Analysis II | Class 5

I L'Hospital's rule

Theorem: Suppose $-\infty \leq a < b \leq +\infty$; $f, g : (a,b) \to \mathbb{R}$
$f, g \in D(a,b)$ $g'(x) \neq 0$ $(x \in (a,b))$ and

either $\lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to a+0} g(x) = 0$ or

$\lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to a+0} g(x) = +\infty$ and

$$\exists \lim_{x \to a+0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}} \text{ (finite or infinite)}$$

Then $\Longrightarrow$ $$\lim_{x \to a+0} \frac{f(x)}{g(x)} = \lim_{x \to a+0} \frac{f'(x)}{g'(x)}.$$

1) Evaluate the following limits:

a) $$\lim_{x \to 0} \frac{\tan x - x}{x - \sin x} = \frac{0}{0} \overset{L'H}{=} \lim_{x \to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} =$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{\cos^2 x \cdot (1 - \cos x)} = \lim_{x \to 0} \frac{\cancel{(1 - \cos x)}(1 + \cos x)}{\cos^2 x \cdot \cancel{(1 - \cos x)}}$$

$$= \lim_{x \to 0} \frac{1 + \cos x}{\cos^2 x} = \frac{1+1}{1} = \boxed{2}$$

b) $\lim_{x\to 0} \frac{a^x - a^{\sin x}}{x^2} = \frac{0}{0} = L'H =$

$\boxed{a>0}$ $= \lim_{x\to 0} \frac{(a^x - a^{\sin x})'}{(x^2)'} =$

$= \lim_{x\to 0} \frac{a^x \cdot \ln a - a^{\sin x} \cdot \ln a \cdot \cos x}{2x} =$

$= \frac{\ln a}{2} \cdot \lim_{x\to 0} \frac{a^x - a^{\sin x} \cdot \cos x}{x} = "\frac{0}{0}"$ STILL $=$

$= L'H$ again $= \frac{\ln a}{2} \cdot \lim_{x\to 0} \frac{(a^x - a^{\sin x} \cdot \cos x)'}{(x)'} =$

$= \frac{\ln a}{2} \cdot \lim_{x\to 0} \left[\frac{a^x \cdot \ln a - a^{\sin x} \cdot \ln a \cdot \cos^2 x + a^{\sin x} \cdot \sin x}{1}\right]$

$= \frac{\ln a}{2} \cdot \left(a^0 \cdot \ln a - a^{\sin 0} \cdot \ln a \cdot \cos^2 0 + a^{\sin 0} \cdot \sin 0\right)$

$= \frac{\ln a}{2} \cdot \underbrace{(\ln a - \ln a)}_{0} = \frac{\ln a}{2} \cdot 0 = \boxed{0}.$

c) $\lim_{x\to 0+0} \ln x \cdot \ln(1-x) = "\infty \cdot 0" = *$

L'H cannot be used here like this, because we need "$\frac{0}{0}$ or $\frac{\infty}{\infty}$" types.

$* = \lim_{x\to 0+0} \frac{\ln(1-x)}{\frac{1}{\ln x}} = "\frac{0}{0}"$ then $= L'H$

$$= \lim_{x\to 0+0} \frac{\frac{1}{1-x}\cdot \cancel{(-1)}}{\cancel{(-)}\frac{1}{\ln^2 x}\cdot \frac{1}{x}} = \lim_{x\to 0+0} \frac{x}{1-x}\cdot \ln^2 x =$$

$$= \underbrace{\lim_{x\to 0+0} \frac{1}{1-x}}_{=1} \cdot \lim_{x\to 0+0} \frac{\ln^2 x}{\frac{1}{x}} \overset{L'H}{=} 1\cdot \lim_{x\to 0+0} \frac{2\ln x\cdot \frac{1}{x}}{-\frac{1}{x^2}}$$

$$" \frac{+\infty}{+\infty} "$$

$$" \frac{-\infty}{-\infty} "$$

$$= \lim_{x\to 0+0} \frac{2\cdot \ln x}{-\frac{1}{x}} \underset{L'H}{=} 2\cdot \lim_{x\to 0+0} \left(\frac{\frac{1}{x}}{\frac{1}{x^2}}\right) =$$

$$= 2 \lim_{x\to 0+0} (x) = 2\cdot 0 = \boxed{0}.$$

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d) $\lim_{x\to +\infty} \left(x\cdot e^{\frac{1}{x}} - x\right) = +\infty\cdot e^{0} - \infty = " +\infty - \infty "$

$$= \lim_{x\to +\infty} x\cdot \left(e^{\frac{1}{x}} - 1\right) = \lim_{x\to +\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \frac{0}{0} =$$

$$= L'H = \lim_{x\to +\infty} \frac{e^{\frac{1}{x}}\cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to +\infty} e^{\frac{1}{x}} = \boxed{1}$$

