What the p-Addicts Did This Week

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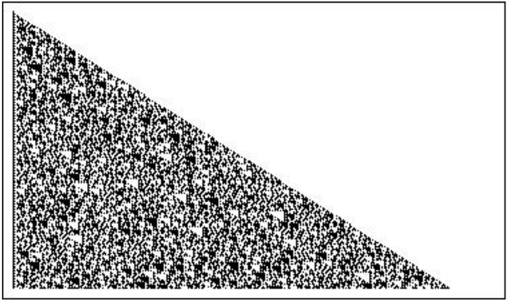
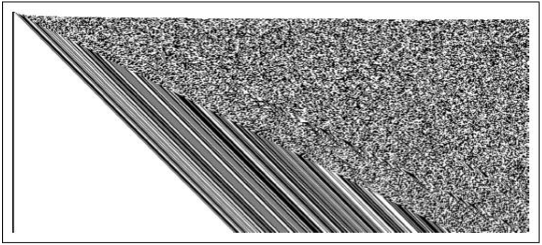
Outline

Eric's Paper about $\{3^n\}$



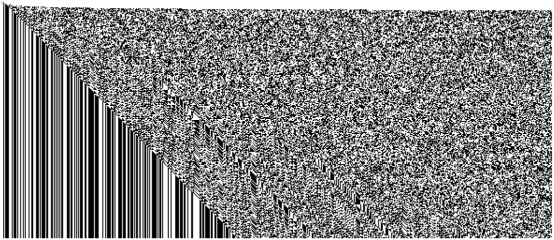
Example

This is a picture of the sequence $\{3^n\}$ in base 2, where the nth row is the binary digits (in black and white) of $\{3^n\}$.



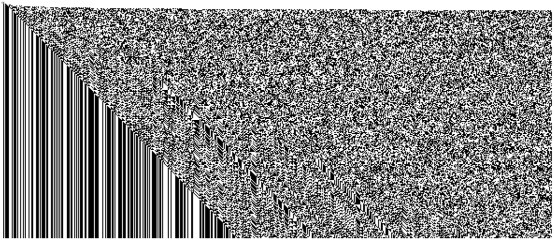
Example

Now we have the sequence $\{3^{2^n}\}$ in base 2, where the nth row is the binary digits of $\{3^{2^n}\}$.



Example

This is the same sequence, but sheared so that the white space is gone and the columns line up. Using p-adic power series, Eric was able to prove that the limit of this sequence was log3.

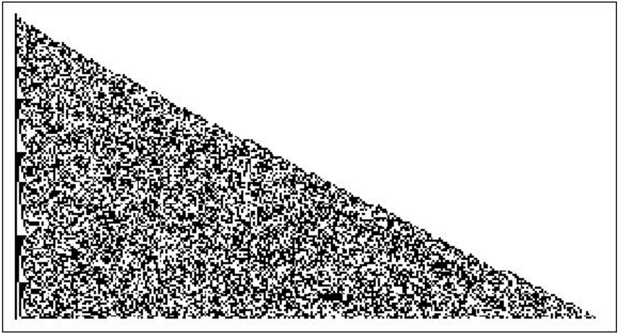


Definition of Convergence

Remember that the definition of convergence is that for every $k > 0$ there is a N such that for $n, m > N$ we have:

$$|C(2^n) - C(2^m)|_2 \leq 2^{-k}$$

Catalan(n)



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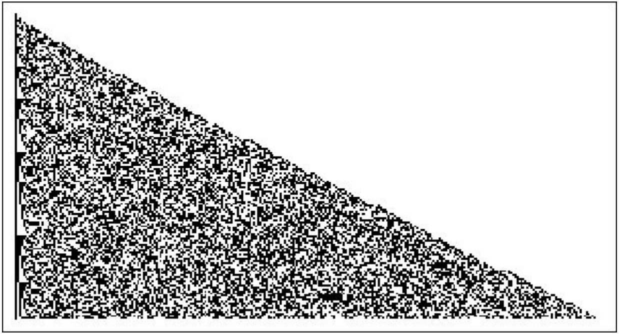
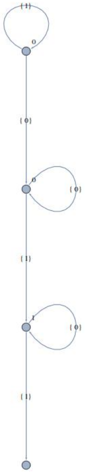


Conjecture: None

Automata



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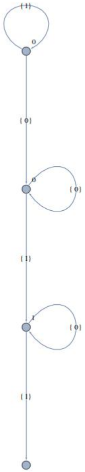

Conjecture: $\{\nu_2(C(2^n))\}$ is constant.

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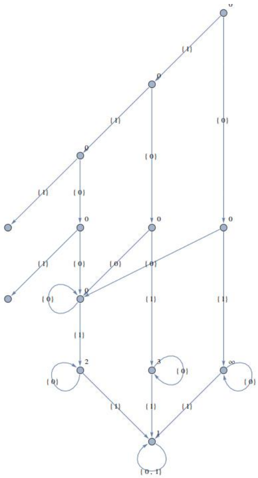

Conjecture: $\{\nu_2(C(2^n) - 2)\}$ is constant.

Convergence of $\{C(2^n)\}$

We used Mathematica to investigate the sequence $\{C(2^n)\}$. It seems to converge 2-adically. Below are the first 25 terms of the sequence in reversed base-2 representation:

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Unfortunately, we couldn't collect enough data to make a conjecture about *what* the limit of the sequence is.

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$$|C(p^n) - C(p^m)|_p \leq p^{-k} \text{ iff}$$

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And we used a fact from an existing paper: that for all $n > k - 1$, $C(2^n) \equiv C(2^{k-1}) \pmod{2^k}$.

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And we have a poorly-understood proof that $\{C(p^n)\}$ converges, using an existing result that has to do with the number of terms in the numerator of $\binom{2p^n}{p^n}$ that are congruent to the elements of the multiplicative group $\mathbb{Z}_{p^k}^{\times}$ modulo p^k. Don't ask.

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The methods we used to prove convergences this week were all number-theoretic. We want to find some useful analytic methods as well.

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Tune-in to next week's episode to find out!