Catalan Numbers Modulo a Prime Power

Larry Shu-Chung Liu

National Hsinchu University of Education, Taiwan

a joint work with Jean C.-C. Yeh (Texas A&M)

In honor of Professor Doron Zeilberger's birthday

Outline

Outline

- The *p-adic order* of an integer n is denoted and defined as

$$\omega_p(n) := \max\{\alpha \in \mathbb{N} : p^\alpha | n\}.$$

- The p*-adic order* of an integer n is denoted and defined as

$$\omega_p(n) := \max\{\alpha \in \mathbb{N} : p^\alpha | n\}.$$

- Let $C(m, n)$ be the number of *total carries* for operating $m + n$.

- The *p-adic order* of an integer n is denoted and defined as

$$\omega_p(n) := \max\{\alpha \in \mathbb{N} : p^\alpha | n\}.$$

- Let $C(m,n)$ be the number of *total carries* for operating $m+n$.

Theorem (Kummer, 1852)

$$\omega_p\left(\binom{m+n}{m}\right) = C(m,n).$$

Let p be a fixed prime and n be any integer. Define that

- $[n]_p := \langle \ldots n_2 n_1 n_0 \rangle_p$ be the representation of n in the base-p system.

Let p be a fixed prime and n be any integer. Define that

- $[n]_p := \langle \dots n_2 n_1 n_0 \rangle_p$ be the representation of n in the base-p system.
- $|\langle \dots n_2 n_1 n_0 \rangle_p| := n_0 + n_1 p + n_2 p^2 + \cdots = n$.

Let p be a fixed prime and n be any integer. Define that

- $[n]_p := \langle \dots n_2 n_1 n_0 \rangle_p$ be the representation of n in the base-p system.
- $|\langle \dots n_2 n_1 n_0 \rangle_p| := n_0 + n_1 p + n_2 p^2 + \cdots = n$.

Theorem (Lucas, 1877)

$$\binom{m}{n} \equiv_p \prod_{i \geq 0} \binom{m_i}{n_i}.$$

Theorem (Anton, 1869; Stickelberger, 1890; Hensel, 1902; and etc)

Given non-negative integers m *and* n*, let* $r = m - n$*. We have*

$$\frac{(-1)^{\omega}}{p^{\omega}}\binom{m}{n} \equiv_p \prod_{i \geq 0} \frac{m_i!}{n_i!\ r_i!},$$

where $\omega = \omega_p(\binom{m}{n})$*.*

Theorem (Anton, 1869; Stickelberger, 1890; Hensel, 1902; and etc)

Given non-negative integers m and n, let $r = m - n$. We have

$$\frac{(-1)^{\omega}}{p^{\omega}}\binom{m}{n} \equiv_p \prod_{i \geq 0} \frac{m_i!}{n_i!\ r_i!},$$

where $\omega = \omega_p(\binom{m}{n})$.

- Notice that

$$r_i \equiv_p m_i - n_i - \kappa(m, n, i-1),$$

where $\kappa(m, n, i-1)$ is the possible borrow from the i-th place to the $(i-1)$-st place when operating $m - n$.

Theorem (Anton, 1869; Stickelberger, 1890; Hensel, 1902; and etc)

Given non-negative integers m and n, let $r = m - n$. We have

$$\frac{(-1)^\omega}{p^\omega}\binom{m}{n} \equiv_p \prod_{i\geq 0} \frac{m_i!}{n_i!\ r_i!},$$

where $\omega = \omega_p(\binom{m}{n})$.

- Notice that

$$r_i \equiv_p m_i - n_i - \kappa(m, n, i-1),$$

  where $\kappa(m, n, i-1)$ is the possible borrow from the i-th place to the $(i-1)$-st place when operating $m - n$.
- Analogous formulae w.r.t. modulus p^k were recently given by Granville and {Davis, Webb}.

Outline

The n-th Catalan number is defined as $c_n = \frac{1}{n+1}\binom{2n}{n}$.

The n-th Catalan number is defined as $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Deutsch and Sagan, 2006)

$$\omega_2(c_n) \;=\; d(n+1)-1 \;=\; d(\alpha),$$

where $d(m) = m_0 + m_1 + \ldots$ *and*
$[n]_2 = \langle [\alpha]_2 \; 0 \; \underbrace{11\ldots1}_{\text{This segment might be empty.}} \rangle_2.$

The n-th Catalan number is defined as $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Deutsch and Sagan, 2006)

$$\omega_2(c_n) \; = \; d(n+1) - 1 \; = \; d(\alpha),$$

where $d(m) = m_0 + m_1 + \ldots$ *and*
$[n]_2 = \langle [\alpha]_2 \; 0 \; \underbrace{11\ldots1}_{\text{This segment might be empty.}} \rangle_2$.

- Postnikov and Sagan generalized this formula for a weighted Catalan number.

The n-th Catalan number is defined as $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Deutsch and Sagan, 2006)

$$\omega_2(c_n) \;=\; d(n+1) - 1 \;=\; d(\alpha),$$

where $d(m) = m_0 + m_1 + \ldots$ *and*
$[n]_2 = \langle [\alpha]_2 \; 0 \; \underbrace{11\ldots1}_{\text{This segment might be empty.}} \rangle_2$.

- Postnikov and Sagan generalized this formula for a weighted Catalan number.
- A general formula of $\omega_p(c_n)$ for any prime p will be given later.

Let $d(n) = n_0 + n_1 + \cdots$ be the digit sum of $[n]_2$.

Let $d(n) = n_0 + n_1 + \cdots$ be the digit sum of $[n]_2$.

Proposition (Eu, Liu and Y.-N. Yeh, 2008)

First of all, $c_n \not\equiv_8 3, 7$ for any n. As for other congruences, we have

$$c_n \equiv_8 \begin{cases} \left.\begin{matrix} 1 \\ 5 \end{matrix}\right\} & \text{if } \; d(\alpha) = 0 \text{ and } \begin{cases} \beta = 0 \text{ or } 1, \\ \beta \geq 2, \end{cases} \\ \left.\begin{matrix} 2 \\ 6 \end{matrix}\right\} & \text{if } \; d(\alpha) = 1 \text{ and } \begin{cases} \alpha = 1, \\ \alpha \geq 2, \end{cases} \\ 4 & \text{if } \; d(\alpha) = 2, \\ 0 & \text{if } \; d(\alpha) \geq 3, \end{cases}$$

where $[n]_2 = \langle [\alpha]_2 \; 0 \; 1^\beta \rangle_2$.

Let $d(n) = n_0 + n_1 + \cdots$ be the digit sum of $[n]_2$.

Proposition (Eu, Liu and Y.-N. Yeh, 2008)

First of all, $c_n \not\equiv_8 3, 7$ *for any* n. *As for other congruences, we have*

$$c_n \equiv_8 \begin{cases} \left.\begin{matrix} 1 \\ 5 \end{matrix}\right\} & \text{if } \; d(\alpha) = 0 \text{ and } \begin{cases} \beta = 0 \text{ or } 1, \\ \beta \geq 2, \end{cases} \\ \left.\begin{matrix} 2 \\ 6 \end{matrix}\right\} & \text{if } \; d(\alpha) = 1 \text{ and } \begin{cases} \alpha = 1, \\ \alpha \geq 2, \end{cases} \\ 4 & \text{if } \; d(\alpha) = 2, \\ 0 & \text{if } \; d(\alpha) \geq 3, \end{cases}$$

where $[n]_2 = \langle [\alpha]_2 \; 0 \; 1^\beta \rangle_2$.

