# Discrete mathematics 1.

## Graphs

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# Basic concepts of graphs

## Definition (graph (undirected))

A triple $G = (\varphi, E, V)$ is called an (undirected) graph, if $E$ and $V$ are sets such that $V \neq \emptyset$, $V \cap E = \emptyset$ and $\varphi\colon E \to \{\{v, v'\}\,|v, v' \in V\}$. $E$ is called the set of edges, $V$ is called the set of vertices (nodes) and $\varphi$ is the incidence function. (The map $\varphi$ assigns to each element of $E$ an unordered pair of elements in $V$.)

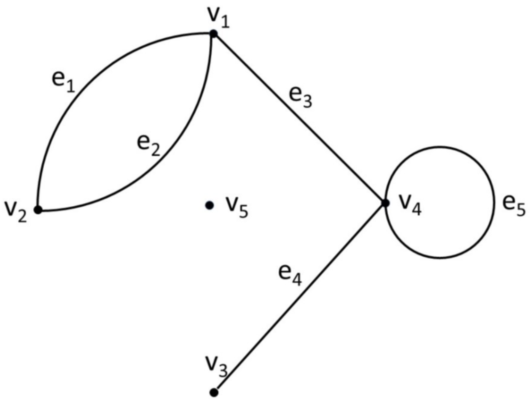
## Definition (incidence of a vertex and an edge)

If $v \in \varphi(e)$ then we say that $e$ is incident to $v$ and $v$ is incident to $e$, or – in other words – $v$ is an endpoint of $e$.

## Definition (incidence relation)

The incidence function determines the so called incidence relation $I \subseteq E \times V$: $(e, v) \in I \Leftrightarrow v \in \varphi(e)$.

# Example




$V = \{v_1, v_2, v_3, v_4, v_5\}$
$E = \{e_1, e_2, e_3, e_4, e_5\}$
$\varphi = \{(e_1, \{v_1, v_2\}), (e_2, \{v_1, v_2\}), (e_3, \{v_1, v_4\}), (e_4, \{v_3, v_4\}), (e_5, \{v_4\})\}$

# Basic concepts of graphs

## Definition (loops, parallel edges, simple graphs)

If an edge is incident only to one vertex then we call this edge a loop.
If $e \neq e'$ and $\varphi(e) = \varphi(e')$ then $e$ and $e'$ are called parallel edges.
If a graph does not contain any loops nor any parallel edges, then this graph is called a simple graph.

## Definition (finite graphs and empty graphs)

If $E$ and $V$ are both finite sets, then the graph is called a finite graph, otherwise it is an infinite graph.
If $E = \emptyset$ then the graph is called an empty graph.

Most graphs considered in informatics are finite, therefore in the rest of this course we are going to study finite graphs.

# Basic concepts of graphs

## Definition (incident edges, adjacent vertices)

The edges $e \neq e'$ are called incident (or in some sources adjacent), if there exists $v \in V$ such that $v \in \varphi(e)$ and $v \in \varphi(e')$. The vertices $v \neq v'$ are adjacent, if there exists $e \in E$ such that $v \in \varphi(e)$ and $v' \in \varphi(e)$.

## Definition (degree of a vertex)

The degree of a vertex $v$ is the number of edges incident to it, counting each loop twice. Notation: $d(v)$ or $deg(v)$.

## Definition (isolated vertex)

If $d(v) = 0$ for some $v \in V$ then $v$ is called an isolated vertex.

# The sum of the degrees of all vertices in a graph

## Theorem (The sum of the degrees of all vertices in a graph)

*In any graph* $G = (\varphi, E, V)$ *we have:*

$$\sum_{v \in V} d(v) = 2|E|.$$

## Proof

*Induction by the number of edges in the graph:*

*Base step: If* $|E| = 0$ *then the values on both sides of the equality are* $0$.

