Discrete mathematics 1.

Combinatorics

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Combinatorics

The goal of combinatorics:

- To organize of elements of finite sets;
- To enumerate the different possible arrangements.

Examples:

- Among any eight people, there are always two who were born on the same day of the week.
- At least how many people do we need to have in a group in order to be certain that two of them have their birthdays on the same day of the year?
- At least how many people do we need to have in a group in order to be certain that two of them were be born in the same month?
- What is the number of all the possible car registration plates /telephone numbers / IP addresses?
- At least how many tickets do we have to complete in order to certainly win the jackpot in the lottery? (In the lottery five numbers from among the numbers $1-90$ are drawn randomly. You can bet by crossing out five from among the numbers $1-90$ on a lottery ticket. You win the jackpot if you crossed exactly those numbers which are drawn later.)

Addition principle (Rule of sum)

Addition principle

Given two finite, disjoint sets:

$A = \{a_1, a_2, \ldots, a_n\}, \quad B = \{b_1, b_2, \ldots, b_m\}.$

in how many ways can we choose one element from A or B?

The possible choices: $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$.

The number of possible choices: $n + m$.

Example

In a pastry shop, they have 3 kinds of sweet pastries (jam roll, cheese cake, coconut cube) and 2 kinds of savory pastries (scones, pretzels). In how many different ways can we choose one sweet or one savory pastry?
Solution: $3 + 2 = 5$.

Multiplication principle (Rule of product)

Multiplication principle

Given two finite, disjoint sets:

$A = \{a_1, a_2, \ldots, a_n\}, \quad B = \{b_1, b_2, \ldots, b_m\}.$

in how many ways can we choose one element from A and B?

The number of possible choices: $n \cdot m$.

$$\begin{array}{c|cccc} & b_1 & b_2 & \ldots & b_m \\ \hline a_1 & (a_1, b_1) & (a_1, b_2) & \ldots & (a_1, b_m) \\ a_2 & (a_2, b_1) & (a_2, b_2) & \ldots & (a_2, b_m) \\ \vdots & & & \ddots & \\ a_n & (a_n, b_1) & (a_n, b_2) & \ldots & (a_n, b_m) \end{array}$$

Example

In a pastry shop, they have 3 kinds of sweet pastries and 2 kinds of savory pastries. In how many different ways can we choose one sweet and one savory pastry? Solution: $3 \cdot 2 = 6$.

Summary

Permutations (without repetition): $n!$, a sequence of n distinct elements, containing each element exactly once (order matters, each element occurs exactly once).

Permutations with repetition: $\frac{(k_1+k_2+\ldots+k_m)!}{k_1!\cdot k_2!\cdot\ldots\cdot k_m!}$, a sequence of length $n=k_1+k_2+\ldots+k_m$, containing the element of kind i exactly k_i times ($1\le i\le m$) (order matters, an element can occur more than once).

Variations without repetition: $n!/(n-k)!$, a sequence of length k chosen from n different elements, containing each element at most once (order matters, an element can occur at most once).

Variations with repetition: n^k, a sequence of length k chosen from n different elements (order matters, an element can occur more than once).

Combinations (without repetition): $\binom{n}{k}$, k-element subset of an n-element set (order does not matter, an element can occur at most once).

Combinations with repetition: $\binom{n+k-1}{k}$, choosing k times from among n elements disregarding the order in which the elements were chosen and allowing for choosing an element more than once (order does not matter, an element can occur more than once).

Permutations (without repetition)

Definition (permutation)

A permutation of a finite set A is a sequence containing each element of A exactly once. (In other words, a possible order of the elements of A.)

An equivalent definition: A permutation of a set A is a bijection $A \to A$.

Theorem (Number of permutations)

The number of permutations of an n-element set is:

$$P_n = n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$$

(Read $n!$ as n factorial). Note: $0! = 1$.

