# Data Mining & Machine Learning

CS37300

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# Independence

- Events A and B are independent iff
  - $P(A \wedge B) = P(A)\, P(B)$
  - Equivalently: $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$
  - *Knowing B happens tells you nothing about whether A happens*
- Random variables X and Y are independent iff
  every event about X is independent of every event about Y.
  - Equivalently: joint distribution $P_{(X,Y)}$ is equal $P_X P_Y$ product of marginal distributions
  - If discrete variables: $P(Y=y,X=x) = P(Y=y)P(X=x)$, or $P(Y=y|X=x) = P(Y=y)$
- Examples
  - Coin flip 1 and coin flip 2?
  - Weather and storm warning?
  - Weather and coin flip=H?

# Example of independent variables

- How to check for independence?
- Joint probability P(X,Y)

| | Y = 1 | Y = 2 | Y = 3 |
|---|---|---|---|
| X = 1 | 0.025 | 0.15 | 0.075 |
| X = 2 | 0.075 | 0.45 | 0.225 |

$P(X=1) = 0.25$

$P(X=2) = 0.75$

$P(Y=1) = 0.1$ $\quad$ $P(Y=2) = 0.6$ $\quad$ $P(Y=3) = 0.3$

- $P(X=1,Y=1) = P(X=1)\,P(Y=1)$ ? $\qquad$ $P(X=2,Y=1) = P(X=2)\,P(Y=1)$ ?
- $P(X=1,Y=2) = P(X=1)\,P(Y=2)$ ? $\qquad$ $P(X=2,Y=2) = P(X=2)\,P(Y=2)$ ?
- $P(X=1,Y=3) = P(X=1)\,P(Y=3)$ ? $\qquad$ $P(X=2,Y=3) = P(X=2)\,P(Y=3)$ ?
- If the answer to the 6 questions above is “Yes”, then X and Y are independent

# Example of independent variables

- How to check for independence?
- Joint probability P(X,Y)

| | Y = 1 | Y = 2 | Y = 3 | |
|---|---|---|---|---|
| X = 1 | 0.025 | 0.15 | 0.075 | → P(X=1) = 0.25 |
| X = 2 | 0.075 | 0.45 | 0.225 | → P(X=2) = 0.75 |
| | ↓ P(Y=1) = 0.1 | ↓ P(Y=2) = 0.6 | ↓ P(Y=3) = 0.3 | |

- 0.025 = 0.25 * 0.1 (Yes) 0.075 = 0.75 * 0.1 (Yes)
- 0.15 = 0.25 * 0.6 (Yes) 0.45 = 0.75 * 0.6 (Yes)
- 0.075 = 0.25 * 0.3 (Yes) 0.225 = 0.75 * 0.3 (Yes)

- The answer to the 6 questions above is “Yes”. **X and Y are independent.**

# Example of independent variables

- How to check for independence?
- Joint probability P(X,Y)

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|---|---|---|---|---|
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| | ↓ P(Y=1) = 0.1 | ↓ P(Y=2) = 0.6 | ↓ P(Y=3) = 0.3 | |

- Quick way to check:
  - In every column, values have proportions 1:3
  - So conditional distribution X given Y=y doesn't depend on y
  - P(X=x|Y=y) = P(X=x)

# Example of dependent variables

- How to check for independence?
- Joint probability P(X,Y)

| | Y = 1 | Y = 2 | Y = 3 | |
|---|---|---|---|---|
| X = 1 | 0.025 | 0.125 | 0.1 | → P(X=1) = 0.25 |
| X = 2 | 0.075 | 0.475 | 0.2 | → P(X=2) = 0.75 |
| | ↓ P(Y=1) = 0.1 | ↓ P(Y=2) = 0.6 | ↓ P(Y=3) = 0.3 | |

- P(X=1,Y=1) = P(X=1) P(Y=1) ?  P(X=2,Y=1) = P(X=2) P(Y=1) ?
- P(X=1,Y=2) = P(X=1) P(Y=2) ?  P(X=2,Y=2) = P(X=2) P(Y=2) ?
- P(X=1,Y=3) = P(X=1) P(Y=3) ?  P(X=2,Y=3) = P(X=2) P(Y=3) ?
- If the answer to the 6 questions above is "Yes", then X and Y are independent

# Example of dependent variables

- How to check for independence?
- Joint probability P(X,Y)

| | Y = 1 | Y = 2 | Y = 3 | |
|---|---|---|---|---|
| X = 1 | 0.025 | 0.125 | 0.1 | → P(X=1) = 0.25 |
| X = 2 | 0.075 | 0.475 | 0.2 | → P(X=2) = 0.75 |
| | ↓ P(Y=1) = 0.1 | ↓ P(Y=2) = 0.6 | ↓ P(Y=3) = 0.3 | |

