Quadrilateral Quality

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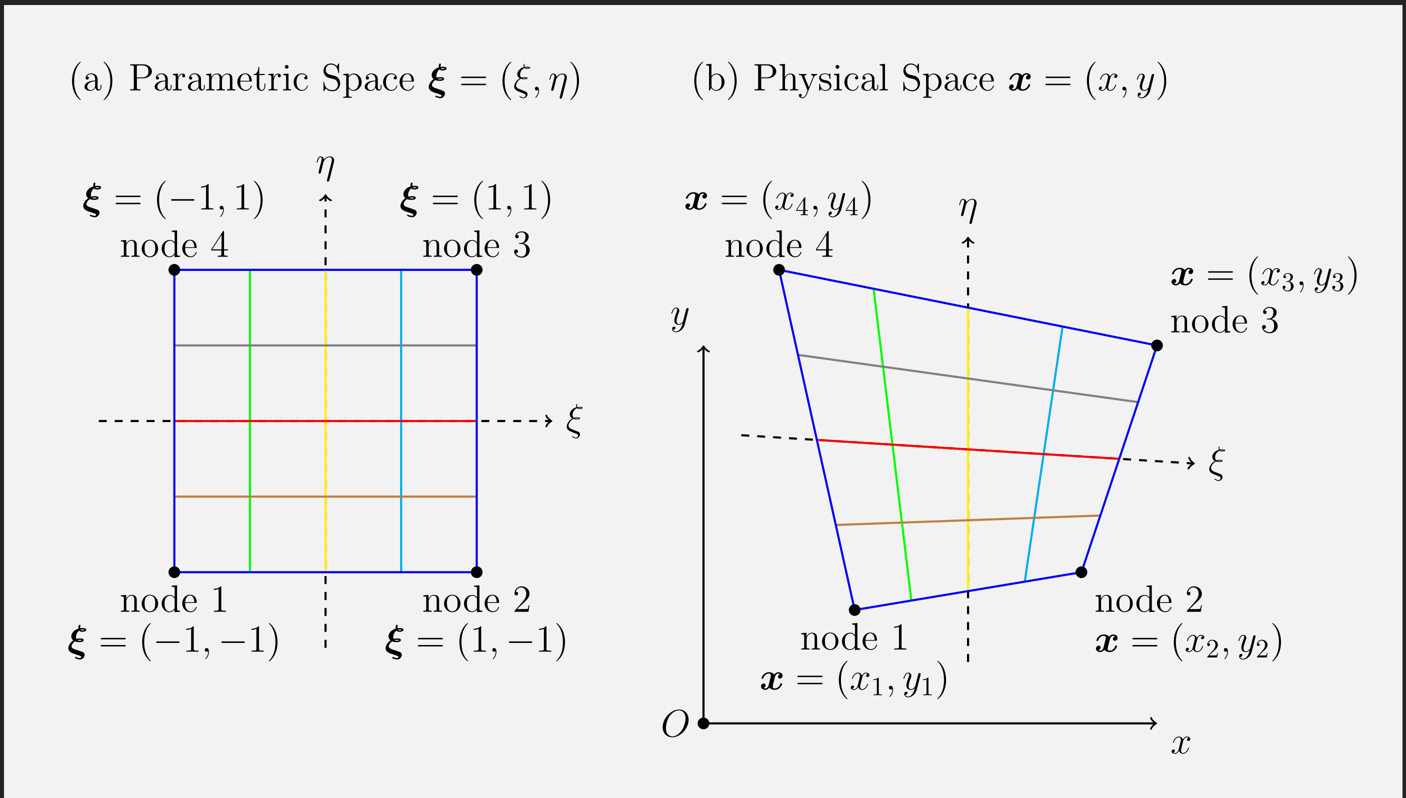

Figure 1: Parametric mapping $\boldsymbol{x} = f(\boldsymbol{\xi})$ from parametric space to physical space.

0.1 Isoparametric Mapping

Let the parametric mapping $f : \boldsymbol{\xi} \mapsto \boldsymbol{x}$ be defined as

$$x(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\, x_a, \tag{1}$$

$$y(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\, y_a, \tag{2}$$

where a nodal **shape function** is defined for each of the four nodes

$$N_1(\xi, \eta) \triangleq \tfrac{1}{4}(1 - \xi)(1 - \eta), \tag{3}$$

$$N_2(\xi, \eta) \triangleq \tfrac{1}{4}(1 + \xi)(1 - \eta), \tag{4}$$

$$N_3(\xi, \eta) \triangleq \tfrac{1}{4}(1 + \xi)(1 + \eta), \tag{5}$$

$$N_4(\xi, \eta) \triangleq \tfrac{1}{4}(1 - \xi)(1 + \eta). \tag{6}$$

0.2 Jacobian

For the quadrilateral element, the Jacobian $\boldsymbol{J}$ is calculated as the partial matrix of derivatives of $\boldsymbol{x} = (x, y)$ with respect to $\boldsymbol{\xi} = (\xi, \eta)$,

$$\boldsymbol{J}(\xi, \eta) \triangleq \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right] = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}. \tag{7}$$

