Problem set 3.: Properties of homogeneous binary relations, equivalence relations and partitions

Properties of homogeneous binary relations

Question 1.
Let $X = \{1, 2, 3\}$. In each of the following examples below decide if the relation ρ on X is reflexive, symmetric, anti-symmetric and/or transitive.
(a) $\rho = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
(b) $\rho = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3)\}$
(c) $\rho = \{(1,2),(1,3),(2,1),(3,1)\}$
(d) $\rho = \{(1,2),(2,3),(3,1)\}$
(e) $\rho = \{(1,2)\}$
(f) $\rho = \{(1,2),(2,1),(2,3),(3,2)\}$
(g) $\rho = \{(1,1),(2,2),(2,3),(3,3)\}$
(h) $\rho = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

Question 2.
(a) Can a relation be both symmetric and anti-symmetric at the same time? Can a relation be both reflexive and irreflexive? Justify your answers.
(b) Prove that if a relation is both symmetric and anti-symmetric then it is also transitive.
(c) Prove that if a relation that is not the empty set is both irreflexive and symmetric, then it is not transitive.

Question 3.
In each of the following examples, decide if the given relation is reflexive, irreflexive, symmetric, anti-symmetric and/or transitive, and find the domain and the range of the relation.
(a) $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \cdot b \text{ is odd}\}$
(b) $S = \{(a, b) \in B \times B \mid$ the surname of a is shorter than the surname of $b\}$ where B is the set of all Dimat 1. students at ELTE.
(c) $T_X = \{(A, B) \in P(X) \times P(X) \mid A \cap B \neq \emptyset\}$ where X is a given set.
(d) $U = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \gcd(a, b) > 1\}$
(e) $V = \{(x, y) \in K \times K| \mid x$ touches y from inside$\}$, where K is the set of all circles in a given plane.

Question 4.
Define the binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$ as follows: for every $m, n \in \mathbb{N}$ let $n\ R\ m$ if and only if the number of common prime divisors of m and n is even. Describe the properties of R (i.e. decide if R is reflexive, transitive, symmetric, anti-symmetric and/or trichotomous).

Question 5.
In each question below give an example for a relation on the set $\{1, 2, 3, 4\}$ satisfying simultaneously all the properties listed in the question:
(a) reflexive and not irreflexive;
(b) anti-symmetric and not symmetric;

(c) symmetric and not anti-symmetric;
(d) both symmetric and anti-symmetric;
(e) neither symmetric nor anti-symmetric;
(f) both reflexive and trichotomous;
(g) not reflexive, not transitive, not symmetric, not anti-symmetric, and not trichotomous

Equivalence relations and partitions

Question 6.
Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For each of the relations defined in parts (a) and (b) below, solve questions (1) and (2).

(a) $\rho = \{(1, 1), (1, 5), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 1), (5, 5)\}$
(b) $\rho = \{(1, 1), (1, 5), (1, 6), (1, 8), (2, 2), (2, 4), (3, 3), (3, 7), (4, 2), (4, 4), (5, 1), (5, 5), (5, 6), (5, 8), (6, 1), (6, 5), (6, 6), (6, 8), (7, 3), (7, 7), (8, 1), (8, 5), (8, 6), (8, 8)\}$

(1) Prove that ρ is an equivalence relation on A.
(2) Write down the partition of A induced by the equivalence relation ρ (in other words: Find the quotient set A/ρ).

Question 7.
In each example below find the equivalence relation on $\{a, b, c, d, e, f\}$ which corresponds to the given partition:
(a) $\{\{a, b, f\}, \{c\}, \{d, e\}\}$
(b) $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e, f\}\}$

Question 8.
In each of the following examples prove that R is an equivalence relation (on the set which R is defined on in the example), and find the equivalence classes of R.
(a) $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + n \text{ páros szám}\}$
(b) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 \text{ osztható 2-vel}\}$
(c) $R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a - b \text{ racionális}\}$
(d) $R = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m^2 - n^2 \text{ osztható 3-mal}\}$
(e) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 + y_1 = x_2 + y_2\}$
(f) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \cdot y_1 = x_2 \cdot y_2\}$

Question 9.
In each question below give an example for a relation on the set $\{1, 2, 3, 4\}$ satisfying simultaneously all the properties listed in the question:
(a) reflexive and not irreflexive;
(b) anti-symmetric and not symmetric;
(c) symmetric and not anti-symmetric;
(d) both symmetric and anti-symmetric;
(e) neither symmetric nor anti-symmetric;
(f) both reflexive and trichotomous;
(g) not reflexive, not transitive, not symmetric, not anti-symmetric, and not trichotomous

Harder and optional questions

Question 10.
Let $R, S \subseteq A \times A$ be symmetric relations. Prove that $R \circ S$ is symmetric if and only if $R \circ S = S \circ R$.

Question 11.
Prove that the intersection of two reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric,transitive) relations is also a(n) reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric,transitive) relation.