- 0.025 = 0.25 * 0.1 (Yes) 0.075 = 0.75 * 0.1 (Yes)
- 0.125 = 0.25 * 0.6 (No) 0.475 = 0.75 * 0.6 (No)
- 0.1 = 0.25 * 0.3 (No) 0.2 = 0.75 * 0.3 (No)
- The answer to at least 1 question above is “No”. **X and Y are NOT independent.**

# Example of dependent variables

- How to check for independence?
- Joint probability P(X,Y)

| | Y = 1 | Y = 2 | Y = 3 | |
|---|---|---|---|---|
| X = 1 | 0.025 | 0.125 | 0.1 | → P(X=1) = 0.25 |
| X = 2 | 0.075 | 0.475 | 0.2 | → P(X=2) = 0.75 |
| | ↓ P(Y=1) = 0.1 | ↓ P(Y=2) = 0.6 | ↓ P(Y=3) = 0.3 | |

- First column has proportions 1:3
- Third column has proportions 1:2
- $P(X=x|Y=y)$ depends on y.
- They can't be independent.

# Mutual Independence

- Multiple events $A_1, A_2, \ldots A_n$ are **(mutually) independent** iff
- Every $I \subset \{1,2,\ldots,n\}$ and $J \subset \{1,2,\ldots,n\}$ have

$$P\left(\bigcap_{i \in I} A_i \cap \bigcap_{j \in J} A_j\right) = P\left(\bigcap_{i \in I} A_i\right) P\left(\bigcap_{j \in J} A_j\right)$$

- Random variables $X_1, X_2, \ldots X_n$ are (mutually) independent iff
- Every event $A_1$ about $X_1$, event $A_2$ about $X_2$,… event $A_n$ about $X_n$
- satisfy that $A_1, A_2, \ldots, A_n$ are mutually independent

# Conditional independence

- Two events *A* and *B* are **conditionally** independent given *C* iff*:*
  - $P(A \wedge B \mid C) = P(A \mid C)\, P(B \mid C)$
  - Equivalently: $P(A \mid B \wedge C) = P(A \mid C)$ or $P(B \mid A \wedge C) = P(B \mid C)$
- Two random variables X and Y are conditionally independent given Z iff:
  - For all events A for X, B for Y, C for Z:
    A and B are conditionally independent given C
  - (discrete variables) Equivalently: $P(X{=}x, Y{=}y \mid Z{=}z) = P(X{=}x \mid Z{=}z)P(Y{=}y \mid Z{=}z)$
- *Note: independence does not imply conditional independence or vice versa*

# Example 1

- **Conditional independence does not imply independence**
- On a random day,
  - A = event that Alice attends a lecture
  - B = event that Bob attends a lecture

$$P(A) = 3/7, \quad P(B) = 0.2$$

  - Given the event D that the day is either Mon, Wed or Fri

$$P(A|D) = 1, \quad P(B|D) = 0.7$$

  - If Alice attends lecture, it's definitely M,W or F i.e. A,D are "duplicates"
  - Alice and Bob don't know each other, $P(A \wedge B \mid D)=P(A|D)P(B|D)$
  - $P(B|A) = 0.7 \neq 0.2 = P(B)$
  - A and B not independent, but are conditionally independent given D

# Example 2

- **Independence does not imply conditional independence**
- Flip 2 coins.
- A = event coin 1 is heads
- B = event coin 2 is heads

$P(A|B) = P(A)$ A and B independent

- C = event exactly one coin was heads: C={HT,TH}

$$P(A|C) = ½, \quad P(B|C) = ½$$

$$P(A \wedge B \mid C) = 0 \neq P(A|C)P(B|C)$$

- A and B not conditionally independent given C

# Expectation

- Denotes the expected value or mean value of a random variable X
- Discrete

$$E[X] = \sum_x x \quad p(x)$$

- Continuous

$$E[X] = \int_x x \quad p(x)dx$$

- Expectation of a function

$$E[aX + b] = a \quad E[X] + b$$

$$E[h(X)] = \sum_x h(x) \quad p(x)$$

$$E[h(X)] = \int_x h(x)p(x)dx$$

# Example

- Let X be a random variable that represents the number of heads which appear when a fair coin is tossed three times.
- X = {0, 1, 2, 3}
- Sample space: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
- **X=0** (TTT), **X=1** (HTT, THT, TTH), **X=2** (HHT, HTH, THH), **X=3** (HHH)
- P(**X=0**) = 1/8; P(**X=1**) = 3/8; P(**X=2**) = 3/8; P(**X=3**) = 1/8
- What is the expected value of X, E[X]?

$$E[X] = (0 \cdot \frac{1}{8}) + (1 \cdot \frac{3}{8}) + (2 \cdot \frac{3}{8}) + (3 \cdot \frac{1}{8})$$
$$= \frac{3}{2}$$

