On p-adic Limits of Subsequences of the Catalan Numbers

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Abstract

Methods for determining p-adic convergence of sequences which are expressible in terms of products of factorials are established. The Catalan sequence is investigated, using these methods, for p-adically convergent subsequences. An infinite class of convergent subsequences of Catalan numbers is found for every prime, and the limits of these subsequences are evaluated.

1 Introduction

1.1 The p-adic numbers

A student familiar with introductory analysis will be familiar with the construction of $\mathbb{R}$ as a completion of $\mathbb{Q}$. In this construction of $\mathbb{R}$, its elements are defined as equivalence classes of sequences in $\mathbb{Q}$ which are Cauchy convergent with respect to the familiar Euclidean distance metric.

The *p-adic field*, denoted $\mathbb{Q}_p$, is a second completion of $\mathbb{Q}$. Instead of the familiar Euclidean metric, it uses a metric induced by the *p-adic norm*.

Definition 1.1. The *p-adic valuation* of an integer n, denoted $\nu_p(n)$, is defined to be the greatest power of p that divides n. For a rational number $x = \frac{a}{b}$, define $\nu_p(x) = \nu_p(|a|) -$

$\nu_p(|b|)$. The *p-adic norm* of x is defined as $|x|_p = p^{-\nu_p(x)}$.

Example 1.2. $\nu_5(35) = 1$, because only one power of 5 divides 35, and $|35|_5 = 5^{-\nu_5(35)} = 5^{-1} = \frac{1}{5}$. $\nu_5(25) = 2$, so $|25|_5 = 5^{-\nu_5(25)} = 5^{-2} = \frac{1}{25}$.

The p-adic metric is defined as the p-adic norm of the difference of two numbers in $\mathbb{Q}_p$. As noted, the completion of $\mathbb{Q}$ under the p-adic metric yields $\mathbb{Q}_p$. A detailed account of the completion of $\mathbb{Q}$ to $\mathbb{Q}_p$ can be found in [FG].

1.2 Convergence in $\mathbb{Z}_p$

The definition of p-adic convergence is analogous to that of convergence with respect to the Euclidean metric.

Definition 1.3 (p-adic Convergence)**.** Given a sequence $\{a_n\} \in \mathbb{Q}_p$, we say that $\{a_n\}$ *converges p-adically* if for all $k \geq 1$, there exists an $N \in \mathbb{N}$ such that for all m, $n > N$,

$$|a_m - a_n|_p \leq p^{-k}.$$

Example 1.4. In $\mathbb{Q}_p$, $\lim_{n\to\infty} p^n = 0$. This is because as n increases, $\nu_p(p^n) = n$ increases, and thus $|p^n|_p = p^{-n}$ tends to 0.

The sequence $\{p^n + 1\}$, however, tends to 1. This is because $\nu_p(p^n + 1) = 0$ for all n, and thus for all n, $|p^n + 1|_p = p^0 = 1$.

Because elements of combinatorial sequences are natural numbers, to investigate the convergence of the sequences it is superfluous to work in $\mathbb{Q}_p$. Instead, one need only work in the completion of $\mathbb{Z}$ under the p-adic metric; this is a subset of $\mathbb{Q}_p$ called the *p-adic integers* (denoted $\mathbb{Z}_p$). It is well-known that $\mathbb{Z}_p$ is a compact subset of $\mathbb{Q}_p$, which is itself a metric space. Thus, every combinatorial sequence has convergent subsequences in $\mathbb{Z}_p$.

Investigating the convergence of these subsequences with respect to the p-adic metric has a few important advantages. The most important of these is that the p-adic metric satisfies a strong-triangle inequality.

Proposition 1.5 (Strong Triangle Inequality)**.** *For all x, $y \in \mathbb{Q}_p$,*

$$|x - y|_p \leq \max\{|x|_p, |y|_p\}.$$

Using the strong triangle inequality, it can be shown that a sequence converges p-adically if and only if its difference sequence converges.

Proposition 1.6 (Convergence Criterion). *In $\mathbb{Q}_p$, a sequence $\{a_n\}$ converges if and only if the sequence $\{a_{n+1} - a_n\}$ converges.*

For proofs of Proposition 1.5 and Proposition 1.6, see [FG] or [SK].

Finally, we note an equivalent statement of the definition of p-adic convergence.

Proposition 1.7 (Equivalent Definition of p-adic Convergence). *In $\mathbb{Q}_p$, a sequence $\{a_n\}$ converges if and only if for all $k \geq 1$, it is eventually constant modulo p^k. Furthermore, $\{a_n\}$ converges to a limit L if and only if for all $k \geq 1$, it is eventually constant to L modulo p^k.*

Proof. Given $k \geq 1$ and sufficiently large m and n,

$$\begin{aligned} |f(n) - f(m)|_p \leq p^{-k} \quad &\text{if and only if} \quad \nu_p(f(n) - f(m)) \geq k \\ &\text{if and only if} \quad f(n) - f(m) \equiv 0 \pmod{p^k} \\ &\text{if and only if} \quad f(n) \equiv f(m) \pmod{p^k}, \end{aligned}$$

proving the first statement of Proposition 1.7. The proof of the second statement is almost identical. □

Note that it is easy to see that $p^n \to 0$ using Theorem 1.7. Given $k \geq 1$, for all $n > k$, $p^n \equiv 0 \pmod{p^k}$.

