2014 MSRI-UP Progress Report for]08 July 2014

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0.1 Monday July 7th

What we like/don't like about our project:

It's hard to know exactly what we like or don't like about the topic, after just the first day. We do like that this topic has room for us to explore, but also has a clear direction that we have some idea of where to start.

4 questions:

- What value does $C(2^n)$ approach as n gets large?
- What tools can we use from module theory to learn about k-kernels?
- What other sequences can we look at from this perspective?
- What tools can we use from Eric's work with the Fibonacci numbers to better understand this sequence?

What we did today:

Today, we first looked at the sequence $\nu_2(C(2^n))$ and saw that it seemed to be constant for $n \geq 1$. We ended up conjecturing that the sequence $\nu_p(C(p^n))$ for prime p was constant as well from trials on Mathematica. We proved this conjecture (see attached pdf for more details on the proof).

Eric suggested we look at the sequence $C(2^n)$ as $n \to \infty$. Writing these numbers out in their base-2 representations for each n, we began to see a pattern in the prefixes of the numbers:

$C(2^0) = 1 = 1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdots = 10000\ldots$
$C(2^1) = 2 = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdots = 01000\ldots$
$C(2^2) = 14 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 \cdots = 011100\ldots$
$C(2^3) = 1430 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 + 0 \cdot 2^5 + 0 \cdot 2^6 + 1 \cdot 2^7 + 1 \cdot 2^8 + 0 \cdot 2^9 + 1 \cdot 2^10 \cdots =$
$01101001101000\ldots$

The limit of the first digit of these expansions is zero because the 2-adic value of the sequence approaches one, so that eventually all of its elements are divisible by 2. Eric has a paper in which he examines the 2-adic values of the sequence $\{3^{2^n}\}$. He finds that, viewed as 2-adic numbers, this sequence converges to 1. This means that the binary expansion of elements of the sequence approach that of one; that is, if we write $3^{2^n} = \sum a_{n_i} 2^i$ and $1 = \sum b_i 2^i (= 1 + 0 \cdot 2^1 + 0 \cdot 2^2 + \cdots)$, then $\lim_{n\to\infty} a_{n_i} = b_i$ for every i. We want to show the analogous result for the sequence $\{C(2^n)\}$. Our data shows that (taking the a_{n_i}s to be coefficient of 2^i for $C(2^n)$) a_{n_i} converges (becomes constant) for the first few values of i.

Claim : $\nu_p(C(p^{nk})) = \begin{cases} 1 & \text{if } p = 2 \text{ and } k = 1 \\ 0 & \text{if } p \text{ is another prime or } k > 1 \end{cases}$

Proof. By the closed form of the catalan numbers, and the formula for the p-adic valuation of factorials we have

$$\begin{aligned}
\nu_{p^k}(C(p^{nk})) &= \nu_{p^k}\left(\frac{1}{p^{nk}+1}\binom{2p^{nk}}{p^{nk}}\right) \\
&= \nu_{p^k}((2p^{nk})!) - 2\nu_{p^k}((p^{nk})!) - \nu_{p^k}(p^{nk}+1) \\
&= 2p^{nk} - S_{p^k}(2p^{nk}) - 2p^{nk} + 2S_{p^k}(p^{nk}) - 0 \\
&= 2S_{p^k}(p^{nk}) - S_{p^k}(2p^{nk}) \\
&= 2 - S_{p^k}(2p^{nk})
\end{aligned}$$

In the case where $p = 2$ and $k = 1$, $S_2(2x) = S_2(x)\ \forall x \in \mathbb{N}$. So we get 1 as the result. Otherwise, we simply multiply the only one in the base p^k representation of p^{nk} by 2 to get $S_{p^k}(2p^{nk}) = 2$ which completes our proof.