# Variance

- Denotes the squared deviation of X from its mean

- Variance

$$Var(X) = E[(X - E[X])^2]$$
$$= E[X^2] - (E[X])^2$$

- Standard deviation

$$\sigma = \sqrt{Var(X)}$$

- Variance of a function

$$Var(aX + b) = a^2 \quad Var(X)$$

$$Var(h(X)) = \sum_x (h(x) - E[h(X)])^2 \quad p(x)$$

# Example

- Let X be a random variable that represents the number of heads which appear when a fair coin is tossed three times.
- X = {0, 1, 2, 3}

$$E[X] = (0 \cdot \frac{1}{8}) + (1 \cdot \frac{3}{8}) + (2 \cdot \frac{3}{8}) + (3 \cdot \frac{1}{8})$$
$$= \frac{3}{2}$$

- What is the variance of X, Var(X)?

$$Var(X) = \left(\left[0 - \frac{3}{2}\right]^2 \cdot \frac{1}{8}\right) + \left(\left[1 - \frac{3}{2}\right]^2 \cdot \frac{3}{8}\right) + \left(\left[2 - \frac{3}{2}\right]^2 \cdot \frac{3}{8}\right) + \left(\left[3 - \frac{3}{2}\right]^2 \cdot \frac{1}{8}\right)$$
$$= \left(\frac{9}{4} \cdot \frac{1}{8}\right) + \left(\frac{1}{4} \cdot \frac{3}{8}\right) + \left(\frac{1}{4} \cdot \frac{3}{8}\right) + \left(\frac{9}{4} \cdot \frac{1}{8}\right)$$
$$= \frac{3}{4}$$

# Common distributions

- Bernoulli
- Binomial
- Multinomial
- Normal

# Bernoulli

- Binary variable $X \in \{0,1\}$ that takes the value of 1 with probability $p \in [0,1]$
  - E.g., Outcome of a fair coin toss is Bernoulli with *p*=0.5. Here *x*=1 means that the coin landed heads up, *x*=0 means the the coin landed tails up

$$P(x) = p^x(1-p)^{1-x}$$

$$E[X] = 1(p) + 0(1-p) = p$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= 1^2(p) + 0^2(1-p) - p^2 \\ &= p(1-p) \end{aligned}$$

# Binomial

- Describes the number of successful outcomes in *n* independent Bernoulli(*p*) trials

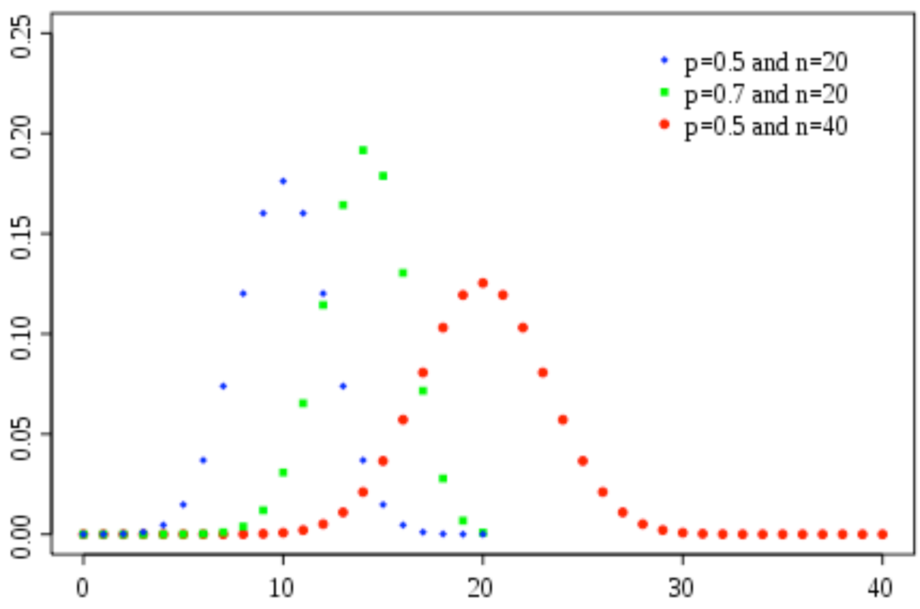
$$X \in \{0, 1, \dots, n\}, \quad p \in [0, 1]$$

  - E.g., Number of heads in a sequence of 10 tosses of a fair coin is Binomial with *n*=10 and *p*=0.5. Here *x* is the number of heads.

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = np$$

$$Var[X] = np(1-p)$$



# Multinomial

- Generalization of binomial to $k$ possible outcomes; outcome $i$ has probability $p_i$ of occurring; $x_i$ is the number of times the $i$-th outcome occurs in $n$ trials
  - E.g., Number of {outs, singles, doubles, triples, homeruns} in a sequence of $n$=10 times at bat is Multinomial. Here $k$=5, $x_1$ is the number of "outs", $p_1$ is the probability of "out", …, $x_5$ is the number of "homeruns", $p_5$ is the probability of "homerun".

