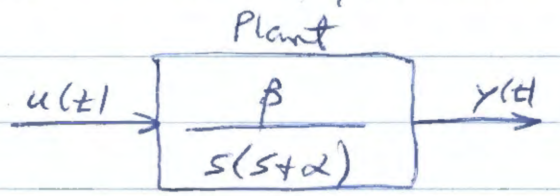
ELE 458

Digital Control Systems

Read sections 1.6, 1.7, 1.8 in book.

Consider a typical positioning system (eg. a motor-driven system):

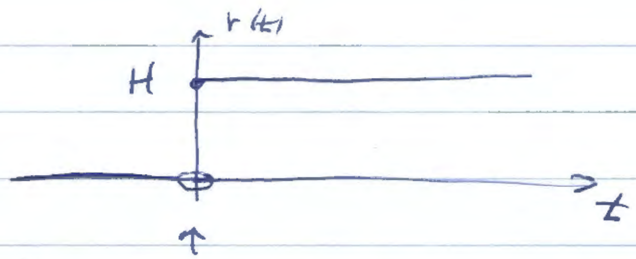
$\alpha, \beta > 0$



The system we want to control is called the "plant"

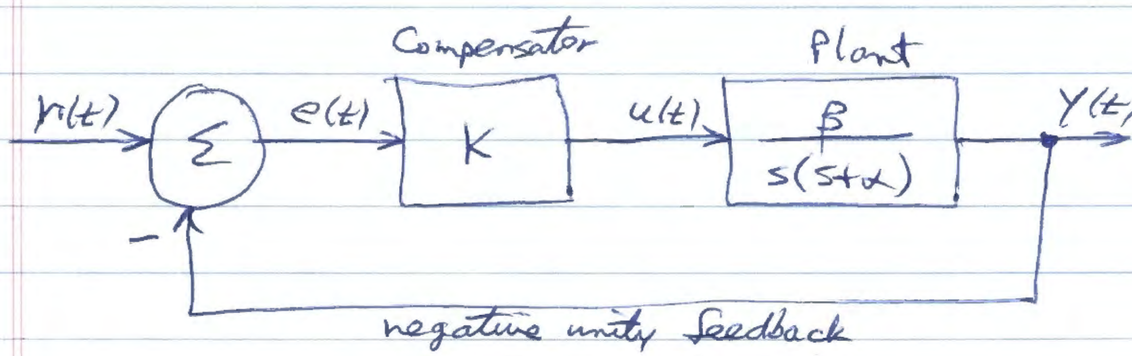
$u(t)$ = voltage applied to motor power amplifier (volts)
$y(t)$ = sensor signal measuring motor position (radians)

Suppose the motor is at initial position $y(0)=0$ and we want to command it to go to H radians. We can use the reference command signal $r(t)$:



the step command is given at time zero.

The simplest type of analog (continuous-time) control system is a negative unity feedback system with proportional control:



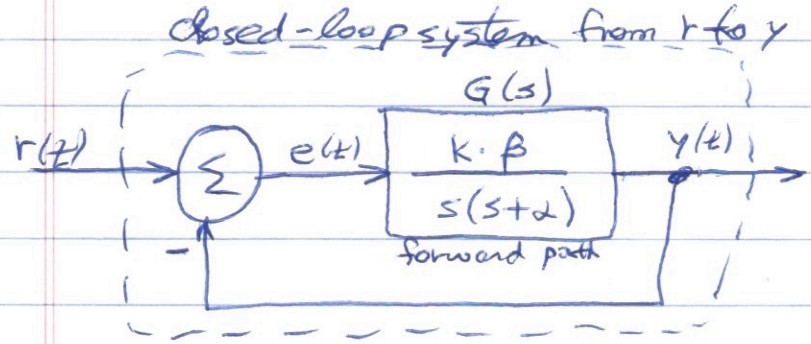
K is a gain (called the proportional gain)

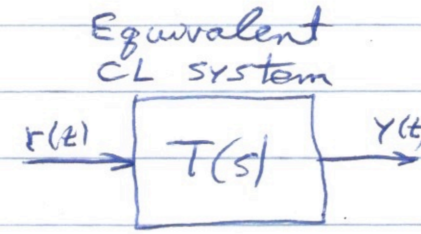
Note that $u(t) = k \cdot e(t)$. The plant input is proportional to the error signal $e(t)$.