$\square$

What we plan to do:

We would like to be able to prove that $C(2^n)$ converges and find the value to which it converges. The problem we had today was that we could only plug in a finite amount of data into Mathematica to confirm our suspicions, so we need to figure out a different approach to proving what the sequence does as n gets large. We are planning to read/skim "P-adic Numbers, P-adic Analysis, and Zeta Functions" by Neal Koblitz and "P-adic Numbers" by Fernando Q. Gouvea to gain some insight.

0.2 Tuesday July 8th

Questions:

- Is the connection between p-adic convergence and congruence modulo a power of p explored in the literature?
- Can we prove the convergence of sequences of Catalan number for a more general class, e.g., sequences of the form $\{C(2^n + k)\}$ for arbitrary k? Are the techniques that we've used/that are available for proving things about $C(2^n)$ applicable to primes besides 2?
- What do these sequences converge to? And can we apply tools that Eric demonstrates in his paper to find the actual limit?

What we did today:

We noticed that there is a simple reformulation of p-adic convergence in terms of congruence modulo a power of p. Let $\{s(n)\}$ be a sequence. To show that it is Cauchy convergent, we can show that for all k there exist sufficiently large m and n such that $|s(m)-s(n)| \leq p^{-k}$. Showing this is equivalent to showing that $\nu_P(s(m)-s(n)) \geq k$, which is equivalent to showing that $s(m) - s(n) \equiv 0 \pmod{p^k}$ or, equivalently, that $s(m) \equiv s(n) \pmod{p^k}$.

Using this latter condition and a couple of known results, we are able to show that the sequences $\{C(2^n - 1)\}$ and $\{C(2^n)\}$ converge.

For the first sequence, we use the fact (shown by Lin in 2011) that for all $k \geq 2$ and all $n \geq k - 1$, the sequence $\{C(2^n - 1)\}$ is constant modulo 2^k. This implies that given an arbitrary $k \geq 1$, for all m and n larger than $k-1$ we have that $C(2^m-1) - C(2^n-1) \equiv 0 \pmod{2^k}$, implying that $|C(2^m-1) - C(2^n-1)| \leq 2^{-k}$, as desired.

For the second sequence, we use a fact shown by Liu and Yeh in 2010. They show that given a number n, we can find a number n' such that for any k, $C(n) \equiv C(n') \pmod{2^k}$. For the special case where n is a sufficiently large (larger than $k-1$) power of 2, say 2^a, we have that $C(2^a) \equiv C(2^{k-1}) \pmod{2^k}$. So if we're given an arbitrary k, for any m and n greater than $k-1$ we have that both $C(2^n)$ and $C(2^m)$ are congruent to $C(2^{k-1})$ modulo 2^k, so that $C(2^n) - C(2^m) \equiv 0 \pmod{2^k}$, which, as we saw, is equivalent to the statement that $|C(2^n) - C(2^m)| \leq 2^{-k}$, our desired result.

What we plan to do:

There are a few papers which discuss Catalan numbers modulo a power of 2. We used results from two of these to prove that the two sequences above converge. We hope that looking more deeply into the methods of these papers will help us show that sequences of the form $\{C(2^n + c)\}$ and those in which 2 is replaced by a different prime converge. We wonder if there are papers which make the above connection between p-adic convergence and congruence classes modulo a power of p. We also wonder whether results which are phrased in terms of congruences might be re-phrased in terms of p-adic convergence.

0.3 Wednesday July 9th

What we did today:

Today was a lot of reading and exploring ways to think about finding the limit of $C(2^n)$. We discussed Eric's process in his paper about 3^{2^n} with him, and gained a lot more insight and depth than we did just reading his paper. We thought about whether or not there could be a way of expressing C(x) as a 2-adic power series, but realized that there could be major complications with trying to match up the constant term and the limit of $C(2^n)$.

What we plan to do: We plan to work with both analytical and number theoretical approaches to find the limit of $C(2^n)$. There are many papers that have been written about Catalan numbers, so understanding those approaches are important too.

Our main problem as of now: For now, we would like to be able to prove that $C(p^n)$

converges for primes p, and find the limits of those sequences as well as the limit of $C(2^n)$.

