Exercise: Discuss and sketch the graph of

$$f(x) := \frac{x^3 + x}{x^2 - 1} \qquad (x \in \mathbb{R} \setminus \{-1, 1\}).$$

Solution:

0. **Rearrangement, simplification:** The factorization of the numerator and the denominator:

$$f(x) = \frac{x \cdot (x^2 + 1)}{(x - 1) \cdot (x + 1)}.$$

Simplification using polynomial division and partial fraction decomposition (easier to differentiate, to determine the limit, ...):

$$f(x) = x \cdot \frac{x^2 + 1}{x^2 - 1} = x \cdot \frac{x^2 - 1 + 2}{x^2 - 1} = x + \frac{2x}{x^2 - 1} = x + \frac{1}{x + 1} + \frac{1}{x - 1}.$$

1. **Continuity, differentiability:** The function is continuous ($f \in C$), differentiable ($f \in D$), two times differentiable ($f \in D^2$), ... (actually, f is infinitely differentiable: $f \in D^\infty$). The first two derivatives:

$$f'(x) = \frac{(3x^2 + 1) \cdot (x^2 - 1) - 2x \cdot (x^3 + x)}{(x^2 - 1)^2} = \frac{x^4 - 4x^2 - 1}{(x^2 - 1)^2},$$

$$f''(x) = \frac{(4x^3 - 8x) \cdot (x^2 - 1)^2 - 4x \cdot (x^2 - 1) \cdot (x^4 - 4x^2 - 1)}{(x^2 - 1)^4} = \frac{4x^3 + 12x}{(x^2 - 1)^3}.$$

2. **Roots:** The only root is $x_0 = 0$, since

$$f(x) = 0 \iff x^3 + x = x \cdot (x^2 + 1) = 0 \iff x = 0.$$

3. **Monotonicity:** The denominator of f' is always positive on the domain, so it is enough to examine the sign of the numerator. Substitute $a := x^2$ in the numerator:

$$x^4 - 4x^2 - 1 = a^2 - 4a - 1,$$

which roots are

$$a_1 = 2 - \sqrt{5}, \quad a_2 = 2 + \sqrt{5}.$$

The factorization of the expression:

$$a^2 - 4a - 1 = (a - a_1) \cdot (a - a_2) = \left(a - \left(2 - \sqrt{5}\right)\right) \cdot \left(a - \left(2 + \sqrt{5}\right)\right),$$

$$x^4 - 4x^2 - 1 = \left(x^2 + \sqrt{5} - 2\right) \cdot \left(x^2 - \left(2 + \sqrt{5}\right)\right).$$

Since

$$x^2 + \sqrt{5} - 2 > 0 \qquad (x \in \mathcal{D}_f),$$

the second term controls the sign of f'. The roots of the second term are

$$x_1 = -\sqrt{2 + \sqrt{5}} \approx -2.058, \quad x_2 = \sqrt{2 + \sqrt{5}} \approx 2.058.$$

Thus, the sign of f' and the monotonicity intervals:
$f'(x) > 0 \iff x < x_1 \vee x > x_2$ ($x \in \mathcal{D}_f$), $f \uparrow$ on the intervals $(-\infty, x_1)$ and (x_2, ∞).
$f'(x) < 0 \iff x_1 < x < x_2$ ($x \in \mathcal{D}_f$), $f \downarrow$ on the intervals $(x_1, -1)$, $(-1, 1)$, and $(1, x_2)$.

4. **Local extrema:** Based on the monotonicity intervals above:

$$f'(x) = 0 \iff x = x_1 \vee x = x_2.$$

The function switch from monotonically increasing to decreasing at point x_0, and from decreasing to increasing at point x_2, so it has a local maximum at x_0 and local minimum at x_1:

$$f(x_{1,2}) = f\left(\pm\sqrt{2+\sqrt{5}}\right) = \pm\sqrt{2+\sqrt{5}} \cdot \frac{\sqrt{5}+1}{2} \approx \pm 3.330.$$

A summary table of the monotonicity intervals and the local extreme values:

x	$(-\infty, x_1)$	x_1	$(x_1, -1)$	-1	$(-1, 1)$	1	$(1, x_2)$	x_2	$(x_2, +\infty)$
f'	$\oplus$	0	$\ominus$	–	$\ominus$	–	$\ominus$	0	$\oplus$
f	$\uparrow$	loc. max.	$\downarrow$	–	$\downarrow$	–	$\downarrow$	loc. min.	$\uparrow$

5. **Concavity, convexity, inflection:** Rearrange f'' as:

$$f''(x) = \frac{4x^3 + 12x}{(x^2-1)^3} = \frac{4x}{x^2-1} \cdot \frac{x^2+3}{(x^2-1)^2}.$$

The second term $\dfrac{x^2+3}{(x^2-1)^2}$ is positive, it is enough to examine the sign of the first term $\dfrac{4x}{x^2-1}$.
Thus, the sign of f'', and the convexity intervals:
$f''(x) > 0 \iff -1 < x < 0 \vee x > 1$, f is convex on the intervals $(-1, 0)$ and $(1, +\infty)$.
$f''(x) < 0 \iff x < -1 \vee 0 < x < 1$, f is concave on the intervals $(-\infty, -1)$ and $(0, 1)$.
The convexity property change at -1, 0, and 1, but 0 is the only inflection point ($f(0) = 0$), because -1 and 1 are outside of the domain.

A summary table of the convexity intervals and inflection:

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
f''	$\ominus$	–	$\oplus$	0	$\ominus$	–	$\oplus$
f	concave	–	convex	inflection	concave	–	convex

6. **Limits at the accumulation point:** The accumulation points outside of the domain:

$$\mathcal{D}'_f = \overline{\mathbb{R}} \implies \mathcal{D}'_f \setminus \mathcal{D}_f = \{-\infty, -1, 1, \infty\}.$$

The limit at $\pm\infty$:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x \cdot \frac{1 + \dfrac{1}{x^2}}{1 - \dfrac{1}{x^2}} = \pm\infty.$$

At -1 and 1, we need to examine the one-sided limits:

$$\lim_{x\to-1\pm} f(x) = \frac{1}{x+1} \cdot \frac{x\cdot(x^2+1)}{x-1} = \pm\infty,$$

$$\lim_{x\to1\pm} f(x) = \frac{1}{x-1} \cdot \frac{x\cdot(x^2+1)}{x+1} = \pm\infty.$$

7. **Graph:**