Quadrilateral Quality

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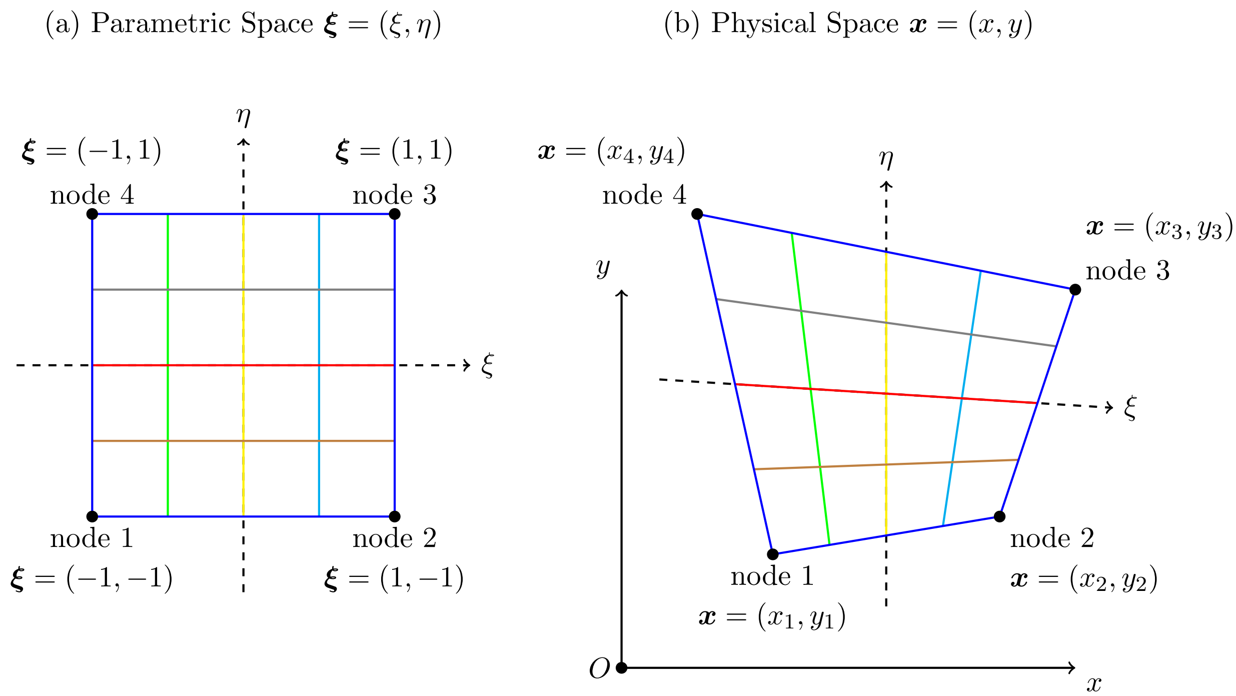

Figure 1: Parametric mapping $\boldsymbol{x} = f(\boldsymbol{\xi})$ from parametric space to physical space.

1 Isoparametric Mapping

Let the parametric mapping $f : \boldsymbol{\xi} \mapsto \boldsymbol{x}$ be defined as

$$x(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\; x_a, \tag{1}$$

$$y(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\; y_a, \tag{2}$$

where a nodal **shape function** is defined for each of the four nodes:

$$N_1(\xi, \eta) \triangleq \tfrac{1}{4}(1-\xi)(1-\eta), \tag{3}$$

$$N_2(\xi, \eta) \triangleq \tfrac{1}{4}(1+\xi)(1-\eta), \tag{4}$$

$$N_3(\xi, \eta) \triangleq \tfrac{1}{4}(1+\xi)(1+\eta), \tag{5}$$

$$N_4(\xi, \eta) \triangleq \tfrac{1}{4}(1-\xi)(1+\eta). \tag{6}$$

The Jacobian $\boldsymbol{J}$ is the partial matrix of derivatives of $\boldsymbol{x}$ with respect to $\boldsymbol{\xi}$, defined for a quadrilateral element as

$$\boldsymbol{J} \triangleq \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}. \tag{7}$$

