# ANALYSIS 2, CLASS 6.

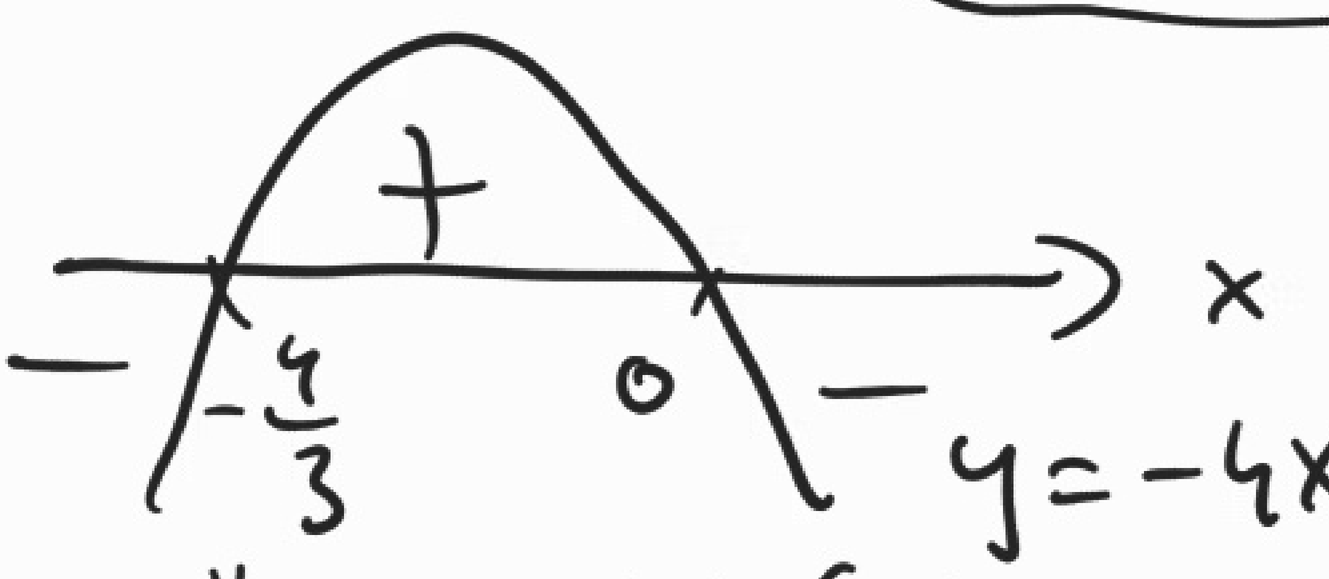
① Discuss and sketch the graph of the following functions:

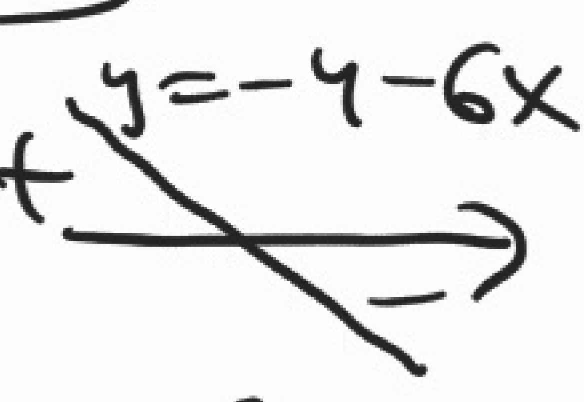
a) $\boxed{f(x) = 2 - 2x^2 - x^3 \quad (x \in \mathbb{R})}$

1. $f(0) = 2$ $\quad f(x) = 0$ we let out...

2. $f'(x) = -4x - 3x^2 = 0$

$x(3x+4) = 0 \iff \boxed{\begin{array}{l} x_1 = 0 \\ x_2 = -\frac{4}{3} \end{array}}$




3. $f''(x) = -4 - 6x = 0 \iff \boxed{x = -\frac{2}{3}}$

4. <u>Limits</u>:

$$\lim_{x \to +\infty} (2 - 2x^2 - x^3) = -\infty$$

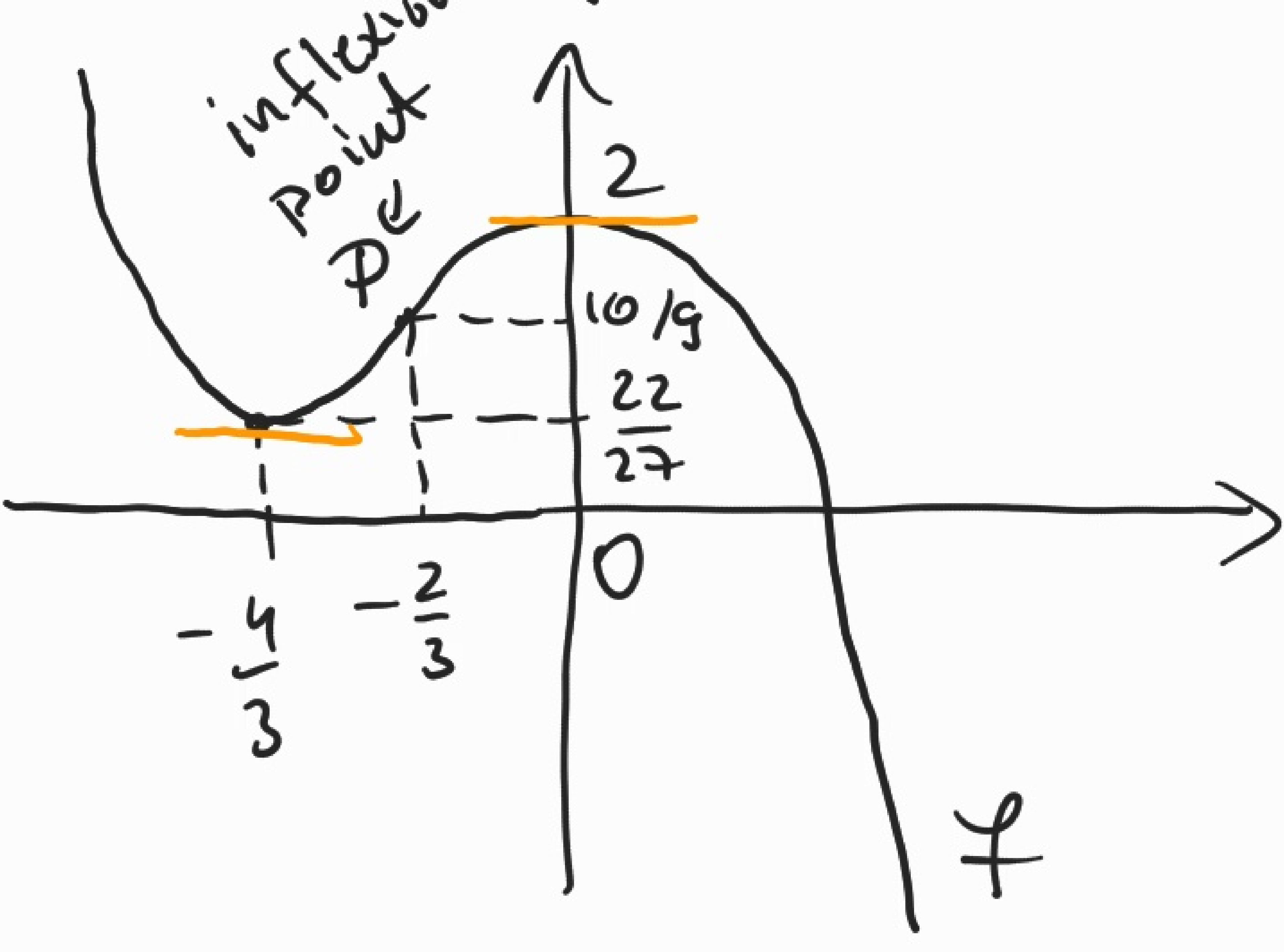
$$\lim_{x \to -\infty} (2 - 2x^2 - x^3) = +\infty$$

5. Table

| $x$ | $-\infty$ | | $-\frac{4}{3}$ | | $-\frac{2}{3}$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f'$ | | $-\ -$ | $0$ | $+\ +$ | $+$ | $+$ | $0$ | $-\ -$ | |
| $f$ | $+\infty$ | $\searrow$ | $\frac{22}{27}$ | $\nearrow$ | | $\nearrow$ | $2$ | $\searrow$ | $-\infty$ |
| $f''$ | | $+\ +$ | $+$ | $+$ | $0$ | $-\ -$ | $-$ | $-\ -$ | |
| conv. | | $\cup$ | | | INFL POINT | | $\cap$ | | |

$$f(0) = 2 \qquad f\left(-\frac{4}{3}\right) = 2 - 2 \cdot \frac{16}{9} + \frac{64}{27} = \frac{22}{27}$$

## 6. GRAPH OF $f$




$f\left(-\frac{2}{3}\right) = \frac{10}{9}$ ; $R_f = \mathbb{R}$ range of values.

5) $\boxed{f(x) := \dfrac{x}{(x+2)^2} \quad (-2 \neq x \in \mathbb{R})}$

(1) Domain of f : $D_f = (-\infty; -2) \cup (-2, \infty)$

2) Intersection with axes :

$x = 0 \Leftrightarrow f(0) = 0$

$y = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow \dfrac{x}{(x+2)^2} = 0$

$\Rightarrow x = 0$

(3) $f'$ and its sign (monoton properties and local min/max)

$$f'(x) = \left(\frac{x}{(x+2)^2}\right)' = \frac{1 \cdot (x+2)^{\cancel{2}} - x \cdot 2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}}$$

$$= \text{SIMPLIFY} = \frac{x+2-2x}{(x+2)^3} \Rightarrow$$

$$f'(x) = \frac{2-x}{(x+2)^3} \quad (x \in \mathbb{R} \setminus \{-2\})$$

$$f'(x) = 0 \iff 2-x=0 \quad \boxed{x=2}$$

$$f'(x) > 0 \iff \begin{cases} 2-x>0 \\ x+2>0 \end{cases} \text{ or } \begin{cases} 2-x<0 \\ x+2<0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$x<2$ and $x>-2$ | $x>2$ and $x<-2$

