Problem set 9.: Binomial theorem, Polynomial theorem, Inclusion-exclusion priciple, Pigeonhole principle

Question 1.

(a) Let $a \in \mathbb{R},\ a \neq 0$. Find the term in the expansion of $\left(\frac{1}{a} + a^2\right)^9$ which does not contain the parameter a .
(b) In the expansion of $(x^7 + 2x^3)^{27}$ find the coefficient of the x^{97} term.
(c) In the expansion of $(x^{11} + 5x^4)^{57}$ find the coefficient of the x^{417} term.
(d) In the expansion of $(6x^8 - 11x^5)^{32}$ find the coefficient of the x^{179} term.

Question 2.

Find the coefficient of each of the terms below in the expansion of $(x_1 + x_2 + x_3 + x_4)^{73}$:
(a) $x_1^{10} \cdot x_2^{23} \cdot x_3^{28} \cdot x_4^{12}$;
(b) $x_1^{9} \cdot x_2^{21} \cdot x_3^{20} \cdot x_4^{23}$;
(c) $x_1^{52} \cdot x_2^{7} \cdot x_3 \cdot x_4^{13}$;
(d) $x_1^{37} \cdot x_2^{11} \cdot x_3^{12} \cdot x_4^{14}$.

Question 3.

Prove that for every real number $x \geq 0$ we have $(x^3 + 7)^{61} \geq 7^{60} \cdot (7 + 61x^3)$.

Question 4.

In a survey, 100 participants were asked about the types of media sources they obtain information from. The number of people mentioning each type of media was as follows: television - 65, radio - 38, newspapers - 39 , television and radio - 20, television and newspapers - 20, radio and newspapers - 9 and television, radio and newspapers - 6. How many of the 100 respondents do not use any of the above sources (television, radio, newspapers)?

Question 5.

A group of 8 friends are going to the movies. In how many different ways can they sit in a row of 8 seats so that neither Anne and Beatrix nor Daniel and Esther are sitting next to each other?

Question 6.

We would like to post 8 different photos to a friend. We have 5 envelopes which are all different. In how many different ways can we put the 8 photos into these envelopes? (We do not necessarily have to use all envelopes.)

Question 7.

(a) Is it true that in any group of 8 children there are always (at least) 2 who were born on the same day of the week?
(b) At a meeting of 34 people each participant has at most 10 acquaintances present at the meeting. Is it true that there must be 4 people at the meeting who have the same number of acquaintances present?
∗ (c) Suppose we are given a square with sides of unit length. We choose 33 arbitrary points inside the square in such a way that no 3 points lie on the same line. Prove that there must be 3

points among the chosen ones such that the triangle determined by these points has an area of at most $\frac{1}{32}$.

Question 8.*
Consider all the strings of length six which can be formed using only the digits 0-9 (repetition is allowed and not all of these digits have to be used). How many of these strings are such that they do not contain the 'substring' 42, i.e. the digits 4 and 2 in this order, next to each other?