Discrete Mathematics I.

Relations

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(Based on Mérai László's slides in Hungarian)

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Relations

Relations

- are a generalization of the concept of functions;
- functions are special type of relations;
- are 'multivalued' functions;
- describe relationships;
- examples: $=, <, \leq, \subseteq$, divisibility, ...

Ordered pair

For any objects $x \neq y$ in the ordered pair (x, y) the order of the objects matters:

- $\{x, y\} = \{y, x\}$
- $(x, y) \neq (y, x)$.

Definition (ordered pair)

The ordered pair (x, y) is defined as the set $\{\{x\}, \{x, y\}\}$; x is the first coordinate and y is the second coordinate of (x, y).

Definition (Cartesian product of sets)

The Cartesian product of two sets X and Y is defined as

$$X \times Y = \{(x, y) \ : \ x \in X, y \in Y\}.$$

Binary relations

Definition (binary relation)

Let X and Y be sets. If $R \subseteq X \times Y$ then we call R a relation from X to Y. If $X = Y$, then we say that R is a relation on X and in this case we say that R is a homogeneous binary relation.

If R is a binary relation, then $(x, y) \in R$ is often written as $x \; R \; y$.

Examples

1. $\mathbb{I}_X = \{(x, x) \in X \times X \ : \ x \in X\}$ is the *identity relation* on the set X.
2. $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \ : \ x|y\}$ is the *divisibility relation*.
3. For a system of sets $\mathscr{F}$, $\{(X, Y) \in \mathscr{F} \times \mathscr{F} \ : \ X \subseteq Y\}$ is the *subset relation* on $\mathscr{F}$.
4. For a function $f : \mathbb{R} \to \mathbb{R}$ the graph $\{(x, f(x)) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$ of the function is a relation on $\mathbb{R}$.

Domain and range of a binary relation

If R is a relation from X to Y, then ($R \subseteq X \times Y$) and $X \subseteq X'$, $Y \subseteq Y'$, then R is also a relation from X' to Y'!

Definition (domain and range)

The domain of a relation $R \subseteq X \times Y$ is the set

$$dmn(R) = \{x \in X \mid \exists y \in Y : (x, y) \in R\},$$

and the range of R is the set

$$rng(R) = \{y \in Y \mid \exists x \in X : (x, y) \in R\}.$$

Examples

1. If $R = \{(x, 1/x^2) \ : \ x \in \mathbb{R}\}$ then $dmn(R) = \{x \in \mathbb{R} \ : \ x \neq 0\}$, $rng(R) = \{x \in \mathbb{R} \ : \ x > 0\}$.
2. If $R = \{(1/x^2, x) \ : \ x \in \mathbb{R}\}$ then $dmn(R) = \{x \in \mathbb{R} \ : \ x > 0\}$, $rng(R) = \{x \in \mathbb{R} \ : \ x \neq 0\}$.

Restrictions and extensions of a binary relation

Definition (restriction and extension of a binary relation)

If $S \subseteq R$ for some binary relations R and S, then we say that R is an extension of S and S is a restriction of R.
Let A be a set. Then the restriction of the binary relation R to A is the relation

$$R|_A = \{(x, y) \in R \ : \ x \in A\}.$$

Example
Let $R = \{(x, x^2) \in \mathbb{R} \times \mathbb{R} \ : \ x \in \mathbb{R}\}$ and $S = \{(\sqrt{x}, x) \in \mathbb{R} \times \mathbb{R} \ : \ x \in \mathbb{R}\}$.
Then R a is an extension of S and S is a restriction of R, furthermore $S = R|_{\mathbb{R}_0^+}$ ($\mathbb{R}_0^+$ is the set of nonnegative real numbers).

Inverse of a binary relation

Definition (inverse of a binary relation)

The inverse of a binary relation R is defined as

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

.

