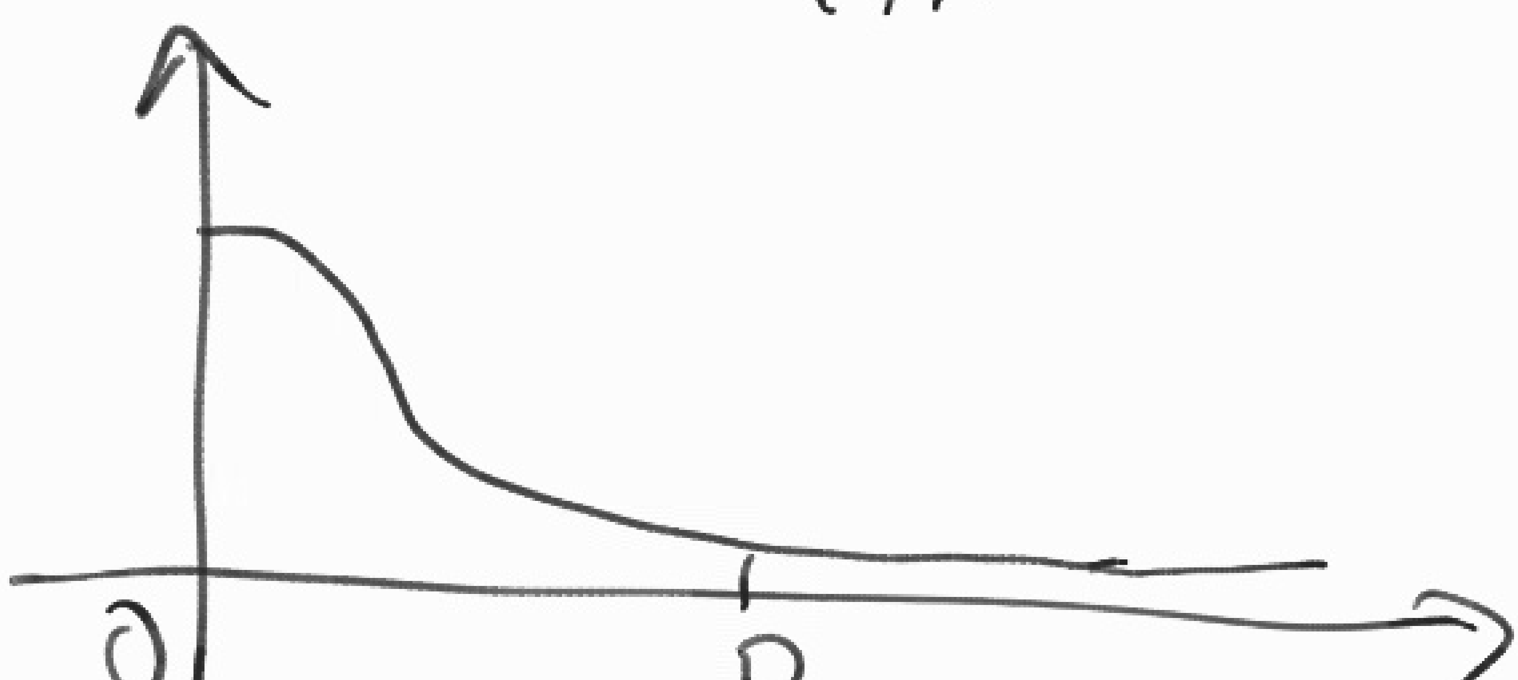
Analysis 2 ; Class 10

① Find the <u>improper</u> integrals:

a) $\displaystyle \boxed{\int_0^{+\infty} \frac{1}{1+x^2}\,dx}$

Here $f(x) = \dfrac{1}{1+x^2}$, $x \in [0, \infty)$



<u>Bounded</u> function on an <u>unbounded</u> interval.

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \int_0^R \frac{1}{1+x^2}\,dx$$

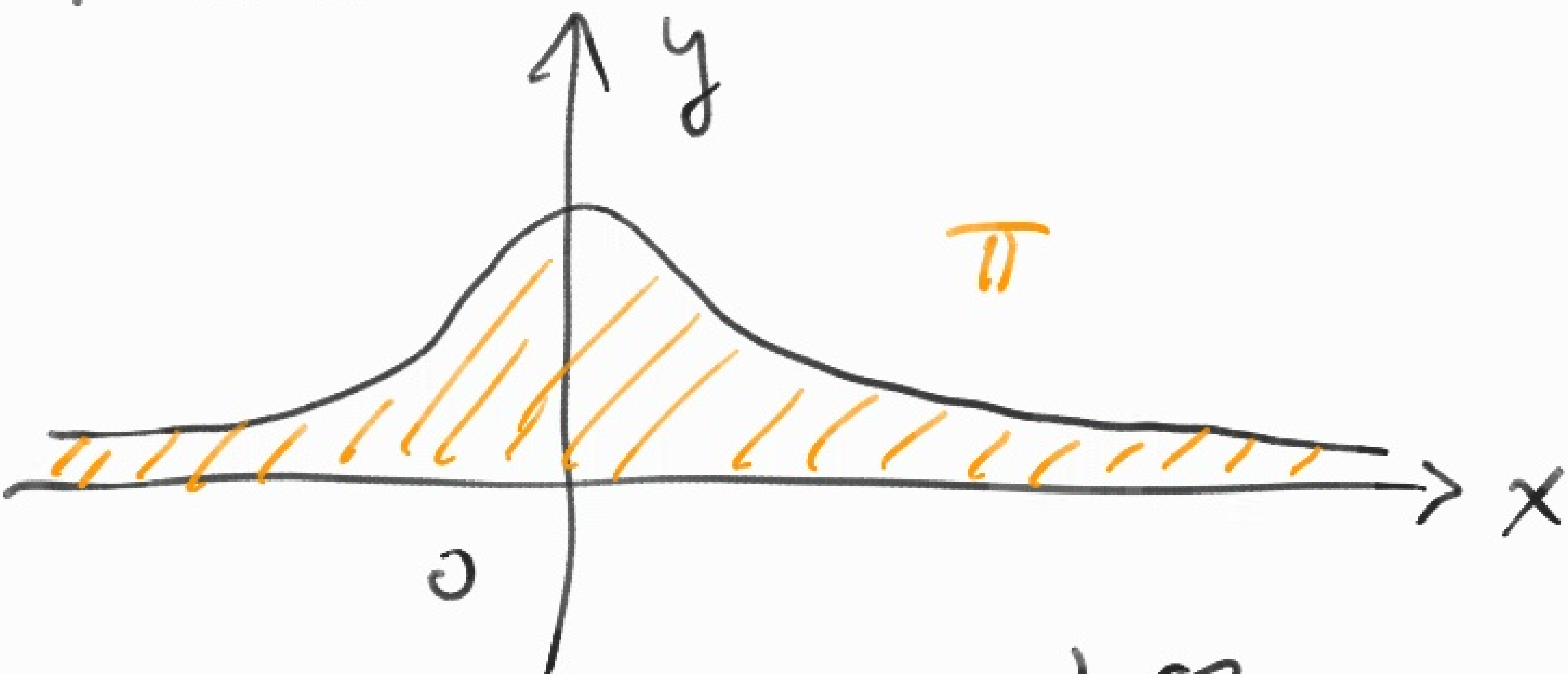
$$= \lim_{R\to+\infty} \Big[\arctan x\Big]_0^R =$$