- e.g. $[83]_2 = \langle 1010011 \rangle_2$ has $[\alpha]_2 = \langle 1010 \rangle_2$, $d(\alpha) = 2$ and $\beta = 2$.

Let $d(n) = n_0 + n_1 + \cdots$ be the digit sum of $[n]_2$.

Proposition (Eu, Liu and Y.-N. Yeh, 2008)

First of all, $c_n \not\equiv_8 3, 7$ for any n. As for other congruences, we have

$$c_n \equiv_8 \begin{cases} \left.\begin{matrix} 1 \\ 5 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 0 \text{ and } \begin{cases} \beta = 0 \text{ or } 1, \\ \beta \geq 2, \end{cases} \\ \left.\begin{matrix} 2 \\ 6 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 1 \text{ and } \begin{cases} \alpha = 1, \\ \alpha \geq 2, \end{cases} \\ 4 & \text{if} \quad d(\alpha) = 2, \\ 0 & \text{if} \quad d(\alpha) \geq 3, \end{cases}$$

where $[n]_2 = \langle [\alpha]_2 \ 0 \ 1^\beta \rangle_2$.

- e.g. $[83]_2 = \langle 1010011 \rangle_2$ has $[\alpha]_2 = \langle 1010 \rangle_2$, $d(\alpha) = 2$ and $\beta = 2$.
- Therefore, $c_{83} \equiv_8 4$.

The Motzkin number can be defined as $M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k$.

Proposition (Eu, Liu and Y.-N. Yeh, 2008)

We have $M_n \equiv_4 0$ if and only if

$$n = (4i+1)4^{j+1} - 1 \quad \text{or} \quad n = (4i+3)4^{j+1} - 2,$$

and $M_n \equiv_4 2$ if and only if

$$n = (4i+1)4^{j+1} - 2 \quad \text{or} \quad n = (4i+3)4^{j+1} - 1,$$

where $i, j \in \mathbb{N}$.

Proposition (Eu, Liu and Y.-N. Yeh, 2008)

The Motzkin number M_n is even if and only if $n = (4i+\varepsilon)4^{j+1} - \delta$ for $i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$. And we never have $M_n \equiv_8 0$. Precisely, we have

$$M_n \equiv_8 \begin{cases} 4 & \text{if } (\varepsilon, \delta) = (1,1) \text{ or } (3,2); \\ 4y+2 & \text{if } (\varepsilon, \delta) = (1,2) \text{ or } (3,1), \end{cases}$$

where y is the number of digit 1's in $[4i + \varepsilon - 1]_2$.

Outline

Main idea I: ω_p, CF_p and $E_{q,t}$

- **Problem 1:** To evaluate congruence for the combinatorial numbers of form $\frac{\prod_{i=1}^{h} M_i}{\prod_{j=1}^{g} N_j}$ (mod $q := p^k$).
  Problem 2: To classify these combinatorial numbers (mod q) according to their congruences.

Main idea I: ω_p, CF_p and $E_{q,t}$

- **Problem 1:** To evaluate congruence for the combinatorial numbers of form $\frac{\prod_{i=1}^{h} M_i}{\prod_{j=1}^{g} N_j}$ (mod $q := p^k$).
  Problem 2: To classify these combinatorial numbers (mod q) according to their congruences.
- $CF_p(n) := \frac{n}{p^{\omega_p(n)}}$, the *cofactor* of n with respect to $p^{\omega_p(n)}$ (or *non-p-cofactor*).

Main idea I: ω_p, CF_p and $E_{q,t}$

- **Problem 1:** To evaluate congruence for the combinatorial numbers of form $\frac{\prod_{i=1}^{h} M_i}{\prod_{j=1}^{g} N_j}$ (mod $q := p^k$).
  Problem 2: To classify these combinatorial numbers (mod q) according to their congruences.
- $CF_p(n) := \frac{n}{p^{\omega_p(n)}}$, the *cofactor* of n with respect to $p^{\omega_p(n)}$ (or *non-p-cofactor*).
- To evaluate a product $M := \prod_{i=1}^{h} M_i$ modulo $q = p^k$, let us consider two cofactors of M, namely

$$
\begin{aligned}
p^{\omega_p(M)} &= p^{\sum_{i=1}^{h} \omega_p(M_i)} \quad \text{and} \\
CF_p(M) &= \prod_{i=1}^{h} CF_p(M_i).
\end{aligned}
$$

Main idea I: ω_p, CF_p and $E_{q,t}$

- We analyze further that

$$\prod_{i=1}^{h} CF_p(M_i) \equiv_q \prod_{t \in \mathbb{Z}_q^*} t^{E_{q,t}(M)},$$

where $\mathbb{Z}_q^* = \{1, 2, \cdots, q-1\} - \{mp \mid m \in \mathbb{N}\}$ and $E_{q,t}(M) := \sum_{i=1}^{h} \chi(CF_p(M_i) \equiv_q t)$.

Main idea I: ω_p, CF_p and $E_{q,t}$

- We analyze further that

$$\prod_{i=1}^{h} CF_p(M_i) \equiv_q \prod_{t \in \mathbb{Z}_q^*} t^{E_{q,t}(M)},$$

  where $\mathbb{Z}_q^* = \{1, 2, \cdots, q-1\} - \{mp \mid m \in \mathbb{N}\}$ and $E_{q,t}(M) := \sum_{i=1}^{h} \chi(CF_p(M_i) \equiv_q t)$.
- We call $E_{q,t}(M)$ the *t-encounter function of modulus q* w.r.t. the product $M := \prod_{i=1}^{h} M_i$.

Main idea I: ω_p, CF_p and $E_{q,t}$

- If $\omega_p(\prod_{i=1}^{h} M_i) \geq k$, then $\prod_{i=1}^{h} M_i \equiv_q 0$; otherwise

$$\prod_{i=1}^{h} M_i \equiv_q p^{\omega_p(\prod_{i=1}^{h} M_i)} \prod_{t \in \mathbb{Z}_q^*} t^{\,E_{q,t}(\prod_{i=1}^{h} M_i)}.$$

Main idea I: ω_p, CF_p and $E_{q,t}$

- If $\omega_p(\prod_{i=1}^h M_i) \geq k$, then $\prod_{i=1}^h M_i \equiv_q 0$; otherwise

$$\prod_{i=1}^{h} M_i \equiv_q p^{\omega_p(\prod_{i=1}^h M_i)} \prod_{t \in \mathbb{Z}_q^*} t^{\,E_{q,t}(\prod_{i=1}^h M_i)}.$$

- The idea can easily apply to $\frac{\prod_{i=1}^h M_i}{\prod_{j=1}^g N_j}$, if this fraction is actually an integer.