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e) $\lim\limits_{x\to+\infty} x^2\cdot e^{-x} = +\infty\cdot 0 = \lim\limits_{x\to+\infty}\left(\frac{x^2}{e^x}\right) =$

$= \frac{+\infty}{+\infty} = L'H = \lim\limits_{x\to+\infty}\frac{2x}{e^x} = \frac{+\infty}{+\infty} = L'H =$

$= \lim\limits_{x\to+\infty}\frac{2}{e^x} = \frac{2}{e^{+\infty}} = \frac{2}{+\infty} = +0 = \boxed{0}$

f) $\lim\limits_{x\to+\infty}\left(\overset{0}{e^{-x}}\cdot\ln x\right) = 0\cdot+\infty = \lim\limits_{x\to+\infty}\frac{\ln x}{e^x} =$

$= \frac{+\infty}{+\infty} = L'H = \lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{e^x} = \lim\limits_{x\to+\infty}\frac{1}{x\cdot e^x} =$

$= \frac{1}{+\infty} = 0$

g) $\lim\limits_{x\to+\infty}\left(\frac{2x-3}{2x+5}\right)^{\overset{0}{2x+1}} \underset{\uparrow}{=} \lim\limits_{x\to+\infty} e^{(2x+1)\cdot\ln\left(\frac{2x-3}{2x+5}\right)} =$

Use $f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\cdot\ln f(x)}$

$\underset{\uparrow}{=} e^{\lim\limits_{x\to+\infty}(2x+1)\cdot\ln\left(\frac{2x-3}{2x+5}\right)} = e^{l}$, where

$\exp\in C$

$l = \lim\limits_{x\to+\infty}\frac{\ln\frac{2x-3}{2x+5}}{\frac{1}{2x+1}} = "\frac{0}{0}" \underset{\uparrow\, L'H}{=} \lim\limits_{x\to+\infty}\frac{\frac{1}{\frac{2x-3}{2x+5}}\cdot\frac{\cancel{2}(2x+5)-(2x-3)\cdot\cancel{2}}{\cancel{(2x+5)^{2}}}}{-\frac{1}{\cancel{(2x+1)^{2}}}\cdot\cancel{2}}$

$-\frac{1}{(2x+1)^2}$

$$= \lim_{x\to+\infty} \frac{\cancel{2x+5}^{\,1}}{2x-3} \cdot \frac{-8}{(2x+5)^{\cancel{2}}} \cdot (2x+1)^2 =$$

$$= (-8) \cdot \lim_{x\to+\infty} \frac{(2x+1)^2}{(2x-3)(2x+5)} = -8 \cdot \frac{4}{4} = \boxed{-8}$$

So the original limit is $e^{-8} = \boxed{\frac{1}{e^8}}$;

II Taylor polinomials and estimations

Theorem: (Taylor's formula with Lagrange reminder): Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$, $f \in D^{n+1}(I)$; $a \in I$ $(n \in \mathbb{N})$; then: $\forall x \in I$ there exists a number c so that:

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1} \quad \text{or}$$

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}, \text{ where}$$

$T_n(x)$ is the n-th Taylor polinomial of f at point a

So:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$(x \in \mathbb{R})$

Exercises:

① Use Taylor polinomials for writing the polinomial $\boxed{f(x) = (2x+1)^3 \quad (x \in \mathbb{R})}$ by the powers of:

a) $(x+1)$;

b) $(x - \frac{1}{2})$;

c) $(x + \frac{1}{2})$;

Sol: f is a 3rd degree polinomial, so we will need the $T_3(x)$.