$$= \lim_{R\to+\infty} \big(\arctan R - \underbrace{\arctan 0}_{0}\big)$$

$$= \lim_{R\to+\infty} (\arctan R) = \frac{\pi}{2} \in \mathbb{R}$$

$\Rightarrow$ So this integral is convergent and its value is $\pi/2$.

Remark since f is even



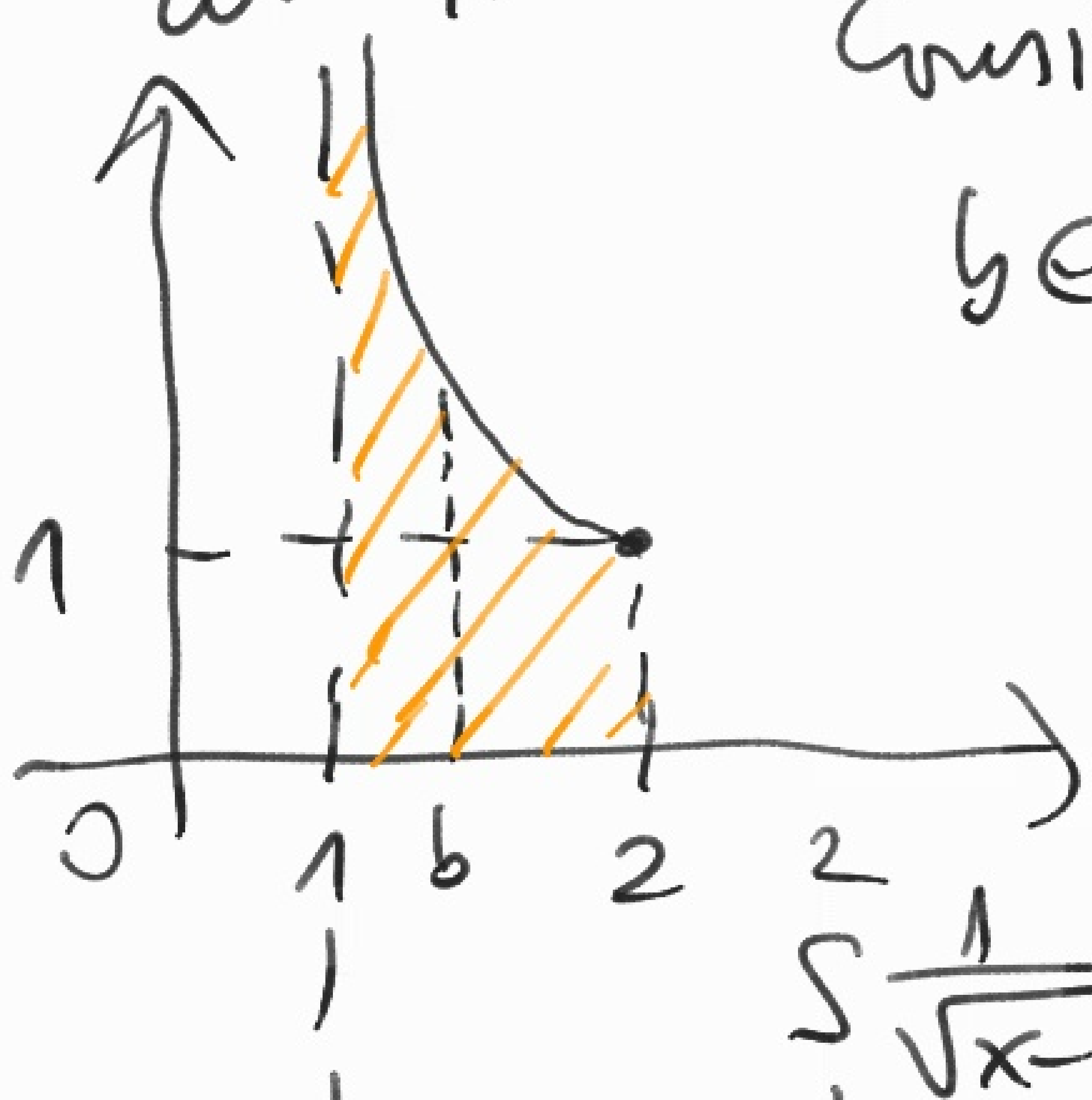
$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = 2\cdot\int_0^{+\infty} \frac{1}{1+x^2}\,dx$$

$$= 2\cdot\frac{\pi}{2} = \underline{\underline{\pi}}$$

$$\boxed{5)\ \int_1^2 \frac{1}{\sqrt{x-1}}\,dx\ ;}$$

<u>Now</u>: $f(x) = \frac{1}{\sqrt{x-1}}$, $x \in [1,2]$

$[1,2]$ is bounded (and closed); f <u>is not bounded</u> at 1.



