

QUANTITATIVE REGULARITY FOR THE NAVIER-STOKES EQUATIONS VIA SPATIAL CONCENTRATION

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Abstract This paper is concerned with quantitative estimates for the Navier-Stokes equations.

First we investigate the relation of quantitative bounds to the behaviour of critical norms near a potential singularity with Type I bound $\|u\|_{L_t^\infty L_x^{3,\infty}} \leq M$. Namely, we show that if T^* is a first blow-up time and $(0, T^*)$ is a singular point then

$$\|u(\cdot, t)\|_{L^3(B_0(R))} \geq C(M) \log\left(\frac{1}{T^* - t}\right), \quad R = O((T^* - t)^{\frac{1}{2}-}).$$

We demonstrate that this potential blow-up rate is optimal for a certain class of potential non-zero backward discretely self-similar solutions.

Second, we quantify the result of Seregin (2012), which says that if u is a smooth finite-energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times (0, 1)$ with

$$\sup_n \|u(\cdot, t_{(n)})\|_{L^3(\mathbb{R}^3)} < \infty \text{ and } t_{(n)} \uparrow 1,$$

then u does not blow-up at $t = 1$.

To prove our results we develop a new strategy for proving quantitative bounds for the Navier-Stokes equations. This hinges on local-in-space smoothing results (near the initial time) established by Jia and Šverák (2014), together with quantitative arguments using Carleman inequalities given by Tao (2019).

Moreover, the technology developed here enables us in particular to give a quantitative bound for the number of singular points in a Type I blow-up scenario.

Keywords Navier-Stokes equations, quantitative estimates, critical norms, Type I blow-up, concentration, Carleman inequalities.

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1. INTRODUCTION

In this paper, we consider the three-dimensional incompressible Navier-Stokes equations

$$(1) \quad \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad u(\cdot, 0) = u_0(x) \quad \text{in } \mathbb{R}^3 \times (0, T),$$

where $T \in (0, \infty]$. It is well known that this system of equations is invariant with respect to the following rescaling

$$(2) \quad (u_\lambda(x, t), p_\lambda(x, t), u_{0\lambda}(x)) := (\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t), \lambda u_0(\lambda x)), \quad \lambda > 0.$$

The question as to whether or not finite-energy solutions¹, with divergence-free Schwartz class initial data, remain smooth for all times is a Millennium Prize problem [14]. The first

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¹Throughout this paper, we say u is a finite-energy solution to the Navier-Stokes equations on $(0, T)$ if $u \in C_w([0, T]; L_\sigma^2(\mathbb{R}^3)) \cap L^2(0, T; \dot{H}^1(\mathbb{R}^3))$ and $\|u(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' \leq \|u(\cdot, 0)\|_{L^2(\mathbb{R}^3)}^2$.

necessary conditions for such a solution to lose smoothness or to ‘blow-up’ at time $T^* > 0$ ² were given in the seminal paper of Leray [26]. In particular, in [26] it is shown that if T^* is a first blow-up time of u then we necessarily have

$$(3) \quad \|u(\cdot, t)\|_{L^p(\mathbb{R}^3)} \geq \frac{C(p)}{(T^* - t)^{\frac{1}{2}(1-\frac{3}{p})}}, \quad \text{for } p \in (3, \infty].$$

The $L^3(\mathbb{R}^3)$ norm is scale-invariant or ‘critical’³ with respect to the Navier-Stokes rescaling. Its role in the regularity theory of the Navier-Stokes equations is much more subtle than that of the subcritical $L^p(\mathbb{R}^3)$ norms with $3 < p \leq \infty$. In particular, it is demonstrated by a elementary scaling argument in [4]⁴ that there *cannot exist* a universal function $f : (0, \infty) \rightarrow (0, \infty)$ such that the following analogue of (3) holds true:

$$(4) \quad \lim_{s \rightarrow 0^+} f(s) = \infty,$$

and if u is a finite-energy solution to the Navier-Stokes equations (with Schwartz class initial data) that first blows-up at $T^* > 0$ then u necessarily satisfies

$$(5) \quad \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} \geq f(T^* - t)$$

for all $t \in [0, T^*)$.

In the celebrated paper [13] of Escauriaza, Seregin and Šverák, it was shown that if a finite-energy solution u first blows-u at $T^* > 0$ then necessarily

$$(6) \quad \limsup_{t \uparrow T^*} \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} = \infty.$$

The proof in [13] is by contradiction. A rescaling procedure or ‘zoom-in’ is performed⁵ using (2) and a compactness argument is applied. This gives a non-zero limit solution to the Navier-Stokes equations that vanishes at the final moment in time. The contradiction is achieved by showing that the limit function must be zero by applying a Liouville type theorem based on backward uniqueness for parabolic operators satisfying certain differential inequalities. By now there are many generalizations of (6) to cases of other critical norms. See, for example, [12], [16], [34] and [47].

Let us mention the arguments in [13] and the aforementioned works are by contradiction and hence are *qualitative*. It is worth noting that the result in [13], together with a proof by contradiction based on the ‘persistence of singularities’ lemma in [35] (specifically Lemma 2.2 in [35]), gives the following. Namely, that there exists an $F : (0, \infty) \rightarrow (0, \infty)$ such that if u is a finite-energy solution to the Navier-Stokes equations then

$$(7) \quad \|u\|_{L^\infty(0,1;L^3(\mathbb{R}^3))} < \infty \Rightarrow \|u\|_{L^\infty(\mathbb{R}^3 \times (\frac{1}{2}, 1))} \leq F(\|u\|_{L^\infty(0,1;L^3(\mathbb{R}^3))}).$$

Such an argument is obtained by a compactness method and gives no explicit⁶ information about F . In a remarkable recent development [45], Tao used a new approach to provide the

²We say that a solution u to the Navier-Stokes equations first blows-up at $T^* > 0$ if $u \in L_{loc}^\infty(0, T^*, L^\infty(\mathbb{R}^3))$ but $u \notin L_{loc}^\infty(0, T^*]; L^\infty(\mathbb{R}^3))$

³We say $(X, \|\cdot\|_X) \subset \mathcal{S}'(\mathbb{R}^3)$ is critical if $u_0 \in X \Rightarrow u_{0\lambda}(x) := \lambda u(\lambda x) \in X$ with X norm equal to that of u_0 .

⁴The argument in [4] is in turn taken from the talk given by G. Seregin. ‘A certain necessary condition of possible blow up for the Navier-Stokes equations’. APDE seminar, University of Sussex, 03 March 2014.

⁵It is worth to note that [13] appears to be the first instance where arguments involving ‘zooming in’ and passage to a limit have been applied to the Navier-Stokes equations.

⁶Throughout this paper we will sometimes use the terminology ‘*effective*’ bounds to describe an explicit quantitative bound. An abstract quantitative bound will sometimes be referred to as ‘*non-effective*’.

first explicit *quantitative* estimates for solutions of the Navier-Stokes equations belonging to the critical space $L^\infty(0, T; L^3(\mathbb{R}^3))$. As a consequence of these quantitative estimates, Tao showed in [45] that if a finite-energy solution u first blows-up at $T^* > 0$ then for some absolute constant $c > 0$

$$(8) \quad \limsup_{t \uparrow T^*} \frac{\|u(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{\left(\log \log \log \frac{1}{T^*-t}\right)^c} = \infty.$$

Since there cannot exist f such that (4)-(5) holds true, at first sight (8) may seem somewhat surprising, though it is not conflicting with such a fact. Notice that

$$\frac{\|u(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{\left(\log \log \log \frac{1}{T^*-t}\right)^c}$$

is not invariant with respect to the Navier-Stokes scaling (2) but is *slightly supercritical*⁷ due to the presence of the logarithmic denominator. Let us also mention that prior to Tao's paper [45], in the presence of axial symmetry, a different slightly supercritical regularity criteria was obtained in [33].

The contribution of our present paper is to develop a new strategy for proving quantitative estimates (see Propositions 2 and 3) for the Navier-Stokes equations, which then enables us to build upon Tao's work [45] to quantify critical norms. Our first Theorem involves applying the backward propagation of concentration stated in Proposition 2 below to give a new necessary condition for solutions to the Navier-Stokes equations to possess a Type I blow-up. Note that if u is a finite-energy solution that first blows-up at $T^* > 0$ we say that T^* is a Type I blow-up if

$$(9) \quad \|u\|_{L^\infty(0, T^*; L^{3,\infty}(\mathbb{R}^3))} \leq M.$$

In the case of a Type I blow-up at T^* the nonlinearity in (2) is heuristically balanced with the diffusion. Despite this, it remains a long standing open problem whether or not Type I blow-ups can be ruled out when M is large. Let us now state our first theorem.

Theorem 1 (rate of blow-up, Type I). *There exists a universal constant $M_0 \in [1, \infty)$ such that for all $M \geq M_0$ and $\delta \in (0, 1)$ the following holds true.*

Assume that u is a mild solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [0, T^)$ with $u \in L_{loc}^\infty([0, T^*); L^\infty(\mathbb{R}^3))$.⁸ Assume that*

- (1) $\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (0, T^*))} \leq M$
- (2) u has a singular point at $(x, t) = (0, T^*)$. In particular $u \notin L_{x,t}^\infty(Q_{(0,T^*)}(r))$ for all sufficiently small $r > 0$.

⁷We say a quantity $F(u, p)$ is supercritical if, for the rescaling (2), we have $F(u_\lambda, p_\lambda) = \lambda^{-\beta} F(u_\lambda, p_\lambda)$ for some $\beta > 0$.

⁸Under these assumptions, u is smooth on the epoch $(0, T)$ for any $T < T^*$ and belongs to $L^\infty((0, T); L^4(\mathbb{R}^3)) \cap L^\infty((0, T); L^5(\mathbb{R}^3))$ by interpolation, which enables us to satisfy the hypothesis needed in Sections 6-7. Furthermore using Lemma 2.4 in [20], it gives that u coincides with all local energy solutions (we refer to footnote 41 for a definition), with initial data $u(\cdot, s)$, on $\mathbb{R}^3 \times (s, T^*)$ for any $0 < s < T^*$. We call such a solution a '**smooth solution with sufficient decay**' on the interval $[0, T]$, for $T < T^*$. A mild solution with Schwartz class initial data and maximal time of existence T^* will be such a solution, and so Theorem 1 applies to that setting. Notice that smoothness is needed here in order to get estimate (10) for all t in the ad hoc interval. The framework of 'smooth solutions with enough decay' is needed to apply Theorem 1 to the setting of Corollary 1, where the solution is not of finite energy.

Then the above assumptions imply that there exists $c(\delta, M, T^*) \in (0, \infty)$, which we will specify in the proof, such that for any $t \in (\max(\frac{T^*}{2}, T^* - c(\delta, M, T^*)), T^*)$ we have

$$(10) \quad \int_{B_0((T^*)^{\frac{1}{2}}(T^*-t)^{\frac{1-\delta}{2}})} |u(x, t)|^3 dx \geq \frac{\log\left(\frac{1}{(T^*-t)^{\frac{\delta}{2}}}\right)}{\exp(\exp(M^{1025}))}.$$

This theorem is proved in Subsection 2.2 below. Notice that in Theorem 1, not only is the rate new but also the fact that the L^3 norm blows up on a ball of radius $O((T^* - t)^{\frac{1}{2}-})$ around *any* Type I singularity. Previously in [27] (specifically Theorem 1.3 in [27]), it was shown that if a solution blows up (without Type I bound) then the L^3 norm blows up on certain *non-explicit* concentrating sets.

The Navier-Stokes scaling symmetry (2) plays a role in considering blow-up ansatzes having certain symmetry properties. In [26], Leray suggested the blow-up ansatz of *backward self-similar solutions*⁹, which are invariant with respect to the Navier-Stokes rescaling. Although the existence of non-zero backward self-similar solutions to the Navier-Stokes equations has been ruled out under general circumstances in [31] and [46], the existence of non-zero backward discretely self-similar solutions remains open. Here we say that u is a backward discretely self-similar solution (λ -DSS) if there exists $\lambda \in (1, \infty)$ such that $u(x, t) = \lambda u(\lambda x, \lambda^2 t)$ for all $(x, t) \in \mathbb{R}^3 \times (-\infty, 0)$. As a corollary to Theorem 1, we show that if there exists a non-zero λ -DSS (having certain decay properties which we will specify), then the localized blow-up rate (10) in Theorem 1 is optimal.

Corollary 1. Suppose $u : \mathbb{R}^3 \times (-\infty, 0) \rightarrow 0$ is a non-zero λ -DSS to the Navier-Stokes equations such that

$$(11) \quad u \in C^\infty(\mathbb{R}^3 \times (-\infty, 0)) \cap C((-\infty, 0); L^p(\mathbb{R}^3)),$$

for some $p \in [3, \infty)$. There exists $M > 1$ such that for every $\delta \in (0, 1)$ there is a $C(\delta, M) \in (0, \infty)$ with the holding true. Namely, for all $t \in [\max(-\frac{1}{2}, -C(\delta, M)), 0)$ we have

$$(12) \quad \frac{\log\left(\frac{1}{(-t)^{\frac{\delta}{2}}}\right)}{\exp(\exp(M^{1025}))} \leq \int_{B_0(1)} |u(x, t)|^3 \leq M^3 \log\left(\frac{2}{\sqrt{-t}}\right).$$

This corollary is proved in Subsection 2.2 below.

In [13], it is shown that if u is a finite-energy solution to the Navier-Stokes equations in $C^\infty(\mathbb{R}^3 \times (0, 1))$, with Schwartz initial data, then

$$(13) \quad \|u\|_{L^\infty((0,1);L^3(\mathbb{R}^3))} < \infty$$

implies that u does not blow-up at time 1 (namely $u \in L_{t,loc}^\infty((0, 1]; L^\infty(\mathbb{R}^3))$). In [36], Seregin refined the assumption (13) to

$$(14) \quad \sup_n \|u(\cdot, t_{(n)})\|_{L^3(\mathbb{R}^3)} < \infty \text{ with } t_{(n)} \uparrow 1.$$

Seregin's result implies that if u is a finite-energy solution that first loses smoothness at $T^* > 0$ then

$$(15) \quad \lim_{t \uparrow T^*} \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} = \infty.$$

⁹We say $u : \mathbb{R}^3 \times (-\infty, 0) \rightarrow \mathbb{R}^3$ is a backward self similar solution if $u(x, t) = \frac{1}{\sqrt{-t}} u\left(\frac{x}{\sqrt{-t}}, 1\right)$ for all $(x, t) \in \mathbb{R}^3 \times (-\infty, 0)$.

This result has been further refined to other wider critical spaces and to domains other than \mathbb{R}^3 . See, for example, [1], [2]-[3], [7] and [28]. All these arguments are qualitative and achieved by contradiction and compactness arguments. It is interesting to note that in contrast to (7) it is not known¹⁰, even abstractly, if there exists a $G : (0, \infty) \rightarrow (0, \infty)$ such that if u is a finite-energy solution of the Navier-Stokes equations belonging to $C^\infty(\mathbb{R}^3 \times (0, 1])$ then

$$(16) \quad \sup_n \|u(\cdot, t_{(n)})\|_{L^3(\mathbb{R}^3)} < \infty \text{ with } t_{(n)} \uparrow 1 \Rightarrow \|u(\cdot, 1)\|_{L^\infty(\mathbb{R}^3)} \leq G\left(\sup_n \|u(\cdot, t_{(n)})\|_{L^3(\mathbb{R}^3)}\right).$$

In our second main theorem, we apply Proposition 3 to fully quantify Seregin's result in [36], which generalizes Theorem 1.2 in [45]. Now let us state our second theorem.

Theorem 2 (main quantitative estimate, time slices; quantification of Seregin's result). *There exists a universal constant $M_1 \in [1, \infty)$. Let $M \in [M_1, \infty)$. We define M^\flat by¹¹*

$$(17) \quad M^\flat := \exp\left(\frac{L_* M^5}{2}\right),$$

for an appropriate constant $L_* \in (0, \infty)$. Let (u, p) be a finite-energy $C^\infty(\mathbb{R}^3 \times (-1, 0))$ solution to the Navier-Stokes equations (1) on $\mathbb{R}^3 \times [-1, 0]$.¹² Assume that there exists $t_{(k)} \in [-1, 0)$ such that

$$(18) \quad t_{(k)} \uparrow 0 \text{ with } \sup_k \|u(\cdot, t_{(k)})\|_{L^3(\mathbb{R}^3)} \leq M.$$

Select any “well-separated” subsequence (still denoted $t_{(k)}$) such that

$$(19) \quad \sup_k \frac{-t_{(k+1)}}{-t_{(k)}} < \exp(-2(M^\flat)^{1223}).$$

Then for

$$(20) \quad j := \lceil \exp(\exp((M^\flat)^{1224})) \rceil + 1,$$

we have the bound

$$(21) \quad \|u\|_{L^\infty\left(\mathbb{R}^3 \times \left(\frac{t_{(j+1)}}{4}, 0\right)\right)} \leq \frac{C_1 M^{-23}}{(-t_{(j+1)})^{\frac{1}{2}}},$$

for a universal constant $C_1 \in (0, \infty)$.

This theorem is proved in Subsection 2.2 below.

¹⁰Assume for contradiction that (16) does not hold. Then we have a sequence of solutions $(u^k)_{k \in \mathbb{N}}$ such that $\|u^{(k)}(\cdot, 1)\|_{L^\infty(\mathbb{R}^3)} \uparrow \infty$ and, for each k , a sequence of associated time slices $t_{(n)}^k \uparrow 1$ such that $\sup_{n,k} \|u^{(k)}(\cdot, t_{(n)}^k)\|_{L^3(\mathbb{R}^3)} = M < \infty$. The main block for the contradiction argument to go through is that the sequence $(t_{(n)}^k)_{n \in \mathbb{N}}$ may be different for distinct indices k .

¹¹In particular, M^\flat is chosen such that the following is true. If (u, p) is a suitable finite-energy solution (defined in Section 1.4 ‘Notations’) to Navier-Stokes on $\mathbb{R}^3 \times [0, T)$ with L^3 initial data $\|u_0\|_{L^3(\mathbb{R}^3)} \leq M$, then $w := u - e^{t\Delta} u_0$ satisfies $\|w(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' \leq (M^\flat)^4 t^{\frac{1}{2}}$ for $t \in (0, T)$ and M larger than a universal constant. See Lemma 6.

¹²Notice that smoothness is needed here to have the energy inequality starting from every time t_k , and not for almost every $t' \in (-1, 0)$, see (43) as would be the case if (u, p) was just a suitable finite-energy solution.

Further applications. Section 4 contains three further applications of the technology developed in the present paper: (i) Proposition 7, a regularity criteria based on an effective¹³ relative smallness condition in the Type I setting, (ii) Corollary 9, an effective bound for the number of singular points in a Type I blow-up scenario, (iii) Proposition 10, a regularity criteria based on an effective relative smallness condition on the L^3 norm at initial and final time. Non-effective quantitative bounds of the above results were previously obtained by compactness methods: for (ii) see [12, Theorem 2], for (iii) see [2, Theorem 4.1 (i)].

1.1. Comparison to previous literature and novelty of our results. Theorems 1 and 2 in this paper follow from new quantitative estimates for the Navier-Stokes equations (Propositions 2 and 3), which build upon recent breakthrough work by Tao in [45]. In particular, Tao shows that for *classical*¹⁴ solutions to the Navier-Stokes equations

(22)

$$\|u\|_{L_t^\infty L_x^3(\mathbb{R}^3 \times (0,1))} \leq A \Rightarrow \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq \exp(\exp(\exp(A^{O(1)})))t^{-\frac{1}{2}} \text{ for } 0 < t \leq 1.$$

Before describing our contribution, we first find it instructive to outline Tao's approach in [45].

Fundamental to Tao's approach for showing (22) is the following fact¹⁵ (see Section 6 in [45]). There exists a universal constant ε_0 such that if u is a classical solution to the Navier-Stokes equations with

$$(23) \quad \|u\|_{L_t^\infty L_x^3(\mathbb{R}^3 \times (0,1))} \leq A$$

$$(24) \quad \text{and } N^{-1}\|P_N u\|_{L_{x,t}^\infty(\mathbb{R}^3 \times (\frac{1}{2}, 1))} < \varepsilon_0 \text{ for all } N \geq N_*,$$

then $\|u\|_{L_{x,t}^\infty(\mathbb{R}^3 \times (\frac{7}{8}, 1))}$ can be estimated explicitly in terms of A and N_* . Related observations were made previously in [11], but in a slightly different context and without the bounds explicitly stated.

In this perspective, Tao's aim is the following:

Tao's goal: Under the scale-invariant assumption (23), if (24) fails for $\varepsilon_0 = A^{-O(1)}$ and $N = N_0$, what is an upper bound for N_0 ?

In [45] (Theorem 5.1 in [45]), it is shown that $N_0 \lesssim \exp \exp \exp(A^{O(1)})$, which implies (22) by means of the quantitative regularity mechanism (24) with $N_* = 2N_0$. We emphasize that since the regularity mechanism (24) is *global*: all quantitative estimates obtained in this way are in terms of globally defined quantities.

The strategy in [45] for showing Tao's goal with $N_0 \lesssim \exp(\exp(\exp(A^{O(1)})))$ can be summarized in four steps. We refer the reader to the Introduction in [45] for more details.

1) Frequency bubbles of concentration (Proposition 3.2 in [45]).

Suppose $\|u\|_{L_t^\infty L_x^3(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq A$ is such that

$$(25) \quad N_0^{-1}|P_{N_0} u(x_0, t_0)| > A^{-O(1)}.$$

Then for all $n \in \mathbb{N}$ there exists $N_n > 0$, $(x_n, t_n) \in \mathbb{R}^3 \times (t_0 - T, t_{n-1})$ such that

$$(26) \quad N_n^{-1}|P_{N_n} u(x_n, t_n)| > A^{-O(1)}$$

¹³See footnote 6 for the definition of ‘effective’ bounds.

¹⁴In [45], these are solutions that are smooth in $\mathbb{R}^3 \times (0, 1)$ and such that all derivatives of u and p lie in $L_t^\infty L_x^2(\mathbb{R}^2 \times (0, 1))$.

¹⁵Let $\varphi \in C_0^\infty(B_0(1))$ with $\varphi \equiv 1$ on $B_0(\frac{1}{2})$. The Littlewood-Paley projection P_N is defined for any $N > 0$ by $\widehat{P_N f}(\xi) := \left(\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right)\widehat{f}(\xi)$.

with

$$(27) \quad x_n = x_0 + O((t_0 - t_n)^{\frac{1}{2}}), N_n \sim |t_0 - t_n|^{-\frac{1}{2}}.$$

2) Localized lower bounds on vorticity (p.37 in [45]). For certain scales $S > 0$ and an ‘epoch of regularity’ $I_S \subset [t_0 - S, t_0 - A^{-\alpha}S]$, where the solution enjoys ‘good’ quantitative estimates on $\mathbb{R}^3 \times I_S$ (in terms of A and S), Tao shows the following: the previous step and $\|u\|_{L_t^\infty L_x^3(\mathbb{R}^3 \times [t_0 - T, t_0])} \leq A$ imply

$$(28) \quad \int_{B_{x_0}(A^\beta S^{\frac{1}{2}})} |\omega(x, t)|^2 dx \geq A^{-\gamma} S^{-\frac{1}{2}} \text{ for all } t \in I_S.$$

Here, α , β and γ are positive universal constants.

3) Lower bound on the L^3 norm at the final moment in time t_0 (p.37-40 in [45]). Using quantitative versions of the Carleman inequalities in [13] (Propositions 4.2-4.3 in [45]), Tao shows that the lower bounds in step 2 can be transferred to a lower bound on the L^3 norm of u at the final moment of time t_0 . The applicability of the Carleman inequalities to the vorticity equation requires the ‘epochs of regularity’ in the previous step and the existence of ‘good spatial annuli’ where the solution enjoys good quantitative estimates. Specifically, Tao shows that step 2 on I_S implies

$$(29) \quad \int_{R_S \leq |x-x_0| \leq R'_S} |u(x, t_0)|^3 dx \geq \exp(-\exp(A^{O(1)})).$$

4) Conclusion: summing scales to bound TN_0^2 . Letting S vary for certain permissible S , the annuli in (29) become disjoint. The sum of (29) over such disjoint permissible annuli is bounded above by $\|u(\cdot, t_0)\|_{L^3(\mathbb{R}^3)}$ and the lower bound due to the summing of scales is $\exp(-\exp(A^{O(1)})) \log(TN_0^2)$. This gives the desired bound on N_0 , namely

$$TN_0^2 \lesssim \exp(\exp(\exp(A^{O(1)}))).$$

Let us emphasize once more that the approach in [45] produces quantitative estimates involving *globally defined quantities*, since the quantitative regularity mechanism (24) is inherently global. We would also like to emphasize that the fact that $\|u\|_{L_t^\infty L_x^3} < A$ is crucial for showing steps 1-2 in the above strategy.

The goal of the present paper is to develop a new robust strategy for obtaining new quantitative estimates of the Navier-Stokes equations, which are then applied to obtain Theorems 1 and 2. The main novelty (which we explain in more detail below) is that our strategy allows us to obtain *local* quantitative estimates and even applies to certain situations where we are outside the regime of scale-invariant controls. For simplicity, we will outline the strategy for the case when $\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M$, before remarking on this strategy for cases without such a scale-invariant control (Theorem 2).

Fundamental to our strategy is the use of local-in-space smoothing near the initial time for the Navier-Stokes equations pioneered by Jia and Šverák in [21] (see also [6] for extensions to critical cases). In particular, the result of [21], together with rescaling arguments from [6], implies the following. If $u : \mathbb{R}^3 \times [t_0 - T, t_0] \rightarrow \mathbb{R}^3$ is a smooth solution with sufficient

decay¹⁶ of the Navier-Stokes equations and $\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M$, then

$$(30) \quad \int_{B_{x_0}(4\sqrt{S^\sharp(M)}^{-1}(t_0 - t_0^*)^{\frac{1}{2}})} |\omega(x, t_0^*)|^2 dx \leq \frac{M^2 \sqrt{S^\sharp(M)}}{(t_0 - t_0^*)^{\frac{1}{2}}}$$

for $t_0^* \in (t_0 - T, t_0)$ implies that

$$(31) \quad \|u\|_{L_{x,t}^\infty(B_{x_0}(\frac{1}{2}\sqrt{S^\sharp(M)}^{-1}(t_0 - t_0^*)^{\frac{1}{2}}) \times (\frac{3}{4}(t_0 - t_0^*) + t_0, t_0))}$$

can be estimated explicitly in terms of M and $t_0 - t_0^*$. Here, $S^\sharp(M) = O(1)M^{-100}$ is as in Theorem 3.

In this perspective, the aim of our strategy is the following

Our goal: If (30) fails for $t_0^* = t_0'$, what is a lower bound for $t_0 - t_0'$?

This is the main aim of Proposition 2. Taking s_0 such that $t_0 - t_0' \geq 2Ts_0$, we can then apply (30)-(31) with $t_0^* = t_0 - Ts_0$. One might think of the main goal of our strategy as a physical space analogy to Tao's goal with

$$N_0 \sim |t_0 - t_0'|^{-\frac{1}{2}}.$$

In contrast to (24), the regularity mechanism (30)-(31) produces quantitative estimates that are in terms of locally defined quantities, which is crucial for obtaining the *localized* results as in Theorem 1. Our strategy for obtaining a lower bound of $t_0 - t_0'$ (see Proposition 2) can be summarized in three steps; see also Figure 1.1.

1) Backward propagation of vorticity concentration (Lemma 4).

Let $\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M$. Suppose $t_0' \in (t_0 - T, t_0)$ is not too close to $t_0 - T$ and is such that

$$(32) \quad \int_{B_{x_0}(4\sqrt{S^\sharp(M)}^{-1}(t_0 - t_0')^{\frac{1}{2}})} |\omega(x, t_0')|^2 dx > \frac{M^2 \sqrt{S^\sharp(M)}}{(t_0 - t_0')^{\frac{1}{2}}}.$$

We show that for all $t_0'' \in (t_0 - T, t_0')$, such that $t_0 - t_0''$ is sufficiently large compared to $t_0 - t_0'$ (in other words t_0'' is well-separated from t_0'), we have

$$(33) \quad \int_{B_{x_0}(4\sqrt{S^\sharp(M)}^{-1}(t_0 - t_0'')^{\frac{1}{2}})} |\omega(x, t_0'')|^2 dx > \frac{M^2 \sqrt{S^\sharp(M)}}{(t_0 - t_0'')^{\frac{1}{2}}}.$$

We refer the reader to Lemma 4 for precise statements for the rescaled/translated situation $\mathbb{R}^3 \times (t_0 - T, t_0) = \mathbb{R}^3 \times (-1, 0)$.

2) Lower bound on localized L^3 norm at the final moment in time t_0 . Using the previous step, together with the same arguments as [45] involving quantitative Carleman inequalities, we show that for certain permissible annuli that

$$(34) \quad \int_{R \leq |x - x_0| \leq R'} |u(x, t_0)|^3 dx \geq \exp(-\exp(M^{O(1)})).$$

We wish to mention that the role of the Type I bound is to show the solution u obeys good quantitative estimates in certain space-time regions, which is needed to apply the Carleman inequalities to the vorticity equation.

¹⁶In this paper, 'smooth solution with sufficient decay' always denotes the notion described in footnote 8.