0.4 Thursday July 10th

Finding the limit of $C(2^n)$:

We thought we'd take the powers series approach to this problem. The plan:

1. Prove that the sequence $C(2^n)$ converges to the same 2-adic number as the sequence $\binom{2^{n+1}}{2^n}$
2. Find a p-analytic continuation (which apparently goes by a different name "p-adic interpolation") for $\binom{2^{n+1}}{2^n}$ using gamma functions
3. Determine the constant term of the power series for this function. This should be the p-adic limit of $C(2^n)$

Executing the plan:

1. Looking at the difference sequence
$$\left\| \frac{1}{2^n+1}\binom{2^{n+1}}{2^n} - \binom{2^{n+1}}{2^n} \right\|_2$$
and factoring out the binomial term (which we have shown to have a 2-adic valuation of 1) we find that the remaining expression $\frac{2^n}{2^n+1}$ goes to zero. Thus, they have the same limit.

2. We have investigated the possibility of a p-adic Γ function Γ_p in Gouvea's introductory p-adic analysis book, Schikhoff's Ultrametric Calculus book, and Koblitz's book. It seems that there is such a function provided one modifies the factorial sequence being interpolated. We cannot proceed until we understand exactly how this modification changes our approach. Also, it seems $p = 2$ is a special case which requires a different approach to interpolation. Since this is the case we are currently interested in, it makes sense to both look into this case more and also explore for other values of p. We have done this in Mathematica to some extent, but nothing particularly insightful has come of it yet.

3. Assuming we find that we indeed can use Γ_p, we will want to know whether or not we can compose analytic functions and get an analytic function. Intuitively it makes sense that we should be able to do so as long as the range of one lies within the radius of convergence of the other.

Questions:

- Why is the modification for Γ_p necessary?
- Why is $p = 2$ special for Γ_p?
- When can we compose p-adic functions? in terms of analyticity...

Proof that $\{\binom{2^{n+1}}{2^n}\}$ converges to the same limit as $\{\frac{1}{2^n+1}\binom{2^{n+1}}{2^n}\}$ 2-adically:

Since we have already proven that $\{C(2^n)\} = \{\frac{1}{2^n+1}\binom{2^{n+1}}{2^n}\}$ converges, if we show that these sequences get closer and closer together as $n \to \infty$ then we can show that they tend towards the same limit (which is still unknown to us).

Proof. Let $k > 0$, then for $N = k - 1$ we want to show that $|\frac{1}{2^n+1}\binom{2^{n+1}}{2^n} - \binom{2^{n+1}}{2^n}|_2 < 2^{-k}$ for $n > N$. But this is equivalent to showing that $\nu_2[(\frac{1}{2^n+1}\binom{2^{n+1}}{2^n} - \binom{2^{n+1}}{2^n})] > k$ for $n > N$.
So for $n > N$, we have:

$$\nu_2[(\frac{1}{2^n+1}\binom{2^{n+1}}{2^n} - \binom{2^{n+1}}{2^n})] =$$
$$\nu_2[(\frac{1}{2^n+1} - 1)\binom{2^{n+1}}{2^n}] =$$
$$\nu_2(\frac{2^n}{2^n+1}) + 1 =$$
$$n + 1 > N + 1 = (k-1) + 1 = k$$

This proves that $\{\binom{2^{n+1}}{2^n}\}$ converges to the same limit as $\{C(2^n)\}$.

□

Proof that $\{C(p^n)\}$ converges:

The proof uses some concepts developed by Liu and Yeh in a 2010 paper. Our strategy is to show that for all k, $\{C(p^n)\}$ is eventually constant modulo p^k. Liu and Yeh allow us to show that, for $n > k$, $C(p^n)$ is eventually congruent to a number that does not depend on n modulo p^k.

We begin with some definitions.

