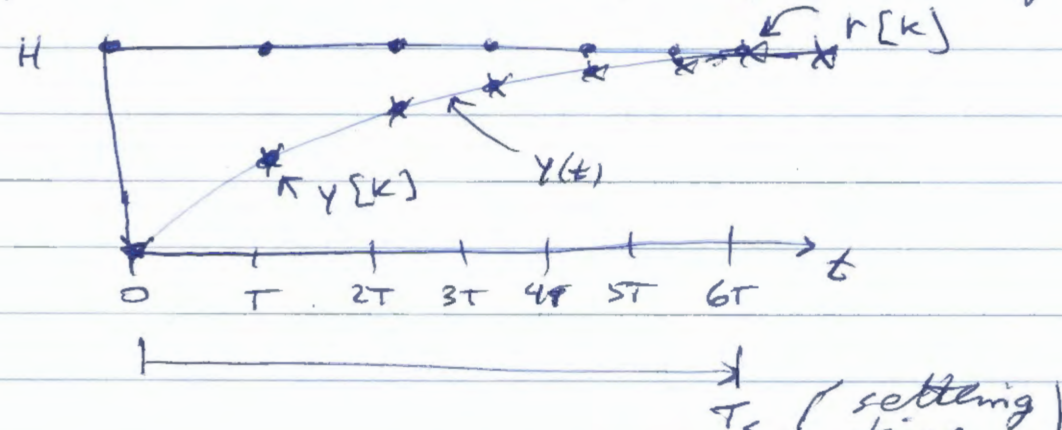
Digital State-Feedback Tracking Systems

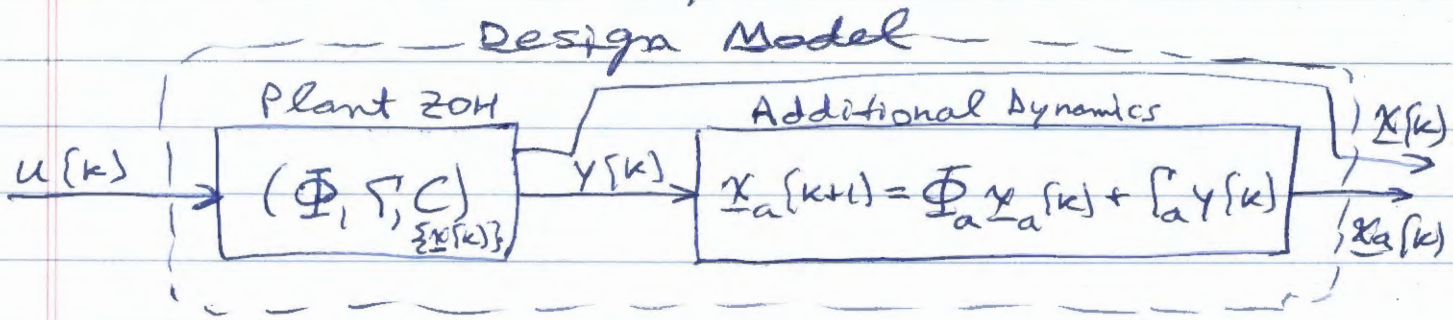
Goal: We want the sampled plant output $y[k]$ to "track" a reference command signal $r[k]$. That is, $y[k] \rightarrow r[k]$ (as k increases)

(Assume a single-input, single-output system.)

Example: If $r[k]$ is a step signal of height H



Derivation: Start with a "design model" which is a cascade of the plant ZOH and an "additional dynamics" block:



Note: $(\underbrace{\Phi_a}_{g\times g}, \underbrace{\Gamma_a}_{g\times 1})$ will be chosen later

The state vector of the design model is:

$$\underset{(n+g)\times 1}{x_d[k]} = \begin{bmatrix} x[k] \\ x_a[k] \end{bmatrix} \begin{matrix} \}\ n \text{ variables} \\ \}\ g \text{ variables} \end{matrix}$$

From the diagram on the previous page

$$\underline{x}[k+1] = \Phi \underline{x}[k] + \Gamma u[k]$$
$$x_a[k+1] = \Phi_a \underline{x}_a[k] + \Gamma_a C \underline{x}[k]$$

Thus, the state-space equation for the design model is

$$\underbrace{\begin{bmatrix} \underline{x}[k+1] \\ \underline{x}_a[k+1] \end{bmatrix}}_{\underline{x}_d[k+1]} = \underbrace{\begin{bmatrix} \Phi & 0 \\ \Gamma_a C & \Phi_a \end{bmatrix}}_{\substack{\Phi_d \\ (n+g)\times(n+g)}} \begin{bmatrix} \underline{x}[k] \\ \underline{x}_a[k] \end{bmatrix} + \underbrace{\begin{bmatrix} \Gamma \\ 0 \end{bmatrix}}_{\substack{\Gamma_d \\ (n+g)\times 1}} u[k]$$

The design model is an $(n+g)$th-order system. Suppose we want to <u>regulate the design model</u> with digital state feedback.

