Honors Project 1:
Newton's Law of Cooling and Mr. Potato Head

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1 Preamble

As Pasadena City College's brightest Calculus II students, we have been tasked by the campus police to solve two problems: when the suspicious hot potatoes have been cooked as well as the mystery of the stolen **_Lancerlite_**. Using our knowledge of differential equations and Newton's Law of Cooling, we present our findings as follows.

2 The Potato Incident

At approximately 10am, we find ourselves upon a peculiar incident. Our suspect Mr. Potato Head had dispatched a number of "hot potatoes" to prevent the hungry students of Math005BH from eating them. Unbeknownst to him, hot potatoes are considered a delicacy amongst the students. As we bide our time waiting for the hot potatoes to cool, we decide to take some measurements to put our differential skills to the test. With enough data points, we should be able to ascertain as to when the potatoes have been cooked!

2.1 Newton's Law of Cooling

As we all know, the rate of cooling for an object is directly proportional to the difference in temperature for an object and it's surroundings. This is otherwise known as Newton's Law of Cooling.

$$\frac{dT}{dt} = k(T - T_s)$$

Variables	Definition
dT	Rate of change in Temperature
dt	Rate of change in time
k	Cooling constant (rate of growth)
T_s	Temperature (of surrounding)

2.2   Solving our Differential Equation

Using Newton's Law of Cooling, our goal is to find a solution for the differential equation. This is done by undoing the differential equation using integrals. Steps are as follows.

$$\frac{dT}{dt} = k(T - T_s) \tag{1}$$

$$\frac{dT}{(T - T_s)} = k(dt) \tag{2}$$

$$\int \frac{dT}{(T - T_s)} = \int k(dt) \tag{3}$$

$$\ln|T - T_s| + C_1 = kt + C_2 \tag{4}$$

$$\ln|T - T_s| = kt + C \tag{5}$$

$$e^{\ln|T - T_s|} = e^{kt + C} \tag{6}$$

$$|T - T_s| = e^c e^{kt} \tag{7}$$

$$|T - T_s| = Ce^{kt} \tag{8}$$

We find our solution to be $\boxed{|T - T_s| = Ce^{kt}}$. Which we can use to solve for k, the rate of growth. As it stands, we have almost all the information we need to start solving for our missing variables (k) and (C). By using the property $(e^{k \cdot 0}, = 1)$ we can find our constant value which inevitably allows us to solve for k. Steps will be detailed below.

2.3   Measurements

Where T_s (room temperature) is considered to be at $73°$:

t (min)	T (potato)	$|T - T_s|$	$\ln(T - T_s)$
0	$148.2°F$	$148.2 - 73.0 = 75.2$	4.32
10	$143.0°F$	70.0	4.25
20	$138.7°F$	65.7	4.19
30	$133.9°F$	60.9	4.11
40	$128.8°F$	55.8	4.02
50	$125.8°F$	52.8	3.97
60	$122.5°F$	49.5	3.90

Where t = time, T = temperature, T_s = surrounding temperature

2.4   Solving for our missing variables

As mentioned previously, now that we have the solution to our differential equation, we can start plugging in values. at $t = 0$, we are able to easily find our constant as such:

$$|T - T_s| = Ce^{kt} \tag{9}$$

$$|148.2 - 73.0| = Ce^{k \cdot 0} \tag{10}$$

$$C(1) = 75.2 \tag{11}$$

We find our constant to be 75.2. Now we can start solving for our rate of growth, k.

Given that $t = 10$ and $C = 75.2$:

$$|143.0 - 73.0| = 75.2e^{k \cdot 10} \tag{12}$$

$$\frac{70}{75.2} = e^{k \cdot 10} \tag{13}$$

$$\ln(.93) = \ln(e^{k \cdot 10}) \tag{14}$$

$$-0.0717 = k \cdot 10 \tag{15}$$

$$-0.0071 = k \tag{16}$$

We find our rate of growth to be approximately $\boxed{k \approx -0.0071}$. We will further verify our findings in the next section below using linear regression modelling.

2.5   Graph

Using linear regression modelling, we interpolate our data onto Desmos. Upon closer insepection, we find our slope to be around -0.00710714 on Desmos, very close to our own calculation of k. This value can also be expressed by rearranging our original equation as follows:

$$ln|T - T_s| = kt + C \tag{17}$$

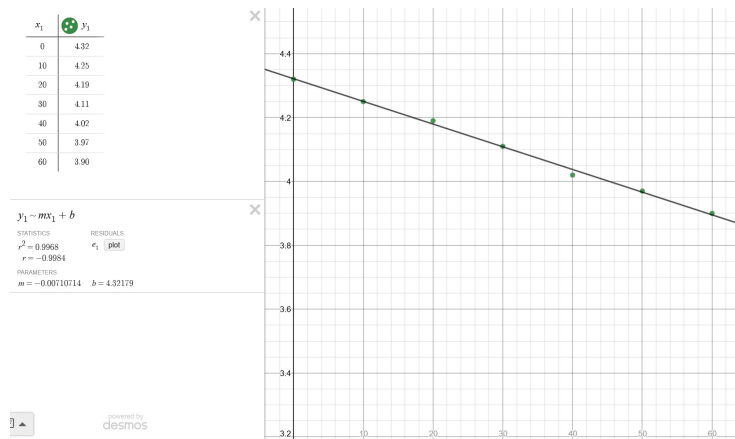$$\frac{ln|T - T_s|}{t} + C = k \tag{18}$$