Definition 1 (Liu and Yeh (2010))**.** For a prime p and a number n, the *cofactor of* n *with respect to* p is defined as

$$CF_p(n) := \frac{n}{p^{\nu_p(n)}}.$$

We next need a concept which Liu and Yeh call the *t-encounter function of modulus* p^k, where $t \in \mathbb{Z}_{p^k}^{\times}$ (i.e., t is such that $\gcd(t, p^k) = 1$; or equivalently, t is not divisible by p).

Definition 2 (Liu and Yeh 2010)**.** For $t \in \mathbb{Z}_{p^k}^{\times}$ and some product $\prod_{i=1}^{a} M_i$, the *t-encounter function of modulus* p^k is defined as

$$E_{p^k,t} := \sum_{i=1}^{a} \chi(CF_p(M_i) \equiv t \bmod p^k).$$

The t-encounter function counts the M_is that are congruent to t modulo p^k. For our purposes, $\displaystyle\prod_{i=1}^{a} M_i$ will be a factorial.

The first fact that we will need is the following congruence reltionship:

$$\frac{\prod M_i}{\prod N_j} \equiv p^{\nu_p(\frac{\prod M_i}{\prod N_j})} \prod_{t \in \mathbb{Z}_{p^k}^{\times}} t^{E_{p^k,t}(\frac{\prod M_i}{\prod N_j})},$$

where $E_{p^k,t}$ of a quotient is the difference of the $E_{p^k,t}$s of the divisor and dividend.

Next, a theorem(Lemma 2.1 in Liu and Yeh's paper) that will ultimately allow us to compute $C(p^n)$ modulo p^k:

Theorem 1 (Liu and Yeh 2010)**.** *Let* $\langle a_r \cdots a_1 a_0 \rangle_p = a_r p^r + \cdots + a_1 p + a_o = n$ *be the* p*-ary expansion of* n*, and let* $d_k(n) = \sum_{i=k}^{r} a_i$*. (*$d_0(n) = d(n)$ *is the usual sum of the coefficients.) Then for* $t \in \mathbb{Z}_{p^k}^{\times}$*,*

$$E_{p^k,t}(n!) = \frac{\langle a_r \cdots a_k a_{k-1} \rangle_p - d_{k-1}(n)}{p-1} + \sum_{i \geq 0} \chi(\langle a_{i+k-1} \cdots a_{i+1} a_i \rangle_p) \geq t).$$

Now we can consider $\{C(p^n)\}$. We showed earlier that $\nu_p(C(p^n)) = 0$, so by the fact just mentioned, we can see that

$$C(p^n) = \frac{1}{p^n+1}\binom{2p^n}{p^n} = \frac{(2p^n)!}{(p^n+1)!(p^n)!} \equiv \prod_{t\in\mathbb{Z}_{p^k}^\times} t^{[E_{p^k,t}((2p^n)!)-E_{p^k,t}((p^n)!)-E_{p^k,t}((p^n+1)!)]}.$$

If we can show that $E_{p^k,t}((2p^n)!) - E_{p^k,t}((p^n)!) - E_{p^k,t}((p^n+1)!)$ is independent of n, then the entire expression on the right-hand side is independent of n, so that $C(p^n)$ is constant modulo p^k for $n > k$, as desired.

First let's examine the first term on the right-hand side of Theorem 1. For $(2p^n)!$, we see that $\frac{\langle a_r\cdots a_k a_{k-1}\rangle_p - d_{k-1}(n)}{p-1} = \frac{2p^{n-k+1}-2}{p-1}$. For both $(p^n)!$ and $(p^n+1)!$, we see that $\frac{\langle a_r\cdots a_k a_{k-1}\rangle_p - d_{k-1}(n)}{p-1} = \frac{p^{n-k+1}-1}{p-1}$. These terms cancel in the expression for $E_{p^k,t}((2p^n)!) - E_{p^k,t}((p^n)!) - E_{p^k,t}((p^n+1)!)$:

$$\frac{2p^{n-k+1}-2}{p-1} + \sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t) -$$

$$\left(\frac{p^{n-k+1}-1}{p-1} + \sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t)\right) -$$

$$\left(\frac{p^{n-k+1}-1}{p-1} + \sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t)\right) =$$

$$\sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t) - \left(\sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t) + \sum_{i\geq 0}\chi(\langle a_{i+k-1}\cdots a_{i+1}a_i\rangle_p) \geq t)\right).$$

(Keep in mind that the a_is vary (albeit only slightly) between $2p^n$, p^n, and p^n+1; they are **not** the same across these three terms.)

