Problem set 1.: Sets

Question 1.
Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set, $A = \{1, 2, 3, 4\}$, $B = \{0, 2, 4, 8\}$ and $C = \{2, 3, 5, 7\}$.

(a) Write down the following sets explicitly, i.e. by listing all their elements:

$A \cap B \qquad B \cup C \qquad A \setminus C \qquad \overline{C}$

(b) Consider the systems of sets $X = \{A, B, C\}$ and $Y = \{\{0, 2, 4, 6, 8\}, \{1, 3, 5, 7, 9\}\}$.
Find the following sets:

$\cap X \qquad \cup X \qquad \cup Y \qquad \cap Y$

(c) Determine the truth value of each of the following statements:

$4 \in B$	$A \subseteq B$	$\{\emptyset\} \subseteq \cup X$	$3 \in A \cap B$
$\{1, 2\} \subseteq A$	$A \in \cup Y$	$A \subseteq \cup Y$	$C \cap \emptyset = \emptyset$
$2 \subseteq A$	$\{2\} \subseteq A$	$2 \in \cup X$	$\{2\} \in \cap X$

Question 2.
Let $\mathcal{A} = \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$. Find the sets $\cup\mathcal{A}$ and $\cap\mathcal{A}$.

Question 3.
Consider the system of sets $X = \{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{0, 2, 3, 7\}\}$. Find the following sets: $\cap X$, $X \cup \{5, 6, 7, 8\}$, $X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\}$, $\cup(X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\})$, $\cap(X \cup \{\{3, 5, 7\}, \{1\}, \{2\}\})$.

Question 4.
Find the sets A, B, C, given that they satisfy the following:
$A \setminus B = \{1, 3, 5\}$, $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$, $(A \cap C) \cup (B \cap C) = \emptyset$, $C \setminus B = \{2, 4\}$ and $(A \cap B) \setminus C = \{6\}$.

Question 5.
Prove that the following equalities are true for any universal set U and sets $A, B, C \subseteq U$ (hence these equalities are identities):

(a) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

(c) $A \cap B = B \cap A$

(d) $(A \cap B) \cap C = A \cap (B \cap C)$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(h) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(i) $A \cup \overline{A} = U$

(j) $A \cap \overline{A} = \emptyset$

(k) $\overline{\overline{A}} = A$

Question 6.
Give an example for sets A, B, C that satisfy all the conditions below:

$$A \cap B \neq \emptyset, \quad A \cap C = \emptyset, \quad (A \cap B) \setminus C = \emptyset.$$

Question 7.
Prove that for any nonempty sets A and B the following equalities hold:
(a) $(A \setminus B) \cap B = \emptyset$
(b) $\big(A \cup \overline{B}\big) \cap \big(\overline{A} \cup \overline{B}\big) = \overline{B}$

Question 8.
Let $A = \{a, b, c, d\}$, $B = \{c, d\}$ and $C = \{a, c, e\}$. Show that then $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. Is this statement true for all sets A, B, C?

Question 9.
Show that the following statements are true for all sets A, B, C:
(a) if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$
(b) if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$
(c) $A \cup (B \cap A) = A$

Question 10.
Write the following expression in its simplest possible form: $(A \cup (A \cap B) \cup (A \cap B \cap C)) \cap (A \cup B \cup C)$.

Question 11.
Prove that the following equalities hold for any universal set U and any sets $A, B, C \subseteq U$ (hence these equalities are identities).
(a) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
(b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(c) $A \setminus \ (A \setminus (B \setminus C)) = A \cap B \cap \overline{C}$

Question 12.
Prove the following identity: $\overline{(\overline{A \cap B} \cup C) \cap \overline{A}} \cup \overline{B} \cup \overline{C} = A \cup \overline{B} \cup \overline{C}$.

Question 13.
Decide which of the following statements are true *for all* sets A, B, C. Prove your statements.
(a) $\overline{A} \cap B = B \setminus A$
(b) $(A \cap B) \setminus C = (A \setminus B) \cap C$
(c) $(A \cup B) \cap (B \setminus A) = (A \cup B) \setminus (A \setminus B)$
(d) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
(e) $(A \cup B) \setminus A = B$
(f) $(A \cup B) \setminus C = A \cup (B \setminus C)$

Question 14.
Prove the following identities.
(a) $A \triangle \emptyset = A$
(b) $A \triangle A = \emptyset$
* (c). $A \triangle (B \triangle C) = (A \triangle B) \triangle C$
* (d). $A \triangle (A \triangle B) = B$

Question 15.
Prove that for any sets A and B we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, where $\mathcal{P}(A)$ denotes the power set of A. What can we say about the truth value of the statement obtained by replacing $\cap$ by $\cup$?

Question 16.
Let A, B, C, D be nonempty sets. Prove that then $A \times B \subseteq C \times D$ holds if and only if $A \subseteq C$ and $B \subseteq D$.

Question 17.
Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:
$A \times A, \quad A \times B, \quad A \times A \times B, \quad B \times A, \quad (A \times A) \times B, \quad A \times (A \times B), \quad A \triangle B, \quad A \triangle C.$

Question 18.
Prove that for any nonempty sets A, B, C the following is true:
$(A \cup B) \times C = (A \times C) \cup (B \times C).$