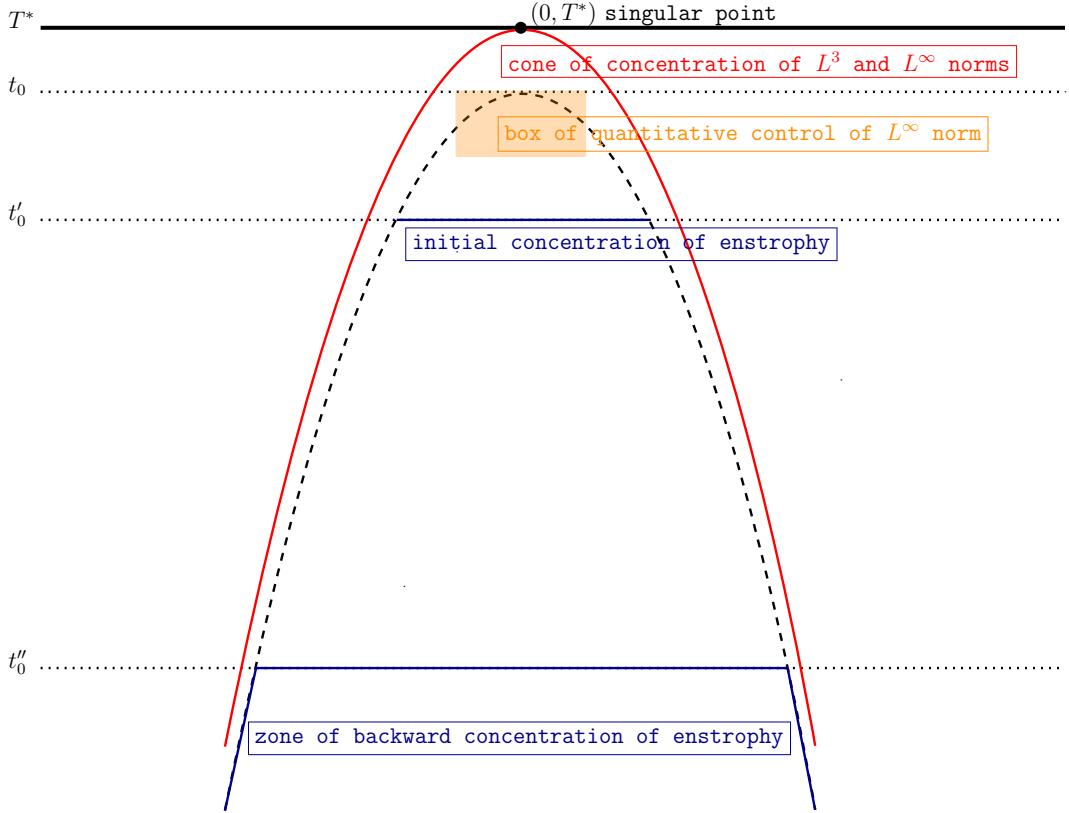


FIGURE 1. Zones of concentration and quantitative regularity

3) Conclusion: summing scales to bound $t_0 - t'_0$ from below. Summing (34) over all permissible disjoint annuli finally gives us the desired lower bound for $t_0 - t'_0$ in Proposition 2. We note that the localized L^3 norm of u at time t_0 plays a distinct role to that of Type I condition described in the previous step. Its sole purpose is to bound the number of permissible disjoint annuli that can be summed, which in turn gives the lower bound of $t_0 - t'_0$. Together with the assumed global Type I assumption, this is essentially why the lower bound in Theorem 1 on the localized L^3 norm near a Type I singularity is a single logarithm and holds at pointwise times.

Although the above relates to the case of Proposition 2 and Theorem 1 where

$$\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M,$$

we stress that the above strategy (with certain adjustments) is robust enough to apply to certain settings without a Type I control (Theorem 2).

Recall that Theorem 2 is concerned with quantitative estimates on $u : \mathbb{R}^3 \times (-1, 0) \rightarrow \mathbb{R}^3$, where we assume

$$(35) \quad t_{(k)} \uparrow 0 \text{ with } \sup_k \|u(\cdot, t_{(k)})\|_{L^3(\mathbb{R}^3)} \leq M.$$

First we remark that the local quantitative regularity statement (30)-(31) remains true (with t_0^* replaced by t_k) if u is a $C^\infty(\mathbb{R}^3 \times (-1, 0])$ finite-energy solution and the Type I condition is replaced by the weaker assumption that $\|u(\cdot, t_{(k)})\|_{L^3(\mathbb{R}^3)} \leq M$. Our goal then becomes the following

Our second goal: If (30) fails for $t_0^* = t_j$ (with $t_0 = 0$ and $T = 1$), what is an upper bound for j ?

In this setting ‘**1) Backward propagation of vorticity concentration**’ still remains valid if a sufficiently well-separated subsequence of $t_{(k)}$ is taken (see Lemma 6 and Proposition 3). To show this we use energy estimates in [38] for solutions to the Navier-Stokes equations with $L^3(\mathbb{R}^3)$ initial data. Such estimates are also central to gain good quantitative control of the solution in certain space-time regions, which are required for applying the quantitative Carleman inequalities. The price one pays in this setting (when compared to the estimates in [45]), is a gain of an additional exponential in the estimates. The reason is the control on the energy of $u(\cdot, t) - e^{t\Delta}u_0$ (with $u_0 \in L^3(\mathbb{R}^3)$) requires the use of Gronwall’s lemma.

In the strategy in [45] the lower bound on vorticity (28), which is needed for getting a lower bound on the localized L^3 norm at t_0 via quantitative Carleman inequalities, is obtained from the frequency bubbles of concentration. In order for this transfer of scale-invariant information to take place, it appears essential that the solution has a scale-invariant control such as $\|u\|_{L_t^\infty L_x^3} \leq A$. In our strategy, we instead work directly with quantities involving vorticity (similar to (28)), which are tailored for the immediate use of quantitative Carleman inequalities. In this way, we crucially avoid any need to transfer scale-invariant information, giving our strategy a certain degree of robustness.

1.2. Final Remarks and Comments. We give some heuristics about the quantitative estimates of the form

$$(36) \quad \|u\|_{L^\infty(\mathbb{R}^3 \times (\frac{1}{2}, 1))} \leq G(\|u\|_X)$$

that one can expect for the Navier-Stokes equations, when a finite-energy solution u solution belongs to certain normed spaces $X \subset \mathcal{S}'(\mathbb{R}^3 \times (0, 1))$.

1.2.1. Subcritical case. Consider a space $X \subset \mathcal{S}'(\mathbb{R}^3 \times (0, 1))$ whose norm $\|\cdot\|_X$ is subcritical¹⁷ (for example $L_{x,t}^{5+\delta}(\mathbb{R}^3 \times (0, 1))$ with $\delta > 0$). If u is a finite-energy solution with a finite subcritical norm on $\mathbb{R}^3 \times (0, 1)$, then it is known that u must be belong to $C^\infty(\mathbb{R}^3 \times (0, 1])$. See, for example, [23]. Moreover, one typically has a quantitative estimate of the form (36) with

$$G(x) = cx^\beta \text{ with } \beta > 0.$$

To demonstrate this, consider u belong to $L_{x,t}^{5+\delta}(\mathbb{R}^3 \times (0, 1))$. An application of Caffarelli, Kohn and Nirenberg’s result [9] (see also Proposition 15) gives that (36) holds true with $G(x) \sim x^{\frac{\delta+5}{\delta}}$. Such a quantitative estimate is invariant with respect to the Navier-Stokes scaling (2). In this context, one could also gain similar quantitative estimates based on parabolic bootstrap arguments applied to the vorticity equation

$$(37) \quad \partial_t \omega - \Delta \omega = \omega \cdot \nabla u - u \cdot \nabla \omega, \quad \omega = \nabla \times u$$

as was done by Serrin in [40].

1.2.2. Critical case. In the subcritical norm case, we saw that seeking estimates of the form (36) that are invariant with respect to the scaling (2), gives a suitable candidate for G that can be realised. The case when the norm $\|\cdot\|_X$ is critical is more subtle, since a scaling argument does not provide a suitable candidate for G . We first mention that the case of sufficiently small critical norms, for example

$$(38) \quad \|u\|_{L^5(\mathbb{R}^3 \times (0, 1))} < \varepsilon_0,$$

¹⁷A quantity $F(u) \in [0, \infty)$ is said to be *subcritical* if, for the rescaling (2), there exists $\alpha > 0$ such that $F(u_\lambda) = \lambda^\alpha F(u)$.

is essentially of a similar category to the subcritical case (though a scaling argument is not applicable). Indeed, a similar argument outlined as before (based on [9], see also Proposition 15) gives that in this case we have (36) with $G(x) \sim x$. Is this consistent with the fact that solutions with small scale-invariant norms exhibit similar behaviour to the linear system and hence are typically expected to satisfy linear estimates.

For obtaining quantitative estimates of the form (36) when the scale-invariant norm is large, it is less clear what the candidate for G might be. This seems to be the case even for large *global* scale-invariant norms that exhibit smallness at small *local* scales¹⁸ (for example $L^5(\mathbb{R}^3 \times (0, 1))$). Such local smallness properties have been utilized to prove *qualitative* regularity by essentially linear methods. See [43], for example.

For the case of a smooth finite-energy solution u having finite scale-invariant $L^5(\mathbb{R}^3 \times (0, 1))$ norm, one way to obtain quantitative estimates¹⁹ is to consider the vorticity equation (37) with initial vorticity $\omega_0 \in L^2(\mathbb{R}^3)$. Performing an energy estimate yields for $t \in [0, 1]$

$$(39) \quad \|\omega(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \omega|^2 dx dt' = \|\omega_0\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^t \int_{\mathbb{R}^3} (\omega \cdot \nabla u) \cdot \omega dx dt',$$

where the second term in right-hand side is due to the *vortex stretching term* $\omega \cdot \nabla u$ in (37). For the case that $u \in L^5(\mathbb{R}^3 \times (0, 1))$, application of Hölder's inequality, Lebesgue interpolation, Sobolev embedding theorems and Young's inequality lead to

$$(40) \quad \|\omega(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \omega|^2 dx dt' \leq \|\omega_0\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|u(\cdot, t')\|_{L^5(\mathbb{R}^3)}^5 \|\omega(\cdot, t')\|_{L^2(\mathbb{R}^3)}^2 dt'.$$

Gronwall's lemma, followed by arguments similar to the subcritical case, yields

$$(41) \quad \|u\|_{L^\infty(\mathbb{R}^3 \times (\frac{1}{2}, 1))} \lesssim \|\omega\|_{L^\infty(0, 1; L^2(\mathbb{R}^3))}^2 \leq \|\omega_0\|_{L^2(\mathbb{R}^3)}^2 \exp(\|u\|_{L^5(\mathbb{R}^3 \times (0, 1))}^5).$$

Though this is not exactly of the form (36), a slightly different argument gives that for any finite-energy solution u in $L^5(\mathbb{R}^3 \times (0, 1))$ we get that (36) holds with $G(x) \sim \exp(O(1)x^5)$. In particular, this can be achieved using L^q energy estimates in [30], the pigeonhole principle and reasoning in the previous subsection.

The above argument (39)-(41) shows that being able to substantially improve upon $G(x) \sim \exp(O(1)x^5)$ would most likely require the utilization of a nonlinear mechanism that reduces the influence of the vortex stretching term $\omega \cdot \nabla u$ in (37). It seems plausible that the discovery of such a mechanism would have implications for the regularity theory of the Navier-Stokes equations (such as Type I blow-ups).

1.3. Outline of the paper. In each of the Sections 2–7, we distinguish between cases where: (i) one assumes a Type I control on the solution and (ii) one assumes a control on the velocity field on time slices only.

In Section 2, we state our main quantitative estimates (Propositions 2 and 3) and we demonstrate how these statements imply the main results of this paper: Theorem 1, Corollary 1 and Theorem 2. Section 3 is devoted to the proof of Propositions 2 and 3. Section 4 contains three further applications of the technology developed in the present paper, in particular Corollary 9 concerning a quantitative bound for the number of singularities in a Type I blow-up scenario. In Section 5, we quantify Jia and Šverák's results regarding

¹⁸In particular, $u \in L^5(\mathbb{R}^3 \times (0, 1)) \Rightarrow \lim_{r \downarrow 0} \|u\|_{L^5(B_0(r) \times (-r^2, 0))} = 0$.

¹⁹For very similar computations, see (for example) Chapter 11 of [25] and references therein.

local-in-space short-time smoothing, which is a main tool for proving the quantitative estimates in Section 3. The main result in Section 5 is Theorem 3. In Section 6, we give a new proof of Tao's result that solutions possess 'quantitative annuli of regularity', which is required for proving our main propositions in Section 3. The central results in Section 6 are Lemma 20 and Lemma 22. Section 7 is concerned with the utilization of arguments from the papers of Leray and Tao to show existence of quantitative epochs of regularity (Lemma 27 and Lemma 29). In Appendix A we recall known results about mild solutions and local energy solutions, and we give pressure formulas. In Appendix B, we recall the quantitative Carleman inequalities proven by Tao.

1.4. Notations.

1.4.1. Universal constants. For universal constants in the statements of propositions and lemmas associated to the Type I case (specifically Proposition 2 and Lemma 4), we adopt the convention of a superscript \sharp . For universal constants in the statements of propositions and lemmas associated to the Type I case (specifically Proposition 3 and Lemma 6), we adopt the convention of a superscript b .

In Lemma 4, Lemma 6 and Section 5, we track the numerical constants arising. Elsewhere in this paper, we adopt the convention that C denotes a positive universal constant which may vary from line to line.

We use the notation $X \lesssim Y$, which means that there exists a positive universal constant C such that $X \leq CY$.

In several places in this paper (notably Section 3 and Appendix B) the notation $O(1)$ is used to denote a positive universal constant and $-O(1)$ denotes a negative universal constant.

Whenever we refer to a quantity (M for example) being 'sufficiently large', we understand this as M being larger than some universal constant that can (in principle) be specified.

1.4.2. Vectors and Domains. For a vector a , a_i denotes the i^{th} component of a .

For $(x, t) \in \mathbb{R}^4$ and $r > 0$ we denote $B_x(r) := \{y \in \mathbb{R}^3 : |y - x| < r\}$ and $Q_{(x,t)}(r) := B_r(x) \times (t - r^2, t)$. Here, $|\cdot|$ denotes the Euclidean metric. As is usual, for $a, b \in \mathbb{R}^3$, $(a \otimes b)_{\alpha\beta} = a_\alpha b_\beta$, and for $A, B \in M_3(\mathbb{R})$, $A : B = A_{\alpha\beta} B_{\alpha\beta}$. Here and in the whole paper we use Einstein's convention on repeated indices. For $F : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we define $\nabla F \in M_3(\mathbb{R})$ by $(\nabla F(x))_{\alpha\beta} := \partial_\beta F_\alpha$.

Let us stress that in Section 5 only we use cubes instead of balls: $B_x(r) = x + (-r, r)^3$. This is for computational convenience, since we track numerical constants in Section 5. We emphasize that the results in Section 5 hold for spherical balls too, with certain universal constants adjusted.

1.4.3. Mild, suitable and finite-energy solutions to the Navier-Stokes equations. Throughout this paper, we refer to $u : \mathbb{R}^3 \times [0, T]$ as a *mild solution* of the Navier-Stokes equations (1) if it satisfies the Duhamel formula:

$$u(x, t) = e^{t\Delta} u(\cdot, 0) + \int_0^t \mathbb{P} \partial_i e^{(t-s)\Delta} u_i(\cdot, s) u_j(\cdot, s) ds,$$

for all $t \in [0, T]$. Here, $e^{t\Delta}$ is the heat semigroup, \mathbb{P} is the projection onto divergence-free vector fields. A mild solution on $[0, T^*)$ is a function that is a mild solution on $[0, T]$ for any $T \in (0, T^*)$.

Let $\Omega \subset \mathbb{R}^3$. We say that (u, p) is a *suitable weak solution* to the Navier-Stokes equations (1) in $\Omega \times (T_1, T)$ if it fulfills the properties described in [18] (Definition 6.1 p.133 in [18]).

We say that u is a *suitable finite-energy solution* to the Navier-Stokes equations on $\mathbb{R}^3 \times (T_1, T)$ if it is a solution to (1) in the sense of distributions and

- $u \in C_w([T_1, T]; L_\sigma^2(\mathbb{R}^3)) \cap L_t^2(T_1, T; \dot{H}^1(\mathbb{R}^3)),$
- it satisfies the global energy inequality

$$(42) \quad \|u(\cdot, t)\|_{L^2}^2 + 2 \int_{T_1}^t \int_{\mathbb{R}^3} |\nabla u|^2 dy ds \leq \|u(\cdot, T_1)\|_{L^2}^2 \quad \text{for all } t \in [T_1, T],$$

- (u, p) is a *suitable weak solution* on $B_1(x) \times (T_1, T)$ for all $x \in \mathbb{R}^3$.

It is known that the above defining properties of suitable weak solutions imply that there exists $\Sigma \subset [T_1, T]$ with full Lebesgue measure $|\Sigma| = T - T_1$ such that

$$(43) \quad \|u(\cdot, t)\|_{L^2}^2 + 2 \int_{t'}^t \int_{\mathbb{R}^3} |\nabla u|^2 dy ds \leq \|u(\cdot, t')\|_{L^2}^2 \quad \text{and} \quad \|\nabla u(\cdot, t')\|_{L^2} < \infty,$$

for all $t \in [t', T]$ and $t' \in \Sigma$.

1.4.4. Lorentz spaces. For a measurable subset $\Omega \subseteq \mathbb{R}^d$ and a measurable function $f : \Omega \rightarrow \mathbb{R}$ we define

$$(44) \quad d_{f, \Omega}(\alpha) := \mu(\{x \in \Omega : |f(x)| > \alpha\}),$$

where μ denotes the Lebesgue measure. The Lorentz space $L^{p,q}(\Omega)$, with $p \in [1, \infty[$, $q \in [1, \infty]$, is the set of all measurable functions g on Ω such that the quasinorm $\|g\|_{L^{p,q}(\Omega)}$ is finite. The quasinorm is defined by

$$(45) \quad \|g\|_{L^{p,q}(\Omega)} := \left(p \int_0^\infty \alpha^q d_{g, \Omega}(\alpha)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}},$$

$$(46) \quad \|g\|_{L^{p,\infty}(\Omega)} := \sup_{\alpha > 0} \alpha d_{g, \Omega}(\alpha)^{\frac{1}{p}}.$$

Notice that there exists a norm, which is equivalent to the quasinorm defined above, for which $L^{p,q}(\Omega)$ is a Banach space. For $p \in [1, \infty)$ and $1 \leq q_1 < q_2 \leq \infty$, we have the following continuous embeddings

$$(47) \quad L^{p, q_1}(\Omega) \hookrightarrow L^{p, q_2}(\Omega)$$

and the inclusion is strict.

2. MAIN QUANTITATIVE ESTIMATES

2.1. Quantitative estimates in the Type I and time slices case.

Proposition 2 (main quantitative estimate, Type I). *There exists a universal constant $M_2 \in [1, \infty)$. Let $M \in [M_2, \infty)$, $t_0 \in \mathbb{R}$ and $T \in (0, \infty)$. There exists $S^\sharp(M) \in (0, \frac{1}{4}]$, such that the following holds. Let (u, p) be a smooth solution with sufficient decay²⁰ to the Navier-Stokes equations (1) in $I = [t_0 - T, t_0]$, which satisfies*

$$(48) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M.$$

²⁰See footnote 8. Notice that by this definition u is bounded up to t_0 .

Assume that there exists $t'_0 \in [t_0 - T, t_0)$ such that t'_0 is not too close to $t_0 - T$ in the sense

$$0 \leq \frac{t_0 - t'_0}{T} < C^\sharp M^{-548}$$

and such that the vorticity concentrates at time t'_0 in the following sense

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(t_0 - t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx > M^2(t_0 - t'_0)^{-\frac{1}{2}} \sqrt{S^\sharp}.$$

Then, we have the following lower bound

$$(49) \quad \frac{t_0 - t'_0}{T} \geq \frac{C^\sharp}{8} M^{-749} \exp \left\{ -\exp(\exp(M^{1024})) \int_{B_0(\exp(M^{1023})T^{\frac{1}{2}})} |u(x, t_0)|^3 dx \right\}.$$

Furthermore, for

$$(50) \quad -s_0 := \frac{C^\sharp}{16} M^{-749} \exp \left\{ -\exp(\exp(M^{1024})) \int_{B_0(\exp(M^{1023})T^{\frac{1}{2}})} |u(x, t_0)|^3 dx \right\},$$

we also have the bound

$$(51) \quad \|u\|_{L^\infty(B_0(C_2 T^{\frac{1}{2}} M^{50} (-s_0)^{\frac{1}{2}}) \times (\frac{s_0 T}{4} + t_0, t_0))} \leq \frac{C_1 M^{-23}}{(-s_0)^{\frac{1}{2}} T^{\frac{1}{2}}},$$

for universal constants $C_1, C_2 \in (0, \infty)$. Here C^\sharp and $S^\sharp(M)$ are the constants given by Lemma 4.

Furthermore, if for fixed $\lambda \in (0, \exp(M^{1023}))$ we additionally assume that

$$(52) \quad \int_{B_0(\lambda T^{\frac{1}{2}})} |u(x, t_0)|^3 dx \geq \frac{3}{2} \exp(-\exp(M^{1024}))$$

then we instead have the lower bound

$$(53) \quad \frac{t_0 - t'_0}{T} \geq \frac{C^\sharp \lambda^2}{8} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(\lambda T^{\frac{1}{2}})} |u(x, t_0)|^3 dx \right\}.$$

Furthermore, for

$$(54) \quad -s_1 := \frac{C^\sharp \lambda^2}{16} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(\lambda T^{\frac{1}{2}})} |u(x, t_0)|^3 dx \right\},$$

we also have the bound

$$(55) \quad \|u\|_{L^\infty(B_0(C_2 T^{\frac{1}{2}} M^{50} (-s_1)^{\frac{1}{2}}) \times (\frac{s_1 T}{4} + t_0, t_0))} \leq \frac{C_1 M^{-23}}{(-s_1)^{\frac{1}{2}} T^{\frac{1}{2}}}.$$

Figure 1.1 illustrates Proposition 2.

Proposition 3 (main quantitative estimate, time slices). *There exists a universal constant $M_1 \in [1, \infty)$. Let $M \in [M_1, \infty)$. We define M^\flat by (17). There exists $S^\flat(M) \in (0, \frac{1}{4}]$, such that the following holds. Let (u, p) be a $C^\infty(\mathbb{R}^3 \times (-1, 0))$ finite-energy solution to*

the Navier-Stokes equations (1) in $I = [-1, 0]$. Assume that there exists $t_{(k)} \in [-1, 0)$ such that

$$(56) \quad t_{(k)} \uparrow 0 \text{ with } \sup_k \|u(\cdot, t_{(k)})\|_{L^3(\mathbb{R}^3)} \leq M.$$

Select any “well-separated” subsequence (still denoted $t_{(k)}$) such that

$$(57) \quad \sup_k \frac{-t_{(k+1)}}{-t_{(k)}} < \exp(-2(M^\flat)^{1223}).$$

For this well-separated subsequence, assume that there exists $j + 1$ such that the vorticity concentrates at time $t_{(j+1)}$ in the following sense

$$(58) \quad \int_{B_0(4\sqrt{S^\flat}^{-1}(-t_{(j+1)})^{\frac{1}{2}})} |\omega(x, t_{(j+1)})|^2 dx > M^2(-t_{(j+1)})^{-\frac{1}{2}} \sqrt{S^\flat}.$$

Then, we have the following upper bound on j

$$(59) \quad j \leq \exp(\exp((M^\flat)^{1224}))$$

Here $S^\flat(M)$ is the constant given by Lemma 6.

2.2. Proofs of the main results. In this section we prove the main results stated in the Introduction.

Proof of Theorem 1. Take $M \geq M_0$. Here, $M_0 := \max(M_2, M_6) \geq 1$ with M_2 being from Proposition 2 and M_6 being from Corollary 12. We prove here a slightly stronger statement than (10), namely

$$(60) \quad \int_{B_0((T^*)^{\frac{1}{2}}(T^*-t)^{\frac{1-\delta}{2}})} |u(x, t)|^3 dx \geq \frac{\log\left(\frac{1}{(T^*-t)^{\frac{\delta}{2}}}\right)}{4M^{1023} \exp(\exp(M^{1024}))}.$$

First we note that [6] (specifically Theorem 2 in [6]), together with assumptions (1)-(2) in the statement of Theorem 1 imply that there exist $S_{BP}(M)$ and $\gamma_{univ} > 0$ such that

$$\int_{B_0(2\sqrt{\frac{T^*-t}{S_{BP}(M)}})} |u(x, t)|^3 dx \geq \gamma_{univ}^3 \text{ for all } t \in (0, T^*).$$

Thus we have

$$(61) \quad \int_{B_0(t^{\frac{1}{2}}(T^*-t)^{\frac{1-\delta}{2}})} |u(x, t)|^3 dx \geq \gamma_{univ}^3 \text{ for all } t \in \left(\max\left(T^* - \left(\frac{T^* S_{BP}(M)}{8}\right)^{\frac{1}{\delta}}, \frac{T^*}{2}\right), T^* \right).$$

Suppose for contradiction that (60) does not hold for some

$$(62) \quad t \in \left(\max\left(\frac{T^*}{2}, T^* - c(\delta, M, T^*)\right), T^* \right),$$

where

$$(63) \quad c(\delta, M, T^*) := \min\left(c(\delta)(T^* M^{-802})^{\frac{2}{\delta}}, \left(\frac{T^* S_{BP}(M)}{8}\right)^{\frac{1}{\delta}}, \exp(2M^{1023})\right)$$

for an appropriate $c(\delta) \in (0, \infty)$. This implies that for M sufficiently large

$$(64) \quad \int_{B_0(t^{\frac{1}{2}}(T^*-t)^{\frac{1-\delta}{2}})} |u(x, t)|^3 dx < \frac{1}{4M^{1023} \exp(\exp(M^{1024}))} \log\left(\frac{1}{(T^*-t)^{\frac{\delta}{2}}}\right).$$

Considering u on $\mathbb{R}^3 \times [0, t]$ and observing Proposition 2, we see that (61) implies that the assumption (52) is satisfied with $\lambda := (T^* - t)^{\frac{1-\delta}{2}}$ and $T := t$. Furthermore, by (62)-(63) we have that $\lambda \in (0, \exp(M^{1023}))$. Hence, we can apply Proposition 2. Namely by (55) for

$$-s_1 := \frac{C^\sharp(T^* - t)^{1-\delta}}{16} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(t^{\frac{1}{2}}(T^*-t)^{\frac{1-\delta}{2}})} |u(x, t)|^3 dx \right\},$$

we have the bound

$$(65) \quad \|u(\cdot, t)\|_{L^\infty(B_0(C_2 t^{\frac{1}{2}} M^{50} (-s_1)^{\frac{1}{2}}))} \leq \frac{C_1 M^{-23}}{(-s_1)^{\frac{1}{2}} t^{\frac{1}{2}}}.$$

Using that $(0, T^*)$ is a singular point of u , the Type I bound on u and Corollary 12, we see that there exists a universal constant C_{univ} such that

$$(66) \quad \|u(\cdot, t)\|_{L^\infty(B_0(\frac{2M^{50}}{C_{univ}}(T^*-t)^{\frac{1}{2}}))} > \frac{C_{univ} M^{-49}}{(T^* - t)^{\frac{1}{2}}}.$$

From our contradiction assumption (which implies (64)) we see that

$$(67) \quad -s_1 > M^{-750}(T^* - t)^{1-\frac{\delta}{2}}$$

Using this and (62) for an appropriate $c(\delta) \in (0, \infty)$, we get that

$$B_0\left(\frac{2M^{50}}{C_{univ}}(T^* - t)^{\frac{1}{2}}\right) \subset B_0\left(\frac{(T^*)^{\frac{1}{2}} C_2 M^{50}}{2^{\frac{1}{2}}} (-s_1)^{\frac{1}{2}}\right)$$

and

$$\frac{2^{\frac{1}{2}} C_1 M^{-23}}{(-s_1)^{\frac{1}{2}} (T^*)^{\frac{1}{2}}} \leq \frac{C_{univ} M^{-49}}{(T^* - t)^{\frac{1}{2}}}.$$

With these two facts, we see that (65) contradicts (66). \square

Proof of Corollary 1. From [10] (specifically Theorem 1.1 in [10]), there exists $M > 1$ such that

$$(68) \quad |u(x, t)| \leq \frac{M}{|x| + \sqrt{-t}} \text{ for all } (x, t) \in (\mathbb{R}^3 \times (-\infty, 0]) \setminus \{(0, 0)\}.$$

Integration of this then immediately gives the upper bound of (12), which in fact holds true for $t \in (-1, 0)$. Next, note that (68) implies

$$(69) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-\infty, 0))} \leq M.$$

We also remark that since u is non-zero and λ -DSS we must have

$$(70) \quad u \notin L_{x,t}^\infty(Q_{(0,0)}(r)) \text{ for all sufficiently small } r.$$

Indeed, suppose for contradiction that $u \in L_{x,t}^\infty(Q_r(0, 0))$ then for any $(x, t) \in \mathbb{R}^3 \times (-\infty, 0)$ we have $(\lambda^{-k}x, \lambda^{-2k}t) \in Q_r(0, 0)$ for all sufficiently large k . Using that u is λ -DSS we have

$$|u(x, t)| = |\lambda^{-k}u(\lambda^{-k}x, \lambda^{-2k}t)| \leq \lambda^{-k}\|u\|_{L^\infty(Q_{(0,0)}(r))} \downarrow 0.$$

We then see that (68)-(70) allow us to apply Theorem 1 on $\mathbb{R}^3 \times (-1, 0)$ to get the lower bound in (12). \square

Proof of Theorem 2. Applying Proposition 3 we see that for $j = \lceil \exp(\exp((M^\flat)^{1224})) \rceil + 1$ we have the contrapositive of (58). In particular,

$$\int_{B_0(4\sqrt{S^\flat}^{-1}(-t_{(j+1)})^{\frac{1}{2}})} |\omega(x, t_{(j+1)})|^2 dx < M^2(-t_{(j+1)})^{-\frac{1}{2}}\sqrt{S^\flat}.$$

Almost identical arguments to those utilized in the proof of Lemma 4, except using the bound (174) instead of (175), give

$$\|u\|_{L^\infty(B_0(C_2 M^{50}(-t_{(j+1)})^{\frac{1}{2}}) \times (\frac{t_{(j+1)}}{4}, 0))} \leq \frac{C_1 M^{-23}}{(-t_{(j+1)})^{\frac{1}{2}}}.$$

Since all estimates are independent of the spatial point where (58) occurs, we conclude that

$$\|u\|_{L^\infty(\mathbb{R}^3 \times (\frac{t_{(j+1)}}{4}, 0))} \leq \frac{C_1 M^{-23}}{(-t_{(j+1)})^{\frac{1}{2}}}.$$

This concludes the proof of the theorem. \square

3. PROOFS OF THE MAIN QUANTITATIVE ESTIMATES

3.1. Backward propagation of concentration. Here we state two pivotal results. These are concerned with backward propagation of concentration in the Type I case and in the time slices case. Figure 1.1 illustrates Lemma 4 below.