1.3 Catalan Numbers

This paper finds p-adic limits of subsequences of the Catalan numbers, C(n). The Catalan numbers are a famous sequence of natural numbers with numerous combinatorial interpretations. For example, they count the number of ways to balance n pairs of parentheses (i.e., such that each open parathesis is closed and each closed parenthesis is opened). For example, 3 pairs of paretheses can be arranged in the following ways.

$$((())),\ ()()(),\ (())(),\ ()(()),\ (()()).$$

Thus, $C(3) = 5$.

The Catalan numbers have a convenient closed form in terms of the central binomial coefficients:

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

We can use this formula to check that $C(3)$ is indeed 5.

$$C(3) = \frac{1}{4}\binom{2\cdot 3}{3} = \frac{6!}{4\cdot 3!^2} = \frac{5\cdot 6}{3!} = 5.$$

Finally, the closed form can be used to derive a recurrence for consecutive Catalan numbers.

$$C(x+1) = \frac{2(2x+1)}{x+2}C(x).$$

2 Finding the p-adic Limit of $C(ap^n)$

In this section,

$$\lim_{n\to\infty} C(ap^n) \tag{2.1}$$

is determined for all $a \in \mathbb{N}$.[1]

Example 2.1. Data generated in Mathematica suggest that $\{C(2^n)\}$ converges. The following graphic shows the binary expansion of $C(2^n)$ for $n = 1, 2, \ldots, 25$. The i^{th} row and j^{th} column gives the coefficient on 2^{j-1} of $C(2^i)$. Coefficients with value 1 are represented by a black dot, those with value 0 by a white dot.

Figure 1: Binary expansions of the first 25 terms of the sequence $C(2^n)$; the power of 2 increases from left to right.

For example, the first row shows the binary representation of $C(1) = 1$. In binary, $1 = 1 + 0\cdot 2 + 0\cdot 2^2 + \cdots$. The coefficient 1 on 2^0 is represented by the black dot in the first

[1]This limit is a p-adic limit, as are all other limits stated in this paper.

column, and the 0 coefficients on the remaining powers of 2 are represented by white dots in the remaining columns.

It is perhaps easiest to see why Figure 1 suggests that $\{C(2^n)\}$ converges by appealing to Proposition 1.6. The binary expansion of $C(2^n) - C(2^{n-1})$ can be obtained by subtracting the $n-1^{st}$ row from the n^{th} row. The resulting binary expansion has a 0 coefficient for all powers of 2 for which the coefficient of $C(2^n)$ agrees with that of $C(2^{n-1})$. As n increases, Figure 1 indicates that the binary expansion of $C(2^n) - C(2^{n-1})$ has a 0 coefficient for an increasingly long string of powers of 2 (starting with 2^0). This indicates that the 2-adic valuation of $C(2^n) - C(2^{n-1})$ is increasing with n, and thus that $|C(2^n) - C(2^{n-1})|$ is tending to 0.

For general a and p, to find the limit of $\{C(ap^n)\}$ it suffices to find the limit of $\{\binom{2ap^n}{ap^n}\}$. This is demonstrated by the following lemma.

Lemma 2.2. *In* $\mathbb{Z}_p$, $\lim\limits_{n\to\infty} C(ap^n) = \lim\limits_{n\to\infty} \binom{2ap^n}{ap^n}$.

Proof. Let $k \geq 1$ be arbitrary. Given $n > k$, note that

$$\left| \frac{1}{ap^n+1}\binom{2ap^n}{ap^n} - \binom{2ap^n}{ap^n} \right|_p < p^{-k} \text{ if and only if } \nu_p\left[\frac{1}{ap^n+1}\binom{2ap^n}{ap^n} - \binom{2ap^n}{ap^n}\right] > k,$$

so it suffices to show the latter. We have

$$\begin{aligned} \nu_p\left[\frac{1}{ap^n+1}\binom{2ap^n}{ap^n} - \binom{2ap^n}{ap^n}\right] &= \nu_p\left[\left(\frac{1}{ap^n+1} - 1\right)\binom{2ap^n}{ap^n}\right] \\ &= \nu_p\left(\frac{ap^n}{ap^n+1}\right) + \nu_p\left[\binom{2ap^n}{ap^n}\right] \\ &\geq n > k, \end{aligned}$$

as desired. $\square$

Thus, the problem of finding the limit of $\{C(ap^n)\}$ can be reduced to that of finding the limit of the sequence of central binomial coefficients $\{\binom{2ap^n}{ap^n}\}$. The elements of this latter sequence can be expressed in terms of the well-known gamma function. On $\mathbb{Z}$, the gamma function is defined to be

$$\Gamma(n) = (n-1)!.$$

We can thus write

$$\binom{2ap^n}{ap^n} = \frac{\Gamma(2ap^n+1)}{(\Gamma(ap^n+1))^2}.$$

But since we are concerned with convergence in $\mathbb{Z}_p$, it will be more useful to write $\binom{2ap^n}{ap^n}$ in terms of a p-adic analog to the gamma function.

