p-adic Limits of Combinatorial Sequences

Alexandra Michel
Mills College

Andrew Miller
Amherst College

Joseph Rennie
Reed College

July 22, 2014

Abstract

The definition of a p-adically Cauchy convergent sequence can be equivalently stated in terms of congruences of elements of that sequence modulo arbitrarily large powers of p: a sequence is p-adically Cauchy convergent if and only if it is eventually constant modulo p^k for all $k \in \mathbb{N}$. Using a result on factorials modulo powers of primes by Granville (1997), a class of p-adically convergent sequences of Catalan numbers is identified and characterized in terms of the p-ary expansion of the sequence elements. The limit of one such sequence, $\{C(2^n)\}$, is found.

1 Introduction

An introductory analysis course describes a completion of $\mathbb{Q}$ to the reals where real numbers are defined as equivalence classes of Cauchy sequences of rationals. There is a second important completion of $\mathbb{Q}$ which, instead of the familiar Euclidean distance metric, uses a metric known as the *p-adic norm*. Define the *p-adic valuation* of an integer n, $\nu_p(n)$, to be the greatest power of p that divides n, and write $\nu_p(n) = k$. For rational numbers $x = \frac{a}{b} \in \mathbb{Q}$, $\nu_p(x) = \nu_p(a) - \nu_p(b)$. The *p-adic norm* of n is then defined as $|x|_p = p^{-\nu_p(x)}$. The notion of Cauchy convergence with respect to the p-adic norm can be defined in a manner analogous to that of Cauchy convergence with respect to the Euclidean norm. The *p-adic numbers* are then the completion of $\mathbb{Q}$ with respect to the p-adic norm.

The aim of this paper is to investigate the convergence of certain sequences of p-adic numbers. In a 2010 paper, E. Rowland [ER] finds one such sequence: $\{3^{2^n}\}$. Rowland looks at $\{3^n\}$ in the 2-adics, which does not converge, before noticing that the subsequence $\{3^{2^n}\}$ shows patterns of converging term by term when its terms are viewed in their base-2 representations. He uses a power series approach to show that the limit is 1, and then finds that $\frac{3^{2^n}-1}{2^n}$ converges to $\log 3$ (expressed as a 2-adic number).

Rowland's process is valuable in the investigation of the Catalan numbers, a sequence which appears in numerous combinatorial problems [RS]. These numbers also have a convenient closed form: $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

2 A Class of p-adically Convergent Sequences

An equivalent formulation of p-adic convergence can be used to find a class of convergent subsequences of the Catalan numbers: to show that a sequence converges p-adically, one can show that its elements are eventually constant modulo arbitrarily large powers of p. This re-formulation is useful because there are existing results on factorials, binomial coefficients, and Catalan numbers modulo powers of primes. We used one such result to prove the following theorem.

Theorem 1. *For all primes p and all $a \in \mathbb{N}$, $\{C(ap^n)\}_{n\geq 0}$ converges p-adically.*

The proof relies on an existing result due to Granville [AG] which allows us to find a formula for $C(ap^n)$ modulo an arbitrary power of p.

Lemma 1 (Granville 1997)**.** *Let n be an integer, and write $n = \gamma_0 + \gamma_1 p + \cdots + \gamma_d p^d$ in base p. For $j \geq 0$ and p^k a power of p, define n_j to be the least positive residue of $\lfloor \frac{n}{p^j} \rfloor \pmod{p^k}$ (so that $n_j = \gamma_j + \gamma_{j+1}p + \cdots + \gamma_{j+k-1}p^{k-1}$). Define $(n_j!)_p$ to be the product of numbers $\leq n_j$ that are coprime with p. Then*

$$n! \equiv p^{\nu_p(n!)} \delta^{\nu_{p^k}(n!)} \prod_{j=\geq 0} (n_j!)_p \pmod{p^k},$$

where $\delta(p,k) = \begin{cases} 1 & \textit{if } p=2 \textit{ and } k \geq 3 \\ -1 & \textit{otherwise} \end{cases}$.