How does this type of control system behave for different values of K?

Can we calculate a "good" value of K?

The previous block diagram may be re-drawn as follows:

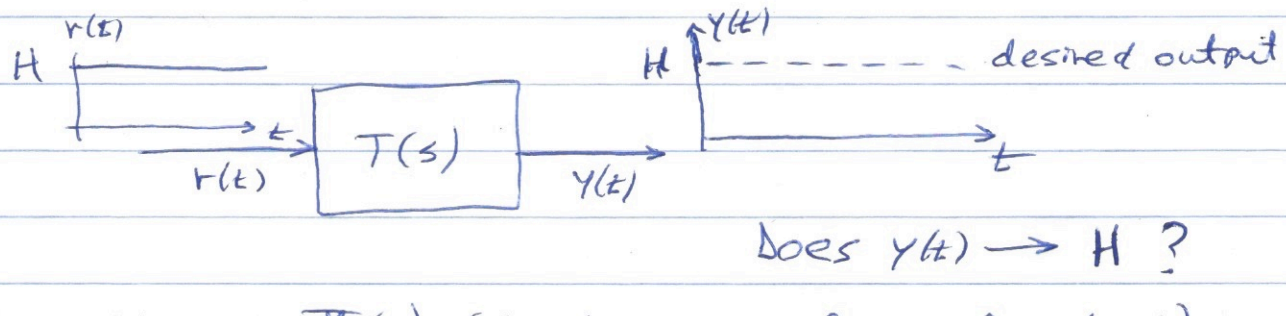




Using the "feedback formula" the system from $r(t)$ to $y(t)$ is

$$T(s) = \frac{G(s)}{1 + G(s)}$$

$$T(s) = \frac{\frac{K\beta}{s(s+\alpha)}}{1 + \frac{K\beta}{s(s+\alpha)}} \times \frac{s(s+\alpha)}{s(s+\alpha)} = \frac{K\beta}{s(s+\alpha)+K\beta} = \frac{K\beta}{s^2+\alpha s+K\beta}$$



Does $y(t) \rightarrow H$?

Recall that $Y(s)$ (Laplace transform of output) equals transfer function times $R(s)$, or

$$Y(s) = T(s)\, R(s)$$

$$R(s) = \frac{H}{s}$$ (see Table 3.2 p. 84)

Also, the Final Value Theorem (Table 3.1 p. 83) says

$$\underbrace{\lim_{t\to\infty} y(t)}_{\text{we want to know if this equals } H} = \underbrace{\lim_{s\to 0} s\, Y(s)}_{\text{calculate this for our problem}}$$

$$= \lim_{s\to 0} \cancel{s} \cdot T(s) \cdot \frac{H}{\cancel{s}}$$

$$= T(0) \cdot H$$

$$= \frac{K \cdot \beta \cdot H}{K \cdot \beta} = H$$

So $y(t) \to H$ as t gets large.

What about the <u>transient response</u> when $y(t)$ is moving from 0 to H?

- How long does it take to get to H?
- Is there overshoot and/or oscillation?

Recall that $y(t)$ is the step response of

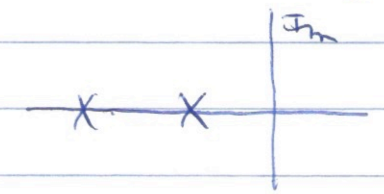
$$T(s) = \frac{K\beta}{s^2 + \alpha s + K\beta}.$$

The <u>poles</u> of this system are the roots of $s^2 + \alpha s + K\beta$.

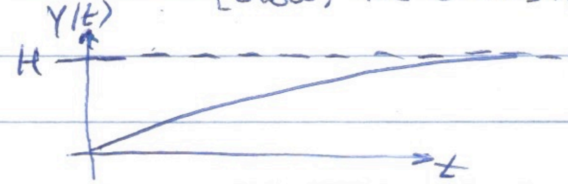
For fixed values of α and β (i.e. a given plant model) the pole locations change as a function of K

There are three possibilies (for fixed α, β):

