P-adic Limits of Combinatorial Sequences

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Welcome

Introduction to the p-adics: $\mathbb{Q}_p$

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Define the *p-adic valuation* of an integer n to be the greatest power of p that divides n: $\nu_p(n) = k$

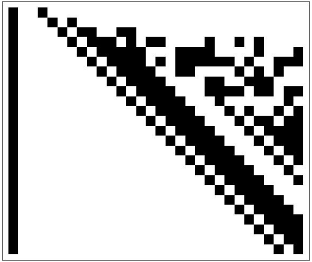
Then the *p-adic norm* of n is defined as $|n|_p = p^{-k}$.

Definition of p-adic convergence

$\forall k \geq 1\ \exists N \geq 1$ such that $\forall m, n \geq N$,

$$|C(f(n)) - C(f(m))|_p \leq p^{-k}.$$

Example ($\{3^{2^n}\}$ converges to 1, 2-adically.)



Example from a paper written by Eric Rowland (2009).

The Catalan Numbers

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Unfortunately, we couldn't collect enough data to make a conjecture about what the limit of the sequence is.

Questions

When does a subsequence of the Catalan numbers converge p-adically?

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What are the limits of these subsequences?

Finding p-adic Convergent Sequences

$\forall k \geq 1$, $\exists N \geq 1$ such that $\forall n, m > N$,

$$\|C(f(m)) - C(f(n))\|_p \leq p^{-k}$$

if and only if

$$\nu_p(C(f(m)) - C(f(n))) \geq k$$

if and only if

$$C(f(m)) - C(f(n)) \equiv 0 \pmod{p^k}$$

if and only if

$$C(f(m)) \equiv C(f(n)) \pmod{p^k}.$$

In words, $\{C(f(n))\}$ converges p-adically if it is eventually constant modulo all powers of p.

Finding p-adic Convergent Sequences

We used this equivalent definition of p-adic convergence and a formula for the Catalan numbers modulo a prime power (derived from a 1997 paper by Granville) to prove that the following class of sequences converges.

Theorem

For all primes p and all $a \in \mathbb{N}$, $\{C(ap^n)\}_{n\geq 0}$ converges p-adically.

The p-adic Gamma Function

Definition (p-adic Gamma Function)

For a prime p and integer n, the *p-adic Gamma function* is defined to be

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ p \nmid k}}^{n-1} k \text{ where } \Gamma_p(0) = 1$$

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$$\Gamma_2(n) = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1) & \text{even n} \\ (-1) \cdot 1 \cdot 3 \cdot 5 \cdots (n-2) & \text{odd n} \end{cases}$$

Elementary Proof of $\lim_{n\to\infty} \Gamma_2(2^n) = 1$

Lemma

$\lim_{n\to\infty} \Gamma_2(2^n)) = 1$ *in the 2-adics.*

Want to show: given $n > m$,

$$\|\Gamma_2(2^n) - \Gamma_2(2^m)\|_2 \leq \frac{1}{2^{m-1}},$$

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$$\begin{aligned}
\Gamma_2(2^n) - \Gamma_2(2^m) &= \prod_{\substack{k=1\\ 2\nmid k}}^{2^n-1} k - \prod_{\substack{k=1\\ 2\nmid k}}^{2^m-1} k \\
&= \Big(\prod_{\substack{k=1\\ 2\nmid k}}^{2^m-1} k\Big)\Big(\Big(\prod_{\substack{k=2^m\\ 2\nmid k}}^{2^n-1} k\Big) - 1\Big).
\end{aligned}$$

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&= (\prod_{\substack{k=1 \\ 2\nmid k}}^{2^m-1} k)((\prod_{\substack{k=2^m \\ 2\nmid k}}^{2^n-1} k) - 1).
\end{aligned}$$

It suffices to show

$$\prod_{\substack{k=2^m \\ 2\nmid k}}^{2^n-1} k \equiv 1 \pmod{2^{m-1}}.$$

Elementary Proof of $\lim_{n\to\infty} \Gamma_2(2^n) = 1$

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\prod_{\substack{k=2^m \\ 2\nmid k}}^{2^n-1} k &= (2^m+1)(2^m+3)\ldots(2^n-1) \\
&\equiv \prod_{k=0}^{2^{n-1}-1-2^{m-1}} (2k+1) \pmod{2^{m-1}}
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&= \begin{matrix} (1) & (3) & \cdots & (2^{m-1}-1) \\ (2^{m-1}+1) & (2^{m-1}+3) & \cdots & (2^m-1) \\ \vdots & & & \\ (2^{n-1}+2^{m-1}+1) & \cdots & \cdots & (2^n-2^m-1) \end{matrix}
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\vdots & & & \\
(2^{n-1}+2^{m-1}+1) & \cdots & \cdots & (2^n-2^m-1)
\end{array} \\
&\equiv [(1)(3)\cdots(2^{m-1}-1)]^{2(2^{n-m}-1)} \pmod{2^{m-1}}.
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$$\|\Gamma_2(2^n) - \Gamma_2(2^m)\|_2 \leq \frac{1}{2^{m-1}} \iff \Gamma_2(2^n) - \Gamma_2(2^m) \equiv 0 \pmod{2^{m-1}}.$$

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This proof strategy generalizes to all p.