Since $C(n) = \frac{(2n)!}{n!(n+1)!}$, it follows that

$$C(n) \equiv \delta^{\nu_{p^k}(C(n))} p^{\nu_p(C(n))} \frac{\prod_{j\geq 0}((2n)_j)!_p}{\prod_{j\geq 0}(n_j)!_p \prod_{j\geq 0}((n+1)_j)!_p} \pmod{p^k}. \tag{1}$$

Theorem 1 uses the case $n = ap^n$. To show that $\{C(ap^n)\}$ is eventually constant modulo p^k, it thus suffices to show that all three components of the right-hand side of (1) (the power of δ, the power of p, and $\mathcal{P}(n) = \frac{\prod_{j\geq 0}((2ap^n)_j)!_p}{\prod_{j\geq 0}(ap^n_j)!_p \prod_{j\geq 0}((ap^n+1)_j)!_p}$) are eventually constant modulo p^k. With this in mind, we are ready to prove the theorem.

Proof. Fix $k \geq 1$. Write $a = \alpha_0 + \alpha_1 p + \cdots + \alpha_m p^m$ in base p ($\alpha_i \neq 0$ for all i), so that $ap^n = \alpha_0 p^n + \cdots + \alpha_m p^{n+m}$ in base p. To show that $\delta^{\nu_{p^k}(C(ap^n))}$ and $p^{\nu_p(C(ap^n))}$ are eventually constant modulo p^k, it is clearly sufficient to show that $\nu_{p^k}(C(ap^n))$ is constant for all n. This is an easy application of Legendre's 1808 result that $\nu_p(n!) = \frac{n-s(n)}{p-1}$, where $s(n)$ is the sum of the base-p coefficients of n, and its proof is left to the reader.

All that remains to show is that $\mathcal{P}(ap^n)$ is eventually constant. This expression can be simplified considerably; it can be shown that

$$(ap^n+1)_j = \begin{cases} ap^n_0 + 1 & \text{if } j = 0 \\ ap^n_j & \text{if } j \neq 0 \end{cases} \tag{2}$$

and that

$$(2ap^n)_j = 2(ap^n_j) \text{ for all } j. \tag{3}$$

To see (2), note that the base-p expansion of ap^n+1 differs from that of ap^n only in that its p^0 coefficient is 1, whereas the p^0 coefficient of the base-p expansion ap^n is 0. The p^0 coefficient is included in $ap^n_j = a_jp^j + a_{j+1}p^{j+1} + \cdots + a_{j+k-1}p^{j+k-1}$ only when $j=0$; thus, $(ap^n+1)_0 = ap^n_0 + 1$ for $j=0$ and $(ap^n+1)_j = ap^n_j$ otherwise.

For (3), simply note that $(2ap^n)_j = \lfloor \frac{2ap^n}{p^j} \rfloor \pmod{p^k} = 2ap^{n-j} \pmod{p^k} = 2\lfloor \frac{ap^n}{p^j} \rfloor \pmod{p^k} = 2ap^n_j$ for all j. Thus

$$\mathcal{P}(ap^n) = \frac{\prod_{j\geq 0}((2ap^n)_j)!_p}{\prod_{j\geq 0}(ap^n_j)!_p \prod_{j\geq 0}((ap^n+1)_j)!_p} = \frac{2}{ap^n_0+1} \cdot \prod_{j\geq 1} \frac{(2ap^n_j)!_p}{((ap^n_j)!_p)^2}.$$

Clearly, this is eventually constant modulo p^k if ap^n_0 and $\prod_{j\geq 1} \frac{(2ap^n_j)!_p}{((ap^n_j)!_p)^2}$ are. It is easy to check that the former is constant for all $n > k$. As for the latter, since it varies with n only if $ap^n_J = \{ap^n_j\}_{j\geq 1}$ does, it is eventually constant modulo p^k if ap^n_J is constant for all sufficiently large n.