$x \in (-2, 2)$ | $x \in \emptyset$

$$f'(x) < 0 \iff x<-2 \text{ or } x>2$$

(4) $f''$ and its sign (convexity)

$$f''(x) = \left(\frac{2-x}{(x+2)^3}\right)' = \frac{-1\cdot(x+2)^{\cancel{3}} - (2-x)\cdot 3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}}$$

$= \text{SIMPLIFY!} = \dfrac{-x-2-6+3x}{(x+2)^4} =$

$= \dfrac{2x-8}{(x+2)^4} = \dfrac{2\cdot(x-4)}{(x+2)^4} \; (\forall x \in \mathbb{R} \setminus \{-2\}),$

$\quad \quad (x+2)^4 \leftarrow \oplus!$

<u>So</u> $f''(x) = 0 \iff \boxed{x=4}$

$f''(x) > 0 \iff x-4>0 \iff$

$\boxed{x>4}$

$f''(x) < 0 \iff x < 4$

(5) limits and asympthotes.

$\lim\limits_{x\to\pm\infty} \dfrac{x}{(x+2)^2} = \dfrac{\pm\infty}{+\infty} = LH = \lim\limits_{x\to\pm\infty} \dfrac{1}{2(x+2)}$

$= \underline{\underline{\pm 0}} \implies \text{Line } \boxed{y=0} \text{ is}$

a horizontal asympthote.

$$\lim_{x \to -2} \frac{x}{(x+2)^2} = \frac{-2}{(\pm 0)^2} = \frac{-2}{+0} = -\infty$$

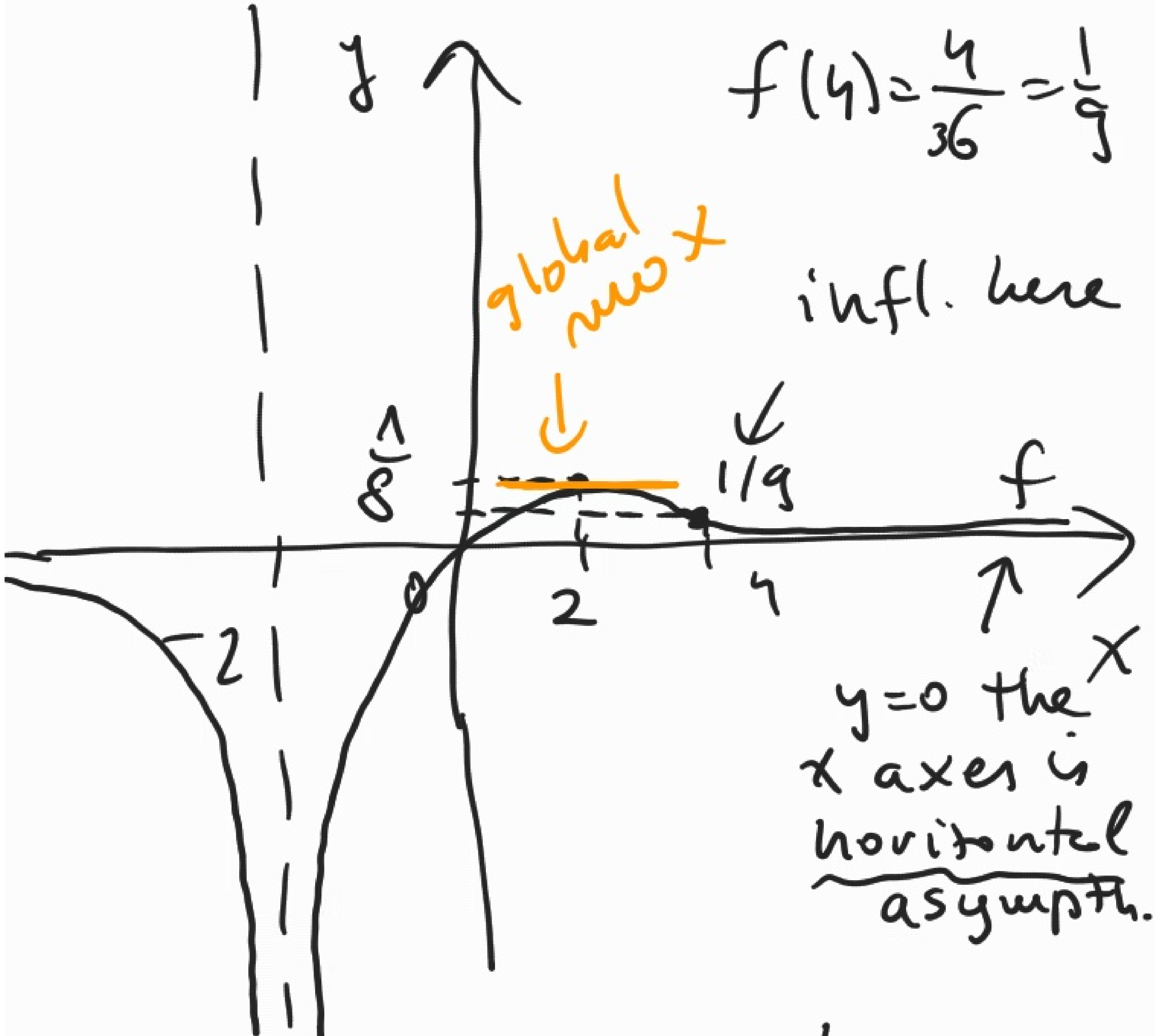
$\Rightarrow$ line $\boxed{x=-2}$ is a vertical asympthote.