Proof

The first element of the sequence can be chosen in n different ways. After this, the second element of the sequence can be chosen in $n-1$ different ways, ... Hence the number of permutations is $n(n-1) \cdot \ldots \cdot 2 \cdot 1$. $\square$

Permutations (without repetition)

Examples

1. In a race 70 runners took part. In how many different orders can they finish? (We assume that everybody completes the race and there are no equal finishes.)
2. For breakfast we can eat
   2 different sandwiches in $2! = 2 \cdot 1 = 2$ different orders.
   3 different sandwiches in $3! = 3 \cdot 2 \cdot 1 = 6$ different orders.
   4 different sandwiches in $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ different orders.
3. A group of 200 students can sign the attendance sheet in $200! = 200 \cdot 199 \cdot 198 \cdot \ldots \cdot 2 \cdot 1 \approx 7,89 \cdot 10^{374}$ different orders.

Permutations with repetition

Example

In an exam 5 students took part and 2 grade 4's and 3 grade 5's were awarded. In how many different orders can we list the results on a paper?

Solution

If we take into account which student obtained each grade (for example, we indicate each student's name next to his/her grade) the there are $(2+3)! = 5!$ possible orders.

If we disregard which student obtained each grade then in the previous calculation we counted each possible order in multiple times:

5	5	5	5	5	5	5	5	5	5	5	5	
5	5	5	5	5	5	5	5	5	5	5	5	
5	5	5	5	5	5	5	5	5	5	5	5	...
4	4	4	4	4	4	4	4	4	4	4	4	
4	4	4	4	4	4	4	4	4	4	4	4	

The grade 5's can be permuted among each other in $3! = 6$ different ways. Similarly, the grade 4's can be permuted $2! = 2$ different ways with each other. Therefore each order of the grades were counted $3!2!$ times. Hence the number of different orders is: $\frac{5!}{2! \cdot 3!} = \frac{120}{2 \cdot 6} = 10$.

Permutions with repetition

Theorem (Number of permutations with repetition)

A permutation with repetition is a sequence of m different kinds of elements containing k_1 number of elements of the first kind, k_2 number of elements of the second kind, ..., k_m number of elements of the m^{th} kind, and the number of these is:

$$\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$$

where $n = k_1 + k_2 + \ldots + k_m$.

Proof

If we distinguish between all elements, then there are $n! = (k_1 + k_2 + \ldots + k_m)!$ possible sequences of the n elements.
However, we do not want to distinguish between elements of the same kind, but for each i we are only interested in the set of positions occupied by the elements of the i^{th} kind. If we fix the k_i positions for the elements of the i^{th} kind, we can permute these elements in these positions in $k_i!$ ways. Hence, in $n!$, each sequence has been counted $k_1! \cdot k_2! \cdot \ldots \cdot k_m!$ times. Therefore the number of permutations with repetition is $\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$.

$\square$

Variations (partial permutations)

Example

In a horse race there are 30 runners. How many different outcomes are possible for the first five places?

Solution

The winner can be chosen in 30 ways, then we can choose the horse for the 2^{nd} place in 29 different ways, ..., we can choose the horse for the 5^{th} place in 26 different ways.
Hence there are $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$ possible outcomes for the first five places.

Definition (variation (or partial permutation))

Let A be a set and $k \in \mathbb{N}$. A sequence of length k of elements in A containing each element of A *at most once*, is called a k-variation without repetition of A.

Variations (partial permutations)

Theorem (Number of variations (without repetition))

Let $k^+ \in \mathbb{N}$. The number of k-variations (without repetition) of an n-element set is

$$V_n^k = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n!/(n-k)!$$

if $k \leq n$ and is 0 otherwise.

Proof

Let $k \leq n$. The first element of the sequence can be chosen in n different ways from set A. After this, the second element can be chosen in $n-1$ different ways (the first element of the sequence cannot be chosen again), then the third element can be chosen in $n-2$ different ways... the k^{th} element can be chosen in $n-k+1$ different ways. Therefore there are $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n!/(n-k)!$ k-variations in total. If $k > n$ then A clearly does not have a k-variation without repetition. $\square$

Variations with repetition

Example

How many different 2-digit numbers can be formed using the digits 1, 2, 3, if not all digits need to be used and the repetition of the digits is allowed?