2 Jacobian

Expanded in terms of shape functions, the Jacobian takes the form

$$\boldsymbol{J}(\xi, \eta) = \frac{1}{4} \begin{bmatrix} -1+\eta & 1-\eta & 1+\eta & -1-\eta \\ -1+\xi & -1-\xi & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{8}$$

The determinant of the Jacobian, $\det(\boldsymbol{J})$, can be found to be

$$\det(\boldsymbol{J}) = c_0 + c_1\xi + c_2\eta, \tag{9}$$

where

$$c_0 = \frac{1}{8}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right], \tag{10}$$

$$c_1 = \frac{1}{8}\left[(x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)\right], \tag{11}$$

$$c_2 = \frac{1}{8}\left[(x_2 - x_3)(y_1 - y_4) - (x_1 - x_4)(y_2 - y_3)\right]. \tag{12}$$

3 Quality

Important *Metrics for Quadrilateral Elements*[1] follow:

- Aspect ratio: Maximum edge length ratios at the quadrilateral center. Citation: Robinson 1987.[2]
- Jacobian metric: Minimum pointwise volume of local map at the four corners and center of quadrilateral. Dimension: L^2. Full range: $(-\infty, \infty)$. Acceptable range: None. Citation: Knupp 2000.[3]
- Scaled Jacobian metric: For a linear element, the minimum Jacobian [at a point] divided by the lengths of the two edge vectors [connecting to that point]. Dimension: L^0. Full range: $[-1.0, 1.0]$. Acceptable range $[0.5, 1.0]$. Citation: Knupp 2000, *op. cit.*.

Following *The Verdict Geometry Quality Library* documentation,[4] we compute the minimum Jacobian as the minimum of the four Jacobians computed at each vertex:[5]

$$J_{\min} \triangleq \min(J_1, J_2, J_3, J_4), \tag{13}$$

and the minimum scaled Jacobian as the minimum of the four Jacobians computed at each vertex divided lengths of the two element edges connected to that vertex[6]

$$\hat{J}_{\min} \triangleq \min\left(\frac{J_1}{e_4 e_1}, \frac{J_2}{e_1 e_2}, \frac{J_3}{e_2 e_3}, \frac{J_4}{e_3 e_4}\right), \tag{14}$$

where the following edge lengths are defined:

$$e_1 \triangleq \| \ \boldsymbol{x}_2 - \boldsymbol{x}_1 \ \|, \tag{15}$$

$$e_2 \triangleq \| \ \boldsymbol{x}_3 - \boldsymbol{x}_2 \ \|, \tag{16}$$

$$e_3 \triangleq \| \ \boldsymbol{x}_4 - \boldsymbol{x}_3 \ \|, \tag{17}$$

$$e_4 \triangleq \| \ \boldsymbol{x}_1 - \boldsymbol{x}_4 \ \| . \tag{18}$$

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[1] See Cubit or Coreform.

[2] Robinson J. CRE method of element testing and the Jacobian shape parameters. Engineering Computations. 1987 Feb 1.

[3] Knupp PM. Achieving finite element mesh quality via optimization of the Jacobian matrix norm and associated quantities. Part II—a framework for volume mesh optimization and the condition number of the Jacobian matrix. International Journal for numerical methods in engineering. 2000 Jul 20;48(8):1165-85. OSTI https://www.osti.gov/servlets/purl/5009.

[4] Knupp PM, Ernst CD, Thompson DC, Stimpson CJ, Pebay PP. The verdict geometric quality library. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA (United States); 2006 Mar 1. OSTI https://www.osti.gov/servlets/purl/901967.

[5] Knupp 2006, *op. cit.* at 42.

[6] Knupp 2006, *op. cit.* at 51.