Consider $b \in (1,2]$ <u>and</u>

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx =$$

$$= \lim_{b \to 1+0} \int_b^2 \frac{1}{\sqrt{x-1}}\,dx =$$

$$= \lim_{b \to 1+0} \left[2\sqrt{x-1} \right]_b^2 =$$

$$= \lim_{b \to 1+0} \left[2 \cdot \sqrt{1} - 2\sqrt{b-1} \right] =$$

$$= 2 - 2 \lim_{b \to 1+0} \sqrt{b-1} =$$

$$= 2 - 2 \cdot 0 = \underline{\underline{2}} \in \mathbb{R}$$

$\Rightarrow$ This integral is convergent with value: 2.

$$\boxed{c)\ \int_0^{+\infty} x\cdot e^{-2x}\,dx =}$$

$$= \lim_{R\to+\infty}\left(\int_0^R x\,\underbrace{e^{-2x}}_{\left(\frac{e^{-2x}}{-2}\right)'}\,dx\right) =$$

$$= \lim_{R\to+\infty}\left[\left[\frac{x\cdot e^{-2x}}{-2}\right]_0^R - \int_0^R (x)'\cdot\frac{e^{-2x}}{-2}\,dx\right] =$$

$$= \lim_{R\to\infty} \left(\frac{Re^{-2R}}{-2} - 0 + \frac{1}{2} \int_0^R e^{-2x}\,dx \right)$$

$$= \lim_{R\to\infty} \left(-\frac{R}{2e^{2R}} + \frac{1}{2} \left[\frac{e^{-2x}}{-2} \right]_0^R \right)$$

$$= \lim_{R\to\infty} \left(-\frac{R}{2e^{2R}} - \frac{1}{4} \underset{\underset{0}{\downarrow}}{\frac{1}{e^{2R}}} + \frac{1}{4} \right) =$$

$$= \underline{\underline{\frac{1}{4}}} \in \mathbb{R} \quad \Big(\underline{\text{since}} \quad \lim_{R\to\infty} \frac{R}{e^{2R}} = \frac{\infty}{\infty} =$$

$$= \lim_{R\to\infty} \frac{1}{2e^{2R}} = 0.\Big)$$
$$\uparrow$$
L'H

So this integral is convergent with value $\frac{1}{4}$.

d) $\boxed{\displaystyle\int_0^{+\infty} x e^{-x^2} dx =}$

$$= \lim_{R\to\infty}\left(\int_0^R x e^{-x^2} dx\right) =$$

$$= \lim_{R\to+\infty} \left[-\frac{1}{2} e^{-x^2} \right]_0^R =$$

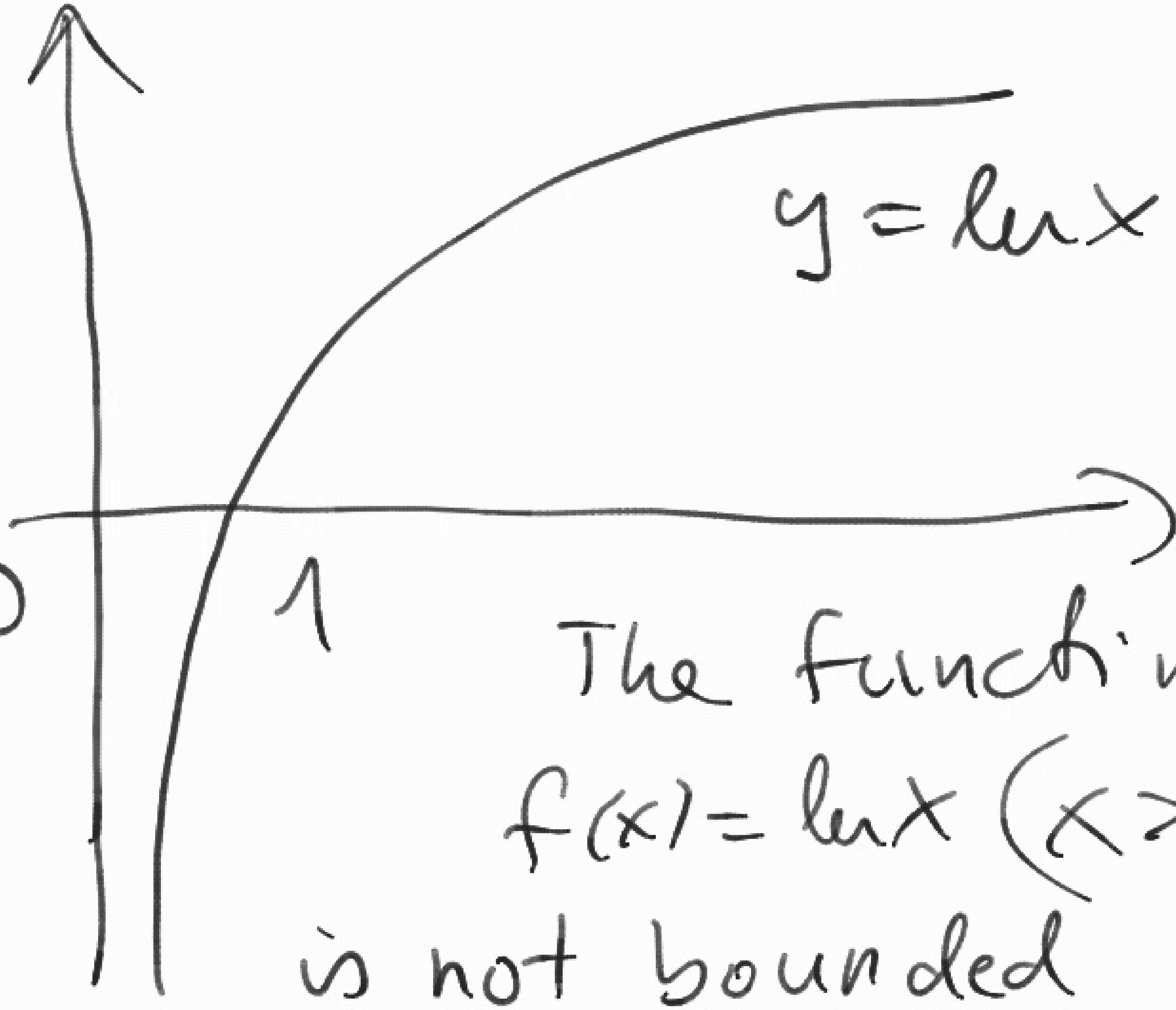
$$= \lim_{R\to\infty} \left(-\frac{1}{2} e^{-R^2} - \left(-\frac{1}{2} e^0 \right) \right) =$$