Lemma 4 (backward propagation of concentration, Type I). *There exists two universal constants $C^\sharp \in (0, \frac{1}{16})$, $M_3 \in [1, \infty)$. For all $M \in [M_3, \infty)$, there exists $S^\sharp(M) \in (0, \frac{1}{4}]$, such that the following holds. Let (u, p) be a ‘smooth solution with sufficient decay’²¹ of the Navier-Stokes equations (1) in $I = [-1, 0]$ satisfying the Type I bound (48). Assume that there exists $t'_0 \in [-1, 0]$ such that t'_0 is not too close to -1 in the sense*

$$0 < -t'_0 < C^\sharp M^{-548}$$

and such that the vorticity concentrates at time t'_0 in the following sense

$$(71) \quad \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx > M^2(-t'_0)^{-\frac{1}{2}}\sqrt{S^\sharp}.$$

Then, the vorticity concentrates in the following sense

$$(72) \quad \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t''_0)^{\frac{1}{2}})} |\omega(x, t''_0)|^2 dx > M^2(-t''_0)^{-\frac{1}{2}}\sqrt{S^\sharp}$$

at any well-separated backward time $t''_0 \in [-1, t'_0]$ such that

$$(73) \quad \frac{-t'_0}{-t''_0} < C^\sharp M^{-548}.$$

Here $S^\sharp(M)$ is defined explicitly by (74) and we have $S^\sharp(M) = O(1)M^{-100}$.

Proof of Lemma 4. The proof is by contradiction. It relies on Theorem 3 below about local-in-space short-time smoothing. We define $S^\sharp \in (0, \frac{1}{4}]$ in the following way:

$$(74) \quad S^\sharp = S^\sharp(M) := S_*(C_{weak}M, 32C_{Sob}(1 + C_{ellip})C_{weak}M),$$

²¹See footnote 8.

where S_* is the constant defined in Theorem 3 (see also the formula (206)), $C_{Sob} \in (0, \infty)$ is the best constant in the Sobolev embedding $H^1(B_0(2)) \subset L^6(B_0(2))$ and $C_{ellip} \in (0, \infty)$ is the best constant in the estimate

$$(75) \quad \|\nabla U(\cdot, 0)\|_{L^2(B_0(2))} \leq C_{ellip} (\|\Omega(\cdot, 0)\|_{L^2(B_0(4))} + \|U(\cdot, 0)\|_{L^2(B_0(4))})$$

for weak solutions to

$$-\Delta U(\cdot, 0) = \nabla \times \Omega(\cdot, 0) \quad \text{in } B_0(4).$$

Furthermore, $C_{weak} \in [1, \infty)$ is a universal constant from the embedding $L^{3,\infty}(\mathbb{R}^3) \subset L^2_{uloc}(\mathbb{R}^3)$. See, for example, Lemma 6.2 in Bradshaw and Tsai's paper [8].

Assume that

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(-t_0'')^{\frac{1}{2}})} |\omega(x, t_0'')|^2 dx \leq M^2 (-t_0'')^{-\frac{1}{2}} \sqrt{S^\sharp}$$

and

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(-t_0')^{\frac{1}{2}})} |\omega(x, t_0')|^2 dx > M^2 (-t_0')^{-\frac{1}{2}} \sqrt{S^\sharp}$$

for times t_0', t_0'' satisfying the condition that they are well-separated (73). Let $r := \sqrt{S^\sharp}^{-1}(-t_0'')^{\frac{1}{2}}$ and rescale in the following way $U(y, s) := ru(ry, r^2s + t_0'')$, $\Omega(y, s) := r^2\omega(ry, r^2s + t_0'')$. We have

$$\begin{aligned} \|U(\cdot, 0)\|_{L^6(B_0(2))} &\leq C_{Sob} (\|U(\cdot, 0)\|_{L^2(B_0(2))} + \|\nabla U(\cdot, 0)\|_{L^2(B_0(2))}) \\ &\leq C_{Sob}(1 + C_{ellip}) (\|U(\cdot, 0)\|_{L^2(B_0(4))} + \|\Omega(\cdot, 0)\|_{L^2(B_0(4))}) \\ &\leq C_{Sob}(1 + C_{ellip})(8^{\frac{3}{2}} + 1)C_{weak}M \\ &\leq 32C_{Sob}(1 + C_{ellip})C_{weak}M. \end{aligned}$$

Here we used the scale-invariant bound (48) and the embedding $L^{3,\infty}(\mathbb{R}^3) \subset L^2_{uloc}(\mathbb{R}^3)$. Taking $M \geq M_3$ sufficiently large, we now apply the bound (175) with $C_{weak}M$ and $N := 32C_{Sob}(1 + C_{ellip})C_{weak}M$. Using $C^\sharp \in (0, \frac{1}{16})$ and (73), we have $-t_0' < \frac{1}{16}(-t_0'')$. Therefore, we have $t_0' \in (t_0'' + \frac{15}{16}S^\sharp r^2, t_0'' + S^\sharp r^2)$, hence

$$\begin{aligned} \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t_0'')^{\frac{1}{2}})} |\omega(x, t_0')|^2 dx &\leq \sup_{t \in (t_0'' + \frac{15}{16}S^\sharp r^2, t_0'' + S^\sharp r^2)} \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t_0')^{\frac{1}{2}})} |\omega(x, t)|^2 dx \\ &\leq \sup_{t \in (t_0'' + \frac{15}{16}S^\sharp r^2, t_0'' + S^\sharp r^2)} \int_{B_0(\frac{1}{4}\sqrt{S^\sharp}^{-1}(-t_0'')^{\frac{1}{2}})} |\omega(x, t)|^2 dx \\ &\leq r^{-1} \sup_{s \in (\frac{15}{16}S^\sharp, S^\sharp)} \int_{B_0(\frac{1}{4})} |\Omega(y, s)|^2 dx \\ &\leq CM^{65}N^{161}(-t_0'')^{-\frac{1}{2}} \\ &\leq CM^{226}(-t_0'')^{-\frac{1}{2}}. \end{aligned}$$

The contradiction follows then for a well chosen universal constant $C^\sharp \in (0, \frac{1}{16})$. \square

Remark 5 (Concentration of the enstrophy). A variant of the proof of Lemma 4 gives the following concentration result for the enstrophy near a Type I singularity.

For all sufficiently large $M \in [1, \infty)$, let $S^\sharp(M) \in (0, \frac{1}{4}]$ be the constant defined by (74). Let (u, p) be a suitable finite-energy solution²² of the Navier-Stokes equations (1) in $I = [-1, 0]$ satisfying the Type I bound (48). Assume that the space-time point $(0, 0)$ is a singularity for u . Then, for all $t' \in \Sigma$, where Σ is a full measure subset of $[-1, 0]$ defined in Subsection 1.4.3, the vorticity concentrates in the following sense

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(-t')^{\frac{1}{2}})} |\omega(x, t')|^2 dx > M^2(-t')^{-\frac{1}{2}}\sqrt{S^\sharp}.$$

Lemma 6 (backward propagation of concentration, time slices). *There exists a universal constant $M_4 \in [1, \infty)$ such that the following holds true. Let $M \in [M_4, \infty)$. We define M^\flat by (17). Fix any $\alpha \geq M^\flat$ and let $t'_0, t''_0 \in [-1, 0)$ be such that*

$$\frac{t''_0}{8\alpha^{201}} < t'_0 < 0.$$

There exists $S^\flat(M) \in (0, \frac{1}{4}]$, such that the following holds. Let (u, p) be a $C^\infty(\mathbb{R}^3 \times (-1, 0))$ finite-energy solution of the Navier-Stokes equations (1) in $I = [-1, 0]$ satisfying

$$(76) \quad \|u(\cdot, t'_0)\|_{L^3} \leq M \text{ and } \|u(\cdot, t''_0)\|_{L^3} \leq M.$$

Suppose further that the vorticity concentrates at time t'_0 in the following sense

$$(77) \quad \int_{B_0(4\sqrt{S^\flat}^{-1}(-t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx > M^2(-t'_0)^{-\frac{1}{2}}\sqrt{S^\flat}.$$

With the additional separation condition that

$$(78) \quad \frac{-t'_0}{-t''_0} < \alpha^{-1051},$$

the above assumptions imply that for any $s_0 \in [t''_0, \frac{t''_0}{8\alpha^{201}}]$ the vorticity concentrates in the following sense

$$(79) \quad \int_{B_0(4(-s_0)^{\frac{1}{2}}\alpha^{106})} |\omega(x, s_0)|^2 dx > \frac{(M+1)^2}{(-s_0)^{\frac{1}{2}}\alpha^{106}}.$$

Here $S^\flat(M) = O(1)M^{-100}$.

Proof. For $s \in [t''_0, 0]$ we decompose u as

$$(80) \quad u(\cdot, s) = e^{(s-t''_0)\Delta}u(\cdot, t''_0) + V(\cdot, s).$$

We then have

$$(81) \quad \|e^{(s-t''_0)\Delta}u(\cdot, t''_0)\|_{L_x^3} \leq M.$$

$$(82) \quad \|e^{(s-t''_0)\Delta}u(\cdot, t''_0)\|_{L_x^4} \leq \frac{CM}{(s_0 - t''_0)^{\frac{1}{8}}}.$$

Furthermore, arguments from [17] imply that

$$(83) \quad \|e^{(t-t''_0)\Delta}u(\cdot, t''_0)\|_{L^5(\mathbb{R}^3 \times (t''_0, \infty))} \leq CM$$

²²See Subsection 1.4.3.

Moreover, similar arguments as those used in Proposition 2.2 of [38]²³ yield that for $s \in [t''_0, 0]$

$$\begin{aligned} & \|V(\cdot, s)\|_{L_x^2}^2 + \int_{t''_0}^s \int_{\mathbb{R}^3} |\nabla V|^2 dx dt \\ & \leq C \int_{t''_0}^s \int_{\mathbb{R}^3} |e^{(t-t''_0)\Delta} u(\cdot, t''_0)|^4 dx dt + C \int_{t''_0}^s \|V(\cdot, t)\|_{L_x^2}^2 \|e^{(t-t''_0)\Delta} u(\cdot, t''_0)\|_{L_x^5}^5 dt. \end{aligned}$$

Using (81)-(83) and Gronwall's lemma, we infer (for M larger than some universal constant) that

$$\|V(\cdot, s)\|_{L_x^2}^2 + \int_{t''_0}^s \int_{\mathbb{R}^3} |\nabla V(x, t)|^2 dx dt \leq C(M^\flat)^4 (s - t''_0)^{\frac{1}{2}}.$$

Here M^\flat is defined by (17) for an appropriate universal constant $L^* \in (0, \infty)$ coming from the Gronwall estimate. In particular, using that $s \in [t''_0, \frac{t''_0}{8\alpha^{201}}]$ we have

$$(84) \quad \|V(\cdot, s)\|_{L_x^2}^2 \leq C 8^{\frac{1}{2}} \alpha^{101} (M^\flat)^4 (-s)^{\frac{1}{2}} < \alpha^{106} (-s)^{\frac{1}{2}}.$$

Here, we used the fact that $\alpha \geq M^\flat$. Next assume for contradiction the under the assumptions of Lemma 6, we have the converse of (79). Namely, there exists $s_0 \in [t''_0, \frac{t''_0}{8\alpha^{201}}]$ such that

$$(85) \quad \int_{B_0(4\alpha^{106}(-s_0)^{\frac{1}{2}})} |\omega(x, s_0)|^2 dx \leq \frac{(M+1)^2}{(-s_0)^{\frac{1}{2}} \alpha^{106}}.$$

Define

$$(86) \quad \lambda := (-s_0)^{\frac{1}{2}} \alpha^{106}$$

and rescale to get $U^\lambda : \mathbb{R}^3 \times (0, \alpha^{-212}) \rightarrow \mathbb{R}^3$ and $P^\lambda : \mathbb{R}^3 \times (0, \alpha^{-212}) \rightarrow \mathbb{R}$. Here,

$$(87) \quad U^\lambda(y, t) := \lambda u(\lambda y, \lambda^2 t + s_0) \quad \text{and} \quad P^\lambda(y, t) := \lambda^2 p(\lambda y, \lambda^2 t + s_0).$$

Using (81) and (84), we see that

$$(88) \quad \|U^\lambda(y, 0)\|_{L_{uloc}^2} \leq C_{Leb} M + 1.$$

Here, $C_{Leb} \in [1, \infty)$ is a universal constant from the embedding $L^3(\mathbb{R}^3) \subset L_{uloc}^2(\mathbb{R}^3)$.

Furthermore, defining $\Omega^\lambda = \nabla \times U^\lambda$, we see that (85) implies that

$$(89) \quad \int_{B_0(4)} |\Omega^\lambda(y, 0)|^2 dy \leq (M+1)^2.$$

Similarly to Lemma 4, we define

$$S^\flat = S^\flat(M) := S_* (C_{Leb} M + 1, 32C_{Sob}(1 + C_{ellip})(C_{Leb} M + 1))$$

where S_* is the constant defined in Theorem 3, $C_{Sob} \in (0, \infty)$ is the best constant in the Sobolev embedding $H^1(B_0(2)) \subset L^6(B_0(2))$ and $C_{ellip} \in (0, \infty)$ is the best constant in the estimate

$$\|\nabla U(\cdot, 0)\|_{L^2(B_0(2))} \leq C_{ellip} (\|\Omega(\cdot, 0)\|_{L^2(B_0(4))} + \|U(\cdot, 0)\|_{L^2(B_0(4))})$$

²³Based on the energy method, Lebesgue interpolation, Sobolev embedding, Hölder's inequality and Young's inequality.

for weak solutions to

$$-\Delta U(\cdot, 0) = \nabla \times \Omega(\cdot, 0) \quad \text{in } B_0(4).$$

Next notice that from (17), if we have for M sufficiently large

$$\alpha^{-212} < (M^\flat)^{-212} < (1 + M)^{-101} < S^\flat.$$

Using (88)-(89), together with a similar reasoning to Lemma 4, we can apply Theorem 3 with $M \geq M_4$ being sufficiently large. Specifically, we apply Remark 11 with $\beta = \alpha^{-212}$. This gives

$$(90) \quad \|\nabla U^\lambda\|_{L_t^\infty L_x^2(B_0(\frac{1}{6}) \times (\frac{255}{256}\alpha^{-212}, \alpha^{-212}))} \leq C_{univ} M^4 \alpha^{265}.$$

This implies that

$$(91) \quad \|\nabla u\|_{L_t^\infty L_x^2(B_0(\frac{(-s_0)^{\frac{1}{2}}\alpha^{106}}{6}) \times (\frac{s_0}{256}, 0))}^2 \leq \frac{C_{univ}^2 M^8 \alpha^{424}}{(-s_0)^{\frac{1}{2}}}.$$

Using that $s_0 < t'_0 < 0$ and that $S^\flat(M) = O(1)M^{-100}$, we have for M sufficiently large that

$$B_0(4\sqrt{S^\flat}^{-1}(-t'_0)^{\frac{1}{2}}) \subset B_0(\frac{1}{6}(-s_0)^{\frac{1}{2}}(M^\flat)^{106}).$$

So for $\frac{s_0}{256} < t'_0$, we see that (91) implies that

$$\int_{B_0(4\sqrt{S^\flat}^{-1}(-t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx \leq \frac{C_{univ}^2 M^8 \alpha^{424}}{(-s_0)^{\frac{1}{2}}} \leq \frac{M^8 \alpha^{525}}{(-t''_0)^{\frac{1}{2}}}.$$

Here, we used $s_0 \in [t''_0, \frac{t''_0}{8\alpha^{201}}]$. Thus,

$$\int_{B_0(4\sqrt{S^\flat}^{-1}(-t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx \leq M^2 (-t'_0)^{-\frac{1}{2}} \sqrt{S^\flat} \times \frac{\alpha^{525} M^6}{\sqrt{S^\flat}} \left(\frac{-t'_0}{-t''_0}\right)^{\frac{1}{2}}.$$

Now,

$$\frac{\alpha^{525} M^6}{\sqrt{S^\flat}} \left(\frac{-t'_0}{-t''_0}\right)^{\frac{1}{2}} \leq C_{univ} M^{56} \alpha^{525} \left(\frac{-t'_0}{-t''_0}\right)^{\frac{1}{2}}.$$

Therefore, if

$$(92) \quad C_{univ} M^{56} \alpha^{525} \left(\frac{-t'_0}{-t''_0}\right)^{\frac{1}{2}} < 1$$

we contradict (77). Thus, if (92) holds we must have that

$$\int_{B_0(4\alpha^{106}(-s_0)^{\frac{1}{2}})} |\omega(x, s_0)|^2 dx > \frac{(M+1)^2}{(-s_0)^{\frac{1}{2}} \alpha^{106}}$$

for all $s_0 \in [t''_0, \frac{t''_0}{8\alpha^{201}}]$ as desired. Note that (78) implies (92). \square

3.2. Proof of the main quantitative estimate in the Type I case. This part is devoted to the proof of Proposition 2. Following Tao [45], the idea of the proof is to transfer the concentration of the enstrophy at times t_0'' far away in the past to large-scale lower bounds for the enstrophy at time t_0 . This is done in Step 1-3 below. The last step, Step 4 below, consists in transferring the lower bound on the enstrophy at time t_0 to a lower bound for the L^3 norm at time t_0 and summing appropriate scales. In Step 5 we sum scales under the additional assumption (52).

Without loss of generality, we now take $t_0 = 0$. We also assume that $T = 1$. The general statement is obtained by scaling. Let $M \in [M_3, \infty)$ where M_3 is a constant in Lemma 4. In the course of the proof we will need to take M larger, always larger than universal constants. Let $u : \mathbb{R}^3 \times [-1, 0] \rightarrow \mathbb{R}^3$ be a ‘smooth solution with sufficient decay’²⁴ of the Navier-Stokes equations (1) in $I = [-1, 0]$ satisfying the Type I bound (48). Assume that there exists $t'_0 \in [-1, 0)$ such that t'_0 is not too close to -1 in the sense

$$0 < -t'_0 < C^\sharp M^{-548}$$

and such that the vorticity concentrates at time t'_0 in the following sense

$$(93) \quad \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t'_0)^{\frac{1}{2}})} |\omega(x, t'_0)|^2 dx > M^2(-t'_0)^{-\frac{1}{2}}\sqrt{S^\sharp},$$

where we recall that $S^\sharp = O(1)M^{-100}$. Lemma 4 then implies that

$$(94) \quad \int_{B_0(4\sqrt{S^\sharp}^{-1}(-t'')^{\frac{1}{2}})} |\omega(x, t'')|^2 dx > M^2(-t'')^{-\frac{1}{2}}\sqrt{S^\sharp}.$$

at any well-separated backward time $t'' \in [-1, t'_0]$ such that²⁵

$$(95) \quad \frac{1}{C^\sharp} M^{548} t'_0 > t''.$$

The rest of the proof relies on the Carleman inequalities of Proposition 34 and Proposition 35. These are the tools used to transfer the concentration information (94) from the time t'' to time 0 and from the small scales $B_0(4\sqrt{S^\sharp}^{-1}(-t'')^{\frac{1}{2}})$ to large scales.

Step 1: quantitative unique continuation. The purpose of this step is to prove the following estimate:

$$(96) \quad T_1^{\frac{1}{2}} e^{-\frac{O(1)M^{149}R^2}{T_1}} \lesssim \int_{-T_1}^{-\frac{T_1}{2}} \int_{B_0(2R) \setminus B_0(R/2)} |\omega(x, t)|^2 dx dt,$$

for all T_1 and R such that

$$(97) \quad \frac{2}{C^\sharp} M^{548} (-t'_0) < T_1 \leq \frac{1}{2} \quad \text{and} \quad R \geq M^{100} \left(\frac{T_1}{2}\right)^{\frac{1}{2}}.$$

Let t''_0 be such that (95) is satisfied with $t'' = \frac{t''_0}{2}$. Let $T_1 := -t''_0$ and $I_1 := (t''_0, t''_0 + \frac{T_1}{2}) = (-T_1, -\frac{T_1}{2}) \subset [-\frac{1}{2}, 0] \subset [-1, 0]$. Thus, we can apply Lemma 27 and Remark 28 with

²⁴See footnote 8.

²⁵Notice that the whole argument of Section 3.2 goes through assuming that (94) holds for almost any $t'' \in [-1, \frac{1}{C^\sharp} M^{548} t'_0]$.

$t_0 = 0$ and $T = 1$. The bound (261) in Remark 28 implies that there exists an epoch of regularity $I''_1 = [t''_1 - T''_1, t''_1] \subset I_1$ such that

$$(98) \quad T''_1 = |I''_1| = \frac{M^{-48}}{4C_{univ}^3} |I_1| = \frac{M^{-48}}{8C_{univ}^3} T_1$$

and for $j = 0, 1, 2$,

$$(99) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I''_1)} \leq \frac{1}{2^{j+1}} |I''_1|^{\frac{-(j+1)}{2}} = \frac{1}{2^{j+1}} (T''_1)^{\frac{-(j+1)}{2}}.$$

Let $T'''_1 := \frac{3}{4}T''_1$ and $s'' \in [t''_1 - \frac{T''_1}{4}, t''_1]$. Let $x_1 \in \mathbb{R}^3$ be such that $|x_1| \geq M^{100}(\frac{T_1}{2})^{\frac{1}{2}}$ and let $r_1 := M^{50}|x_1| \geq M^{150}(\frac{T_1}{2})^{\frac{1}{2}}$. Notice that for M large enough

$$(100) \quad r_1 := M^{50}|x_1| \geq M^{150}\left(\frac{T_1}{2}\right)^{\frac{1}{2}} \geq M^{99} \cdot 4\sqrt{S^\sharp}^{-1}(-t''_0)^{\frac{1}{2}}$$

and

$$r_1^2 \geq 4000T'''_1.$$

We apply the second Carleman inequality, Proposition 35 (quantitative unique continuation), on the cylinder $\mathcal{C}_1 = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t \in [0, T'''_1], |x| \leq r_1\}$ to the function $w : \mathbb{R}^3 \times [0, T'''_1] \rightarrow \mathbb{R}^3$, defined by for all $(x, t) \in \mathbb{R}^3 \times [0, T'''_1]$,

$$w(x, t) := \omega(x_1 + x, s'' - t).$$

Notice that the quantitative regularity (99) and the vorticity equation (37) implies that on \mathcal{C}_1

$$|(\partial_t + \Delta)w| \leq \frac{3}{16}T'''_1^{-1}|w| + \frac{\sqrt{3}}{4}T'''_1^{-\frac{1}{2}}|\nabla w|,$$

so that (312) is satisfied with $S = S_1 := T'''_1$ and $C_{Carl} = \frac{16}{3}$. Let

$$\hat{s}_1 = \frac{T'''_1}{20000}, \quad \check{s}_1 = M^{-150}T'''_1.$$

For M sufficiently large we have $0 < \check{s}_1 \leq \hat{s}_1 \leq \frac{T'''_1}{10000}$. Hence by (314) we have

$$(101) \quad Z_1 \lesssim e^{-\frac{r_1^2}{500\check{s}_1}} X_1 + (\hat{s}_1)^{\frac{3}{2}} \left(\frac{e\hat{s}_1}{\check{s}_1} \right)^{\frac{O(1)r_1^2}{\check{s}_1}} Y_1,$$

where

$$\begin{aligned} X_1 &:= \int_{s'' - T'''_1}^{s''} \int_{B_{x_1}(M^{50}|x_1|)} ((T'''_1)^{-1}|\omega|^2 + |\nabla\omega|^2) dx ds, \\ Y_1 &:= \int_{B_{x_1}(M^{50}|x_1|)} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx, \\ Z_1 &:= \int_{s'' - \frac{T'''_1}{20000}}^{s'' - \frac{T'''_1}{10000}} \int_{B_{x_1}(\frac{M^{50}|x_1|}{2})} ((T'''_1)^{-1}|\omega|^2 + |\nabla\omega|^2) e^{-\frac{|x-x_1|^2}{4(s''-s)}} dx ds. \end{aligned}$$

We first use the concentration (94) for times $s \in [s'' - \frac{T'''_1}{10000}, s'' - \frac{T'''_1}{20000}]$ to bound Z_1 from below. By (100), we have

$$B_0(4\sqrt{S^\sharp}^{-1}(-s)^{\frac{1}{2}}) \subset B_{x_1}(2|x_1|) \subset B_{x_1}\left(\frac{M^{50}|x_1|}{2}\right)$$

for all $s \in [s'' - \frac{T_1'''}{10000}, s'' - \frac{T_1'''}{20000}]$ and for M sufficiently large. We have

$$\begin{aligned}
Z_1 &\gtrsim \int_{s'' - \frac{T_1'''}{20000}}^{s'' - \frac{T_1'''}{10000}} \int_{B_0(4\sqrt{S^{\sharp}}^{-1}(-s)^{\frac{1}{2}})} (T_1''')^{-1} |\omega(x, s)|^2 dx ds e^{-\frac{O(1)|x_1|^2}{T_1'''}} \\
&\gtrsim \int_{s'' - \frac{T_1'''}{10000}}^{s'' - \frac{T_1'''}{20000}} M^{-48} (-s)^{-\frac{1}{2}} ds (T_1'')^{-1} e^{-\frac{O(1)|x_1|^2}{T_1''}} \\
&\gtrsim M^{-48} \frac{T_1'''}{(-s'' + \frac{T_1'''}{10000})^{\frac{1}{2}}} (T_1'')^{-1} e^{-\frac{O(1)|x_1|^2}{T_1''}} \\
&\gtrsim M^{-48} (T_1)^{-\frac{1}{2}} e^{-\frac{O(1)|x_1|^2}{T_1''}} \\
&\gtrsim M^{-48} (M^{48} T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)|x_1|^2}{T_1''}} \\
&= M^{-72} (T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)|x_1|^2}{T_1''}}.
\end{aligned}$$

Second, we bound from above X_1 . We rely on the quantitative regularity (99) to obtain

$$X_1 \lesssim (T_1'')^{-2} M^{150} |x_1|^3.$$

Hence,

$$\begin{aligned}
e^{-\frac{r_1^2}{500s_1}} X_1 &\lesssim (T_1'')^{-2} M^{150} |x_1|^3 e^{-\frac{O(1)M^{100}|x_1|^2}{T_1''}} \\
&\lesssim (T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)M^{100}|x_1|^2}{T_1''}}.
\end{aligned}$$

Third, for Y_1 we decompose and estimate as follows

$$\begin{aligned}
Y_1 &:= \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx \\
&\quad + \int_{B_{x_1}(M^{50}|x_1|) \setminus B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx \\
&\lesssim M^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \right. \\
&\quad \left. + \int_{B_{x_1}(M^{50}|x_1|) \setminus B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 e^{-\frac{O(1)M^{150}|x_1|^2}{T''_1}} dx \right) \\
&\lesssim M^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx + M^{150}|x_1|^3 (T''_1)^{-2} e^{-\frac{O(1)M^{150}|x_1|^2}{T''_1}} \right) \\
&\lesssim M^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx + (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{150}|x_1|^2}{T''_1}} \right),
\end{aligned}$$

where we used the quantitative regularity (99). Hence,

$$\begin{aligned}
(\hat{s}_1)^{\frac{3}{2}} \left(\frac{e\hat{s}_1}{\check{s}_1} \right)^{\frac{O(1)r_1^2}{\check{s}_1}} Y_1 &\lesssim (T''_1)^{\frac{3}{2}} e^{\frac{O(1)M^{100}|x_1|^2}{T''_1} \log(\frac{eM^{150}}{200000})} Y_1 \\
&\lesssim M^{225} e^{\frac{O(1)M^{101}|x_1|^2}{T''_1}} \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \\
&\quad + M^{225} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{150}|x_1|^2}{T''_1}}.
\end{aligned}$$

Gathering these bounds and combining with (101) yields

$$\begin{aligned}
M^{-72} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)|x_1|^2}{T''_1}} &\lesssim (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{100}|x_1|^2}{T''_1}} \\
&\quad + M^{225} e^{\frac{O(1)M^{101}|x_1|^2}{T''_1}} \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx + M^{225} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{150}|x_1|^2}{T''_1}}.
\end{aligned}$$

Using (98) and $|x_1| \geq M^{100}(\frac{T_1}{2})^{\frac{1}{2}}$, we see that for M sufficiently large

$$M^{-297} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{101}|x_1|^2}{T''_1}} \lesssim \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx.$$

Hence, for all $s'' \in I''_1 = [t''_1 - \frac{T''_1}{4}, t''_1]$, for all $|x_1| \geq M^{100}(\frac{T_1}{2})^{\frac{1}{2}}$,

$$\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \gtrsim M^{-297} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)M^{101}|x_1|^2}{T''_1}}.$$

Let $R \geq M^{100}(\frac{T_1}{2})^{\frac{1}{2}}$ and $x_1 \in \mathbb{R}^3$ be such that $|x_1| = R$. Integrating in time $[t''_1 - \frac{T_1''}{4}, t''_1]$ yields the estimate

$$\begin{aligned} M^{-321} e^{O(1)M^{349}} T_1^{\frac{1}{2}} e^{-\frac{2O(1)M^{149}R^2}{T_1}} &\lesssim M^{-321} T_1^{\frac{1}{2}} e^{-\frac{O(1)M^{149}R^2}{T_1}} \\ &\lesssim \int_{t''_1 - \frac{T_1''}{4}}^{t''_1} \int_{B_0(2R) \setminus B_0(R/2)} |\omega(x, t)|^2 dx dt \end{aligned}$$

which yields the claim (96) of Step 1.