Solution

The first digit of the number can be chosen in 3 different ways:

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

The second digit of the number can be chosen again in 3 different ways:

$$\begin{matrix} 11 & 21 & 31 \\ 12 & 22 & 32 \\ 13 & 23 & 33 \end{matrix}$$

The total number of possibilities:

$$3 \cdot 3 \quad = 9$$

Variations with repetition

Definition (variation with repetition)

Let A be a set and $k \in \mathbb{N}$. A sequence of length k consisting of some elements of A (an element may occur more than once), is called a k-variation with repetition of A.

Theorem (Number of variations with repetition)

Let $n, k \in \mathbb{N}$. The number of k-variations with repetition of an n-element set is: n^k.

Proof

The first element of the sequence can be chosen in n different ways, then the second element can be chosen also in n different ways . . . . Therefore the sequence of length k can be chosen in n^k different ways. $\square$

Variations with repetition

Examples

1. How many different 10-digit numbers can be formed using the digits $1-9$, if not all the digits have to be used and the repetition of digits is allowed?

   Each of the 10 digits can be chosen in 9 different ways (independently of each other), hence there are 9^{10} such numbers.

2. How many subsets does an n-element set have?

   Let $A=\{a_1,a_2,\dots,a_n\}$. To each subset S of A we can assign a $0-1$ sequence of length n: for each $1\le i\le n$ let the i^{th} element of the sequence be 1 if S contains a_i and 0 otherwise.
   $\emptyset\leftrightarrow(0,0,\dots,0)$, $\{a_1,a_3\}\leftrightarrow(1,0,1,0,\dots,0)$, ...,
   $A\leftrightarrow(1,1,\dots,1)$
   How many $0-1$ sequences of length n are there? 2^n.

Combinations

Definition (combination (without repetition))

Let $k \in \mathbb{N}$. A k-element subset of a set A is called a k-combination of A.

Theorem (Number of combinations (without repetition))

Let $n, k \in \mathbb{N}$. The number of k-combinations of an n-element set is:

$$C_n^k = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

if $k \leq n$ and is 0 otherwise.

Proof

First choose k elements from among the n elements by taking into account the order. This way there are $\frac{n!}{(n-k)!}$ different choices. If we now disregard the order of the elements then each subset of k elements have been counted $k!$ times, because this is how many ways we can arrange k elements into order. Hence, by dividing by $k!$ we obtain that the number of k-element subsets is $\frac{n!}{k! \cdot (n-k)!}$. $\square$

Combinations

Examples

1. In how many different ways can you complete a lottery ticket (we need to select 5 numbers from among the numbers $1-90$)?

$$\binom{90}{5} = \frac{90!}{5!\cdot 85!} = \frac{90\cdot 89\cdot 88\cdot 87\cdot 86}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 43\,949\,268$$

2. How many $0-1$ sequences of length 12 exist containing exactly seven 1's?

$$\binom{12}{7} = \frac{12!}{7!5!}$$

Combinations with repetition

Example

In a post office 4 different types of postcards are sold. In how many different ways can we buy 12 postcards?

$$\binom{12+4-1}{12} = \binom{15}{12} = \frac{15!}{12!3!}$$

Definition (combination with repetition)

A *k*-combination with repetition (or *k*-multiset) from a set A is a selection of k (not necessarily distinct) elements from A, where repetition is allowed and the order does not matter.

Comment: In a combination with repetition what matters only is *how many times* each element has been chosen.

Combinations with repetition

Theorem (Number of combinations with repetition)

The number of k-combinations with repetition of an n-element set is:

$$\binom{n+k-1}{k}.$$

Proof

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then each k-combination with repetition of A can be represented by a $0-1$-sequence:

$$\underbrace{1,1,\ldots,1}_{\substack{\text{number of } a_1\text{'s}\\ \text{chosen}}},0,\underbrace{1,1,\ldots,1}_{\substack{\text{number of } a_2\text{'s}\\ \text{chosen}}},0,\ldots,0,\underbrace{1,1,\ldots,1}_{\substack{\text{number of } a_n\text{'s}\\ \text{chosen}}}.$$

This sequence contains k 1's (number of elements chosen) and $(n-1)$ 0's (number of separators). Hence the length of the sequence is $n-1+k$. There are $\binom{n+k-1}{k}$ such sequences in total, because this is how many ways we can choose the k positions from among $n+k-1$ positions, for the 1's. Therefore the number of k-combinations with repetition of A is $\binom{n+k-1}{k}$. $\square$

Combinations with repetition

Examples

1. In a cake shop they sell 5 different types of cakes. We would like to buy 8 cakes. In how many different ways can we do this?

