Day 2: Departures from OLS Assumptions and ANOVA

Outline

- Recap
- One-Way ANOVA
- Two-Way ANOVA

From Last Time...

Under certain conditions, we have the following results:

- $\hat{\beta} = (X^T X)^{-1} X^T Y$ is an unbiased estimator for regression coefficients β with variance $\sigma^2 (X^T X)^{-1}$.
- $S^2 = \frac{(Y - X\hat{\beta})^T (Y - X\hat{\beta})}{n-r}$ is an unbiased estimator for σ^2.
- If X has full rank, then $\hat{\beta}$ is the BLUE.
- If ϵ are normally distributed, then $\hat{\beta} \sim \mathcal{N}\left((X^T X)^{-1} X^T Y,\ \sigma^2 (X^T X)^{-1}\right)$.

From Last Time...

These results depend on four assumptions:

- Errors are unbiased: $E[\epsilon] = 0$.
- Errors have constant variance (homoscedasticity).
- Errors are uncorrelated.
- Errors are normally distributed (only necessary for distributional results).

Departures from Assumptions

What Happens When Assumptions Don't Hold?

There are a series of situations where the usual assumptions do not apply, and it is helpful to know how OLS estimates behave in those circumstances.

- Underfitting
- Overfitting
- Misspecified Covariance Matrix
- Non-normality

Underfitting

Underfitting means that the model was fit with too few explanatory variables. This can be represented as fitting

$$E[Y] = X\beta$$

when the true model generating the data is

$$E[Y] = X\beta + Z\gamma.$$

Underfitting

In this case,

$$\begin{aligned} E[\hat{\beta}] &= E[(X^T X)^{-1} X^T Y] \\ &= (X^T X)^{-1} X^T (X\beta + Z\gamma) \\ &= (X^T X)^{-1} X^T X\beta + (X^T X)^{-1} X^T Z\gamma \\ &= \beta + (X^T X)^{-1} X^T Z\gamma, \end{aligned}$$

so $\hat{\beta}$ is a biased estimator for β.

Underfitting

The covariance matrix for $\hat{\beta}$ is still the same:

$$\begin{aligned} V[\hat{\beta}] &= V[(X^T X)^{-1} X^T Y] \\ &= (X^T X)^{-1} X^T V[Y] X (X^T X)^{-1} \\ &= \sigma^2 (X^T X)^{-1} \end{aligned}$$

However, we find that underfitting causes S^2 to be biased upwards.

Overfitting

Overfitting involves including too many explanatory variables in the model, like fitting

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

when the true model is

$$Y = X_1\beta_1 + \epsilon$$

Overfitting

We can show that $\hat{\beta}$ is still unbiased:

$$
\begin{aligned}
E[\hat{\beta}] &= (X^T X)^{-1} X^T E[Y] \\
&= (X^T X)^{-1} X^T X_1 \beta_1 \\
&= (X^T X)^{-1} X^T \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} \\
&= (X^T X)^{-1} X^T X \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Overfitting

We can also show that the error variance estimate S^2 is unbiased, but the covariance matrix $Var(\hat{\beta})$ is higher than the true variance.

Misspecified Covariance

One way of misspecifying the covariance matrix is by assuming that $Var(\epsilon) = \sigma^2 I$ when it is actually

$$Var(\epsilon) = \sigma^2 V.$$

In this case,

$$\begin{aligned} Var(\hat{\beta}) &= Var((X^T X)^{-1} X^T Y) \\ &= (X^T X)^{-1} X^T Var(Y) X (X^T X)^{-1} \\ &= (X^T X)^{-1} X^T (\sigma^2 V) X (X^T X)^{-1} \\ &= \sigma^2 (X^T X)^{-1} X^T V X (X^T X)^{-1} \\ &\neq \sigma^2 (X^T X)^{-1} \end{aligned}$$

In most cases, S^2 is also a biased estimator for σ^2.

Non-Normality

If we have correctly specified the model $Y = X\beta + \epsilon$ with $E[\epsilon] = 0$ and $Var(\epsilon) = \sigma^2 I$, but incorrectly assume that the errors are normally distributed:

- $\hat{\beta}$ is unbiased for β
- $Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$
- If the sample size is large enough, then $\hat{\beta} \approx \mathcal{N}(\beta, \sigma^2 (X^T X)^{-1})$.

