Discrete mathematics 1.

Complex numbers

Juhász Zsófia
jzsofia@inf.elte.hu
jzsofi@gmail.com
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Department of Computer Algebra

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Extension of number sets

- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, \ldots\}$
  There is no natural number $x \in \mathbb{N}$ such that $x + 2 = 1$!
  On $\mathbb{N}$ subtraction is not defined for all numbers.
- **Integers:** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  In $\mathbb{Z}$ subtraction is always possible: $x = -1$.
  There is no integer $x \in \mathbb{Z}$ such that $x \cdot 2 = 1$!
  On $\mathbb{Z}$ division is not defined by all numbers.
- **Rational numbers:** $\mathbb{Q} = \left\{ \frac{p}{q} : \ p, q \in \mathbb{Z}, q \neq 0 \right\}$
  We can divide by any nonzero number in $\mathbb{Q}$: $x = \frac{1}{2}$.
  There is no rational number $x \in \mathbb{Q}$ such that $x^2 = 2$!
  Taking the square root of a rational number $\mathbb{Q}$ does not always produce a rational number, not even in the case of a nonnegative rational number.
- **Real numbers:** $\mathbb{R}$.
  There is no real number $x \in \mathbb{R}$ such that $x^2 = -1$.

  Since: If $x \geq 0$ then $x^2 \geq 0$.
  If $x < 0$ then $x^2 = (-x)^2 > 0$.

Extension of number sets

Among complex numbers the equation $x^2 = -1$ can be solved!

Applications of complex numbers:

- solving equations;
- geometry;
- physics (fluid dynamics, quantum mechanics, relativity theory);
- computer graphics, quantum computers.

Introducing complex numbers

Definition (imaginary unit)

Let i be a solution to the equation $x^2 = -1$; i is called the imaginary unit.

We would like to extend the operations of addition and multiplication from the set of real numbers to a larger set containing i, while keeping the 'usual rules' of calculation and adding the rule: $i^2 = -1$. E.g.:

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$

.

Definition of complex numbers (informal definition)

Definition (complex numbers)

The expressions of the form $a + bi$ where $a, b \in \mathbb{R}$, are called complex numbers with addition and multiplication defined as:

- addition: $(a + bi) + (c + di) = a + c + (b + d)i$.
- multiplication: $(a + bi)(c + di) = ac - bd + (ad + bc)i$.

The set of all complex numbers is denoted by $\mathbb{C}$. The form $a + bi$ where $a, b \in \mathbb{R}$ is called the algebraic form (or Cartesian or rectangular form) of a complex number.

Definition (real part and imaginary part of a complex number)

Let $z = a + bi$ ($a, b \in \mathbb{R}$) be a complex number. Then the real part of z is $Re(z) = a \in \mathbb{R}$ and the imaginary part of z is $Im(z) = b \in \mathbb{R}$.

- **Note:** $Im(z) \neq bi$
- The complex numbers of the form $a + 0 \cdot i$ are the real numbers. The complex numbers of the form $0 + bi$ are called pure imaginary numbers.
- Two complex numbers with algebraic forms $a + bi$ and $c + di$ are equal: $a + bi = c + di$, if and only if $a = c$ and $b = d$.

The definition of complex numbers (formal definition)

Definition (formal definition of complex numbers)

The set $\mathbb{C}$ of complex numbers is the set $\mathbb{R} \times \mathbb{R}$ together with the following operations:

- addition: $(a, b) + (c, d) = (a + c, d + b)$;
- multiplication: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

The two definitions of complex numbers are equivalent: $a + bi \leftrightarrow (a, b)$, e.g. $i \leftrightarrow (0, 1)$.
The format $a + bi$ is more convenient for manual calculations.
The format (a, b) is more convenient for use with computers.

There is no need to introduce further numbers:

Theorem (Fundamental Theorem of Algebra; no proof required)

Let $n \in \mathbb{N}^+$. Then for every $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$, there exists $z \in \mathbb{C}$ such that $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$ (i.e. the polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ has a root in $\mathbb{C}$.)