6) Table

| $x$ | $-\infty$ | | $-2$ | | $2$ | | $4$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ $-$ | ‖ | $+$ $+$ | $0$ | $-$ | | $-$ $-$ | |
| $f(x)$ | $-0$ | $\searrow$ | $-\infty$ ‖ $-\infty$ | $\nearrow$ | $1/8$ | $\searrow$ | | $\searrow$ | $+0$ |
| $f''(x)$ | | $-$ $-$ | ‖ | $-$ $-$ | | $-$ | $0$ | $+$ $+$ | |
| conv. | | concave | ‖ | concave | | concave | inflection | convex | |

$f(2) = 1/8$

(7) Graph of f:




vertical asympthote

$R_f = (-\infty; 1/8]$.

c) $f(x) = 5x^3 - 4x^4 \quad (x \in \mathbb{R})$

$\underline{2}$) Int. with axes:

$x = 0 \Rightarrow f(0) = 0$

$y = 0 \Leftrightarrow f(x) = 5x^3 - 4x^4 = 0 \Leftrightarrow$

$x^3(5 - 4x) = 0 \Leftrightarrow x_1 = 0$ or $x_2 = \frac{5}{4}$.

(1) Domain: $D_f = (-\infty, +\infty)$

(3) $f'$ and its sign:

$f'(x) = 15x^2 - 16x^3 = 0 \Leftrightarrow$

$\underbrace{x^2}_{+10}(15 - 16x) = 0 \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = \frac{15}{16} \end{cases}$.

$f'(x) > 0 \Leftrightarrow 15 - 16x > 0 \Leftrightarrow$

$x < \frac{15}{16}$.

$f'(x)<0 \iff 15-16x<0 \iff$

$x>\frac{15}{16}$

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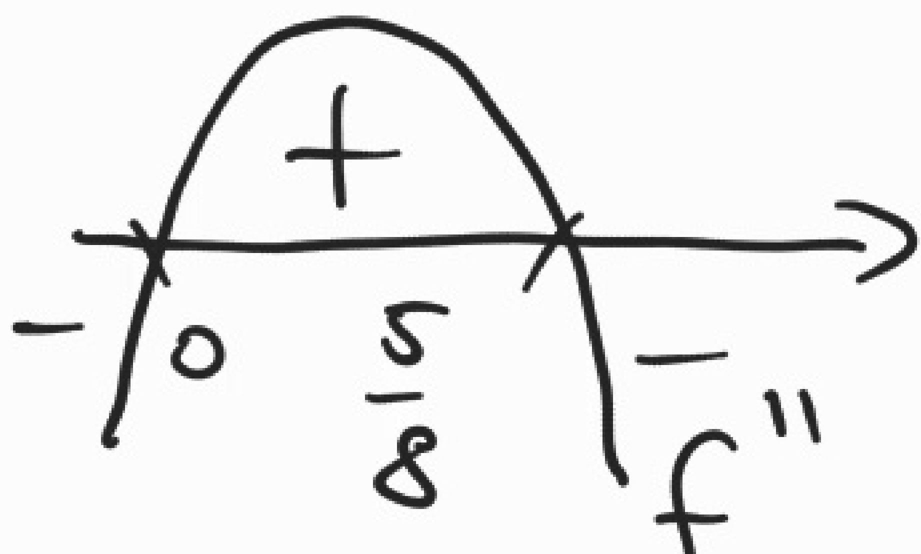
(4) <u>$f''$ and its sign</u>:

$f''(x)=30x-48x^2 \Rightarrow$

$f''(x)=0 \iff 2x(15-24x)=0$

$x_1=0$ or $x_2=\frac{15}{24}=\frac{5}{8}$




(5) <u>Limits</u>: $\lim_{x\to\pm\infty}(5x^3-4x^4)=$

$$= \lim_{x\to\pm\infty} x^4 \cdot \left(\frac{5}{x} - 4\right) = +\infty \cdot (0-4) =$$

$$= \boxed{-\infty} \; ;$$

(6) Table.

