The Wonderful Arithmetic Aspects of Elementary Functions

Carl F. Gauss
Göttingen University

Leonhard Euler
Basel College

Emmy Nöether
Berlin Academy

Abstract

We present a sequence of polynomials in $\mathbb{Q}[x]$, arising from a simple family of rational functions, that approximates uniformly the classical function $\arctan x$ on $[0,1]$ (and hence, via standard identities, on all of $\mathbb{R}$). The sequence is attractive and interesting for several reasons including its rate of convergence, its simplicity, its rational approximations to π, and because of its similarities with Hermite-interpolating polynomials.

Introduction

The Taylor series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

was discovered by the Scotsman James Gregory in 1671 ([B, Ch. 12]). It is not hard to show that the series converges uniformly to $\arctan x$ on $[-1,1]$; thus, the series produces the following sequence of Taylor polynomials in $\mathbb{Q}[x]$ that converges uniformly to $\arctan x$ on $[-1,1]$:

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

Like the Taylor polynomials for several other classical functions, e.g., $\cos x$, $\sin x$, and e^x, this sequence of polynomials is very easy to describe and work with; but unlike those Taylor polynomials with factorials in the denominators of their coefficients, it does not converge rapidly for all "important" values of x. In particular, it converges extremely slowly to $\arctan x$ when $|x|$ is near 1. For example, if $x = 0.95$, we would need to use T_{28}, a polynomial of degree 57, to get three decimal places of accuracy for $\arctan(0.95)$; if $x = 1$, we would need to use T_{500}, a polynomial of degree 1001, to get three decimal places of accuracy for $\arctan 1$.

Indeed, for $x \in [0,1]$

$$\begin{aligned} \arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k}\,dt \\ &= \int_0^x \sum_{k=0}^{n} (-1)^k t^{2k} + \sum_{k=n+1}^{\infty} (-1)^k t^{2k}\,dt \\ &= T_n(x) + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt; \end{aligned}$$

therefore, $|\arctan x - T_n(x)| = \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \geq \int_0^x \frac{t^{2n+2}}{2}\,dt = \frac{x^{2n+3}}{2(2n+3)}$. Thus, as $x \to 1$, $T_n(x)$ cannot approximate $\arctan x$ any better than $\frac{1}{2(2n+3)} = \frac{1}{2(\text{degree } T_n)+4}$. The same holds true for x values near -1. It is only fair to note that the sequence $\{T_n(x)\}$ converges to $\arctan x$ reasonably fast for x values near 0.

The purpose of this article is to present another very elementary, easily-described sequence of polynomials in $\mathbb{Q}[x]$ that approximates $\arctan x$ uniformly on $[0,1]$ and which does so much more rapidly than the sequence of Taylor polynomials centered at 0. We note that such an approximating sequence provides, via the identities $\arctan x = -\arctan(-x) = \frac{\pi}{2} - \arctan(\frac{1}{x})$, a method of approximating $\arctan x$ for all $x \in \mathbb{R}$. The approximating sequence arises from the simple family of rational functions $\left\{\frac{x^{4m}(1-x)^{4m}}{1+x^2}\right\}_{m\in\mathbb{N}}$.

Results

Indeed, because of Theorem 1, the sequence $\{h_m\}$ is easily programmable and gives functions that will approximate arctan to a prescribed accuracy for $x \in [0,1]$. Appendix A contains a Mathematica program that computes the necessary $h_m(x)$ to approximate $\arctan x$, $x \in [0,1]$, to "numberofdigits" decimal places of accuracy. The program computes the h_m much faster than the direct computation of the coefficients, via solving a system of linear equations, suggested by the result of Theorem 1. For example, to find a polynomial h_m of degree $8m$ such that $h_m(0) = \arctan 0$, $h_m^{(n)}(0) = \arctan^{(n)}(0)$ and $h_m^{(n)}(1) = \arctan^{(n)}(1)$ for $1 \leq n \leq 4m$, we solved for the coefficients of h_m by solving the system of linear equations (the $8m+1$ coefficients being the unknown variables), generated by the conditions on h_m. For $m = 5,\ 10,\ 15$, and 20, on a 733 MHz Pentium III computer running Mathematica 4.1, this method yielded computation times of $0.82,\ 13.63,\ 88.1$ and 332.08 seconds respectively. The program above (without the "Print" statements) computed the h_m in $0.0,\ 0.06,\ 0.11$ and 0.11 seconds respectively.