The basic properties of operations on $\mathbb{C}$

Based on the definitions it is easy to verify that addition and multiplication on $\mathbb{C}$ satisfy the following properties:

Proposition (Basic properties of operations on $\mathbb{C}$)

Properties of addition

1. *Associativity:* $\forall a, b, c \in \mathbb{C} : (a + b) + c = a + (b + c)$.
2. *Commutativity:* $\forall a, b \in \mathbb{C} : a + b = b + a$.
3. *Neutral element (zero element):* $\exists 0 \in \mathbb{C}$ (*zero element*) *such that* $\forall a \in \mathbb{C} : 0 + a = a + 0 = a$.
4. *Additive inverse (opposite):* $\forall a \in \mathbb{C} : \exists -a \in \mathbb{C}$ (*opposite of* a) *such that* $a + (-a) = (-a) + a = 0$.

The basic properties of operations on $\mathbb{C}$

Proposition (Basic properties of operations on $\mathbb{C}$)

Properties of multiplication

1. *Associativity:* $\forall a, b, c \in \mathbb{C} : (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. *Commutativity:* $\forall a, b \in \mathbb{C} : a \cdot b = b \cdot a$.
3. *Unit element:* $\exists 1 \in \mathbb{C}$ *(unit element) such that* $\forall a \in \mathbb{C} : 1 \cdot a = a \cdot 1 = a$.
4. *Multiplicative inverse (reciprocal):* $\forall a \in \mathbb{C} \setminus \{0\}\ \exists a^{-1} = \frac{1}{a} \in \mathbb{C}$ *(reciprocal of* a*) for which* $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Distributivity

$\forall a, b, c \in \mathbb{C} : a(b + c) = ab + ac$ *(and* $(a + b)c = ac + bc$*)*

Corollary:

- Because of the above properties, the algebraic structure $(\mathbb{C}, +, \cdot)$ is a so called *field* (just like $(\mathbb{R}, +, \cdot)$ and $(\mathbb{Q}, +, \cdot)$).
- Informally we can say that we can calculate with complex numbers ‘in the same way’ as with real numbers (in sums and products we can ‘move’ the brackets; the order of the terms in a sum and of the factors in a product can be changed; brackets can be expanded by the distributive property etc.) with the additional rule: $i^2 = -1$.

Calculating with complex numbers: absolute value, conjugate

Definition (absolute value of a complex number)

The absolute value of a complex number z with algebraic form $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

In particular, if z is a real number, then $z = a$ and its absolute value is the 'usual' absolute value of a real number: $|z| = |a| = \sqrt{a^2}$.

Proposition (Hw)

For any complex number z:

1. $|z| \geq 0$,
2. $|z| = 0 \Leftrightarrow z = 0$.

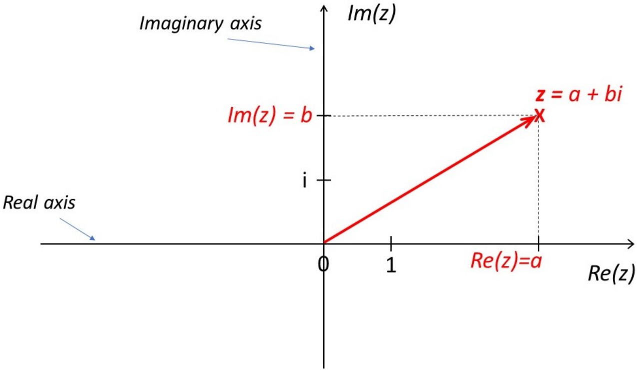
Definition (conjugate of a complex number)

The conjugate of a complex number z with algebraic form $z = a + bi$ is $\overline{z} = a - bi$.

Representing complex numbers in the Complex plane (Gaussian plane, Argand diagram)

Complex numbers can be represented in the complex plane (Gaussian plane, Argand diagram):

- $z = a + bi \leftrightarrow (a, b)$
- bijecion (one-to-one correspondence) between $\mathbb{C}$ and the points (or position vectors) of the plane.