Main idea I: ω_p, CF_p and $E_{q,t}$

- If $\omega_p(\prod_{i=1}^h M_i) \geq k$, then $\prod_{i=1}^h M_i \equiv_q 0$; otherwise

$$\prod_{i=1}^{h} M_i \equiv_q p^{\omega_p(\prod_{i=1}^h M_i)} \prod_{t \in \mathbb{Z}_q^*} t^{E_{q,t}(\prod_{i=1}^h M_i)}.$$

- The idea can easily apply to $\frac{\prod_{i=1}^h M_i}{\prod_{j=1}^g N_j}$, if this fraction is actually an integer.
- Since $c_n = \frac{1}{n+1}\binom{2n}{n}$, we have

$$\begin{aligned} c_n &\equiv_q p^{-\omega_p(n+1)+\omega_p((2n)!)-2\omega_p(n!)} \\ &\quad \times \frac{1}{CF_p(n+1)} \prod_{t \in \mathbb{Z}_q^*} t^{E_{q,t}((2n)!)-2E_{q,t}(n!)}. \end{aligned}$$

Main idea II

- Let $q = p^k$. We have a bijection T_q as follows

$$\begin{aligned} T_q : (\mathbb{Z}^*_{2^k}, \times_q) &\rightarrow (C_2 \times C_{2^{k-2}}, +) \quad \text{for } k \geq 2; \\ T_q : (\mathbb{Z}^*_{p^k}, \times_q) &\rightarrow (C_{p^{k-1}(p-1)}, +) \quad \text{for an odd prime } p. \end{aligned}$$

Main idea II

- Let $q = p^k$. We have a bijection T_q as follows

$$
\begin{aligned}
T_q : (\mathbb{Z}^*_{2^k}, \times_q) &\rightarrow (C_2 \times C_{2^{k-2}}, +) \quad \text{for } k \geq 2; \\
T_q : (\mathbb{Z}^*_{p^k}, \times_q) &\rightarrow (C_{p^{k-1}(p-1)}, +) \quad \text{for an odd prime } p.
\end{aligned}
$$

- Let $A = C_2 \times C_{2^{k-2}}$ or $A = C_{p^{k-1}(p-1)}$. If we want to use A, then we need to consider

$$
CF_p(\prod_{i=1}^{h} M_i) \equiv_q T_q^{-1}\left(\sum_{t \in \mathbb{Z}^*_q} T_q(t) E_{q,t}(\prod_{i=1}^{h} M_i)\right) \quad \text{or}
$$

$$
T_q\left(CF_p(\prod_{i=1}^{h} M_i) \pmod{q}\right) \equiv_A \sum_{y \in A} y \; E_{q,T^{-1}(y)}(\prod_{i=1}^{h} M_i).
$$

Main idea II

- Let $q = p^k$. We have a bijection T_q as follows

$$\begin{aligned} T_q : (\mathbb{Z}^*_{2^k}, \times_q) &\rightarrow (C_2 \times C_{2^{k-2}}, +) \quad \text{for } k \geq 2; \\ T_q : (\mathbb{Z}^*_{p^k}, \times_q) &\rightarrow (C_{p^{k-1}(p-1)}, +) \quad \text{for an odd prime } p. \end{aligned}$$

- Let $A = C_2 \times C_{2^{k-2}}$ or $A = C_{p^{k-1}(p-1)}$. If we want to use A, then we need to consider

$$CF_p(\prod_{i=1}^{h} M_i) \equiv_q T_q^{-1}\left(\sum_{t \in \mathbb{Z}_q^*} T_q(t) E_{q,t}(\prod_{i=1}^{h} M_i)\right) \quad \text{or}$$

$$T_q\left(CF_p(\prod_{i=1}^{h} M_i) \qquad\qquad \right) \equiv_A \sum_{y \in A} y\ E_{q,T^{-1}(y)}(\prod_{i=1}^{h} M_i).$$

Recall that $d(n) = n_0 + n_1 + \cdots$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$.
Define $d_k(n) = n_k + n_{k+1} + \cdots$.

Lemma

Let $q = p^k$, $t \in \mathbb{Z}_q^*$ and $[n]_p = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$. We have

$$\begin{aligned} \omega_p(n!) &= \frac{n - d(n)}{p-1}, \\ E_{q,t}(n!) &= \frac{|\langle n_r \ldots n_{k-1} \rangle_p| - d_{k-1}(n)}{p-1}, \\ &+ \sum_{i \geq 0} \chi(|\langle n_{i+k-1} \ldots n_{i+1} n_i \rangle_p| \geq t). \end{aligned}$$

Recall that $d(n) = n_0 + n_1 + \cdots$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$.
Define $d_k(n) = n_k + n_{k+1} + \cdots$.

Lemma

Let $q = p^k$, $t \in \mathbb{Z}_q^*$ *and* $[n]_p = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$. *We have*

$$\begin{aligned} \omega_p(n!) &= \frac{n - d(n)}{p-1}, \\ E'_q(n!) &= \frac{|\langle n_r \ldots n_{k-1} \rangle_p| - d_{k-1}(n)}{p-1}, \\ E''_{q,t}(n!) &= \sum_{i \geq 0} \chi(|\langle n_{i+k-1} \ldots n_{i+1} n_i \rangle_p| \geq t). \end{aligned}$$

Recall that $d(n) = n_0 + n_1 + \cdots$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$.
Define $d_k(n) = n_k + n_{k+1} + \cdots$.

Lemma

Let $q = p^k$, $t \in \mathbb{Z}_q^*$ and $[n]_p = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$. We have

$$\begin{aligned} \omega_p(n!) &= \frac{n - d(n)}{p-1}, \\ E'_q(n!) &= \frac{|\langle n_r \ldots n_{k-1} \rangle_p| - d_{k-1}(n)}{p-1}, \\ E''_{q,t}(n!) &= \sum_{i \geq 0} \chi(|\langle n_{i+k-1} \ldots n_{i+1} n_i \rangle_p| \geq t). \end{aligned}$$

where $\mathcal{S}(n) = \{|\langle n_{k+i-1} \ldots n_{i+1} n_i \rangle_p| : \text{except } 0\}_{i \geq 0}$ is a multi-set and $\#(\mathcal{S}, T)$ is the number of elements (with multiplicity) in $\mathcal{S}$ belonging to T.

Recall that $d(n) = n_0 + n_1 + \cdots$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$.
Define $d_k(n) = n_k + n_{k+1} + \cdots$.

Lemma

Let $q = p^k$, $t \in \mathbb{Z}_q^*$ and $[n]_p = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$. We have

$$\begin{aligned} \omega_p(n!) &= \frac{n - d(n)}{p-1}, \\ E'_q(n!) &= \frac{|\langle n_r \ldots n_{k-1} \rangle_p| - d_{k-1}(n)}{p-1}, \\ E''_{q,t}(n!) &= \#(\mathcal{S}(n), [t, q-1]). \end{aligned}$$

where $\mathcal{S}(n) = \{|\langle n_{k+i-1} \ldots n_{i+1} n_i \rangle_p| : \text{except } 0\}_{i \geq 0}$ is a multi-set and $\#(\mathcal{S}, T)$ is the number of elements (with multiplicity) in $\mathcal{S}$ belonging to T.

- Therefore, $CF_p(n!) \equiv_q \left(\prod_{t \in \mathbb{Z}_q^*} t\right)^{E'_q(n!)} \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}(n!)}$.

- Therefore, $CF_p(n!) \equiv_q \left(\prod_{t\in\mathbb{Z}_q^*} t\right)^{E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}(n!)}$.
- For $p = 2$ and $k \geq 3$, we have

$$\sum_{y\in C_2\times C_{2^{k-2}}} y \equiv (0,0); \text{ equivalently, } \prod_{t\in\mathbb{Z}_{2^k}^*} t \equiv_q 1;$$

So $E'_q(n!)$ is useless for evaluating $CF_p(n!) \pmod{2^k}$.