$$= \lim_{R\to\infty} \left(\frac{1}{2} - \underbrace{\frac{1}{2e^{R^2}}}_{\to 0\ (R\to\infty)} \right) =$$

$$= \underline{\underline{\frac{1}{2}}} \in \mathbb{R} \implies \int_0^{+\infty} x e^{-x^2}\, dx$$

is convergent and $= \frac{1}{2}$.

c) $\boxed{\int_0^1 \ln x \, dx =}$



The function $f(x) = \ln x \ (x > 0)$ is not bounded at $0+0 \Rightarrow$

$$\int_0^1 \ln x \, dx = \lim_{\varepsilon \to 0+0} \int_\varepsilon^1 \ln x \, dx$$

$$= \lim_{\varepsilon \to 0+0} \int_\varepsilon^1 (x)' \cdot \ln x \, dx \overset{i \cdot P}{=}$$

$$= \lim_{\varepsilon \to 0+0} \left(\left[x \ln x \right]_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x} dx \right)$$

$$= \lim_{\varepsilon \to 0+0} \left(1 \cdot \underbrace{\ln 1}_{=0} - \varepsilon \cdot \ln \varepsilon - \int_\varepsilon^1 1 \, dx \right)$$

$$= \lim_{\varepsilon \to 0+0} \left(-\varepsilon \ln \varepsilon - \left[x \right]_\varepsilon^1 \right) =$$

$$= \lim_{\varepsilon \to 0+0} \left(-\varepsilon \ln \varepsilon - 1 + \underset{\downarrow \atop 0}{\varepsilon} \right)$$

$$= -\lim_{\varepsilon \to 0+0} (\varepsilon \cdot \ln \varepsilon) - 1 =$$

$$= \underline{\underline{-1}} \in \mathbb{R} \quad \text{since}$$

$$\lim_{\varepsilon \to 0+0} \varepsilon \cdot \ln \varepsilon = 0 \, (-\infty) =$$

$$= \lim_{\varepsilon \to 0+0} \frac{\ln \varepsilon}{\frac{1}{\varepsilon}} = \frac{-\infty}{+\infty} = L'H$$

$$= \lim_{\varepsilon \to 0+0} \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon \to 0+0} (-\varepsilon) = \underline{\underline{0}}$$

$$\Rightarrow \boxed{\int_0^1 \ln x \, dx = -1}$$

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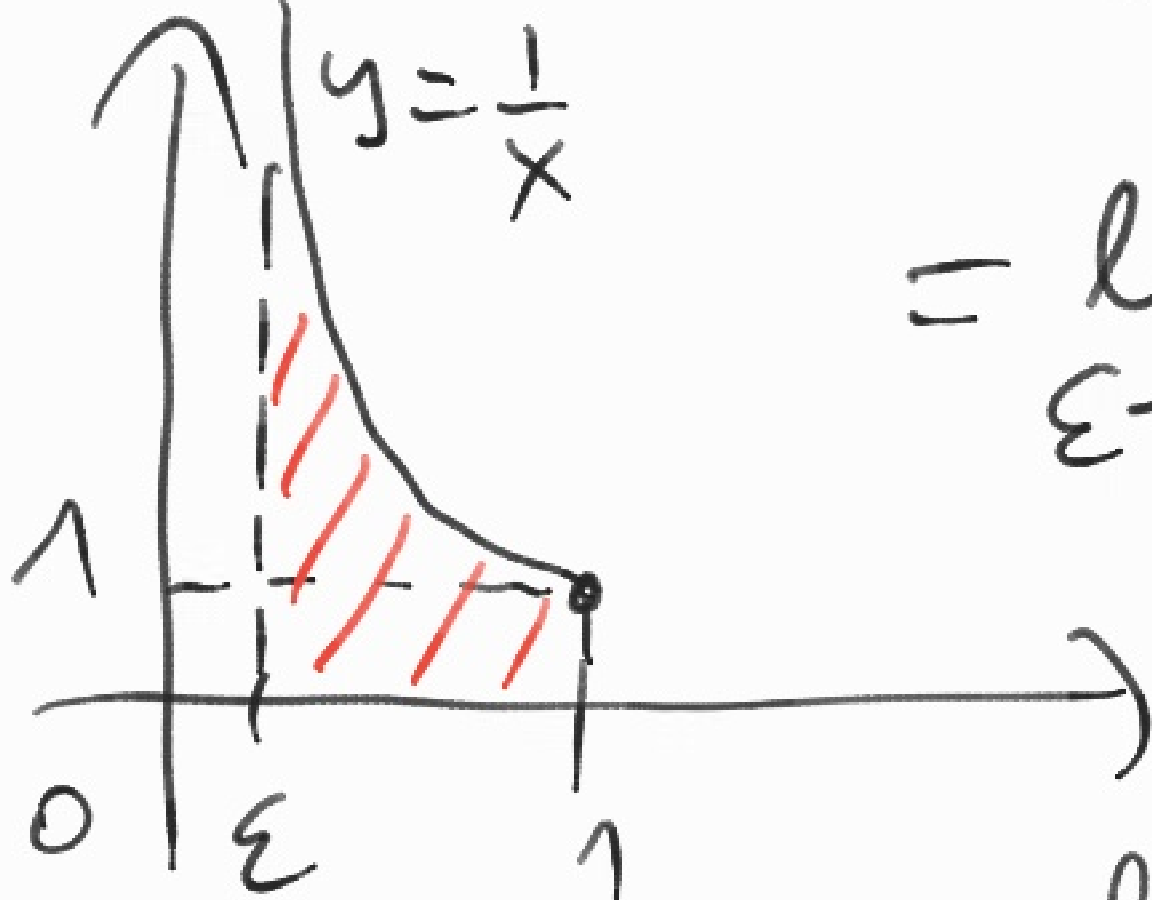
f) $$\int_0^1 \frac{1}{x} \, dx = \lim_{\varepsilon \to 0+0} \int_\varepsilon^1 \frac{1}{x} \, dx$$