Step 2: quantitative backward uniqueness. The goal of this step and Step 3 below is to prove the following claim:

$$(102) \quad T_2^{-\frac{1}{2}} \exp(-\exp(M^{1021})) \lesssim \int_{B_0\left(\frac{3}{4}C(100)M^{1000}R'_2\right) \setminus B_0(2R'_2)} |\omega(x, 0)|^2 dx,$$

for all $\frac{8}{C^\sharp} M^{749}(-t'_0) < T_2 \leq 1$ and M sufficiently large. Here, R_2, R'_2 and $C(100)$ are as in (104)-(106). This is the key estimate for Step 4 below and the proof of Proposition 2.

We apply here the results of Section 6 for the quantitative existence of an annulus of regularity. Although the parameter μ in Section 6 is any positive real number, here we need to take μ sufficiently large in order to have a large enough annulus of quantitative regularity, and hence a large r_+ below in the application of the first Carleman inequality Proposition 34. To fix the ideas, we take $\mu = 100$.²⁶ Let T_1 and T_2 such that

$$(103) \quad \frac{8}{C^\sharp} M^{548+201}(-t'_0) \leq T_2 \leq 1 \quad \text{and} \quad T_1 := \frac{T_2}{4M^{201}}.²⁷$$

Let

$$(104) \quad R_2 := K^\sharp(T_2)^{\frac{1}{2}},$$

for a universal constant $K^\sharp \geq 1$ to be chosen sufficiently large below. In particular it is chosen in Step 3 such that (123) holds, which makes it possible to absorb the upper bound (122) of X_3 in the left hand side of (120). By Corollary 21, for $M \geq M_1(100)$ there exists a scale

$$(105) \quad 2R_2 \leq R'_2 \leq 2R_2 \exp(C(100)M^{1020})$$

and a good cylindrical annulus

$$(106) \quad \mathcal{A}_2 := \{R'_2 < |x| < c(100)M^{1000}R'_2\} \times \left(-\frac{T_2}{32}, 0\right)$$

such that for $j = 0, 1$,

$$(107) \quad \begin{aligned} \|\nabla^j u\|_{L^\infty(\mathcal{A}_2)} &\leq 2^{\frac{j+1}{2}} \bar{C}_j C(100) M^{-300} T_2^{-\frac{j+1}{2}}, \\ \|\nabla \omega\|_{L^\infty(\mathcal{A}_2)} &\leq 2^{\frac{3}{2}} \bar{C}_2 C(100) M^{-300} T_2^{-\frac{3}{2}}. \end{aligned}$$

We apply now the quantitative backward uniqueness, Proposition 34 to the function $w : \mathbb{R}^3 \times [0, \frac{T_2}{M^{201}}] \rightarrow \mathbb{R}^3$ defined by for all $(x, t) \in \mathbb{R}^3 \times [0, \frac{T_2}{M^{201}}]$,

$$w(x, t) = \omega(x, -t).$$

²⁶More specifically, we see that μ is chosen so that $10\mu > 350$ in order to obtain (111) from (109) and (110).

²⁷The reason for this is to ensure we can apply Step 1 to get a lower bound (110) for Z_2 .

An important remark is that although we have a large cylindrical annulus of quantitative regularity \mathcal{A}_2 , we apply the Carleman estimate on a much smaller annulus, namely

$$(108) \quad \tilde{\mathcal{A}}_2 := \left\{ 4R'_2 < |x| < \frac{c(100)}{4} M^{1000} R'_2 \right\} \times \left(-\frac{T_2}{M^{201}}, 0 \right).$$

Choosing M sufficiently large such that $2\bar{C}_j C(100) M^{-300} \leq 1$ and $2^{\frac{3}{2}} \bar{C}_2 C(100) M^{-300} \leq 1$, we see that the bounds (107) imply that the differential inequality (309) is satisfied with $S = S_2 := \frac{T_2}{M^{201}}$ and $C_{Carl} = M^{201}$. Take

$$r_- = 4R'_2, \quad r_+ = \frac{1}{4} c(100) M^{1000} R'_2.$$

Then,

$$B_0(160R'_2) \setminus B_0(40R'_2) = B_0(40r_-) \setminus B_0(10r_-) \subset \left\{ 40R'_2 < |x| < \frac{c(100)}{8} M^{1000} R'_2 \right\}$$

on condition that M is sufficiently large: one needs $c(100) M^{1000} > 1280$. Note also that

$$r_-^2 = 16(R'_2)^2 \geq 64R'_2 = 64(K^\sharp)^2 T_2 > 64T_2 > 4SC_{Carl}.$$

By (311), we get

$$(109) \quad Z_2 \lesssim e^{-\frac{O(1)M^{1000}(R'_2)^2}{T_2}} (X_2 + e^{\frac{O(1)M^{2000}(R'_2)^2}{T_2}} Y_2),$$

where

$$\begin{aligned} X_2 &:= \int_{-\frac{T_2}{M^{201}}}^0 \int_{r_- \leq |x| \leq r_+} e^{\frac{4|x|^2}{T_2}} (M^{201} T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt, \\ Y_2 &:= \int_{r_- \leq |x| \leq r_+} |\omega(x, 0)|^2 dx, \\ Z_2 &:= \int_{-\frac{T_2}{4M^{201}}}^{10r_-} \int_{10r_- \leq |x| \leq \frac{r_+}{2}} (M^{201} T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt. \end{aligned}$$

Thanks to the separation condition (103) and to the fact that for M large enough (104) implies

$$20r_- \geq 10R'_2 \geq 20R_2 = 20K^\sharp T_2^{\frac{1}{2}} \geq M^{100} \left(\frac{T_2}{8M^{201}} \right)^{\frac{1}{2}} = M^{100} \left(\frac{T_1}{2} \right)^{\frac{1}{2}},$$

we can apply the concentration result of Step 1, taking there $T_1 = \frac{T_2}{4M^{201}} = \frac{S_2}{4}$ and $R = 20r_-$. By (96) we have that

$$(110) \quad Z_2 \gtrsim M^{201} \left(\frac{T_2}{4M^{201}} \right)^{\frac{1}{2}} e^{-\frac{O(1)M^{350}(R'_2)^2}{T_2}} T_2^{-1} \gtrsim T_2^{-\frac{1}{2}} e^{-\frac{O(1)M^{350}(R'_2)^2}{T_2}}.$$

Therefore, one of the following two lower bounds holds

$$(111) \quad T_2^{-\frac{1}{2}} \exp \left(\frac{O(1)M^{1000}(R'_2)^2}{T_2} \right) \lesssim X_2,$$

$$(112) \quad T_2^{-\frac{1}{2}} \exp(-\exp(M^{1021})) \lesssim e^{-\frac{O(1)M^{2000}(R'_2)^2}{T_2}} T_2^{-\frac{1}{2}} \lesssim Y_2,$$

where we used the upper bound (105) for (112). The bound (112) can be used directly in Step 4 below. On the contrary, if (111) holds more work needs to be done to transfer the lower bound on the enstrophy at time 0. This is the objective of Step 3 below.

Step 3: a final application of quantitative unique continuation. Assume that the bound (111) holds. We will apply the pigeonhole principle three times successively in order to end up in a situation where we can rely on the quantitative unique continuation to get a lower bound at time 0. We first remark that this with the definition (108) of the annulus $\tilde{\mathcal{A}}_2$ implies the following lower bound

$$\begin{aligned} & T_2^{-\frac{1}{2}} \exp \left(\frac{O(1)M^{1000}(R'_2)^2}{T_2} \right) \\ & \lesssim \int_{-\frac{T_2}{M^{201}}}^0 \int_{4R'_2 \leq |x| \leq \frac{1}{4}c(100)M^{1000}R'_2} e^{\frac{4|x|^2}{T_2}} (M^{201}T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt. \end{aligned}$$

By the pigeonhole principle, there exists

$$(113) \quad 8R'_2 \leq R_3 \leq \frac{1}{2}c(100)M^{1000}R'_2$$

such that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{4R_3^2}{T_2} \right) \lesssim \int_{-\frac{T_2}{M^{201}}}^0 \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt.$$

Using the bounds (107), we have that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{4R_3^2}{T_2} \right) \lesssim \int_{-\frac{T_2}{M^{201}}}^{-\exp(-\frac{8R_3^2}{T_2})T_2} \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt.$$

By the pigeonhole principle, there exists

$$(114) \quad \frac{1}{2} \exp \left(-\frac{8R_3^2}{T_2} \right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}}$$

such that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{5R_3^2}{T_2} \right) \lesssim \int_{2t_3}^{t_3} \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt.$$

We finally cover the annulus $B_0(R_3) \setminus B_0(\frac{R_3}{2})$ with

$$O(1) \frac{R_3^3}{(-t_3)^{\frac{3}{2}}} \lesssim \frac{R_3^3}{T_2^{\frac{3}{2}}} \exp \left(\frac{12R_3^2}{T_2} \right) \lesssim \exp \left(\frac{13R_3^2}{T_2} \right)$$

balls of radius $(-t_3)^{\frac{1}{2}}$, and apply the pigeonhole principle a third time to find that there exists $x_3 \in B_0(R_3) \setminus B_0(\frac{R_3}{2})$ such that

$$(115) \quad T_2^{-\frac{1}{2}} \exp \left(-\frac{18R_3^2}{T_2} \right) \lesssim \int_{2t_3}^{t_3} \int_{B_{x_3}((-t_3)^{\frac{1}{2}})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt.$$

We apply now the second Carleman inequality, Proposition 35, to the function $w : \mathbb{R}^3 \times [0, -20000t_3] \rightarrow \mathbb{R}^3$ defined by for all $(x, t) \in \mathbb{R}^3 \times [0, -20000t_3]$,

$$w(x, t) = \omega(x + x_3, -t).$$

Let $S_3 := -20000t_3$. We take²⁸

$$(116) \quad r_3 := 1000R_3 \left(-\frac{t_3}{T_2} \right)^{\frac{1}{2}}, \quad \hat{s}_3 = \check{s}_3 = -t_3.$$

Notice that due to (104)-(105) and (113), we have that

$$(117) \quad r_3^2 = 10^6 R_3^2 \left(-\frac{t_3}{T_2} \right) \geq (2.56 \times 10^8)(K^\sharp)^2(-t_3) \geq 4000S_3 = (8 \times 10^7)(-t_3),$$

$$(118) \quad \frac{r_3}{2} \geq 8000R_2 \left(-\frac{t_3}{T_2} \right)^{\frac{1}{2}} = 8000K^\sharp(-t_3)^{\frac{1}{2}} > (-t_3)^{\frac{1}{2}},$$

so that (313) is satisfied. Furthermore, from (114) we have

$$\frac{|x_3|}{2} \geq \frac{R_3}{4} \geq 1000R_3 \left(\frac{1}{M^{201}} \right)^{\frac{1}{2}} \geq r_3.$$

Thus

$$(119) \quad \begin{aligned} B_{x_3}((-t_3)^{\frac{1}{2}}) &\subset B_{x_3}(\frac{r_3}{2}) \subset B_{x_3}(r_3) \subset B_{x_3}\left(\frac{|x_3|}{2}\right) \\ &\subset \left\{ \frac{R_3}{4} < |y| < \frac{3}{2}R_3 \right\} \subset \left\{ 2R'_2 < |y| < \frac{3}{4}c(100)M^{1000}R'_2 \right\}. \end{aligned}$$

Moreover,

$$0 \leq \hat{s}_3 = \check{s}_3 = -t_3 \leq -2t_3 = \frac{S_3}{10^4}.$$

By (114), we see that for M large enough $S_3 \leq \frac{T_2}{32}$, hence the bounds (107) imply that the differential inequality (309) is satisfied on $B_0(r) \times [0, S]$ with $S = S_3$, $r = r_3$ and $C_{Carl} = 1$. Therefore, by (314) we have

$$(120) \quad Z_3 \leq C_{univ} e^{\frac{r_3^2}{500t_3}} X_3 + C_{univ} (-t_3)^{\frac{3}{2}} e^{-\frac{O(1)r_3^2}{t_3}} Y_3,$$

where

$$\begin{aligned} X_3 &:= \int_{-S_3}^0 \int_{B_{x_3}(r_3)} (S_3^{-1}|\omega|^2 + |\nabla \omega|^2) dx dt, \quad Y_3 := \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 (-t_3)^{-\frac{3}{2}} e^{\frac{|x-x_3|^2}{4t_3}} dx, \\ Z_3 &:= \int_{2t_3}^{t_3} \int_{B_{x_3}(\frac{r_3}{2})} (S_3^{-1}|\omega|^2 + |\nabla \omega|^2) e^{\frac{|x-x_3|^2}{4t}} dx dt. \end{aligned}$$

Using (115) and $T_2^{-1} \leq S_3^{-1}$ we have

$$(121) \quad T_2^{-\frac{1}{2}} \exp\left(-\frac{18R_3^2}{T_2}\right) \lesssim \int_{2t_3}^{t_3} \int_{B_{x_3}((-t_3)^{\frac{1}{2}})} (T_2^{-1}|\omega|^2 + |\nabla \omega|^2) e^{\frac{|x-x_3|^2}{4t}} dx dt \leq Z_3$$

Using the bounds (107) along with (114), we find that

$$(122) \quad \begin{aligned} C_{univ} e^{\frac{r_3^2}{500t_3}} X_3 &\lesssim S_3^{-2} r_3^3 e^{\frac{r_3^2}{500t_3}} \lesssim (-t_3)^{-\frac{1}{2}} e^{\frac{r_3^2}{1000t_3}} \lesssim T_2^{-\frac{1}{2}} e^{\frac{4R_3^2}{T_2}} e^{\frac{r_3^2}{1000t_3}} \\ &\lesssim T_2^{-\frac{1}{2}} e^{-\frac{996R_3^2}{T_2}} \lesssim T_2^{-\frac{1}{2}} e^{-\frac{18R_3^2}{T_2}} e^{-\frac{978R_3^2}{T_2}} \leq C'_{univ} T_2^{-\frac{1}{2}} e^{-\frac{18R_3^2}{T_2}} e^{-978 \cdot 256(K^\sharp)^2}. \end{aligned}$$

²⁸We follow here an idea of Tao which enables to remove one exponential from the final estimate. This idea appears on his blog <https://terrytao.wordpress.com/2019/08/15/quantitative-bounds-for-critically-bounded-solutions-to-the-navier-stokes-equations/> in a comment dated December 28, 2019. See also footnote 29 and (124).

We choose K^\sharp sufficiently large so that

$$(123) \quad C'_{univ} e^{-978 \cdot 256(K^\sharp)^2} \leq \frac{1}{2},$$

where $C'_{univ} \in (0, \infty)$ is the universal constant appearing in the last inequality of (122). Therefore, the term in the right hand side of (122) is negligible with respect to the lower bound (121) of Z_3 . Combining now (120) with the lower bound (121), we obtain²⁹

$$\begin{aligned} T_2^{-\frac{1}{2}} \exp\left(-\frac{18R_3^2}{T_2}\right) &\lesssim \exp\left(-\frac{O(1)r_3^2}{t_3}\right) \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx \\ &\lesssim \exp\left(O(1)\frac{R_3^2}{T_2}\right) \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx. \end{aligned}$$

Hence,

$$T_2^{-\frac{1}{2}} \exp\left(-O(1)\frac{R_3^2}{T_2}\right) \lesssim \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx.$$

Using (104), (119) and the upper bound

$$R_3 \leq \frac{1}{2}c(100)M^{1000}R'_2 \leq c(100)M^{1000} \exp(C(100)M^{1020})R_2,$$

it follows that

$$(125) \quad T_2^{-\frac{1}{2}} \exp(-\exp(M^{1021})) \lesssim \int_{B_0\left(\frac{3}{4}C(100)M^{1000}R'_2\right) \setminus B_0(2R'_2)} |\omega(x, 0)|^2 dx.$$

Step 4, conclusion: summing the scales and lower bound for the global L^3 norm. The key estimate is (102). From (104)-(105), we see that the volume of $B_0\left(\frac{3}{4}C(100)M^{1000}R'_2\right) \setminus B_0(2R'_2)$ is less than or equal to $T_2^{\frac{3}{2}} \exp(M^{1021})$. By the pigeonhole principle, there exists $i \in \{1, 2, 3\}$ and

$$x_4 \in B_0\left(\frac{3}{4}C(100)M^{1000}R'_2\right) \setminus B_0(2R'_2) \text{ such that } |\omega_i(x_4, 0)| \geq 2T_2^{-1} \exp(-\exp(M^{1022})).$$

Let $r_4 := T_2^{\frac{1}{2}} \exp(-\exp(M^{1022}))$. Using (104)-(106), we see that $B_{r_4}(x_4) \times \{0\} \subset \mathcal{A}_2$. Thus the quantitative estimate (107) gives that

$$|\omega_i(x, 0)| \geq T_2^{-1} \exp(-\exp(M^{1022})) \text{ in } B_{r_4}(x_4)$$

²⁹Here one notices a key advantage of taking r_3 to depend linearly on $-t_3$ as in (116). Otherwise the trivial bound

$$(124) \quad \frac{r_3^2}{-t_3} \leq \frac{R_3^2}{4(-t_3)} \lesssim \exp\left(\frac{8R_3^3}{T_2}\right) \frac{R_3^2}{T_2},$$

where we used the lower bound (114) on $-t_3$, would lead one more exponential in the final estimate. Taking r_3 as in (116) is Tao's idea; see footnote 28.

and that $\omega_i(x, 0)$ has constant sign in $B_{r_4}(x_4)$. This along with Hölder's inequality yields that

$$\begin{aligned} T_2^{-1} \exp(-\exp(M^{1022})) &\leq \left| \int_{B_0(1)} \omega_i(x_4 - r_4 z, 0) \varphi(z) dz \right| \\ &\leq r_4^{-1} \left| \int_{B_0(1)} u(x_4 - r_4 z, 0) \nabla \times \varphi(z) dz \right| \\ &\leq r_4^{-2} \|u\|_{L^3(B_0(C(100)M^{1000}R'_2) \setminus B_0(R'_2))} \|\nabla \times \varphi\|_{L^{\frac{3}{2}}(B_0(1))} \end{aligned}$$

for a fixed positive $\varphi \in C_c^\infty(B_0(1))$. Hence, using (104)-(105) we get

$$(126) \quad \int_{B_0(\exp(M^{1023})T_2^{\frac{1}{2}}) \setminus B_0(T_2^{\frac{1}{2}})} |u(x, 0)|^3 dx \geq \exp(-\exp(M^{1023})),$$

for all $\frac{8}{C^\sharp} M^{749}(-t'_0) \leq T_2 \leq 1$. Next we divide into two cases.

Case 1: $-t'_0 > \frac{C^\sharp}{8} M^{-749} \exp(-2M^{1023})$

In this case, we use (126) with $T_2 = 1$ to immediately get (for M greater than a sufficiently large universal constant)

$$-t'_0 \geq \frac{C^\sharp}{8} M^{-749} \exp \left\{ -\exp(\exp(M^{1024})) \int_{B_0(\exp(M^{1023}))} |u(x, 0)|^3 dx \right\}.$$

Case 2: $-t'_0 \leq \frac{C^\sharp}{8} M^{-749} \exp(-2M^{1023})$

In this case we sum (126) on the

$$k := \lfloor \frac{1}{2} M^{-1023} \log(\frac{C^\sharp}{8} M^{-749} (-t'_0)^{-1}) \rfloor \geq 1$$

scales T_2 ,

$$\begin{aligned} \left(\frac{8}{C^\sharp} M^{749} (-t'_0) \right)^{\frac{1}{2}} &\leq \exp(M^{1023}) \left(\frac{8}{C^\sharp} M^{749} (-t'_0) \right)^{\frac{1}{2}} \\ &\leq \dots \leq \exp(kM^{1023}) \left(\frac{8}{C^\sharp} M^{749} (-t'_0) \right)^{\frac{1}{2}} \leq 1, \end{aligned}$$

we obtain that

$$\begin{aligned} &\exp(-\exp(M^{1024})) \log(\frac{C^\sharp}{8} M^{-749} (-t'_0)^{-1}) \\ &\leq \int_{B_0(\exp(M^{1023})) \setminus B_0((\frac{8}{C^\sharp} M^{749} (-t'_0))^{\frac{1}{2}})} |u(x, 0)|^3 dx \\ &\leq \int_{\mathbb{R}^3} |u(x, 0)|^3 dx. \end{aligned}$$

This gives

$$-t'_0 \geq \frac{C^\sharp}{8} M^{-749} \exp \left\{ -\exp(\exp(M^{1024})) \int_{B_0(\exp(M^{1023}))} |u(x, 0)|^3 dx \right\},$$

which was also obtained in Case 1 and hence applies in all cases. Defining

$$-s_0 := \frac{C^\sharp}{16} M^{-749} \exp \left\{ -\exp(\exp(M^{1024})) \int_{B_0(\exp(M^{1023}))} |u(x, 0)|^3 dx \right\},$$

we see we have the contrapositive of (93). In particular,

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(-s_0)^{\frac{1}{2}})} |\omega(x, s_0)|^2 dx \leq M^2 (-s_0)^{-\frac{1}{2}} \sqrt{S^\sharp}.$$

almost identical arguments to those utilized in the proof of Lemma 4, except using the bound (174) instead of (175), give

$$\|u\|_{L^\infty(B_0(C_2 M^{50}(-s_0)^{\frac{1}{2}}) \times (\frac{s_0}{4}, 0))} \leq \frac{C_1 M^{-23}}{(-s_0)^{\frac{1}{2}}}.$$

Step 5, conclusion: summing of scales under additional assumption (52).

Case 1: $-t'_0 > \frac{C^\sharp \lambda^2}{8} M^{-749} \exp(-6M^{1023})$

In this case, we use the additional assumption (52) to immediately get

$$-t'_0 > \frac{C^\sharp \lambda^2}{8} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(\lambda)} |u(x, 0)|^3 dx \right\}.$$

Case 2: $-t'_0 \leq \frac{C^\sharp \lambda^2}{8} M^{-749} \exp(-6M^{1023})$

First notice that in this case

$$M^{-1023} \log \left(\frac{C^\sharp \lambda^2}{8(-t'_0)} M^{-749} \right) \geq 6$$

which implies

$$(127) \quad k + 1 := \lfloor \frac{1}{2} M^{-1023} \log \left(\frac{C^\sharp \lambda^2}{8(-t'_0)} M^{-749} \right) \rfloor \geq 2,$$

$$(128) \quad k + 1 \geq \frac{1}{4} M^{-1023} \log \left(\frac{C^\sharp \lambda^2}{8(-t'_0)} M^{-749} \right)$$

and

$$(129) \quad \exp((k + 1) M^{1023}) \left(\frac{8}{C^\sharp} M^{749} (-t'_0)^{\frac{1}{2}} \right) \leq \lambda < \exp(M^{1023}).$$

In this case we sum (126) on the $k + 1 \geq 2$ scales T_2 ,

$$\begin{aligned} \left(\frac{8}{C^\sharp} M^{749} (-t'_0)^{\frac{1}{2}} \right)^{\frac{1}{2}} &\leq \exp(M^{1023}) \left(\frac{8}{C^\sharp} M^{749} (-t'_0)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\leq \dots \leq \exp(k M^{1023}) \left(\frac{8}{C^\sharp} M^{749} (-t'_0)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq 1. \end{aligned}$$

Using (128)-(129) we obtain

$$\begin{aligned}
& \exp(-\exp(M^{1024})) \frac{1}{4} M^{-1023} \log(\frac{C^\sharp \lambda^2}{8} M^{-749} (-t'_0)^{-1}) \\
& \leq \int_{B_0(\lambda) \setminus B_0\left(\left(\frac{8}{C^\sharp} M^{749} (-t'_0)\right)^{\frac{1}{2}}\right)} |u(x, 0)|^3 dx \\
& \lesssim \int_{\mathbb{R}^3} |u(x, 0)|^3 dx.
\end{aligned}$$

This gives

$$-t'_0 \geq \frac{C^\sharp \lambda^2}{8} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(\lambda)} |u(x, 0)|^3 dx \right\},$$

which was also obtained in Case 1 and hence applies in all cases.

Defining

$$-s_1 := \frac{C^\sharp \lambda^2}{16} M^{-749} \exp \left\{ -4M^{1023} \exp(\exp(M^{1024})) \int_{B_0(\lambda)} |u(x, 0)|^3 dx \right\},$$

we see we have the contrapositive of (93). In particular,

$$\int_{B_0(4\sqrt{S^\sharp}^{-1}(-s_1)^{\frac{1}{2}})} |\omega(x, s_1)|^2 dx \leq M^2 (-s_1)^{-\frac{1}{2}} \sqrt{S^\sharp}.$$

almost identical arguments to those utilized in the proof of Lemma 4, except using the bound (174) instead of (175), give

$$\|u\|_{L^\infty(B_0(C_2 M^{50} (-s_1)^{\frac{1}{2}}) \times (\frac{s_1}{4}, 0))} \leq \frac{C_1 M^{-23}}{(-s_1)^{\frac{1}{2}}}.$$

This concludes the proof of Proposition 2.

3.3. Proof of the main estimate in the time slices case. We give the full proof of Proposition 3 for the sake of completeness. Notice that the proof follows the same scheme as the proof of Proposition 2. In most estimates M is simply replaced by M^\flat , albeit with slightly different powers. However, since in Step 2 below concentration is needed on the very small time interval $[-\frac{T_2}{4(M^\flat)^{201}}, 0]$, some care is needed when applying Lemma 29 on the epoch of quantitative regularity and Lemma 6 on the backward propagation of concentration.

Let $M \in [M_4, \infty)$ where M_4 is a constant in Lemma 6. In the course of the proof we will need to take M larger, always larger than universal constants. Let $u : \mathbb{R}^3 \times [-1, 0] \rightarrow \mathbb{R}^3$ be a $C^\infty(\mathbb{R}^3 \times (-1, 0))$ finite-energy solution to the Navier-Stokes equations (1) in $I = [-1, 0]$. Assume that there exists $t_{(k)} \in [-1, 0]$ such that

$$(130) \quad t_{(k)} \uparrow 0 \quad \text{with} \quad \sup_k \|u(\cdot, t_{(k)})\|_{L^3(\mathbb{R}^3)} \leq M.$$

Selecting any “well-separated” subsequence (still denoted $t_{(k)}$) such that³⁰

$$(131) \quad \sup_k \frac{-t_{(k+1)}}{-t_{(k)}} < \exp(-2(M^\flat)^{1223}).$$

³⁰This separation condition is stronger than that of Lemma 6. This stronger condition is needed to sum disjoint annuli in Step 4 below.