$$x_i \in \{0,1,\ldots,n\}, \quad p_i \in [0,1], \quad \sum_{i=1}^{k} x_i = n, \quad \sum_{i=1}^{k} p_i = 1$$

$$P(x_1, \ldots x_k) = \binom{n}{x_1, \ldots x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

$$E[X_i] = np_i \qquad Var(X_i) = np_i(1 - p_i)$$

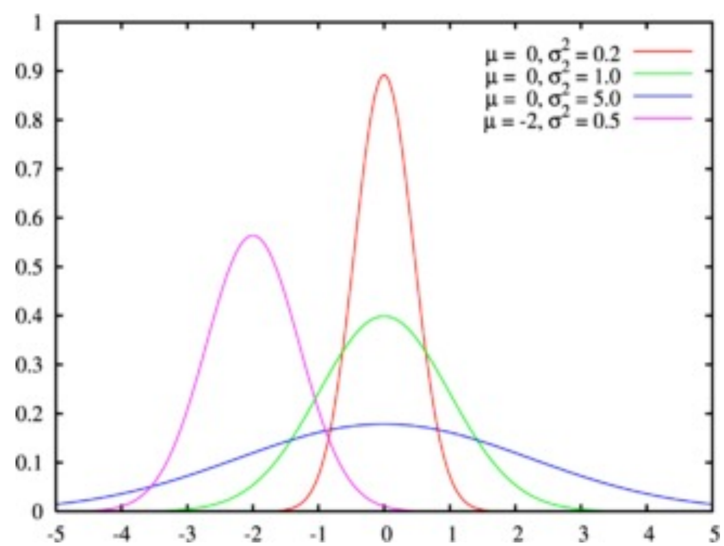
# Normal (Gaussian)

- Important distribution gives well-known bell shape
- Central limit theorem:
  - Distribution of $\sqrt{n}$ times the average of $n$ independent zero-mean samples becomes normally distributed as $n \rightarrow \infty$, regardless of the distribution of the underlying population

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$



# Likelihood function

- A random variable $\underline{x}$ has **parameters** $\theta$ and probability $P(\underline{x};\theta)$

  e.g., Bernoulli: $\theta = p$ , $P(x;\theta) = p^{x}(1-p)^{1-x}$

  multinomial: $\theta = (p_1, \ldots, p_k)$ , $P(\underline{x};\theta) = \binom{n}{x_1, \ldots, x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$

- Assume we have *n* **independent** samples $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$

- Define the dataset $D = \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$

- The likelihood function represents the probability of the dataset *D* as a function of the model parameters $\theta$

$$L(D;\theta) = P(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n;\theta) \underset{\text{by independence}}{=} \prod_{i=1}^{n} P(\underline{x}_i;\theta)$$

# Likelihood function

- The likelihood function represents the probability of the dataset *D* as a function of the model parameters $\theta$

$$L(D;\theta) = P(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n; \theta) = \prod_{i=1}^{n} P(\underline{x}_i; \theta)$$

- Gives **relative probability of data given a parameter**
- We can compare two values $\theta$ and $\theta'$ by comparing their likelihoods
- We say that $\theta$ is better for explaining the dataset *D* than $\theta'$ if

$$L(D;\theta) > L(D;\theta')$$

# Maximum likelihood estimation (MLE)

- Most widely used method of parameter estimation
- **Intuition:** a $\theta$ with higher likelihood explains better the data
- "Learn" the best parameters $\theta$ that maximizes likelihood:

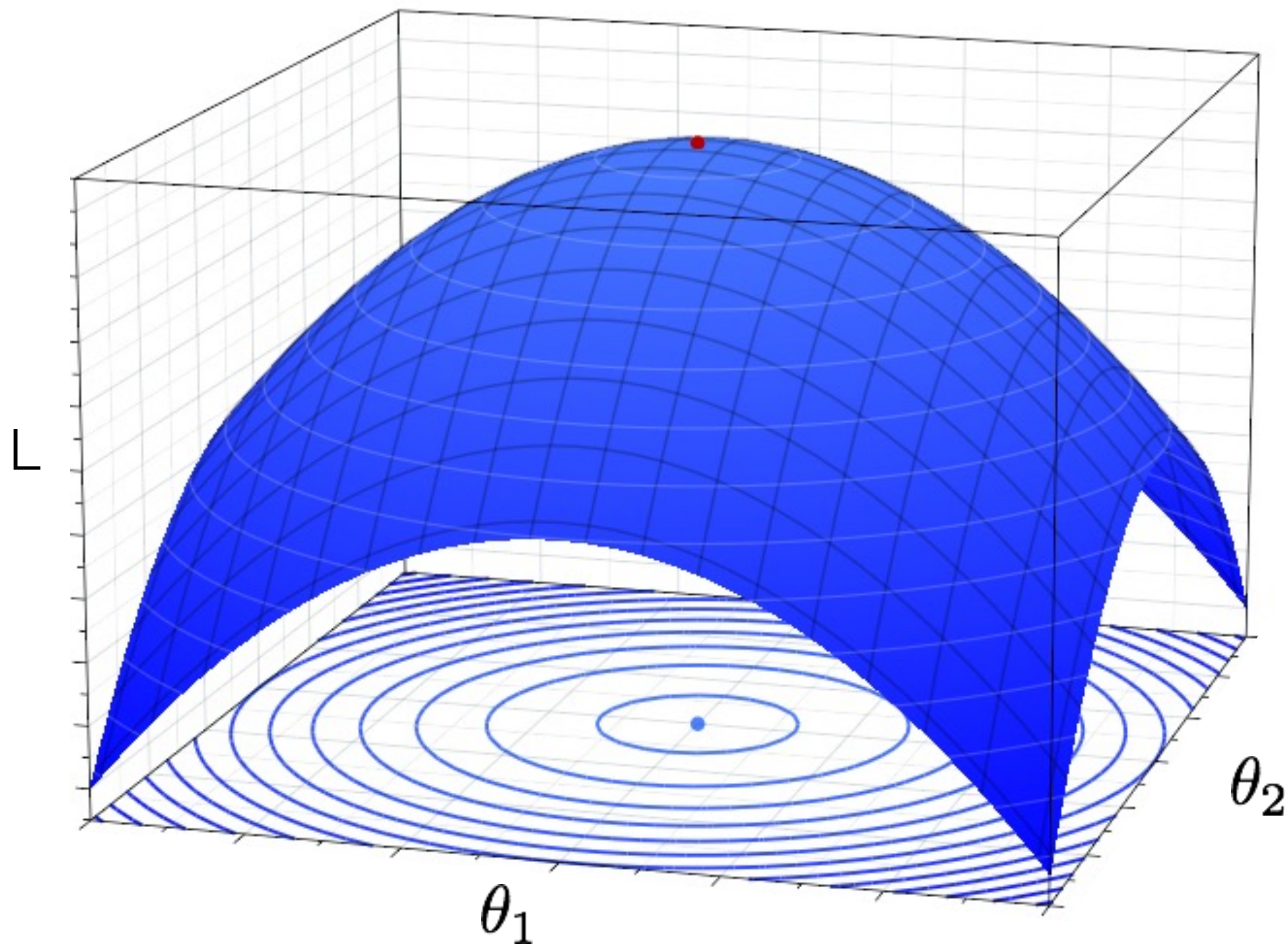
$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \; L(D;\theta)$$

- Often easier to work with log-likelihood:

$$l(D;\theta) = \log L(D;\theta) = \log \prod_{i=1}^{n} P(\underline{x}_i;\theta) = \sum_{i=1}^{n} \log P(\underline{x}_i;\theta)$$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \; l(D;\theta)$$

# Likelihood surface




**If the log-likelihood surface is concave, we can often determine the parameters that maximize the function analytically**

# Maximum Likelihood Estimation (MLE) for Bernoulli

- For a Bernoulli r.v. $x_i \in \{0,1\}$ , $\theta = p$ , $P(x_i;\theta) = p^{x_i}(1-p)^{1-x_i}$
- Clearly: $\log P(x_i;\theta) = x_i \log p + (1-x_i)\log(1-p)$
- The **log-likelihood function** is:

$$
\begin{aligned}
l(D;\theta) &= \sum_{i=1}^{n} \log P(\underline{x}_i;\theta) \\
&= \sum_{i=1}^{n} \left( x_i \log p + (1-x_i)\log(1-p) \right) \\
&= \left( \sum_{i=1}^{n} x_i \right) \log p + \left( n - \sum_{i=1}^{n} x_i \right) \log(1-p)
\end{aligned}
$$

- Recall that the **MLE** is:

$$
\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \; l(D;\theta)
$$

# Maximum Likelihood Estimation (MLE) for Bernoulli

- For a Bernoulli r.v. $x_i \in \{0,1\}$ , $\theta = p$ , $P(x_i;\theta) = p^{x_i}(1-p)^{1-x_i}$
- The **log-likelihood function** is:

$$l(D;\theta) = \left(\sum_{i=1}^{n} x_i\right)\log p + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-p)$$

- Recall that the **MLE** is:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ l(D;\theta)$$

- We can maximize $l(D;\theta)$ by taking derivative equal to zero:

$$\frac{\partial l(D;\theta)}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0 \qquad \text{then} \qquad \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

- The MLE $\hat{\theta} = \hat{p}$ is the **proportion of ones in the dataset**. This is intuitive since the parameter $\theta = p = \mathrm{E}[X]$ is the **expected proportion of ones**.