The simplicity of the p-ary expansions of $2p^n$, p^n, and p^n+1 allows us to write this expression in a simpler way, as

$$\sum_{i=0}^{k-1}\chi(2p^i \geq t > p_i) - \sum_{i=0}^{k-1}\chi(p_i \geq t) + \chi(t \neq 1),$$

or, to see more clearly what each term is counting, as

$$|\{i : 1 < \frac{t}{p^i} \leq 2\}| + |\{i : \frac{t}{p^i} \leq 1\}| + \chi(t \neq 1).$$

The key detail is that this expression does **not** depend on n. We thus have that

$$C(p^n) \equiv \prod_{t \in \mathbb{Z}_{p^k}^{\times}} t^{|\{i:1<\frac{t}{p^i}\leq 2\}|+|\{i:\frac{t}{p^i}\leq 1\}|+\chi(t\neq 1)},$$

and, since the right-hand side of this congruence is constant for $n > k$, we have that $C(p^n)$ converges.

0.5 Resources

Papers we've looked at:

- We used *Catalan Numbers Modulo* 2^k (2010) by Liu and Yeh first to prove that $\{C(2^n)\}$ converges and then to prove that $\{C(p^n)\}$ converges. (The proofs use different theorems from the paper.)
- We used a 2011 paper by Lin (*Odd Catalan Numbers Modulo* 2^k) to prove that $\{C(2^n - 1)\}$ converges. This was a quick corollary to one of Lin's results.
- Since we began to focus on showing that $C(p^n) \equiv C(p^m) \pmod{p^k}$ as an equivalent condition to showing p-adic convergence, we've looked at papers exploring congruences of Catalan numbers modulo a prime power. We've compiled some such papers in our Dropbox folder: https://www.dropbox.com/home/MSRI-UP

Books we've looked at:

- Experimental Number Theory by Fernando Rodriguez Villegas
- koblitz
- gouvea
- p-adic vs reals book

Abstract: The definition of a p-adically Cauchy convergent sequence can be equivalently stated in terms of congruences of elements of that sequence modulo arbitrarily large powers

of p: a sequence is p-adically Cauchy convergent if and only if it is eventually constant modulo p^k for all $k \in \mathbb{N}$. Using results on factorials modulo powers of primes by Lin and Yeh (2010) and Granville (1997), we identify a class of p-adically convergent sequences of Catalan numbers in terms of the p-ary expansion of the sequence elements. An extension of our results to a more general class of sequences involving factorials and binomial coefficients is also considered.

0.6 Monday, July 14

Where we have gotten stuck

Today we realized that our group was going in a bunch of different directions at once. We wanted to prove properties using number theoretic approaches, while simultaneously investigating the hopes of a power series that could illustrate the sequence of Catalan numbers in the p-adics. Trying to express the catalan numbers in terms of the Γ_p function has also been hard, because it doesn't represent $n!$ the same way as it does in the usual Γ function, but we are trying to figure out the direct relation between $C(n)$ and Γ_p in order to use the Mahler series expansion.

What we looked at today

We can use some results from a 1997 paper by Andrew Granville to identify a class of sequences of Catalan numbers which converges p-adically. (The class includes $\{C(p^n))\}$.) The main result gives a formula for factorials modulo an arbitrary power of p:

Proposition 1 (Granville 1997)**.** *For any integer n and prime power p^k, we have*

$$n! \equiv p^{\nu_p(n!)} \delta^{\nu_{p^k}(n!)} \prod_{j=\geq 0} (N_j!)_p \pmod{p^k}$$

.