For this well-separated subsequence, assume that there exists $j + 1$ such that the vorticity concentrates at time $t_{(j+1)}$ in the following sense

$$(132) \quad \int_{B_0(4\sqrt{S^\flat}^{-1}(-t_{(j+1)})^{\frac{1}{2}})} |\omega(x, t_{(j+1)})|^2 dx > M^2(-t_{(j+1)})^{-\frac{1}{2}}\sqrt{S^\flat}.$$

where we recall that $S^\flat = O(1)M^{-100}$. Fix $k \in \{1, 2, \dots, j\}$. Note that (131) implies that for M sufficiently large

$$\frac{-t_{(j+1)}}{-t_{(k)}} < (M^\flat)^{-1051}.$$

Lemma 6 then implies that the vorticity concentrates in the following sense

$$(133) \quad \int_{B_0(4(-s)^{\frac{1}{2}}(M^\flat)^{106})} |\omega(x, s)|^2 dx > \frac{(M+1)^2}{(-s)^{\frac{1}{2}}(M^\flat)^{106}}.$$

for any

$$(134) \quad s \in \left[t_{(k)}, \frac{t_{(k)}}{8(M^\flat)^{201}} \right].$$

Step 1: quantitative unique continuation. The purpose of this step is to prove the following estimate:

$$(135) \quad T_1^{\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{965}R^2}{T_1}} \lesssim \int_{-T_1}^{-\frac{T_1}{2}} \int_{B_0(2R) \setminus B_0(R/2)} |\omega(x, t)|^2 dx dt,$$

for all T_1, s_0 and R such that

$$(136) \quad s_0 \in \left[\frac{t_{(k)}}{2}, \frac{t_{(k)}}{4(M^\flat)^{201}} \right] \quad T_1 := -s_0 \quad \text{and} \quad R \geq (M^\flat)^{100} \left(\frac{T_1}{2} \right)^{\frac{1}{2}}.$$

Here, $k \in \{1, \dots, j\}$ is fixed. Let $I_1 := (-T_1, -\frac{T_1}{2}) \subset [\frac{t_{(k)}}{2}, \frac{t_{(k)}}{8(M^\flat)^{201}}] \subset [-1, 0]$. The bound (288) in Remark 30 implies that there exists an epoch of regularity $I_1'' = [t_1'', T_1''] \subset I_1$ such that

$$(137) \quad T_1'' = |I_1''| = \frac{(M^\flat)^{-864}}{4C_4^3} |I_1| = \frac{(M^\flat)^{-864}}{8C_4^3} T_1$$

and for $j = 0, 1, 2$,

$$(138) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I_1'')} \leq \frac{1}{2^{j+1}} |I_1''|^{-\frac{(j+1)}{2}} = \frac{1}{2^{j+1}} (T_1'')^{-\frac{(j+1)}{2}}.$$

Let $T_1''' := \frac{3}{4}T_1''$ and $s'' \in [t_1'' - \frac{T_1''}{4}, t_1'']$. Let $x_1 \in \mathbb{R}^3$ be such that $|x_1| \geq (M^\flat)^{100}(\frac{T_1}{2})^{\frac{1}{2}}$ and let $r_1 := (M^\flat)^{50}|x_1| \geq (M^\flat)^{150}(\frac{T_1}{2})^{\frac{1}{2}}$. Notice that for M large enough

$$(139) \quad \frac{(M^\flat)^7|x_1|}{2} \geq \frac{(M^\flat)^{107}}{2} \left(\frac{T_1}{2} \right)^{\frac{1}{2}} \geq 4(-s_0)^{\frac{1}{2}}(M^\flat)^{106}$$

and

$$r_1^2 \geq 4000T_1'''.$$

We apply the second Carleman inequality, Proposition 35 (quantitative unique continuation), on the cylinder $\mathcal{C}_1 = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t \in [0, T_1'''], |x| \leq r_1\}$ to the function $w : \mathbb{R}^3 \times [0, T_1'''] \rightarrow \mathbb{R}^3$, defined by for all $(x, t) \in \mathbb{R}^3 \times [0, T_1''']$,

$$w(x, t) := \omega(x_1 + x, s'' - t).$$

Notice that the quantitative regularity (138) and the vorticity equation (37) implies that on \mathcal{C}_1

$$|(\partial_t + \Delta)w| \leq \frac{3}{16}T_1''''^{-1}|w| + \frac{\sqrt{3}}{4}T_1''''^{-\frac{1}{2}}|\nabla w|,$$

so that (312) is satisfied with $S = S_1 := T_1'''$ and $C_{Carl} = \frac{16}{3}$. Let

$$\hat{s}_1 = \frac{T_1'''}{20000}, \quad \check{s}_1 = (M^\flat)^{-150}T_1'''.$$

For M sufficiently large we have $0 < \check{s}_1 \leq \hat{s}_1 \leq \frac{T_1'''}{10000}$. Hence by (314) we have

$$(140) \quad Z_1 \lesssim e^{-\frac{r_1^2}{500\check{s}_1}} X_1 + (\hat{s}_1)^{\frac{3}{2}} \left(\frac{e\hat{s}_1}{\check{s}_1} \right)^{\frac{O(1)r_1^2}{\check{s}_1}} Y_1,$$

where

$$\begin{aligned} X_1 &:= \int_{s'' - T_1'''}^{s''} \int_{B_{x_1}((M^\flat)^{50}|x_1|)} ((T_1''')^{-1}|\omega|^2 + |\nabla\omega|^2) dx ds, \\ Y_1 &:= \int_{B_{x_1}((M^\flat)^{50}|x_1|)} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx, \\ Z_1 &:= \int_{s'' - \frac{T_1'''}{20000}}^{s'' - \frac{T_1'''}{10000}} \int_{B_{x_1}(\frac{(M^\flat)^{50}|x_1|}{2})} ((T_1''')^{-1}|\omega|^2 + |\nabla\omega|^2) e^{-\frac{|x-x_1|^2}{4(s'' - s)}} dx ds. \end{aligned}$$

We first use the concentration (133) for times

$$s \in \left[s'' - \frac{T_1'''}{10000}, s'' - \frac{T_1'''}{20000} \right] \subset \left(s_0, \frac{s_0}{2} \right) \subset \left(t_{(k)}, \frac{t_{(k)}}{8(M^\flat)^{201}} \right)$$

to bound Z_1 from below. By (139), we have

$$B_0(4(-s)^{\frac{1}{2}}(M^\flat)^{106}) \subset B_0(4(-s_0)^{\frac{1}{2}}(M^\flat)^{106}) \subset B_0\left(\frac{(M^\flat)^7|x_1|}{2}\right) \subset B_{x_1}((M^\flat)^7|x_1|)$$

for all $s \in [s'' - \frac{T_1'''}{10000}, s'' - \frac{T_1'''}{20000}]$ and for M sufficiently large. Hence, we have

$$\begin{aligned}
Z_1 &\gtrsim \int_{s'' - \frac{T_1'''}{10000}}^{s'' - \frac{T_1'''}{20000}} \int_{s'' - \frac{T_1'''}{10000} B_0(4(-s)^{\frac{1}{2}}(M^b)^{106})}^{(T_1''')^{-1}|\omega(x, s)|^2} dx ds e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1'''}} \\
&\gtrsim \int_{s'' - \frac{T_1'''}{10000}}^{s'' - \frac{T_1'''}{20000}} (M^b)^{-106} (-s)^{-\frac{1}{2}} ds (T_1'')^{-1} e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1''}} \\
&\gtrsim (M^b)^{-106} \frac{T_1'''}{(-s'' + \frac{T_1'''}{10000})^{\frac{1}{2}}} (T_1'')^{-1} e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1''}} \\
&\gtrsim (M^b)^{-106} (T_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1''}} \\
&\gtrsim (M^b)^{-106} ((M^b)^{864} T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1''}} \\
&= (M^b)^{-538} (T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)(M^b)^{14}|x_1|^2}{T_1''}}.
\end{aligned}$$

Second, we bound from above X_1 . We rely on the quantitative regularity (138) to obtain

$$X_1 \lesssim (T_1'')^{-2} (M^b)^{150} |x_1|^3.$$

Hence,

$$\begin{aligned}
e^{-\frac{r_1^2}{500s_1}} X_1 &\lesssim (T_1'')^{-2} (M^b)^{150} |x_1|^3 e^{-\frac{O(1)(M^b)^{100}|x_1|^2}{T_1''}} \\
&\lesssim (T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)(M^b)^{100}|x_1|^2}{T_1''}}.
\end{aligned}$$

Third, for Y_1 we decompose and estimate as follows

$$\begin{aligned}
Y_1 &:= \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx \\
&\quad + \int_{B_{x_1}((M^\flat)^{50}|x_1|) \setminus B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 (\check{s}_1)^{-\frac{3}{2}} e^{-\frac{|x-x_1|^2}{4\check{s}_1}} dx \\
&\lesssim (M^\flat)^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \right. \\
&\quad \left. + \int_{B_{x_1}((M^\flat)^{50}|x_1|) \setminus B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 e^{-\frac{O(1)(M^\flat)^{150}|x_1|^2}{T''_1}} dx \right) \\
&\lesssim (M^\flat)^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx + (M^\flat)^{150} |x_1|^3 (T''_1)^{-2} e^{-\frac{O(1)(M^\flat)^{150}|x_1|^2}{T''_1}} \right) \\
&\lesssim (M^\flat)^{225} (T''_1)^{-\frac{3}{2}} \left(\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx + (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{150}|x_1|^2}{T''_1}} \right),
\end{aligned}$$

where we used the quantitative regularity (138). Hence,

$$\begin{aligned}
(\hat{s}_1)^{\frac{3}{2}} \left(\frac{e\hat{s}_1}{\check{s}_1} \right)^{\frac{O(1)r_1^2}{\check{s}_1}} Y_1 &\lesssim (T''_1)^{\frac{3}{2}} e^{\frac{O(1)(M^\flat)^{100}|x_1|^2}{T''_1} \log(\frac{e(M^\flat)^{150}}{20000})} Y_1 \\
&\lesssim (M^\flat)^{225} e^{\frac{O(1)(M^\flat)^{101}|x_1|^2}{T''_1}} \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \\
&\quad + (M^\flat)^{225} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{150}|x_1|^2}{T''_1}}.
\end{aligned}$$

Gathering these bounds and combining with (140) yields

$$\begin{aligned}
(M^\flat)^{-538} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{14}|x_1|^2}{T''_1}} &\lesssim (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{100}|x_1|^2}{T''_1}} \\
&\quad + (M^\flat)^{225} e^{\frac{O(1)(M^\flat)^{101}|x_1|^2}{T''_1}} \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \\
&\quad + (M^\flat)^{225} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{150}|x_1|^2}{T''_1}},
\end{aligned}$$

which implies

$$(M^\flat)^{-763} (T''_1)^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{101}|x_1|^2}{T''_1}} \lesssim \int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx.$$

Hence, for all $s'' \in I_1'' = [t_1'' - \frac{T_1''}{4}, t_1'']$, for all $|x_1| \geq (M^\flat)^{100}(\frac{T_1}{2})^{\frac{1}{2}}$,

$$\int_{B_{x_1}(\frac{|x_1|}{2})} |\omega(x, s'')|^2 dx \gtrsim (M^\flat)^{-763}(T_1'')^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{101}|x_1|^2}{T_1''}}.$$

Let $R \geq (M^\flat)^{100}(\frac{T_1}{2})^{\frac{1}{2}}$ and $x_1 \in \mathbb{R}^3$ be such that $|x_1| = R$. Integrating in time $[t_1'' - \frac{T_1''}{4}, t_1'']$ yields the estimate

$$(M^\flat)^{-763}(T_1'')^{\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{101}R^2}{T_1''}} \lesssim \int_{t_1'' - \frac{T_1''}{4}}^{t_1''} \int_{B_0(2R) \setminus B_0(R/2)} |\omega(x, t)|^2 dx dt$$

which yields the claim (135) of Step 1.

Step 2: quantitative backward uniqueness. The goal of this step and Step 3 below is to prove the following claim:

$$(141) \quad T_2^{-\frac{1}{2}} \exp(-\exp((M^\flat)^{1221})) \lesssim \int_{B_0\left(\frac{3(M^\flat)^{1200}}{4}R'_2\right) \setminus B_0(2R'_2)} |\omega(x, 0)|^2 dx,$$

for $T_2 = -t_{(k)}$ with $k \in \{1, \dots, j\}$. Here, R_2 and R'_2 are as in (143)-(144). This is the key estimate for Step 4 below and the proof of Proposition 3.

We apply here the results of Section 6 for the quantitative existence of an annulus of regularity. Although the parameter μ in Section 6 is any positive real number, here we need to take μ sufficiently large in order to have a large enough annulus of quantitative regularity, and hence a large r_+ below in the application of the first Carleman inequality, Proposition 34. To fix the ideas, we take $\mu = 120$. Let T_1 and T_2 such that

$$(142) \quad T_2 := -t_{(k)} \quad \text{and} \quad T_1 := \frac{T_2}{4(M^\flat)^{201}}.$$

Let

$$(143) \quad R_2 := K^\flat(T_2)^{\frac{1}{2}}$$

for a universal constant $K^\flat \geq 1$ to be chosen sufficiently large below. In particular it is chose such that (162) holds. By Corollary 23 applied on the epoch $(t_{(k)}, 0)$, for $M \geq M_2(120)$ there exists a scale

$$(144) \quad 2R_2 \leq R'_2 \leq 2R_2 \exp(C(120)(M^\flat)^{1220})$$

and a good cylindrical annulus

$$(145) \quad \mathcal{A}_2 := \{R'_2 < |x| < (M^\flat)^{1200}R'_2\} \times \left(-\frac{T_2}{32}, 0\right)$$

such that for $j = 0, 1$,

$$(146) \quad \begin{aligned} \|\nabla^j u\|_{L^\infty(\mathcal{A}_2)} &\leq 2^{j+1}\bar{C}_j(M^\flat)^{-360}T_2^{-\frac{j+1}{2}}, \\ \|\nabla\omega\|_{L^\infty(\mathcal{A}_2)} &\leq 2^{\frac{3}{2}}\bar{C}_2(M^\flat)^{-360}T_2^{-\frac{3}{2}}. \end{aligned}$$

We apply now the quantitative backward uniqueness, Proposition 34 to the function $w : \mathbb{R}^3 \times [0, \frac{T_2}{(M^\flat)^{201}}] \rightarrow \mathbb{R}^3$ defined by for all $(x, t) \in \mathbb{R}^3 \times [0, \frac{T_2}{(M^\flat)^{201}}]$,

$$w(x, t) = \omega(x, -t).$$

An important remark is that although we have a large cylindrical annulus of quantitative regularity \mathcal{A}_2 , we apply the Carleman estimate on a much smaller annulus, namely

$$(147) \quad \tilde{\mathcal{A}}_2 := \left\{ 4R'_2 < |x| < \frac{(M^\flat)^{1200}}{4} R'_2 \right\} \times \left(-\frac{T_2}{(M^\flat)^{201}}, 0 \right).$$

The reason for this is to ensure we can apply Step 1 to get a lower bound (149) for Z_2 .

Choosing M sufficiently large such that $2\bar{C}_j(M^\flat)^{-360} \leq 1$ and $2^{\frac{3}{2}}\bar{C}_2(M^\flat)^{-360} \leq 1$, we see that the bounds (146) imply that the differential inequality (309) is satisfied with $S = S_2 := \frac{T_2}{(M^\flat)^{201}}$ and $C_{Carl} = (M^\flat)^{201}$. Take

$$r_- = 4R'_2, \quad r_+ = \frac{1}{4}(M^\flat)^{1200}R'_2.$$

Then,

$$B_0(160R'_2) \setminus B_0(40R'_2) = B_0(40r_-) \setminus B_0(10r_-) \subset \left\{ 40R'_2 < |x| < \frac{1}{8}(M^\flat)^{1200}R'_2 \right\}$$

on condition that M is sufficiently large: one needs $(M^\flat)^{1200} > 1280$. By (311), we get

$$(148) \quad Z_2 \lesssim e^{-\frac{O(1)(M^\flat)^{1200}(R'_2)^2}{T_2}} (X_2 + e^{\frac{O(1)(M^\flat)^{2400}(R'_2)^2}{T_2}} Y_2),$$

where

$$\begin{aligned} X_2 &:= \int_{-\frac{T_2}{(M^\flat)^{201}}}^{r_-} \int_{r_- \leq |x| \leq r_+} e^{\frac{4|x|^2}{T_2}} ((M^\flat)^{201} T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt, \\ Y_2 &:= \int_{r_- \leq |x| \leq r_+} |\omega(x, 0)|^2 dx, \\ Z_2 &:= \int_{-\frac{T_2}{4(M^\flat)^{201}}}^0 \int_{10r_- \leq |x| \leq \frac{r_+}{2}} ((M^\flat)^{201} T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt. \end{aligned}$$

For M large enough (143) implies

$$20r_- \geq 10R'_2 \geq 20R_2 = 20K^\flat(T_2)^{\frac{1}{2}} \geq (M^\flat)^{100} \left(\frac{T_2}{8(M^\flat)^{201}} \right)^{\frac{1}{2}} = (M^\flat)^{100} \left(\frac{T_1}{2} \right)^{\frac{1}{2}}.$$

Hence, we can apply the concentration result of Step 1, taking $T_1 = \frac{T_2}{4(M^\flat)^{201}} = \frac{-t_{(k)}}{4(M^\flat)^{201}} = \frac{S_2}{4}$ and $R = 20r_-$. By (135) we have that

$$(149) \quad Z_2 \gtrsim (M^\flat)^{201} \left(\frac{T_2}{4(M^\flat)^{201}} \right)^{\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{1166}(R'_2)^2}{T_2}} T_2^{-1} \gtrsim T_2^{-\frac{1}{2}} e^{-\frac{O(1)(M^\flat)^{1166}(R'_2)^2}{T_2}}.$$

Therefore, one of the following two lower bounds holds

$$(150) \quad T_2^{-\frac{1}{2}} \exp \left(\frac{O(1)M^{1200}(R'_2)^2}{T_2} \right) \lesssim X_2,$$

$$(151) \quad T_2^{-\frac{1}{2}} \exp(-\exp((M^\flat)^{1221})) \lesssim e^{-\frac{O(1)(M^\flat)^{2400}(R'_2)^2}{T_2}} T_2^{-\frac{1}{2}} \lesssim Y_2,$$

where we used the upper bound (144) for (151). The bound (151) can be used directly in Step 4 below. On the contrary, if (150) holds more work needs to be done to transfer the lower bound on the enstrophy at time 0. This is the objective of Step 3 below.

Step 3: a final application of quantitative unique continuation. Assume that the bound (150) holds. We will apply the pigeonhole principle three times successively in order to end

up in a situation where we can rely on the quantitative unique continuation to get a lower bound at time 0. We first remark that this with the definition (147) of the annulus $\tilde{\mathcal{A}}_2$ implies the following lower bound

$$\begin{aligned} & T_2^{-\frac{1}{2}} \exp \left(\frac{O(1)(M^\flat)^{1200}(R'_2)^2}{T_2} \right) \\ & \lesssim \int_{-\frac{T_2}{(M^\flat)^{201}}}^0 \int_{4R'_2 \leq |x| \leq \frac{(M^\flat)^{1200}}{4} R'_2} e^{\frac{4|x|^2}{T_2}} ((M^\flat)^{201} T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt. \end{aligned}$$

By the pigeonhole principle, there exists

$$(152) \quad 8R'_2 \leq R_3 \leq \frac{1}{2}(M^\flat)^{1200} R'_2$$

such that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{4R_3^2}{T_2} \right) \lesssim \int_{-\frac{T_2}{(M^\flat)^{201}}}^0 \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt.$$

Using the bounds (146), we have that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{4R_3^2}{T_2} \right) \lesssim \int_{-\frac{T_2}{(M^\flat)^{201}}}^{-\exp(-\frac{8R_3^2}{T_2}) T_2} \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt.$$

By the pigeonhole principle, there exists

$$(153) \quad \frac{1}{2} \exp \left(-\frac{8R_3^2}{T_2} \right) T_2 \leq -t_3 \leq \frac{T_2}{(M^\flat)^{201}}$$

such that

$$T_2^{-\frac{1}{2}} \exp \left(-\frac{5R_3^2}{T_2} \right) \lesssim \int_{2t_3}^{t_3} \int_{B_0(R_3) \setminus B_0(\frac{R_3}{2})} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt.$$

We finally cover the annulus $B_0(R_3) \setminus B_0(\frac{R_3}{2})$ with

$$O(1) \frac{R_3^3}{(-t_3)^{\frac{3}{2}}} \lesssim \frac{R_3^3}{T_2^{\frac{3}{2}}} \exp \left(\frac{12R_3^2}{T_2} \right) \lesssim \exp \left(\frac{13R_3^2}{T_2} \right)$$

spheres of radius $(-t_3)^{\frac{1}{2}}$, and apply the pigeonhole principle a third time to find that there exists $x_3 \in B_0(R_3) \setminus B_0(\frac{R_3}{2})$ such that

$$(154) \quad T_2^{-\frac{1}{2}} \exp \left(-\frac{18R_3^2}{T_2} \right) \lesssim \int_{2t_3}^{t_3} \int_{B_{x_3}((-t_3)^{\frac{1}{2}})} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt.$$

We apply now the second Carleman inequality, Proposition 35, to the function $w : \mathbb{R}^3 \times [0, -20000t_3] \rightarrow \mathbb{R}^3$ defined by for all $(x, t) \in \mathbb{R}^3 \times [0, -20000t_3]$,

$$w(x, t) = \omega(x + x_3, -t).$$

Let $S_3 := -20000t_3$. We take³¹

$$(155) \quad r_3 := 1000R_3 \left(-\frac{t_3}{T_2} \right)^{\frac{1}{2}}, \quad \hat{s}_3 = \check{s}_3 = -t_3.$$

Notice that due to (143)-(144) and (152), we have that

$$(156) \quad r_3^2 = 10^6 R_3^2 \left(-\frac{t_3}{T_2} \right) \geq (2.56 \times 10^8)(K^\flat)^2(-t_3) \geq 4000S_3 = (8 \times 10^7)(-t_3),$$

$$(157) \quad \frac{r_3}{2} \geq 8000R_2 \left(-\frac{t_3}{T_2} \right)^{\frac{1}{2}} = 8000K^\flat(-t_3)^{\frac{1}{2}} > (-t_3)^{\frac{1}{2}},$$

so that (313) is satisfied. Furthermore, from (153) we have

$$\frac{|x_3|}{2} \geq \frac{R_3}{4} \geq 1000R_3 \left(\frac{1}{(M^\flat)^{201}} \right)^{\frac{1}{2}} \geq r_3.$$

Thus

$$(158) \quad \begin{aligned} B_{x_3}((-t_3)^{\frac{1}{2}}) &\subset B_{x_3}(\frac{r_3}{2}) \subset B_{x_3}(r_3) \subset B_{x_3}\left(\frac{|x_3|}{2}\right) \\ &\subset \left\{ \frac{R_3}{4} < |y| < \frac{3}{2}R_3 \right\} \subset \left\{ 2R'_2 < |y| < \frac{3(M^\flat)^{1200}}{4}R'_2 \right\}. \end{aligned}$$

Moreover,

$$0 \leq \hat{s}_3 = \check{s}_3 = -t_3 \leq -2t_3 = \frac{S_3}{10^4}.$$

By (153), we see that for M large enough $S_3 \leq \frac{T_2}{32}$, hence the bounds (146) imply that the differential inequality (309) is satisfied with $S = S_3$ and $C_{Carl} = 1$. Therefore, by (314) we have

$$(159) \quad Z_3 \leq C_{univ} e^{\frac{r_3^2}{500t_3}} X_3 + C_{univ} (-t_3)^{\frac{3}{2}} e^{-\frac{O(1)r_3^2}{t_3}} Y_3,$$

where

$$\begin{aligned} X_3 &:= \int_{-S_3}^0 \int_{B_{x_3}(r_3)} (S_3^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt, \quad Y_3 := \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 (-t_3)^{-\frac{3}{2}} e^{\frac{|x-x_3|^2}{4t_3}} dx, \\ Z_3 &:= \int_{2t_3}^{t_3} \int_{B_{x_3}(\frac{r_3}{2})} (S_3^{-1}|\omega|^2 + |\nabla\omega|^2) e^{\frac{|x-x_3|^2}{4t}} dx dt. \end{aligned}$$

Using (154) and $T_2^{-1} \leq S_3^{-1}$ we have

$$(160) \quad T_2^{-\frac{1}{2}} \exp\left(-\frac{18R_3^2}{T_2}\right) \lesssim \int_{2t_3}^{t_3} \int_{B_{x_3}((-t_3)^{\frac{1}{2}})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) e^{\frac{|x-x_3|^2}{4t}} dx dt \leq Z_3$$

Using the bounds (146) along with (153), we find that as in (122),

$$(161) \quad C_{univ} e^{\frac{r_3^2}{500t_3}} X_3 \lesssim T_2^{-\frac{1}{2}} e^{-\frac{996R_3^2}{T_2}} \leq C'_{univ} e^{-\frac{18R_3^2}{T_2}} e^{-978 \cdot 256(K^\flat)^2}.$$

We choose K^\flat sufficiently large such that

$$(162) \quad C'_{univ} e^{-978 \cdot 256(K^\flat)^2} \leq \frac{1}{2},$$

³¹As in the proof of Proposition 2 above, we follow here again Tao's idea; see footnotes 28 and 29.

where $C'_{univ} \in (0, \infty)$ is the constant appearing in the last inequality of (161). Combining now (159) with the lower bound (160), we obtain

$$\begin{aligned} T_2^{-\frac{1}{2}} \exp\left(-\frac{18R_3^2}{T_2}\right) &\lesssim \exp\left(-\frac{O(1)r_3^2}{t_3}\right) \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx \\ &\lesssim \exp\left(O(1)\frac{R_3^2}{T_2}\right) \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx. \end{aligned}$$

Hence,

$$T_2^{-\frac{1}{2}} \exp\left(-O(1)\frac{R_3^2}{T_2}\right) \lesssim \int_{B_{x_3}(r_3)} |\omega(x, 0)|^2 dx.$$

Using (143), (158) and the upper bound

$$R_3 \leq \frac{1}{2}(M^\flat)^{1200} R'_2 \leq (M^\flat)^{1200} \exp(C(120)(M^\flat)^{1220}) R_2,$$

it follows that

$$(163) \quad T_2^{-\frac{1}{2}} \exp(-\exp((M^\flat)^{1221})) \lesssim \int_{B_0\left(\frac{3(M^\flat)^{1200}}{4}R'_2\right) \setminus B_0(2R'_2)} |\omega(x, 0)|^2 dx.$$

This together with the bounds (144) and (152) for R_3 proves the claim (141).

Step 4, conclusion: summing the scales and lower bound for the global L^3 norm.

The key estimate is (141). From (143)-(144), we see that the volume of the annulus $B_0\left(\frac{3(M^\flat)^{1200}}{4}R'_2\right) \setminus B_0(2R'_2)$ is less than or equal to $T_2^{\frac{3}{2}} \exp((M^\flat)^{1221})$. By the pigeon-hole principle, there exists $i \in \{1, 2, 3\}$ and

$$x_4 \in B_0\left(\frac{3(M^\flat)^{1200}}{4}R'_2\right) \setminus B_0(2R'_2) \text{ such that } |\omega_i(x_4, 0)| \geq 2T_2^{-1} \exp(-\exp((M^\flat)^{1222})).$$

Let $r_4 := T_2^{\frac{1}{2}} \exp(-\exp((M^\flat)^{1222}))$. Using (143)-(145), we see that $B_{r_4}(x_4) \times \{0\} \subset \mathcal{A}_2$. Thus the quantitative estimate (146) gives that

$$|\omega_i(x, 0)| \geq T_2^{-1} \exp(-\exp((M^\flat)^{1222})) \text{ in } B_{r_4}(x_4)$$

and that $\omega_i(x, 0)$ has constant sign in $B_{r_4}(x_4)$. This along with Hölder's inequality yields that

$$\begin{aligned} T_2^{-1} \exp(-\exp((M^\flat)^{1222})) &\leq \left| \int_{B_0(1)} \omega_i(x_4 - r_4 z, 0) \varphi(z) dz \right| \\ &\leq r_4^{-1} \left| \int_{B_0(1)} u(x_4 - r_4 z, 0) \nabla \times \varphi(z) dz \right| \\ &\leq r_4^{-2} \|u\|_{L^3(B_0((M^\flat)^{1200}R'_2) \setminus B_0(R'_2))} \|\nabla \times \varphi\|_{L^{\frac{3}{2}}(B_0(1))} \end{aligned}$$

for a fixed non-negative $\varphi \in C_c^\infty(B_0(1))$. Recalling (142)-(144) we conclude that,

$$(164) \quad \int_{B_0\left(\exp((M^\flat)^{1223})(-t_{(k)})^{\frac{1}{2}}\right) \setminus B_0\left((-t_{(k)})^{\frac{1}{2}}\right)} |u(x, 0)|^3 dx \geq \exp(-\exp((M^\flat)^{1223})),$$

for all $k \in \{1, \dots, j\}$. Note that (131) implies that for distinct k the spatial annuli in (164) are disjoint. Summing (164) over such k we obtain that

$$\begin{aligned} & \exp(-\exp((M^\flat)^{1223}))j \\ & \leq \int_{B_0(\exp((M^\flat)^{1223})(-t_1)^{\frac{1}{2}}) \setminus B_0((-t_{(j)})^{\frac{1}{2}})} |u(x, 0)|^3 dx \\ & \leq \int_{\mathbb{R}^3} |u(x, 0)|^3 dx. \end{aligned}$$

This gives

$$j \leq \exp(\exp((M^\flat)^{1223})) \int_{\mathbb{R}^3} |u(x, 0)|^3 dx \leq \exp(\exp((M^\flat)^{1224})).$$

This concludes the proof of Proposition 3.

4. FURTHER APPLICATIONS

4.1. Effective regularity criteria based on the local smallness of the $L^{3,\infty}$ at blow-up time.

Proposition 7. *For all $M \in [1, \infty)$ sufficiently large the following result holds true. Consider a suitable finite-energy solutions (u, p) to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ that satisfies the following Type I bound*

$$\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-1, 0))} \leq M.$$

Assume

$$(165) \quad \limsup_{r \rightarrow 0} \|u(\cdot, T^*)\|_{L^{3,\infty}(B_0(r))} \leq \exp(-\exp(M^{1023})).$$

Then, $(0, T^*)$ is a regular point.

Proof of Proposition 7. We argue by contradiction and assume $(0, T^*)$ is a singular point. The proof relies on two ingredients: (i) the concentration of the enstrophy norm near a Type I singularity, see Remark 5, (ii) the transfer of concentration at backward times to a lower bound at final time in Section 3.2. Contrary to the proof of Proposition 2 no summing of scales argument is required.

Without loss of generality, we assume that u solves Navier-Stokes on $\mathbb{R}^3 \times (-1, 0)$, that $(0, 0)$ is a singular point of u and that it satisfies the Type I bound $\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-1, 0))} \leq M$. First note that by Lebesgue interpolation (see Lemma 2.2 in [29] for example) we have that any suitable finite-energy solution with Type I bound is a mild solution on $\mathbb{R}^3 \times [-1, 0]$ with

$$(166) \quad u \in L_{x,t}^4(\mathbb{R}^3 \times (-1, 0)).$$

By Remark 5 and following Step 1-3 in Section 3.2, see in particular footnote 25, we can prove that

$$T_2^{-\frac{1}{2}} \exp(-\exp(M^{1021})) \lesssim \int_{B_0(\exp(M^{1021})(T_2)^{\frac{1}{2}}) \setminus B_0((T_2)^{\frac{1}{2}})} |\omega(x, 0)|^2 dx,$$

for all $0 < T_2 \leq 1$ and M sufficiently large. Here we used that $u \in L_{x,t}^4(\mathbb{R}^3 \times (-1, 0)) \cap L_t^\infty L^{3,\infty}(\mathbb{R}^3 \times (-1, 0))$, which allows an application of Corollary 21 and Lemma 27 in the course of following Steps 1-3.