   Here $n = 5$ and $k = 8$:

   $$\binom{5+8-1}{8} = \binom{12}{8} = \frac{12!}{8! \cdot 4!} = 495$$

2. In how many different ways can we distribute 11 identical candies among 5 children?

   For each candy we choose one from among the 11 children who to give the candy to. We choose from among the 11 children 5 times: the order in which we choose the children does not matter and any child can be chosen more than once (what matters only is *how many times* each child has been chosen). Combination with repetition where $n = 11$ and $k = 5$:

   $$\binom{11+5-1}{5} = \binom{15}{5} = \frac{15!}{5! \cdot 10!}$$

Summary

Permutations (without repetition): $n!$, a sequence of n distinct elements (order matters, each element occurs exactly once).

Permutations with repetition: $\dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!\cdot k_2!\cdot\ldots\cdot k_m!}$, a sequence of length $n=k_1+k_2+\ldots+k_m$, containing the element of kind i exactly k_i times $(1\le i\le m)$ (order matters, an element can occur more than once).

Variations without repetition: $n!/(n-k)!$, a sequence of length k chosen from n different elements, containing each element at most once (order matters, an element can occur at most once).

Variations with repetition: n^k, a sequence of length k chosen from n different elements (order matters, an element can occur more than once).

Combinations (without repetition): $\dbinom{n}{k}$, k-element subset of an n-element set (order does not matter, an element can occur at most once).

Combinations with repetition: $\dbinom{n+k-1}{k}$, choosing k-times from among n elements disregarding the order of the choices and allowing for repeated elements (order does not matter, element can occur more than once).

Binomial theorem

Theorem (Binomial theorem)

For any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

$$(x+y)^n = (x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)$$

After multiplying out the brackets we obtain a sum of terms of the form $x^k y^{n-k}$. For a given k the term $x^k y^{n-k}$ occurs in the sum as many times as many different ways we can choose those k pairs of brackets out of the n pairs of brackets from which we select the x term. $\square$

Definition (binomial coefficient)

The numbers $\binom{n}{k}$ ($n, k \in \mathbb{N}$, $k \leq n$) are called binomial coefficients.

Binomial coefficients

Theorem (Some properties of the binomial coefficients)

For every $n, k \in \mathbb{N}$, $k \leq n$ we have:

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *if* $n \geq 1$ *and* $k \geq 1$.

Proof

1. *The number of $0-1$ sequences of length n containing exactly k number of 1's equals the number of $0-1$ sequences of length n containing exactly $n-k$ number of 0's. The former number equals $\binom{n}{k}$, whereas the latter one is $\binom{n}{n-k}$, therefore $\binom{n}{k} = \binom{n}{n-k}$.*
2. *The number of those $0-1$ sequences of length n which contain exactly k number of 1's is equal to $\binom{n}{k}$. Among the $0-1$ sequences of length n containing exactly k number of 1's there are $\binom{n-1}{k-1}$ sequences starting with 1 and there are $\binom{n-1}{k}$ sequences starting with 0. Hence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.*

$\square$

Pascal's triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}: \ \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

n	$\binom{n}{k}$	$(x+y)^n$
0	1	1
1	1 1	$x+y$
2	1 2 1	$x^2+2xy+y^2$
3	1 3 3 1	$x^3+3x^2y+3xy^2+y^3$
4	1 4 6 4 1	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$
5	1 5 10 10 5 1	$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

Polynomial theorem

Example

Expand the following:

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz$. $(x+y+z)^3 = \ldots$

Theorem (Polynomial theorem)

For any r and $n \in \mathbb{N}$ we have

$$(x_1+x_2+\ldots+x_r)^n = \sum_{\substack{i_1+i_2+\ldots+i_r=n \\ i_1,i_2,\ldots,i_r\in\mathbb{N}}} \frac{n!}{i_1!\cdot i_2!\cdot\ldots\cdot i_r!} x_1^{i_1}\cdot x_2^{i_2}\cdot\ldots\cdot x_r^{i_r}.$$

Proof

$(x_1+x_2+\ldots+x_r)^n = (x_1+x_2+\ldots+x_r)(x_1+x_2+\ldots+x_r)\cdots(x_1+x_2+\ldots+x_r)$.