Example
$R^{-1} = \{(x^2, x) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$, $S^{-1} = \{(x, \sqrt{x}) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$

Image and inverse image of a set

Definition (image and inverse image of a set)

Let $R \subseteq X \times Y$ be a binary relation and A be a set. The image of A under the relation R is the set $R(A) = \{y \in Y \mid \exists x \in A : (x, y) \in R\}$. The inverse image or preimage of a set B is $R^{-1}(B)$, that is the image of B under the relation R^{-1}.

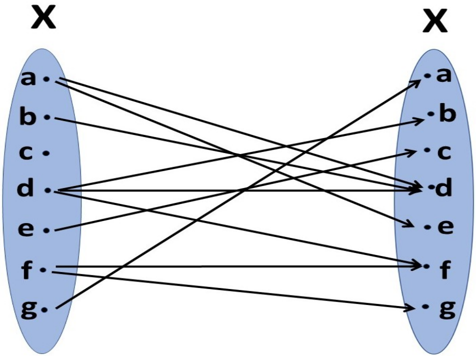
Examples

Let $R = \{(x^2, x) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$ and $S = \{(x, \sqrt{x}) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{R}\}$. Then

- $R(\{9\}) = \{-3, +3\}$ (using a shorter notation: $R(9) = \{-3, +3\}$),
- $S(9) = \{+3\}$.

Example

Let R be the relation on the set $X = \{a, b, c, d, e, f, g\}$ shown in the diagram below.



Then:

- $dmn(R) = \{a, b, d, e, f, g\}$.
- $rng(R) = \{a, b, c, d, e, f, g\} = X$.
- $R|_{\{a,b,c,d\}} = \{(a, d), (a, e), (b, d), (d, b), (d, d), (d, f)\}$.

Composition of binary relations

Definition (composition of binary relations)

Let R and S be binary relations. The composition of R and S is the binary relation defined as:

$$R \circ S = \{(x, y) | \exists z : (x, z) \in S, (z, y) \in R\}.$$

In the composition we write the relations ´from right to left':

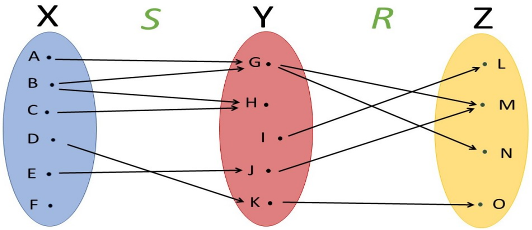
Example

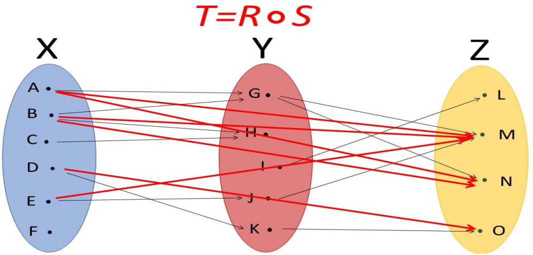
Let $R_{\sin} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \sin x = y\}$ and $S_{\log} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \log x = y\}$.

Then
$R_{\sin} \circ S_{log} = \{(x, y) | \exists z : \log x = z, \sin z = y\}$
$= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \sin \log x = y\}$.

Composition of relations: example

Example: Let $S \subseteq X \times Y$ and $R \subseteq Y \times Z$ be two relations. Consider the composition $T = R \circ S$:





Properties of composition

Proposition (Properties of the composition of relations)

Let R, S *and* T *be binary relations. Then*

1. $R \circ (S \circ T) = (R \circ S) \circ T$ *(composition is associative).*
2. $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

Proof

1. $(x, w) \in R \circ (S \circ T) \Leftrightarrow \exists z : (x, z) \in S \circ T \wedge (z, w) \in R \Leftrightarrow \exists z\ \exists y : (x, y) \in T \wedge (y, z) \in S \wedge (z, w) \in R \Leftrightarrow \exists y\ \exists z : (x, y) \in T \wedge (y, z) \in S \wedge (z, w) \in R \Leftrightarrow \exists y : (x, y) \in T \wedge (y, w) \in R \circ S \Leftrightarrow (x, w) \in (R \circ S) \circ T$
2. $(z, x) \in (R \circ S)^{-1} \Leftrightarrow (x, z) \in R \circ S \Leftrightarrow \exists y : (x, y) \in S \wedge (y, z) \in R \Leftrightarrow \exists y : (y, x) \in S^{-1} \wedge (z, y) \in R^{-1} = (z, x) \in S^{-1} \circ R^{-1}$.