# Maximum Likelihood Estimation (MLE) for Bernoulli

```
import numpy as np
def example_bernoulli(n):
  z = np.random.randint(0,2,n)
  return 1.0/n * np.sum(z)

>>> example_bernoulli(10)
0.8
>>> example_bernoulli(100)
0.44
>>> example_bernoulli(10000)
0.5138
```

Returns n random integers >= 0 and < 2, each value with equal probability.
In this case (0 or 1) then p = 0.5 in the Bernoulli distribution

Computes average

From the terminal, use your Career account to start a session:

    ssh username@data.cs.purdue.edu

From the terminal:

    python

- Linear algebra review

# Vectors

- A vector is a matrix with several rows and one column

$$a = \begin{bmatrix} 5 \\ 7 \\ 1 \\ 4 \end{bmatrix} = (5,7,1,4)^T$$

- Notation: $a \in \mathbb{R}^m$

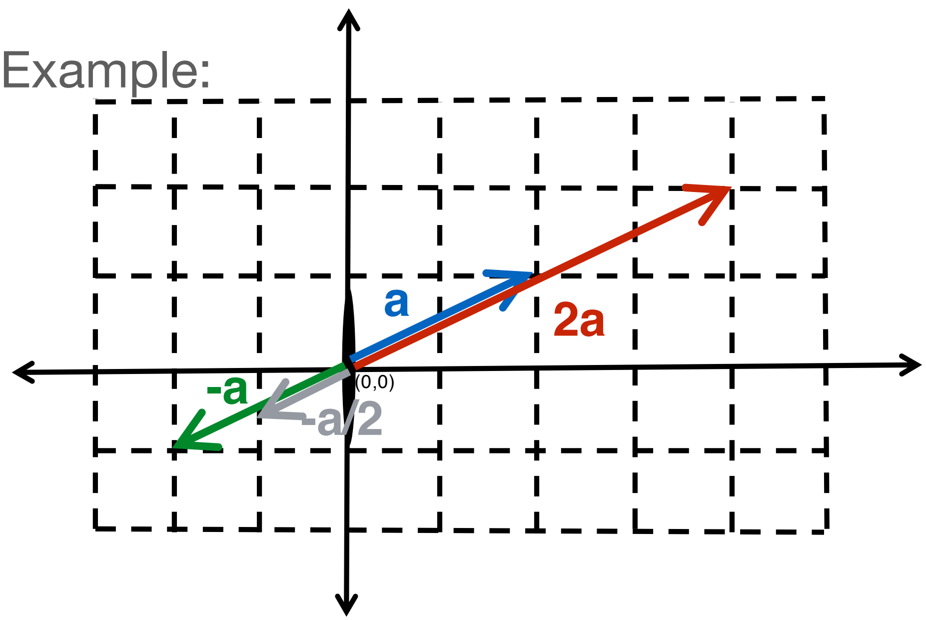
# Vector: multiplication by scalar

- A scalar c is a real value
- Multiply/divide all entries of vector a by the scalar c

$$(ca)_i = ca_i$$

$$(a/c)_i = a_i / c$$

- Example:




$$a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad 2\text{a} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$-a = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad -a/2 = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$$

# Vector: addition and subtraction

- a and b have the same number of rows

$$a = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

$$(a+b)_i = a_i + b_i$$

- Add corresponding entries in a and b

$$a + b = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}$$

$$(a-b)_i = a_i - b_i$$

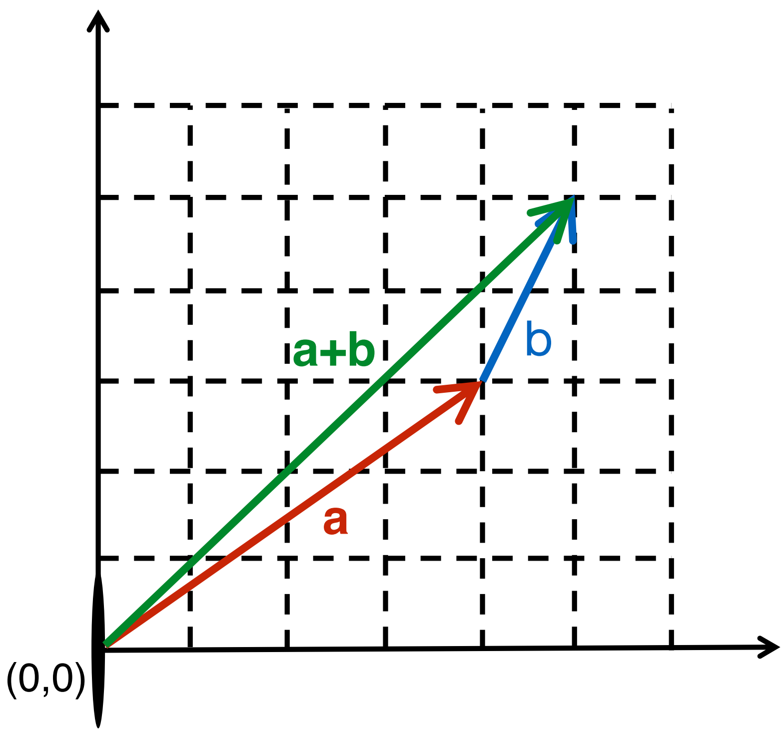
- Subtract corresponding entries in a and b