- Therefore, $CF_p(n!) \equiv_q \left(\prod_{t\in\mathbb{Z}_q^*} t\right)^{E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}(n!)}$.
- For $p = 2$ and $k \geq 3$, we have

$$\sum_{y\in C_2\times C_{2^{k-2}}} y \equiv (0,0); \text{ equivalently, } \prod_{t\in\mathbb{Z}^*_{2^k}} t \equiv_q 1;$$

So $E'_q(n!)$ is useless for evaluating $CF_p(n!)$ (mod 2^k).

- For an odd prime p, we have

$$\sum_{y\in C_{p^{k-1}(p-1)}} y \equiv p^{k-1}(p-1)/2; \text{ equivalently, } \prod_{t\in\mathbb{Z}^*_{p^k}} t \equiv_q -1.$$

We only care about the parity of $E'_q(n!)$.

Outline

- We use the bijection $T_q : \mathbb{Z}^*_{2^k} \to C_2 \times C_{q/4}$ to study $c_n \pmod{2^k}$. Define $T_q(t) := (b(t), u_q(t))$.

- We use the bijection $T_q : \mathbb{Z}^*_{2^k} \to C_2 \times C_{q/4}$ to study $c_n \pmod{2^k}$. Define $T_q(t) := (b(t), u_q(t))$.
- $b(t) = \begin{cases} 0 & \text{if } t \equiv_4 1; \\ 1 & \text{if } t \equiv_4 3. \end{cases}$

- We use the bijection $T_q : \mathbb{Z}_{2^k}^* \to C_2 \times C_{q/4}$ to study $c_n \pmod{2^k}$. Define $T_q(t) := (b(t), u_q(t))$.
- $b(t) = \begin{cases} 0 & \text{if } t \equiv_4 1; \\ 1 & \text{if } t \equiv_4 3. \end{cases}$
- $b(CF_2(n!)) \equiv_2 r(n) + n_0 + n_1 \equiv_2 zr(n) + n_1$,
  where $r(n)$ is the number of runs of 1 in $[n]_2$ and
  $zr(n)$ is the number of runs of 0.

- We use the bijection $T_q : \mathbb{Z}^*_{2^k} \to C_2 \times C_{q/4}$ to study $c_n \pmod{2^k}$. Define $T_q(t) := (b(t), u_q(t))$.
- $b(t) = \begin{cases} 0 & \text{if } t \equiv_4 1; \\ 1 & \text{if } t \equiv_4 3. \end{cases}$
- $b(CF_2(n!)) \equiv_2 r(n) + n_0 + n_1 \equiv_2 zr(n) + n_1$, where $r(n)$ is the number of runs of 1 in $[n]_2$ and $zr(n)$ is the number of runs of 0.
- $$\begin{aligned} u_q(CF_2(n!)) &\equiv_{q/4} \sum_{\substack{3 \le t \le s \le q-1 \\ t\text{: odd}}} \#(\mathcal{S}(n), \{s\})\ u_q(t) \\ &= \sum_{s \in [3, q-2]} \#(\mathcal{S}(n), \{s\}) \sum_{t \in [3,s]_{\text{odd}}} u_q(t). \end{aligned}$$

- We use the bijection $T_q : \mathbb{Z}^*_{2^k} \to C_2 \times C_{q/4}$ to study $c_n \pmod{2^k}$. Define $T_q(t) := (b(t), u_q(t))$.
- $b(t) = \begin{cases} 0 & \text{if } t \equiv_4 1; \\ 1 & \text{if } t \equiv_4 3. \end{cases}$
- $b(CF_2(n!)) \equiv_2 r(n) + n_0 + n_1 \equiv_2 zr(n) + n_1$, where $r(n)$ is the number of runs of 1 in $[n]_2$ and $zr(n)$ is the number of runs of 0.
- $$\begin{aligned} u_q(CF_2(n!)) &\equiv_{q/4} \sum_{\substack{3 \le t \le s \le q-1 \\ t:\ \text{odd}}} \#(\mathcal{S}(n), \{s\})\ u_q(t) \\ &= \sum_{s \in [3, q-2]} \#(\mathcal{S}(n), \{s\}) \sum_{t \in [3,s]_{\text{odd}}} u_q(t). \end{aligned}$$
- We built a table for $\sum_{t \in [3,s]_{\text{odd}}} u_q(t)$ or $\prod_{t \in [3,s]_{\text{odd}}} t$ according to $s \in [3, q-2]$ to study $c_n \pmod{16}$ and $c_n \pmod{64}$.

Lemma

Let m be an integer such that $[m]_2$ is obtained by either extending or truncating some runs of 0 or 1 of length $\geq k-1$ in $[n]_2$ to be different length but still $\geq k-1$. We have

$$b(CF_2(n!)) = b(CF_2(m!)) \text{ and } u_q(CF_2(n!)) = u_q(CF_2(m!)),$$

and then

$$CF_2(n!) \equiv_q CF_2(m!).$$

Lemma

Let m be an integer such that $[m]_2$ is obtained by either extending or truncating some runs of 0 or 1 of length $\geq k-1$ in $[n]_2$ to be different length but still $\geq k-1$. We have

$$b(CF_2(n!)) = b(CF_2(m!)) \text{ and } u_q(CF_2(n!)) = u_q(CF_2(m!)),$$

and then

$$CF_2(n!) \equiv_q CF_2(m!).$$

Pf. $b(CF_2(n!)) \equiv_2 zr(n) + n_1 \equiv_2 zr(m) + m_1 \equiv_2 b(CF_2(m!))$.

Lemma

Let m be an integer such that $[m]_2$ is obtained by either extending or truncating some runs of 0 or 1 of length $\geq k-1$ in $[n]_2$ to be different length but still $\geq k-1$. We have

$$b(CF_2(n!)) = b(CF_2(m!)) \text{ and } u_q(CF_2(n!)) = u_q(CF_2(m!)),$$

and then

$$CF_2(n!) \equiv_q CF_2(m!).$$

Pf. $b(CF_2(n!)) \equiv_2 zr(n) + n_1 \equiv_2 zr(m) + m_1 \equiv_2 b(CF_2(m!))$.

$$u_q(CF_2(n!)) \equiv_{q/4} \sum_{s \in [3, q-3]_{\text{odd}}} \#(\mathcal{S}(n), \{s, s+1\}) \sum_{t \in [3,s]_{\text{odd}}} u_q(t)$$

is independent on the numbers of $\langle 0^{k-1} \rangle_2$ and $\langle 1^{k-1} \rangle_2$ in $[n]_2$.

Given $n \in \mathbb{N}$, let $\bar{n}$ be the integer such that $[\bar{n}]_2$ is obtained by the following rules.

a. When the rightmost run of 0 in $[n]_2$ is of length $\geq k+1$, let us truncate it to be length k, otherwise keep it the same.

Given $n \in \mathbb{N}$, let $\bar{n}$ be the integer such that $[\bar{n}]_2$ is obtained by the following rules.

a. When the rightmost run of 0 in $[n]_2$ is of length $\geq k+1$, let us truncate it to be length k, otherwise keep it the same.
b. For any other run of 0 or 1 of $[n]_2$ with length $\geq k$, truncate them to be length $k-1$.

Theorem (Liu and Yeh, 2010)

Let $n, k \in \mathbb{N}$ with $k \geq 3$. We have

$$c_n \equiv_{2^k} \begin{cases} c_{\bar{n}} & \text{for } d(\alpha) \leq k-1, \text{ and} \\ 0 & \text{for } d(\alpha) \geq k. \end{cases}$$

Given $n \in \mathbb{N}$, let $\bar{n}$ be the integer such that $[\bar{n}]_2$ is obtained by the following rules.

a. When the rightmost run of 0 in $[n]_2$ is of length $\geq k+1$, let us truncate it to be length k, otherwise keep it the same.
b. For any other run of 0 or 1 of $[n]_2$ with length $\geq k$, truncate them to be length $k-1$.