Let $r \in (0, 1]$. Define $T_2 := r^2 \exp(-2M^{1023})$. Following Step 4 of Section 3.2 and using Hunt's inequality in Proposition 26 instead of Hölder's inequality, we then obtain that

$$(167) \quad \|u(\cdot, 0)\|_{L^{3,\infty}(B_0(r))} \geq \|u(\cdot, 0)\|_{L^{3,\infty}(B_0(\exp(M^{1023})(T_2)^{\frac{1}{2}}) \setminus B_0(T_2^{\frac{1}{2}}))} \geq 2 \exp(-\exp(M^{1023})).$$

This contradicts (165). \square

4.2. Estimate for the number of singular points in a Type I scenario. The technology developed in the present paper also enables us to give an effective bound for the number singularities in a Type I scenario. The following proposition and its corollary are effective versions of the results by Choe, Wolf and Yang [12] and Seregin [37].

Proposition 8. *Let $M \in [1, \infty)$ be sufficiently large and define*

$$(168) \quad \varepsilon(M) := \exp(-4 \exp(M^{1023})).$$

For all suitable finite-energy solutions³² (u, p) to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ that satisfy the following Type I bound

$$\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-1, 0))} \leq M,$$

the following result holds.

Let $x_0 \in \mathbb{R}^3$. Assume that there exists $r \in (0, \exp(M^{1021}))$,

$$(169) \quad \frac{1}{|B_{x_0}(r)|} \left| \left\{ x \in B_{x_0}(r) : |u(x, 0)| \geq \frac{\varepsilon(M)}{r} \right\} \right| \leq \varepsilon(M).$$

Then $(x_0, 0)$ is a regular space-time point.

This result is a variant of Theorem 1 in [12] and Proposition 1.3 in [37]. Our contribution is to provide the explicit formula (168) for $\varepsilon(M)$ in terms of M .

Corollary 9. *Let $T^* \in (0, \infty)$ and $M \in [1, \infty)$ be sufficiently large. Assume that (u, p) is a suitable finite-energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [0, T^*]$ that satisfies the following Type I bound*

$$\|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (0, T^*))} \leq M.$$

Then u has at most $\exp(\exp(M^{1024}))$ blow-up points at time T^ .*

Proof of Corollary 9. We follow here the argument of [37]. Without loss of generality we can assume that u is defined on $[-1, 0]$ rather than $[0, T^*]$. Let σ denote the set of all singular points at time 0. We take a finite collection of p points

$$(170) \quad x_1, \dots, x_p \in \sigma.$$

There exists $r \in (0, \exp(M^{1021}))$ such that $B_{x_i}(r) \cap B_{x_j}(r) = \emptyset$ for all $i \neq j$. Then, Proposition 8 implies that

$$\begin{aligned} |B_0(1)|\varepsilon(M)^4 p &< \sum_{i=1}^p \left(\frac{\varepsilon(M)}{r} \right)^3 \left| \left\{ x \in B_{x_i}(r) : |u(x, 0)| \geq \frac{\varepsilon(M)}{r} \right\} \right| \\ &\leq \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-1, 0))}^3 \leq M^3. \end{aligned}$$

This yields the result. \square

³²For a definition of *suitable finite-energy solutions* we refer to Section 1.4 ‘Notations’.

Proof of Proposition 8. Without loss of generality we assume that $x_0 = 0$. As in the proof of Proposition 7, we assume for contradiction that $(0, 0)$ is a singular point. Using verbatim reasoning as in the proof of Proposition 7, we see that the outcome of Step 1-3 in Section 3.2 holds, in particular estimate (102), which holds for all $0 < T_2 \leq 1$.

Arguing as in Step 4, and using the same notation, we get that there exists

$$x_4 \in B_0\left(\frac{3}{4}C(100)M^{1000}R'_2\right) \setminus B_0(2R'_2)$$

such that for $r_4 := T_2^{\frac{1}{2}} \exp(-\exp(M^{1022}))$,

$$\begin{aligned} \exp(-\exp(M^{1023})) &\leq \int_{B_{x_4}(r_4)} |u(x, 0)|^3 dx \\ &\leq T_2^{\frac{3}{2}} \exp(-3 \exp(M^{1022})) \sup_{B_{x_4}(r_4)} |u(x, 0)|^3. \end{aligned}$$

Hence, there exists $x_5 \in B_{x_4}(r_4)$ such that

$$|u(x_5, 0)| \geq 2T_2^{-\frac{1}{2}} \exp(-\frac{1}{3} \exp(M^{1023})).$$

By estimate (107) and the choice of M sufficiently large, we have $\|\nabla u\|_{L^\infty(\mathcal{A}_2)} \leq 1$ in the good annulus. Hence, for $r_5 := T_2^{\frac{1}{2}} \exp(-\exp(M^{1023}))$, the ball $B_{x_5}(r_5)$ is contained in \mathcal{A}_2 and

$$(171) \quad |u(x, 0)| \geq T_2^{-\frac{1}{2}} \exp(-\exp(M^{1023})) \quad \text{in } B_{x_5}(r_5),$$

for all $0 < T_2 \leq 1$. For $r := T_2^{\frac{1}{2}} \exp(M^{1021})$, we have $B_{x_5}(r_5) \subset \mathcal{A}_2 \subset B_0(r)$ and

$$(172) \quad |u(x, 0)| \geq \frac{\exp(-\exp(M^{1023}))}{r}.$$

Subsequently,

$$\begin{aligned} \frac{1}{|B_0(r)|} \left| \left\{ x \in B_0(r) : |u(x, 0)| \geq \frac{\exp(-\exp(M^{1023}))}{r} \right\} \right| \\ \geq \left(\frac{r_5}{r} \right)^3 > \exp(-4 \exp(M^{1023})). \end{aligned}$$

This holds for $r = T_2^{\frac{1}{2}} \exp(M^{1021})$ and every $0 < T_2 \leq 1$, which contradicts our assumption (169) on $u(\cdot, 0)$. \square

4.3. Effective regularity criteria based on the relative smallness of the L^3 norm at the final moment in time. Here we prove an effective regularity criteria for (u, p) a solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ based on the relative smallness of $\|u(\cdot, 0)\|_{L^3}$ vs. $\|u(\cdot, -1)\|_{L^3}$. A *non-effective* version of this result (without explicit quantitative bounds) is in [2, Theorem 4.1 (i)].

Proposition 10. *For all sufficiently large $M \in [1, \infty)$, we define M^b by (17). Let (u, p) be a suitable finite-energy solution to the Navier-Stokes equations (1) on $\mathbb{R}^3 \times [-1, 0]$. Assume that*

$$\|u(\cdot, -1)\|_{L^3(\mathbb{R}^3)} \leq M.$$

If

$$\|u(\cdot, 0)\|_{L^3(B_0(\exp((M^b)^{1221})) \setminus B_0(1))} \leq \exp(-\exp((M^b)^{1223})),$$

then $(0, 0)$ is a regular point.

Proof. Assume for contradiction that $(0, 0)$ is a singular point. Since (u, p) is a suitable finite-energy solution, there exists $\Sigma \subset (-1, 0)$ such that $|\Sigma| = 1$ and

- $\|\nabla u(\cdot, t')\|_{L^2(\mathbb{R}^3)} < \infty$ for all $t' \in \Sigma$,
- u satisfies the energy inequality on $[t', 0]$.

Then, arguing in a similar way as in the proof of Lemma 6, we show that for any $s_0 \in [-1, -\frac{1}{8\alpha^{201}}] \cap \Sigma$ the vorticity concentrates in the following sense,

$$\int_{B_0(4(-s_0)^{\frac{1}{2}}(M^\flat)^{106})} |\omega(x, s_0)|^2 dx > \frac{(M+1)^2}{(-s_0)^{\frac{1}{2}}(M^\flat)^{106}}.$$

Using $|\Sigma| = 1$ and then following Step 1-3 of Section 3.3 with one time scale, we obtain

$$(173) \quad T_2^{-\frac{1}{2}} \exp(-\exp((M^\flat)^{1221})) \lesssim \int_{B_0\left(\frac{3(M^\flat)^{1200}}{4}R'_2\right) \setminus B_0(2R'_2)} |\omega(x, 0)|^2 dx,$$

for

$$T_2 = 1, \quad R_2 := K^\flat(T_2)^{\frac{1}{2}}, \quad 2R_2 \leq R'_2 \leq 2R_2 \exp(C(120)(M^\flat)^{1220})$$

for K^\flat chosen such that (162) holds. Reasoning as in Step 4 of Section 3.3, we then obtain

$$\int_{B_0(\exp((M^\flat)^{1221})) \setminus B_0(1)} |u(x, 0)|^3 dx \geq 2 \exp(-\exp((M^\flat)^{1223})).$$

This concludes the proof. \square

5. MAIN TOOL 1: LOCAL-IN-SPACE SHORT-TIME SMOOTHING

The role of the next result is central in our paper.

Theorem 3 (local-in-space short-time smoothing). *There exists three universal constants C_* , M_5 , $N_1 \in [1, \infty)$. For all $M \geq M_5$, $N \geq N_1$, there exists a time $S_*(M, N) \in (0, \frac{1}{4}]$ such that the following holds. Consider an initial data u_0 satisfying the global control*

$$\|u_0\|_{L^2_{uloc}(\mathbb{R}^3)} \leq M, \quad \|u_0\|_{L^2(B_{\bar{x}}(1))} \xrightarrow{|\bar{x}| \rightarrow \infty} 0,$$

and, in addition, $u_0 \in L^6(B_0(2))$ with

$$\|u_0\|_{L^6(B_0(2))} \leq N.$$

Then, for any global-in-time added ‘global-in-time’ local energy solution³³ (u, p) to (1) with initial data u_0 we have the estimate

$$(174) \quad \|u\|_{L^\infty(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))} \leq C_* M^8 N^{19},$$

$$(175) \quad \|\nabla u\|_{L_t^\infty L_x^2(B_0(\frac{1}{4}) \times (\frac{15}{16}S_*, S_*))} \leq C_* M^{40} N^{98}.$$

Moreover, there is an explicit formula for S_* , see (206), and $S_*(M, N) = O(1)M^{-30}N^{-70}$.

Remark 11. As a conclusion to the hypothesis in the above Theorem, one can also obtain general version of (175). Specifically, for a local energy solution with $\beta \in (0, S_*)$, we get

$$(176) \quad \begin{aligned} \|\nabla u\|_{L_t^\infty L_x^2(B_0(\frac{1}{6}) \times (\frac{255}{256}\beta, \beta))} \\ \leq C_* \beta^{-\frac{3}{4}} \left(MN^2 + MN\beta^{-\frac{1}{4}} + M\beta^{-\frac{1}{2}} + M^2 + (N^2 + N\beta^{-\frac{1}{4}})^2 \right). \end{aligned}$$

³³We recall that the definition of a ‘local energy solution’ is given in footnote 8.

We will require this more general estimate. The computations producing it are identical to those used to show Theorem 3 and hence are omitted.

Corollary 12. *There exists three universal constants C_{**} , $M_6 \in [1, \infty)$. For all $M \geq M_6$ there exists a time $S_{**}(M) \in (0, \frac{1}{4}]$ with $S_{**}(M) = O(1)M^{-100}$ (given explicitly by (180)) such that the following holds. Suppose (u, p) is a ‘smooth solution with sufficient decay’³⁴ on $\mathbb{R}^3 \times [0, T']$ for any $T' \in (0, T)$ and satisfies*

$$(177) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times [0, T])} \leq M.$$

Furthermore, suppose there exists $t \in (0, T)$ such that

$$(178) \quad \|u(\cdot, t)\|_{L^\infty(B_0(2\sqrt{S_{**}}^{-1}(T-t)^{\frac{1}{2}}))} \leq \frac{M\sqrt{S_{**}}}{(T-t)^{\frac{1}{2}}}.$$

Then we conclude that

$$(179) \quad \|u\|_{L^\infty(B_0(\frac{1}{2}\sqrt{S_{**}}^{-1}(T-t)^{\frac{1}{2}}) \times (t + \frac{3}{4}(T-t), T)))} \leq \frac{C_{**}M^{27}\sqrt{S_{**}}}{(T-t)^{\frac{1}{2}}}.$$

Proof. We define $S_{**} \in (0, \frac{1}{4}]$ in the following way:

$$(180) \quad S_{**} = S_{**}(M) := S_*(C_{weak}M, |B_0(2)|^{\frac{1}{6}}M),$$

where S_* is the constant defined in Theorem 3 (see also the formula (206)). Define

$$r := \sqrt{S_{**}}^{-1}(T-t)^{\frac{1}{2}}$$

and rescale

$$(181) \quad U(y, s) := ru(ry, r^2s + t) \text{ for } (y, s) \in \mathbb{R}^3 \times (0, S_{**}).$$

Then assumptions (177)-(178) imply that

$$\|U(\cdot, 0)\|_{L^\infty(B_0(2))} \leq M \quad \text{and} \quad \|U\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (0, S_{**}))} \leq M.$$

Hence we have,

$$(182) \quad \|U(\cdot, 0)\|_{L^6(B_0(2))} \leq M|B_0(2)|^{\frac{1}{6}}$$

and

$$(183) \quad \|U(\cdot, 0)\|_{L_{uloc}^2(\mathbb{R}^3)} \leq C_{weak}M.$$

Here, $C_{weak} \in [1, \infty)$ is a universal constant from the embedding $L^{3,\infty}(\mathbb{R}^3) \subset L_{uloc}^2(\mathbb{R}^3)$. We then apply Theorem 3 to U and then rescale according to (181). This gives (179) as desired. \square

Theorem 3 is proven in Section 5.2 below. It relies on an ε -regularity result for suitable weak solutions³⁵ to the perturbed Navier-Stokes equations

$$(184) \quad \partial_t v - \Delta v + \nabla q = -v \cdot \nabla v - a \cdot \nabla v - \nabla \cdot (a \otimes v), \quad \nabla \cdot v = 0, \quad \nabla \cdot a = 0$$

around a subcritical drift $a \in L^m(Q_{(0,0)}(1))$, $m > 5$. We recover the result of Jia and Šverák [21, Theorem 2.2] by a Caffarelli, Kohn and Nirenberg scheme [9] already used in [6] for critical drifts. We also point out here that local-in-space short-time regularity estimates near locally critical initial data were recently proved in [22] using compactness arguments. Contrary to the critical case, here we can prove boundedness directly.

³⁴See footnote 8.

³⁵For a definition of suitable weak solutions for (184), we refer to [6, Definition 1].

Theorem 4 (epsilon-regularity around a subcritical drift). *There exists $C_{***} \in (0, \infty)$, for all $m \in (5, \infty]$, there exists $\varepsilon_*(m) \in (0, \infty)$ such that the following holds for all $\varepsilon \in (0, \varepsilon_*(m))$. Take any $a \in L^m(Q_{(0,0)}(1))$ and any suitable weak solution (v, q) to (184) satisfying*

$$(185) \quad \sup_{-1 < s < 0} \int_{B_0(1)} |v(x, s)|^2 dx + \int_{Q_{(0,0)}(1)} |\nabla v|^2 dx ds \leq \varepsilon^{\frac{5}{9}}.$$

Assume that

$$(186) \quad \|a\|_{L^m(Q_{(0,0)}(1))} \leq \varepsilon^{\frac{1}{9}},$$

$$(187) \quad \int_{Q_{(0,0)}(1)} |v|^3 + |q|^{\frac{3}{2}} dx ds \leq \varepsilon.$$

Then,

$$(188) \quad \sup_{(\bar{x}, t) \in Q_{(0,0)}(\frac{1}{2})} \sup_{r \in (0, \frac{1}{4}]} \int_{Q_{(\bar{x}, t)}(r)} |v|^3 dx ds \leq C_{***} \varepsilon^{\frac{2}{3}}.$$

This theorem is proved in Section 5.1 below. Notice that the smallness on the large-scale quantity (187) in $Q_{(0,0)}(1)$ is transferred to the L^∞ bound (192). In the following statement, we remove the smallness assumption (186) on the drift.

Corollary 13. *Let $m \in (5, \infty]$. Let C_{***} and ε_* be given by Theorem 4. For all $\varepsilon \in (0, \min(\varepsilon_*, 2^{-9}))$, for all $N \in [1, \infty)$, for all $a \in L^m(Q_{(0,0)}(1))$ and any suitable weak solution (v, q) to (184) satisfying*

$$(189) \quad \sup_{-1 < s < 0} \int_{B_0(1)} |v(x, s)|^2 dx + \int_{Q_{(0,0)}(1)} |\nabla v|^2 dx ds \leq N^{-\frac{1}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{6-\frac{5}{m}}{1-\frac{5}{m}}}.$$

Assume that

$$(190) \quad \|a\|_{L^m(Q_{(0,0)}(1))} \leq N,$$

$$(191) \quad \int_{Q_{(0,0)}(1)} |v|^3 + |q|^{\frac{3}{2}} dx ds \leq N^{-\frac{2}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{11-\frac{45}{m}}{1-\frac{5}{m}}}.$$

Then,

$$(192) \quad \|v\|_{L^\infty(Q_{(0,0)}(\frac{1}{2}))} \leq C_{***}^{\frac{1}{3}} N^{\frac{1}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{1-\frac{10}{m}}{1-\frac{5}{m}}}.$$

Proof of Corollary 13. We use a scaling argument as in [21, Theorem 2.2]. Let $(x_0, t_0) \in Q_{(0,0)}(\frac{1}{2})$ and define $R_0 \in (0, \infty)$ as

$$(193) \quad R_0 = N^{-\frac{1}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{1}{1-\frac{5}{m}}}.$$

Notice that due to $0 < \varepsilon < \min(\varepsilon_*, 2^{-9})$, we have $R_0 < \frac{1}{2}$, so that the following rescaling is well defined: for all $(y, s) \in Q_{(0,0)}(1)$,

$$V(y, s) := R_0 v(x_0 + R_0 y, t_0 + R_0^2 s), \quad Q(y, s) := R_0^2 q(x_0 + R_0 y, t_0 + R_0^2 s).$$

Then (V, Q) is a suitable weak solution to (184) with a drift b defined by

$$b(y, s) := R_0 a(x_0 + R_0 y, t_0 + R_0^2 s).$$

We have by our choice of R_0 in (193)

$$\|b\|_{L^m(Q_{(0,0)}(1))} \leq R_0^{1-\frac{5}{m}} \|a\|_{L^m(Q_{(x_0,t_0)}(R_0))} \leq R_0^{1-\frac{5}{m}} \|a\|_{L^m(Q_{(0,0)}(1))} \leq R_0^{1-\frac{5}{m}} N \leq \varepsilon^{\frac{1}{9}}$$

for the drift,

$$\begin{aligned} & \sup_{-1 < s < 0} \int_{B_0(1)} |V(y, s)|^2 dy + \int_{Q_{(0,0)}(1)} |\nabla V|^2 dyds \\ & \leq R_0^{-1} \left(\sup_{t_0 - R_0^2 < s < t_0} \int_{B_{x_0}(R_0)} |v(x, s)|^2 dx + \int_{Q_{(x_0,t_0)}(R_0)} |\nabla v|^2 dxds \right) \\ & \leq R_0^{-1} \left(\sup_{-1 < s < 0} \int_{B_0(1)} |v(x, s)|^2 dx + \int_{Q_{(0,0)}(1)} |\nabla v|^2 dxds \right) \\ & \leq R_0^{-1} N^{-\frac{1}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{6-\frac{5}{m}}{1-\frac{5}{m}}} \leq \varepsilon^{\frac{5}{9}} \end{aligned}$$

for the local energy and finally

$$\begin{aligned} & \int_{Q_{(0,0)}(1)} |V|^3 + |Q|^{\frac{3}{2}} dyds \\ & \leq R_0^{-2} \int_{Q_{(x_0,t_0)}(R_0)} |v|^3 + |q|^{\frac{3}{2}} dxds \\ & \leq R_0^{-2} \int_{Q_{(0,0)}(1)} |v|^3 + |q|^{\frac{3}{2}} dxds \\ & \leq R_0^{-2} N^{-\frac{2}{1-\frac{5}{m}}} \varepsilon^{\frac{1}{9} \cdot \frac{11-\frac{45}{m}}{1-\frac{5}{m}}} = \varepsilon. \end{aligned}$$

Therefore, (185), (186) and (187) are satisfied for (V, Q) , and hence,

$$\sup_{(\bar{x}, t) \in \overline{Q_{(0,0)}}(\frac{1}{2})} \sup_{r \in (0, \frac{1}{4}]} \int_{Q_{(\bar{x}, t)}(r)} |V|^3 dyds \leq C_{***} \varepsilon^{\frac{2}{3}}.$$

Rescaling, this gives

$$\sup_{(\bar{x}, t) \in Q_{(x_0,t_0)}(\frac{R_0}{2})} \sup_{r \in (0, \frac{R_0}{4}]} \int_{Q_{(\bar{x}, t)}(r)} |v|^3 dxds \leq C_{***} \varepsilon^{\frac{2}{3}} R_0^{-3},$$

hence the bound (192) by taking the supremum over $(x_0, t_0) \in Q_{(0,0)}(\frac{1}{2})$. This concludes the proof. \square

5.1. Sketch of the proof of Theorem 4. The proof follows almost verbatim the one of Theorem 3 in [6], provided the following modifications are made. We propagate the following

two bounds: for $r_k = 2^{-k}$,

$$(A_k) \quad \frac{1}{r_k^2} \int_{Q(\bar{x}, t)(r_k)} |v(x, s)|^3 dx ds + \frac{1}{r_k^{1+\kappa}} \int_{Q(\bar{x}, t)(r_k)} |q - (q)_{r_k}(s)|^{\frac{3}{2}} dx ds \leq \varepsilon^{\frac{2}{3}} r_k^3,$$

$$(B_k) \quad \sup_{t-r_k < s < t} \int_{B_{\bar{x}}(r_k)} |v(x, s)|^2 dx + \int_{Q(\bar{x}, t)(r_k)} |\nabla v|^2 dx ds \leq C_B \varepsilon^{\frac{2}{3}} r_k^3,$$

for a universal constant $C_B \in (0, \infty)$ chosen sufficiently large and $\kappa(m) \in (0, \infty)$ such that

$$0 < \kappa < \min \left(2, 3 - \frac{15}{m} \right).$$

One takes advantage of the subcriticality of the drift a in the following way:

$$\|a\|_{L^5(Q(\bar{x}, t)(r_k))} \lesssim r_k^{1-\frac{5}{m}} \|a\|_{L^m(Q(\bar{x}, t)(r_k))} \lesssim \varepsilon^{\frac{1}{9}} r_k^{1-\frac{5}{m}}.$$

This plays a key role in the estimate of I_4 and I_5 in Step 3, J_2 and J_4 in Step 4, using the same notations as in the proof of [6, Theorem 3]. The restriction $\kappa < 2$ comes from handling J_5 and J_6 , while the restriction $\kappa < 3 - \frac{15}{m}$ comes from bounding J_2 and J_4 .

5.2. Proof of Theorem 3. We fix $n = 6$ and $m = \frac{5n}{3} = 10$ in this proof. Let C_{***} and $\varepsilon_* = \varepsilon_*(10)$ be given by Theorem 4. Let also $k_0 = k_0(6)$ and $K_0 = K_0(6)$ be given by Proposition 31.

Let $M, N \in [1, \infty)$. Let $u_0 \in L^2_{uloc}(\mathbb{R}^3)$ such that $\|u_0\|_{L^2(B_{\bar{x}}(1))} \xrightarrow{|\bar{x}| \rightarrow \infty} 0$. We assume in addition that $u_0 \in L^6(B_0(2))$. Moreover,

$$\|u_0\|_{L^2_{uloc}(\mathbb{R}^3)} \leq M, \quad \|u_0\|_{L^6(B_0(2))} \leq N.$$

Let u be any local energy solution to (1) with such a data u_0 . The goal is to prove the local-in-space short-time smoothing for u stated in Theorem 3.

Step 1: decomposition of the initial data.

Lemma 14. *Let $u_0 \in L^2_{uloc}(\mathbb{R}^3)$ with, in addition, $u_0|_{B_0(2)} \in L^6(\mathbb{R}^3)$. Then, there exists a universal constant $K_2 \in [1, \infty)$, there exists $u_{0,a} \in L^6_\sigma(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$, $\text{supp}(u_{0,a}) \subset B_0(2)$, and $u_{0,b} \in L^2_{uloc}(\mathbb{R}^3)$ such that the following holds:*

$$\begin{aligned} u_0 &= u_{0,a} + u_{0,b}, \quad u_{0,a} = u_0 \text{ on } B_0\left(\frac{3}{2}\right), \quad \|u_{0,a}\|_{L^6} \leq K_2 \|u_0\|_{L^6(B_0(2))}, \\ \|u_{0,a}\|_{L^2} &\leq K_2 \|u_0\|_{L^2(B_0(2))} \quad \text{and} \quad \|u_{0,b}\|_{L^2_{uloc}} \leq K_2 \|u_0\|_{L^2_{uloc}}. \end{aligned}$$

Proof. The proof is standard using Bogovskii's operator [15, Chapter III.3]. We refer to [6] for a detailed proof. \square

Step 2: control of the local energy of the perturbation. We use the decomposition given by Lemma 14 for u_0 as above. Let a be the mild solution given by Proposition 31 associated to the data $u_{0,a} \in L^6(\mathbb{R}^3)$. The mild solution a exists at least on the time interval $(0, S_{mild}^a)$, where

$$S_{mild}^a := k_0 N^{-4}.$$

Moreover since $u_0 \in L^2_\sigma(\mathbb{R}^3)$, the mild solution a can be constructed to be a weak Leray-Hopf solution on $\mathbb{R}^3 \times (0, S_{mild}^a)$ and we have the global energy control³⁶

$$(194) \quad \sup_{s \in (0, S_{mild}^a)} \int_{\mathbb{R}^3} \frac{|a(x, s)|^2}{2} dx + \int_0^{S_{mild}^a} \int_{\mathbb{R}^3} |\nabla a|^2 dx ds \leq K'_0 N^2,$$

³⁶This can be inferred from arguments similar to [5, Section 3.1].

with $K'_0 \in [1, \infty)$ a universal constant. This and Calderón-Zygmund theory implies

$$(195) \quad \|q_a\|_{L^{\frac{5}{3}}(\mathbb{R}^3 \times (0, S_{mild}^a))} \leq K''_0 N^2,$$

where q_a is the pressure associated to a . Moreover, since u is a local energy solution with the initial data u_0 , Proposition 33 implies the following control of the local energy

$$(196) \quad \sup_{s \in (0, S_{locen}^u)} \sup_{\bar{x} \in \mathbb{R}^3} \int_{B_{\bar{x}}(1)} \frac{|u(x, s)|^2}{2} dx + \sup_{\bar{x} \in \mathbb{R}^3} \int_0^{S_{locen}^u} \int_{B_{\bar{x}}(1)} |\nabla u(x, s)|^2 dx ds \leq K_1 M^2,$$

where $S_{locen}^u(N) := k_1 \min(M^{-4}, 1)$. As a consequence, the perturbation $v = u - a$ is a local energy solution to (184)

$$(197) \quad \begin{aligned} & \sup_{s \in (0, S^v)} \sup_{\bar{x} \in \mathbb{R}^3} \int_{B_{\bar{x}}(1)} \frac{|v(x, s)|^2}{2} dx + \sup_{\bar{x} \in \mathbb{R}^3} \int_0^{S^v} \int_{B_{\bar{x}}(1)} |\nabla v(x, s)|^2 dx ds \\ & \leq K'_1(N^2 + M^2), \end{aligned}$$

where $K'_1 \in [1, \infty)$ is a universal constant and

$$(198) \quad \begin{aligned} S^v &= S^v(M, N) := \min\left(\frac{1}{4}, S_{mild}^a, S_{locen}^u\right) \\ &= \min\left(\frac{1}{4}, k_0 N^{-4}, k_1 M^{-4}, k_1\right). \end{aligned}$$

Moreover, we have the following pressure estimate

$$(199) \quad \|q - C_0(t)\|_{L^{\frac{5}{3}}(B_0(\frac{3}{2}) \times (0, S^v))} \leq K'_1(M^2 + N^2),$$

with a universal constant $K'_1 \in [1, \infty)$. This bound follows from (195), (305) and (308).