Definition 2.3 (p-adic Gamma Function)**.** Let p be prime, and $x \in \mathbb{Z}_p$. The *p-adic gamma function*, $\Gamma_p(x)$, is defined to be the unique continuous p-adic interpolation of the function taking the following values over $\mathbb{N}$.

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ p\nmid k}}^{n-1} k \ \text{, and} \quad \Gamma_p(0) = 1.$$

For a detailed exposition of the p-adic gamma function, including a proof of its existence and uniqueness, see [FG]. The following proposition can be used to prove Lemma 2.5, which expresses $\binom{2ap^n}{ap^n}$ in terms of the p-adic gamma function.

Proposition 2.4. *For all primes p and all $n \in \mathbb{N}$,*

$$n! = \left\lfloor \frac{n}{p} \right\rfloor ! \Gamma_p(n+1)(-1)^{n+1} p^{\left\lfloor \frac{n}{p} \right\rfloor}.$$

Proof. We have

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{k=1 \\ p\nmid k}}^{n} k = \frac{(-1)^{n+1}(n)!}{\prod_{\substack{k=1 \\ p\mid k}}^{n-1} k} = \frac{(-1)^{n+1}(n)!}{p^{\left\lfloor \frac{n}{p} \right\rfloor} \left\lfloor \frac{n}{p} \right\rfloor !}.$$

Solving for $n!$ gives the result. ∎

Lemma 2.5. *For all primes p and all $a \in \mathbb{N}$,*

$$\binom{2ap^n}{ap^n} = \binom{2a}{a} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}.$$

Proof. We first use Theorem 2.4 to express $(ap^n)!$ which gives

$$(ap^n)! = (ap^{n-1})! \Gamma_p(ap^n+1)(-1)^{ap^n+1} p^{ap^{n-1}}. \tag{2.2}$$

This is a first-order recursion on n. It can be used to show via induction that

$$(ap^n)! = a! p^{\frac{ap^n - a}{p-1}} (-1)^{\sum_{i=1}^{n} ap^i + 1} \prod_{i=1}^{n} \Gamma_p(ap^i+1). \tag{2.3}$$

For the base case ($n = 0$), we have $(ap^0)! = a! = a!p^0$.

For the inductive step, assume that (2.3) holds when $n = k$. Then

$$\begin{aligned}(ap^{k+1})! &= (ap^k)!\Gamma_p(ap^{k+1}+1)(-1)^{ap^{k+1}+1}p^{ap^k} \\ &= \left[a!p^{\frac{ap^k-a}{p-1}}(-1)^{\sum_{i=1}^k ap^i+1}\prod_{i=1}^{k}\Gamma_p(ap^i+1)\right]\Gamma_p(ap^{k+1}+1)(-1)^{ap^{k+1}+1}p^{ap^k} \\ &= a!p^{\frac{ap^{k+1}-a}{p-1}}(-1)^{\sum_{i=1}^{k+1} ap^i+1}\prod_{i=1}^{k+1}\Gamma_p(ap^i+1),\end{aligned}$$

completing the induction and proving that (2.3) holds for all n. It can thus be shown that

$$\binom{2ap^n}{ap^n} = \frac{(2ap^n)!}{(ap^n)!^2} = \frac{(2a)!(-1)^n}{(a!)^2}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i+1)}{\Gamma_p(ap^i+1)^2} = \binom{2a}{a}\prod_{i=1}^{n}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2},$$

the desired result. $\square$

Lemma 2.2 and Lemma 2.5 imply that

$$C(ap^n) \to \binom{2ap^n}{ap^n} \to \binom{2a}{a}\prod_{i=1}^{\infty}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}, \tag{2.4}$$

if the latter converges. To show this, one more lemma is needed.

Lemma 2.6. *Let p be prime and let $a \in \mathbb{N}$. In $\mathbb{Z}_p$, $\lim_{n\to\infty}\Gamma_p(ap^n) = 1$.*

Given Lemma 2.6, Equation (2.4), stated here as a theorem, can be proven.

Theorem 2.7 (Limits of Catalan Subsequences)**.** *For all primes p and all $a \in \mathbb{Z}$, the p-adic limit of $C(ap^n)$ exists and is given by*

$$\lim_{n\to\infty} C(ap^n) = \binom{2a}{a}\prod_{i=1}^{\infty}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}.$$

Note: An elementary proof that $\{C(ap^n)\}$ converges (not that it approaches the stated limit) is given in an Appendix (Section 6).

Proof. By Theorem 2.2 and Theorem 2.5, it suffices to show that

$$\prod_{i=1}^{\infty}\frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$$

converges.

