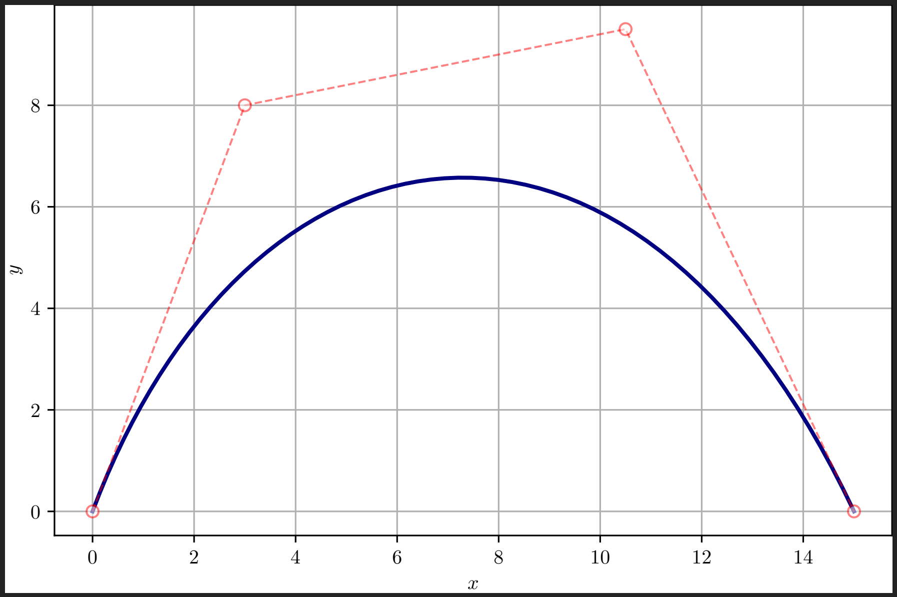

Figure 2.1: Reproduction of [Piegl and Tiller, 1997] Figure 3.1. Source code on GitHub.

Example 19.
Reproduction of [Piegl and Tiller, 1997] Figure 3.2:
A cubic $(p = 3)$ B-spline curve with knot vector composed of 11 knots $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10}\ \rangle = \langle\ 0,\ 0,\ 0,\ 0,\ 0.25,\ 0.50,\ 0.75,\ 1,\ 1,\ 1,\ 1\ \rangle$, produce four elements (four non-zero knot spans), and seven control points $\langle \boldsymbol{P}_i \rangle_{i=0}^{n} = [[-14, 0], [0, 0], [0, 13], [15, 13], [20, -1.5], [9, -10], [0, -5]]$ to create the 2D B-spline curve in Fig. 2.2. □

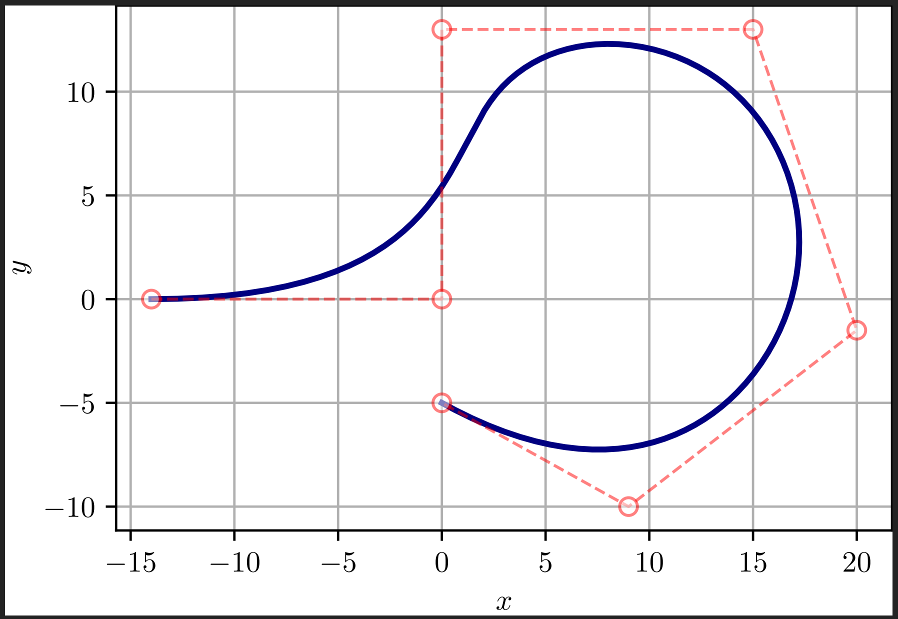

Figure 2.2: Reproduction of [Piegl and Tiller, 1997] Figure 3.2. Source code on GitHub.

Example 20.
Reproduction of [Cottrell et al., 2009] Figure 2.20 (left):
The eight B-spline quadratic basis functions ($p = 2$) for the knot vector composed of 11 knots $\boldsymbol{T} = \langle\ t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9,\ t_{10}\ \rangle = \langle\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4,\ 4,\ 5,\ 5,\ 5\ \rangle$ produce five elements (five non-zero knot spans) which as shown in Fig. 1.19. These basis function are used with the eight control points $\langle \boldsymbol{P}_i \rangle_{i=0}^{n} = [[0, 1], [1, 0], [2, 0], [2, 2], [4, 2], [5, 4], [2, 5], [1, 3]]$ to create the 2D B-spline curve in Fig. 2.3. $\square$

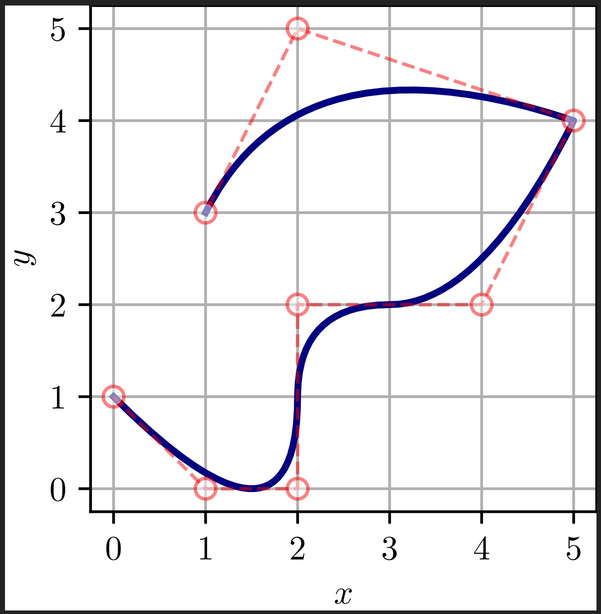

Figure 2.3: Reproduction of [Cottrell et al., 2009] Figure 2.20 (left). Source code on GitHub.

Chapter 3

B-Spline Surfaces

Chapter 4

Acknowledgements

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