P-adic Limits of Combinatorial Sequences

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Welcome to the p-adic $\mathbb{Z}\mathcal{O}\mathbb{N}\exists$

Introduction to the p-adics: $\mathbb{Q}_p$

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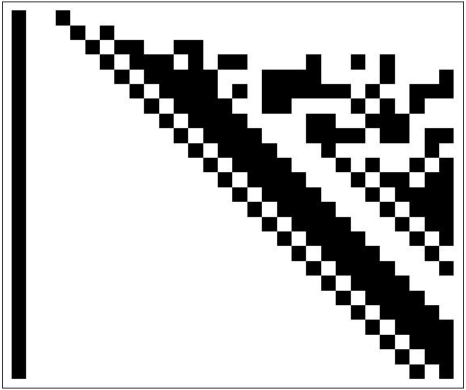
Define the *p-adic valuation* of an integer n to be the greatest power of p that divides n: $\nu_p(n) = k$

Then the *p-adic norm* of n is defined as $|n|_p = p^{-k}$.

P-adic Convergence

For example: $3^{2^n} \to 1$



The Catalan Numbers

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- What are the limits of these subsequences?

An equivalent definition of p-adic convergence

$\forall k \geq 1\ \exists N \geq 1$ such that $\forall m, n \geq N$,

$$|C(f(n)) - C(f(m))|_p \leq p^{-k}$$

if and only if

$\forall k \geq 1\ \exists N \geq 1$ such that $\forall m, n \geq N$,

$$C(f(n)) \equiv C(f(m)) \pmod{p^k}.$$

(In other words, $\{C(f(n))\}$ converges if and only if it is eventually constant modulo arbitrarily large powers of p.)

A useful result

Using a theorem due to Granville, we can derive that

$$C(f(n)) \equiv (-1)^{\nu_{p^k}(C(f(n)))} p^{\nu_p(C(f(n)))} \prod_{j \geq 1} \frac{(2f(n)_j)!_p}{((f(n)_j)!_p)^2} \pmod{p^k}.$$

Let's look at $\prod_{j \geq 1} \frac{(2f(n)_j)!_p}{((f(n)_j)!_p)^2} \pmod{p^k}$.

When is $\prod_{j\geq 1} \frac{(2f(n)_j)!_p}{((f(n)_j)!_p)^2} \pmod{p^k}$ constant?

Writing $f(n) = a_{n,0} + a_{n,1}p + \cdots + a_{n,n}p^n$, we have

$$f(n)_j = a_{n,j} + a_{n,j+1}p + \cdots + a_{n,j+k-1}p^{k-1}.$$

$\prod_{j\geq 1} \frac{(2f(n)_j)!_p}{((f(n)_j)!_p)^2} \pmod{p^k}$ **varies with** n **only if the** $f(n)_j$**s do.**

Define $f(n)_J = \{f(n)_j\}_{j\geq 1}$. Our question is this:

For which $f(n)$ **do we have, for all** k **and all sufficiently large** m **and** n**,** $f(m)_J = f(n)_J$**?**

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Both $f(50)_j$ and $f(75)_j$ are 0 for most values of j.

When $j = 50$ we get $f(50)_{50} = 1$.
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We get $f(50)_J = f(75)_J = \{1, p, p^2, p^3\}$.

Generalizing the example

The sequence in the example, $f(n) = p^n$, can be generalized as follows.

Theorem

Let p be prime, and fix $l \geq 0$. Let α be a vector of l numbers arbitrarily chosen from $\{1, \ldots, \frac{p-1}{2}\}$. Write $\alpha = \langle \alpha_0, \ldots, \alpha_l \rangle$.

For $n \geq l$, define $f(n) = \alpha \cdot \langle p^{n-l}, p^{n-l+1}, \ldots, p^n \rangle$, so that $f(n) = \sum_{i=n-l}^{n} a_i p^i$, with $a_i = \alpha_{i-n-l}$. Then $\{C(f(n))\}$ converges p-adically.

Finding a Limit

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The Catalan numbers can be expressed in terms of factorials:

$$C(2^n) = \frac{1}{(2^n+1)} \frac{2^{n+1}!}{2^n!^2} = \frac{(2^{n+1})!}{(2^n+1)!2^n!}$$

And we've already proven that if $\{\frac{1}{(2^n+1)} \frac{2^{n+1}!}{2^n!^2}\} \to L$ 2-adically,

$\{\frac{2^{n+1}!}{2^n!^2}\} \to L$ as well.

The p-adic Gamma Function

The p-adic Gamma function is defined as:

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k=1 \\ p \nmid k}}^{n-1} k \text{ where } \Gamma_p(0) = 1$$

So then,

$$\Gamma_2(n) = \begin{cases} \prod_{\substack{k=1 \\ 2 \nmid k}}^{n-1} k = (1 \cdot 3 \cdot 5 \cdots (n-1)) & \text{even n} \\ (-1)^n \prod_{\substack{k=1 \\ 2 \nmid k}}^{n-1} k = (-1)(1 \cdot 3 \cdot 5 \cdots (n-2)) & \text{odd n} \end{cases}$$

Finding the Limit

We have $C(2^n) \equiv \frac{2\Gamma_2(2^{n+1})}{\prod_{i=1}^{n}\Gamma_2(2^i)} \pmod{2^k}$.

The numerator approaches 2, and $\prod_{i=k}^{n}\Gamma_2(2^i) \equiv_{2^k} 1$
So this simplifies to $C(2^n) \equiv_{2^k} \frac{2}{1^k 3^{k-1}(5\cdot 7)^{k-2}(9\cdot 11\cdot 13\cdot 15)^{k-3}\cdots}$

Since we know this sequence coverges to L, any subsequence of it will converge as well. Replace $k \to 2^k$

Finding the Limit

Focusing on the denominator, we have
$1^{2^k}3^{2^k-1}(5 \cdot 7)^{2^k-2}(9 \cdot 11 \cdot 13 \cdot 15)^{2^k-3} \cdots$

For all primes p, $\{p^{2^k}\}$ converges to 1, so we can write

$$L = 2 \cdot 3(5 \cdot 7)^2(9 \cdot 11 \cdot 13 \cdot 15)^3 \cdots .$$

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We will attempt to...

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- Expand the class of sequences that we can show converge p-adically.
- Find limits of these sequences; e.g., $\{C(p^n)\}$.
- Find similar results for other combinatorial sequences, e.g.

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Questions?

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