To do so, fix $k \geq 1$, and let $n > k$ be arbitrary. Then

$$\begin{aligned}
\left| \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} - \prod_{i=1}^{n-1} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} \right|_p &= \left| \left(\prod_{i=1}^{n-1} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} \right) \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2} - 1 \right) \right|_p \\
&= \left| \prod_{i=1}^{n-1} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} \right|_p \cdot \left| \frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2} - 1 \right|_p \\
&= 1 \cdot \left| \frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2} - 1 \right|_p \to 0,
\end{aligned}$$

where $\lim\limits_{n\to\infty} \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2} - 1 \right) = 0$ because by Lemma 2.6, $\Gamma_p(2ap^n) \to 1$ and $\Gamma_p(ap^n) \to 1$. □

We conclude this section by proving Lemma 2.6.

Proof of Theorem 2.6. By Proposition 1.7, to prove Lemma 2.6 it thus suffices to prove that for all $k \geq 1$ and all sufficiently large n,

$$\Gamma_p(ap^n) \equiv 1 \pmod{p^k}. \tag{2.5}$$

To verfiy this, taking $n > k$ will suffice. For such n,[2]

$$\begin{aligned}
\Gamma_p(ap^n) &= (-1)^{ap^n} \begin{pmatrix} (1) & \cdots & (p^n - 1) \\ (p^n + 1) & \cdots & (2p^n - 1) \\ \vdots & & \\ ((a-1)p^n + 1) & \cdots & (ap^n - 1) \end{pmatrix} \\
&\equiv (-1)^{ap^n} \left((1) \cdots (p^n - 1) \right)^a \pmod{p^k} \\
&= (-1)^{ap^n} \begin{pmatrix} (1) & \cdots & (p^k - 1) \\ (p^k + 1) & \cdots & (2p^k - 1) \\ \vdots & & \\ (p^{k+1} - p^k + 1) & \cdots & (p^{k+1} - 1) \\ (p^{k+1} + 1) & \cdots & (p^{k+1} + p^k - 1) \\ \vdots & & \\ (p^n - p^k + 1) & \cdots & (p^n - 1) \end{pmatrix}^a \pmod{p^k}
\end{aligned}$$

[2]The arrays here are not matricies. Each row is a product, and the rows are being multiplied together. They are displayed in this way because it makes it easier to see what the terms of $\Gamma_p(ap^n)$ are equivalent to modulo p^k.

$$\equiv \quad (-1)^{ap^n}\left((1)\cdots(p^k-1)\right)^{ap^{n-k}} \pmod{p^k}$$
$$\equiv \quad (-1)^{ap^n}(p^k-1)^{ap^{n-k}} \equiv (-1)^{ap^n}(-1)^{ap^{n-k}} \equiv (-1)^{ap^{n-k}(1+p^k)} \equiv 1 \pmod{p^k}.$$

The second to last equivalence is due to the fact that the factors of the product $(1)\cdots(p^k-1)$ are precisely the elements of the multiplication group $(\mathbb{Z}/(p^k\mathbb{Z}))^*$. After multiplying inverses, p^k-1, which is its own inverse, remains. The last equivalence follows because either ap^{n-k} or $1+p^k$ is even. This proves Equation 2.5.

As was noted at the beginning of the proof, Equation 2.5 implies that for arbitrary k and all $n > k$,

$$\Gamma_p(ap^n) - 1 \equiv 0 \pmod{p^k},$$

so that

$$\nu_p(\Gamma_p(ap^n) - 1) \geq k,$$

and finally

$$|\Gamma_p(ap^n) - 1|_p \leq p^{-k}.$$

□

Example 2.8. This example demonstrates Lemma 2.6 for the case where $a = 1$ and $p = 2$. For an arbitrary $k \geq 1$ and $n > k$,

$$\Gamma_p(2^n) = (-1)^{2^n}(1)(3)\cdots(2^n-3)(2^n-1).$$

The power of (-1) clearly evaluates to 1. This expression can be further expanded by writing

$$\Gamma_p(2^n) = (-1)^{2^n}\overbrace{(1)(3)\cdots(2^k-1)}\overbrace{(2^k+1)(2^k+3)\cdots(2^{k+1}-1)}\cdots\overbrace{(2^{n-1}+1)(2^{n+1}+3)\cdots(2^n-1)}.$$

As the braces indicate, the product can be divided into sections containing $\frac{2^k}{2} = 2^{k-1}$ terms. There are $\frac{2^n}{2^k} = 2^{n-k}$ such sections. The first is $(1)(3)\cdots(2^k-1)$, and the rest are all of the form

$$(2^l+1)(2^l+3)\cdots(2^{l+1}-1),$$

where l runs from k to $n-1$. Thus each section is equivalent to

$$(1)(3)\cdots(2^k-1) \pmod{2^k},$$

and we have that

$$\Gamma_p(2^n) \equiv \left((1)(3)\cdots(2^k-1)\right)^{2^{n-k}} \pmod{p^k}.$$

As noted in the proof of Lemma 2.6, the product $(1)(3)\cdots(2^k-1)$ contains precisely the elements of $\mathbb{Z}\setminus 2^k\mathbb{Z}$. Now

$$\big((1)(3)\cdots(2^k-1)\big)^2 \equiv 1 \pmod{p^k},$$

because when one copy of $(1)(3)\cdots(2^k-1)$ is multiplied by a second copy, every element of the group is multipled by its inverse, yielding 1. Thus, since in the expression for $\Gamma_p(2^n)$ the product $(1)(3)\cdots(2^k-1)$ is raised to a multiple of 2, we get that $\Gamma_p(2^n)\equiv 1 \pmod{p^k}$ for all $k\geq 1$ and all $n>k$. By Proposition 1.7, this implies that $\Gamma_p(2^n)\to 1$.