*Inductive step: Assume that the statement is true when* $|E| = n$, *for some* $n \in \mathbb{N}$. *Let* $G$ *be a graph with* $n + 1$ *edges. By deleting one edge of* $G$ *we obtain a graph* $G'$ *with* $n$ *edges. By our inductive hypothesis, the statement is true for* $G'$. *If we now add to* $G'$ *the edge deleted earlier from* $G$, *the values on both sides of the equality will increase by* $2$, *hence the equality remains true. Therefore the statement also holds for* $G$.

# Basic concepts of graphs

### Definition (deleting edges from a graph)

Let $G = (\varphi, E, V)$ be a graph and $E' \subseteq E$. The graph obtained by deleting the set of edges $E'$ from $G$ is the graph $G' = (\varphi|_{E \setminus E'}, E \setminus E', V)$.

### Definition (deleting vertices from a graph)

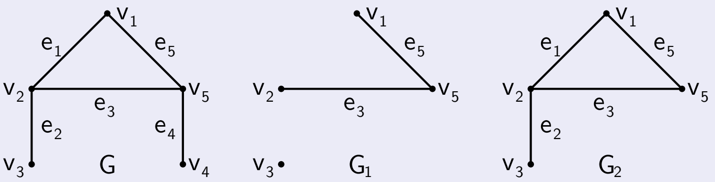
Let $G = (\varphi, E, V)$ be a graph and $V' \subseteq V$. Denote by $E'$ the set of those edges in $E$ which are incident to at least one vertex in $V'$. The graph obtained by deleting the set of vertices $V'$ from $G$ is the graph $G' = (\varphi|_{E \setminus E'}, E \setminus E', V \setminus V')$.

# Basic concepts of graphs

## Definition (subgraphs, supergraphs)

A graph $G' = (\varphi', E', V')$ is called a subgraph of a graph $G = (\varphi, E, V)$, if $E' \subseteq E$, $V' \subseteq V$ and $\varphi' \subseteq \varphi$. In this case we also say that $G$ is a supergraph of $G'$.
If $E'$ contains all those edges of $G$ which have both endpoints in $V'$, then $G'$ is called a subgraph spanned (or induced) by $V'$.

## Example



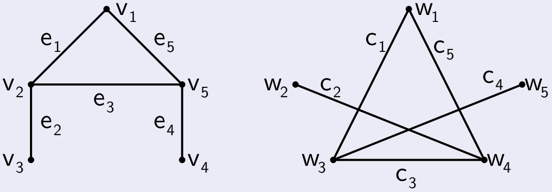

$G_1$ is a subgraph, but not a spanned subgraph of $G$ and $G_2$ is a spanned subgraph of $G$.

# Basic concepts of graphs

## Definition (isomorphic graphs)

Two graphs $G = (\varphi, E, V)$ and $G' = (\varphi', E', V')$ are said to be isomorphic to each other, if there exist bijections $f: E \to E'$ and $g: V \to V'$ such that for every $e \in E$ and $v \in V$, $e$ is incident to $v$ if and only if $f(e)$ is incident to $g(v)$.

## Example




Suitable bijections $f$ and $g$:

$f = \{(e_1, c_5), (e_2, c_2), (e_3, c_3), (e_4, c_4), (e_5, c_1)\}$

$g = \{(v_1, w_1), (v_2, w_4), (v_3, w_2), (v_4, w_5), (v_5, w_3)\}$

# Basic concepts of graphs

## Definition (complete graphs)

A simple graph in which any two vertices are adjacent is called a complete graph. The complete graph with $n$ ($n \in \mathbb{N}^+$) vertices is denoted by $K_n$.

## Comment

It is easy to show that for any $n \in \mathbb{N}^+$, all complete graphs on $n$ vertices are isomorphic, hence $K_n$ is unique up to graph isomorphism.

## Proposition (The number of edges in $K_n$)

*For every $n \in \mathbb{N}^+$, $K_n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.*

## Definition (regular graphs)

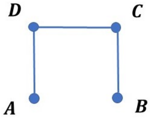
If the degree of every vertex in a graph is equal to $n$ for some $n \in \mathbb{N}$ then the graph is called $n$-regular. A graph is called regular, if it is $n$-regular for some $n \in \mathbb{N}$.