Let's define the notation used in the proposition. This "delta" function is easy to define: $\delta(p,k) = 1$ if $p = 2$ and $k \geq 3$, and -1 otherwise.

Defining $(N_j)!_p$ is a bit more complicated. To do so, first expand n base p: write $n = a_0 + a_1 p + \cdots + a_r p^r$. For $j \geq 0$, N_j is defined to be the least positive residue of $\lfloor \frac{n}{p^j} \rfloor$ modulo p^k. This can be expressed simply: $N_j = a_j + a_{j+1}p + \cdots + a_{j+k-1}p^{k-1}$. Then $(N_j)!_p$ is defined to be the product of those integers $\leq n$ that are not divisible by p.

Using Proposition 1, we can see that (for $p \neq 2$)

$$C(n) = \frac{(2n)!}{(n+1)!n!} \equiv (-1)^{\nu_{p^k}(C(n))} p^{\nu_p(C(n))} \frac{\prod_{j\geq 0}((2N)_j)!_p}{\prod_{j\geq 0}(N_j)!_p \prod_{j \geq 0}((N+1)_j)!_p} \pmod{p^k}.$$

We want to find sequnces of Catalan numbers ({C(f(n))} for which this expression is eventually constant; note that it is eventually constant if and only if all three of its components are (the power of -1, the power of p, and the quotient of infinite products).

So let $\{C(f(n))\}$ be an arbitrary sequence of Catalan numbers. Ultimately, we want to find formula for the p-ary expansion of $f(n)$ (e.g., $f(n) = p^n + p^{n-1} + p^{n-2}$) such that $\{C(f(n))\}$ converges. For now, write $f(n) = a_0 + a_1 p + \cdots + a_n p^n$. ($N_j$ will now correspond to $f(n)$.)

By adding the restriction that $a_i \leq \frac{p-1}{2}$ for all $i = 0, \ldots, n$, we can simplify $\frac{\prod_{j\geq 0}((2N)_j)!_p}{\prod_{j\geq 0}(N_j)!_p \prod_{j\geq 0}((N+1)_j)!_p}$ considerably. Firstly, note that because $a_0 \neq p-1$, the p-ary expansion of $f(n)+1$ differs from that of $f(n)$ only in that its p^0 coefficient is $a_0 + 1$. So if $j > 0$, we have that $N_j = a_j + a_{j+1}p + \cdots + a_{j+k-1}p^{k-1} = (N+1)_j$, and if $j = 0$, we have that $(N+1)_0 = (a_0+1) + a_1 p + \cdots + a_{k-1}p^{k-1} = N_0 + 1$. Thus we have that

$$\prod_{j\geq 0}((N+1)_j)!_p = (N_0+1)\prod_{j\geq 1}(N_j)!_p.$$

Secondly, note that our restriction implies that for every j, $(2N)_j = 2N_j$. We can thus write

$$\frac{\prod_{j\geq 0}((2N)_j)!_p}{\prod_{j\geq 0}(N_j)!_p \prod_{j\geq 0}((N+1)_j)!_p} = \frac{2}{N_0+1} \cdot \prod_{j\geq 1} \frac{(2N_j)!_p}{((N_j)!_p)^2}.$$

This expression is eventually constant modulo arbitrarily large powers of p if and only if N_0 and $\prod_{j\geq 1} \frac{(2N_j)!_p}{((N_j)!_p)^2}$ are. It is easy to see when this holds for N_0, which is eventually constant if and only if the first k coefficients ("first" meaning starting with the least index) of the p-ary expansion of the sequence elements are eventually constant.