Substituting $x(\xi, \eta)$ and $y(\xi, \eta)$ with shape function equations (1)-(2) and expanding terms, the Jacobian takes the form

$$\boldsymbol{J}(\xi, \eta) = \frac{1}{4} \begin{bmatrix} -1+\eta & 1-\eta & 1+\eta & -1-\eta \\ -1+\xi & -1-\xi & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{8}$$

The determinant of the Jacobian, $\det(\boldsymbol{J})$, can be found to be

$$\det(\boldsymbol{J}(\xi, \eta)) = c_0 + c_1\xi + c_2\eta, \tag{9}$$

where

$$c_0 = \frac{1}{8}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right], \tag{10}$$

$$c_1 = \frac{1}{8}\left[(x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)\right], \tag{11}$$

$$c_2 = \frac{1}{8}\left[(x_2 - x_3)(y_1 - y_4) - (x_1 - x_4)(y_2 - y_3)\right]. \tag{12}$$

0.3 Quality

We follow *The Verdict Geometry Quality Library* documentation[1] and implementation[2] for the definitions of quality metrics. The SNL Cubit help manual is also helpful.[3]

0.3.1 Preliminaries

Let the four edge vectors and their respective lengths be defined as

$$\boldsymbol{e}_1 \triangleq \boldsymbol{x}_2 - \boldsymbol{x}_1, \qquad \ell_1 \triangleq \| \ \boldsymbol{e}_1 \ \|, \tag{13}$$

$$\boldsymbol{e}_2 \triangleq \boldsymbol{x}_3 - \boldsymbol{x}_2, \qquad \ell_2 \triangleq \| \ \boldsymbol{e}_2 \ \|, \tag{14}$$

$$\boldsymbol{e}_3 \triangleq \boldsymbol{x}_4 - \boldsymbol{x}_3, \qquad \ell_3 \triangleq \| \ \boldsymbol{e}_3 \ \|, \tag{15}$$

$$\boldsymbol{e}_4 \triangleq \boldsymbol{x}_1 - \boldsymbol{x}_4, \qquad \ell_4 \triangleq \| \ \boldsymbol{e}_4 \ \| \, . \tag{16}$$

The two (non-normalized) *principal axes* are the defined though vector addition of the two opposing side lengths

$$\boldsymbol{X} \triangleq \boldsymbol{e}_1 - \boldsymbol{e}_3 = (\boldsymbol{x}_2 - \boldsymbol{x}_1) - (\boldsymbol{x}_4 - \boldsymbol{x}_3), \tag{17}$$

$$\boldsymbol{Y} \triangleq \boldsymbol{e}_2 - \boldsymbol{e}_4 = (\boldsymbol{x}_3 - \boldsymbol{x}_2) - (\boldsymbol{x}_1 - \boldsymbol{x}_4). \tag{18}$$

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[1] Knupp PM, Ernst CD, Thompson DC, Stimpson CJ, Pebay PP. The verdict geometric quality library. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA (United States); 2006 Mar 1. OSTI https://www.osti.gov/servlets/purl/901967.

[2] See https://github.com/Kitware/VTK/blob/master/ThirdParty/verdict/vtkverdict/ and in particular, the `quad_scaled_jacobian` function in the `V_QuadMetric.cpp` implementation.

[3] See https://cubit.sandia.gov/files/cubit/16.04/help_manual/WebHelp/cubithelp.htm

At each vertex, there is a normal vector and its respective normalized unit vector

$$\boldsymbol{N}_1 \triangleq \boldsymbol{e}_4 \times \boldsymbol{e}_1, \qquad \hat{\boldsymbol{n}}_1 \triangleq \boldsymbol{N}_1 \,/\, \| \; \boldsymbol{N}_1 \; \|, \tag{19}$$

$$\boldsymbol{N}_2 \triangleq \boldsymbol{e}_1 \times \boldsymbol{e}_2, \qquad \hat{\boldsymbol{n}}_2 \triangleq \boldsymbol{N}_2 \,/\, \| \; \boldsymbol{N}_2 \; \|, \tag{20}$$

$$\boldsymbol{N}_3 \triangleq \boldsymbol{e}_2 \times \boldsymbol{e}_3, \qquad \hat{\boldsymbol{n}}_3 \triangleq \boldsymbol{N}_3 \,/\, \| \; \boldsymbol{N}_3 \; \|, \tag{21}$$

$$\boldsymbol{N}_4 \triangleq \boldsymbol{e}_3 \times \boldsymbol{e}_4, \qquad \hat{\boldsymbol{n}}_4 \triangleq \boldsymbol{N}_4 \,/\, \| \; \boldsymbol{N}_4 \; \| \,. \tag{22}$$

