$A = 23.1$	rad/sec^2
$C = 25.0$	sec^{-1}
$D = 2,633.$	rad/(volt-sec^2)
$n = 495$	rad/m
$g = 9.81$	m/sec^2

Table 3.9 The parameter values for (3.166) and (3.167) corresponding to the hardware system at the University of Rhode Island.

loop around the motor: it not only reduces the effect of friction on the motor, but it also decouples the pendulum and motor dynamics.

The resulting equation is then

$$\dot{\omega}_m + C\omega_m = Dv_d \tag{3.165}$$

for some constants C and D. This is just a one time-constant system. The step response of this system has a time constant equal to $1/C$ and a steady-state value of DV_d/C for a voltage step equal to V_d volts. Thus the parameters C and D can be obtained from a step response of the cart. The parameters A, B, C, n and g (gravity), are all that is needed to specify the behavior of the cart/pendulum system.

In fact, (3.159) and (3.165) can be used to write down a state-space description of the cart/pendulum system if the input and state variables are chosen as follows:

u = D/A voltage from computer
x_1 = θ (pendulum position)
x_2 = $\dot{\theta}$ (pendulum velocity)
x_3 = θ_m (motor position)
x_4 = ω_m (motor velocity)

With these definitions, (3.159) and (3.165) can be written as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -A\sin(x_1) - \frac{A}{ng}\cos(x_1)(-Cx_4 + Du) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -Cx_4 + Du.
\end{aligned} \tag{3.166}$$

The numerical values for the parameters in (3.166) which correspond to the hardware system used at the University of Rhode Island are shown in Table 3.9.

Linearized models can be obtained from (3.166) for the case when the pendulum is hanging down, and also when it is pointing up. When the pendulum is hanging down, we define the first state variable to be $x_1 = \theta$, and consider only small deviations about zero. A general linearization technique is given in the Appendix to this Chapter. Here, we simply note that for values of x_1 near zero, $\sin x_1 \approx x_1$ and $\cos x_1 \approx 1$.

When the pendulum is pointing up, we define the first state variable to be $x_1 = \theta - \pi$, and consider only small deviations about the vertical $\theta = \pi$. For this range of θ, $\sin x_1 \approx -x_1$ and $\cos x_1 \approx 0$. Using these approximations, (3.166) becomes a linear state-space model for either the hanging or inverted pendulum. These linear equations are shown below,

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \pm A & 0 & 0 & \mp\frac{AC}{ng} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -C \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ \pm\frac{AD}{ng} \\ 0 \\ D \end{bmatrix}u(t) \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}(t)
\end{aligned} \tag{3.167}$$

where the top plus or minus signs are used to describe the inverted pendulum, and the bottom signs are used to describe the hanging pendulum. For future reference, we give the linearized state-space model for the inverted pendulum using the parameter values from Table 3.9:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 23.1 & 0 & 0 & -0.1189 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -25.0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 12.5 \\ 0 \\ 2633 \end{bmatrix}u(t) \tag{3.168}$$

3.7 Chapter Summary

In this chapter we considered transfer function and state-space models for continuous- and discrete-time systems. We briefly reviewed Laplace and z transforms. These results were used to find a necessary condition for the step response of a standard, second-order, continuous-time system and the step response of a discrete-time sampled data system to be equal at sampling instants. The condition was that the poles of the discrete-time system had to be related to the poles of the continuous-time system by

$$z_{1,2} = e^{s_{1,2}T}$$

where T is the sampling interval. This result is shown to be true in general in Chapter 4.

Canonical form state-space models were introduced as a way to obtain state-space models from arbitrary transfer functions. A given transfer function has an infinitely many different state-space models. Given one state-space model, another state-space model can be obtained using a linear transformation matrix. A linear transformation of a state-space model results in a new state-space model which has the same input, output, and transfer function as the original model.

We presented solutions to the state equations for continuous- and discrete-time systems and used these solutions to develop formulas for the impulse responses in terms of the state-space matrices. When two state-space models are connected, the resulting system can be described by a state-space model. We presented state-space models for parallel, cascade, and feedback connections of systems. Finally, state-space and transfer function models for some simple systems were presented, and a state-space model for a pendulum-on-a-cart system was derived.