# Complement of a graph

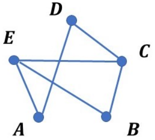
## Definition (complement of a simple graph)

The complement of a simple graph $G$ is the simple graph $\overline{G}$ which has the same set of vertices as $G$ and in which two (distinct) vertices are connected by an edge if and only if they are not connected in $G$.

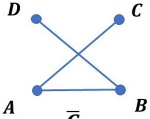
## Examples



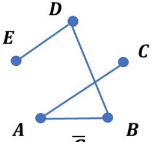

$G_1$




$G_2$




$\overline{G}_1$




$\overline{G}_2$

# Basic concepts of graphs

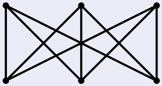
## Definition (bipartite graphs)

A graph $G = (\varphi, E, V)$ bipartite graph, if $V$ can be expressed as the union of two disjoint sets $V'$ and $V''$ such that for every edge $e$ in $E$ one endpoint of $e$ is in $V'$ and the other endpoint of $e$ is in $V''$.

## Definition (the graphs $K_{m,n}$)

Let $m, n \in \mathbb{N}^+$. The simple bipartite graph in which $|V'| = m$ and $|V''| = n$ and every vertex in $V'$ is adjacent to every vertex in $V''$, is denoted by $K_{m,n}$.

## Example



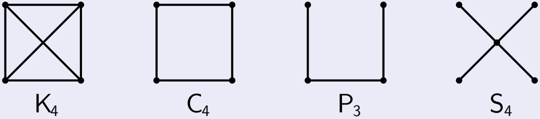
$K_{3,3}$

# Further special graphs

## Definition (cycle graphs, path graphs, stars)

For every $n \in \mathbb{N}^+$ the cycle graph $C_n$ on $n$ vertices (or n-cycle or n-gon) is a simple graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and $n$ edges $e_1, e_2, \ldots, e_n$ such that for every $1 \leq i \leq n-1 : \varphi(e_i) = \{v_i, v_{i+1}\}$ and $\varphi(e_n) = \{v_n, v_1\}$.
For every $n \in \mathbb{N}$ the path graph $P_n$ is the graph obtained by deleting one edge from the graph $C_{n+1}$.
For every $n \in \mathbb{N}^+$ the star graph $S_n$ is the graph $K_{n,1}$. ($S_0$ can also be defined as a graph consisting of a single vertex and containing no edges.)

## Examples



# Basic concepts of graphs

## Definition (walk)

Let $G = (\varphi, E, V)$ be a graph, $n \in \mathbb{N}$. A walk of length $n$ from vertex $v_0$ to $v_n$ is a sequence

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$$

where

- $v_j \in V \quad 0 \leq j \leq n$,
- $e_k \in E \quad 1 \leq k \leq n$,
- $\varphi(e_m) = \{v_{m-1}, v_m\} \quad 1 \leq m \leq n$.

If $v_0 = v_n$, then the walk is a closed walk, otherwise it is an open walk.

## Definition (trail/line)

If a walk does not contain repeated edges, then it is called a trail (or line). According to the above definition, a trail can be an open trail or a closed trail.

# Basic concepts of graphs

## Definition (path)

If a walk does not contain repeated vertices then it is called a path.

## Comments

- Every path is also a trail.
- A walk of length $0$ is also a path consisting of a single vertex.
- A walk of length $1$ is a path if and only if the single edge contained by it is not a loop.

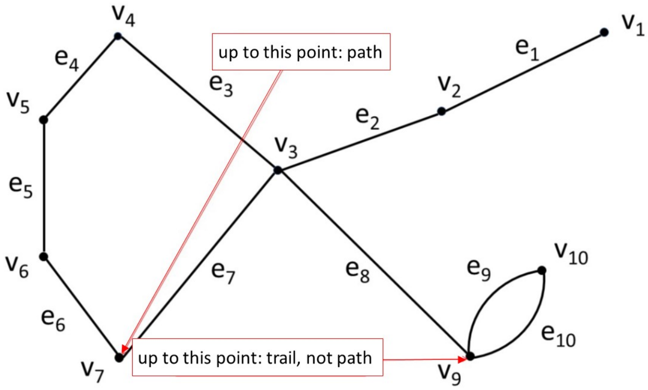
## Definition (circuit)

A circuit is a closed trail of length $\geq 1$.