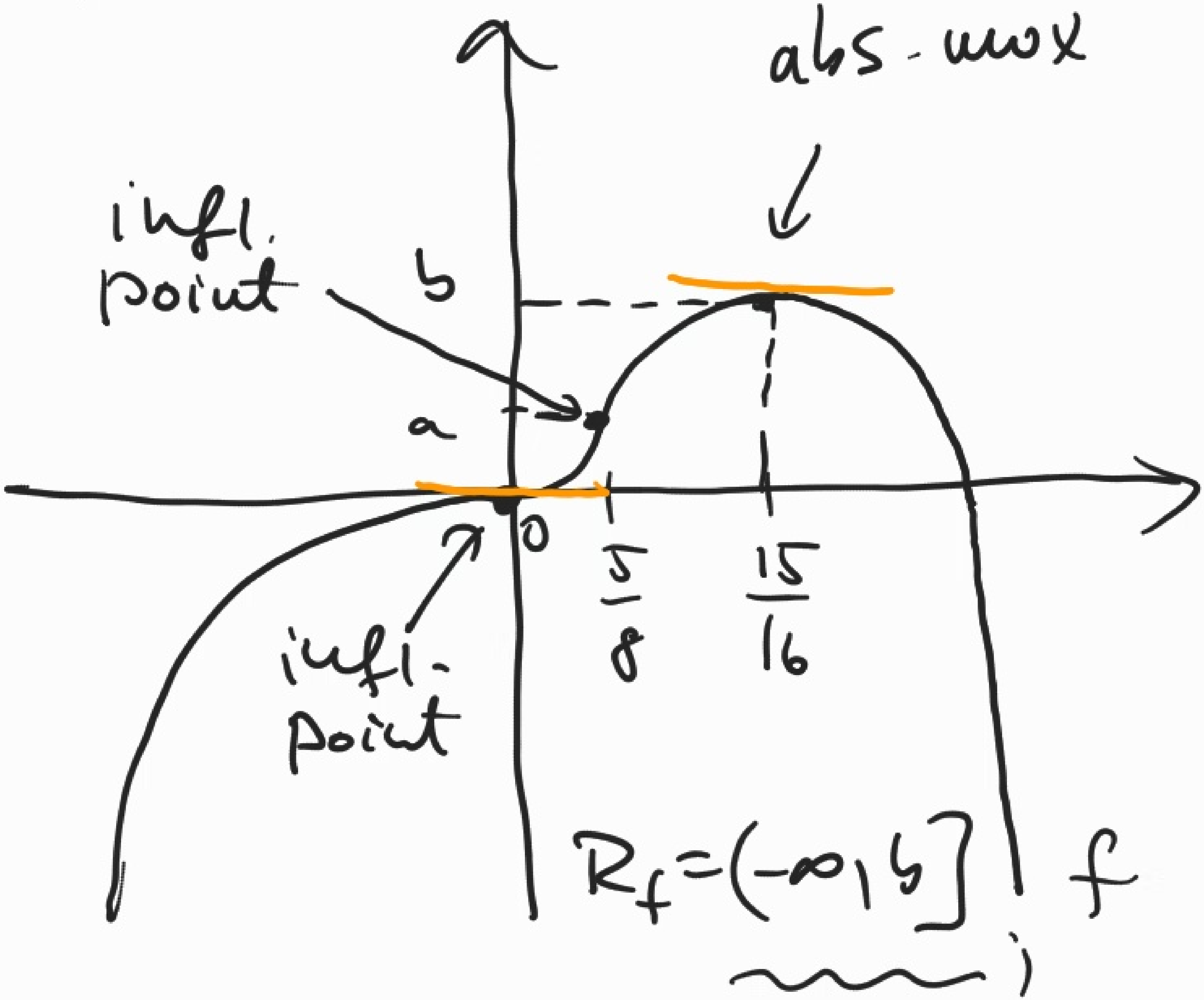
| $x$ | $-\infty$ | | $0$ | | $\frac{5}{8}$ ($= \frac{10}{16}$) | | $\frac{15}{16}$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $+$ $+$ | $0$ | $+$ | $+$ | $+$ | $0$ | $-$ $-$ $-$ | |
| $f(x)$ | $-\infty$ | $\nearrow$ | $0$ | | $\nearrow$ | | $a$ | $\searrow$ | $-\infty$ |
| $f''(x)$ | | $-$ $-$ | $0$ | $+$ $+$ | $0$ | | $-$ $-$ $-$ $-$ | | |
| conv. | | $\frown$ | infl. | $\smile$ | infl. | | $\frown$ | | |

$$f\left(\frac{15}{16}\right) = 5\cdot\left(\frac{15}{16}\right)^3 - 4\cdot\left(\frac{15}{16}\right)^4 = \left(\frac{15}{16}\right)^3 \cdot \frac{5}{4} = a$$

$$f\left(\frac{5}{8}\right)=5\cdot\left(\frac{5}{8}\right)^3-4\cdot\left(\frac{5}{8}\right)^4=\left(\frac{5}{8}\right)^3\cdot\left(5-4\cdot\frac{5}{8}\right)=$$

$$=\left(\frac{5}{8}\right)^3\left(5-\frac{5}{2}\right)=\left(\frac{5}{8}\right)^3\cdot\frac{5}{2}=:b$$