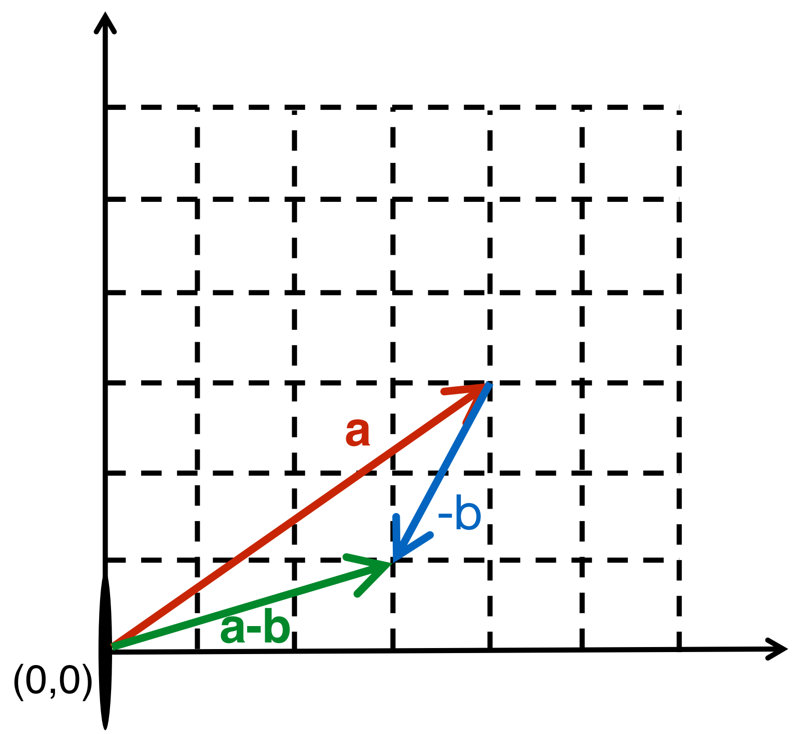
$$a - b = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

# Vector: addition and subtraction

- Geometrically…

$$a = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad a + b = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad a - b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



# Vector: inner product

- Using matrix multiplication notation:

$$a \bullet b = a^T b = \sum_{k=1}^{m} a_k b_k$$

$$a \in \mathbb{R}^m \quad b \in \mathbb{R}^m$$

- a and b have the same number of rows: ,

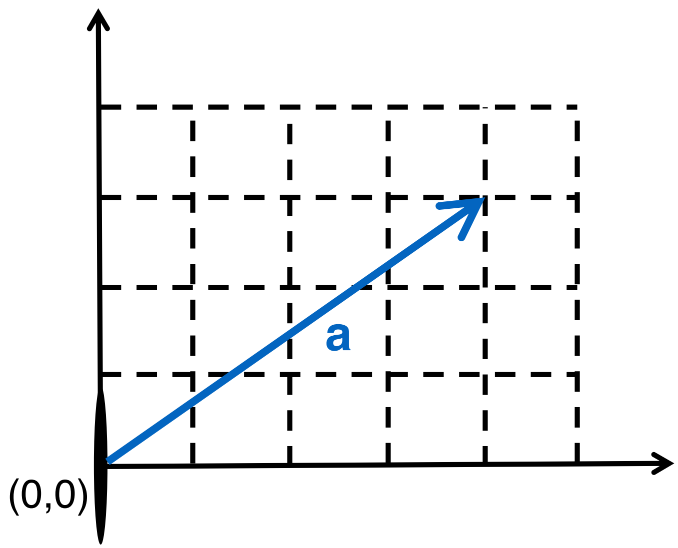
$$a = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix}$$

$$a \bullet b = 3 \times 1 + 2 \times (-7) + 4 \times 3 = 1$$

# Vector: Euclidean norm

- The norm of $a \in \mathbb{R}^m$ is $\|a\| = \sqrt{a \bullet a} = \sqrt{a_1^2 + a_2^2 + \ldots + a_m^2}$

- Example




$$a = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \|a\| = \sqrt{3^2 + 4^2} = 5$$