## Definition (cycle)

A cycle is a circuit which contains no repeated vertices apart from the first and last vertices, which are identical.

# Example




path: $v_1, e_1, v_2, e_2, v_3, \dots, v_6, e_6, v_7$;
line, but not a path: $v_1, e_1, v_2, e_2, v_3, e_3, v_4, \dots, e_7, v_3, e_8, v_9$;
cycle: $v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7, e_7, v_3$.

# Basic concepts of graphs

## Proposition (Creating a path from a walk)

*Given any walk between two distinct vertices $v$ and $v'$ of a graph, we can obtain a path from $v$ to $v'$ by deleting suitable vertices and edges of the walk.*

## Proof

*Consider the walk:*

$$v = v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n = v'.$$

*If this walk does not contain any repeated vertices, then it is already a path. Otherwise we have $v_i = v_j$ for some $i < j$. By deleting the part*

$$e_{i+1}, v_{i+1}, e_{i+2}, v_{i+2}, \ldots, v_{j-1}, e_j, v_j$$

*from the walk we obtain a shorter walk from $v$ to $v'$. Repeat this step until there are no repeated vertices in the walk. The process will finish in finite number of steps, since the length of the walk reduces in each step. When the process stops there are no repeated vertices, hence we obtained a path.*

# Basic concepts of graphs

## Definition (connected graphs)

A graph is said to be connected if there is a walk (or, equivalently, there is a path) between any pair of vertices of the graph.

For a graph $G = (\varphi, E, V)$ define the relation $\sim$ on $V$: let $v \sim v'$ if and only if there exists a walk (or, equivalently, there is a path) from $v$ to $v'$ in $G$.

Since $\sim$ is an equivalence relation (Why?), the set of corresponding equivalence classes will be a partition of $V$.

## Definition (components of a graph)

The subgraphs spanned by these equivalence classes are called the components of the graph.

## Comment

A graph is connected if and only if it consists of only one component.

# Trees

## Definition (tree)

A graph is called a tree if it is connected and contains no cycle (in other words: it is an acyclic graph).

## Theorem (Equivalent characterisations of trees 1.)

*For a simple graph $G$ the following conditions are equivalent:*

1. *$G$ is a tree;*
2. *$G$ is connected, and by deleting any of its edges, the remaining subgraph is not connected (i.e. $G$ is a minimally connected graph);*
3. *any vertices $v$ and $v'$ in $G$ there is exactly one path from $v$ to $v'$;*
4. *$G$ contains no cycles, but by adding any new edge to $G$, the new graph will contain a cycle (i.e. $G$ is a maximally acyclic graph).*

## Structure of the proof

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$

# Trees

## Proof

(1) $\Rightarrow$ (2)

*By the definition of a tree, $G$ is connected. The other part of the statement we show by proof by contradiction. Suppose there is an edge $e$ (denote its endpoints by $v$ and $w$) in the graph such that after deleting $e$ the remaining subgraph is connected. Then in the remaining subgraph there is a path from $v$ to $w$: $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = w$. Adding $e$ and $v$ to this path we obtain a cycle in $G$: $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = w, e, v$.*

(2) $\Rightarrow$ (3)

*Let $v$ and $w$ be vertices in $G$. Since $G$ is connected, there is at least one path from $v$ to $w$. We show that there cannot exist two different paths from $v$ to $w$, by proof by contradiction: Suppose there exist 2 different paths from $v$ to $w$:*
$v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = w$ *and*
$v = v_0, e'_1, v'_1, e'_2, \ldots, v'_{m-1}, e'_m, v'_m = w$. *Let $k$ be the smallest index such that $v_k \neq v'_k$. (Why does such a $k$ exist?) By deleting the edge $e_k$ from $G$ we obtain a connected subgraph, because the walk $v_{k-1}, e_k, v_k$ can be replaced by the walk $v_{k-1}, e'_k, v'_k, \ldots, e'_m, v'_m = w, e_n, v_{n-1}, e_{n-1}, v_{n-2}, \ldots, v_{k+1}, e_{k+1}, v_k$. ↯*