What about $\prod_{j\geq 1} \frac{(2N_j)!_p}{((N_j)!_p)^2}$? Remember that $(N_j)!_p$ is the product of those numbers that are less than N_j and coprime with p. So since p is fixed, $\prod_{j\geq 1} \frac{(2N_j)!_p}{((N_j)!_p)^2}$ varies between elements of $\{C(f(n)\}$ if and only if the N_js do; that is, if we define $N_J = \{N_j\}_{j\geq 1}$, then $\prod_{j\geq 1} \frac{(2N_j)!_p}{((N_j)!_p)^2}$ is eventually constant if and only if for all sufficienty large m and n, $M_J = N_J$. Notice that this equality can hold even if there exist values of j for which $M_j \neq N_j$.

Since showing $M_j = N_j$ is thus sufficient to show that $\frac{\prod_{j\geq 0}((2N)_j)!_p}{\prod_{j\geq 0}(N_j)!_p \prod_{j\geq 0}((N+1)_j)!_p}$ is eventually constant, and is therefore a crucial step on the way to showing that $\{C(f(n))\}$ converges, we should try to understand what it means for N_J to be equal to M_J.

To do so, let's look at an example. For $n \geq 2$, define $f(n) = p^n + 2p^{n-1} + 3p^{n-2}$. (Note that given the restrictions we've placed on the coefficients, we need $p \geq 7$.) Let $k = 10$, and suppose that $n = 102$, so that $f(n) = a_{102}p^{102} + a_{101}p^{101} + a_{100}p^{100}$, with $a_{102} = 1$, $a_{101} = 2$, and $a_{100} = 3$. Then for $j = 1, \ldots, 90$, $N_j = a_j + a_{j+1}p + \cdots + a_{j+9}p^9 = 0$, since none of these N_j include a term that has a_{100}, a_{101}, or a_{102} as its coefficient. But then $N_{91} = 3p^9$, $N_{92} = 3p^8 + 2p^9$, $N_{93} = 3p^7 + 2p^8 + p^9$, and so on until $N_{101} = 2 + p$ and $N_{102} = 1$. For $j > 102$, we again have $N_j = 0$.

Now consider $m = 1002$. We get $N_j = 0$ for $j = 1, \ldots, 990$, because none of these N_j include a term with $a_{1000}(= 3)$, $a_{1001}(= 2)$ or $a_{1002}(= 1)$ as its coefficient. But then $N_{991} = 3p^9$, $N_{992} = 3p^8 + 2p^9$, $N_{993} = 3p^7 + 2p^8 + p^9$, and so on until $N_{1001} = 2 + p$ and $N_{1002} = 1$. For $j > 1002$, we again have $N_j = 0$. So, in running j from 1 to infinity, we get the same set of numbers for both $n = 102$ and $m = 1002$ (even though, in this case, N_j never equals M_j). We have that $N_J = M_J$.

Why did this choice of $f(n)$ work? There was nothing special about the coefficients that we chose; we only needed to adhere to the restrictions established above. It was important, however, that each $f(n)$ had the same number of terms (so, for example, $f(n) = 1+p+\cdots+p^n$ wouldn't work), and we also needed the set and order of the coefficients to remain constant between terms (so, for example, $p^2 + 2p^3$ and $2p^4 + p^5$ could not both have been values of $f(n)$).

Our discussion suggests the following conjecture:

Proposition 2. *Let p be an arbitrary prime, and let $l \geq 0$ be arbitrary. Let α be a vector of l numbers arbitrarily chosen from $\{1, \ldots, \frac{p-1}{2}\}$ (write $\alpha = \langle \alpha_0, \ldots, \alpha_l \rangle$). For $n \geq l$, define $f(n) = \alpha \cdot \langle p^{n-l}, p^{n-l+1}, \ldots, p^n \rangle$ (so that $f(n) = \sum_{i=n-l}^{n} a_i p^i$, with $a_i = \alpha_{i-n-l}$). Then $\{C(f(n))\}$ converges p-adically.*

What we want to investigate moving forward:

- Find the 2-adic limit of $C(2^n)$
- Refine our process to find more combinatorial sequences that have p-adic limits or converge p-adically
- Further analyze the Γ_p function and its Mahler expansion
- Try using Newton's method and/or computing the coefficients of the Mahler series to find the limit of $C(2^n)$