Theorem (Liu and Yeh, 2010)

Let $n, k \in \mathbb{N}$ with $k \geq 3$. We have

$$c_n \equiv_{2^k} \begin{cases} c_{\bar{n}} & \text{for } d(\alpha) \leq k-1, \text{ and} \\ 0 & \text{for } d(\alpha) \geq k. \end{cases}$$

Example. $c_{\langle 100001111\rangle_2} \equiv_8 6, \quad c_{\langle 1111111\rangle_2} \equiv_{16} 13,$
$c_{\langle 100011\rangle_2} \equiv_8 6, \quad c_{\langle 111\rangle_2} \equiv_{16} 13.$

Theorem (Liu and Yeh, 2010)

Let $n \in \mathbb{N}$ *and* $q = 2^k$ *with* $k \geq 2$. *Then we have*

$$c_n \equiv_q (-1)^{zr(\alpha)} 2^{d(\alpha)} 5^{u_q(CF_2(c_n))}.$$

In particular, when $k = 2$ *we have*

$$c_n \equiv_4 (-1)^{zr(\alpha)} 2^{d(\alpha)}.$$

Proposition (Liu and Yeh, 2010)

Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have

$$c_n \equiv_{16} \begin{cases} \left.\begin{array}{c} 1 \\ 5 \\ 13 \end{array}\right\} & \text{if} \quad d(\alpha) = 0 \text{ and } \begin{cases} \beta \le 1, \\ \beta = 2, \\ \beta \ge 3, \end{cases} \\ \left.\begin{array}{c} 2 \\ 10 \end{array}\right\} & \text{if} \quad d(\alpha) = 1,\ \alpha = 1 \text{ and } \begin{cases} \beta = 0 \text{ or } \beta \ge 2, \\ \beta = 1, \end{cases} \\ \left.\begin{array}{c} 6 \\ 14 \end{array}\right\} & \text{if} \quad d(\alpha) = 1,\ \alpha \ge 2 \text{ and } \begin{cases} (\alpha = 2, \beta \ge 2) \text{ or } (\alpha \ge 3, \beta \le 1), \\ (\alpha = 2, \beta \le 1) \text{ or } (\alpha \ge 3, \beta \ge 2), \end{cases} \\ \left.\begin{array}{c} 4 \\ 12 \end{array}\right\} & \text{if} \quad d(\alpha) = 2 \text{ and } \begin{cases} zr(\alpha) \ne 1, \\ zr(\alpha) = 1, \end{cases} \\ 8 & \text{if} \quad d(\alpha) = 3, \\ 0 & \text{if} \quad d(\alpha) \ge 4. \end{cases}$$

where $[n]_2 = \langle [\alpha]_2\ 0\ 1^\beta \rangle_2$.

We also completely classified $c_n \pmod{64}$. Here we only post the classification for odd congruences.

Proposition (Liu and Yeh, 2010)

Let $n \in \mathbb{N}$ with $d(\alpha) = 0$, i.e. $n = 2^\beta - 1$. Then we have

$$c_n \equiv_{64} \begin{cases} 1 & \textit{if} \quad \beta = 0 \textit{ or } 1; \\ 5 & \textit{if} \quad \beta = 2; \\ 45 & \textit{if} \quad \beta = 3; \\ 61 & \textit{if} \quad \beta = 4; \\ 29 & \textit{if} \quad \beta \geq 5. \end{cases}$$

Moreover, any number in $[1, 63]_{odd} - \{1, 5, 29, 45, 61\}$ can never be the congruence of $c_n \pmod{64}$.

After observing all odd congruences from modulus 4 up to modulus 1024, once we conjectured the following property. This property was proved recently.

Theorem (Lin, 2010)

Let $k \geq 2$. Only $k-1$ different odd congruences $c_n \pmod{2^k}$ exist, and they are

$$c_{2^m-1} \pmod{2^k}$$

for $m = 1, 2, \ldots, k-1$.

After observing all odd congruences from modulus 4 up to modulus 1024, once we conjectured the following property. This property was proved recently.

Theorem (Lin, 2010)

Let $k \geq 2$. Only $k-1$ different odd congruences $c_n \pmod{2^k}$ exist, and they are

$$c_{2^m-1} \pmod{2^k}$$

for $m = 1, 2, \ldots, k-1$.

Example There are only 6 congruences $c_n \pmod{128}$ with odd value:

$$\begin{aligned} c_{\langle 1\rangle_2} &\equiv_{128} 1, & c_{\langle 11\rangle_2} &\equiv_{128} 5, \\ c_{\langle 111\rangle_2} &\equiv_{128} 45, & c_{\langle 1111\rangle_2} &\equiv_{128} 125, \\ c_{\langle 11111\rangle_2} &\equiv_{128} 29, & c_{\langle 111111\rangle_2} &\equiv_{128} 93. \end{aligned}$$

Outline

- Let $\kappa_p(m,n;i) := \left\lfloor \frac{|\langle m_i \ldots m_0 \rangle_p| + |\langle n_i \ldots n_0 \rangle_p|}{p^{i+1}} \right\rfloor$ $(= 0$ or $1)$ be the possible *carry* from the i-th to the $(i+1)$-st places for $m+n$ in the base-p system.

- Let $\kappa_p(m,n;i) := \left\lfloor \frac{|\langle m_i \dots m_0\rangle_p| + |\langle n_i \dots n_0\rangle_p|}{p^{i+1}} \right\rfloor$ $(= 0$ or $1)$ be the possible *carry* from the i-th to the $(i+1)$-st places for $m+n$ in the base-p system.
- Let $C_p(m,n;i) := \sum_{j \geq i} \kappa_p(m,n;j)$ and $C_p(m,n) = C_p(m,n;0)$ which is the number of total carries.

- Let $\kappa_p(m,n;i) := \left\lfloor \frac{|\langle m_i \ldots m_0 \rangle_p| + |\langle n_i \ldots n_0 \rangle_p|}{p^{i+1}} \right\rfloor$ $(= 0$ or $1)$ be the possible *carry* from the i-th to the $(i+1)$-st places for $m+n$ in the base-p system.
- Let $C_p(m,n;i) := \sum_{j \geq i} \kappa_p(m,n;j)$ and $C_p(m,n) = C_p(m,n;0)$ which is the number of total carries.
- $d(m+n) = d(m) + d(n) - (p-1)C_p(m,n)$; $\omega_p((m+n)!) = \frac{m+n-d(m)-d(n)}{p-1} + C_p(m,n)$.

- Let $\kappa_p(m,n;i) := \left\lfloor \frac{|\langle m_i \dots m_0\rangle_p| + |\langle n_i \dots n_0\rangle_p|}{p^{i+1}} \right\rfloor$ $(= 0$ or $1)$ be the possible *carry* from the i-th to the $(i+1)$-st places for $m+n$ in the base-p system.
- Let $C_p(m,n;i) := \sum_{j \geq i} \kappa_p(m,n;j)$ and $C_p(m,n) = C_p(m,n;0)$ which is the number of total carries.
- $d(m+n) = d(m) + d(n) - (p-1)C_p(m,n)$; $\omega_p((m+n)!) = \frac{m+n-d(m)-d(n)}{p-1} + C_p(m,n)$.