The next section uses the fact that $\lim\limits_{n\to\infty} C(ap^n)$ is known to find $\lim\limits_{n\to\infty} C(ap^n+r)$ for all $r\in\mathbb{Z}$.

3 Finding the Limit of $C(ap^n+r)$

Given that $\lim\limits_{n\to\infty} C(ap^n)$ is known, it is not hard to find $\lim\limits_{n\to\infty} C(ap^n+r)$ for all $r\in\mathbb{Z}$. The latter limit is thus presented as a corollary to Theorem 2.7.

Corollary 3.1. *Let $r\in\mathbb{Z}$ and let $L=\lim\limits_{n\to\infty} C(ap^n)$. Then*

$$\lim_{n\to\infty} C(ap^n+r) = \begin{cases} C(r)\cdot L & \text{if } r>0\\ -\frac{1}{2}L & \text{if } r=-1 \text{ and } p\neq 2\\ 0 & \text{if } r<-1. \end{cases}$$

Proof. Each case will be proven using induction and the recurrence

$$C(x+1)=\frac{2(2x+1)}{x+2}C(x).$$

Begin with the $r>0$ case. For the base case ($r=1$), we have

$$C(ap^n+1) \quad = \quad \frac{2(2ap^n+1)}{ap^n+2}C(ap^n)\to \frac{2}{2}\cdot L = C(1)\cdot L,$$

as desired.

For the inductive step, suppose that $C(ap^n+r)\to C(r)\cdot L$. Then

$$C(ap^n+r+1)=\frac{2(2(ap^n+r)+1)}{ap^n+r+2}C(ap^n+r)\to\frac{2(2r+1)}{r+2}C(r)\cdot L = C(r+1)\cdot L,$$

proving the $r > 0$ case.

For the cases for which $r < 0$, rewrite the recurrence as

$$C(x) = \frac{(x+2)}{2(2x+1)} C(x+1).$$

If $r = -1$, then

$$C(ap^n - 1) = \frac{ap^n - 1 + 2}{2(2(ap^n - 1) + 1)} C(ap^n) = \frac{ap^n + 1}{4ap^n - 2} C(ap^n) \to -\frac{1}{2} \cdot L,$$

as desired.

For the base case of the $r < -1$ case, we have

$$C(ap^n - 2) = \frac{ap^n - 2 + 2}{2(2(ap^n - 2) + 1)} C(ap^n - 1) = \frac{ap^n}{4ap^n - 6} C(ap^n - 1) \to \frac{0}{-6} \cdot \frac{-1}{2} \cdot L = 0.$$

Now suppose that $C(ap^n - r) = 0$. Then for the inductive step,

$$C(ap^n - r - 1) = \frac{ap^n - r - 1 + 2}{2(2(ap^n - r - 1) + 1)} C(ap^n - r) = \frac{ap^n - r + 1}{4ap^n - 4r - 3} C(ap^n - r) \to \frac{r-1}{4r+3} \cdot 0 = 0,$$

completing the $r < -1$ case and proving the theorem. □

We note two interesting consequences of Corollary 3.1. First, since $\lim_{n\to\infty} \frac{C(ap^n + r)}{C(ap^n)} = C(r)$ even when $r < 0$, it suggests a definition of $C(n)$ for $n < 0$. Such a defintion would, for example, give $C(-1) = -1/2$.

Secondly, Corollary 3.1 implies that $C(n)$ does not converge p-adically. This is because for a fixed a we can choose distinct values of r that yield convergent subsequences with different limits.

Proposition 3.2. *For any prime p, $\{C(n)\}$ does not converge p-adically.*

Proof. Given a prime p, suppose that $\{C(n)\}$ converges p-adically. Then every infinite subsequence of $\{C(n)\}$ converges to the same limit. But consider the two subsequences $\{C(p^n + 1)\}$ and $\{C(p^n + 2)\}$. By Theorem 3.1,

$$\lim_{n\to\infty} \left(\frac{C(p^n + 1)}{C(p^n + 2)} \right) = \frac{C(1)}{C(2)} = \frac{1}{2} \neq 1,$$

contradicting that all subsequences of $\{C(n)\}$ approach the same limit. □

4 An Alternative Way of Stating the Limit of $C(ap^n)$

Theorem 2.7 showed that

$$\lim_{n\to\infty} C(ap^n) = \binom{2a}{a} \prod_{i=1}^{\infty} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}. \tag{4.1}$$

The goal of this section is to find a more illuminating expression of these limits. Hence, we arrive at the following proposition.