$f(x) = (2x+1)^3$

$f'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$

$f''(x) = 12(2x+1)^1 \cdot 2 = 24(2x+1)$

$f'''(x) = 24 \cdot 2 = 48$

$f^{(4)}(x) = 0.$

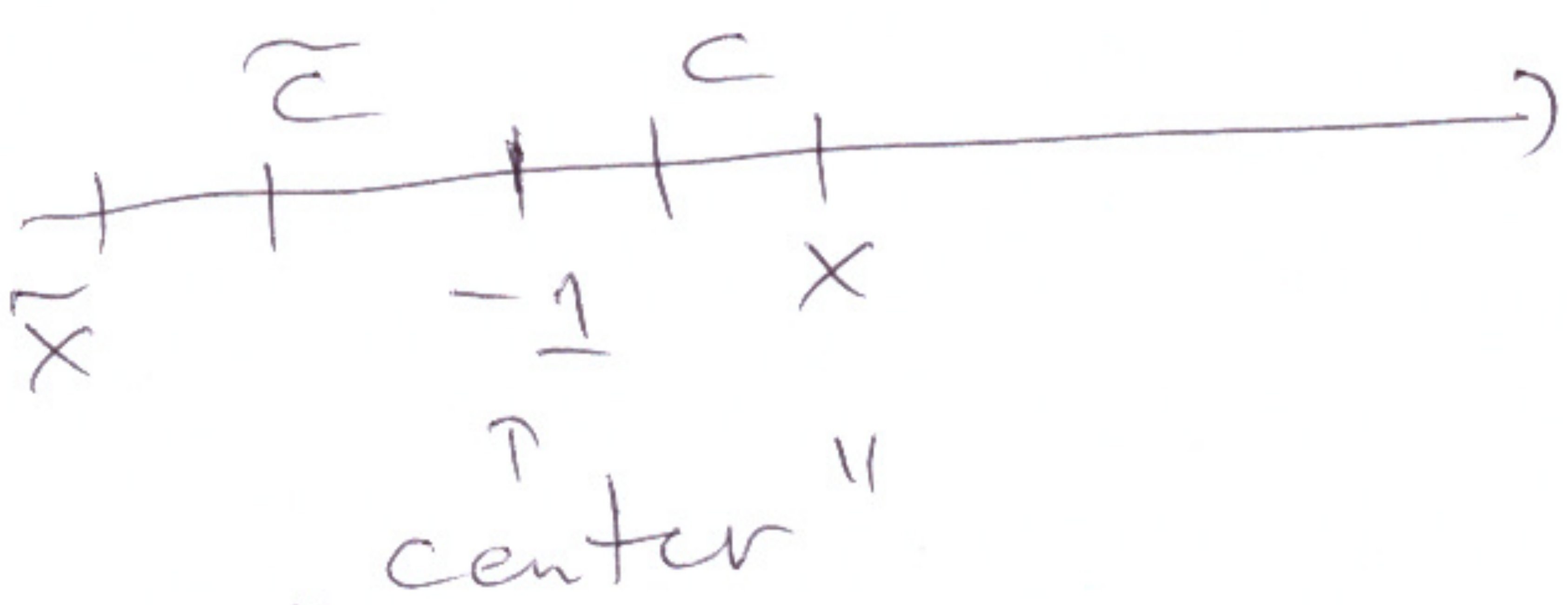
For a) $\boxed{a=-1}$ $\quad x-a = x+1$ so $a=-1$

$\Rightarrow$ Taylor formula:

$$(2x+1)^3 = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 +$$

$$+ \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(c)}{4!}(x+1)^4$$

with c between -1 and x



Now $f^{(4)}(c) = 0$ so no error term!

$f(-1) = -1$
$f'(-1) = 6$
$f''(-1) = -24$
$f'''(-1) = 48$

$$(2x+1)^3 = -1 + 6(x+1) - \frac{24}{2}(x+1)^2$$

$$+ \frac{48}{6}(x+1)^3$$

So

$$\boxed{(2x+1)^3 = -1 + 6(x+1) - 12(x+1)^2 + 8(x+1)^3}$$

b) For $(x-\frac{1}{2})$ powers $\boxed{a=\frac{1}{2}}$

$$T_3(x) = f\left(\tfrac{1}{2}\right) + f'\left(\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right) + \frac{f''\left(\frac{1}{2}\right)}{2!}\left(x-\tfrac{1}{2}\right)^2 +$$