Theorem

We have

$$\omega_p(c_n) = C_p(n,n) - \beta \; = \; C_p(n,n;\beta),$$

where $[n]_p = \langle \dots \widetilde{(p-1)} \; (p-1)^{\beta} \rangle_p$.

- Let $\kappa_p(m,n;i) := \left\lfloor \frac{|\langle m_i \dots m_0 \rangle_p| + |\langle n_i \dots n_0 \rangle_p|}{p^{i+1}} \right\rfloor$ $(= 0$ or $1)$ be the possible *carry* from the i-th to the $(i+1)$-st places for $m+n$ in the base-p system.
- Let $C_p(m,n;i) := \sum_{j \geq i} \kappa_p(m,n;j)$ and $C_p(m,n) = C_p(m,n;0)$ which is the number of total carries.
- $d(m+n) = d(m) + d(n) - (p-1)C_p(m,n)$; $\omega_p((m+n)!) = \frac{m+n-d(m)-d(n)}{p-1} + C_p(m,n)$.

Theorem

We have

$$\omega_p(c_n) = C_p(n,n) - \beta \;=\; C_p(n,n;\beta),$$

where $[n]_p = \langle \dots \widetilde{(p-1)} \; (p-1)^\beta \rangle_p$.

If $p = 2$, then $C_2(n,n;\beta) = d(\alpha)$, where $[n]_2 = \langle [\alpha]_2 \; 0 \; 1^\beta \rangle_2$.
$\Rightarrow$ Theorem of Deutsch and Sagan.

Let $\oplus_q$ be the operator of addition over ring $\mathbb{Z}_q$, and $z(m,n;i) = |\langle m_{i+k-1} \dots m_i \rangle p| \oplus_q |\langle n_{i+k-1} \dots n_i \rangle p|$. Then

$$\langle (m+n)_{i+k-1} \dots (m+n)_i \rangle p = [z(m,n;i) \oplus_q \kappa(m,n;i-1)]_p.$$

Let $\oplus_q$ be the operator of addition over ring $\mathbb{Z}_q$, and $z(m,n;i) = |\langle m_{i+k-1} \dots m_i \rangle p| \oplus_q |\langle n_{i+k-1} \dots n_i \rangle p|$. Then

$$\langle (m+n)_{i+k-1} \dots (m+n)_i \rangle p = [z(m,n;i) \oplus_q \kappa(m,n;i-1)]_p.$$

Lemma (From now on p is an odd prime.)

$$\begin{aligned} E'_q((m+n)!) &= \frac{|\langle \dots m_{k-1} \rangle_p| + |\langle \dots n_{k-1} \rangle_p| - d_{k-1}(m) - d_{k-1}(n)}{p-1} \\ &\quad + C(m,n;k-1); \\ E''_{q,t}((m+n)!) &= \sum_{i \geq 0} \left[\begin{array}{c} \chi(z(m,n;i) \geq t) \\ + \chi(z(m,n;i) = t-1)\, \kappa(m,n;i-1) \end{array} \right] \\ &\quad - \sigma(m,n), \end{aligned}$$

where $\sigma(m,n)$ be # of i st. $z(m,n;i) = q-1$ and $\kappa(m,n;i-1) = 1$.

ecause $\sigma(m,n)$ is independent on t, let's modify E'_q and $E''_{q,t}$.

Let $\oplus_q$ be the operator of addition over ring $\mathbb{Z}_q$, and $z(m,n;i) = |\langle m_{i+k-1} \dots m_i \rangle p| \oplus_q |\langle n_{i+k-1} \dots n_i \rangle p|$. Then

$$\langle (m+n)_{i+k-1} \dots (m+n)_i \rangle p = [z(m,n;i) \oplus_q \kappa(m,n;i-1)]_p.$$

Lemma (From now on p is an odd prime.)

$$\begin{aligned} E'_q((m+n)!) &= \frac{|\langle \dots m_{k-1} \rangle_p| + |\langle \dots n_{k-1} \rangle_p| - d_{k-1}(m) - d_{k-1}(n)}{p-1} \\ &\quad + C(m,n;k-1) - \sigma(m,n); \\ E''_{q,t}((m+n)!) &= \sum_{i \geq 0} \left[\begin{array}{c} \chi(z(m,n;i) \geq t) \\ + \chi(z(m,n;i) = t-1)\, \kappa(m,n;i-1) \end{array} \right] \end{aligned}$$

Because $\sigma(m,n)$ is independent on t, let's modify E'_q and $E''_{q,t}$.

Let $\oplus_q$ be the operator of addition over ring $\mathbb{Z}_q$, and $z(m,n;i) = |\langle m_{i+k-1}\dots m_i\rangle p| \oplus_q |\langle n_{i+k-1}\dots n_i\rangle p|$. Then

$$\langle (m+n)_{i+k-1}\dots(m+n)_i\rangle p = [z(m,n;i) \oplus_q \kappa(m,n;i-1)]_p.$$

Lemma (final version)

$$\begin{aligned} E'_q((m+n)!) &\equiv_2 \sum_{j\geq 0}(m_{2j+k}+n_{2j+k}) + C(m,n;k-1) - \sigma(m,n); \\ E''_{q,t}((m+n)!) &= \#(\mathcal{S}^+,[t,q-1]) + \#(\bar{\mathcal{S}}^+,\{t-1\}), \end{aligned}$$

where $\mathcal{S}^+ = \{z(m,n;i) \ : \ z(m,n;i) \neq 0\}_{i\geq 0}$, $\bar{\mathcal{S}}^+ = \{z \in \mathcal{S}^+ : \kappa(m,n;i-1) = 1$ *and* $z_0 \neq p-1\}$, *and* $\sigma(m,n)$ *is the number of* i *such that* $z(m,n;i) = q-1$ *and* $\kappa(m,n;i-1) = 1$

- $$CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)} (-1)^{E'_q((n+n)!) - 2E'_q(n!)} \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}.$$

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \dots (p-1)\ (p-1)^{\beta}\rangle_p = \langle \dots n_\beta\ (p-1)^{\beta}\rangle_p$. So $[n+1]_p = \langle \dots n_{\beta+1}(n_\beta+1)\ 0^{\beta}\rangle_p$ and then $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1}\dots n_{\beta+1}(n_\beta+1)\rangle_p|$.

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)}$
  $\times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \dots (p-1)\ (p-1)^\beta \rangle_p = \langle \dots n_\beta\ (p-1)^\beta \rangle_p$.
  So $[n+1]_p = \langle \dots n_{\beta+1}(n_\beta+1)\ 0^\beta \rangle_p$ and then
  $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \dots n_{\beta+1}(n_\beta+1)\rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n,n;k-1) - \bar{\sigma}$.

  where $\bar{\sigma} = |\{i : \langle n_{i+k-1} \dots n_i \rangle = \langle (\frac{p-1}{2})^{k-1} \rangle$
  and $\kappa(n,n;i-1) = 1\}|$.