Lemma 1. Define $p_1(x) = 4-4x^2+5x^4-4x^5+x^6$ and $p_m(x) = x^4(1-x)^4 p_{m-1}(x) + (-4)^{m-1} p_1(x)$ for $m \geq 2$. Then

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) + \frac{(-4)^m}{1+x^2}, \text{ for all } m \in \mathbb{N}.$$

Proof. We argue by induction on m. A computation shows that $\frac{x^4(1-x)^4}{1+x^2} = p_1(x) - \frac{4}{1+x^2}$. Assume $\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) + \frac{(-4)^m}{1+x^2}$. Now

$$\begin{aligned} \frac{x^{4(m+1)}\,(1-x)^{4(m+1)}}{1+x^2} &= x^4(1-x)^4\Big(p_m(x) + \frac{(-4)^m}{1+x^2}\Big) \\ &= x^4(1-x)^4 p_m(x) + (-4)^m \frac{x^4(1-x)^4}{1+x^2} \\ &= x^4(1-x)^4 p_m(x) + (-4)^m \Big(p_1(x) - \frac{4}{1+x^2}\Big) \\ &= x^4(1-x)^4 p_m(x) + (-4)^m p_1(x) + \frac{(-4)^{m+1}}{1+x^2} \\ &= p_{m+1}(x) + \frac{(-4)^{m+1}}{1+x^2}. \end{aligned}$$

□

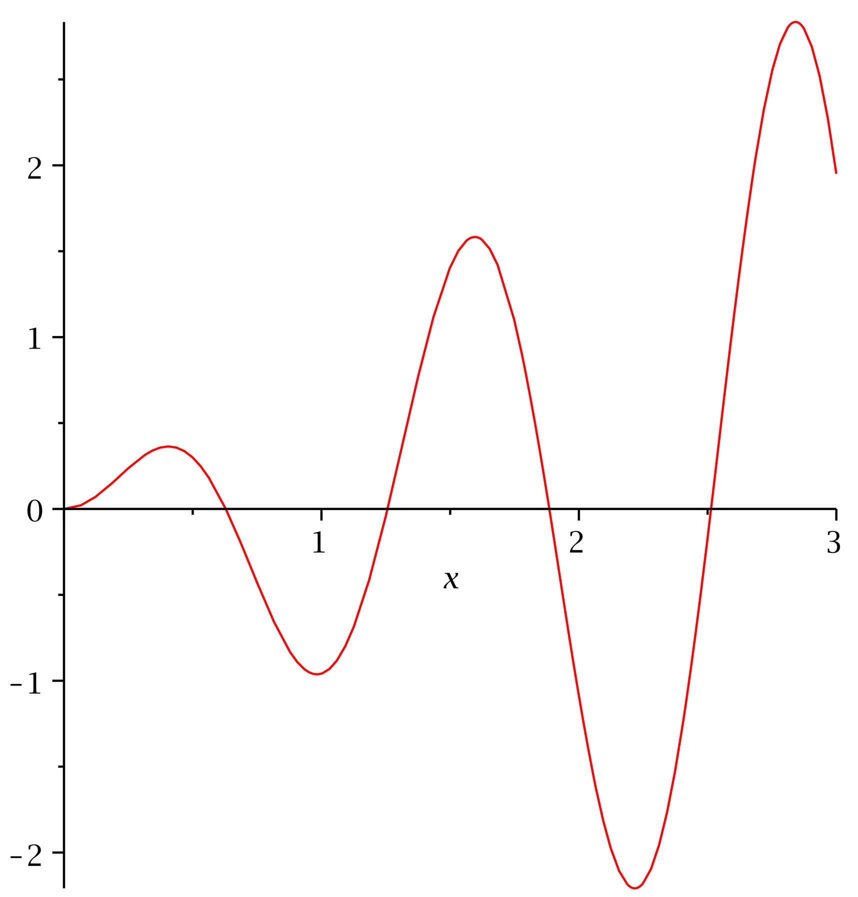

Figure 1: An example graphic

Conclusion

Here are my primary conclusions.

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□

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Acknowledgements

This research was conducted during the 2014 Mathematical Sciences Research Institute Undergraduate Program (MSRI-UP) in Berkeley, CA under the direction of Prof. Victor H. Moll, Tulane University. MSRI-UP is supported by the National Science Foundation (grant No. DMS-1156499) and the National Security Agency (grant No. H-98230-13-1-0262). We would like to thank ...

Contact Information: Carl F. Gauss: gauss@msri.org; Leonhard Euler: euler@basel.edu; Emmy Nöether: noether@berlin.de