(7) Graph:




$R_f=(-\infty, b]$;

d) $\boxed{f(x) := x \cdot \ln x \quad (x>0)}$

(1) $D_f = (0; +\infty)$

(2) $\nexists f(0)$, $0 \notin D_f$ <u>and</u>

$f(x) = 0 \iff x \cdot \ln x = 0 \iff$

$x_1 = 0 \notin (0; +\infty)$ or $\ln x = 0$

$\iff \underline{x_2 = e^0 = 1} \in (0; +\infty)$

(3) $f'(x) = (x \cdot \ln x)' = 1 \cdot \ln x +$

$+ x \cdot \frac{1}{x} = \underline{\ln x + 1} \quad (x>0)$

$f'(x) = 0 \iff \ln x + 1 = 0 \iff$

$\ln x = -1 \iff \underline{x = e^{-1} = \frac{1}{e}}$

$f'(x) > 0 \iff \ln x + 1 > 0 \iff$

$\ln x > -1 = \ln e^{-1} \iff \boxed{x > e^{-1}}$

$f'(x) < 0 \iff 0 < x < e^{-1}$

(4) $f''(x) = (\ln x + 1)' = \frac{1}{x} > 0$

$\forall x \in (0, +\infty)$

(5) $\lim\limits_{x \to 0+0} (x \cdot \ln x) = 0 \cdot (-\infty) =$

$= \lim\limits_{x \to 0+0} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{+\infty} \overset{\text{l'H}}{=}$

$= \lim\limits_{x \to 0+0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim\limits_{x \to 0+0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x \to 0+0} (-x) = \boxed{-0}$

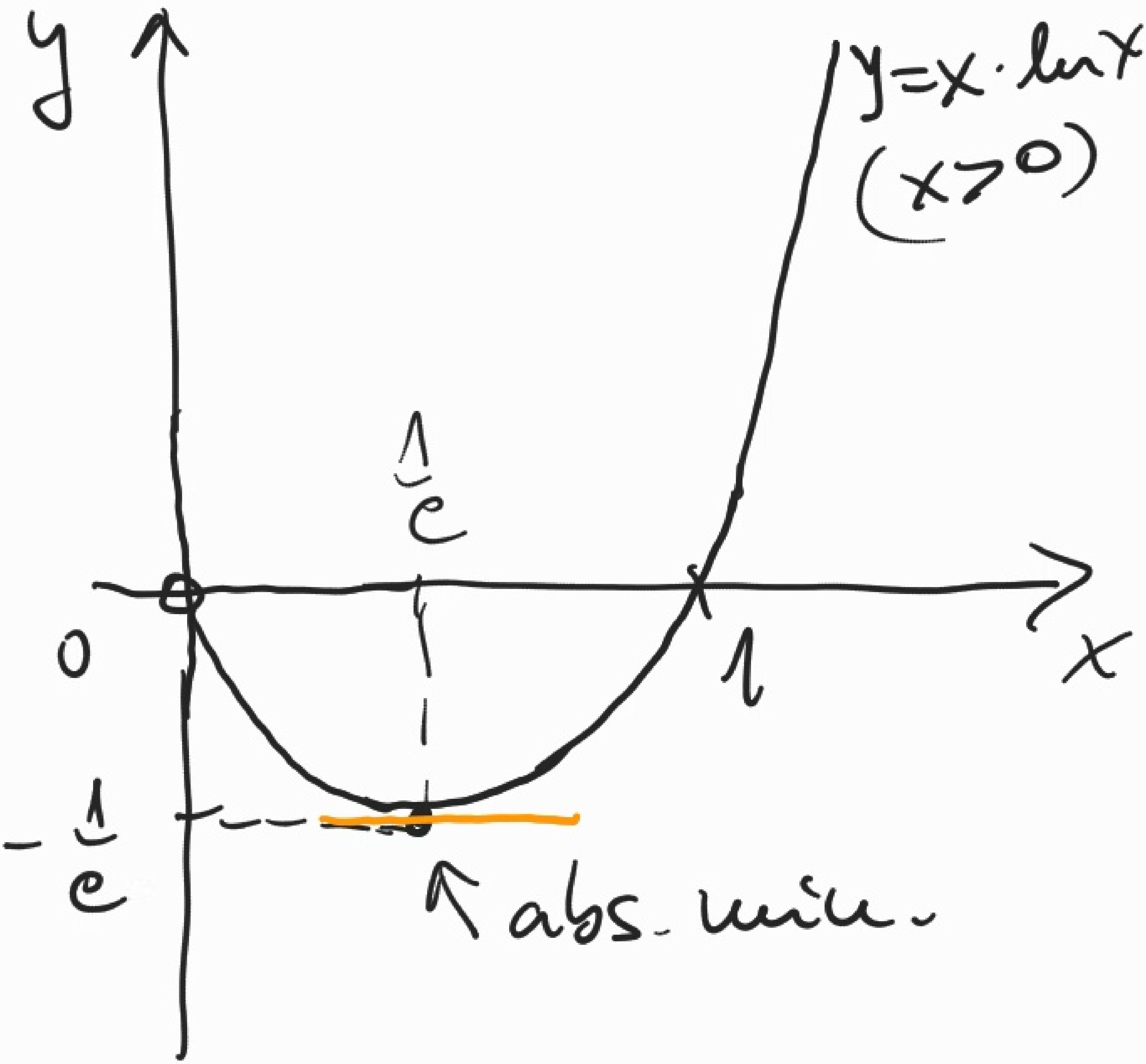
and $\lim\limits_{x \to +\infty} x \cdot \ln x = +\infty$.