- $$CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}.$$
- Let $[n]_p = \langle \ldots (p-1)\ (p-1)^\beta \rangle_p = \langle \ldots n_\beta\ (p-1)^\beta \rangle_p$. So $[n+1]_p = \langle \ldots n_{\beta+1}(n_\beta+1)\ 0^\beta \rangle_p$ and then $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \ldots n_{\beta+1}(n_\beta+1)\rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n,n;k-1) - \bar{\sigma}$.

  where $$\bar{\sigma} = |\{i : \langle n_{i+k-1} \ldots n_i \rangle = \langle (\tfrac{p-1}{2})^{k-1} \rangle \text{ and } \kappa(n,n;i-1) = 1\}|.$$

  However, once $\langle (\frac{p-1}{2})^{k-1} \rangle$ with $\kappa(n,n;i-1)$ appears in $[n]_p$, we have $C_p(n,n;\beta) \ge k$; and then $c_n \equiv_q 0$.

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)}$
$\times \underbrace{\prod_{t\in\mathbb{Z}_q^*}} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \ldots (p-1)\ (p-1)^\beta \rangle_p = \langle \ldots n_\beta\ (p-1)^\beta \rangle_p$.
So $[n+1]_p = \langle \ldots n_{\beta+1}(n_\beta+1)\ 0^\beta \rangle_p$ and then
$CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \ldots n_{\beta+1}(n_\beta+1) \rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n,n;k-1) - \bar{\sigma}$.

where $\bar{\sigma} = |\{i : \langle n_{i+k-1} \ldots n_i \rangle = \langle (\frac{p-1}{2})^{k-1} \rangle$
and $\kappa(n,n;i-1) = 1\}|$.

However, once $\langle (\frac{p-1}{2})^{k-1} \rangle$ with $\kappa(n,n;i-1)$ appears in $[n]_p$,
we have $C_p(n,n;\beta) \geq k$; and then $c_n \equiv_q 0$.

Therefore, $\bar{\sigma}$ is irrelevant for those $c \not\equiv_q 0$.

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \ldots \widetilde{(p-1)}\ (p-1)^\beta \rangle_p = \langle \ldots n_\beta\ (p-1)^\beta \rangle_p$. So $[n+1]_p = \langle \ldots n_{\beta+1}(n_\beta+1)\ 0^\beta \rangle_p$ and then $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \ldots n_{\beta+1}(n_\beta+1)\rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n,n;k-1) - \bar{\sigma}$.

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \ldots \overbrace{(p-1)\ (p-1)^{\beta}} \rangle_p = \langle \ldots n_\beta\ (p-1)^{\beta}\rangle_p$.
  So $[n+1]_p = \langle \ldots n_{\beta+1}(n_\beta + 1)\ 0^{\beta}\rangle_p$ and then
  $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \ldots n_{\beta+1}(n_\beta+1)\rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n, n; k-1) - \bar{\sigma}$.
- By last lemma, we need $\mathcal{S}^+ = \{z(n,n;i)\ :\ z(n,n;i) \neq 0\}_{i\geq 0}$,
  $\bar{\mathcal{S}}^+ = \{z \in \mathcal{S}^+\ :\ \kappa(n,n;i-1) = 1$ and $z_0 \neq p-1\}$ and
  $\mathcal{S} := \{|\langle n_{k+i-1} \ldots n_{i+1} n_i\rangle_p|$ except $0\}_{i\geq 0}$.
  Note that $\mathcal{S}^+ = \{2s \pmod q\ :\ s \in \mathcal{S}\}$.

- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)}(-1)^{E'_q((n+n)!)-2E'_q(n!)} \times \prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$.
- Let $[n]_p = \langle \dots \overbrace{(p-1)\ (p-1)}^{\beta} \rangle_p = \langle \dots n_\beta\ (p-1)^\beta \rangle_p$. So $[n+1]_p = \langle \dots n_{\beta+1}(n_\beta+1)\ 0^\beta \rangle_p$ and then $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \dots n_{\beta+1}(n_\beta+1)\rangle_p|$.
- $E'_q((n+n)!) \equiv_2 C_p(n,n;k-1) - \bar{\sigma}$.
- By last lemma, we need $\mathcal{S}^+ = \{z(n,n;i)\ :\ z(n,n;i) \neq 0\}_{i\geq 0}$, $\bar{\mathcal{S}}^+ = \{z \in \mathcal{S}^+\ :\ \kappa(n,n;i-1) = 1$ and $z_0 \neq p-1\}$ and $\mathcal{S} := \{|\langle n_{k+i-1} \dots n_{i+1} n_i \rangle_p|$ except $0\}_{i\geq 0}$.
  Note that $\mathcal{S}^+ = \{2s \pmod q\ :\ s \in \mathcal{S}\}$.

  To study $\prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$, we need $\mathcal{S}^+$, $\bar{\mathcal{S}}^+$ and $\mathcal{S}$. But $\mathcal{S}^+ = 2\mathcal{S}$ and $\bar{\mathcal{S}}^+ \subseteq \mathcal{S}^+$, we shall derive a formula using only $\mathcal{S}$.

Finally, we get

$$\begin{aligned}
&\prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)} \\
\equiv_q \quad &\prod_{u\in[1,q-1]} \Big(\prod_{t\in[1,2u \pmod q)]\cap\mathbb{Z}_q^*} t\Big)^{\#(\mathcal{S},\{u\})} \\
&\times \prod_{u\in[1,q-1]} \Big(\prod_{t\in[u+1,q-1]\cap\mathbb{Z}_q^*} t^2\Big)^{\#(\mathcal{S},\{u\})} \\
&\times \prod_{\substack{u\in[1,q-1] \\ 2u+1\in\mathbb{Z}_q^*}} (2u+1 \pmod q))^{\#(\bar{\mathcal{S}},\{u\})},
\end{aligned}$$

where $\bar{\mathcal{S}} = \{u = |\langle n_{k+i-1}\dots n_{i+1}n_i\rangle_p| \ : \ u \neq 0, u_0 \neq (p-1)/2$ and $\kappa(n,n;i-1) = 1\}_{i\geq 0}$.
Note $\bar{\mathcal{S}} \subseteq \mathcal{S}$ and $\bar{\mathcal{S}}^+ = \{2u \pmod q \ : \ u \in \bar{\mathcal{S}}\}$.

- The formula $\prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}$ on the last page only depends on $\mathcal{S}$ and $\bar{\mathcal{S}}$, and no more $\mathcal{S}^+$ and $\bar{\mathcal{S}}^+$.

- The formula $\prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$ on the last page only depends on $\mathcal{S}$ and $\bar{\mathcal{S}}$, and no more $\mathcal{S}^+$ and $\bar{\mathcal{S}}^+$.
- To evaluate $\prod_{t\in\mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!)-2E''_{q,t}(n!)}$ for a particular prime power q, we need to construct a table according to $u \in \mathbb{Z}_q^*$ for the following three values:

$$\begin{aligned}
A &= \prod_{t\in[1,2u \pmod q]\cap\mathbb{Z}_q^*} t, \\
B &= (\prod_{t\in[u+1,q-1]\cap\mathbb{Z}_q^*} t)^2 \quad \text{and} \\
C &= 2u+1 \pmod q.
\end{aligned}$$

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?
- $[3212]_3 = \langle 11101222 \rangle_3$. So $\beta = 3$.

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?
- $[3212]_3 = \langle 11101222 \rangle_3$. So $\beta = 3$.
- Then $\omega_p(c_{3212}) = C_p(n, n; \beta) = 1$.

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?
- $[3212]_3 = \langle 11101222 \rangle_3$. So $\beta = 3$.
- Then $\omega_p(c_{3212}) = C_p(n, n; \beta) = 1$.
- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)} (-1)^{E'_q((n+n)!)}$
  $\times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}$.