Calculating with complex numbers: opposite, subtraction

Definition (opposite of a complex number)

The opposite of a complex number z is the complex number denoted by $-z$ such that $z + (-z) = 0$.

Proposition (Opposite of a complex number; proof is hw)

The opposite of a complex number z with algebraic form $z = a + bi$ is the complex number with algebraic form $-z = -a - bi$.

Definition (subtraction of complex numbers)

The difference of complex numbers z and w is defined as:

$$z - w = z + (-w)$$

Calculating with complex numbers: reciprocal

Definition (reciprocal of a nonzero complex number)

The reciprocal of a nonzero complex number z is the number $z^{-1} = \frac{1}{z}$ such that $z \cdot z^{-1} = 1$.

By the definition of multiplication it is easy to show that every nonzero complex number has a reciprocal.

Using the reciprocal, we can define division by nonzero complex numbers:

Definition (division by nonzero complex numbers)

The quotient of two complex numbers z and $w \neq 0$ is:

$$\frac{z}{w} = z \cdot \frac{1}{w}.$$

Calculating with complex numbers: reciprocal, division

What is $\frac{2+3i}{1+i}$ in algebraic form?

Idea: Similar to the rationalization of the denominator in fractions of real numbers:

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1^2-\sqrt{2}^2} =$$

$$= \frac{1-\sqrt{2}}{1-2} = -1+\sqrt{2}$$

.

Multiply both the numerator and the denominator **by the conjugate of the denominator:**

$$\frac{2+3i}{1+i} = \frac{2+3i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{5+i}{1^2-i^2} = \frac{5+i}{1-(-1)} = \frac{5+i}{2} = \frac{5}{2}+\frac{1}{2}i$$

.

Why did this method work? When multiplying the denominator $1+i$ by its conjugate $1-i$, the result (the new denominator) is a real number.

Calculating with complex numbers: reciprocal, division

Lemma

For any complex number z we have $z \cdot \overline{z} = |z|^2$ (hence $z \cdot \overline{z}$ is a real number).

Proof

Let $z = a + bi$ be the algebraic form of z. Then
$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$.

Hence:

Proposition (Calculating the quotient in algebraic form)

Let $z, w \in \mathbb{C}$, $w \neq 0$. Then the quotient $\frac{z}{w}$ in algebraic form can be found as:

$$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}}$$

.

Proof

Let $z = a + bi$ and $w = c + di$ $(a, b, c, d \in \mathbb{R})$. Then
$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$.

Calculating with complex numbers

Theorem (Properties of conjugation and the absolute value of complex numbers; proof is hw.)

Let z and w be complex numbers. Then:

1. $\overline{\overline{z}} = z$;
2. $\overline{z + w} = \overline{z} + \overline{w}$;
3. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$;
4. $z + \overline{z} = 2Re(z)$;
5. $z - \overline{z} = 2Im(z) \cdot i$;
6. $z \cdot \overline{z} = |z|^2$;
7. *if* $z \neq 0$ *then* $z^{-1} = \frac{\overline{z}}{|z|^2}$;
8. $|0| = 0$ *and if* $z \neq 0$ *then* $|z| > 0$;
9. $|\overline{z}| = |z|$;
10. $|z \cdot w| = |z| \cdot |w|$;
11. $|z + w| \leq |z| + |w|$ *(triangle-inequality).*

Calculating with complex numbers

Theorem

$\vdots$

10. $|z \cdot w| = |z| \cdot |w|$;

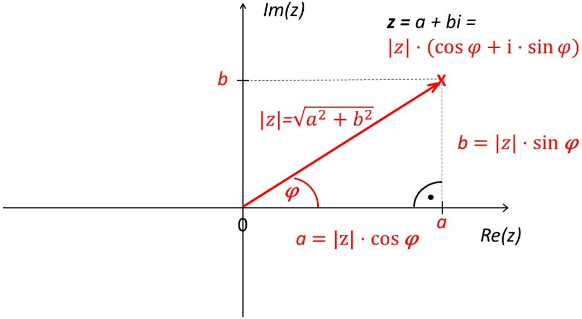
$\vdots$

Proof

$|z \cdot w|^2 = z \cdot w \cdot \overline{z \cdot w} = z \cdot w \cdot \overline{z} \cdot \overline{w} = z \cdot \overline{z} \cdot w \cdot \overline{w} = |z|^2 \cdot |w|^2 = (|z| \cdot |w|)^2.$

The polar form of complex numbers

Let $z = a + bi \in \mathbb{C}$ $(a, b \in \mathbb{R})$, $z \neq 0$.