Figure 1: Our data points plotted using Desmos.

Furthermore, we verify our derived solution for Newton's Law of Cooling by plotting T (temperature) over t against our own linear equation of $\boxed{|T - T_s| = Ce^{kt}}$.
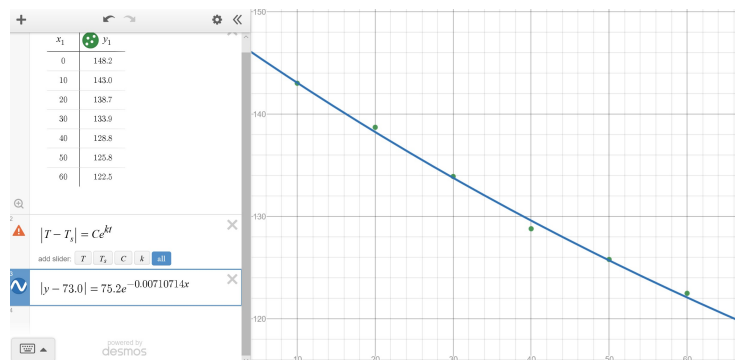

Figure 2: Graph of Temperature over time against our derived equation

The graphs reflects a direct correlation between the collected data points and the derived equation (as shown in blue).

2.6   Calculations

With all of our missing variables (C, k) found, we can proceed to find out just exactly when our potato was cooked (t). Now, given that a fully cooked potato starts at 208°F, we solve for t using our equation as follows:

$$|T - T_s| = Ce^{kt} \tag{19}$$

$$|208 - 73| = 75.2e^{-0.0071 \cdot t} \tag{20}$$

$$|135| = 75.2e^{-0.0071 \cdot t} \tag{21}$$

$$\frac{135}{75.2} = e^{-0.0071 \cdot t} \tag{22}$$

$$\ln\left(\frac{135}{75.2}\right) = \ln\left(e^{-0.0071 \cdot t}\right) \tag{23}$$

$$1.8 = -0.0071 \cdot t \tag{24}$$

$$t = -252.85 \tag{25}$$

We find that the time elapsed since the potato was 208°F was approximately 253 minutes ago. This is roughly around 4 hours and 13 minutes upon first measuring the temperature of the potato at 10am, making it 5:47am. Now, either our suspect Mr. Potato Head is an incredibly early riser or something has gone terribly wrong with our findings. We can confirm by evaluating other test points.

Given $t = 30$:

$$T_{30} - 73 = 75.2e^{-.0071 \cdot (30)} \tag{26}$$
$$T_{30} - 73 = 75.2(.808) \tag{27}$$
$$T_{30} = 73 + 60.77 \tag{28}$$
$$T_{30} = 133.77 \tag{29}$$

Given $t = 60$:

$$T_{60} - 73 = 75.2e^{-.0071 \cdot (60)} \tag{30}$$
$$T_{60} - 73 = 75.2(.653) \tag{31}$$
$$T_{60} = 73 + 49.11 \tag{32}$$
$$T_{60} = 122.11 \tag{33}$$

Both answers support our collected measurements!

2.7   Analysis

Now, while our calculations seem to agree, the time of cooking for the potato seems a bit far-fetched. For the temperature of this particular potato to last so long is somewhat hard to believe but there may be other factors at play we can consider.

We postulate: Will the number of collected points affect the overall estimated growth rate? As evident from the figure to the left, leaving all but two points still results in an approximate growth rate of -0071. Thus our findings lead us to believe that it is entirely possibly for the potatoes to have been cooked 4 hours prior to their discovery.