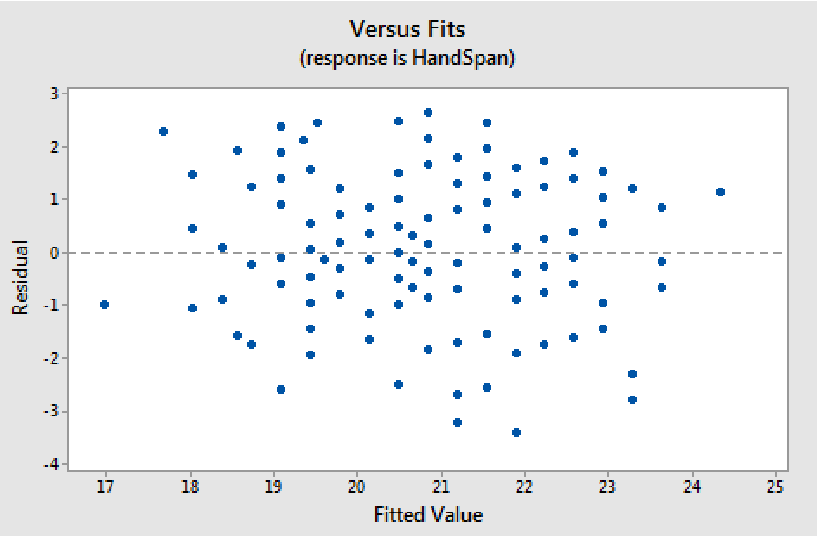
How Can We Check Assumptions?

We are able to check the validity of some assumptions empirically. The main tools we use are:

- Residual Plots
- Normal Q-Q Plots

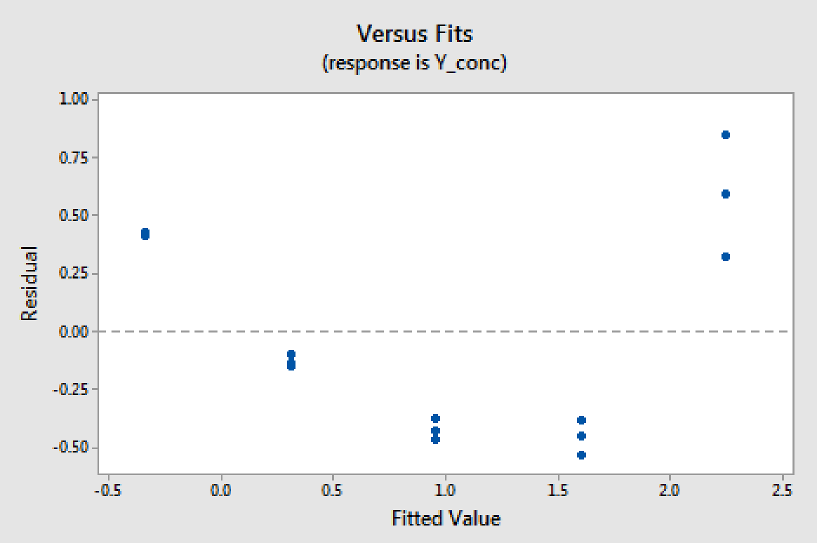
Residual Plots

Comparing the model residuals to the fitted values allows us to check whether the linearity and homoscedasticity assumptions are correct.