Proposition 4.1. *The limits in* (4.1) *can be written as*

$$\binom{2a}{a} \prod_{\substack{i=1 \\ p\nmid i}}^{\infty} i^{2\left\lfloor \log_p(i/a)\right\rfloor - \lfloor \log(i/2a)\rfloor}.$$

The proof of Proposition 4.1 requires a lemma similar to Lemma 2.6.

Lemma 4.2. *Let p be prime and let $a \in \mathbb{N}$. In $\mathbb{Z}_p$,*

$$\lim_{n\to\infty} \left(\Gamma_p(ap^n)\right)^n = 1.$$

Proof. The proof of Lemma 2.6 showed that for all $k \geq 1$,

$$\Gamma_p(ap^n) \equiv (-1)^{ap^n}((1)\cdots(p^k-1))^{ap^{n-k}} \equiv 1 \pmod{p^k},$$

Thus $(\Gamma_p(ap^n))^n \equiv 1^n \equiv 1 \pmod{p^k}$, so $(\Gamma_p(2ap^n)^n \to 1$, proving Lemma 4.2. $\square$

Theorem 4.1 can now be proven.

Proof of Proposition 4.1. The goal is to prove that

$$\prod_{i=1}^{\infty} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} = \prod_{\substack{i=1 \\ p\nmid i}}^{\infty} i^{2\left\lfloor \log_p(i/a)\right\rfloor - \lfloor \log_p(i/2a)\rfloor}.$$

We have

$$\prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(2ap^i)^2} = \frac{(1\cdots(2ap-1))^n((2ap+1)\cdots(2ap^2-1))^{n-1}\cdots((2ap^{n-1}+1)\cdots(2ap^n-1))}{[(1\cdots(ap-1))^n((ap+1)\cdots(ap^2-1))^{n-1}\cdots((ap^{n-1}+1)\cdots(ap^n-1))]^2},$$

Factoring out a copy of each factor raised to n, we thus have

$$\left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2}\right)^n \frac{(1\cdots(2ap-1))^0((2ap+1)\cdots(2ap^2-1))^{-1}\cdots((2ap^{n-1}+1)\cdots(2ap^n-1))^{n-1}}{[(1\cdots(ap-1))^0((ap+1)\cdots(ap^2-1))^{-1}\cdots((ap^{n-1}+1)\cdots(ap^n-1))^{n-1}]^2}.$$

The factor on the right has a nice form as the product of coprime numbers raised to logarithmically increasing powers. The whole expression is written as follows.

$$\left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2}\right)^n \frac{\prod\limits_{\substack{i=1\\ p\nmid i}}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor}\right)}{\prod\limits_{\substack{i=1\\ p\nmid i}}^{2ap^n-1}\left(i^{\left\lfloor \log_p(i/2a)\right\rfloor}\right)} = \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2}\right)^n \frac{\prod\limits_{\substack{i=1\\ p\nmid i}}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor-\left\lfloor \log_p(i/2a)\right\rfloor}\right)}{\prod\limits_{\substack{i=ap^n+1\\ p\nmid i}}^{2ap^n-1}\left(i^{\left\lfloor \log_p(i/2a)\right\rfloor}\right)}$$

$$= \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2}\right)^n \frac{\prod\limits_{\substack{i=ap+1\\ p\nmid i}}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor-\left\lfloor \log_p(i/2a)\right\rfloor}\right)}{\prod\limits_{\substack{i=ap^n+1\\ p\nmid i}}^{2ap^n-1}\left(i^{n-1}\right)}$$

$$= \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^2}\right)^n \left(\frac{\Gamma_p(ap^n)}{\Gamma_p(2ap^n)}\right)^{n-1} \prod_{\substack{i=ap+1\\ p\nmid i}}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor-\left\lfloor \log_p(i/2a)\right\rfloor}\right)$$

$$= \left(\frac{\Gamma_p(2ap^n)}{\Gamma_p(ap^n)^{n+1}}\right) \prod_{\substack{i=ap+1\\ p\nmid i}}^{ap^n-1}\left(i^{2\left\lfloor \log_p(i/a)\right\rfloor-\left\lfloor \log_p(i/2a)\right\rfloor}\right)$$

The result follows from Lemma 4.2 and Lemma 2.6. □

Using Proposition 4.1, $\lim\limits_{n\to\infty} C(2^n)$ (see Example 2.1) can be expressed nicely as a product numbers coprime to 2 raised to logarithmically increasing powers.

Example 4.3. In $\mathbb{Z}_p$, $\lim\limits_{n\to\infty} C(2^n) = 2\cdot 3\cdot (5\cdot 7)^2\cdot (9\cdot 11\cdot 13\cdot 15)^3\cdots$. More simply, The limit is an infinite product consisting of blocks of 2^n consecutive odd numbers raised to $n+1$-st power.

5 Conclusion

Combinatorial sequences, while they may not have limits, are integer sequences, and as such they have convergent subsequences by compactness of the p-adic integers. Sometimes the form of these limits can be difficult to characterize explicitly. In the case of the Catalan numbers, the sequence does not converge p-adically. However, we have an infinite class of increasing subsequences which have limits.