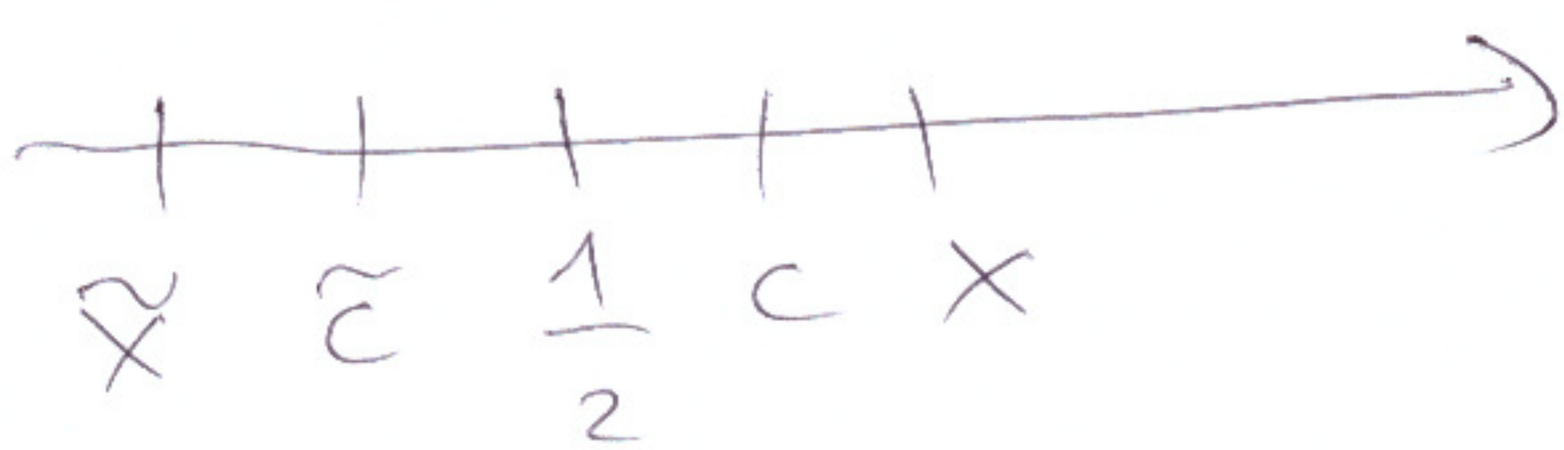
$$+ \frac{f'''\left(\frac{1}{2}\right)}{3!}\left(x-\tfrac{1}{2}\right)^3 =$$

$$= 8 + 24\left(x-\tfrac{1}{2}\right) + \frac{48}{2}\left(x-\tfrac{1}{2}\right)^2 + \frac{48}{6}\left(x-\tfrac{1}{2}\right)^3$$

So

$$\boxed{(2x+1)^3 = 8 + 24\left(x-\tfrac{1}{2}\right) + 24\left(x-\tfrac{1}{2}\right)^2 + 8\left(x-\tfrac{1}{2}\right)^3}$$

no error here: $\quad f^{(4)}(c) = 0$



c) HOMEWORK

② $f(x) = \ln(1+x)$ $(x > -1)$

So $I = (-1; +\infty)$;

a) Find $T_2(x)$ centered at $a = 0$.

b) Estimate the error of estimation $f(x) \approx T_2(x)$ if $x > 0$ and for $-1 < x < 0$

c) Approximate $\ln 2$ by means of T_2 and estimate the error.

Sol:

a) $T_2(x) = f(0) + f'(0) \cdot x + \dfrac{f''(0)}{2} \cdot x^2$

$f(x) = \ln(1+x) \Rightarrow f(0) = \ln 1 = 0$

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1} \Rightarrow f'(0) = 1$

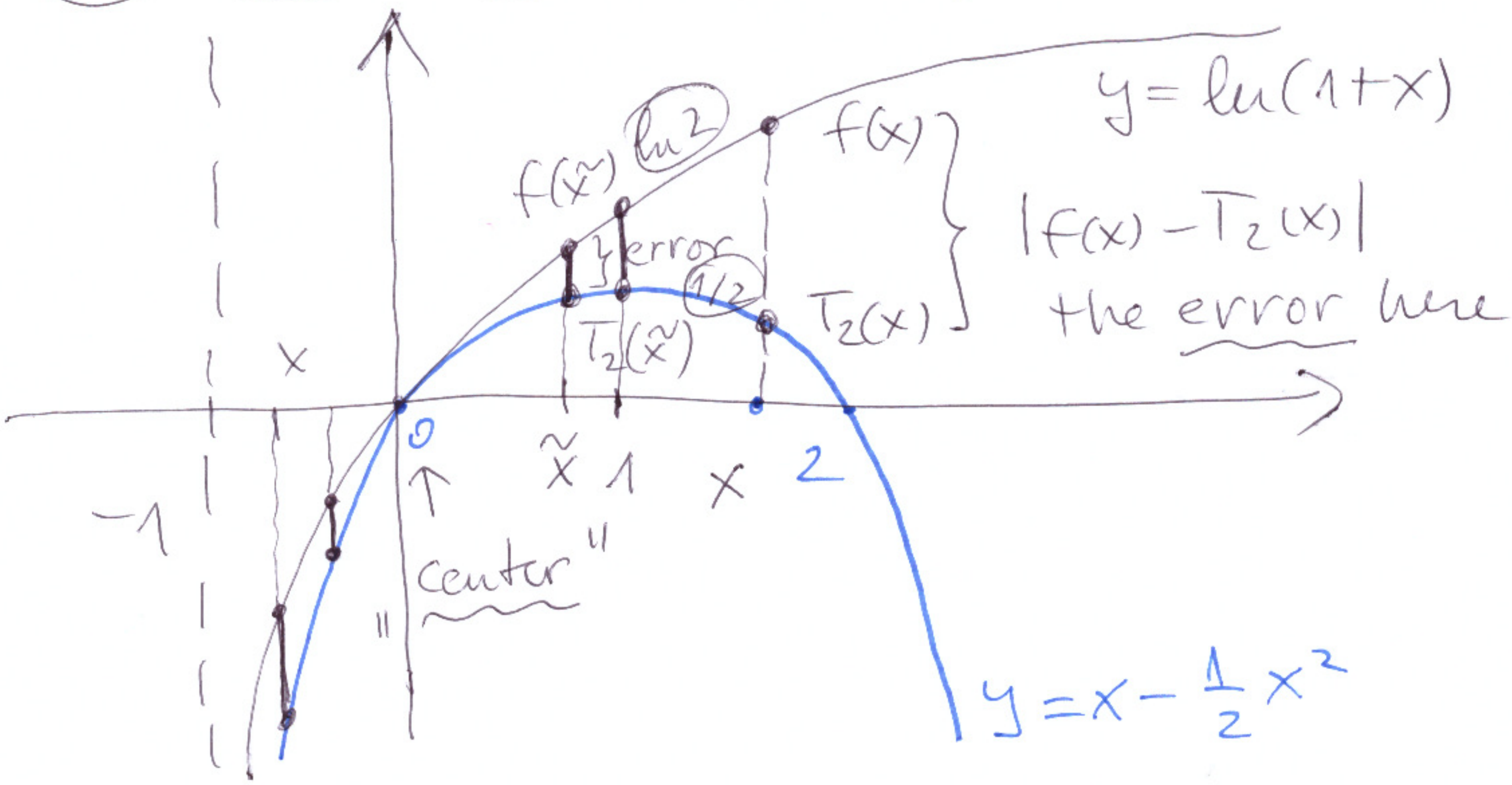
$f''(x) = -\dfrac{1}{(1+x)^2} = -(1+x)^{-2}$ $\quad f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\quad f'''(c) = \dfrac{2}{(1+c)^3}$

So:

$$T_2(x) = 0 + 1\cdot x - \frac{1}{2}x^2$$

$$\boxed{T_2(x) = x - \frac{1}{2}x^2 \quad (x\in\mathbb{R})}$$



b) $\left|f(x) - T_2(x)\right| = \left|\ln(1+x) - \left(x - \frac{1}{2}x^2\right)\right| =$

$= \left|\ln(1+x) - x + \frac{1}{2}x^2\right| =$ (Taylor formula) $=$

$= \left|\frac{f'''(c)}{3!}\cdot(x)^3\right| = \frac{1}{6}\cdot\frac{2}{\left|(1+c)^3\right|}\cdot|x|^3$

with c between 0 and x

So if $x > 0 \Rightarrow$

$0 < c < x$

$$|f(x) - T_2(x)| \le \frac{1}{3}\cdot\frac{1}{(1+0)^3}\cdot|x|^3 = \underline{\underline{\frac{|x|^3}{3}}}$$

If $\boxed{-1 < x < c < 0} \Longrightarrow$

$$|f(x) - T_2(x)| \le \frac{1}{3}\cdot\frac{1}{\underbrace{(1+x)^3}_{\oplus}}\cdot\overbrace{|x|^3}^{<1} <$$

$$< \frac{1}{3(1+x)^3}.$$

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c) $\ln 2 = \ln(1+1)$ <u>so</u>