The coefficient of the term $x_1^{i_1}x_2^{i_2}\ldots x_r^{i_r}$ is equal to:

$$\binom{n}{i_1}\binom{n-i_1}{i_2}\binom{n-i_1-i_2}{i_3}\cdots\binom{n-i_1-i_2-\ldots-i_{r-1}}{i_r} =$$

$$\frac{n!}{i_1!(n-i_1)!}\,\frac{(n-i_1)!}{i_2!(n-i_1-i_2)!}\cdots\frac{(n-i_1-i_2-\ldots-i_{r-1})!}{i_r!(n-i_1-\ldots-i_{r-1}-i_r)!} = \frac{n!}{i_1!\cdot i_2!\cdots i_r!} \qquad \square$$

Polynomial theorem

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{\substack{i_1+i_2+\ldots+i_r=n \\ i_1, i_2, \ldots, i_r \in \mathbb{N}}} \frac{n!}{i_1! i_2! \cdots i_r!} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$$

$(x + y + z)^3 = \ldots$

i_1	i_2	i_3	$\frac{3!}{i_1! i_2! i_3!}$	$(x+y+z)^3 =$
3	0	0	$\frac{3!}{3!0!0!} = 1$	x^3
2	1	0	$\frac{3!}{2!1!0!} = 3$	$+3x^2y$
2	0	1	$\frac{3!}{2!0!1!} = 3$	$+3x^2z$
1	2	0	$\frac{3!}{1!2!0!} = 3$	$+3xy^2$
1	1	1	$\frac{3!}{1!1!1!} = 6$	$+6xyz$
1	0	2	$\frac{3!}{1!0!2!} = 3$	$+3xz^2$
0	3	0	$\frac{3!}{0!3!0!} = 1$	$+y^3$
0	2	1	$\frac{3!}{0!2!1!} = 3$	$+3y^2z$
0	1	2	$\frac{3!}{0!1!2!} = 3$	$+3yz^2$
0	0	3	$\frac{3!}{0!0!3!} = 1$	$+z^3$

Pigeonhole principle

Pigeonhole principle

If $n+1$ items are put into n containers, then there must be a container that contains at least two items.

Examples

1. In any group of eight people there must be at least two who were born on the same day of the week.

2. If we choose any five (different) numbers from the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, there will always be two numbers among them which add up to 9.

   Consider the sets $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$. By the Pigeonhole principle, among the five numbers chosen from A there will be two numbers which belong to the same set, hence their sum is 9.

Inclusion-exclusion principle

Example

How many positive integers less than 1000 exist which are not divisible by 2, nor by 3, nor by 5?

First consider the following question: How many positive integers less than 1000 exist which are divisible by 2, by 3 or by 5?

$A_1 = \{1 \leq n \leq 999 : 2|n\} \to |A_1| = \left\lfloor \frac{999}{2} \right\rfloor;$
$A_2 = \{1 \leq n \leq 999 : 3|n\} \to |A_2| = \left\lfloor \frac{999}{3} \right\rfloor;$
$A_3 = \{1 \leq n \leq 999 : 5|n\} \to |A_3| = \left\lfloor \frac{999}{5} \right\rfloor.$

Similarly, $|A_1 \cap A_2| = \left\lfloor \frac{999}{2\cdot 3} \right\rfloor$, $|A_1 \cap A_3| = \left\lfloor \frac{999}{2\cdot 5} \right\rfloor$, $|A_2 \cap A_3| = \left\lfloor \frac{999}{3\cdot 5} \right\rfloor$, $|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor$.