Slick Proof of $\lim \Gamma_2(2^n) = 1$

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$$\lim_{n\to\infty} \Gamma_2(2^n) = \Gamma_2(\lim_{n\to\infty} 2^n) = \Gamma_2(0) = 1.$$

This proof (clearly) also generalizes to all p.

What is the limit of $C(ap^n)$?

Remember the formula for Catalan numbers:

$$C(ap^n) = \frac{1}{ap^n + 1}\binom{2ap^n}{ap^n} = \frac{(2ap^n)!}{(ap^n+1)!(ap^n)!}$$

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We have shown that $\binom{2p^n}{p^n}$ and $\frac{1}{p^n+1}\binom{2p^n}{p^n}$ converge to the same limit p-adically. This doesn't change when we add a constant a.

What is the limit of $C(ap^n)$?

To find this limit, we want to express $\binom{2ap^n}{ap^n}$ in terms of the p-adic gamma function.

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{k=1 \\ p\nmid k}}^{n} k = \frac{(-1)^{n+1}(n)!}{\prod_{\substack{k=1 \\ p\mid k}}^{n-1} k} = \frac{(-1)^{n+1}(n)!}{p^{\left\lfloor \frac{n}{p} \right\rfloor} \left\lfloor \frac{n}{p} \right\rfloor !}$$

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$$\begin{aligned} \Rightarrow (ap^n)! &= (ap^{n-1})! \Gamma_p(ap^n + 1)(-1)^{ap^n+1} p^{ap^{n-1}} \\ &= a! p^{\frac{ap^n - a}{p-1}} (-1)^{ap^n+1} \prod_{i=1}^{n} \Gamma_p(ap^i) \end{aligned}$$

What is the limit of $C(ap^n)$?

Using the above equation, we get

$$\frac{(2ap^n)!}{(ap^n)!^2} = \frac{(2a)!\cancel{p^{\frac{2ap^n-2a}{p-1}}}}{(a!)^2\cancel{p^{\frac{2(ap^n-a)}{p-1}}}}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} = \frac{(2a)!}{(a!)^2}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$$

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New problem: look at $\lim\limits_{n\to\infty}\frac{(2a)!}{(a!)^2}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$ p-adically.

Example: $p = 2$

We found a very *special* case of this limit when $p = 2$.

$$\binom{2a2^n}{a2^n} = \binom{a2^{n+1}}{a2^n} = \frac{(a2^{n+1})!}{(a2^n)!^2} = \frac{(2a)!}{(a!)^2} \prod_{i=1}^{n} \frac{\Gamma_2(a2^{i+1})}{\Gamma_2(a2^i)^2}$$

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$$\prod_{i=1}^{n}\frac{\Gamma_2(a2^{i+1})}{\Gamma_2(a2^i)^2} = \frac{\Gamma_2(4a)\cdot\Gamma_2(8a)\cdot\Gamma_2(16a)\cdots\Gamma_2(2^n a)\cdot\Gamma_2(2^{n+1}a)}{\Gamma_2(2a)^2\cdot\Gamma_2(4a)^2\cdot\Gamma_2(8a)^2\cdot\Gamma_2(16a)^2\cdots\Gamma_2(2^n a)^2}$$

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$$\begin{aligned}\prod_{i=1}^{n} \frac{\Gamma_2(a2^{i+1})}{\Gamma_2(a2^i)^2} &= \frac{\cancel{\Gamma_2(4a)} \cdot \cancel{\Gamma_2(8a)} \cdot \cancel{\Gamma_2(16a)} \cdots \cancel{\Gamma_2(2^n a)} \cdot \Gamma_2(2^{n+1}a)}{\Gamma_2(2a)^2 \cdot \Gamma_2(4a)^{\cancel{2}} \cdot \Gamma_2(8a)^{\cancel{2}} \cdot \Gamma_2(16a)^{\cancel{2}} \cdots \Gamma_2(2^n a)^{\cancel{2}}} \\ &= \frac{\Gamma_2(2^{n+1}a)}{\Gamma_2(2a)} \prod_{i=1}^{n} \frac{1}{\Gamma_2(a2^i)}\end{aligned}$$

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So far we have,

$$\binom{a2^{n+1}}{a2^n} = \frac{(a2^{n+1})!}{(a2^n)!^2} = \frac{(2a)!\Gamma_2(2^{n+1}a)}{(a!)^2\Gamma_2(2a)} \prod_{i=1}^{n} \frac{1}{\Gamma_2(a2^i)}$$

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By the previous Lemma, $\Gamma_2(2^{n+1}a) \to 1$, 2-adically.