We put $x = 1$ into b) $\Longrightarrow$

$$|\ln 2 - T_2(1)| = \left|\ln 2 - \frac{1}{2}\right| = \frac{1}{3}\cdot\frac{1}{|1+c|^3}\cdot 1$$

$T_2(1) = 1 - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$

with

So $\boxed{\ln 2 \approx \frac{1}{2}}$ $\boxed{0 < c < 1}$

$$\Rightarrow \underline{\underline{|\ln 2 - T_2(1)|}} = \frac{1}{3}\cdot\frac{1}{(1+c)^3} < \frac{1}{3}\frac{1}{(1+0)^3} = \underline{\underline{\frac{1}{3}}}.$$

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③ Estimate the error of the approx.:

$$\tan x \approx x + \frac{x^3}{3} \quad \left(x \in \left(-\frac{1}{10}, \frac{1}{10}\right)\right)$$

From here we can see: $x - a = x \Rightarrow$
$\boxed{a = 0}$

$f(x) = \tan x \quad x \in \left(-\frac{1}{10}, \frac{1}{10}\right)$

$$f'(x) = \frac{1}{\cos^2 x} = (\cos x)^{-2}$$

$$\boxed{\begin{array}{l} f(0) = 0 \\ f'(0) = 1 \end{array}}$$

$$f''(x) = -2(\cos x)^{-3}\cdot(-\sin x) = 2\sin x\cdot(\cos x)^{-3}$$

$$f'''(x) = 2\cos x\,(\cos x)^{-3} + 2\sin x\cdot(-3)\cdot(\cos x)^{-4}\cdot(-\sin x) =$$

$$= 2(\cos x)^{-2} + 6\sin^2 x(\cos x)^{-4} =$$

$$= 2(\cos x)^{-2} + 6(1-\cos^2 x)(\cos x)^{-4} =$$

$$= 2(\cos x)^{-2} + 6(\cos x)^{-4} - 6(\cos x)^{-2} =$$

$$= 6(\cos x)^{-4} - 4(\cos x)^{-2}$$

$$\boxed{\begin{array}{l} f''(0) = 0 \\ f'''(0) = 2 \end{array}}$$

$$f^{(4)}(x) = -24(\cos x)^{-5}\cdot(-\sin x) + 8(\cos x)^{-3}(-\sin x)$$

So:

$$\tan x = \underbrace{f(0)}_{0} + \underbrace{f'(0)}_{1} x + \frac{\overbrace{f''(0)}^{0}}{2!} x^2 + \frac{\overbrace{f'''(0)}^{2}}{3!} x^3 +$$

$$+ \frac{f^{(4)}(c)}{4!} x^4$$ with some c between 0 and x

So:

$$\boxed{T_3(x) = x + \frac{x^3}{3}}$$

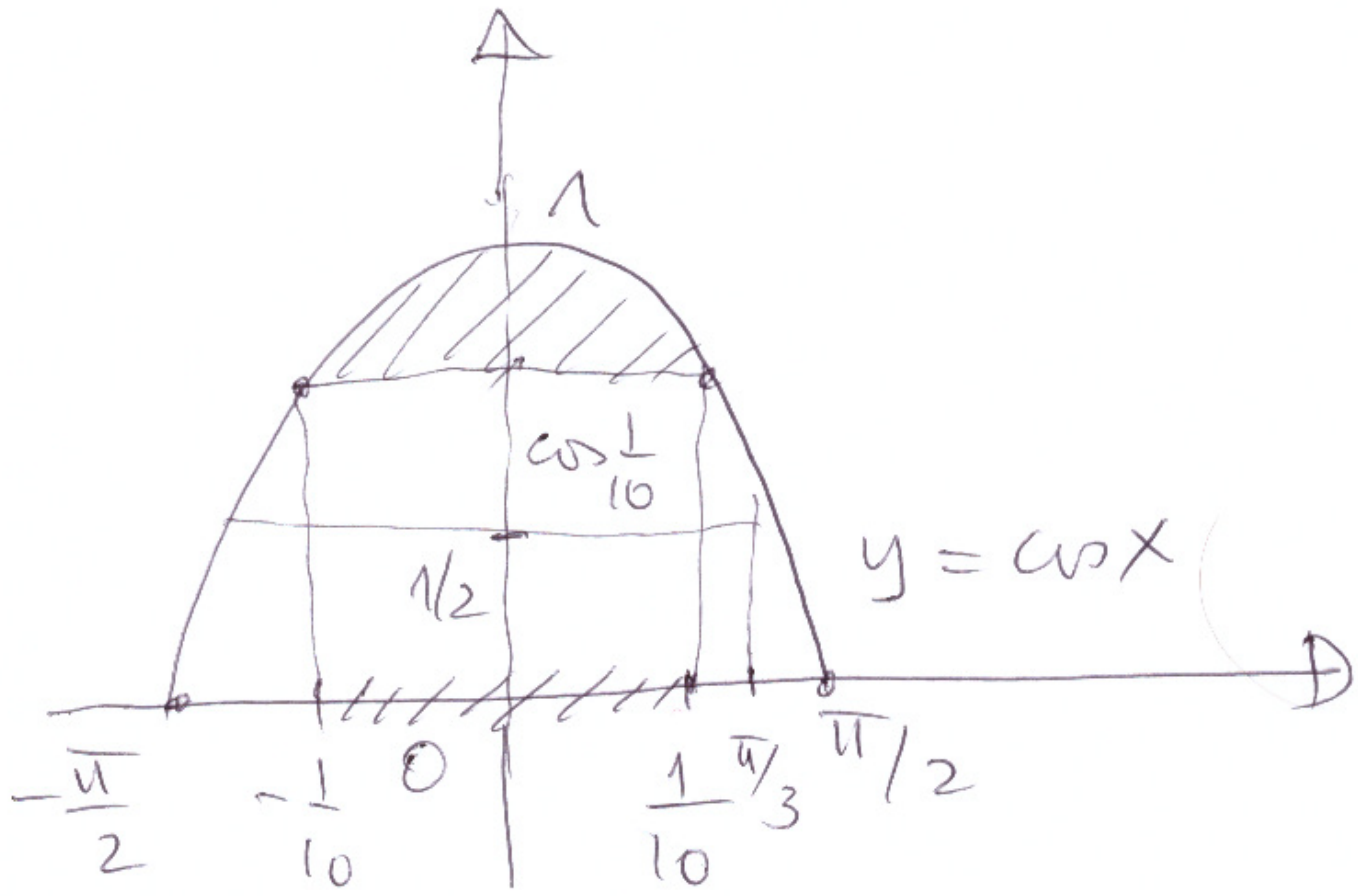
Error of approximation:

$$|f(x) - T_3(x)| = \left|\tan x - \left(x + \frac{x^3}{3}\right)\right| =$$ $|x| < \frac{1}{10}$

$$= \left|\frac{f^{(4)}(c)}{4!} x^4\right| = \underbrace{\left|\frac{24 \sin c}{(\cos c)^5} - \frac{8 \sin c}{(\cos c)^3}\right|}_{} \cdot \frac{x^4}{24} \le *$$

$$\underset{\Delta \text{ ineq.}}{\le} \frac{24 \cdot \overbrace{|\sin c|}^{\le 1}}{|\cos c|^5} + \frac{8 \overbrace{|\sin c|}^{\le 1}}{|\cos c|^3}$$

$$* \le \left(\frac{24}{|\cos c|^5} + \frac{8}{|\cos c|^3}\right) \cdot \frac{1}{24} \cdot \left(\frac{1}{10}\right)^4 < \left[\frac{24}{\left(\cos\frac{1}{10}\right)^5} + \frac{8}{\left(\cos\frac{1}{10}\right)^3}\right] \frac{1}{24 \cdot 10^4}$$



We can use that $\cos x > \frac{1}{2}$ if $x \in \left(-\frac{1}{10}, \frac{1}{10}\right)$

<u>So</u>

$$\left|\tan x - \left(x + \frac{x^3}{3}\right)\right| < \left[\frac{24}{\left(\frac{1}{2}\right)^5} + \frac{8}{\left(\frac{1}{2}\right)^3}\right] \cdot \frac{1}{24 \cdot 10^4} =$$

$$= \frac{\overset{3}{\cancel{24}} \cdot 32 + \overset{1}{\cancel{8}} \cdot 8}{\underset{3}{\cancel{24}} \cdot 10^4} = \frac{96 + 8}{3 \cdot 10^4} = \underline{\underline{\frac{104}{30000}}}.$$