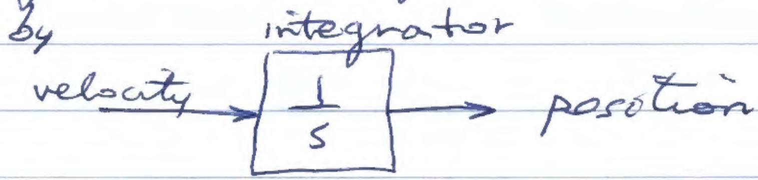
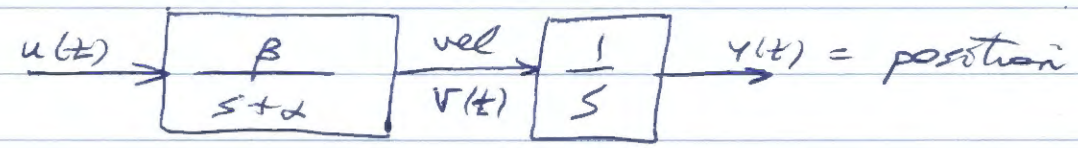
Recall Laplace Transform formulas:

	function	Laplace Transform
	$f(t)$	$F(s)$
The "dot" means $\frac{d}{dt}$ →	$\dot{f}(t)$	$sF(s)$ (if $f(0)=0$)
	$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s)$

Thus, integrating velocity to get position is represented by



Combining the integrator with the previous velocity transfer function yields



or $\dfrac{\beta}{s(s+\alpha)}$ for the complete positioning system.

State-Space Models (See Section 3.3 in book)

Look at first transfer function (velocity):

$$\frac{V(s)}{U(s)} = \frac{\beta}{s+\alpha} \Rightarrow (s+\alpha)V(s) = \beta U(s)$$

$$sV(s) + \alpha V(s) = \beta U(s)$$

take inverse Laplace transforms ↓ ↓ ↓

$$\boxed{\dot{V}(t) + \alpha V(t) = \beta u(t)} \quad (1)$$

Look at integrator transfer function:

$$\frac{Y(s)}{V(s)} = \frac{1}{s} \Rightarrow sY(s) = V(s)$$

I.L.T.

$$\boxed{\dot{y}(t) = v(t)} \quad (2)$$

Define <u>state variables</u>: $x_1(t) = y(t)$

$x_2(t) = v(t)$

Then (1) and (2) become

from (2): $\dot{x}_1(t) = \dot{y}(t) = v(t) = x_2(t)$

or

$$\dot{x}_1(t) = x_2(t) \quad (3)$$

from (1) $\dot{x}_2(t) = \dot{v}(t) = -\alpha v(t) + \beta u(t)$

or $\dot{x}_2(t) = -\alpha x_2(t) + \beta u(t)$ (4)

Define the <u>state vector</u> $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

(3) and (4) can then be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \beta \end{bmatrix}}_{B} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0 \cdot u(t)$$

In general, for an nth-order system,

$$\dot{\underline{x}}(t) = \underbrace{A}_{n \times n} \underline{x}(t) + \underbrace{B}_{n \times 1} u(t)$$

$$y(t) = \underbrace{C}_{1 \times n} \underline{x}(t) + \underbrace{D}_{1 \times 1} u(t)$$

state-space model is (A, B, C, D)

Discrete-Time State-Space Models
(Section 3.4)

The *input* is a sequence of numbers, $u[k]$
$k = 0, 1, 2, \ldots$

The *output* is a sequence of numbers, $y[k]$
The *state vector* is a sequence of vectors

State update equation: $\underset{n\times 1}{\underline{x}[k+1]} = \underset{n\times n}{A}\,\underset{n\times 1}{\underline{x}[k]} + \underset{n\times 1}{B}\,\underset{1\times 1}{u[k]}$

Output equation:

$$\underset{1\times 1}{y[k]} = \underset{1\times n}{C}\,\underset{n\times 1}{\underline{x}[k]} + \underset{1\times 1}{D}\,\underset{1\times 1}{u[k]}$$

State-Space Model:

$$u[k] \longrightarrow \boxed{\begin{array}{l} \underline{x}[k+1] = A\underline{x}[k] + Bu[k] \\ y[k] = C\underline{x}[k] + Du[k] \end{array}} \longrightarrow y[k]$$

The state update equation is a
first-order, vector-valued difference equation
Consider a special case when $u[k] = 0$ for all
k and $\underline{x}[0] = \underline{x}_0$ is a given vector of numbers
initial state vector

What is the *solution* to $\underline{x}[k+1] = A\underline{x}[k]$?

Compute:
$$\underline{x}[1] = A\underline{x}[0] = A\underline{x}_0$$
$$\underline{x}[2] = A\underline{x}[1] = A\cdot A\underline{x}_0 = A^2\underline{x}[0]$$
$$\underline{x}[3] = A\underline{x}[2] = A\cdot A^2\underline{x}[0] = A^3\underline{x}[0]$$
$$\vdots$$
$$\underbrace{\underline{x}[k] = A^k\underline{x}_0}_{\text{This is the solution}}, \qquad A^k = \underbrace{A\cdot A\cdots A}_{k\text{ factors}}$$

Example $A = \begin{bmatrix} .8 & 0 \\ 0 & .1 \end{bmatrix}, \quad \underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A^2 = \begin{bmatrix} .8 & 0 \\ 0 & .1 \end{bmatrix} \cdot \begin{bmatrix} .8 & 0 \\ 0 & .1 \end{bmatrix} = \begin{bmatrix} .8^2 & 0 \\ 0 & .1^2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} .8^3 & 0 \\ 0 & .1^3 \end{bmatrix}$$

$$A^k = \begin{bmatrix} .8^k & 0 \\ 0 & .1^k \end{bmatrix}.$$

Note: In general, for a diagonal matrix

$$\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}^k = \begin{bmatrix} d_1^k & & & 0 \\ & d_2^k & & \\ & & \ddots & \\ 0 & & & d_n^k \end{bmatrix}$$

so

$$\underline{x}[k] = A^k \underline{x}_0 = \begin{bmatrix} .8^k \\ .1^k \end{bmatrix}$$

or $x_1[k] = .8^k, \quad x_2[k] = .1^k$

Note $\underline{x}[k] \to \underline{0}$ as k increases.

$x_2[k] \to 0$ much faster than $x_1[k]$

$x_2 = *$

$x_1 = \cdot$