The only *mystery* now is what $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{1}{\Gamma_2(a2^i)}$ is.

Example $p = 2$

The idea behind $\prod_{i=1}^{n} \frac{1}{\Gamma_2(a2^i)}$:

$$\frac{1}{(1\cdot 3\cdot 5\cdots(2a-1))(1\cdot 3\cdot 5\cdots(2a-1)\cdots(4a-1))\cdots\cdots(1\cdot 3\cdot 5\cdots(2a-1)\cdots(4a-1)\cdots(2^n a-1))}$$

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$$\frac{1}{(1{\cdot}3{\cdot}5\cdots(2a-1))^n((2a+1)\cdots(4a-1))^{n-1}((4a+1)\cdots(8a-1))^{n-2}\cdots((2^{n-1}a+1)\cdots(2^n a-1))}$$

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$$\frac{((2a+1)\cdots(4a-1))((4a+1)\cdots(8a-1))^2\cdots\cdots((2^{n-1}a+1)\cdots(2^n a-1))^n}{(1{\cdot}3{\cdot}5\cdots(2a-1)(2a+1)\cdots(4a-1)(4a+1)\cdots(8a-1)(8a+1)\cdots\cdots(2^n a-1))^n}$$

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$$\frac{((2a+1)\cdots(4a-1))((4a+1)\cdots(8a-1))^2\cdots\cdots((2^{n-1}a+1)\cdots(2^n a-1))^n}{\Gamma_2(2^n a)^n}$$

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If we take the limit as $n \to \infty$, the denominator is 1 by our lemma, and the numerator becomes an infinite product of powers of odd numbers. i.e.

$$\lim_{n\to\infty} \binom{a2^{n+1}}{a2^n} = \frac{(2a)!}{\Gamma_2(2a)(a!)^2} \prod_{j=0}^{\infty} (2j+1)^{\left\lfloor \log_2\left(\frac{2j+1}{a}\right)\right\rfloor}$$

$p = 2$ and $a = 1$

Looking at the simplest case, we have

$$\lim_{n\to\infty} \binom{2^{n+1}}{2^n} = \frac{2}{\Gamma_2(2)} \prod_{j=0}^{\infty} (2j+1)^{\lfloor \log_2(2j+1) \rfloor}$$
$$= 2 \cdot 3 \cdot (5 \cdot 7)^2 (9 \cdot 11 \cdot 13 \cdot 15)^3 \cdots$$

Simplifying $\lim_{n\to\infty} C(ap^n)$

One should always generalize
-Carl Jacobi

$$\lim_{n\to\infty} \binom{2a}{a} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} = ?$$

Simplifying $\prod_{i=1}^{n} \Gamma_p(ap^i)$

$\prod_{i=1}^{n} \Gamma_p(ap^i)$ is given by

$(1 \cdots (ap-1))^n((ap+1) \cdots (ap^2-1))^{n-1} \cdots ((ap^{n-1}+1) \cdots (ap^n-1))$

which is simply:

$$\Gamma_p(ap^n)^n \prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1} (i^{-\lfloor \log_p(i/2a) \rfloor})$$

Simplifying $\binom{2ap^n}{ap^n}$

$$\binom{2a}{a} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$$

Simplifying $\binom{2ap^n}{ap^n}$

$$
\begin{aligned}
& \binom{2a}{a} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} \\
= \quad & \binom{2a} {a} \frac{\Gamma_p(2ap^n)^n}{\Gamma_p(ap^n)^{2n}} \frac{\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1} \left(i^{2\lfloor \log_p(i/a) \rfloor}\right)}{\prod_{i=ap+1 \text{ and } p\nmid i}^{2ap^n-1} \left(i^{\lfloor \log_p(i/2a) \rfloor}\right)}
\end{aligned}
$$

