In general, the solution to

$\underline{x}[k+1] = A\underline{x}[k]$ is $\underline{x}[k] = A^k \underline{x}_0$.

Under what conditions does $\underbrace{\underline{x}[k] \to \underline{0}}$?
stable decay from initial state.

Consider the eigenvalue decomposition of A:

$$\underset{n\times n}{A} = \underbrace{U}_{n\times n}\underbrace{\Lambda}_{n\times n}\underbrace{U^{-1}}_{n\times 1}.$$

Columns of U are the eigenvectors of A

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

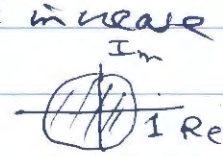
λ_i are the eigenvalues of A, which are real numbers or complex-conjugate pairs

Consider
$$\begin{aligned} A^3 &= A \cdot A \cdot A \\ &= (U\Lambda U^{-1})(U\Lambda U^{-1})(U\Lambda U^{-1}) \\ &= U\Lambda \underbrace{U^{-1}U}_{=I}\Lambda \underbrace{U^{-1}U}_{=I}\Lambda U^{-1} \\ &= U\Lambda^3 U^{-1}, \quad \Lambda^3 = \begin{bmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 \end{bmatrix} \end{aligned}$$

So
$$\underline{x}[k] = A^k \underline{x}[0] = \underbrace{U}_{\text{numbers}} \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} \underbrace{U^{-1}\underline{x}_0}_{\text{numbers}}$$

$\boxed{\text{If } |\lambda_i| < 1}$ then $\Lambda^k \to 0$ (zero matrix) as k increase

thus $\underline{x}[k] \to \underline{0}$



That is, all the eigenvalues of A must be inside a unit circle in complex plane to get stable decay from initial state

Necessary Eigenvalue Knowledge

0. The eigenvalue decomposition of an $n \times n$ matrix is

$$\underset{n\times n}{A} = \underbrace{U}_{n\times n} \cdot \underbrace{\Lambda}_{n\times n} \cdot \underbrace{U^{-1}}_{n\times n}$$

where the columns of U are the eigenvectors of A and the diagonal elements of Λ are the eigenvalues of A.

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

1. If A comes from a discrete-time or a continuous-time state-space model (A, B, C, D), the eigenvalues of A are the poles of the system.

2. An $n \times n$ matrix A has n eigenvalues, which are either real numbers or conjugate pairs of complex numbers.

3. a 1×1 matrix is its own eigenvalue.
   Example: $A = 4 \Rightarrow \text{eig}(A) = 4$

4. If A is a diagonal matrix
$$A = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$
then the eigenvalues of A are the real numbers $d_1, d_2, \ldots, d_n$.

Recall the state differential equation for a continuous-time state-space model:

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,u(t), \quad \underline{x}(0) = \underline{x}_0, \text{ a given vector of numbers}$$

Suppose $u(t) = 0$ for $t \geq 0$.
Then

$$\dot{\underline{x}}(t) = A\,\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0 .$$

The solution to this vector-valued differential equation is

$$\underbrace{\underline{x}(t)}_{n\times 1} = \underbrace{\boxed{e^{At}}}_{n \times n}^{\text{matrix exponential}} \underbrace{\underline{x}_0}_{n\times 1}$$

The $n\times n$ matrix exponential function e^{At} is defined in terms of the eigenvalue decomposition of A:

$$A = U \Lambda U^{-1}, \quad e^{At} = U e^{\Lambda t} U^{-1}$$

$$= U \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}$$

Note: $e^{\lambda_i t} \to 0$ if $\text{real}(\lambda_i) < 0$.

If $\text{real}(\lambda_i) < 0$ for all λ_i (i.e. all poles in left half plane)

then $e^{At} \to$ zero matrix and $\underbrace{\underline{x}(t) \to \underline{0}}_{\text{stable decay from initial state.}}$