The limits of these subsequences appear to resist evaluation by any standard means (such as power series expansions, or continuity). However, we have evaluated the p-adic limit of the subsequence $C(ap^n)$, and even more generally $C(ap^n + r)$, where a is a constant and $r \in \mathbb{Z}$. The limits of these sequences can be written as an infinite product of numbers which don't divide p, raised to powers increasing logarithmically.

5.1 Open Problems

It remains an open problem to characterize all convergent subsequences of Catalan numbers as well as to find the limits of these subsequences. The methods used to answer these questions will no doubt present their utility in a similar analysis of other combinatorial sequences. Furthermore, it is unknown whether or not the limits established here are transcendental over the rational numbers.

6 Appendix: An Elementary Proof that $\{C(ap^n)\}$ Converges

Proposition 1.7 states that to show that a sequence $\{f(n)\}$ converges p-adically, it suffices to show that its elements are eventually constant modulo arbitrarily large powers of p. This equivalent definition of p-adic convergence is useful because there are existing results on factorials, binomial coefficients, and Catalan numbers modulo powers of primes. One such result is used to prove

Theorem 6.1. *For all primes p and all $a \in \mathbb{N}$, $\{C(ap^n)\}_{n\geq 0}$ converges p-adically.*

The proof of Theorem 6.1 relies on a 1997 result due to Granville.

Theorem 6.2 (Granville 1997)**.** *Let n be an integer, and write $n = \gamma_0 + \gamma_1 p + \cdots + \gamma_d p^d$ in base p. For $j \geq 0$ and p^k a power of p, define n_j to be the least positive residue of $\lfloor \frac{n}{p^j} \rfloor$ $\pmod{p^k}$ (so that $n_j = \gamma_j + \gamma_{j+1}p + \cdots + \gamma_{j+k-1}p^{k-1}$). Define $(n_j!)_p$ to be the product of numbers $\leq n_j$ that are coprime with p. Then*

$$n! \equiv p^{\nu_p(n!)}(\delta(p,k))^{\nu_{p^k}(n!)} \prod_{j=\geq 0} (n_j!)_p \pmod{p^k},$$

$$\text{where } \delta(p,k) = \begin{cases} 1 & \text{if } p=2 \text{ and } k \geq 3 \\ -1 & \text{otherwise.} \end{cases}$$

Since $C(n) = \frac{(2n)!}{n!(n+1)!}$, applying Theorem 6.2 to $C(n)$ yields

$$C(n) \equiv \delta^{\nu_{p^k}(C(n))} p^{\nu_p(C(n))} \overbrace{\frac{\prod_{j\geq 0}((2n)_j)!_p}{\prod_{j\geq 0}(n_j)!_p \prod_{j\geq 0}((n+1)_j)!_p}}^{\mathcal{P}(n)} \pmod{p^k}. \tag{6.1}$$

Theorem 6.1 uses the case $n = ap^n$. To show that $\{C(ap^n)\}$ is eventually constant modulo p^k, it thus suffices to show that all three components of the right-hand side of (Equation 6.1) (the power of δ, the power of p, and $\mathcal{P}(n)$) are eventually constant modulo p^k.

Proof of Theorem 6.1. Fix $k \geq 1$. Write $a = \alpha_0 + \alpha_1 p + \cdots + \alpha_m p^m$ in base p ($\alpha_i \neq 0$ for all i), so that $ap^n = \alpha_0 p^n + \cdots + \alpha_m p^{n+m}$ in base p. To show that $\delta^{\nu_{p^k}(C(ap^n))}$ and $p^{\nu_p(C(ap^n))}$ are eventually constant modulo p^k, it is clearly sufficient to show that $\nu_{p^k}(C(ap^n))$ is constant for all n. This is an easy application of Legendre's 1808 result that $\nu_p(n!) = \frac{n-s(n)}{p-1}$, where $s(n)$ is the sum of the base-p coefficients of n. We have

$$\begin{aligned}
\nu_{p^k}(C(ap^n)) = \nu_{p^k}\left(\frac{(2ap^n)!}{((ap^n)!)^2}\right) &= \nu_{p^k}((2ap^n)!) - 2\nu_{p^k}(n!) \\
&= \frac{2ap^n - s(2ap^n)}{p-1} - 2\frac{ap^n - s(ap^n)}{p-1} \\
&= \frac{2ap^n - s(2ap^n}{p-1} - \frac{2ap^n - 2s(ap^n)}{p-1} \\
&= \frac{2s(ap^n) - s(2ap^n)}{p-1},
\end{aligned}$$

which does not vary with n.