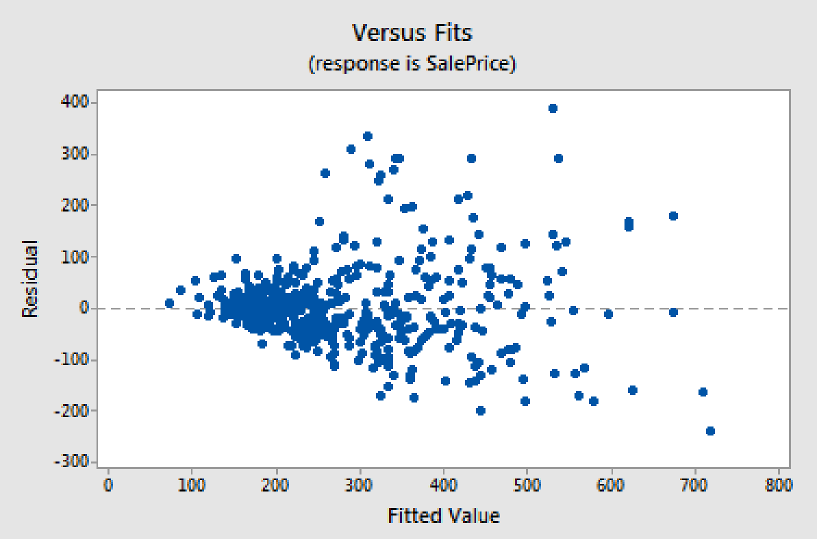
Residual Plots

Here we see a case where linearity does not hold:



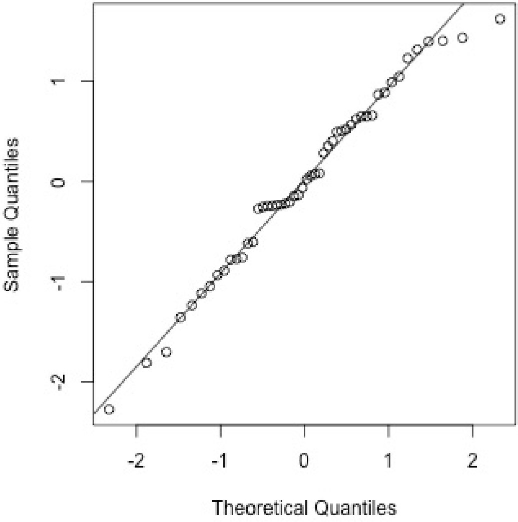
Residual Plots

Here, homoscedasticity does not hold:



Normal Q-Q Plots

Normal Q-Q plots allow us to verify the normality of the residuals:



Domain Knowledge

Domain knowledge can be the most important way to verify the validity of the model

- Do the variables included make sense?
- Are there variables that should be included that are not?
- Is a linear model a reasonable assumption?

ANOVA

Regression with Categorical Predictors

We usually consider linear regression being used with continuous explanatory variables, but sometimes we have variables that indicate belonging to a certain group.

- Analysis of Variance (ANOVA) models are essentially linear regression models where the explanatory variables are indicator variables.
- Analysis of Covariance (ANCOVA) models are regression models using both continuous and indicator variables as explanatory variables.

One-Way ANOVA

One-way ANOVA models are used in situations where we want to compare several independent samples. We have data where we assume:

- We have I independent samples
- Each sample is allowed to have a different number of observations, denoted J_i

The simplest way to describe this data is with the model:

$$Y_{ij} = \theta_i + \epsilon_{ij}.$$

Here, we can assume that $E[\epsilon_{ij}] = 0$, so $E[Y_{ij}] = \theta_i$. This means that θ_i represents the mean of the ith sample.

Alternative Parameterization

Another common way to parameterize the one-way ANOVA model is:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}.$$

One-Way ANOVA Assumptions

The validity of the ANOVA results depend on the following assumptions:

- $E[\epsilon_{ij}] = 0$.
- $Var(\epsilon_{ij}) = \sigma_i^2 < \infty$ and $Cov(\epsilon_{ij}, \epsilon_{i'j'}) = 0$ if $i \neq i'$ or $j \neq j'$.
- $\sigma_i^2 = \sigma^2$ for all i: samples have constant variances (homoscedasticity).
- ϵ_{ij} are independent and normally distributed.

These assumptions are the same as the OLS assumptions.

Inference Using ANOVA

One-way ANOVA models are traditionally used to test the hypotheses:

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_I$$
$$H_1 : \theta_i \neq \theta_j \text{ for some } i \neq j.$$

One-Way ANOVA Table

Variation	DF	SSE	MSE	F Statistic
Between Groups	$I-1$	$SSB = \sum_i J_i(\bar{y}_i - \bar{y})^2$	$MSB = \frac{SSB}{I-1}$	$F = \frac{MSB}{MSW}$
Within Groups	$N-I$	$SSW = \sum_j \sum_i (y_{ij} - \bar{y}_i)^2$	$MSW = \frac{SSW}{N-I}$	
Total	$N-1$	$SST = \sum_j \sum_i (y_{ij} - \bar{y})^2$		