Step 3: smallness of the local energy in short time. Let $\phi \in C_c^\infty(\mathbb{R}^3)$ be a cut-off function such that

$$(200)$$

$$0 \leq \phi \leq 1, \quad \text{supp } \phi \subset B_0(\frac{3}{2}), \quad \phi = 1 \text{ on } B_0(1) \quad \text{and} \quad |\nabla(\phi^2)| + |\Delta(\phi^2)| \leq K_3,$$

where $K_3 \in [1, \infty)$. We estimate the local energy

$$E(t) := \sup_{s \in (0, t)} \int_{\mathbb{R}^3} |v(x, t)|^2 \phi^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \phi|^2 \phi^2 dx ds$$

for all $t \in (0, S^v)$. The local energy inequality gives

$$E(t) \leq I_1 + \dots + I_6,$$

with

$$\begin{aligned} I_1 &= \int_0^t \int_{\mathbb{R}^3} |v|^2 \Delta(\phi^2) dx ds, \quad I_2 = \int_0^t \int_{\mathbb{R}^3} |v|^2 v \cdot \nabla(\phi^2) dx ds, \\ I_3 &= 2 \int_0^t \int_{\mathbb{R}^3} (q - C_0(t)) v \cdot \nabla(\phi^2) dx ds, \quad I_4 = -2 \int_0^t \int_{\mathbb{R}^3} (a \cdot \nabla v) \cdot v \phi^2 dx ds, \\ I_5 &= 2 \int_0^t \int_{\mathbb{R}^3} (a \otimes v) : \nabla v \phi^2 dx ds \quad \text{and} \quad I_6 = 2 \int_0^t \int_{\mathbb{R}^3} (a \otimes v) : v \otimes \nabla(\phi^2) dx ds. \end{aligned}$$

Let $t \in (0, S^v)$. Let us estimate each term in the right hand side. For that purpose, we rely on the bounds (197) for the local energy and (199) for the pressure. For the terms involving only v , we have using that $|B_0(\frac{3}{2})| = 3^3$,

$$|I_1| \leq 3^3 K_3 \cdot 2K'_1(M^2 + N^2)t,$$

and

$$\begin{aligned} |I_2| &\leq 3^{3+\frac{3}{10}} K_3 (3^3 \cdot 2K'_1)^{\frac{3}{2}} (M^2 + N^2)^{\frac{3}{2}} t^{\frac{1}{10}} \\ &= 2^{\frac{3}{2}} \cdot 3^{\frac{39}{5}} K_3 K'_1^{\frac{3}{2}} (M^2 + N^2)^{\frac{3}{2}} t^{\frac{1}{10}}. \end{aligned}$$

For the terms involving a and v we use (301) in Proposition 31, more precisely the bound $\|a\|_{L^{10}(\mathbb{R}^3 \times (0, S_{mild}^a))} \leq K_0 N$. This in turn implies the controls

$$\begin{aligned} |I_4 + I_5 + I_6| &\leq 3\|a\|_{L^{10}} 3^3 E(t) (3^3 t)^{\frac{1}{5} - \frac{1}{10}} \\ &\leq 2 \cdot 3^{\frac{43}{10}} K_0 K_3 K'_1 N (M^2 + N^2) t^{\frac{1}{10}}. \end{aligned}$$

For I_6 , we used $t \in (0, S^v) \subset (0, \frac{1}{4})$.

Finally, we estimate the term involving the pressure

$$\begin{aligned} |I_3| &\leq 2K_3 \|q - C_0(t)\|_{L^{\frac{5}{3}}(B_0(\frac{3}{2}))} \|v\|_{L^{\frac{10}{3}}(B_0(\frac{3}{2}))} (3^3 t)^{\frac{1}{10}} \\ &\leq 2 \cdot 3^{\frac{9}{5}} K_3 K'_1^{\frac{3}{2}} (M^2 + N^2)^{\frac{3}{2}} t^{\frac{1}{10}}. \end{aligned}$$

Finally, we get the following estimate: there exists a universal constant $K_* \in [1, \infty)$ such that for all $t \in (0, 1]$,

$$(201) \quad E(t) \leq K_* (M^2 + N^2)^{\frac{3}{2}} t^{\frac{1}{10}}.$$

Notice that K_* can be taken as

$$(202) \quad K_* := 4 \max(2 \cdot 3^3 K_3 K'_1, 2^{\frac{3}{2}} \cdot 3^{\frac{39}{5}} K_3 K'_1^{\frac{3}{2}}, 2 \cdot 3^{\frac{43}{10}} K_0 K'_1),$$

where K_0 is defined in Proposition 31, K'_1 in (197) and (199), and K_3 is the constant in (200).

Step 4: boundedness of the perturbation. Let $\varepsilon \in (0, \min(\varepsilon_*, 2^{-9}))$. Our objective is now to apply the ε -regularity result Corollary 13 in order to get the boundedness of the perturbation. As in [21] and [6], we extend v by 0 in the past. The extension v is a suitable weak solution on $B_0(1) \times (-\infty, S^v)$ to the Navier-Stokes equations (184) with a drift a defined to be the zero extension of a to $\mathbb{R}^3 \times (-\infty, 0)$. The bound on the local energy (201) is crucial here, as is emphasized in [21]. Notice that the extended a is not a mild solution to the Navier-Stokes system (1) on $\mathbb{R}^3 \times (-\infty, S_{mild}^a)$ but in Corollary 13 this fact is not required. We have the bound

$$\|a\|_{L^{10}(\mathbb{R}^3 \times (-\infty, S_{mild}^a))} \leq K_0 N.$$

By the control (197) on the local energy and (199) on the pressure, we have

$$\begin{aligned} (203) \quad \int_{t-1}^t \int_{B_0(1)} |v|^3 + |q - C_0(t)|^{\frac{3}{2}} dx ds &= \int_0^t \int_{B_0(1)} |v|^3 + |q - C_0(t)|^{\frac{3}{2}} dx ds \\ &\leq 2K'_1^{\frac{3}{2}} (M^2 + N^2)^{\frac{3}{2}} (2^3 t)^{\frac{1}{10}}. \end{aligned}$$

Therefore, in order to apply Corollary 13, we choose $S_* = S_*(M, N) \in (0, S^v)$ sufficiently small such that

$$(204) \quad \begin{aligned} K_*(M^2 + N^2)^{\frac{3}{2}} S_*^{\frac{1}{10}} &\leq (K_0 N)^{-2} \varepsilon^{\frac{11}{9}}, \\ 2^{\frac{13}{10}} K'_1 S_*^{\frac{3}{2}} (M^2 + N^2)^{\frac{3}{2}} S_*^{\frac{1}{10}} &\leq (K_0 N)^{-4} \varepsilon^{\frac{13}{9}}. \end{aligned}$$

Conditions (204) imply that (189) and (191) are satisfied on $B_0(1) \times (S_* - 1, S_*)$. According to (204), we define $S_* = S_*(M, N)$ in the following way

$$(205) \quad S_* := \min \left(S^v, \frac{2^{-\frac{3}{2}} \varepsilon^{\frac{130}{9}}}{K_0^{40} M^{30} N^{70}} \min \left(\frac{1}{2^{13} K'_1^{15}}, \frac{1}{K_*^{10}} \right) \right)$$

Notice that for M, N sufficiently large we have

$$(206) \quad \begin{aligned} S_* &= \frac{2^{-\frac{3}{2}} \varepsilon^{\frac{130}{9}}}{K_0^{40} M^{30} N^{70}} \min \left(\frac{1}{2^{13} K'_1^{15}}, \frac{1}{K_*^{10}} \right) \\ &= O(M^{-30} N^{-70}). \end{aligned}$$

For the rest of the proof we take this definition of S_* . It follows from (188) that

$$(207) \quad \|u - a\|_{L^\infty(Q_{(0,S_*)}(\frac{1}{2}))} = \|v\|_{L^\infty(Q_{(0,S_*)}(\frac{1}{2}))} \leq C_{***}^{\frac{1}{3}} N^2.$$

Combining this estimate with (301) enables to obtain for all $\beta \in (0, S_*)$

$$(208) \quad \|u\|_{L^\infty(B_0(\frac{1}{2}) \times (\beta, S_*))} \leq C_{***}^{\frac{1}{3}} N^2 + \frac{K_0 N}{\beta^{\frac{1}{4}}}.$$

which implies estimate (174) as a particular case.

Step 5: estimates of the gradient of the perturbation. In this step, we take $\beta = \frac{3S_*}{4}$. Our goal is to prove the following claim

$$(209) \quad \sup_{\bar{x} \in B_0(\frac{1}{4})} \sup_{t \in (\frac{15}{16} S_*, S_*)} \int_{B_{\bar{x}}(\frac{\sqrt{S_*}}{4})} |\nabla u(x, t)|^2 dx \leq C M^{34} N^{80},$$

for a universal constant $C \in [1, \infty)$. Estimate (209) implies estimate (175) in the theorem by a covering argument. Let $\bar{x} \in B_0(\frac{1}{4})$. Notice that $B_{\bar{x}}(\frac{\sqrt{S_*}}{2}) \subset B_0(\frac{1}{3})$ for M, N sufficiently large. Without loss of generality, we assume that $\bar{x} = 0$. We bootstrap the regularity of u in the parabolic cylinder

$$Q_{(0,S_*)}(\frac{\sqrt{S_*}}{2}) = B_0(\frac{\sqrt{S_*}}{2}) \times (\frac{3}{4} S_*, S_*)$$

using the local maximal regularity for the non-stationary Stokes system. We first zoom in on the parabolic cylinder $Q_{(0,S_*)}(\frac{\sqrt{S_*}}{2})$ and define

$$U(y, s) := r u(r y, r^2 s + S_*), \quad P(y, s) := r^2 (p(r y, r^2 s + S_*) - C_0(r^2 s + S_*)),$$

for all $(y, s) \in Q_{(0,0)}(1)$, where $r := \frac{\sqrt{S_*}}{2}$. By the estimate (174), we have

$$(210) \quad \begin{aligned} \|U\|_{L^\infty(Q_{(0,0)}(1))} &\leq \frac{C_*}{2} M^8 N^{19} S_*^{\frac{1}{2}} \\ &\leq C M^{-7} N^{-16}. \end{aligned}$$

Moreover, the local energy estimate (196) implies that

$$\|\nabla U\|_{L^2(Q_{(0,0)}(1))} = r^{-\frac{1}{2}} \|\nabla u\|_{L^2(Q_{(0,S_*)}(r))} \leq 2^{\frac{1}{2}} K_1^{\frac{1}{2}} M S_*^{-\frac{1}{4}} \leq C M^9 N^{18},$$

For the pressure, we decompose $p - C_0(t)$ according to (305). We have according to the estimates in Proposition 33

$$\begin{aligned} \|p_{nonloc}\|_{L^2(Q_{(0,S_*)}(r))} &\leq \|p_{nonloc}\|_{L^2(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))} \\ &\leq S_*^{\frac{1}{2}} \|p_{nonloc}\|_{L^\infty(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))} \\ &\leq S_*^{\frac{1}{2}} K_1 M^2, \end{aligned}$$

on the one hand, and

$$\begin{aligned} \|p_{loc}\|_{L^2(Q_{(0,S_*)}(r))} &\leq \|p_{loc}\|_{L^2(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))} \\ &\leq C\|u\|_{L^4(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))}^2 \\ &\leq CS_*^{\frac{1}{2}} \|u\|_{L^\infty(B_0(\frac{1}{2}) \times (\frac{3}{4}S_*, S_*))}^2 \\ &\leq CS_*^{\frac{1}{2}} M^{16} N^{39} \end{aligned}$$

using (306) and Calderón-Zygmund on the other hand. Hence,

$$\begin{aligned} \|P\|_{L^2(Q_{(0,0)}(1))} &= r^{-\frac{1}{2}} \|p - C(t)\|_{L^2(Q_{(0,S_*)}(r))} \leq 2^{\frac{1}{2}} S_*^{-\frac{1}{4}} (S_*^{\frac{1}{2}} K_1 M^2 + CS_*^{\frac{1}{2}} M^{16} N^{39}) \\ &\leq CM^9 N^{22}. \end{aligned}$$

Notice that these are rough bounds, but they are enough for our purposes. Therefore,

$$(211) \quad \|\nabla U\|_{L^2(Q_{(0,0)}(1))} + \|P\|_{L^2(Q_{(0,0)}(1))} \leq CM^9 N^{22}.$$

Using the local maximal regularity [39, Proposition 2.4] leads to

$$\begin{aligned} &\|\partial_t U\|_{L^2(Q_{(0,0)}(\frac{3}{4}))} + \|\nabla^2 U\|_{L^2(Q_{(0,0)}(\frac{3}{4}))} + \|\nabla P\|_{L^2(Q_{(0,0)}(\frac{3}{4}))} \\ &\leq C \left(\|U\|_{L^\infty(Q_{(0,0)}(1))} \|\nabla U\|_{L^2(Q_{(0,0)}(1))} \right. \\ &\quad \left. + \|U\|_{L^2(Q_{(0,0)}(1))} + \|\nabla U\|_{L^2(Q_{(0,0)}(1))} + \|P\|_{L^2(Q_{(0,0)}(1))} \right) \\ &\leq CM^9 N^{22}, \end{aligned}$$

where we used the bounds (210) and (211). A simple local energy estimate for ∇U then leads to the bound

$$\|\nabla U\|_{L_t^\infty L_x^2(Q_{(0,0)}(\frac{1}{2}))} \leq CM^9 N^{22}.$$

Scaling back to the original variables leads to (209) and concludes the proof.

6. MAIN TOOL 2: QUANTITATIVE ANNULI OF REGULARITY

In this section we prove that a solution u , satisfying the hypothesis of Propositions 2-3, enjoys good quantitative bounds in certain spatial annuli. This was crucially used in the aforementioned propositions in two places. Namely for applying the Carleman inequalities (Propositions 34-35), as well as in ‘Step 4’ for transferring the lower bound of the vorticity to a lower bound on the localized L^3 norm.

In the context of classical solutions to the Navier-Stokes equations in $L_t^\infty L_x^3(\mathbb{R}^3 \times (t_0 - T, t_0))$, a related version was proven by Tao in [45] using a delicate analysis of local enstrophies from [44]. Our proof is somewhat different and elementary (though we use the ‘pigeonhole principle’ as in [45]), instead we utilize known ε -regularity criteria.

Proposition 15 ([9] and [24, Theorem 30.1]). *There exists absolute constants $\varepsilon_0^* > 0$ and $C_{CKN} \in (0, \infty)$ such that if (u, p) is a suitable weak solution to the Navier-Stokes equations on $Q_{(0,0)}(1)$ and for some $\varepsilon_0 \leq \varepsilon_0^*$*

$$(212) \quad \int_{Q_{(0,0)}(1)} |u|^3 + |p|^{\frac{3}{2}} dxdt \leq \varepsilon_0$$

then one concludes that

$$(213) \quad u \in L^\infty(Q_{(0,0)}(1/2)) \text{ with } \|u\|_{L^\infty(Q_{(0,0)}(\frac{1}{2}))} \leq C_{CKN} \varepsilon_0^{\frac{1}{3}}.$$

We require the following proposition, which is a quantitative version of Serrin's result [40]. Since the procedure is the same as that described in [40], we omit the proof.

Proposition 16. *Suppose $u \in L^\infty(Q_{(0,0)}(1/2))$ and $\omega := \nabla \times u \in L^2(Q_{(0,0)}(1/2))$ is such that (u, p) is a distributional solution to the Navier-Stokes equations in $Q_{(0,0)}(1/2)$. Furthermore, suppose*

$$(214) \quad \|u\|_{L^\infty(Q_{(0,0)}(1/2))} < 1$$

and

$$(215) \quad \|\omega\|_{L^2(Q_{(0,0)}(1/2))} < 1.$$

There exists universal constants $C'_k \in (0, \infty)$ with $k = 0, 1$, such that the above assumptions imply that for $k = 0, 1$

$$(216) \quad \|\nabla^k \omega\|_{L^\infty(Q_{(0,0)}(1/3))} \leq C'_k (\|u\|_{L^\infty(Q_{(0,0)}(1/2))} + \|\omega\|_{L^2(Q_{(0,0)}(1/2))}).$$

Remark 17. If we instead use the framework of *suitable weak solutions* in the above proposition, we can use the time integrability of the pressure to gain space-time Hölder continuity of all spatial derivatives of u in $Q_{(0,0)}(1/3)$. The vorticity equation then implies w , $\partial_t w$, ∇w and $\nabla^2 w$ are continuous in space and time in $Q_{(0,0)}(1/3)$.

Proposition 18. *There exists absolute constants $\varepsilon_1^* \in (0, 1)$ and $C''_k \in (0, \infty)$, $k = 0, 1, 2$, such that if (u, p) is a suitable weak solution to the Navier-Stokes equations on $Q_{(0,0)}(1)$ and for some $\varepsilon_1 \leq \varepsilon_1^*$*

$$(217) \quad \int_{Q_{(0,0)}(1)} |u|^3 + |p|^{\frac{3}{2}} dxdt \leq \varepsilon_1$$

then one concludes that for $j = 0, 1$

$$(218) \quad \nabla^j u \in L^\infty(Q_{(0,0)}(1/4)) \text{ with } \|\nabla^j u\|_{L^\infty(Q_{(0,0)}(\frac{1}{4}))} \leq C''_j \varepsilon_1^{\frac{1}{3}}$$

and

$$(219) \quad \nabla \omega \in L^\infty(Q_{(0,0)}(1/4)) \text{ with } \|\nabla \omega\|_{L^\infty(Q_{(0,0)}(\frac{1}{4}))} \leq C''_2 \varepsilon_1^{\frac{1}{3}}.$$

Proof. Let $\varepsilon_1^* \in (0, \min(1, \varepsilon_0^*))$. Notice that ε_1^* still to be determined. Then for $\varepsilon_1 \leq \varepsilon_1^*$ we can apply Proposition 15 to get

$$(220) \quad \|u\|_{L^\infty(Q_{(0,0)}(\frac{1}{2}))} \leq C_{CKN} \varepsilon_1^{\frac{1}{3}} \leq C_{CKN} (\varepsilon_1^*)^{\frac{1}{3}}$$

The local energy inequality for suitable weak solutions to the Navier-Stokes equations implies that

$$\int_{Q_{(0,0)}(\frac{1}{2})} |\nabla u|^2 dxdt \leq C \left(\int_{Q(1)} |u|^3 dxdt \right)^{\frac{2}{3}} + C \int_{Q(1)} |u|^3 + |p|^{\frac{3}{2}} dxdt.$$

Thus using (217) and the fact that $\varepsilon_1^* < 1$, we get that

$$(221) \quad \int_{Q(0,0)(\frac{1}{2})} |\omega|^2 dxdt \leq C_{univ} \varepsilon_1^{\frac{2}{3}} \leq C_{univ} (\varepsilon_1^*)^{\frac{2}{3}}.$$

So defining

$$(222) \quad \varepsilon_1^* := \min \left(\epsilon_0^*, 1, (2C_{univ})^{-\frac{3}{2}}, (2C_{CKN})^{-3} \right),$$

we get that (220)-(221) imply

$$(223) \quad \|u\|_{L^\infty(Q(0,0)(1/2))} < 1 \quad \text{and} \quad \|\omega\|_{L^2(Q(0,0)(1/2))} < 1.$$

Applying Proposition 16, together with (220)-(221), gives that for $k = 0, 1$

$$(224) \quad \|\nabla^k \omega\|_{L^\infty(Q(\frac{1}{3}))} \leq C'_k (\|u\|_{L^\infty(Q(0,0)(\frac{1}{2}))} + \|\omega\|_{L^2(Q(0,0)(\frac{1}{2}))}) \leq C''_k \varepsilon_1^{\frac{1}{3}}.$$

Using $-\Delta u = \nabla \times \omega$, known elliptic theory, (220) and (224) gives

$$\begin{aligned} \|\nabla u\|_{L^\infty(Q(0,0)(\frac{1}{4}))} &\leq C (\|u\|_{L^\infty(Q(0,0)(\frac{1}{3}))} + \|\omega\|_{L^\infty(Q(0,0)(\frac{1}{3}))} + \|\nabla \omega\|_{L^\infty(Q(0,0)(\frac{1}{3}))}) \\ &\leq C''_1 \varepsilon_1^{\frac{1}{3}}. \end{aligned} \quad \square$$

Proposition 19 (annulus of regularity, general form). *For all $\mu > 0$ there exists $\lambda_0(\mu) > 1$ such that the following holds true. For all $\lambda \geq \lambda_0(\mu)$, $R \geq 1$ and for every solution (u, p) to the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ that is a suitable weak solution on $Q_{(x_*, 0)}(1)$ for any $x_* \in \mathbb{R}^3$ and satisfies*

$$(225) \quad \int_{-1}^0 \int_{\mathbb{R}^3} |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} dxdt \leq \lambda < \infty,$$

there exists $R''(u, p, \lambda, \mu, R)$ with

$$(226) \quad 2R \leq R'' \leq 2R \exp(2\mu\lambda^{\mu+2})$$

and universal constants $\bar{C}_j \in (0, \infty)$ for $j = 0, 1, 2$ such that for $j = 0, 1$

$$(227) \quad \|\nabla^j u\|_{L^\infty(\{R'' < |x| < \frac{\lambda\mu}{4} R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_j \lambda^{-\frac{3\mu}{10}}$$

and

$$(228) \quad \|\nabla \omega\|_{L^\infty(\{R'' < |x| < \frac{\lambda\mu}{4} R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_2 \lambda^{-\frac{3\mu}{10}}.$$

Proof. Fix any $R \geq 1$ and $\mu > 1$. With these choices, take $\lambda > \lambda_0(\mu) \geq 1$. Here, λ_0 is to be determined. Then

$$\sum_{k=0}^{\infty} \int_{-(\lambda^\mu)^k R}^0 \int_{(\lambda^\mu)^k R < |x| < (\lambda^\mu)^{k+1} R} |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} dxdt \leq \lambda.$$

by the pigeonhole principle, there exists $k_0 \in \{0, 1, \dots, \lceil \lambda^{\mu+1} \rceil\}$ such that

$$\int_{-(\lambda^\mu)^{k_0} R}^0 \int_{(\lambda^\mu)^{k_0} R < |x| < (\lambda^\mu)^{k_0+1} R} |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} dxdt \leq \lambda^{-\mu}.$$

Define $R' := R\lambda^{\mu k_0}$. Then

$$(229) \quad R \leq R' \leq R \exp(2\mu\lambda^{\mu+2})$$

and

$$(230) \quad \int_{-1}^0 \int_{\substack{R' < |x| < \lambda^\mu R'}} |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} dx dt \leq \lambda^{-\mu}$$

Impose the restriction $\lambda_0(\mu) > 4^{\frac{1}{\mu}}$ and define

$$(231) \quad A := \{x : R' + 1 < |x| < \lambda^\mu R' - 1\}.$$

By Hölder's inequality we have

$$\begin{aligned} \sup_{x_* \in A} \int_{-1}^0 \int_{B(x_*, 1)} |u|^3 + |p|^{\frac{3}{2}} dx dt &\leq C_{univ} \left(\sup_{x_* \in A} \int_{-1}^0 \int_{B_{x_*}(1)} |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} dx dt \right)^{\frac{9}{10}} \\ &\leq C_{univ} \lambda^{-\frac{9\mu}{10}}. \end{aligned}$$

Defining

$$(232) \quad \lambda_0(\mu) := \max \left(2 \cdot 4^{\frac{1}{\mu}}, \left(\frac{2C_{univ}}{\varepsilon_1^*} \right)^{\frac{10}{9\mu}} \right),$$

the inequality $\lambda \geq \lambda_0(\mu)$ implies that

$$\sup_{x_* \in A} \int_{-1}^0 \int_{B_{x_*}(1)} |u|^3 + |p|^{\frac{3}{2}} dx dt \leq C_{univ} \lambda^{-\frac{9\mu}{10}} < \varepsilon_1^*.$$

Thus, we can apply Proposition 18 to get that for $j = 0, 1$

$$\sup_{x_* \in A} \|\nabla^j u\|_{L^\infty Q_{(x_*, 0)}(\frac{1}{4})} \leq C_j'' C_{univ}^{\frac{1}{3}} \lambda^{-\frac{3\mu}{10}} \quad \text{and} \quad \sup_{x_* \in A} \|\nabla \omega\|_{L^\infty Q_{(x_*, 0)}(\frac{1}{4})} \leq C_2'' C_{univ}^{\frac{1}{3}} \lambda^{-\frac{3\mu}{10}}.$$

Hence,

$$\|\nabla^j u\|_{L^\infty(A \times (-\frac{1}{16}, 0))} \leq C_j'' C_{univ}^{\frac{1}{3}} \lambda^{-\frac{3\mu}{10}} \quad \text{and} \quad \|\nabla \omega\|_{L^\infty(A \times (-\frac{1}{16}, 0))} \leq C_2'' C_{univ}^{\frac{1}{3}} \lambda^{-\frac{3\mu}{10}}.$$

Finally, note that (232) and $\lambda \geq \lambda_0$ imply that $\lambda^\mu R' - 1 > \frac{\lambda^\mu}{2} R' > 2R' > R' + 1$. Defining $R'' := 2R'$ and using $\{x : R'' < |x| < \frac{\lambda^\mu}{4} R''\} \subset A$, we see that the above estimates readily give the desired conclusion. \square

Bearing in mind (268), the energy estimates (266), (272) and Calderón-Zygmund estimates for the pressure, the following Lemma is obtained as an immediate corollary to the above Proposition. We also use the known fact that mild solutions to the Navier-Stokes equations in $L_{x,t}^4(\mathbb{R}^3 \times (0, T))$ are suitable weak solutions on $\mathbb{R}^3 \times (0, T)$, which can be seen by using a mollification argument along with Calderón-Zygmund estimates for the pressure.

Lemma 20 (annulus of regularity, Type I). *For all $\mu > 0$ there exists $M_1(\mu) > 1$ such that the following holds true. For all $M \geq M_1(\mu)$, $R \geq 1$ and for every mild solution (u, p) of the Navier-Stokes equations on $\mathbb{R}^3 \times [-2, 0]$ satisfying*

$$(233) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-2, 0))} \leq M$$

and

$$(234) \quad u \in L_{x,t}^4(\mathbb{R}^3 \times (-2, 0)),$$

there exists $R''(u, p, M, \mu, R)$ with

$$(235) \quad 2R \leq R'' \leq 2R \exp(C(\mu)M^{10(\mu+2)})$$

and universal constants $\bar{C}_j \in (0, \infty)$ for $j = 0, 1, 2$ such that for $j = 0, 1$

$$(236) \quad \|\nabla^j u\|_{L^\infty(\{R'' < |x| < c(\mu)M^{10\mu}R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_j C(\mu) M^{-3\mu}$$

and

$$(237) \quad \|\nabla \omega\|_{L^\infty(\{R'' < |x| < c(\mu)M^{10\mu}R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_2 C(\mu) M^{-3\mu}.$$

A simple rescaling gives the following corollary, which is directly used in the proof of Proposition 2.

Corollary 21. *Let $S \in (0, \infty)$. For all $\mu > 0$, let $M_1(\mu) > 1$ be given by Lemma 20. For all $M \geq M_1(\mu)$, $R \geq S^{\frac{1}{2}}$ and for every mild solution (u, p) of the Navier-Stokes equations on $\mathbb{R}^3 \times [-S, 0]$ satisfying*

$$(238) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-S, 0))} \leq M$$

and

$$(239) \quad u \in L_{x,t}^4(\mathbb{R}^3 \times (-S, 0)),$$

there exists $R''(u, p, M, \mu, R)$ with (235) and universal constants \bar{C}_j for $j = 0, 1, 2$ such that for $j = 0, 1$

$$(240) \quad \|\nabla^j u\|_{L^\infty(\{R'' < |x| < c(\mu)M^{10\mu}R''\} \times (-\frac{S}{32}, 0))} \leq 2^{\frac{j+1}{2}} \bar{C}_j C(\mu) M^{-3\mu} S^{-\frac{j+1}{2}}$$

and

$$(241) \quad \|\nabla \omega\|_{L^\infty(\{R'' < |x| < c(\mu)M^{10\mu}R''\} \times (-\frac{S}{32}, 0))} \leq 2^{\frac{2}{3}} \bar{C}_2 C(\mu) M^{-3\mu} S^{-\frac{3}{2}}.$$

Bearing in mind (295), the energy estimate in footnote 11 (see also Lemma 6) and Calderón-Zygmund estimates for the pressure, the following Lemma is obtained as an immediate corollary to Proposition 19.

Lemma 22 (annulus of regularity, time slices). *For all $\mu > 0$ there exists $M_2(\mu) > 1$ such that the following holds true. For all $M \geq M_2(\mu)$, $R \geq 1$ and for every suitable finite-energy solution³⁷ (u, p) of the Navier-Stokes equations on $\mathbb{R}^3 \times [-1, 0]$ satisfying*

$$(242) \quad \|u(\cdot, -1)\|_{L^3(\mathbb{R}^3)} \leq M$$

and with M^\flat defined by (17), there exists $R''(u, p, M, \mu, R)$ with

$$(243) \quad 2R \leq R'' \leq 2R \exp(C(\mu)(M^\flat)^{10(\mu+2)})$$

and universal constants \bar{C}_j for $j = 0, 1, 2$ such that for $j = 0, 1$

$$(244) \quad \|\nabla^j u\|_{L^\infty(\{R'' < |x| < (M^\flat)^{10\mu}R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_j (M^\flat)^{-3\mu}$$

and

$$(245) \quad \|\nabla \omega\|_{L^\infty(\{R'' < |x| < (M^\flat)^{10\mu}R''\} \times (-\frac{1}{16}, 0))} \leq \bar{C}_2 (M^\flat)^{-3\mu}.$$

A simple rescaling gives the following corollary, which is directly used in the proof of Theorem 2.

³⁷See Section 1.4 ‘Notations’ for a definition of *suitable finite-energy solutions*.

Corollary 23. *Let $S \in (0, \infty)$. For all $\mu > 0$, let $M_2(\mu) > 1$ and M^\flat be given by Lemma 22. For all $M \geq M_2(\mu)$, $R \geq S^{\frac{1}{2}}$ and for every suitable finite-energy solution (u, p) of the Navier-Stokes equations on $\mathbb{R}^3 \times [-S, 0]$ satisfying*

$$(246) \quad \|u(\cdot, -S)\|_{L^3(\mathbb{R}^3)} \leq M,$$

there exists $R''(u, p, M, \mu, R)$ with

$$(247) \quad 2R \leq R'' \leq 2R \exp(C(\mu)(M^\flat)^{10(\mu+2)})$$

and universal constants \bar{C}_j for $j = 0, 1, 2$ such that for $j = 0, 1$

$$(248) \quad \|\nabla^j u\|_{L^\infty(\{R'' < |x| < (M^\flat)^{10\mu} R''\} \times (-\frac{S}{32}, 0))} \leq 2^{\frac{j+1}{2}} \bar{C}_j (M^\flat)^{-3\mu} S^{-\frac{j+1}{2}}$$

and

$$(249) \quad \|\nabla \omega\|_{L^\infty(\{R'' < |x| < (M^\flat)^{10\mu} R''\} \times (-\frac{S}{32}, 0))} \leq 2^{\frac{3}{2}} \bar{C}_2 (M^\flat)^{-3\mu} S^{-\frac{3}{2}}.$$

Remark 24. According to Remark 17, in Proposition 19-Corollary 23 we have that w , $\partial_t w$, ∇w and $\nabla^2 w$ are continuous in space and time on the annuli considered. This remark is needed to apply the first Carleman inequality, Proposition 34, in Section 3 and 4.