- Distance between two vectors a and b is $\|a - b\|$

- If $\|a\| = 1$ then $a$ is called **unitary**

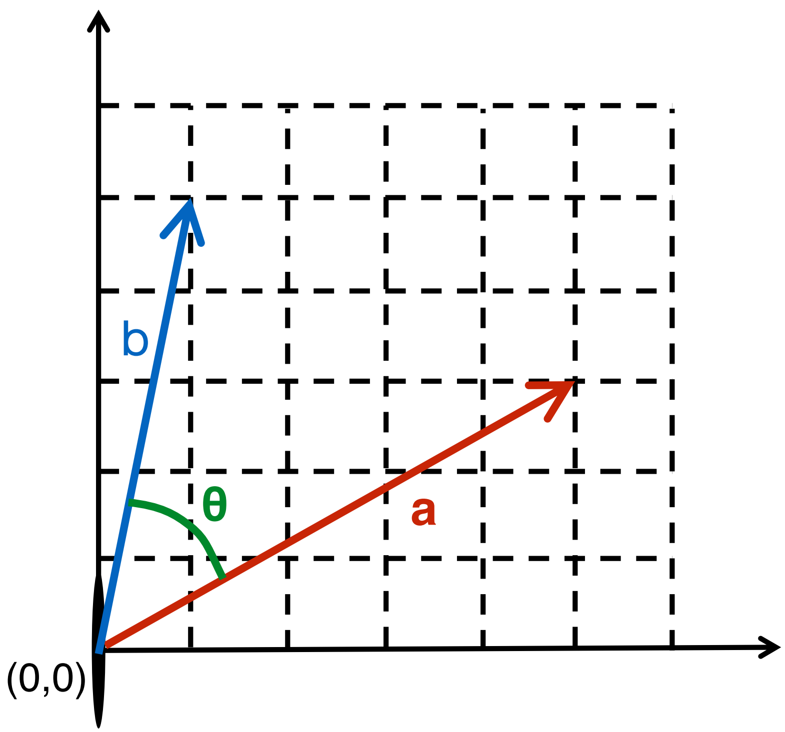
$$a / \|a\|$$

$$a = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

# Vector: inner product

- The cosine of the angle between two vectors can be found by using norms and the inner product

$$\cos\theta = \frac{a \bullet b}{\|a\| \times \|b\|} = \left(\frac{a}{\|a\|}\right) \bullet \left(\frac{b}{\|b\|}\right)$$




$$a = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

(Goodfellow, 2016)

# Matrices

- A matrix is a 2-D array of numbers:

Row

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

Column

- Example notation for type and shape:

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}$$

# Matrix: addition and subtraction

- A and B have the same number of rows and columns

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 & 0 \\ 5 & 7 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$

- Add corresponding entries in A and B

$$A+B = \begin{bmatrix} 7 & 4 & 1 \\ 6 & 9 & 2 \\ -5 & 7 & 6 \end{bmatrix}$$

2+5

5+1

$$(A-B)_{i,j} = A_{i,j} - B_{i,j}$$

- Subtract corresponding entries in A and B

$$A-B = \begin{bmatrix} -3 & 2 & 1 \\ -4 & -5 & -2 \\ 5 & 1 & 4 \end{bmatrix}$$

2-5

5-1

# Matrix: multiplication

- Number of columns of A = number of rows of B

$$(AB)_{i,j} = \sum_k A_{i,k} B_{k,j}$$

- Example:

$$A = \begin{bmatrix} 3 & 1 & -2 & 4 \\ -2 & 4 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & -3 \\ 2 & 3 & 2 \\ -1 & 2 & -4 \end{bmatrix}$$

$$3 \times 2 + 1 \times 5 - 2 \times 3 + 4 \times 2 = 13$$

$$AB = \begin{bmatrix} 5 & 13 & -20 \\ 14 & 22 & -10 \end{bmatrix}$$

# Matrix: multiplication by scalar

- A scalar c is a real value
- Multiply/divide all entries of matrix A by the scalar c

$$(cA)_{i,j} = cA_{i,j}$$

$$(A/c)_{i,j} = A_{i,j}/c$$

- Example:

$$A = \begin{bmatrix} 4 & 5 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}, \quad 3A = \begin{bmatrix} 12 & 15 \\ 0 & -6 \\ 9 & 18 \end{bmatrix}, \quad A/2 = \begin{bmatrix} 2 & 2.5 \\ 0 & -1 \\ 1.5 & 3 \end{bmatrix}$$

# Matrix: transpose

- Rows become columns, columns become rows

$$(A^T)_{i,j} = A_{j,i}$$

- Example:

$$A = \begin{bmatrix} 3 & 1 & -2 & 4 \\ -2 & 4 & 2 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & -2 \\ 1 & 4 \\ -2 & 2 \\ 4 & 0 \end{bmatrix}$$

- Multiplication property: $(AB)^T = B^T A^T$

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix}$$

- If $A = A^T$ then A is called **symmetric**

# Identity matrix and Inverse

- **Identity** matrix has 1s in the diagonals and 0s everywhere else

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- For any vector x, we have

$$Ix = x$$

$$A^{-1}A = I$$

- Matrix **inverse**:

- A matrix cannot be inverted if:
  - More rows than columns

# Identity matrix and Inverse

- Example

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ -2 & 1 & 7 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -27 & 19 & 5 \\ 16 & -11 & -3 \\ -10 & 7 & 2 \end{bmatrix}$$

- Several languages provide functions/methods for computing the inverse (*We will not go into these details.*)

# Functions and gradients

- We can define a function $f(x)$ of a vector $x \in \mathbb{R}^m$

- The **gradient** has the derivatives with respect to each entry:

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_m \end{bmatrix} \in \mathbb{R}^m$$

- Example:

$$f(x) = 5e^{x_2} + x_3 e^{x_1}, \qquad \nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \partial f / \partial x_3 \end{bmatrix} = \begin{bmatrix} x_3 e^{x_1} \\ 5e^{x_2} \\ e^{x_1} \end{bmatrix}$$