Taking $n > k$ will suffice. Given such an n, write $ap^n = a_np^n + a_{n+1}p^{n+1} + \cdots + a_{n+m}p^{n+m}$, where $a_{n+i} = \alpha_i$ for $i \in \{0,\ldots,m\}$. For $j \in \mathbb{N} \setminus \{n-k+1,\ldots,n+m\}$, $ap^n_j = 0$, since none of a_n through a_{n+m} (the non-zero coefficients of the base-p expansion of ap^n) appears as a coefficient of ap^n_j for any such j. Thus there are $n+m-(n-k) = m+k$ values of j for which ap^n_j is non-zero (crucially, this number does not depend on n). Running j from $n-k+1$ to $n+m$, we get that $ap^n_J = \{\alpha_0p^{k-1}, \alpha_0p^{k-2}+\alpha_1p^{k-1}, \ldots, \alpha_{m-1}+\alpha_mp, \alpha_m\}$. None of the elements of this set depends on n, as desired. □

The following example is meant to give the reader a better sense of the previous paragraph.

Example 1. Suppose that $a=1$, so that $ap^n = p^n$. Fix $k=3$. For a given $n>3$, the base-p expansion of $p^n = a_np^n = 1\cdot p^n$ has only one non-zero coefficient, so for all $j \geq 1$, $p^n_j = a_j + a_{j+1}p + a_{j+2}p^2$ will have at most one non-zero term. If none of j, $j+1$, or $j+2$ is n, then p^n_j=0; thus, $p^n_j = 0$ for all $j \in \mathbb{N} \setminus \{n-2, n-1, n\}$. For the remaining values of j, we have tht $p^n_{n-2} = p^2$, $p^n_{n-1} = p$, and $p^n_n = 1$, so that $p^n_J = \{1, p, p^2\}$. The cardinality of this set, $3 = 0+3 = m+k$, does not depend on n, and neither do its elements.

Notice that taking $n > k = 3$ is necessary because if $n=2$, for instance, $p^2_1 = p$, $p^2_2 = 1$, and $p^2_j = 0$ for all $j > 2$. Thus $p^2_J = \{1, p\}$; p^2 is excluded from p^2_J because there are no j for which a_{j+2} is non-zero.

3 The 2-adic Limit of $\{C(2^n)\}$

The sequence of Catalan numbers of powers of 2 converges by Theorem 1. The 2-adic Gamma function can be used to find its limit.

Definition 1 (p-adic Gamma Function)**.**

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ p\nmid k}}^{n-1} k \text{ where } \Gamma_p(0) = 1$$

Lemma 2. *In the 2-adics,* $\lim_{n\to\infty} (\Gamma_2(2^n))^n = 1$

Proof. The 2-adic Gamma Function is a continuous function in the 2-adics [NK], and so is x^n. Therefore $\lim_{n\to\infty} (\Gamma_2(2^n))^n = (\Gamma_2(\lim_{n\to\infty} 2^n))^n$. Because as $n \to \infty$ the sequence $\{2^n\}$ acquires more and more powers of 2, $\lim_{n\to\infty} 2^n = 0$ in $\mathbb{Z}_2$.Then this means that $\lim_{n\to\infty} \Gamma_2(2^n) = \Gamma_2(0) = 1$. □

Lemma 3. *Let* $C'(p^n) = \{\binom{p^{n+1}}{p^n}\}$ *and* $C(p^n) = \{\frac{1}{p^n+1}\binom{p^{n+1}}{p^n}\}$*. Then* $C'(p^n)$ *converges to the same limit as* $C(p^n)$ *p-adically.*