- The length of the vector (a, b) is: $\sqrt{a^2 + b^2} = |z|$.
- Denote by φ the angle from the positive real axis to the vector (a, b) (comment: this angle is not unique, because integer multiples of 2π can be added to it).

The coordinates a and b expressed in terms of $|z|$ and φ:

$$a = |z| \cdot \cos\varphi, \quad b = |z| \cdot \sin\varphi$$

The polar form of complex numbers

Definition (polar form)

The polar form of a nonzero complex number $z \in \mathbb{C}$ is:

$$z = |z|(\cos\varphi + i\sin\varphi).$$

Note:

- The polar form of zero is usually not used, because the angle could be any real number.
- The polar form is not unique (because the angle is not unique): $|z|(\cos\varphi + i\sin\varphi) = |z|(\cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi))$.

Definition (argument)

The argument of a nonzero $z \in \mathbb{C}$ is the angle $\varphi = arg(z) \in [0, 2\pi)$ such that $z = |z|(\cos\varphi + i\sin\varphi)$.

- $z = a + bi$ algebraic form;
- $z = |z|(\cos\varphi + i\sin\varphi)$ polar form. Here $a = |z|\cos\varphi$, $b = |z|\sin\varphi$.

Converting from algebraic form to polar form

Given the algebraic form $z = a + bi \neq 0$ we would like to determine the polar form of a nonzero complex number.

$$a + bi = |z|(\cos\varphi + i\sin\varphi)$$

Given a and b we are looking for $|z|$ and φ.

- $|z| = \sqrt{a^2 + b^2}$.
- Finding φ:

$$\left.\begin{array}{l} a = |z|\cos\varphi \\ b = |z|\sin\varphi \end{array}\right\}$$

If $a \neq 0$ then $\tan\varphi = \frac{b}{a}$, and so

$$\varphi = \begin{cases} \frac{\pi}{2}, & \text{if } a = 0 \text{ and } b > 0; \\ \frac{3\pi}{2}, & \text{if } a = 0 \text{ and } b < 0; \\ \arctan\frac{b}{a}, & \text{if } a > 0; \\ \arctan\frac{b}{a} + \pi, & \text{if } a < 0. \end{cases}$$

De Moivre's formulas

Theorem (De Moivre's formulas)

Let $z, w \in \mathbb{C}$ *be nonzero complex numbers:* $z = |z|(\cos\varphi + i\sin\varphi)$, $w = |w|(\cos\psi + i\sin\psi)$, *and let* $n \in \mathbb{N}^+$. *Then*

1. $zw = |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi))$;
2. $\frac{z}{w} = \frac{|z|}{|w|} \cdot (\cos(\varphi-\psi) + i\sin(\varphi-\psi))$;
3. $z^n = |z|^n(\cos n\varphi + i\sin n\varphi)$.

The angles are added, subtracted, multiplied by n.

Geometric meaning

Multiplication by a nonzero complex number $z \in \mathbb{C}$ acts on the complex plane like an enlargement by a scale factor of $|z|$ together with a rotation by an angle of $arg(z)$ around the origin.