(6) Table:

| $x$ | $0$ | | | $\frac{1}{e}$ | | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $-$ | $0$ | $+$ | $+$ | $+$ |
| $f(x)$ | $0$ | ↘ | | $-\frac{1}{e}$ | | ↗ | $+\infty$ |
| $f''(x)$ | | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| conv | | | | $\cup$ | | | |

$$f\left(\frac{1}{e}\right) = \frac{1}{e} \cdot \underbrace{\ln\frac{1}{e}}_{-1} = -\frac{1}{e}$$

(7) Graph:




$$R_f = \left[-\frac{1}{e}, +\infty\right);$$

$\square$

e) $\boxed{f(x) = \frac{e^x}{x+3} \quad (x \in \mathbb{R} \setminus \{-3\})}$

(1) $D_f = (-\infty, -3) \cup (-3; +\infty)$

(2) $f(0) = \frac{e^0}{3} = \frac{1}{3}$

$f(x) = 0 \iff \frac{e^x}{x+3} = 0$ no sol. here

(3) $f'(x) = \left(\frac{e^x}{x+3}\right)' = \frac{e^x \cdot (x+3) - e^x \cdot 1}{(x+3)^2}$

$= \frac{e^x \cdot (x+3-1)}{(x+3)^2} = \frac{e^x \cdot (x+2)}{(x+3)^2}$ for

all $x \in \mathbb{R} \setminus \{-3\}$.

$$f'(x)=0 \Leftrightarrow \frac{\overbrace{e^x}^{+}(x+2)}{\underbrace{(x+3)^2}_{+}}=0 \Leftrightarrow$$

$$x+2=0 \Leftrightarrow \boxed{x=-2}$$

$$f'(x)>0 \Leftrightarrow \quad x+2>0 \quad \boxed{x>-2}$$

$$f'(x)<0 \Leftrightarrow \quad x+2<0 \quad \boxed{\begin{matrix} x<-2 \\ x\neq-3 \end{matrix}}$$

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(4) $f''(x)=\left(\dfrac{e^x(x+2)}{(x+3)^2}\right)'=$

$$=\frac{\left(e^x(x+2)\right)'\cdot(x+3)^2-(x+2)e^x\cdot 2(x+3)}{(x+3)^4}$$

= SIMPLIFY !!! =

$$= \frac{\cancel{(x+3)} \cdot \left[ \left[ e^x (x+2) + e^x \cdot 1 \right] (x+3) - 2e^x (x+2) \right]}{(x+3)^{\cancel{4}\, 3}}$$

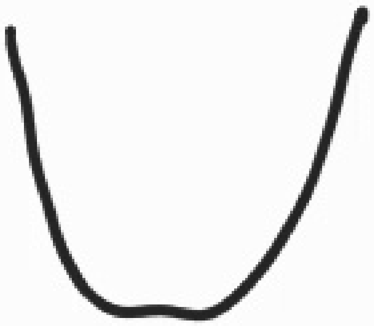
$$= \frac{e^x \cdot \left[ (x+3)^2 - 2x - 4 \right]}{(x+3)^3} =$$

$$= \frac{e^x \left( x^2 + 6x + 9 - 2x - 4 \right)}{(x+3)^3} =$$

$$= \underbrace{\frac{e^x \left( x^2 + 4x + 5 \right)}{(x+3)^3}}, \quad \left( x \in \mathbb{R} \setminus \{-3\} \right)$$

Since $x^2 + 4x + 5 = 0$

$D = 16 - 20 < 0 \Rightarrow$



$y = x^2 + 4x + 5 > 0$

$(\forall x \in \mathbb{R})$

**So** $f''(x) \neq 0 \quad (\forall x \in \mathbb{R} \setminus \{-3\})$

$f''(x) > 0 \iff x+3 > 0 \quad \boxed{x > -3}$

$f''(x) < 0 \iff x+3 < 0 \quad \boxed{x < -3}$

(5) Limits:

$$\lim_{x \to +\infty} \frac{e^x}{x+3} = \frac{+\infty}{+\infty} = L'H =$$

$$\lim_{x \to +\infty} \frac{e^x}{1} = e^{+\infty} = \boxed{+\infty}$$

$$\lim_{x \to -\infty} \frac{e^x}{x+3} = \frac{e^{-\infty}}{-\infty} = \boxed{-0}$$

$$\lim_{x \to -3} \frac{e^x}{x+3} = \frac{e^{-3}}{\pm 0}$$

$$\lim_{x \to -3-0} \frac{e^x}{x+3} = \frac{e^{-3}}{-0} = \underline{\underline{-\infty}}$$