Proof. Let $k > 0$, then for $N = k - 1$ we want to show that $\left\|\frac{1}{p^n+1}\binom{p^{n+1}}{p^n} - \binom{p^{n+1}}{p^n}\right\|_p < p^{-k}$ for $n > N$. But this is equivalent to showing that $\nu_p[(\frac{1}{p^n+1}\binom{p^{n+1}}{p^n} - \binom{p^{n+1}}{p^n})] > k$ for $n > N$. So for $n > N$, we have:

$$\begin{aligned}
\nu_p[(\frac{1}{p^n+1}\binom{p^{n+1}}{p^n} - \binom{p^{n+1}}{p^n})] &= \nu_p[(\frac{1}{p^n+1} - 1)\binom{p^{n+1}}{p^n}] \\
&= \nu_p(\frac{p^n}{p^n+1}) + 1 \\
&= n + 1 \\
&> N + 1 \\
&= (k-1) + 1 = k
\end{aligned}$$

□

Theorem 2. *In* $\mathbb{Z}_2$*,* $\lim_{n\to\infty} C(2^n) = 2 \cdot 3 \cdot (5 \cdot 7)^2 \cdot (9 \cdot 11 \cdot 13 \cdot 15)^3 \cdots$.

Proof. Lemma 3 tells us that instead of looking at $\lim_{n\to\infty} C(2^n)$, we can instead look at $\lim_{n\to\infty} C'(2^n)$ and it will yield the same limit. Applying the well known binomial coefficient identity $\binom{2n}{n} = \frac{2^{2n}\Gamma_2(2n)}{n!}$ and the formula for the 2-adic valuation of factorials, we get

$$\begin{aligned}
C'(2^n) &\to \frac{2^{2^n}}{2^{2^n-1}\Pi_{i=0}^n \Gamma_2(2^i)} \\
&= \frac{2}{3^{n-1} \cdot (5 \cdot 7)^{n-2} \cdot (9 \cdot 11 \cdot 13 \cdot 15)^{n-3} \cdots} \\
&= \frac{2 \cdot 3 \cdot (5 \cdot 7)^2 \cdot (9 \cdot 11 \cdot 13 \cdot 15)^3 \cdots}{(3 \cdot 5 \cdot 7 \cdots)^n}
\end{aligned}$$

By Lemma 2 the denominator goes to 1.

□

4 Conclusion

Combinatorial sequences, while they may not have limits, are integer sequences, and as such they have convergent subsequences. Sometimes the form of these sequences can be difficult to characterize explicitly. In the case of the Catalan numbers, we have a sufficient condition, but by no means a necessary condition. Furthermore, the limits of these subsequences appear to resist evaluation by any standard means (such as power series expansions, Mahler expansions, and continuity). However, we have evaluated the limit of the subsequence of powers of two, in the 2-adic metric, as an infinite product of odd numbers raised to powers increasing logarithmically.

4.1 Open Problems

It remains an open problem to characterize all converging subsequences of Catalan numbers as well as to find the limits of these subsequences, including those which we have already characterized. The methods used to answer these questions will no doubt present their utility in a similar analysis of other combinatorial sequences.

5 Acknowledgments

This work was carried out during the 2014 Mathematical Sciences Research Institute Undergraduate Program (MSRI-UP) which is funded by the National Science Foundation (grant No. DMS-1156499) and the National Security Agency (grant No. H-98230-13-1-0262). We would like to thank Dr. Victor Moll, Dr. Herbert Medina, Dr. Eric Rowland, Asia Wyatt and our peers in MSRI-UP 2014 for their support throughout this process.

References

[ER] Rowland, Eric. *"Regularity Vs. Complexity in the Binary Representation of* 3^n*"*. arXiv preprint arXiv:0902.3257, 2010.

[AG] Granville, Andrew. *"Binomial coefficients modulo prime powers"*. Canadian Mathematical Society Conference Proceedings, Vol 20, pages 253-275. 1997.

[FG] Gouvea, Fernando Q. *p-adic Numbers: An Introduction*. Second Edition. Springer, 2003.

[NK] Koblitz, Neal. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Second Edition. Springer, 1984

[RS] Stanley, Richard. *Enumerative Combinatorics*. Vol 1. Second Edition. Cambridge University Press, 2011.