$$= \lim_{\varepsilon \to 0+0} \left[\ln x \right]_\varepsilon^1 = \lim_{\varepsilon \to 0+0} \Big(\overset{0}{\overset{\shortparallel}{\ln 1}} - \ln \varepsilon \Big) = +\infty \notin \mathbb{R}$$

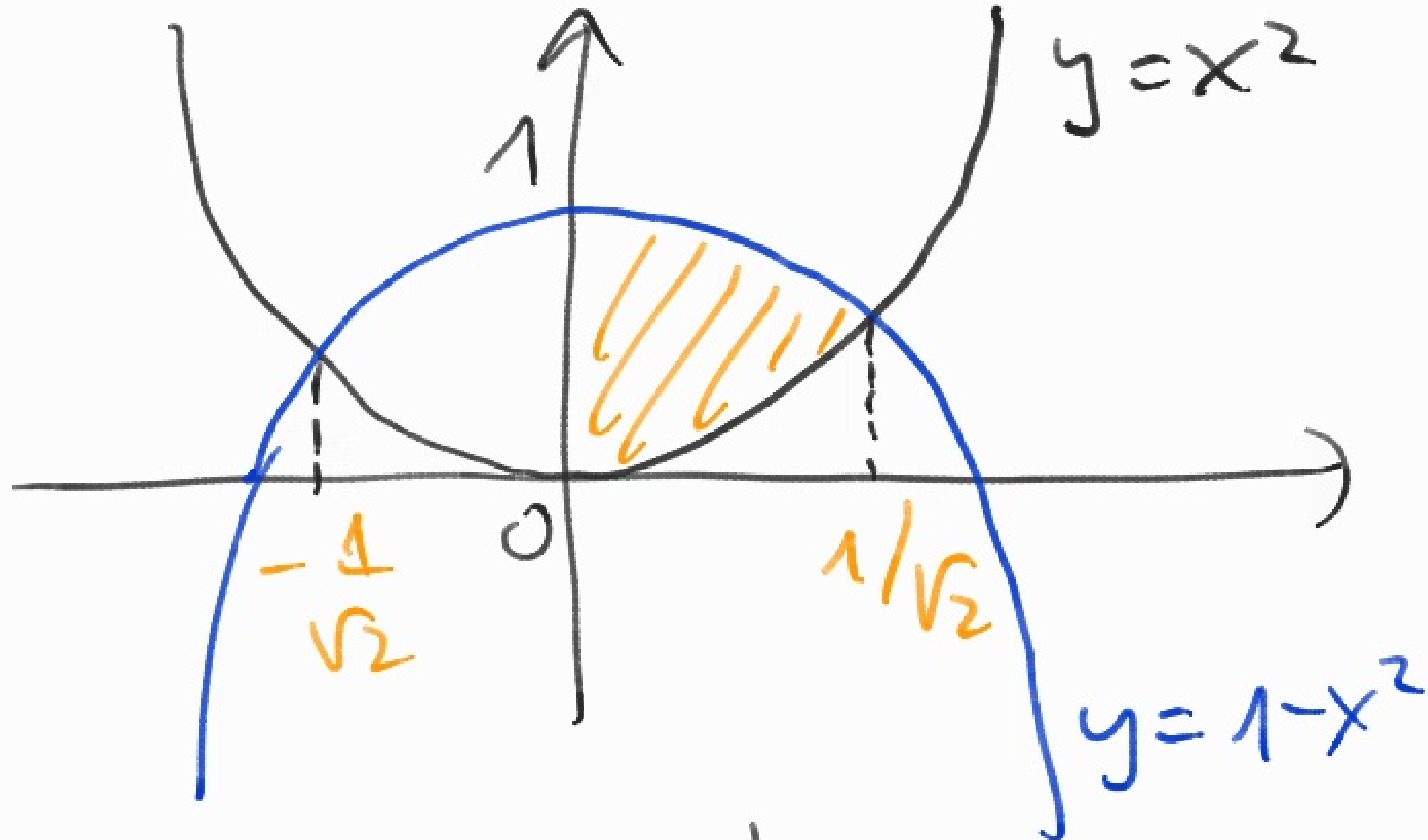
So $\int_0^1 \frac{1}{x}\,dx$ is divergent

(2) Applications of integrals.