Properties of homogeneous binary relations

Example

Relations: $=$, $<$, $\leq$, $|$, $\subseteq$, $T = \{(x, y) : x, y \in \mathbb{R}, |x - y| < 1\}$.

Definition (properties of homogeneous binary relations)

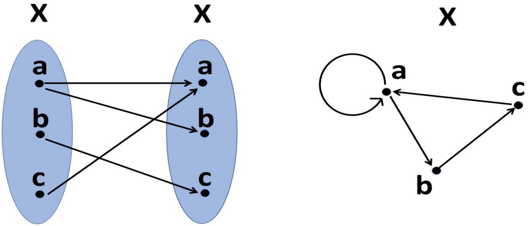
Let R be a relation on X. Then:

1. R transitive if $\forall x, y, z \in X : (x\ R\ y \wedge y\ R\ z) \Rightarrow x\ R\ z$; ($=$, $<$, $\leq$, $|$, $\subseteq$)
2. R symmetric if $\forall x, y \in X : x\ R\ y \Rightarrow y\ R\ x$; ($=$, T)
3. R anti-symmetric if $\forall x, y \in X : (x\ R\ y \wedge y\ R\ x) \Rightarrow x = y$;($=$, $\leq$, $\subseteq$)
4. R strictly anti-symmetric if $\forall x, y \in X : x\ R\ y \Rightarrow \neg y\ R\ x$; ($<$)
5. R reflexive if $\forall x \in X : x\ R\ x$; ($=$, $\leq$, $|$, $\subseteq$, T)
6. R irreflexive if $\forall x \in X : \neg x\ R\ x$; ($<$)
7. R trichotomous if $\forall x, y \in X$ *exactly one of* the following three statements is true: $x = y$, $x\ R\ y$ or $y\ R\ x$; ($<$)
8. R dichotomous if $\forall x, y \in X$ *at least one of* the following two statements holds (perhaps both): $x\ R\ y$ or $y\ R\ x$. ($\leq$)

Properties of homogeneous binary relations

The reflexive, trichotomous and dichotomous properties of a relation also depend on the underlying set:
For example, $\{(x,x) \in \mathbb{R}\times\mathbb{R}, x\in\mathbb{R}\} \subseteq \mathbb{R}\times\mathbb{R} \subseteq \mathbb{C}\times\mathbb{C}$ considered as a relation on $\mathbb{R}$ is reflexive, but as a relation on $\mathbb{C}$, it is not reflexive.

Example



transitive	×	strictly anti-symmetric	×	trichotomous	×
symmetric	×	reflexive	×	dichotomous	×
anti-symmetric	✓	irreflexive	×		

Equivalence relations, equivalence classes

Definition (equivalence relation)

Let X be a set. A binary relation R on X is called an equivalence relation if it is reflexive, symmetric and transitive.

Examples

1. $=$;
2. $\forall x, y \in \mathbb{N}$: $x \sim y$ if and only if $5|(x-y)$.

Definition (equivalence class of an element)

Let $\sim$ be an equivalence relation on a set X. The equivalence class $\tilde{x} = [x]$ of an element x in X is the set of those elements of X which are $\sim$-related to x, that is:

$$[x] = \{y \in X \mid y \sim x\}.$$

Partitions of sets

Definition (partition of a set)

Let $X \neq \emptyset$ be a set. A system $\mathscr{P}$ of subsets of X is called a partition of X (or quotient set of X) if:

- the elements of $\mathscr{P}$ are nonempty,
- $\mathscr{P}$ is a pairwise disjoint system and
- $\cup\mathscr{P} = X$.

