P-adic Limits of Catalan Subsequences

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Abstract

Methods for determining p-adic convergence of sequences which are expressible in terms of products of factorials are established. The Catalan sequence is investigated, using these methods, for p-adically convergent subsequences. An infinite class of convergent subsequences of Catalan numbers is found for every prime, and the limits of these subsequences are evaluated.

Introduction

A student familiar with introductory analysis will be familiar with the construction of $\mathbb{R}$ as a completion of $\mathbb{Q}$. In this construction of $\mathbb{R}$, its elements are defined as equivalence classes of sequences in $\mathbb{Q}$ which are Cauchy convergent with respect to the familiar Euclidean distance metric.

The *p-adic field*, denoted $\mathbb{Q}_p$, is a second completion of $\mathbb{Q}$. Instead of the familiar Euclidean metric, it uses a metric induced by the *p-adic norm*.

Definition 1. The *p-adic valuation* of an integer n, denoted $\nu_p(n)$, is defined to be the greatest power of p that divides n. For a rational number $x = \frac{a}{b}$, define $\nu_p(x) = \nu_p(|a|) - \nu_p(|b|)$. The *p-adic norm* of x is defined as $|x|_p = p^{-\nu_p(x)}$.

The p-adic metric is defined as the p-adic norm of the difference of two numbers in $\mathbb{Q}_p$. As noted, the completion of $\mathbb{Q}$ under the p-adic metric yields $\mathbb{Q}_p$. A detailed account of the completion of $\mathbb{Q}$ to $\mathbb{Q}_p$ can be found in [FG].

The definition of p-adic convergence is analogous to that of convergence with respect to the Euclidean metric.

Example 1. In $\mathbb{Q}_p$, $\lim_{n\to\infty} p^n = 0$. This is because as n increases, $\nu_p(p^n) = n$ increases, and thus $|p^n|_p = p^{-n}$ tends to 0. The sequence $\{p^n + 1\}$ tends to 1. This is because the sequence $\{(p^n + 1) - 1\}$ tends to 0.

Because elements of combinatorial sequences are natural numbers, to investigate the convergence of the sequences it is superfluous to work in $\mathbb{Q}_p$. Instead, one need only work in the completion of $\mathbb{Z}$ under the p-adic metric; this is a subset of $\mathbb{Q}_p$ called the *p-adic integers* (denoted $\mathbb{Z}_p$). It is well-known that $\mathbb{Z}_p$ is a compact subset of $\mathbb{Q}_p$, which is itself a metric space. Thus, every combinatorial sequence has convergent subsequences in $\mathbb{Z}_p$. Furthermore, convergence in p-adic fields is easier to determine than in euclidean space due to the following

Lemma 1 (Strong Triangle Inequality)**.** *For all x, $y \in \mathbb{Q}_p$,*

$$|x - y|_p \leq \max\{|x|_p, |y|_p\}.$$

Using the strong triangle inequality, it can be shown that a sequence converges p-adically if and only if its difference sequence converges.

Corollary 1 (Convergence Criterion)**.** *In $\mathbb{Q}_p$, a sequence $\{a_n\}$ converges if and only if the sequence $\{a_{n+1} - a_n\}$ converges.*

For proofs of Proposition 1 and Proposition 1, see [FG] or [SK].

Methods and Results

We determine for all $a \in \mathbb{N}$:

$$\lim_{n\to\infty} C(ap^n)$$

This problem can be reduced to that of finding the limit of the sequence of central binomial coefficients $\{\binom{2ap^n}{ap^n}\}$. The elements of this latter sequence can be expressed in terms of the well-known p-adic gamma function.

Definition 2 (p-adic Gamma Function)**.** Let p be prime, and $x \in \mathbb{Z}_p$. The *p-adic gamma function*, $\Gamma_p(x)$, is defined to be the unique continuous p-adic interpolation of the function taking the following values over $\mathbb{N}$.

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1\\ p\nmid k}}^{n-1} k \text{ , and } \Gamma_p(0) = 1.$$

The following lemma expresses $\binom{2ap^n}{ap^n}$ in terms of the p-adic gamma function.

Lemma 2. *For all primes p and all $a \in \mathbb{N}$,*

$$\binom{2ap^n}{ap^n} = \binom{2a}{a} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}.$$

Expressing a combinatorial sequence in terms of Γ_p is very powerful because we can invoke its continuity to get useful results such as the following

Lemma 3. *Let p be prime and let $a \in \mathbb{N}$. In $\mathbb{Z}_p$, $\lim_{n\to\infty} \Gamma_p(ap^n) = 1$.*

Our main results follow (although not trivially) from Lemma 3.

Theorem 1 (Limits of Catalan Subsequences)**.** *For all primes p and all $a \in \mathbb{Z}$, the p-adic limit of $C(ap^n)$ exists and is given by*

$$\lim_{n\to\infty} C(ap^n) = \binom{2a}{a} \prod_{i=1}^{\infty} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}. = \binom{2a}{a} \prod_{\substack{i=1\\ p\nmid i}}^{\infty} i^{2\lfloor \log_p(i/a)\rfloor - \lfloor \log(i/2a)\rfloor}.$$

Given that $\lim_{n\to\infty} C(ap^n)$ is known, it is not hard to find $\lim_{n\to\infty} C(ap^n + r)$ for all $r \in \mathbb{Z}$. The latter limit is thus presented as a corollary to Theorem 1.

Corollary 2. *Let $r \in \mathbb{Z}$ and let $L = \lim_{n\to\infty} C(ap^n)$. Then*

$$\lim_{n\to\infty} C(ap^n + r) = \begin{cases} C(r) \cdot L & \text{if } r > 0 \\ -\frac{1}{2}L & \text{if } r = -1 \text{ and } p \neq 2 \\ 0 & \text{if } r < -1. \end{cases}$$

One interesting consequence is that Corollary 2 implies that $C(n)$ does not converge p-adically. This is because for a fixed a we can choose distinct values of r that yield convergent subsequences with different limits. Another is in the following

Example 2. In $\mathbb{Z}_p$, $\lim_{n\to\infty} C(2^n) = 2 \cdot 3 \cdot (5 \cdot 7)^2 \cdot (9 \cdot 11 \cdot 13 \cdot 15)^3 \cdots$. This is an infinite product consisting of blocks of 2^n consecutive odd numbers raised to the $n + 1^{st}$ power.

Conclusion

Combinatorial sequences, while they may not have limits, are integer sequences, and as such they have convergent subsequences by compactness of the p-adic integers. Sometimes the form of these limits can be difficult to characterize explicitly. In the case of the Catalan numbers, the sequence does not converge p-adically. However, we have an infinite class of increasing subsequences which have limits.

The limits of these subsequences appear to resist evaluation by any standard means (such as power series expansions, or continuity). However, we have evaluated the p-adic limit of the subsequence $C(ap^n)$, and even more generally $C(ap^n + r)$, where a is a constant and $r \in \mathbb{Z}$. The limits of these sequences can be written as an infinite product of numbers which don't divide p, raised to powers increasing logarithmically.

It remains an open problem to generalize (let alone characterize) which subsequences of Catalan numbers converge as well as to find the limits of these subsequences. The methods used here will no doubt present their utility in attempting to do this as well as in a similar analysis of other combinatorial sequences.

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