(1) Areas

Determine the area bounded by the curves $y=x^2$ and $y=1-x^2$

Sol: First sketch the graphs of these curves



Find the intersection points: <u>solve</u> $\begin{cases} y = x^2 \\ y = 1-x^2 \end{cases}$

$\Rightarrow x^2 = 1-x^2 \Rightarrow 2x^2 = 1$

$x^2 = \frac{1}{2} \Leftrightarrow x_{1,2} = \pm\frac{1}{\sqrt{2}}$

The area is symmetric on axis y so

$$A = 2 \cdot \int_0^{1/\sqrt{2}} \left((1-x^2) - x^2 \right) dx$$

<u>upper</u> funct'ion

<u>lower</u> funct.

$$= 2 \cdot \int_0^{1/\sqrt{2}} (1 - 2x^2)\, dx =$$

$$= 2 \cdot \left[x - 2\frac{x^3}{3} \right]_0^{1/\sqrt{2}} =$$

$$= 2 \left(\frac{1}{\sqrt{2}} - 2 \cdot \frac{1}{3} \cdot \left(\frac{1}{\sqrt{2}} \right)^3 - 0 \right) =$$

$$= 2 \left(\frac{1}{\sqrt{2}} - \frac{1}{3} \frac{1}{\sqrt{2}} \right) = \boxed{\frac{4}{3\sqrt{2}}}$$

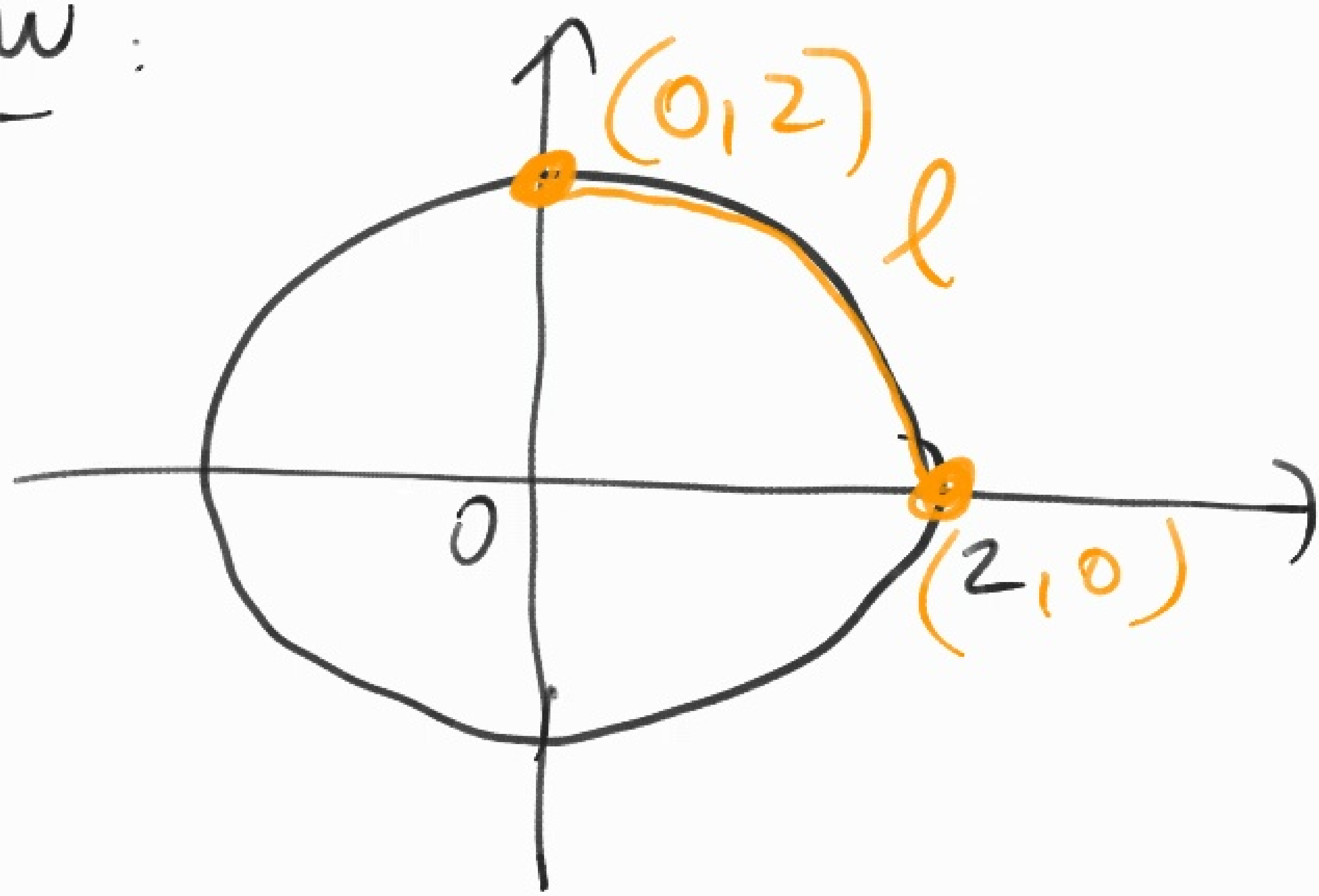
(2) Arc length of the graph of a function f:

Compute the arc length of the graph of f: $f(x)=\sqrt{4-x^2}$, bounded by the points $(0,2)$ and $(2,0)$.

Review. If $f \in C^1[a,b]$

$$\Rightarrow \exists\, l = \int_a^b \sqrt{1+(f'(x))^2}\,dx.$$

Now:



$$y=\sqrt{4-x^2}\quad,\ l=?$$

$$f(x)=\sqrt{4-x^2}\quad\Rightarrow$$

$$f'(x)=\frac{1}{2\sqrt{4-x^2}}\cdot(-2x)=\underline{\underline{-\frac{x}{\sqrt{4-x^2}}}}$$

$$\Rightarrow l=\int_0^2\sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\,dx=$$

$$= \int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}}\, dx =$$

$$= \int_0^2 \sqrt{\frac{4}{4-x^2}}\, dx =$$

$$= 2 \cdot \int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx =$$

= (is improper integral the function is not bounded at 2) =

$$= 2 \cdot \lim_{b \to 2-0} \int_0^b \frac{1}{\sqrt{4-x^2}}\,dx =$$

$$= 2 \lim_{b \to 2-0} \frac{1}{2} \cdot \int_0^b \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$= \lim_{b \to 2-0} \left[\frac{\arcsin\left(\frac{x}{2}\right)}{\frac{1}{2}} \right]_0^b =$$

$$= 2 \lim_{b \to 2-0} \left(\arcsin\left(\frac{b}{2}\right) - \underbrace{\arcsin 0}_{= 0} \right)$$

$$= 2 \arcsin 1 = 2 \cdot \pi/2 = \boxed{\pi}$$

Remark: We know, by elementory geometrie, that the arc length of a circle of radius R is $2\pi R$; Now the whole circle's length is $2\pi \cdot 2 = 4\pi \implies$

$$\ell = \frac{1}{4} \cdot 4\pi = \pi.$$

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(3) Volume of rotation solids