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?
- $[3212]_3 = \langle 11101222 \rangle_3$. So $\beta = 3$.
- Then $\omega_p(c_{3212}) = C_p(n, n; \beta) = 1$.
- $$CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)} (-1)^{E'_q((n+n)!)} \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}.$$
- In general, $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \ldots n_{\beta+1}(n_\beta + 1) \rangle_p|$. In this case, $CF_q(3213) \equiv_{27} |\langle 102 \rangle_p| = 11$.

Example: $c_{3212} \pmod{27}$

- Let $p = 3$, $k = 3$ and $n = 3212$. What is $c_{3212} \pmod{27}$?
- $[3212]_3 = \langle 11101222 \rangle_3$. So $\beta = 3$.
- Then $\omega_p(c_{3212}) = C_p(n, n; \beta) = 1$.
- $CF_q(c_n) \equiv_q \frac{1}{CF_q(n+1)} (-1)^{E'_q((n+n)!)} \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}$.
- In general, $CF_q(n+1) \equiv_q |\langle n_{\beta+k-1} \dots n_{\beta+1}(n_\beta + 1) \rangle_p|$. In this case, $CF_q(3213) \equiv_{27} |\langle 102 \rangle_p| = 11$.
- $E'_q((n+n)!) \equiv_2 C_p(n, n; k-1) = 2 \equiv_2 0$.

Here is the table for evaluating $\prod_{t\in\mathbb{Z}_{27}^*} t^{E''_{27,t}((2n)!)-2E''_{27,t}(n!)}$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
A	2	8	13	26	17	25	14	8	1	2	17	13	26	$\cdots$
B	1	7	7	19	4	4	10	1	1	10	7	7	1	$\cdots$
C		5	7		11	13		17	19		23	25		$\cdots$

$$
\begin{aligned}
A &= \prod_{t\in[1,2u \pmod q]\cap\mathbb{Z}_q^*} t, \\
B &= \Big(\prod_{t\in[u+1,q-1]\cap\mathbb{Z}_q^*} t\Big)^2 \quad \text{and} \\
C &= 2u+1 \pmod q.
\end{aligned}
$$

Here is the table for evaluating $\prod_{t \in \mathbb{Z}_{27}^*} t^{E''_{27,t}((2n)!) - 2E''_{27,t}(n!)}$.

u	$\cdots$	14	15	16	17	18	19	20	21	22	23	24	25	26
A	$\cdots$	2	8	13	26	17	25	14	8	1	2	17	13	26
B	$\cdots$	4	4	19	1	1	19	7	7	10	4	4	1	1
C	$\cdots$	2	4		8	10		14	16		20	22		26

$$A = \prod_{t \in [1, 2u \pmod{q}] \cap \mathbb{Z}_q^*} t,$$

$$B = \Big(\prod_{t \in [u+1, q-1] \cap \mathbb{Z}_q^*} t \Big)^2 \quad \text{and}$$

$$C = 2u + 1 \pmod{q}.$$

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.

We have $CF_{27}(c_{3212}) \equiv_{27} \frac{1}{11}(-1)^0 \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times \prod_{t \in \mathbb{Z}_q^*} t^{E''_{q,t}((n+n)!) - 2E''_{q,t}(n!)}$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 001 \rangle_3| = 1$ corresponding to 2 in the table.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 011 \rangle_3| = 4$ corresponding to 8.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 111 \rangle_3| = 13$ corresponding to 26.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 110 \rangle_3| = 12$ with $\kappa = 1$ corresponding to 7.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \cdot 7 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 101 \rangle_3| = 10$ corresponding to 20. ($\kappa = 1$ but $u_0 = 1$)

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \cdot 7 \cdot 20 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 012 \rangle_3| = 5$ with $ka = 1$ corresponding to 19.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \cdot 7 \cdot 20 \cdot 19 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 122 \rangle_3| = 17$ with $ka = 1$ corresponding to 19.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \cdot 7 \cdot 20 \cdot 19 \cdot 19 \times \cdots)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.
So $u = |\langle 222 \rangle_3| = 26$ corresponding to 26.

We have $CF_{27}(c_{3212}) \equiv_{27} 5 \times (2 \cdot 8 \cdot 26 \cdot 7 \cdot 20 \cdot 19 \cdot 19 \cdot 26)$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and $\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.

We have $CF_{27}(c_{3212}) \equiv_{27} 4$.

When $\kappa = 1$ and $u_0 \neq 1$, we shall use $A \cdot B \cdot C$; otherwise $A \cdot B$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	

14	15	16	17	18	19	20	21	22	23	24	25	26
8	5	4	26	17	16	17	2	10	8	14	13	26
16	20		19	8		22	5		25	11		1

For our case, $[3212]_3 = \langle 11101222 \rangle_3$ and
$\mathcal{S} = \{001, 011, 111, 110, 101, 012, 122, 222\}$.

We have $CF_{27}(c_{3212}) \equiv_{27} 4$. Then $c_{3212} \equiv_{27} 3^1 \times 4 = 12$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13
$A \cdot B$	2	2	10	8	14	19	5	8	1	20	11	10	26
$A \cdot B \cdot C$		10	16		19	4		1	19		10	7	■

u	14	15	16	17	18	19	20	21	22	23	24	25	26
$A \cdot B$	8	5	4	26	17	16	17	2	10	8	14	13	26
$A \cdot B \cdot C$	16	20		19	8		22	5		25	11		1

Lemma

$A \cdot B \equiv_q \pm 1$ and $A \cdot B \cdot C \pm 1$ for both $u = (q-1)/2$ and $u = q - 1$.

Given $n \in \mathbb{N}$, let $\bar{n}$ be the integer such that $[\bar{n}]_2$ is obtained by the following rules.

a. If the rightmost 0 run and $p-1$ run of form the $[n]_2 = \langle \dots 0^{\beta'}(p-1)^{\beta} \rangle_p$ with $\beta, \beta' \geq k+1$, let us truncate their to length k, otherwise keep it the same.

b. For any other run of 0 and $(p-1)/2$ of $[n]_2$ with length $\geq k+1$, truncate them to be length of the same parity as k or $k-1$.

Theorem

Let $n, k \in \mathbb{N}$ with $k \geq 3$. We have

$$c_n \equiv_{2^k} \begin{cases} c_{\bar{n}} & \text{for } d(\alpha) \leq k-1, \text{ and} \\ 0 & \text{for } d(\alpha) \geq k. \end{cases}$$

Example.

$$
\begin{aligned}
c_{\langle 3330006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 30006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 3006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 306\rangle_7} &\equiv_{49} 15,\\
c_{\langle 3066\rangle_7} &\equiv_{49} 1,\\
c_{\langle 30666\rangle_7} &\equiv_{49} 1,\\
c_{\langle 33306\rangle_7} &\equiv_{49} 15,\\
c_{\langle 3306\rangle_7} &\equiv_{49} 34.
\end{aligned}
$$

Example.

$$\begin{aligned}
c_{\langle 3330006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 30006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 3006\rangle_7} &\equiv_{49} 43,\\
c_{\langle 306\rangle_7} &\equiv_{49} 15,\\
c_{\langle 3066\rangle_7} &\equiv_{49} 1,\\
c_{\langle 30666\rangle_7} &\equiv_{49} 1,\\
c_{\langle 33306\rangle_7} &\equiv_{49} 15,\\
c_{\langle 3306\rangle_7} &\equiv_{49} 34.
\end{aligned}$$

The End!