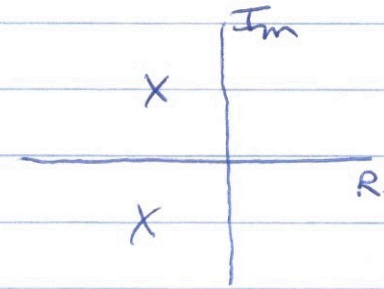
(a) The poles could be real and unequal



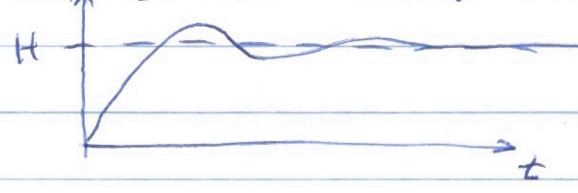
Response is <u>overdamped</u>
[Slow, no overshoot]



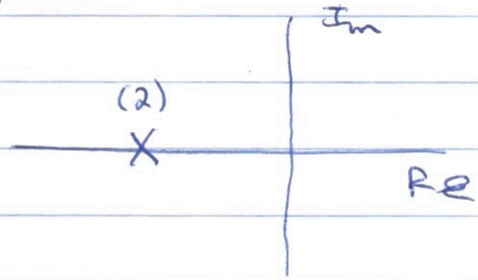
(b) The poles could be a complex-conjugate pair



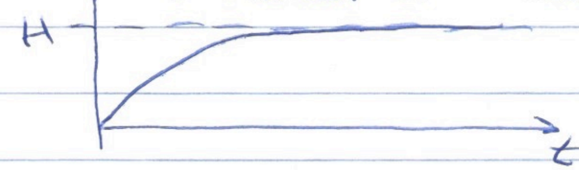
Response is <u>underdamped</u>
[fast, overshoot, oscillatory]



(c) Could have a double pole



Response is <u>critically damped</u>
[fast as possible with no overshoot or oscillation]



By selecting a value for the proportional gain, K, we can create a control system with any of these responses.

General principle: write down what you have and set it equal to what you want!

We have the denominator of $T(s)$: $s^2 + \alpha \cdot s + K \cdot \beta$

We want $(s+p)^2$, which has a double pole at $-p$. (The number p is unknown)

$(s+p)(s+p) = s^2 + 2ps + p^2$

Set

$$s^2 + \alpha s + K\beta = s^2 + 2ps + p^2$$

Equate corresponding powers of s:

s^2: $1 = 1$

s: $\alpha = 2p \Rightarrow p = \frac{\alpha}{2}$ (the repeated pole will occur at $s = -\frac{\alpha}{2}$)

constant: $K\beta = p^2$

$$K = \frac{p^2}{\beta} = \boxed{\frac{\alpha^2}{4\beta}}$$

this gain value gives critical damping

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Look at the roots of the denominator polynomial $s^2 + \alpha s + K\beta$ (use quadratic formula) (CL poles).

$$\text{poles} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4K\beta}}{2}$$

$$-\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2}$$

$K = 0 \Rightarrow$ poles are $-\alpha, 0$

$0 < K < \frac{\alpha^2}{4\beta} \Rightarrow$ poles are real and unequal (overdamped)

$K = \frac{\alpha^2}{4\beta} \Rightarrow$ repeated real poles at $-\frac{\alpha}{2}$ (critically damped)

$K > \frac{\alpha^2}{4\beta} \Rightarrow$ complex conjugate poles with real part $-\frac{\alpha}{2}$ (underdamped)