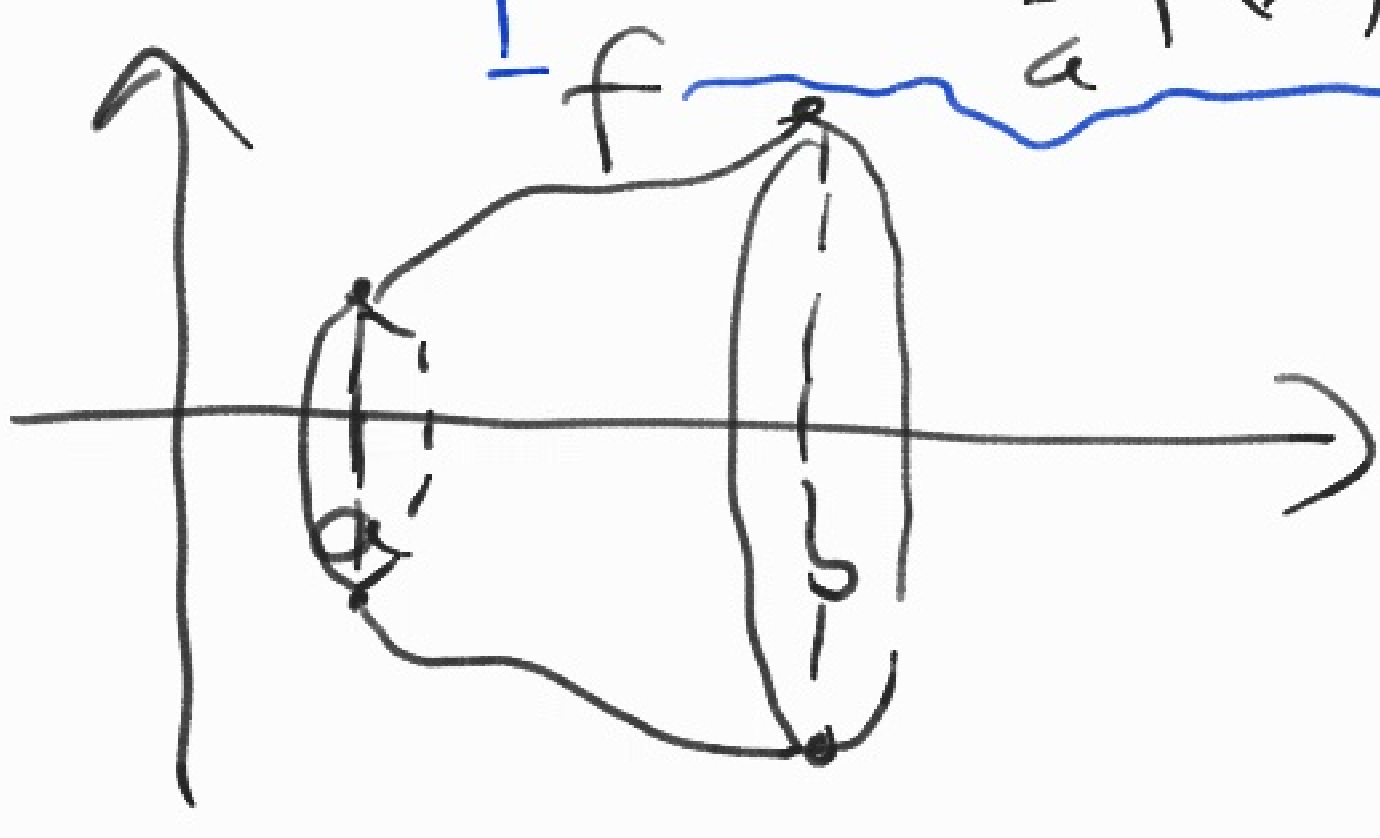
Rotate the curve

$$y = e^{2x} \quad (0 \le x \le 2)$$

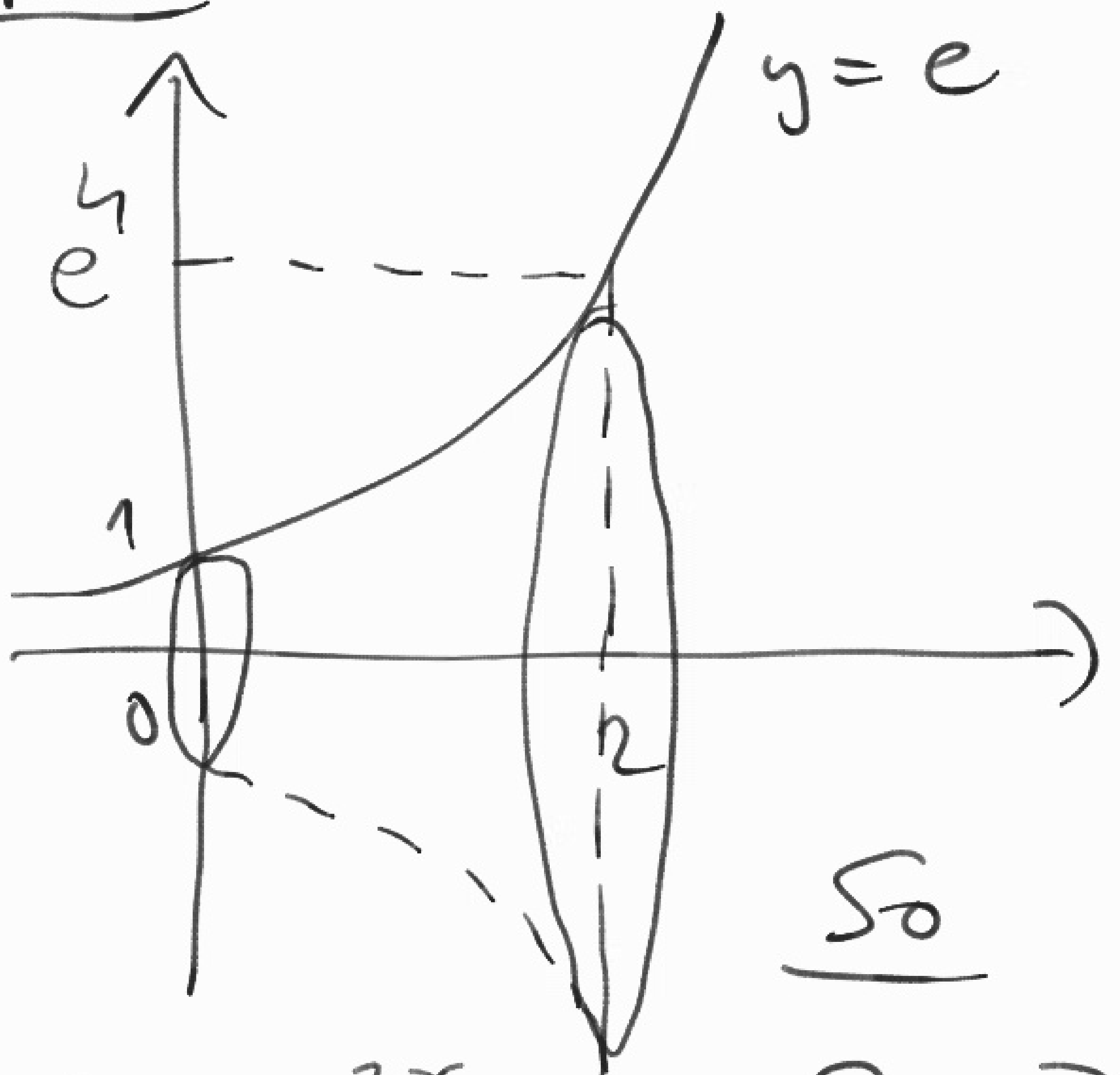
around x-axis. Determine the volume of this solid of revolution.

Sol: Remark: If $f \in C[a,b]$

$$f \ge 0 \Rightarrow \boxed{V = \pi \int_a^b f^2(x)\,dx}$$



Now:



So

$$f(x)=e^{2x},\ x\in[0,2] \Rightarrow$$

$$f\in C[0,2],\ f\geq 0 \Rightarrow$$

$$V=\pi\cdot\int_0^2 \left(e^{2x}\right)^2 dx =$$

$$= \pi \int_0^2 e^{4x}\,dx = \pi \left[\frac{e^{4x}}{4}\right]_0^2 =$$

$$= \frac{\pi}{4}\left[e^8 - e^0\right] = \frac{\pi(e^8-1)}{4}$$