7. MAIN TOOL 3: QUANTITATIVE EPOCHS OF REGULARITY

In this section, we prove that a solution satisfying the hypothesis of Propositions 2-3 enjoys good quantitative estimates in certain time intervals. In the literature, these are commonly referred to as ‘epochs of regularity’. Such a property is crucially used in ‘Step 1’ of the above propositions, when applying a quantitative Carleman inequality based on unique continuation (Proposition 35).

To show the existence of such epochs of regularity, we follow Leray’s approach in [26]. In particular, we utilize arguments involving existence of mild solutions for subcritical data and weak-strong uniqueness. We provide full details for the reader’s convenience.

We first recall a result known as ‘O’Neil’s convolution inequality’ (Theorem 2.6 of O’Neil’s paper [32]).

Proposition 25. *Suppose $1 < p_1, p_2, r < \infty$ and $1 \leq q_1, q_2, s \leq \infty$ are such that*

$$(250) \quad \frac{1}{r} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$$

and

$$(251) \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}.$$

Suppose that

$$(252) \quad f \in L^{p_1, q_1}(\mathbb{R}^d) \text{ and } g \in L^{p_2, q_2}(\mathbb{R}^d).$$

Then

$$(253) \quad f * g \in L^{r, s}(\mathbb{R}^d) \text{ with}$$

$$(254) \quad \|f * g\|_{L^{r, s}(\mathbb{R}^d)} \leq 3r \|f\|_{L^{p_1, q_1}(\mathbb{R}^d)} \|g\|_{L^{p_2, q_2}(\mathbb{R}^d)}.$$

We will also use an inequality that we will refer to as ‘Hunt’s inequality’. The statement below and its proof can be found in Hunt’s paper [19] (Theorem 4.5, p.271 of [19]).

Proposition 26. Suppose that $1 \leq p, q, r \leq \infty$ and $1 \leq s_1, s_2 \leq \infty$. Furthermore, suppose that p, q, r, s_1 and s_2 satisfy the following relations:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

and

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}.$$

Then the assumption that $f \in L^{p,s_1}(\Omega)$ and $g \in L^{q,s_2}(\Omega)$ implies that $fg \in L^{r,s}(\Omega)$, with the estimate

$$(255) \quad \|fg\|_{L^{r,s}(\Omega)} \leq C(p, q, s_1, s_2) \|f\|_{L^{p,s_1}(\Omega)} \|g\|_{L^{q,s_2}(\Omega)}.$$

Lemma 27 (epoch of regularity, Type I). *There exists a universal constant $C_3 \in [1, \infty)$ such that the following holds. Suppose $u : \mathbb{R}^3 \times [t_0 - T, t_0] \rightarrow \mathbb{R}^3$ and $p : \mathbb{R}^3 \times [t_0 - T, t_0] \rightarrow \mathbb{R}$ is a mild solution³⁸ of the Navier-Stokes equations. Furthermore, assume for some $M \geq 1$ that*

$$(256) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (t_0 - T, t_0))} \leq M$$

and

$$(257) \quad u \in L_{x,t}^4(\mathbb{R}^3 \times (t_0 - T, t_0)).$$

Then for all intervals $I \subset [t_0 - \frac{T}{2}, t_0]$ there exists a subinterval $I' \subset I$ such that the following holds true. Namely,

$$(258) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I')} \leq C_3 M^{18} |I|^{-\frac{(j+1)}{2}}$$

for $j = 0, 1, 2$ and

$$(259) \quad |I'| \geq C_3^{-1} M^{-12} |I|.$$

Remark 28 (estimates for applying Carleman inequalities (Type I)). Let $I'' \subset I'$ be such that

$$|I''| = \frac{M^{-36}}{4C_3^2} |I'|.$$

Then

$$|I'|^{-1} = \frac{M^{-36}}{4C_3^2} |I''|^{-1}.$$

Using (256)-(258), together with the fact that C_3 and $M \in [1, \infty)$, we see that

$$(260) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I'')} \leq \frac{1}{2^{j+1}} |I''|^{-\frac{(j+1)}{2}}$$

for $j = 0, 1, 2$ and

$$(261) \quad |I''| \geq \frac{M^{-48}}{4C_3^3} |I|.$$

Proof. The first part of the proof closely follows arguments in Tao's paper [45]. The only difference in the first part of the proof is that we exploit the above facts regarding Lorentz spaces. For completeness, we give full details.

As observed by Tao in [45], (256)-(259) are invariant with respect to the Navier-Stokes scaling and time translation. So we can assume without loss of generality

$$(262) \quad I = [0, 1] \subset [t_0 - \frac{T}{2}, t_0] \Rightarrow [-1, 1] \subset [t_0 - T, t_0]$$

³⁸See Subsection 1.4.3 of Section 1.4 ‘Notations’.

Step 1: a priori energy estimates. Clearly we have from the standing assumptions that

$$(263) \quad \|u\|_{L_t^\infty L_x^{3,\infty}(\mathbb{R}^3 \times (-1,1))} \leq M.$$

On $\mathbb{R}^3 \times (-1,1)$ we have

$$(264) \quad u = e^{(t+1)\Delta} u(\cdot, -1) + w, \quad w := - \int_{-1}^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(\cdot, s) ds.$$

It is known that $e^{t\Delta} \mathbb{P} \nabla \cdot$ has an associated convolution kernel K . Furthermore, from Solonnikov's paper [42], this satisfies the estimate

$$(265) \quad |\partial_t^m \nabla^j K(x, t)| \leq \frac{C(m, j)}{(|x|^2 + t)^{2+\frac{j}{2}+m}}, \quad \text{for } j, m = 0, 1, \dots$$

Thus we may apply O'Neil's convolution inequality (Proposition 25) with $r = s = 2$, $q_1 = p_1 = \frac{6}{5}$, $p_2 = \frac{3}{2}$ and $q_2 = \infty$. This and Hunt's inequality (Proposition 26) gives that for $t \in [-1, 1]$

$$\|w(\cdot, t)\|_{L_x^2} \leq \int_{-1}^t \frac{\|u \otimes u(\cdot, s)\|_{L_x^{\frac{3}{2}, \infty}}}{(t-s)^{\frac{3}{4}}} ds \leq M^2(t+1)^{\frac{1}{4}}.$$

Thus,

$$(266) \quad \|w\|_{L_t^\infty L_x^2(\mathbb{R}^3 \times (-1,1))} \leq CM^2.$$

Using O'Neil's convolution inequality once more gives

$$(267) \quad \|e^{(t+1)\Delta} u(\cdot, -1)\|_{L_x^\infty} \leq \frac{CM}{(t+1)^{\frac{1}{2}}},$$

$$(268) \quad \|e^{(t+1)\Delta} u(\cdot, -1)\|_{L_x^{\frac{10}{3}}} \leq \frac{CM}{(t+1)^{\frac{1}{20}}},$$

$$(269) \quad \|e^{(t+1)\Delta} u(\cdot, -1)\|_{L_x^4} \leq \frac{CM}{(t+1)^{\frac{1}{8}}},$$

and

$$(270) \quad \|e^{(t+1)\Delta} u(\cdot, -1)\|_{L_x^{6,2}} \leq \frac{CM}{(t+1)^{\frac{1}{4}}}.$$

Next, we see that w satisfies

$$(271) \quad \partial_t w - \Delta w + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot w = 0, \quad w(\cdot, -1) = 0.$$

Using (257), (267) and (269) we infer the following. Namely, $w \in C([0, 1]; L_\sigma^2(\mathbb{R}^3)) \cap L^2(0, 1; \dot{H}^1(\mathbb{R}^3))$ and satisfies the energy equality³⁹ for $t \in [0, 1]$:

$$\begin{aligned} & \frac{1}{2} \|w(\cdot, t)\|_{L_x^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' \\ &= \frac{1}{2} \|w(\cdot, 0)\|_{L_x^2}^2 + \int_0^t \int_{\mathbb{R}^3} e^{(t'+1)\Delta} u(\cdot, -1) \otimes (w + e^{(t'+1)\Delta} u(\cdot, -1)) : \nabla w dx dt'. \end{aligned}$$

³⁹This can be shown by utilizing arguments in [25] (Lemma 7.2 in [25]) and [41] (Theorem 2.3.1 in [41]), for example.

Using Hölder's inequality followed by Young's inequality we see that

$$\begin{aligned} \|w(\cdot, t)\|_{L_x^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' &\leq \|w(\cdot, 0)\|_{L_x^2}^2 + C \int_0^t \int_{\mathbb{R}^3} |e^{(t'+1)\Delta} u(\cdot, -1)|^4 dx dt' \\ &\quad + C \int_0^t \|w(\cdot, t')\|_{L_x^2}^2 \|e^{(t'+1)\Delta} u(\cdot, -1)\|_{L_x^\infty}^2 dt'. \end{aligned}$$

Using this, together with (266)-(269) and the fact $M > 1$, we obtain

$$(272) \quad \int_0^1 \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' \leq C_{univ} M^6.$$

Here, C_{univ} is a universal constant.

Step 2: higher integrability via weak-strong uniqueness. From (272), the pigeonhole principle, the Sobolev embedding theorem and (270), there exists $t_1 \in [0, \frac{1}{2}]$ such that

$$(273) \quad \|u(\cdot, t_1)\|_{L^6(\mathbb{R}^3)} \leq C_{univ} M^3.$$

This, (256) and Lebesgue interpolation (see Lemma 2.2 in [29] for example) implies that $u_0 \in L_\sigma^4(\mathbb{R}^3)$. This and (273) allows us to apply Proposition 31 and Remark 32. In particular, there exists $C'_{univ} \in (0, \infty)$ and a mild solution $U : \mathbb{R}^3 \times [t_1, t_1 + \frac{C'_{univ}}{M^{12}}] \rightarrow \mathbb{R}^3$ to the Navier-Stokes equations, with initial data $u(\cdot, t_1)$, which satisfies the following properties. Specifically,

$$(274) \quad \|U\|_{L_t^\infty L_x^6(\mathbb{R}^3 \times [t_1, t_1 + \frac{C'_{univ}}{M^{12}}])} \leq CM^3$$

and

$$(275) \quad U \in L_t^\infty L_x^4(\mathbb{R}^3 \times [t_1, t_1 + \frac{C'_{univ}}{M^{12}}]).$$

Let $W : \mathbb{R}^3 \times [t_1, t_1 + \frac{C'_{univ}}{M^{12}}] \rightarrow \mathbb{R}^3$ be defined by

$$(276) \quad W := u - U = - \int_{t_1}^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u - U \otimes U)(\cdot, s) ds \quad \text{for } t \in [t_1, t_1 + \frac{C'_{univ}}{M^{12}}].$$

Using (257) and (275), we see that

$$(277) \quad W \in C\left([t_1, t_1 + \frac{C'_{univ}}{M^{12}}]; L_\sigma^2(\mathbb{R}^3)\right) \cap L_t^2\left(t_1, t_1 + \frac{C'_{univ}}{M^{12}}; \dot{H}^1(\mathbb{R}^3)\right)$$

and W satisfies the energy equality for $t \in [t_1, t_1 + \frac{C'_{univ}}{M^{12}}]$:

$$(278) \quad \|W(\cdot, t)\|_{L_x^2}^2 + 2 \int_{t_1}^t \int_{\mathbb{R}^3} |\nabla W|^2 dx dt' = 2 \int_{t_1}^t \int_{\mathbb{R}^3} U \otimes W : \nabla W dx dt'.$$

Using this, (274) and known weak-strong uniqueness arguments from [26], we infer that $W \equiv 0$ on $\mathbb{R}^3 \times [t_1, t_1 + \frac{C'_{univ}}{M^{12}}]$. Using this together with (274), we get that for $\tau(s) := t_1 + \frac{sC'_{univ}}{M^{12}}$:

$$(279) \quad \|u\|_{L_t^\infty L_x^6(\mathbb{R}^3 \times (\tau(0), \tau(1)))} \leq CM^3.$$

Using that the pressure is given by a Riesz transform acting on $u \otimes u$, we can apply Calderón-Zygmund to get that the pressure p associated to u satisfies

$$(280) \quad \|p\|_{L_t^\infty L_x^3(\mathbb{R}^3 \times (\tau(0), \tau(1)))} \leq CM^6.$$

Step 3: higher derivative estimates. Here, the arguments differ from those utilized in [45]. Fix any $x \in \mathbb{R}^3$ and $t \in [\tau(\frac{1}{2}), \tau(1)]$. Take any $r \in (0, \sqrt{\frac{C'_{univ}}{2M^{12}}}]$, which ensures that $t - r^2 \in [\tau(0), \tau(1)]$. Using this and (279)-(280) we see that

$$(281) \quad \frac{1}{r^2} \int_{Q_{(x,t)}(r)} |u|^3 + |p|^{\frac{3}{2}} dx dt' \leq C''_{univ} r^{\frac{3}{2}} M^9 = C''_{univ} (rM^6)^{\frac{3}{2}}.$$

Taking

$$r = r_0 := \frac{1}{M^6} \min \left(\sqrt{\frac{C'_{univ}}{2}}, \frac{\epsilon_{CKN}^{\frac{2}{3}}}{(C''_{univ})^{\frac{2}{3}}} \right),$$

we can then apply the Caffarelli-Kohn-Nirenberg theorem [9] to get that for $j = 0, 1, \dots$

$$\sup_{(x,t) \in \mathbb{R}^3 \times [\tau(\frac{1}{2}), \tau(1)]} |\nabla^j u(x, t)| \leq \frac{C}{r_0^{j+1}} \simeq C(j)(M^6)^{j+1}.$$

This concludes the proof. \square

Lemma 29 (epoch of regularity, time slices). *There exists a universal constant $C_4 \in [1, \infty)$ such that the following holds. Suppose $u : [-1, 0] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $p : [-1, 0] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a suitable finite-energy solution to the Navier-Stokes equations. Furthermore, assume for some $M \geq 1$ and $t_0 \in [-1, 0)$ that*

$$(282) \quad \|u(\cdot, t_0)\|_{L_x^3} \leq M$$

and u satisfies the energy inequality (43) starting from $t' = t_0$.

We define M^\flat as in (17). Fix any $\alpha \geq M^\flat$ and let

$$(283) \quad s_0 \in \left[\frac{t_0}{2}, \frac{t_0}{4\alpha^{201}} \right].$$

Define

$$(284) \quad I := \left[s_0, \frac{s_0}{2} \right].$$

There exists a subinterval $I' \subset I$ such that the following holds true. Namely,

$$(285) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I')} \leq C_4 \alpha^{324} |I|^{-\frac{(j+1)}{2}}$$

for $j = 0, 1, 2$ and

$$(286) \quad |I'| \geq C_4^{-1} \alpha^{-216} |I|.$$

Remark 30 (estimates for applying Carleman inequalities, time slices). Let $I'' \subset I'$ be such that

$$|I''| = \frac{\alpha^{-648}}{4C_4^2} |I'|.$$

Then

$$|I'|^{-1} = \frac{\alpha^{-648}}{4C_4^2} |I''|^{-1}.$$

Using (285)-(286), together with the fact that C_4 and $M \in [1, \infty)$, we see that

$$(287) \quad \|\nabla^j u\|_{L_t^\infty L_x^\infty(\mathbb{R}^3 \times I'')} \leq \frac{1}{2^{j+1}} |I''|^{-\frac{(j+1)}{2}}$$

for $j = 0, 1, 2$ and

$$(288) \quad |I''| \geq \frac{\alpha^{-864}}{4C_4^3} |I|.$$

Proof. Define,

$$\hat{t} := \frac{t_0}{s_0} - 1.$$

Note that (283) implies that

$$(289) \quad \hat{t} \in (1, 4\alpha^{201} - 1).$$

By appropriate scalings and translations, we can assume without loss of generality that $u : \mathbb{R}^3 \times (0, \hat{T}) \rightarrow \mathbb{R}^3$, for some $\hat{T} \in (0, \infty)$ ⁴⁰ and

$$(290) \quad \|u(\cdot, 0)\|_{L^3(\mathbb{R}^3)} \leq M,$$

$$(291) \quad \|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(y, s)|^2 dy ds \leq \|u(\cdot, 0)\|_{L^2}^2$$

and

$$(292) \quad I := \left[\hat{t}, \hat{t} + \frac{1}{2} \right] \subset (1, \hat{T}).$$

On $\mathbb{R}^3 \times (0, \hat{T})$ we have

$$(293) \quad u = e^{t\Delta} u(\cdot, 0) + w.$$

Using O’Neil’s convolution inequality once more gives for all $t \in (0, \infty)$,

$$(294) \quad \|e^{t\Delta} u(\cdot, 0)\|_{L_x^3} \leq CM,$$

$$(295) \quad \|e^{t\Delta} u(\cdot, 0)\|_{L_x^{10}} \leq \frac{CM}{t^{\frac{1}{20}}},$$

$$(296) \quad \|e^{t\Delta} u(\cdot, 0)\|_{L_x^4} \leq \frac{CM}{t^{\frac{1}{8}}},$$

and

$$(297) \quad \|e^{t\Delta} u(\cdot, 0)\|_{L_x^{6,2}} \leq \frac{CM}{t^{\frac{1}{4}}}.$$

Furthermore, arguments from [17] imply that

$$(298) \quad \|e^{t\Delta} u(\cdot, 0)\|_{L^5(\mathbb{R}^3 \times (0, \infty))} \leq CM.$$

Moreover, similar arguments as those used in Proposition 2.2 of [38] yield that for $t \in (0, \hat{T})$,

$$\begin{aligned} & \|w(\cdot, t)\|_{L_x^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' \\ & \leq C \int_0^t \int_{\mathbb{R}^3} |e^{t\Delta} u(\cdot, 0)|^4 dx dt' + C \int_0^t \|w(\cdot, t')\|_{L_x^2}^2 \|e^{t'\Delta} u(\cdot, 0)\|_{L_x^5}^5 dt'. \end{aligned}$$

⁴⁰The time \hat{T} is the image of 0 by the scalings and translations. Its precise value does not matter at all, since the proof is carried out on the time interval $[0, \hat{t} + \frac{1}{2}]$.

Note that the energy inequality for w , which is used to produce this estimate, can be justified rigorously using (298) and similar arguments as those used in Proposition 14.3 in [24].

Using (294)-(298) and Gronwall's lemma, we infer that

$$(299) \quad \sup_{0 < t < \hat{t} + \frac{1}{2}} \|w(\cdot, t)\|_{L_x^2}^2 + \int_0^{\hat{t} + \frac{1}{2}} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt' \leq (M^\flat)^4 (\hat{t} + \frac{1}{2})^{\frac{1}{2}} < 2\alpha^{105}.$$

Here we used (289). Now let $\Sigma \subset [0, \hat{T}]$ be such that (43) is satisfied for all $t \in [t', \hat{T}]$ and $t' \in \Sigma$. Since u is a suitable finite-energy solution we have that $|\Sigma| = \hat{T}$. Furthermore, Σ can be chosen without loss of generality such that

$$\int_{\mathbb{R}^3} |\nabla w(x, t')|^2 dx < \infty \text{ for all } t' \in \Sigma.$$

Using (299), the Sobolev embedding theorem, the pigeonhole principle and (297), we see that there exists $t_1 \in [\hat{t}, \hat{t} + \frac{1}{4}] \cap \Sigma$ such that

$$(300) \quad \|u(\cdot, t_1)\|_{L^6}^2 \leq C\alpha^{105}.$$

Making use of the fact that u satisfies the energy inequality starting from t_1 and (300), we can utilize similar arguments to those used in Lemma 27 replacing M by α^{18} . \square

APPENDIX A. AUXILIARY RESULTS

We first state the existence result of mild solutions with subcritical data.

Proposition 31 ([48, 17]). *Let $n \in (3, \infty)$. There exists $k_0(n) \in (0, \infty)$, $K_0(n) \in [1, \infty)$ such that the following holds. For all $u_0 \in L_\sigma^n(\mathbb{R}^3)$, we define*

$$S_{mild}(u_0) := k_0 \|u_0\|_{L^n}^{-\frac{2n}{n-3}} \in (0, \infty).$$

There exists a unique mild solution $a \in C([0, S_{mild}); L^n) \cap L^\infty((0, S_{mild}); L^n)$ with initial data u_0 such that

$$(301) \quad \begin{aligned} \sup_{t \in (0, S_{mild})} & \left(\|a(\cdot, t)\|_{L^n} + t^{\frac{3}{2n}} \|a(\cdot, t)\|_{L^\infty} + t^{\frac{1}{2}} \|\nabla a(\cdot, t)\|_{L^n} + t^{\frac{1}{2} + \frac{3}{2n}} \|\nabla a(\cdot, t)\|_{L^\infty} \right) \\ & + \|a\|_{L^{\frac{5n}{3}}(\mathbb{R}^3 \times (0, S_{mild}))} \leq K_0 \|u_0\|_{L^n}. \end{aligned}$$

Remark 32. For $U, V : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3$ define

$$(302) \quad B(U, V)(\cdot, t) := \int_0^t \mathbb{P} \partial_i e^{(t-s)\Delta} U_i(\cdot, s) V_j(\cdot, s) ds.$$

Using (265), one has the estimate

$$\begin{aligned} \|B(U, V)\|_{L_t^\infty L_x^4(\mathbb{R}^3 \times [0, T])} + \|B(V, U)\|_{L_t^\infty L_x^4(\mathbb{R}^3 \times [0, T])} \\ \leq cT^{\frac{1}{4}} \|U\|_{L_t^\infty L_x^6(\mathbb{R}^3 \times [0, T])} \|V\|_{L_t^\infty L_x^4(\mathbb{R}^3 \times [0, T])}. \end{aligned}$$

Using this, one can show that for $u_0 \in L_\sigma^6(\mathbb{R}^3) \cap L_\sigma^4(\mathbb{R}^3)$ and k_0 sufficiently small the following *persistency* property holds true. Namely, the mild solution in Proposition 31 satisfies

$$(303) \quad \sup_{t \in (0, S_{mild})} \|a(\cdot, t)\|_{L^4} \leq 2\|u_0\|_{L^4}$$

in addition to (301) with $n = 6$. This is utilized in the proof of Lemma 27.

The next result is the local energy bound for local energy solutions.⁴¹

Proposition 33 ([20, Lemma 2.1], [24]). *There exist two universal constants $k_1 \in (0, \infty)$, $K_1 \in [1, \infty)$ such that the following holds. For all $M \in (0, \infty)$, we define*

$$S_{locen}(M) := k_1 \min(M^{-4}, 1) \in (0, \infty).$$

For all $u_0 \in L^2_{uloc}(\mathbb{R}^3)$ with $\|u_0\|_{L^2(B_{\bar{x}}(1))} \xrightarrow{|\bar{x}| \rightarrow \infty} 0$, for all local energy solution (u, p) to (1) with initial data u_0 , if

$$\sup_{\bar{x} \in \mathbb{R}^3} \int_{B_{\bar{x}}(1)} |u_0(x)|^2 dx \leq M^2,$$

then

$$(304) \quad \sup_{s \in (0, S_{locen})} \sup_{\bar{x} \in \mathbb{R}^3} \int_{B_{\bar{x}}(1)} \frac{|u(x, s)|^2}{2} dx + \sup_{\bar{x} \in \mathbb{R}^3} \int_0^{S_{locen}} \int_{B_{\bar{x}}(1)} |\nabla u(x, s)|^2 dx ds \leq K_1 M^2.$$

Moreover, we have the following decomposition of the pressure: for all $\bar{x} \in \mathbb{R}^3$ and $t \in (0, S_{locen})$, there exists $C_{\bar{x}}(t) \in \mathbb{R}$ such that⁴²

$$(305) \quad p(x, t) - C_{\bar{x}}(t) = -\frac{1}{3}|u(x, t)|^2 + p_{loc}(x, t) + p_{nonloc}(x, t)$$

for all $(x, t) \in B_{\bar{x}}(\frac{3}{2}) \times (0, S_{locen})$, with

$$(306) \quad p_{loc}(x, t) = - \int_{\mathbb{R}^3} K_{ij}(x - y) \varphi(y) u_i(y, t) u_j(y, t) dy$$

and

$$(307) \quad p_{nonloc}(x, t) = - \int_{\mathbb{R}^3} (K_{ij}(x - y) - K_{ij}(\bar{x} - y))(1 - \varphi(y)) u_i(y, t) u_j(y, t) dy.$$

Here, $\varphi \in C_0^\infty(B_{\bar{x}}(4))$ (with $\varphi \equiv 1$ on $B_{\bar{x}}(3)$) and $K_{ij}(x) := \partial_i \partial_j \left(\frac{1}{|x|} \right)$.

Moreover, we have the estimate

$$(308) \quad \|p_{loc}\|_{L^{\frac{5}{3}}(B_{\bar{x}}(\frac{3}{2}) \times (0, S_{locen}))} + \|p_{nonloc}\|_{L^\infty(B_{\bar{x}}(\frac{3}{2}) \times (0, S_{locen}))} \leq K_1 M^2.$$

APPENDIX B. CARLEMAN INEQUALITIES

The two statements below are taken directly from [45]. The first Carleman inequality in Proposition 34 corresponds to a quantitative backward uniqueness result.

Proposition 34 (first Carleman inequality [45, Proposition 4.2]). *Let $C_{Carl} \in [1, \infty)$, $S \in (0, \infty)$, $0 < r_- < r_+$ and we define the space-time annulus*

$$\mathcal{A} := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t \in [0, S], r_- \leq |x| \leq r_+\}.$$

⁴¹Notice that ‘local energy solutions’ to the Navier-Stokes equations, are sometimes described in the literature as ‘Lemarié-Rieusset solutions’ or ‘Leray solutions’. They were conceived by Lemarié-Rieusset in [24]. In our paper, whenever we refer to ‘local energy solutions’, we mean in the sense of Definition 2.1 in [20]. Notice, in particular, that suitable finite-energy solutions (defined in Section 1.4 ‘Notations’) are local energy solutions.

⁴²This decomposition is also valid for p_{loc} defined in $B_{\bar{x}}(\frac{1}{2}) \times (0, S_{locen})$ instead of $B_{\bar{x}}(\frac{3}{2}) \times (0, S_{locen})$ as stated here. The constant $C_{\bar{x}}(t)$ has to be adapted.

Let $w : \mathcal{A} \rightarrow \mathbb{R}^3$ be such that w , $\partial_t w$, ∇w and $\nabla^2 w$ are continuous in space and time and such that w satisfies the differential inequality

$$(309) \quad |(\partial_t + \Delta)w| \leq C_{Carl}^{-1} S^{-1} |w| + C_{Carl}^{-\frac{1}{2}} S^{-\frac{1}{2}} |\nabla w| \quad \text{on } \mathcal{A}.$$

Assume

$$(310) \quad r_-^2 \geq 4C_{Carl}S.$$

Then we have the following bound

$$(311) \quad \int_0^{\frac{S}{4}} \int_{10r_- \leq |x| \leq \frac{r_+}{2}} (S^{-1}|w|^2 + |\nabla w|^2) dxdt \lesssim C_{Carl}^3 e^{-\frac{r_- \cdot r_+}{4C_{Carl}S}} (X + e^{\frac{2r_+^2}{C_{Carl}S}} Y),$$

where

$$X := \iint_{\mathcal{A}} e^{\frac{2|x|^2}{C_{Carl}S}} (S^{-1}|w|^2 + |\nabla w|^2) dxdt, \quad Y := \int_{r_- \leq |x| \leq r_+} |w(x, 0)|^2 dx.$$

The second Carleman inequality in Proposition 35 below corresponds to a quantitative unique continuation result.

Proposition 35 (second Carleman inequality [45, Proposition 4.3]). *Let $C_{Carl} \in [1, \infty)$, $S \in (0, \infty)$, $r > 0$ and we define the space-time cylinder*

$$\mathcal{C} := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t \in [0, S], |x| \leq r\}.$$

Let $w : \mathcal{C} \rightarrow \mathbb{R}^3$ such that w , $\partial_t w$, ∇w and $\nabla^2 w$ are continuous in space and time and such that w satisfies the differential inequality

$$(312) \quad |(\partial_t + \Delta)w| \leq C_{Carl}^{-1} S^{-1} |w| + C_{Carl}^{-\frac{1}{2}} S^{-\frac{1}{2}} |\nabla w| \quad \text{on } \mathcal{C}.$$

Assume

$$(313) \quad r^2 \geq 4000S.$$

Then, for all $0 < \check{s} \leq \hat{s} < \frac{S}{10000}$ one has the bound

$$(314) \quad \int_{\hat{s}}^{2\hat{s}} \int_{|x| \leq \frac{r}{2}} (S^{-1}|w|^2 + |\nabla w|^2) e^{-\frac{|x|^2}{4t}} dxdt \lesssim e^{-\frac{r^2}{500\check{s}}} X + (\hat{s})^{\frac{3}{2}} \left(\frac{e\hat{s}}{\check{s}} \right)^{\frac{O(1)r^2}{\check{s}}} Y,$$

where

$$X := \int_0^S \int_{|x| \leq r} (S^{-1}|w|^2 + |\nabla w|^2) dxdt, \quad Y := \int_{|x| \leq r} |w(x, 0)|^2 (\check{s})^{-\frac{3}{2}} e^{-\frac{|x|^2}{4\check{s}}} dx.$$

Proposition 34 and Proposition 35 are proved in [45] for smooth functions. The proof works under the weaker smoothness assumption stated here. This is used in Section 4 in particular, where the results are stated for suitable finite-energy solutions.

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