Plant used in HW 1: $\beta = 2{,}400$

$\alpha = 22$ so plant

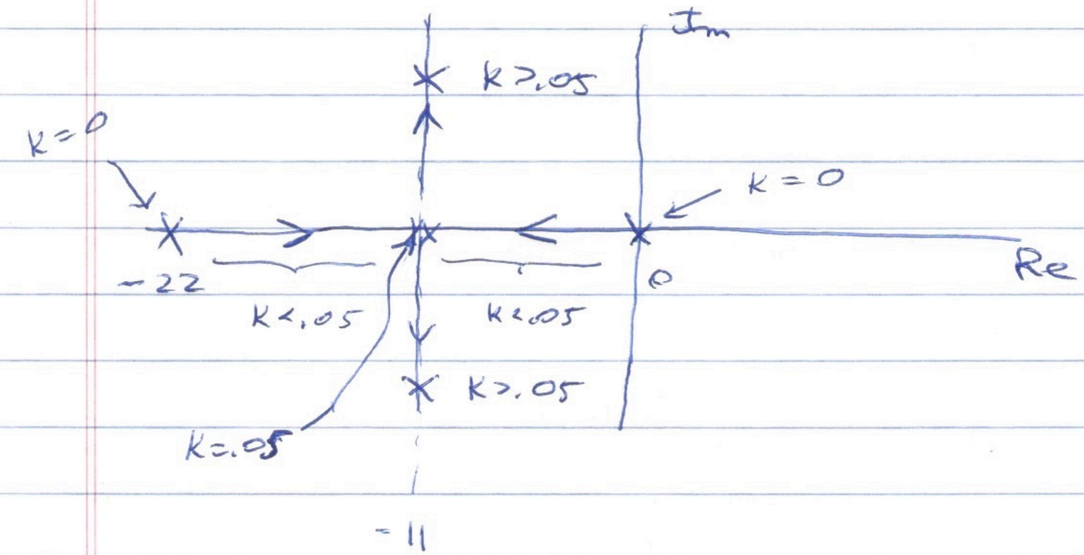
is $\dfrac{2{,}400}{s(s+22)}$

With $K = \dfrac{\alpha^2}{4\beta} = \dfrac{(22)^2}{4 \cdot 2{,}400}$

$K = 0.05 \Rightarrow$ CL system has double pole at -11.

Denominator of $T(s)$ is $s^2 + \alpha \cdot s + K \cdot \beta$

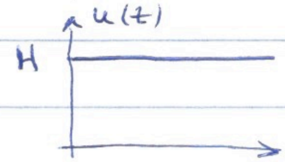
CL Poles as a function of gain:

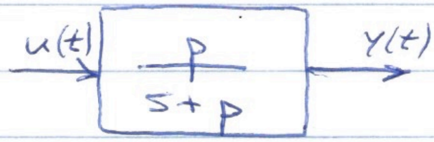


Relationship Between Pole Locations and Settling Time

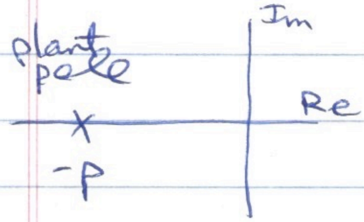
<u>Def</u> The 1% settling time T_s is the time at which the step response enters and stays within $\pm 1\%$ of its desired value, H.

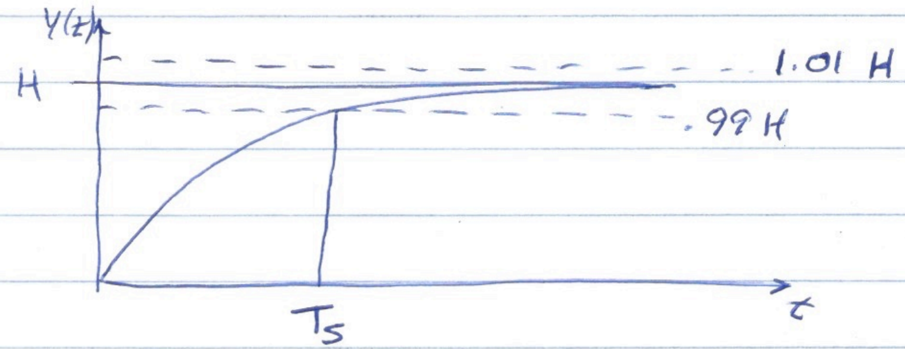
Consider a first-order system with a step input:





$$y(t) = H\left(1 - e^{-pt}\right) \quad \text{(see Table 3.2)}$$





The settling time is found as follows:

$$y(t)\Big|_{t=T_s} = .99\,H$$

or

$$\cancel{H}\left(1 - e^{-p \cdot T_s}\right) = .99\,\cancel{H}$$

$$0.01 = e^{-p \cdot T_s}$$

$$\ln(0.01) = -p \cdot T_s$$

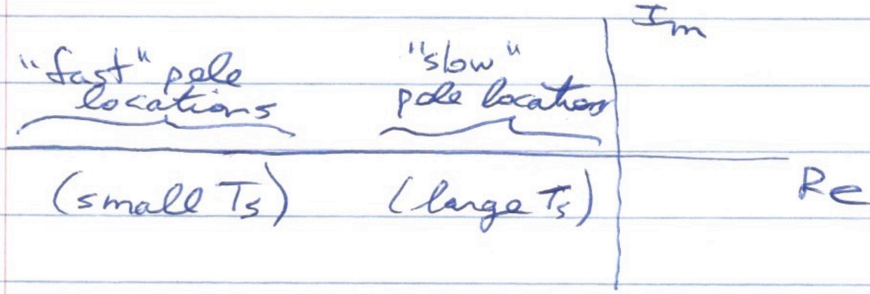
$$T_s = \frac{-\ln(0.01)}{p} = \boxed{\frac{4.62}{p}}$$

settling time for first-order system with pole at $-p$.

Note: I have rounded
$-\ln(0.01) = 4.6052$
up to a nice number 4.62

When p is a large number T_s is small ("fast" system).

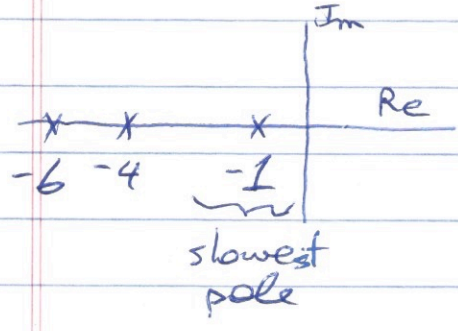
When p is a small number T_s is large ("slow" system).



How about higher-order systems?

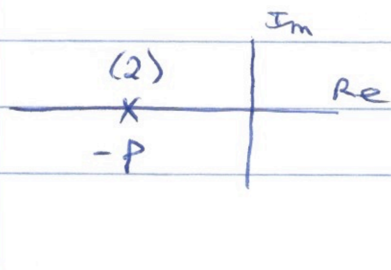
For systems with more than one pole, the settling time is <u>approximately</u> equal to the settling time of the slowest pole.

<u>Example</u>: 3^{rd}-order system



$$T_s \approx \frac{4.62}{1} = 4.62 \text{ seconds}$$

For a <u>double pole</u> at $-p$ there is a 45% increase in settling time:

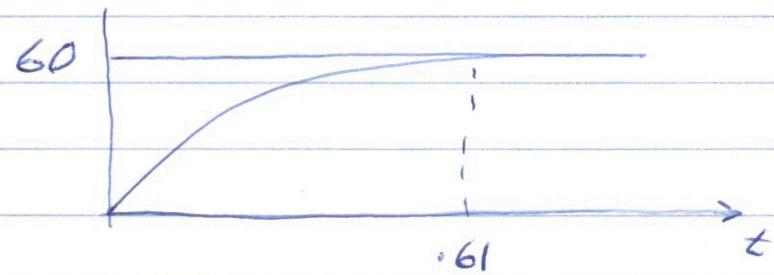


$$T_s \approx \frac{4.62}{p} \times 1.45$$

For the HW 1 proportional control system the closed-loop system has a double pole at -11. Therefore the settling time is

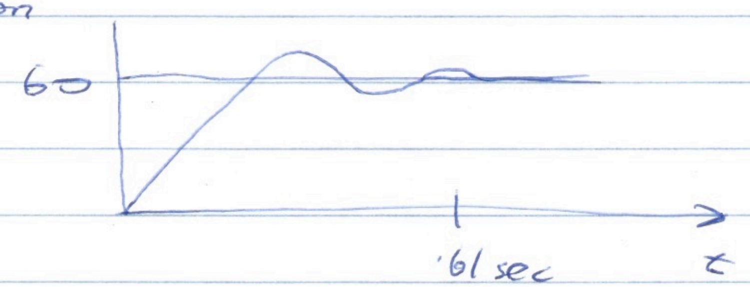
$$T_s = \frac{4.62}{11} \times 1.45 = .61 \text{ sec}$$

We expect the following step response:



for $k = 0.05$

If we make the proportional gain $K > 0.05$ the system will be "faster" but will have overshoot and oscillation



Can we get a faster system without overshoot?

Also, the hardware plant has the constraint that

$$|u(t)| \le 5$$ (plant input signal must be less than 5 volts)