Simplifying $\binom{2ap^n}{ap^n}$

$$\binom{2a}{a}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$$

$$= \binom{2a}{a}\frac{\Gamma_p(2ap^n)^n}{\Gamma_p(ap^n)^{2n}}\frac{\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor}\right)}{\prod_{i=ap+1 \text{ and } p\nmid i}^{2ap^n-1}\left(i^{\left\lfloor \log_p(i/2a)\right\rfloor}\right)}$$

$$= \binom{2a}{a}\frac{\Gamma_p(2ap^n)^n}{\Gamma_p(ap^n)^{2n}}\frac{\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor-\left\lfloor \log_p(i/2a)\right\rfloor}\right)}{\prod_{i=ap^n+1 \text{ and } p\nmid i}^{2ap^n-1}\left(i^{\left\lfloor \log_p(i/2a)\right\rfloor}\right)}$$

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= \quad & \binom{2a}{a} \frac{\Gamma_p(2ap^n)^n}{\Gamma_p(ap^n)^{2n}} \frac{\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1} \left(i^{2\left\lfloor \log_p(i/a) \right\rfloor - \left\lfloor \log_p(i/2a) \right\rfloor}\right)}{\prod_{i=ap^n+1 \text{ and } p\nmid i}^{2ap^n-1} \left(i^{\left\lfloor \log_p(i/2a) \right\rfloor}\right)} \\
= \quad & \binom{2a}{a} \frac{\Gamma_p(2ap^n)^n}{\Gamma_p(ap^n)^{2n}} \frac{\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1} \left(i^{2\left\lfloor \log_p(i/a) \right\rfloor - \left\lfloor \log_p(i/2a) \right\rfloor}\right)}{\prod_{i=ap^n+1 \text{ and } p\nmid i}^{2ap^n-1} \left(i^{n-1}\right)}
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\end{aligned}
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Simplifying $\lim_{n\to\infty} \binom{2ap^n}{ap^n}$

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= \quad & \binom{2a}{a}\lim_{n\to\infty}\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1}\left(i^{2\lfloor \log_p(i/a)\rfloor - \lfloor \log_p(i/2a)\rfloor}\right)
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= \ & \binom{2a}{a}\lim_{n\to\infty}\prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor - \left\lfloor \log_p(i/2a)\right\rfloor}\right)
\end{aligned}
$$

Question : When does this limit exist?

An Example

For $p = 2$ we get

$$\lim_{n\to\infty} C(a2^n) \quad = \quad \binom{2a}{a} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{2\lfloor \log_2(i/a)\rfloor - \lfloor \log_2(i/2a)\rfloor}\right)$$

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\lim_{n\to\infty} C(a2^n) &= \binom{2a}{a} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{2\lfloor \log_2(i/a)\rfloor - \lfloor \log_2(i/2a)\rfloor}\right) \\
&= \binom{2a}{a} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor + 1}\right)
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&= \binom{2a}{a} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor + 1}\right) \\
&= \binom{2a}{a} \lim_{n\to\infty} \frac{\Gamma_2(a2^n)}{\Gamma_2(2a)} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor}\right)
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&= \binom{2a}{a} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor + 1}\right) \\
&= \binom{2a}{a} \lim_{n\to\infty} \frac{\Gamma_2(a2^n)}{\Gamma_2(2a)} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor}\right) \\
&= \binom{2a}{a} \frac{1}{\Gamma_2(2a)} \lim_{n\to\infty} \prod_{i=2a+1 \text{ and i odd}}^{a2^n-1} \left(i^{\lfloor \log_2(i/a)\rfloor}\right)
\end{aligned}$$

Proof of Convergence of $C(ap^n)$

We first reduce to the problem of determining the disk of convergence for

$$\lim_{n\to\infty} \prod_{i=ap+1 \text{ and } p\nmid i}^{ap^n-1} \left(i^{2\lfloor \log_p(i/a)\rfloor - \lfloor \log_p(i/2a)\rfloor}\right)$$

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Taking the difference of consecutive terms in the sequence,

$$K_n(\prod_{i=ap^{n-1}+1 \text{ and } p\nmid i}^{ap^n-1} (i^{2\lfloor \log_p(i/a)\rfloor - \lfloor \log_p(i/2a)\rfloor}) - 1)$$

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$$K_n(\prod_{i=ap^{n-1}+1 \text{ and } p\nmid i}^{ap^n-1} (i^{2\lfloor \log_p(i/a)\rfloor - \lfloor \log_p(i/2a)\rfloor}) - 1)$$

The product can be written as

$$(\frac{\Gamma_p(ap^n)}{\Gamma_p(ap^{n-1})})^{n-2}\frac{\Gamma_p(a(p-1)p^{n-1})}{\Gamma_p(ap^{n-1})}$$

which converges to 1.

Summary

$C(ap^n)$ can be "solved" by expressing its limit in terms of p-adic gamma functions. Γ_p

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All of these sequence converge and can be written as a product of integers coprime with p to powers ascending logarithmically.

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Similar methods should work for other combinatorial sequences which are factorials of multiples of prime powers.

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Questions for us?

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