The elements of $\mathscr{P}$ are called the blocks or cells of the partition $\mathscr{P}$.

Examples

1. a partition of $\mathbb{R}$: $\{\{a\} : a \in \mathbb{R}\}$
2. another partition of $\mathbb{R}$: $\{\{a \in \mathbb{R} : |a| = r\} : r \in \mathbb{R}_0^+\}$
3. a partition of $X = \{a, b, c, d, e, f, g\}$: $\{\{a, c\}, \{b\}, \{e\}, \{d, f, g\}\}$

Partitions induced by equivalence relations

Theorem (Partitions induced by equivalence relations)

Let $\sim$ be an equivalence relation on a set $X \neq \emptyset$. Then the set of all equivalence classes of $\sim$: $\{[x] \mid x \in X\}$ is a partition of X. This partition is called the quotient set of X and is denoted by $X/\sim$.

Proof (For sake of completeness; **not required** for the exam)

Let $\sim$ be an equivalence relation on X. We need to show that $X/\sim = \{[x] : x \in X\}$ is a partition of X.

- *As $\sim$ is reflexive, $x \in [x]$ and so*
  - $\cup\{[x] : x \in X\} = X$ *and*
  - $[x] \neq \emptyset$
- *We show that if $[x] \neq [y]$ for some $x, y \in X$ then $[x] \cap [y] = \emptyset$. Suppose $[x] \cap [y] \neq \emptyset$ for some $x, y \in X$ and let $z \in [x] \cap [y]$. As $z \in [x]$, hence $z \sim x$, which – by symmetry of $\sim$ – implies $x \sim z$. Similarly, $z \in [y]$ implies $z \sim y$. If $x_1 \in [x]$, then $x_1 \sim x$, hence by transitivity of $\sim$, $x_1 \sim x \wedge x \sim z \Rightarrow x_1 \sim z$, and so $x_1 \sim z \wedge z \sim y \Rightarrow x_1 \sim y \Rightarrow x_1 \in [y]$.*

  It can be shown similarly that for every $y_1 \in [y]$, $y_1 \in [x]$ holds. Therefore $[x] = [y]$.

Equivalence relations determined by partitions

Theorem (The equivalence relation determined by a partition)

Let $\mathscr{P}$ be a partition of a nonempty set X. Then the relation

$$R = \{(x, y) \in X \times X \mid x \text{ belongs to the same cell of } \mathscr{P} \text{ as } y\}$$

is an equivalence relation, and the partition induced by R is $\mathscr{P}$.

Proof (For sake of completeness; **not required** for the exam)

- *R is reflexive: every $x \in X$ clearly belongs to the same cell as itself, hence $x \; R \; x$.*
- *R is symmetric: if $(x, y) \in R$ then x belongs to the same cell as y, hence y belongs to the same cell as x and so $(y, x) \in R$.*
- *R transitive: if $(x, y), (y, z) \in R$ then x belongs to the same cell as y and y belongs to the same cell as z, hence x belongs to the same cell as z and so $(x, z) \in R$.*

Equivalence relations and partitions

For a nonempty set X, the equivalence relations on X and the partitions of X can be put into a one-to-one correspondence with each other: they mutually determine each other.

Examples (equivalence relations and the corresponding partitions)

- $=$ on $\mathbb{R} \leftrightarrow \{\{a\} : a \in \mathbb{R}\}$;
- $\forall x, y \in \mathbb{R}$: $x \sim y$ iff $|x| = |y| \leftrightarrow \{\{x, -x\} : x \in \mathbb{R}\}$.
- Let two lines in a plane be $\sim$-related iff they are parallel to each other. Then the equivalence classes of $\sim$ can be identified with the different directions in the plane.
- Let two line segments of a given plane be $\sim$-related iff they are congruent to each other. Then the equivalence classes of $\sim$ yield the notion of length of the line segments in the plane.