# Trees

## Proof

(3) $\Rightarrow$ (4)

*We show that $G$ contains no cycle, by proof by contradiction: Suppose there is a cycle $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v$. Then there exist two different paths from $v_1$ to $v$: $v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v$ and $v_1, e_1, v_0 = v$. $\lightning$*

*Showing that $G$ is maximally acyclic: If the newly added edge $e$ is a loop and $v$ is its endpoint, then $v, e, v$ is a cycle. Otherwise $e$ has two distinct endpoints $v$ and $w$. Let $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = w$ be the path from $v$ to $w$ in $G$. By adding the edge $e$ and vertex $v$ to this path we obtain the cycle: $v = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = w, e, v$.*

(4) $\Rightarrow$ (1)

*By our assumption in (4), $G$ contains no cycle. It remains to show that $G$ is connected, i.e. for any vertices $v$ and $w$ there is a path in $G$. Add an edge $e$ with endpoints $v$ and $w$ to the graph. The resulting new graph will contain a cycle with the edge $e$ in it (Why?): $w, e, v, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, w$. Then $v, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, w$ is a path from $v$ to $w$.*

# Trees

## Lemma (Vertices of degree 1 in finite acyclic graphs)

*If a finite graph $G$ does not contain a cycle and contains at least one edge then there are at least $2$ vertices with degree $1$ in $G$.*

## Proof

*Since $G$ is finite, among all paths in $G$ there is at least one path $P$ of maximal length (i.e. a path $P$ such that there is no path longer than $P$ in $G$). Let $P$ be: $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$. As $G$ contains at least one edge, the length of $P$ is at least $1$, hence $v_0 \neq v_n$.*
*We show that $deg(v_0) = deg(v_n) = 1$. Proof by contradiction: Suppose that there is an edge $e \neq e_1$ which is incident to $v_0$. Then the other endpoint $v_0$ of $e$ must lie on $P$, because otherwise $v_0, e, v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ would be a path longer than $P$, a contradiction. Hence $v_0 = v_k$ for some vertex $v_k$ on $P$. Then $v_k, e, v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$ is a cycle, which is also a contradiction. Therefore there is no edge $e \neq e_1$ incident to $v_0$ and so $deg(v_0) = 1$. Similarly $deg(v_n) = 1$.*

# Trees

## Theorem (Equivalent characterisations of trees 2 - using the number of edges)

*For a simple graph $G$ on $n$ vertices ($n \in \mathbb{N}^+$) the following conditions are equivalent:*

1. *$G$ is a tree;*
2. *$G$ contains no cycles (i.e. acyclic) and it has $n-1$ edges;*
3. *$G$ is connected and it has $n-1$ edges.*

## Proof

*If $n=1$ then the statement is clearly true. (Why?)*
*$(1) \Rightarrow (2)$: Proof by induction on $n$: Suppose the statement is true for some $n \in \mathbb{N}^+$. Let $G$ be a tree with $n+1$ vertices. Then $G$ has a vertex $v$ of degree $1$. (Why?) Delete vertex $v$ from $G$. The new graph $G'$ is clearly acyclic. It is also connected: since $v$ can be contained only as an endpoint in any path in $G$, hence for any vertices $v'$ and $v''$ in $G'$, the path between $v'$ and $v''$ in $G$ cannot contain $v$, and so it is also a path in $G'$. Therefore $G'$ is connected, hence a tree, and so by our inductive hypothesis it has $n-1$ edges, and so $G$ has $n$ edges.*

