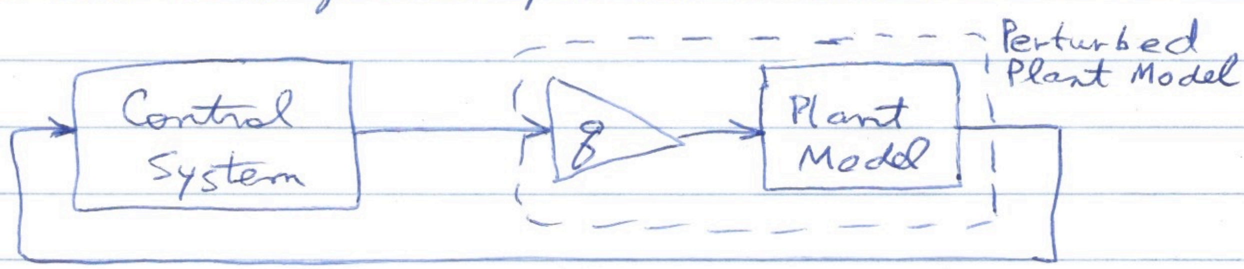
More on Stability Margins for a Single-Input Control System

This topic was introduced previously (see Lecture Notes pp. 34-35) ...

Given a plant model (e.g. a state-space model) we can design a control system such that the closed-loop system is stable. However, if the control system is connected to a perturbed plant model, how large a perturbation can the control system tolerate while maintaining stability?

Previously we considered gain perturbations. Consider now phase perturbations:



where $g = e^{-j\phi}$ represents ϕ radians of (unmodeled) phase lag.

When $\phi = 0$, $g = 1$ and the closed-loop system is stable by design. The closed-loop system will remain stable as ϕ goes from 0 up to a value ϕ_{max}. (The system will also remain stable for ϕ going to $-\phi_{max}$).

ϕ_{max} is called the phase margin and is reported in degrees. We want $\phi_{max} \geq 30°$ (could ask for $\phi_{max} \geq 45°$)

Obtaining State-Space Models

(Read Section 3.3.5 with this title)

We will consider three different types of given information from which we want to obtain an equivalent state-space model.

The first type of given information is a transfer function or interconnection of transfer functions.

Given a general nth-order transfer function model:

$$u(t) \longrightarrow \boxed{\frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}} \longrightarrow y(t)$$

we can obtain a state-space model using a canonical form (there are several).
We will use <u>observable canonical form</u> (OCF)
The OCF state-space model (A, B, C, D) is

$$\underset{n\times n}{A} = \begin{array}{cc} & \begin{array}{ccccc} & & n=2 & n=3 & \end{array} \\ n=1 & \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \cdots & - & 0 \end{bmatrix} \end{array}, \quad \underset{n\times 1}{B} = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ \vdots \\ b_n - a_n b_0 \end{bmatrix}$$

$$\underset{1\times n}{C} = [1 \ 0 \ \cdots \ 0], \quad \underset{1\times 1}{D} = b_0$$

Example: $\dfrac{2400}{s(s+22)} = \dfrac{\overset{b_0}{0}s^2 + \overset{b_1}{0}s + \overset{b_2}{2400}}{s^2 + \underset{a_1}{22}s + \underset{a_2}{0}}$

The OCF state-space model is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -22 & 1 \\ 0 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 2400 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} \cdot u$$

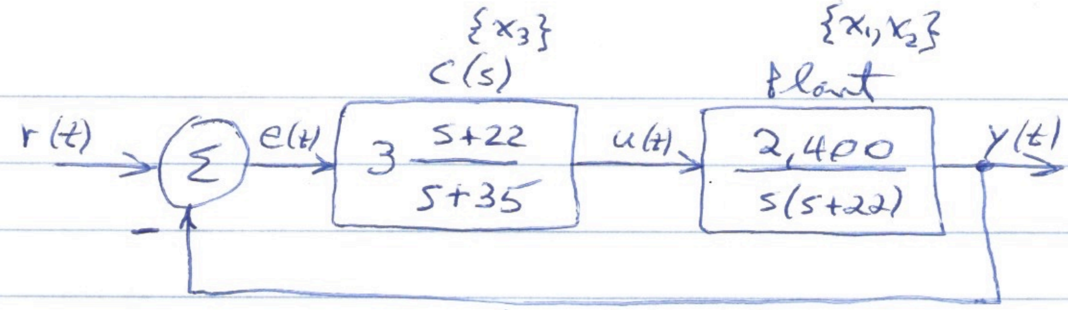
Note this property of OCF models:
if $b_0 = 0$ then $D = 0$ and the output equation becomes

$$y = C\underline{x} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1$$

In other words, the output of an OCF model equals the first state variable when $b_0 = 0$.

Example
Consider the following interconnection of transfer functions. The state variables assigned to each block are shown at the top of the block. OCF canonical form will be used to represent each block.



$\boxed{\text{OCF for Plant}}$: $$\frac{\overset{b_0}{0s^2}+\overset{b_1}{0s}+\overset{b_2}{2400}}{s^2+\underset{a_1}{22s}+\underset{a_2}{0}}$$

(1) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -22 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2400 \end{bmatrix} \underset{\text{block input}}{u(t)}$$

(2) $$\underbrace{y(t)}_{\text{block output}} = \underbrace{x_1}_{\text{first state variable because } b_0 = 0.}$$

$\boxed{\text{OCF for } C(s)}$ $$\frac{\overset{b_0}{3s}+\overset{b_1}{66}}{s+\underset{a_1}{35}}$$

(3) $$\dot{x}_3 = -35\,x_3 + (66-35*3)\,\underbrace{e(t)}_{\text{block input}}$$

(4) $$\underbrace{u(t)}_{\text{block output}} = x_3 + 3\,\underbrace{e(t)}_{\text{block input}}$$

Note that $e(t) = r(t) - y(t)$

$e(t) = r(t) - x_1(t)$ (5)

Plug (5) into (3) and (4)

Then plug (4) into (1)

5 into 3

$$\dot{x}_3 = -35x_3 - 39(r - x_1)$$
$$= 39x_1 - 35x_3 - 39r \qquad (6)$$

5 into 4

$$u = x_3 + 3(r - x_1)$$
$$= -3x_1 + x_3 + 3r \qquad (7)$$

7 into 1 $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -22 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2400 \end{bmatrix} (-3x_1 + x_3 + 3r)$$

or

$$\dot{x}_1 = -22x_1 + x_2$$
$$\dot{x}_2 = -7200x_1 + 2400x_3 + 7200r$$

Combine with 6 to get

System from r to y

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -22 & 1 & 0 \\ -7200 & 0 & 2400 \\ 39 & 0 & -35 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 7200 \\ -39 \end{bmatrix}}_{B} r$$

From 2

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{D}{0} \cdot r$$

If we wanted the system from r to u, for example, it would have the same A and B as the system from r to y but the output equation (see 7) would be

$$u = \underbrace{\begin{bmatrix} -3 & 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{3}_{D} \cdot r$$