Partial orders and orders

Definition ((partial) order and (partially) ordered set)

- A binary relation on a set X is called a partial order if it is reflexive, transitive and anti-symmetric. (Notations: $\leq$, $\preceq$, $\ldots$)
- If $\preceq$ is a partial order on a set X then the pair $(X; \preceq)$ is called a partially ordered set.
- If for some $x, y \in X$ $x \preceq y$ or $y \preceq x$ holds, then x and y are said to be comparable. (If x and y are comparable for every $x, y \in X$ then the relation is dichotomous.)
- A binary relation on a set X is called an order or a total order if it is reflexive, transitive, anti-symmetric and dichotomous. (In other words, an order is a dichotomous partial order: a partial order such that every pair of elements are comparable.)

Examples

- On $\mathbb{R}$ the standard $\leq$ is an order: $\forall x, y \in \mathbb{R} : x \leq y$ or $y \leq x$.
- The divisibility relation $\mid$ on $\mathbb{N}$ is a partial order, but not an order: $4 \not| 5$, $5 \not| 4$.
- The subset relation $\subseteq$ on $X = \mathscr{P}(\{a, b, c\})$ (the power set of $\{a, b, c\}$) is a partial order, but not an order: $\{a\} \not\subseteq \{b, c\}$, $\{b, c\} \not\subseteq \{a\}$.

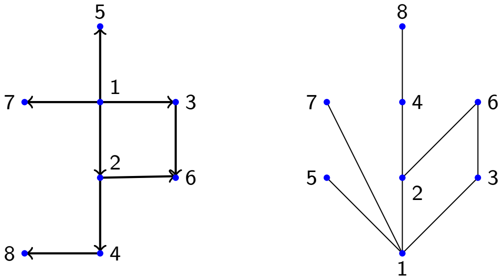
Hasse-diagram of a partially ordered set

Definition (immediate predecessor)

Let $(X; \preceq)$ be a partially ordered set. If for some $x, y \in X$ we have $x \prec y$, but $\nexists\, z \in X$ such that $x \prec z \prec y$, then x is an immediate predecessor of y (or x immediately predecedes y).

In a Hasse-diagram of a partially ordered set $(X; \preceq)$ the elements of the set are represented by 'dots'; for every $x, y \in X$ we draw a directed edge ('arrow') from x to y if and only if x is an immediate predecessor of y. Sometimes they use undirected edges ('lines') instead of directed edges and in this case the smaller element has to be placed vertically lower that the greater one, in the diagram.

Example: Consider $X = \{1, 2, \ldots, 8\}$ with the divisibility relation:



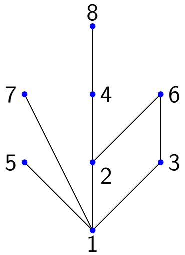
Least, greatest, minimal and maximal element(s)

Definition (least, greatest, minimal and maximal element(s))

An element x in a partially ordered set $(X; \preceq)$ is called a
least element iff $\forall y \in X: \ x \preceq y$;
greatest element iff $\forall y \in X: \ y \preceq x$;
minimal element iff $\neg\exists y \in X: x \neq y, \ y \preceq x$;
maximal element iff $\neg\exists y \in X: x \neq y, \ x \preceq y$.

Consider $X = \{1, 2, \ldots, 8\}$ with the divisibility relation:

least element: 1,
greatest element: does not exist,
minimal element: 1,
maximal elements: 5, 6, 7, 8.



Strict partial orders

Definition (strict partial order)

A binary relation on a set X is called a strict partial order if it is transitive and irreflexive. (Notations: $<$, $\prec$, $\dots$)
A trichotomous strict partial order is called a strict order.

Examples

- The relation $<$ on $\mathbb{R}$ is a strict order: $\forall x, y \in \mathbb{R}:$ exactly one of the following three conditions holds: $x = y$, $x < y$ and $y < x$.
- The proper subset $\subsetneq$ relation is a strict partial order on $X = \mathscr{P}(\{a, b, c\})$, but not a strict order: none of the statements $\{a\} = \{b, c\}$, $\{a\} \subsetneq \{b, c\}$ and $\{b, c\} \subsetneq \{a\}$ is true.