The number of positive integers less than 100 divisible by 2 or by 3or by 5:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| =$$

$$= \left\lfloor \frac{999}{2} \right\rfloor + \left\lfloor \frac{999}{3} \right\rfloor + \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{999}{2\cdot 3} \right\rfloor - \left\lfloor \frac{999}{2\cdot 5} \right\rfloor - \left\lfloor \frac{999}{3\cdot 5} \right\rfloor + \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor.$$

Inclusion-exclusion principle

Example continued

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| =$$

$$= \left\lfloor \frac{999}{2} \right\rfloor + \left\lfloor \frac{999}{3} \right\rfloor + \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{999}{2\cdot 3} \right\rfloor - \left\lfloor \frac{999}{2\cdot 5} \right\rfloor - \left\lfloor \frac{999}{3\cdot 5} \right\rfloor + \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor =$$

$$= 499 + 333 + 199 - 166 - 99 - 66 + 33 = 733.$$

The number of positive integers less than 1000, not divisible by 2, nor by 3, nor by 5: $999 - 733 = 266$

Inclusion-exclusion principle

Theorem (Inclusion-exclusion principle)

Let A_1, A_2, ..., A_n be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i_1<i_2\le n} |A_{i_1}\cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le n} |A_{i_1}\cap A_{i_2}\cap A_{i_3}| - \dots$$

$$+(-1)^{n+1}|A_1\cap A_2\cap\ldots\cap A_n|$$

Introducing the notation:

$$S_r = \sum_{1\le i_1<i_2<\ldots<i_r\le n} |A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_r}| \text{ (ahol } 1\le r\le n)$$

the Inclusion-exclusion principle can be written in the following form:

$$\left|\bigcup_{i=1}^{n} A_i\right| = S_1 - S_2 + \cdots + (-1)^{n+1} S_n$$

Inclusion-exclusion principle

Introducing the notation: $S_r = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|$ (ahol $1 \le r \le n$)

the Inclusion-exclusion principle can be written in the following form:

$$\left| \bigcup_{i=1}^{n} A_i \right| = S_1 - S_2 + \cdots + (-1)^{n+1} S_n$$

Proof

Let $x \in \bigcup_{i=1}^{n} A_i$ be arbitrary. We show that x is counted exactly once in the expression $S_1 - S_2 + \ldots + (-1)^{n+1} S_n$. Denote by t the number of those sets among $A_1, \ldots, A_n$ which contain x, and denote the sets containing x by $A_{j_1}, \ldots, A_{j_t}$.

Note that for any $1 \le r \le n$ we counted x in S_r as many times as many different ways r sets can chosen from among the sets $A_{j_1}, \ldots, A_{j_t}$, that is $\binom{t}{r}$ times. Therefore in $S_1 - S_2 + \ldots + (-1)^{n+1} S_n$ we counted x $\binom{t}{1} - \binom{t}{2} + \ldots (-1)^{t+1} \binom{t}{t}$ times.

Inclusion-exclusion principle

Proof (Proof continued)

Calculate the value of $\binom{t}{1} - \binom{t}{2} + \ldots + (-1)^{t+1}\binom{t}{t}$.

Idea: Apply the Binomial theorem for $x = 1$ *and* $y = -1$*:*

$0 = (1-1)^t = \binom{t}{0}1^t(-1)^0 + \binom{t}{1}1^{t-1}(-1)^1 + \ldots \binom{t}{t}1^0(-1)^t =$
$\binom{t}{0} - \binom{t}{1} + \binom{t}{2} - \ldots + \binom{t}{t}(-1)^t = 1 - \binom{t}{1} + \binom{t}{2} - \ldots + \binom{t}{t}(-1)^t$.

Rearranging the equation $0 = 1 - \binom{t}{1} + \binom{t}{2} - \ldots + \binom{t}{t}(-1)^t$*:*

$\binom{t}{1} - \binom{t}{2} + \ldots + \binom{t}{t}(-1)^{t+1} = 1$.

Hence x *was counted once in the expressions* $S_1 - S_2 + \ldots + (-1)^{n+1}S_n$.