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b) $f(x) = \sin x$, $x \in [0, \pi]$

Volume of the rotation solid = ?

$f \in C[0, \pi]$, $f \geq 0 \Rightarrow$

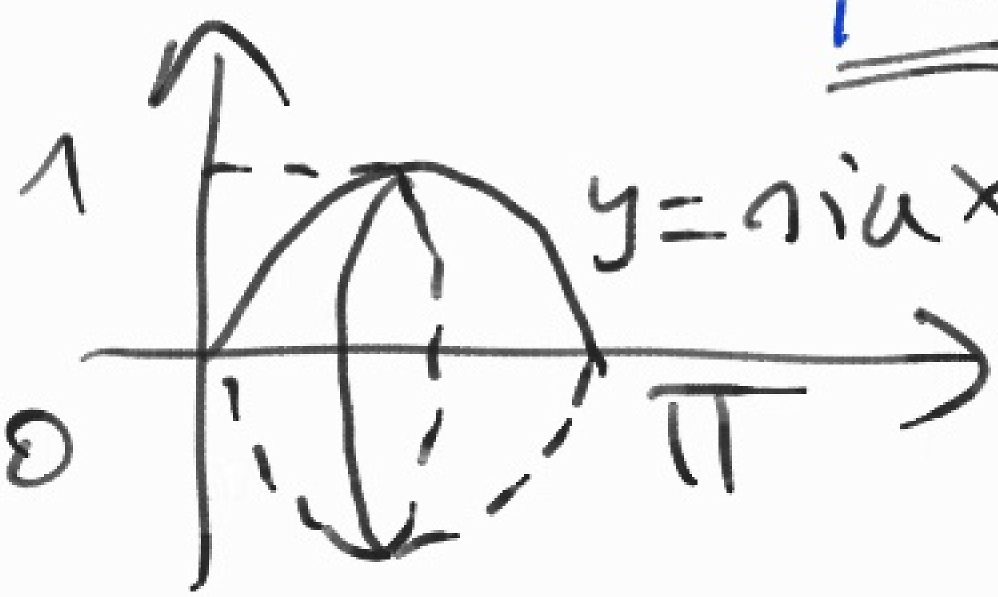
$$V = \pi \int_0^{\pi} \sin^2 x\,dx =$$

$$= \pi \int_0^{\pi} \frac{1-\cos 2x}{2}\,dx =$$

$$= \pi \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi} =$$

$$= \pi \cdot \left[\frac{\pi}{2} - \overbrace{\frac{\sin 2\pi}{4}}^{0} - 0 \right]$$

$$= \pi \cdot \frac{\pi}{2} = \boxed{\frac{\pi^2}{2}}$$



THE END.