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The p-adic numbers

An introductory analysis course describes the completion of $\mathbb{Q}$ to the reals, in which real numbers are defined as equivalence classes of Cauchy sequences of rationals. There is a second important completion of $\mathbb{Q}$ which, instead of the familiar Euclidean distance metric, uses a metric known as the *p-adic norm.* If we define the *p-adic valuation* of an integer n to be the greatest power of p that divides n (and write $\nu_p(n) = k$, where k is this greatest power of p), then the *p-adic norm* of n is defined as $|n|_p = p^{-k}$. The notion of Cauchy convergence with respect to the p-adic norm can be defined in a manner analogous to that of Cauchy convergence with respect to the Euclidean norm. The *p-adic numbers* are then the completion of $\mathbb{Q}$ with respect to the p-adic norm.

The aim of our work is to investigate the convergence of sequences of p-adic numbers. In a 2009 paper, Eric Rowland finds one such sequence: $\{3^{2^n}\}$ He looks at $\{3^n\}$, which does not converge, before noticing that the subsequence $\{3^{2^n}\}$ showed patterns of converging term by term when its terms were viewed in their base-2 representations. He used a power series approach to show that the limit was $\log 3$ (expressed as a 2-adic number). We've been using a similar process to investigate the Catalan numbers, a sequence which appears in numerous combinatorial problems. These numbers have a convenient closed form: $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

Our Results

We first investigated the p-adic valuation of Catalan numbers of the form $C(p^n)$, finding that $\nu_p(C(p^n))$ is constant for all primes p. We used Mathematica to look at the p-adic valuation of the terms of the sequence $\nu_p(C(p^n))$ for various p. These sequences were constant for every p that we checked, and were able to prove this using the closed form of the catalan numbers (stated above) and the multiplication formula for p-adic valuation (i.e., that $\nu_p(a \cdot b) = \nu_p(a) + \nu_p(b)$).

We used Eric's method to examine whether the full sequence of Catalan numbers showed patterns of converging 2-adically, by examining the base-2 representations of the terms of

$C(n)$. This did not converge at all, but the subsequence $C(2^n)$ did. We have been working towards finding its limit by way of comparing $C(2^n)$ to the 2-adic Gamma function, Γ_2. So far, we have been able to show that the limit we are searching for $\lim_{n\to\infty} C(2^n)$ is also $\lim_{n\to\infty} \frac{2\Gamma_2(2^{n+1})}{\prod_{i=1}^{n} \Gamma_2(2^i)}$. In this quotient, we have found that the numerator approaches 2, and the denominator approaches some 2-adic number that is unclear. We hope to find what the denominator approaches, so that then we can find the limit of $C(2^n)$.

On another note, we have also been able to use an equivalent formulation of p-adic convergence to find a class of convergent sequences of Catalan numbers. To show that a sequence converges p-adically, one can show that its elements are eventually constant modulo arbitrarily large powers of p. Noticing this re-formulation has been beneficial because there are existing results on factorials, binomial coefficients, and Catalan numbers modulo arbitrary powers of p.

One such result, from a 1997 paper by Andrew Granville, has been particularly useful. It gives a formula for factorials modulo an arbitrary power of p, and since the n^{th} Catalan number can be written as $\frac{(2n)!}{(n+1)!(n)!}$, we were able to find an analogous formula for $C(n)$ modulo an arbitrary power of p. We used this formula to obtain the following result.

Proposition 1. *Let $l \geq 0$ be arbitrary. Let $\alpha = \langle \alpha_0, \ldots, \alpha_l \rangle$, with the α_is chosen from $\{1, \ldots, \frac{p-1}{2}\}$. For $n \geq l$, define $f(n) = \alpha \cdot \langle p^{n-l}, p^{n-l+1}, \ldots, p^n \rangle$. Then $\{C(f(n))\}$ converges p-adically.*

Future Work

Our primary goal is to find the limit of $\{C(2^n))\}$. We would also like to extend the class of sequences that we can show converge. Granville's result gives us three criteria that must hold if a sequence of Catalan numbers is to converge. We would like to focus specifically on these criteria in hopes of determining precisely which sequences satisfy all three.