Martingale betting system

The Martingale system entails doubling bets until a win eventually arrives. Here we will work out what occurs when a ceiling prevents unlimited doubling. The simulations demonstrate that any ceiling dooms the strategy, but simulations are stochastic and don't reveal the details we'll examine here about the expected winnings with different ceilings.

Let's think about what betting with a ceiling looks like. A $10 starting bet and a low ceiling of $40 implies the following possibilities:

Win $10
Lose $10 → Win $20
Lose $10 → Lose $20 → Win $40
Lose $10 → Lose $20 → Lose $40

The process resets to a $10 bet after all of these, making them independent courses of betting. So the most straightforward way to start finding the expected value of this betting process is by calculating the relative frequency of each of these six values (+10, -10, +20, -20, +40, -40). For the moment the starting bet will be 1 unit to improve clarity.

Let the probability of winning any bet be p. The probability of losing is therefore $1 - p$. To start thinking about the frequency of each bet, it would be easiest to begin with the simplest scenario, in which the ceiling is equal to the starting bet.

Let L1 be the fraction of all bets occupied by a loss in the first round of betting; let W1 be the fraction occupied by a win in the first round.

L1 $= 1 - p$
W1 $= p$

The expected value per bet is E = W1 − L1 $= p - (1 - p) = 2p - 1$.

Now we'll make it more complicated, with a ceiling twice the starting bet. Let L2 be the fraction of bets occupied by a loss in the second round of betting; let W2 be the fraction occupied by a win in the second round. This pattern will continue.

L1 $= x$
L2 $= x(1 - p)$
W1 $= xp/(1 - p)$
W2 $= xp$

The expressions above summarize what we know. L1 is defined as an unknown x, and because a loss in the second round depends on a loss in the first round, L2 is x multiplied by $1-p$, the probability of losing. A win in the second round also depends on an initial loss, so W2 is x multiplied by p, the probability of winning. And a win in the first round, W1, is x multiplied by the ratio of wins to losses.

Now let's find x. We know these expressions sum to 1:

$$x + x(1-p) + xp/(1-p) + xp = 1$$

$$x(1 + (1-p) + p/(1-p) + p) = 1$$

$$x = \frac{1}{2 + p/(1-p)} = \frac{1-p}{2(1-p)+p} = \frac{1-p}{2-p}$$

We can now express {L1, L2, W1, W2} purely in terms of p:

L1 $= (1-p)/(2-p)$
L2 $= (1-p)^2/(2-p)$
W1 $= p/(2-p)$
W2 $= p(1-p)/(2-p)$

The expected value per bet is E = W1 - L1 + 2W2 - 2L2:

$$E = \frac{p-(1-p)+2p(1-p)-2(1-p)^2}{2-p} = \frac{2p-1+2(1-p)(p-(1-p))}{2-p}$$

$$= \frac{2p-1+2(1-p)(2p-1)}{2-p} = \frac{(2p-1)(1+2(1-p))}{2-p} = \frac{(2p-1)(2p-3)}{p-2}$$

We can now move on to the initially suggested ceiling which is 4x the value of the starting bet. A pattern will become clear which will allow us to derive a general expression for any ceiling.

L1 $= x$
L2 $= x(1-p)$
L3 $= x(1-p)^2$
W1 $= xp/(1-p)$
W2 $= xp$
W3 $= xp(1-p)$

Solve for x:

$$x(p/(1-p)+1+p+(1-p)+p(1-p)+(1-p)^2)=1$$

$$x=\frac{1-p}{p+(1-p)+p(1-p)+(1-p)^2+p(1-p)^2+(1-p)^3}$$

$$=\frac{1-p}{1+(1-p)+(1-p)^2}=\frac{1-p}{p^2-3p+3}$$

Express {L1, L2, L3, W1, W2, W3} in terms of p:

L1 $=(1-p)/(p^2-3p+3)$
L2 $=(1-p)^2/(p^2-3p+3)$
L3 $=(1-p)^3/(p^2-3p+3)$
W1 $=p/(p^2-3p+3)$
W2 $=p(1-p)/(p^2-3p+3)$
W3 $=p(1-p)^2/(p^2-3p+3)$

The expected value per bet is $E=$ W1 - L1 + 2W2 - 2L2 + 4W3 - 4L3:

$$E=\frac{p-(1-p)+2p(1-p)-2(1-p)^2+4p(1-p)^2-4(1-p)^3}{p^2-3p+3}$$

$$=\frac{2p-1+2(1-p)(2p-1)+4(1-p)^2(2p-1)}{p^2-3p+3}$$

$$=\frac{(2p-1)(1+2(1-p)+4(1-p)^2)}{p^2-3p+3}=\frac{(2p-1)(4p^2-10p+7)}{p^2-3p+3}$$

Close attention to the algebra makes two patterns clear — one each for the numerator and the denominator. We can bypass all this work for each individual ceiling by creating a general expression.

The variable f will refer to the number of times one must double the starting bet to reach the ceiling. So the value of the ceiling is $v=s(2^f)$.

The following expresses the expected value per bet of the Martingale strategy given a probability of winning p and starting bet s:

$$E = \frac{s\ (2p-1)\ \sum_{n=0}^{f} (2(1-p))^n}{\sum_{n=0}^{f} (1-p)^n}$$

This formula is implemented in the R and Python scripts below the main section of code.

Notably, $E = 0$ when $p = .5$ and $E < 0$ when $p < .5$. The Martingale system is incapable of reversing the fortunes of anyone on the wrong end of an unfair game of chance.

The expression when betting without a ceiling is very simple: $E = sp$. Why? Without a ceiling, the strategy is guaranteed to eventually triumph. No matter how many losing bets are placed, the winning bet at the end of a losing streak yields a profit equal to the starting bet. So naturally, the expected value per bet is the starting bet divided by the average number of bets needed to win. For example, $p = .25$ implies a victory once every four bets on average, so the long-run value of the strategy is the starting bet divided by four. This is equivalent to multiplying by p.

When $s = \$10$ and $p = .48$, the expected value per bet E is the following:

f	E	v
0	-\$0.40	\$10
1	-\$0.54	\$20
2	-\$0.70	\$40
3	-\$0.88	\$80
4	-\$1.08	\$160
5	-\$1.30	\$320
6	-\$1.53	\$640
7	-\$1.78	\$1 280
8	-\$2.04	\$2 560
9	-\$2.31	\$5 120
10	-\$2.59	\$10 240
15	-\$4.19	\$327 680
20	-\$6.14	\$10 485 760
25	-\$8.51	\$335 544 320
27	-\$9.59	\$1 342 177 280
28	-\$10.17	\$2 684 354 560
30	-\$11.39	\$10 737 418 240
∞	+\$4.80	

A final note of interest: expected value per bet is more neatly expressed when p instead refers to the probability of losing:

$$E = \frac{s\ (1 - 2p)\ \sum_{n=0}^{f} (2p)^n}{\sum_{n=0}^{f} p^n}$$