Functions

Definition (function)

A binary relation $f \subseteq X \times Y$ is called a function (or map, mapping, transformation, operator) if

$$\forall x, y, y' : (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'.$$

If f is a function then for $(x, y) \in f$, the notations $f(x) = y$, $f : x \mapsto y$ and $f_x = y$ are also used and y is called the value of the function f at (argument) x .

Examples

- The relation $f = \{(x, x^2) \in \mathbb{R} \times \mathbb{R}\}$ is a function: $f(x) = x^2$.
- The inverse relation $f^{-1} = \{(x^2, x) \in \mathbb{R} \times \mathbb{R}\}$ of f is not a function: $(4, 2), (4, -2) \in f^{-1}$.
- The Fibonacci sequence F_n defined as: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$: $0, 1, 1, 2, 3, 5, 8, \ldots$ The the relation $F \subseteq \mathbb{N} \times \mathbb{N}$ is a function; the value of F at n is $F(n) = F_n$.

Functions: the set of functions $X \to Y$

Definition (set of functions $X \to Y$)

Let X and Y be sets. The set of all functions $f \subseteq X \times Y$ is denoted by $X \to Y$, hence the notation $f \in X \to Y$ can be also used. If $dmn(f) = X$, then we can also write $f : X \to Y$ (but this notation can be used only when $dmn(f) = X$).

Note: If $f : X \to Y$ then $dmn(f) = X$ and $rng(f) \subseteq Y$.

Example

Let $f(x) = \sqrt{x}$. Then

- $f \in \mathbb{R} \to \mathbb{R}$, but we cannot write $f : \mathbb{R} \to \mathbb{R}$.
- $f : \mathbb{R}_0^+ \to \mathbb{R}$.
- $f : \mathbb{R}_0^+ \to \mathbb{C}$.

Functions: injective, surjective and bijective functions

Definition (injective, surjective and bijective functions)

A function $f : X \to Y$ is called

- injective if $\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$;
- surjective if $rng(f) = Y$;
- bijective if it is both injective and surjective.

Note: A function f is injective if and only if the relation f^{-1} is a function.

Examples

- The function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^2$ is not injective and not surjective: $f(-1) = f(1)$, $rng(f) = \mathbb{R}_0^+$.
- The function $f : \mathbb{R} \to \mathbb{R}_0^+$, $f : x \mapsto x^2$ is not injective, but surjective.
- The function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $f : x \mapsto x^2$ is injective and surjective, hence bijective.

Note: Whether a function $f : X \to Y$ is surjective or not, depends on Y. If $Y \subsetneq Y'$, then $rng(f) \subseteq Y \subsetneq Y'$, hence the function $f : X \to Y'$ cannot be surjective.

Functions: permutations

Definition (permutations on a set)

Let X be a set. A bijective function $f : X \to X$ is called a permutation of X.

Examples

- Let $X = \{1, 2, \dots, n\}$. Then the number of permutations of X is $n!$.
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a permutation of the set of real numbers.
- The function $f(x) = x^2$ is not a permutation of $\mathbb{R}$: it is not injective and not surjective.

Composition of functions

Reminder

Composition of relations: $R \circ S = \{(x, z) | \exists y : (x, y) \in S \wedge (y, z) \in R\}$.

function: A relation f is a function, if $(x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$.