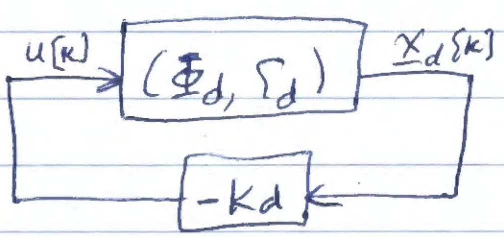
The rule of thumb for choosing the sampling interval becomes:

$$T = \min\left(\frac{T_s}{20(n+g)}, \frac{\pi}{5\beta_{max}}\right)$$

>> spoles = ... choose $n+g$ CL poles using the usual rules

>> zpoles = exp(T*spoles)

>> K_d = place $(\underbrace{\text{phid}}_{\Phi_d}, \underbrace{\text{gammad}}_{\Gamma_d}, \text{zpoles})$



$$u[k] = \underbrace{-K_d}_{1\times(n+g)} \underbrace{\underline{x}_d[k]}_{(n+g)\times 1}$$

Recall that $\underline{x}_d[k] = \begin{bmatrix} \underline{x}[k] \\ \underline{x}_a[k] \end{bmatrix} \begin{matrix} \}n \\ \}q \end{matrix}$

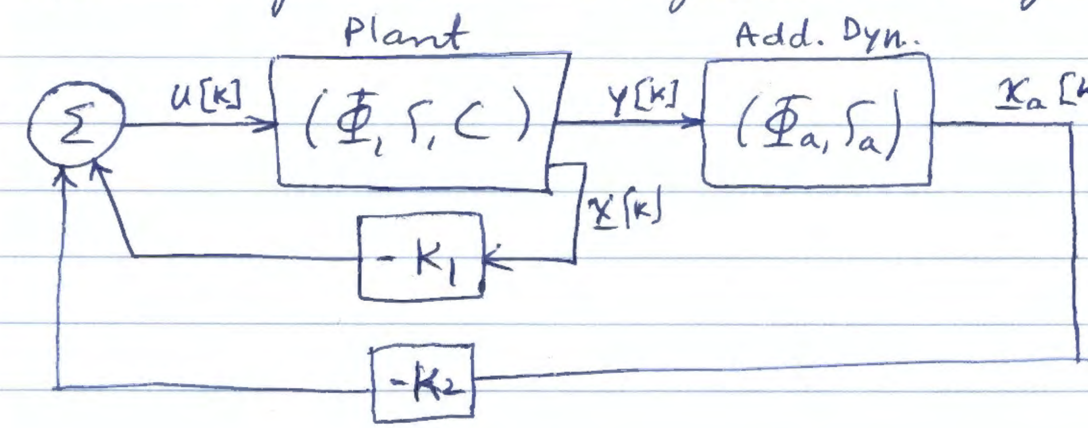
Partition the feedback gain vector into two pieces:

$$\underset{1\times(n+q)}{K_d} = [\, \underbrace{K_1}_{1\times n} \;\; \underbrace{K_2}_{1\times q} \,]$$

Then

$$u[k] = -K_d\, \underline{x}_d[k] = -K_1\, \underline{x}[k] - K_2\, \underline{x}_a[k]$$

Block diagram of the regulated design model:

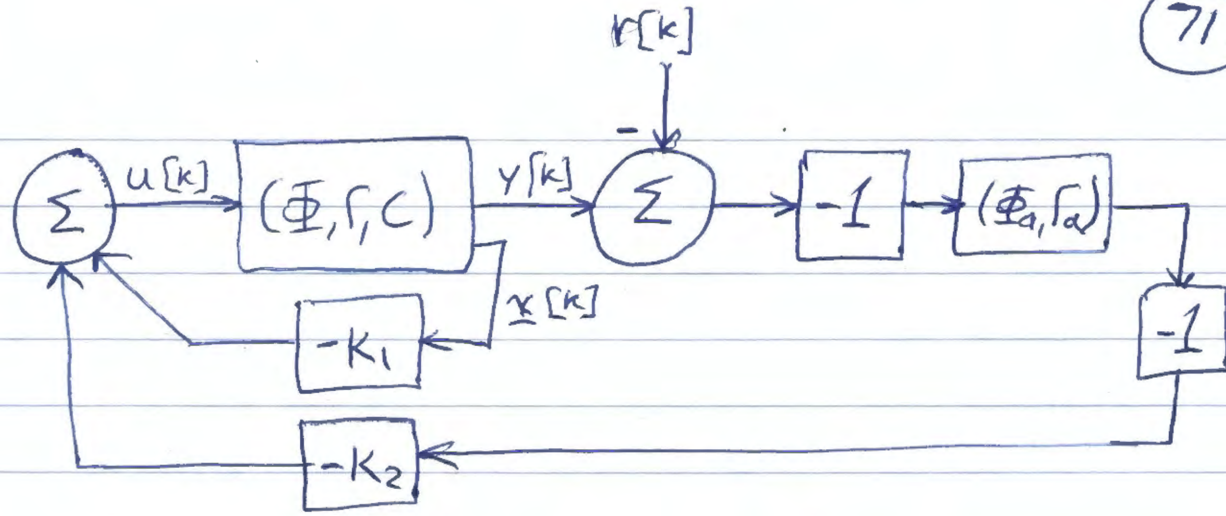


This is a stable closed-loop system with $n+q$ poles located at the numbers specified in zpoles.

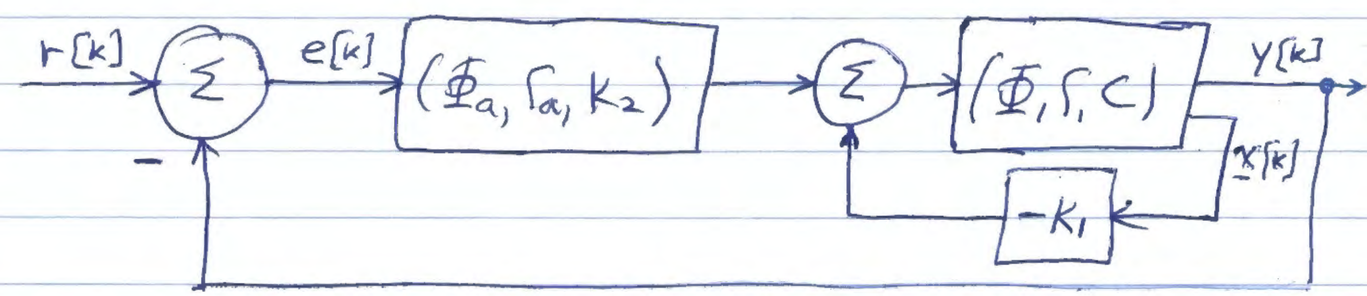
The next step is to introduce a reference command signal $r[k]$ into the above diagram. External signals added to a feedback loop do not change the closed-loop pole locations.

We also insert two minus signs in a way that does not change the feedback loop.

$$(-1)\cdot(-1) = 1$$



- Combine $-K_2$ with -1 to get K_2
- "Push" the other minus sign back through the summing junction and change the signs on the two incoming signals.
- Combine K_2 with (Φ_a, Γ_a) to get a complete state-space block (Φ_a, Γ_a, K_2)
- Rotate the above diagram counterclockwise to get r[k] on the left



Note: the poles of this CL system are still located at zpoles (stable with a T_s-second settling time)

How to choose the additional dynamics for a SISO plant. [Easily generalizes to MIMO.]

We want to guarantee $y[k] \rightarrow r[k]$ } zero steady-state error even with perturbations to the plant model.

Condition: to track $r[k]$ with zero steady-state error, the matrix Φ_a must have eigenvalues equal to the poles of the z transform of $r[k]$.

If $r[k]$ is a step of height H, z transform is $\dfrac{H}{z-1}$ ← 1 pole at $z=1$.

So Φ_a must have one eigenvalue equal to 1. A 1x1 matrix is its own eigenvalue, so choose $\Phi_a = 1$ ($g=1$, additional dynamics use one state variable)

The second requirement is that (Φ_a, Γ_a) is controllable, which implies that Γ_a be a nonzero number. Choose $\Gamma_a = 1$.

$(\Phi_a = 1, \Gamma_a = 1)$ is a digital integrator.

So the additional dynamics block is,

$$e[k] \longrightarrow \boxed{(\underbrace{1, 1}_{\text{digital integrator}}, \underset{\uparrow\ \text{integral gain}}{K_2})} \longrightarrow v[k]$$

$\frac{1}{s}$

$\dot{x} = 0x + 1 \cdot u$

$A = 0 \quad B = 1$

$\Phi_a = e^{A \cdot T} = e^0 = 1$

so pole at $s=0 \Rightarrow$ pole at $z=1$