# Trees

## Proof

*$(2) \Rightarrow (3)$: Proof by induction on $n$: Suppose the statement is true for some $n \in \mathbb{N}^+$. Let $G$ be an acyclic graph with $n+1$ vertices and $n$ edges. It is sufficient to show that $G$ is connected. The graph $G$ contains a vertex $v$ of degree $1$. (Why?) Delete $v$ from $G$. The resulting graph $G'$ is also acyclic and has $n$ vertices and $n-1$ edges. Hence, by our inductive hypothesis $G'$ is connected. Between any vertices $v'$ and $v''$ in $G'$ there is a path in $G'$, which is also a path in $G$. From any vertex $v'$ in $G'$ we can obtain a path in $G$ to $v$ if we consider the path in $G'$ from $v'$ to the vertex adjacent to $v$ in $G$ and to this path we add the edge incident to $v$, and $v$.*

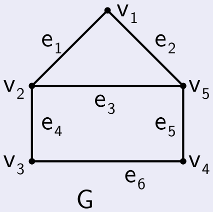
*$(3) \Rightarrow (1)$: Let $G$ be a graph satisfying Condition (3). It is sufficient to show that $G$ is acyclic. Proof by contradiction: Suppose $G$ contains a cycle. Then by deleting any edge of a cycle in $G$ we obtain a connected graph. (Why?) Repeat this step, while the graph still has a cycle. The process halts after finite steps (Why?), when the new graph $T$ is a tree. If we omitted $k > 0$ edges during the process, then $T$ has $n-1-k < n-1$ edges. Because of the implication $(1) \Rightarrow (2)$, $T$ has $n-1$ edges, a contradiction.* ↯

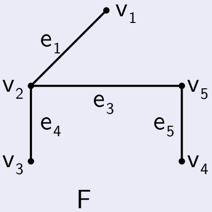
# Spanning trees

## Definition (spanning tree)

A subgraph $T$ of a graph $G$ is called a spanning tree of $G$, if $T$ is a tree and $T$ contains all vertices of $G$.

## Example



# Spanning tree

## Proposition (Existence of a spanning tree)

*Every finite connected graph has a spanning tree.*

## Proof

*Let $G$ be a finite, connected graph. If $G$ contains a cycle then delete an edge from one of the cycles in $G$. The new graph is still connected. (Why?) Repeat this step until the graph becomes acyclic. This process will terminate in finite number of steps, since the number of edges in the graph decreases in each step. When the process stops, the graph obtained is a spanning tree of $G$.*

# Spanning tree

## Proposition (A lower bound on the number of cycles)

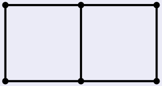
*A finite connected graph $G = (\varphi, E, V)$ contains at least $|E| - |V| + 1$ cycles with pairwise different sets of edges.*

## Proof

*Let $T$ be a spanning tree of $G$. Then $T$ has $|V| - 1$ edges. Denote by $E'$ the set of those edges in $G$ which are not in $T$. If we add any edge $e \in E'$ to $T$ there will be a cycle $K_e$ in the new graph $F'$ (Because $T$ is a maximally acyclic graph), which is also a cycle in $G$. The cycle $K_e$ contains the edge $e$ (Why?) and if $e \neq e' \in E'$ then $K_{e'}$ does not contain $e$. This way we obtain $|E'| = |E| - |V| + 1$ cycles with pairwise different sets of edges.*

## Comment

The above lower bound does not necessarily give the exact number of cycles in the graph ($3 > 7 - 6 + 1 = 2$).



# Forests, spanning forests

## Definition (forest, spanning forest)

An acyclic graph is called a forest.
A subgraph $F$ of a graph $G$ containing one spanning tree from each component of $G$ is called a spanning forest of $G$.

## Proposition (Spanning forests in finite graphs)

*Every finite graph has a spanning forest.*

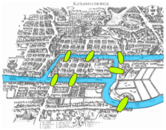
## Proposition (Number of edges in a finite forest)

*The number of edges in a finite forest $F$ equals the number of vertices in $F$ minus the number of components in $F$.*

## Comment

Among all (not necessarily connected) graphs, forests and spanning forests play a similar role to those of trees and spanning trees among connected graphs.

# Euler trail



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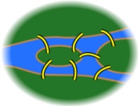
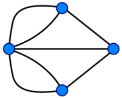


$\longrightarrow$



## Definition (Euler trail)

A trail that contains all edges of a graph is called an Euler trail. (An Euler trail can be an open Euler trail or a closed Euler trail, depending on whether it is an open or a closed trail.)

