On p-adic Limits of Combinatorial Sequences

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What is the limit of $C(ap^n)$

To find this limit, we want to express $\binom{2ap^n}{ap^n}$ in terms of the p-adic gamma function.

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{k=1 \\ p \nmid k}}^{n} k = \frac{(-1)^{n+1}(n)!}{\prod_{\substack{k=1 \\ p \mid k}}^{n-1} k} = \frac{(-1)^{n+1}(n)!}{p^{\frac{n}{p}} \frac{n}{p}!}$$

$\Rightarrow n! = \frac{n}{p}! \Gamma_p(n+1)(-1)^{n+1} p^{\frac{n}{p}}$

$$\Rightarrow (ap^n)! = (ap^{n-1})! \Gamma_p(ap^n+1)(-1)^{ap^n+1} p^{ap^{n-1}}$$

$$= a! p^{\frac{ap^n - a}{p-1}} (-1)^{ap^n+1} \prod_{i=1}^{n} \Gamma_p(ap^i)$$

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What is the limit of $C(ap^n)$?

Using the above equation, we get

$$\frac{(2ap^n)!}{(ap^n)!^2} = \frac{(2a)!\cancel{p^{\frac{2ap^n-2a}{p-1}}}}{(a!)^2\cancel{p^{\frac{2(ap^n-a)}{p-1}}}} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2} = \frac{(2a)!}{(a!)^2} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$$

New problem: look at $n \to \infty \frac{(2a)!}{(a!)^2} \prod_{i=1}^{n} \frac{\Gamma_p(2ap^i)}{\Gamma_p(ap^i)^2}$ p-adically.

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