At the center of the element, there is principal axis normal as well

$$\boldsymbol{N}_c \triangleq \boldsymbol{X} \times \boldsymbol{Y}, \qquad \hat{\boldsymbol{n}}_c \triangleq \boldsymbol{N}_c \,/\, \| \; \boldsymbol{N}_c \; \| \,. \tag{23}$$

There are four contributions to the quadrilateral area from each of the four nodal areas

$$\alpha_1 \triangleq \boldsymbol{N}_1 \cdot \hat{\boldsymbol{n}}_c, \tag{24}$$

$$\alpha_2 \triangleq \boldsymbol{N}_2 \cdot \hat{\boldsymbol{n}}_c, \tag{25}$$

$$\alpha_3 \triangleq \boldsymbol{N}_3 \cdot \hat{\boldsymbol{n}}_c, \tag{26}$$

$$\alpha_4 \triangleq \boldsymbol{N}_4 \cdot \hat{\boldsymbol{n}}_c. \tag{27}$$

When all vertices of the quadrilateral are in the same plane, as is the case for a 2D quadrilateral finite element, all unit norms are the same

$$\hat{\boldsymbol{n}}_1 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_2 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_3 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_4 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_c. \tag{28}$$

and the four contributions to the quadrilateral areas reduce to

$$\alpha_1 \xrightarrow{\text{2D}} \| \ \boldsymbol{N}_1 \ \|, \tag{29}$$
$$\alpha_2 \xrightarrow{\text{2D}} \| \ \boldsymbol{N}_2 \ \|, \tag{30}$$
$$\alpha_3 \xrightarrow{\text{2D}} \| \ \boldsymbol{N}_3 \ \|, \tag{31}$$
$$\alpha_4 \xrightarrow{\text{2D}} \| \ \boldsymbol{N}_4 \ \| . \tag{32}$$

0.3.2 Signed Area

The **signed area** SA is defined as the average of all nodal area contributions:

$$\text{SA} \triangleq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}. \tag{33}$$

The metric dimension is L^2 and the idea (unit square) value is `1.0`.

0.3.3 Aspect Ratio

The **aspect ratio** AR is defined as the maximum edge length ratios taken at the quadrilateral center.[4] This can be expressed in terms of the norms of the principal axes as

$$\text{AR} = \max\left(\frac{\| \ \boldsymbol{X} \ \|}{\| \ \boldsymbol{Y} \ \|}, \frac{\| \ \boldsymbol{Y} \ \|}{\| \ \boldsymbol{X} \ \|},\right) \tag{34}$$

[4]Robinson J. CRE method of element testing and the Jacobian shape parameters. Engineering Computations. 1987 Feb 1.

Alternatively, the perimeter length multiplied by the maximum side length, divided by four times the area to define a triangle aspect ratio that is meaningful for quadrilaterals,[5] with dimension L^0 and acceptable range $[1.0, 1.3]$.

0.3.4 Minimum Jacobian

The **Minimum Jacobian** $J_{\min}$ is defined as the minimum pointwise area of local map at the four corners and center of quadrilateral[6]

$$J_{\min} \triangleq \min\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4\right). \tag{35}$$

0.3.5 Minimum Scaled Jacobian

The **Minimum Scaled Jacobian** $\hat{J}_{\min}$ is the minimum nodal area divided by the lengths of the two edge vector connecting that point[7]

$$\hat{J}_{\min} \triangleq \min\left(\frac{\alpha_1}{\ell_4\ell_1}, \frac{\alpha_2}{\ell_1\ell_2}, \frac{\alpha_3}{\ell_2\ell_3}, \frac{\alpha_4}{\ell_3\ell_4}\right), \tag{36}$$

When all normals are in the same plane, the Minimum Scaled Jacobian reduces to

$$\hat{J}_{\min} \xrightarrow{\text{2D}} \min\left(\sin\theta_1, \sin\theta_2, \sin\theta_3, \sin\theta_4\right), \tag{37}$$

[5]Knupp 2006, *op. cit.* at 38.
[6]Knupp 2006, *op. cit.* at 42.
[7]Knupp 2006, *op. cit.* at 51.

where θ_1 is the angle between $\boldsymbol{e}_4$ and $\boldsymbol{e}_1$, θ_2 with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, θ_3 with $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, and θ_4 with $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$.

The dimension is L^0. The full range is $[-1.0, 1.0]$. The acceptable range is typically taken as $[0.3, 1.0]$ in the generous case and $[0.5, 1.0]$ in the more restricted case.