## Comment

Since a trail does not contain repeated edges, an Euler trail contains every edge of the graph exactly once.

# Euler trail

## Theorem (Existence of a closed Euler trail)

*A finite connected graph contains a closed Euler trail if and only if the degree of every vertex in the graph is even.*

## Proof

$\Rightarrow$: *Let* $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_0$ *be a closed Euler trail in the graph. Following the Euler trail we 'enter' each vertex* $v$ *the same number of times as we 'leave' it. Hence the trail contains an even number of edges incident to any vertex* $v$ *(counting loops twice), and so the degree of every vertex is even.*

# Euler trail

## Proof

*$\Leftarrow$: The proof is constructive. First consider an arbitrary closed trail (a closed trail certainly exists in any graph, for example a trail containing no edges, just a single vertex), call it $T$. As any closed trail, $T$ contains an even number of edges incident to any vertex of the graph (counting each loop twice). (Why?)*

*If the current closed trail $T$ does not contain all edges of the graph, then – because the graph is connected – there is a vertex $w$ in our closed trail, which has incident edges not included in $T$. Start a new trail $T'$ from $w$ leaving $w$ on an edge not used in $T$ and proceed always on unused edges. As every vertex has an even number of edges not used in $T$, we can only get stuck when returning to $w$. Consider the following new trail: starting at $w$ move along $T$, then after arriving back at $w$ at the end, move along $T'$, at the end of which arriving back at $w$. This way we obtain a closed trail longer than $T$. Hence, by repeating this expansion step, after a finite number of expansions we obtain a closed Euler trail.*

# Hamiltonian path, Hamiltonian cycle

## Definition (Hamiltonian path)

If a path in a graph contains all vertices of the graph then we call it a Hamiltonian path.

## Comment

Since a path does not contain repeated vertices, a Hamiltonian path contains each vertex of the graph exactly once.

## Definition (Hamiltonian cycle, Hamiltonian graph)

A Hamiltonian cycle in a graph is a cycle that contains all vertices of the graph. A graph is called a Hamiltonian graph if it contains a Hamiltonian cycle.

## Theorem (Dirac, NP)

*If in a simple graph $G = (\varphi, E, V)$, $|V| > 2$, and the degree of every vertex is at least $|V|/2$, then the graph contains a Hamiltonian cycle.*

# Planar graphs

## Definition (planar graph)

A graph $G$ is called a planar graph, if it can be drawn in (with a more technical word 'embedded in') the plane ($\mathbb{R}^2$) in such a way that its edges intersect only at their endpoints. (In other words, it can be drawn on the plane in such a way that no edges cross each other, except for at the vertices.) Such a drawing is called a plane graph representation or planar embedding of the graph.

## Comment

Not all graphs are planar, i.e. not all graphs can be embedded into $\mathbb{R}^2$ (not even all finite graphs are planar). However every finite graph can be embedded into $\mathbb{R}^3$.

# Planar graphs

## Definition (faces in a planar representation)

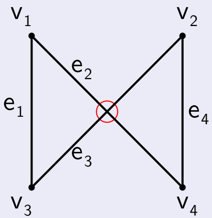
Given a planar embedding of a graph $G$, a face is a region – i.e. connected subset – of the plane surrounded by edges of $G$, (i.e. a face is a set of points, such that between any two points of a face, there is a line (curve) in the plane, such that it does not cross any of the edges (or vertices) of $G$). A face can be unbounded, and in that case it is an external face, otherwise it is an internal face.

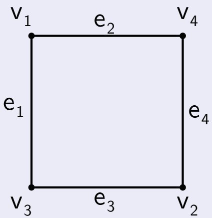
## Comment

The status of internal/external face is not significant: An external face can become an internal face in a different planar embedding. However, the number of faces is independent from the embedding.