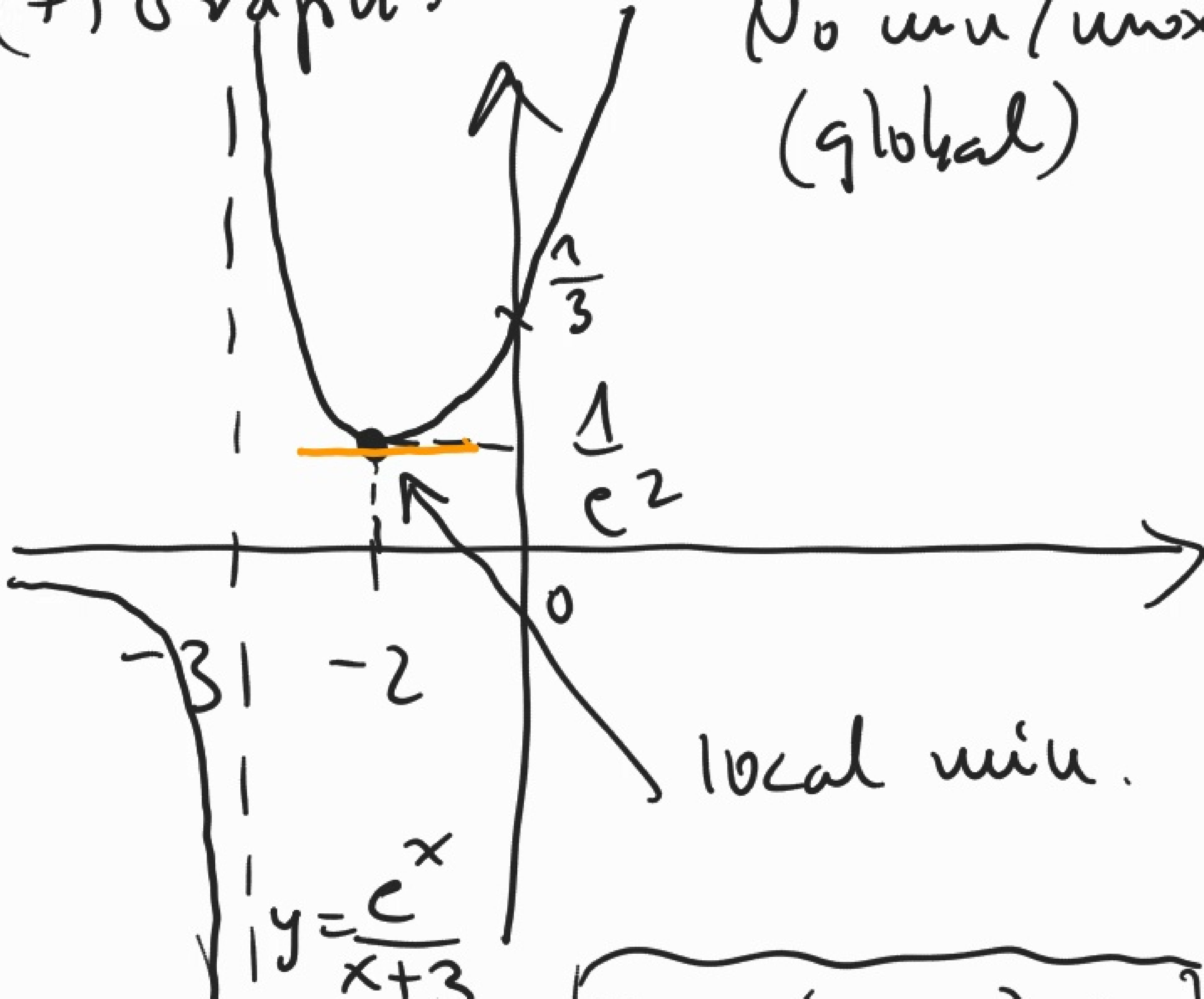
$$\lim_{x \to -3+0} \frac{e^x}{x+3} = \frac{e^{-3}}{+0} = \underline{\underline{+\infty}}$$

(6) Table:

| $x$ | $-\infty$ | | $-3$ | | $-2$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-\ -$ | \|\| | $-\ -$ | $0$ | $+\ +\ +$ | |
| $f(x)$ | $-0$ | $\searrow$ | $-\infty$ \|\| $+\infty$ | $\searrow$ | $e^{-2}$ | $\nearrow$ | $+\infty$ |
| $f''(x)$ | | $-\ -\ -$ | \|\| | $+\ +$ | | $+\ +\ +$ | |
| Conv | | $\frown$ | \|\| | | $\smile$ | | |

(7) Graph:

No min/max (global)




vertical asympthote

$$\boxed{R_f = (-\infty, 0) \cup \left[\frac{1}{e^2}; +\infty\right)}$$