Some other considerations: Another way that the data could have been skewed is based on the temperature of the surroundings. Since we found our potatoes to be well secured its foil wrapper, we are left to think about the role of the surrounding temperature within our calculations. We are entirely under the assumption that our potato has been sitting in a room of $73°F$. Were this not the case, it is also entirely possible that the potato could have possibly been transported from different surroundings with different temperatures. Either way, at this point it is mere conjecture and is left up to the reader's own imagination. We now move on to our second problem at hand.
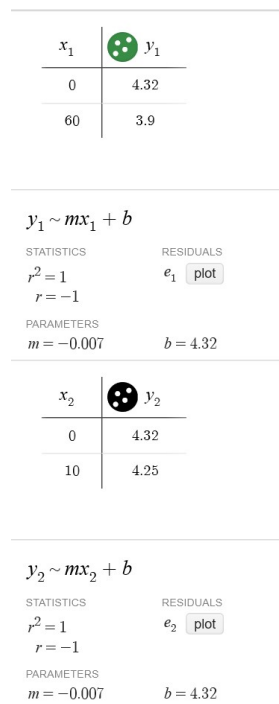
x_1		y_1
0		4.32
60		3.9

$y_1 \sim mx_1 + b$

STATISTICS   RESIDUALS
$r^2 = 1$ e_1 plot
$r = -1$
PARAMETERS
$m = -0.007$ $b = 4.32$

x_2		y_2
0		4.32
10		4.25

$y_2 \sim mx_2 + b$

STATISTICS   RESIDUALS
$r^2 = 1$ e_2 plot
$r = -1$
PARAMETERS
$m = -0.007$ $b = 4.32$

Figure 3: Testing regression with less data points.

3 The Mystery of the Missing Lancerlite

As the faithful students of MATH005BH, it is our responsibility to solve once and for all the mystery of the missing *Lancerlite*. While there isn't much we can do about the missing *Lancerlite*, we can however determine how many hours until the *Lancerlite* becomes explosive.

3.1 Exponential Decay

We will yet again derive our solution, but this time using our knowledge of exponential decay. Given that the rate of growth/decay is proportional the the remaining mass, we express this relationship as follows:

$$\frac{dM}{dt} = k(M_0)$$

Variables	Definition
dM	Rate of change in mass
dt	Rate of change in time
k	Rate of decay
M_0	Initial Mass

3.2 Solving our Differential Equation

Just like in Newton's Law of Cooling, our goal is to find a solution for the given differential equation. This is done by undoing the differential equation using integrals. Steps are as follows.

$$\frac{dM}{dt} = k(M_0) \tag{34}$$

$$\frac{dM}{M_0} = k(dt) \tag{35}$$

$$\int \frac{dM}{M_0} = \int k(dt) \tag{36}$$

$$\ln |M_0| + C_1 = kt + C_2 \tag{37}$$

$$\ln |M_0| = kt + C \tag{38}$$

$$e^{\ln |M_0|} = e^{kt+C} \tag{39}$$

$$|M_0| = e^c e^{kt} \tag{40}$$

$$|M_0| = Ce^{kt} \tag{41}$$

We find our solution to be $\boxed{|M_0| = Ce^{kt}}$.

3.3 Solving for the missing variables

Now that we have a solution to our differential equation, we can now use it to solve for k, the rate of growth. Very much like the first problem, we start by solving for our constant at $t = 0$.

Given the assumption that mass of the *Lancerlite* is at 1 (100%) at $t = 0$:

$$1 = Ce^{k(0)} \tag{42}$$

$$1 = C(1) \tag{43}$$

Since we find that $\boxed{C = 1}$ our constant be cancelled out from the original equation:

$$M = e^{kt} \tag{44}$$

To determine the rate of growth, we need to be able to compare another test point from our initial mass. Luckily, our suspect was kind enough to give us a clue: "The half-life of *Lancerlite* (in hours) is equal to one fifth the number formed by the last two digits of the first year that Jackie Robinson attended PCC." Being the resourceful students that we are, we find that that exact year to be 1937. Using that information we get the half-life in hours to be $\frac{37}{5} = 7.4$ hours!. We can now solve for our rate of growth, using $M_{7.4} = .50$.

Given that the half-life represents 50% of the mass, we take M to be 0.5:

$$M = e^{kt} \tag{45}$$

$$.50 = e^{k7.4} \tag{46}$$

$$\ln(.50) = \ln(e^{7.4k}) \tag{47}$$

$$-0.69 = 7.4k \tag{48}$$

$$k = -0.093 \tag{49}$$

Now that k is solved for, we can now fully solve for our missing variable t. We were informed by the authorities that the *Lancerlite* becomes explosive at 35% of its original amount. To calculate the time it takes to decay up to that point,

we reuse our equation:

$$M = e^{kt} \tag{50}$$

$$.35 = e^{-0.093t} \tag{51}$$

$$\ln(.35) = \ln(e^{7.4k}) \tag{52}$$

$$-1.04 = -0.093k \tag{53}$$

$$t = 11.28 \tag{54}$$

At approximately 11 hours and 17 minutes from the time the *Lancerlite* was stolen, it would have decayed to about 35% of its mass and become explosive! Given that the *Lancerlite* was removed from its laboratory storage 30 minutes after the potatoes were cooked (5:47am), that would only give the authorities until around 5:30pm to nab that potato fiend! Oh, wait a minute..