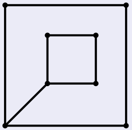
# Planar graphs

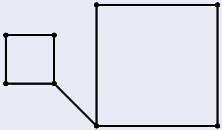
## Example




## Example





# Planar graphs

## Theorem (Euler's formula)

*Let $G = (\varphi, E, V)$ be a connected planar graph. Then for any planar embedding of $G$:*

$$|E| + 2 = |V| + f$$

*where $f$ denotes the number of faces in the planar embedding.*

## Proof

*Suppose there is a cycle in $G$. By deleting an edge of the cycle, two faces are merged, so both $f$ and $|E|$ is reduced by $1$. In the end, we obtain a tree for which the equation holds (Why?).*

# Planar graphs

## Proposition (Upper bound on the number of edges in simple planar graphs)

*For any simple, connected, planar graph $G = (\varphi, E, V)$ with $|V| \geq 3$: $|E| \leq 3|V| - 6$.*

## Proof

*In case $|V| = 3$ there are two such graphs: $P_2$ and $C_3$, both of which satisfy the inequality.*
*Assume $|V| > 3$. Then $|E| \geq 3$ (Why?). Since $G$ is simple, every face is surrounded by at least $3$ edges. Therefore if we count the number of edges surrounding each face and then add up these numbers for all faces then the sum obtained will be $\geq 3f$. As every edge separates at most two regions, in this sum every edge was counted at most twice, hence this sum is $\leq 2|E|$. Therefore $2|E| \geq 3f$. Expressing $f$ from Euler's formula and substituting for it we obtain $2|E| \geq 3(|E| + 2 - |V|)$, which after rearrangement yields $|E| \leq 3|V| - 6$.*

## Comment

The theorem holds for disconnected graphs as well, since it can be made a connected planar graph by adding edges.

# Planar graphs

## Proposition (Lower bound on the minimal degree in simple planar graphs)

*If* $G = (\varphi, E, V)$ *is a simple planar graph, then*

$$\delta = \min_{v \in V} d(v) \leq 5.$$

## Proof

*We can assume* $|V| \geq 3$ *(Why?).*
*Proof by contradiction: Suppose* $\delta \geq 6$*. Then* $6|V| \leq 2|E|$ *(Why?), furthermore, using our previous theorem,* $2|E| \leq 6|V| - 12$*, implying* $6|V| \leq 6|V| - 12$*, a contradiction.*

## Comment

There exists 5-regular simple planar graph.

# Planar graphs

## Proposition

$K_{3,3}$ *is not a planar graph.*

## Proof

*Proof by contradiction: Suppose that on the contrary,* $K_{3,3}$ *is a planar graph. Denote by* $f$ *the number of faces in its planar embeddings. Since* $|E| = 9$ *and* $|V| = 6$, *by Euler's formula* $f = 5$ *must hold. Since it is a simple bipartite graph, each face is surrounded by at least* $4$ *edges (Why?), and every edge separates at most* $2$ *faces, so* $4f \leq 2|E|$, *implying* $20 \leq 18$, *a contradiction.*

## Proposition

$K_5$ *is not a planar graph.*

## Proof

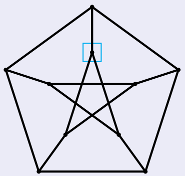
*Proof by contradiction: Suppose that on the contrary,* $K_5$ *is a planar graph. Since* $|E| = 10$ *and* $|V| = 5$, *by the theorem about the upper bound on the number of edges in simple planar graphs* $10 \leq 3 \cdot 5 - 6 = 9$, *a contradiction.*

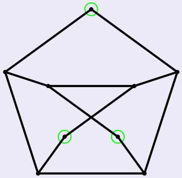
# Planar graphs

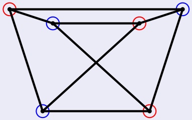
## Definition (topological isomorphy of graphs)

The finite graphs $G$ and $G'$ are topologically isomorphic if they can be converted into each other applying the following transformation or its inverse a finite number of times: delete a vertex with degree two and connect its neighboring vertices with an edge.

## Example







## Theorem (Kuratowski (NP))

*A simple and finite graph is a planar graph if and only if it has no subgraph topologically isomorphic to $K_5$ or $K_{3,3}$.*