Theorem (Properties of the composition of functions)

1. *If f and g are functions, then the relation $g \circ f$ is also a function.*
2. *If f and g functions, then $(g \circ f)(x) = g(f(x))$.*
3. *If f and g injective functions, then $g \circ f$ is also an injective function.*
4. *If $f : X \to Y$ and $g : Y \to Z$ surjective functions, then $g \circ f : X \to Z$ is also a surjective function.*

Proof

1. *Let $(x, z) \in g \circ f$ and $(x, z') \in g \circ f$. Then $\exists y : (x, y) \in f, (y, z) \in g$ and $\exists y' : (x, y') \in f, (y', z') \in g$. Since f is a function, $y = y'$, and since g is a function, $z = z'$.*

Composition of functions: proof of Theorem continued

Proof (continued)

2. *Let* $(g \circ f)(x) = z$. *Then there exists* y *such that* $(x, y) \in f \wedge (y, z) \in g$. *Since* f *and* g *are functions, hence* $f(x) = y$ *and* $g(y) = z$, *and so* $g(f(x)) = z$.
3. *Let* $(g \circ f)(x) = (g \circ f)(x')$, *that is* $g(f(x)) = g(f(x'))$. *As* g *is injective, hence* $f(x) = f(x')$. *As* f *is injective, hence* $x = x'$.
4. *Hw.*

Operations

Definition (unary and binary operations)

Let X be a set. A function $* : X \times X \to X$ is called a binary operation on X. We often write $x * y$ instead of $*(x, y)$.
A function $* : X \to X$ is called a unary operation on X.

Examples

- On $\mathbb{R}$, $+$ and $\cdot$ are binary operations and $x \mapsto -x$ (opposite) is a unary operation.
- On $\mathbb{R}$ division $\div$ is not an operation, because $dmn(\div) \neq \mathbb{R} \times \mathbb{R}$.
- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ division $\div$ is a binary, $x \mapsto \frac{1}{x}$ (reciprocal) is a unary operation.

Operations

An operation on a finite set can be defined by its operation table.

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

XOR	T	F
T	F	T
F	T	F

	$\neg$
T	F
F	T

Definition (operations with functions)

Let X and Y be sets, $*$ an operation on Y and $f, g : X \to Y$ be functions. Then :

$$\forall x \in X : (f * g)(x) = f(x) * g(x).$$

Example

For the functions $\sin, \cos : \mathbb{R} \to \mathbb{R}$ we have: $(\sin + \cos)(x) = \sin x + \cos x$ $\forall x \in X$.

Properties of binary operations

Definition (associative and commutative operations)

A binary operation $*: X \times X \to X$ is

- associative if $\forall a, b, c \in X : (a * b) * c = a * (b * c)$;
- commutative if $\forall a, b \in X : a * b = b * a$.

Examples

- Addition and multiplication are associative and commutative operations on $\mathbb{R}$.
- The composition of functions is an associative operation: $(f \circ g) \circ h = f \circ (g \circ h)$.
- The composition of $\mathbb{R} \to \mathbb{R}$ functions is not commutative: $f(x) = x + 1, g(x) = x^2$: $(f \circ g)(x) = x^2 + 1 \neq (x + 1)^2 = (g \circ f)(x)$.
- Division is not an associative operation on $\mathbb{R}^*$: $(a \div b) \div c = \frac{a}{bc} \neq \frac{ac}{b} = a \div (b \div c)$

Operation-preserving mappings

Definition (operation-preserving mapping)

Let X and Y be sets with binary operations $*$ and $\diamond$, respectively. A function $f : X \to Y$ is operation-preserving if $\forall x_1, x_2 \in X$:

$$f(x_1 * x_2) = f(x_1) \diamond f(x_2).$$

Examples

- Consider $X = \mathbb{R}$ with the operation of addition $+$ and $Y = \mathbb{R}^+$ with the operation of multiplication $\cdot$.
  Then for any $a \in \mathbb{R}^+$ the function $x \mapsto a^x$ is operation-preserving:
  $\forall x_1, x_2 \in \mathbb{R} : a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$.
- Consider $X = Y = \mathbb{R}$ with the operation of addition $+$.
  Then $x \mapsto -x$ is operation-preserving:
  $\forall x_1, x_2 \in \mathbb{R} : -(x_1 + x_2) = (-x_1) + (-x_2)$.