Thus, all that remains to show is that $\mathcal{P}(ap^n)$ is eventually constant. This expression can be simplified considerably by showing that

$$(ap^n+1)_j = \begin{cases} ap^n_0 + 1 & \text{if } j = 0 \\ ap^n_j & \text{if } j \neq 0 \end{cases} \tag{6.2}$$

and that

$$(2ap^n)_j = 2(ap^n_j) \text{ for all } j. \tag{6.3}$$

To verify Equation 6.2, note that the base-p expansion of ap^n+1 differs from that of ap^n only in that its p^0 coefficient is 1, whereas the p^0 coefficient of the base-p expansion ap^n is 0. The p^0 coefficient is included in $ap^n_j = a_j p^j + a_{j+1}p^{j+1} + \cdots + a_{j+k-1}p^{j+k-1}$ only when $j=0$; thus, $(ap^n+1)_0 = ap^n_0 + 1$ for $j=0$ and $(ap^n+1)_j = ap^n_j$ otherwise.

To verify Equation 6.3, simply note that for all j

$$(2ap^n)_j = \lfloor \frac{2ap^n}{p^j} \rfloor \pmod{p^k} = 2ap^{n-j} \pmod{p^k} = 2\lfloor \frac{ap^n}{p^j} \rfloor \pmod{p^k} = 2ap^n_j.$$

Applying Equation 6.2 and Equation 6.3 to $\mathcal{P}(ap^n)$ gives

$$\mathcal{P}(ap^n) = \frac{\prod_{j\ge 0}((2ap^n)_j)!_p}{\prod_{j\ge 0}(ap^n_j)!_p \prod_{j\ge 0}((ap^n+1)_j)!_p} = \frac{2}{ap^n_0+1} \cdot \overbrace{\prod_{j\ge 1} \frac{(2ap^n_j)!_p}{((ap^n_j)!_p)^2}}^{\mathcal{P}'(ap^n)}.$$

Clearly, this is eventually constant modulo p^k if ap^n_0 and $\mathcal{P}'(ap^n)$ are. It is easy to check that the former is constant for all $n > k$. $\mathcal{P}'(ap^n)$ varies with n only if the set $\{ap^n_j\}_{j\ge 1}$ does. Define

$$ap^n_J = \{ap^n_j\}_{j\ge 1}.$$

Then $\mathcal{P}'(ap^n)$ is eventually constant modulo p^k if ap^n_J is constant for all sufficiently large n.

To prove this, it suffices to take $n > k$. Given such an n, write $ap^n = a_n p^n + a_{n+1}p^{n+1} + \cdots + a_{n+m}p^{n+m}$, where $a_{n+i} = \alpha_i$ for $i \in \{0, \dots, m\}$. For $j \in \mathbb{N} \setminus \{n-k+1, \dots, n+m\}$, $ap^n_j = 0$, since none of a_n through a_{n+m} (the non-zero coefficients of the base-p expansion of ap^n) appears as a coefficient of ap^n_j for any such j. Thus there are $n+m-(n-k) = m+k$ values of j for which ap^n_j is non-zero (crucially, this number does not depend on n). Running j from $n-k+1$ to $n+m$, we get that $ap^n_J = \{\alpha_0 p^{k-1}, \alpha_0 p^{k-2} + \alpha_1 p^{k-1}, \dots, \alpha_{m-1} + \alpha_m p, \alpha_m\}$. None of the elements of this set depends on n, as desired. □

Retracing the steps of the proof, showing that ap^n_J is eventually constant modulo an arbitrary power of p (say p^k) was sufficient to show that $\mathcal{P}'(ap^n)$, and thus $\mathcal{P}(ap^n)$, is eventually constant modulo p^k. This was was needed to prove our original objective, that Equation 6.1 is eventually constant modulo p^k. Furthermore, recall that this is sufficient to show convergence because for all k and sufficiently large m and n,

$$|f(n) - f(m)|_p \le p^{-k} \text{ if and only if } f(n) \equiv f(m) \pmod{p^k}.$$

Showing that ap_J^n is eventually constant modulo p^k is thus crucial step of the proof. It is also its most difficult step. The following example is meant to give the reader a better sense of ap_J^n, and of why it is eventually constant, by way of the sequence $\{C(p^n)\}$.

Example 6.3. Suppose that $a = 1$, so that $ap^n = p^n$. Fix $k = 3$. For a given $n > 3$, the base-p expansion of $p^n = a_n p^n = 1 \cdot p^n$ has only one non-zero coefficient, so for all $j \geq 1$, $p_j^n = a_j + a_{j+1}p + a_{j+2}p^2$ will have at most one non-zero term. If none of j, $j+1$, or $j+2$ is n, then p_j^n=0; thus, $p_j^n = 0$ for all $j \in \mathbb{N} \setminus \{n-2, n-1, n\}$. For the remaining values of j, we have tht $p_{n-2}^n = p^2$, $p_{n-1}^n = p$, and $p_n^n = 1$, so that $p_J^n = \{1, p, p^2\}$. The cardinality of this set, $3 = 0 + 3 = m + k$, does not depend on n, and neither do its elements.

Notice that taking $n > k = 3$ is necessary because if $n = 2$, for instance, $p_1^2 = p$, $p_2^2 = 1$, and $p_j^2 = 0$ for all $j > 2$. Thus $p_J^2 = \{1, p\}$; p^2 is excluded from p_J^2 because there are no j for which a_{j+2} is non-zero.

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