Proof

1. $$\begin{aligned} zw &= |z|(\cos\varphi + i\sin\varphi)\cdot|w|(\cos\psi + i\sin\psi) = \\ &= |z||w|(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\cos\varphi\sin\psi + \sin\varphi\cos\psi)) = \\ &\quad \textit{Hence by the trigonometric addition formulas:} \\ &= |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi)) \end{aligned}$$

Trigonometric addition formulas:
$\cos(\varphi+\psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi$
$\sin(\varphi+\psi) = \cos\varphi\sin\psi + \sin\varphi\cos\psi$

- The absolute value of the product: $|zw| = |z||w|$.
- The argument of the product:
  - if $0 \le arg(z) + arg(w) < 2\pi$, akkor $arg(zw) = arg(z) + arg(w)$;
  - if $2\pi \le arg(z) + arg(w) < 4\pi$ then $arg(zw) = arg(z) + arg(w) - 2\pi$.

The functions sin, cos are periodic with a period 2π, for finding the argument of the product, we may need to reduce the sum of the arguments.

Roots of complex numbers

Definition (n^{th} roots of a complex number)

Let $n \in \mathbb{N}^+$ and $z \in \mathbb{C}$. The n^{th} roots of z are those complex numbers w for which $w^n = z$.

Theorem (Formula of the n^{th} roots of a complex number)

Let $z = |z|(\cos\varphi + i\sin\varphi)$, $n \in \mathbb{N}^+$. The n^{th} roots of z are:

$$w_k = \sqrt[n]{|z|}(\cos(\frac{\varphi}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\varphi}{n} + \frac{2k\pi}{n}))$$

$k = 0, 1, \ldots, n-1$.

The following fact will be used in the proof of the theorem:
Two complex numbers given in polar forms $z = |z|(\cos\varphi + i\sin\varphi)$ and $w = |w|(\cos\psi + i\sin\psi)$ are equal:

$$|z|(\cos\varphi + i\sin\varphi) = |w|(\cos\psi + i\sin\psi),$$

if and only if:

- $|z| = |w|$ and
- $\varphi = \psi + 2k\pi$ for some $k \in \mathbb{Z}$.

Roots of complex numbers

Theorem (Formula of the n^{th} roots of a complex number)

Let $z = |z|(\cos\varphi + i\sin\varphi)$, $n \in \mathbb{N}^+$. *The* n^{th} *roots of* z *are:*

$$w_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right)$$

$k = 0, 1, \ldots, n-1$.

Proof

By De Moivre's formula, for any complex number $w = |w|(\cos\psi + i\sin\psi)$ *we have* $w^n = |w|^n(\cos n\psi + i\sin n\psi)$.
Hence $w^n = z$ *is equivalent to* $|w|^n(\cos n\psi + i\sin n\psi) = |z|(\cos\varphi + i\sin\varphi)$, *which holds if and only if:*

- $|w|^n = |z| \Leftrightarrow |w| = \sqrt[n]{|z|}$ *and*
- $n\psi = \varphi + 2k\pi$ *for some* $k \in \mathbb{Z} \Leftrightarrow \psi = \frac{\varphi}{n} + \frac{2k\pi}{n}$ *for some* $k \in \mathbb{Z}$.

If $k \in \{0, 1, \ldots, n-1\}$, *then we obtain distinct complex numbers.*

Example

Example

Find the 6^{th} roots (w) of $\frac{1-i}{\sqrt{3}+i}$.

$1-i=\sqrt{2}(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})=\sqrt{2}(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})$

$\sqrt{3}+i=2(\frac{\sqrt{3}}{2}+i\frac{1}{2})=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

Since $\frac{7\pi}{4}-\frac{\pi}{6}=\frac{19\pi}{12}$, hence: $\frac{1-i}{\sqrt{3}+i}=\frac{1}{\sqrt{2}}(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12})$.

So the 6^{th} roots are:

$w_k=\frac{1}{\sqrt[12]{2}}(\cos\frac{19\pi+24k\pi}{72}+i\sin\frac{19\pi+24k\pi}{72})\ :\ k=0,1,\ldots,5$

Complex roots of unity

Definition (n^{th} roots of unit)

For any $n \in \mathbb{N}^+$ the n^{th} roots of 1 are called the n^{th} roots of unity. (I.e. the complex numbers ϵ satisfying $\epsilon^n = 1$.)

Using the formula of the n^{th} roots of a complex number we obtain the following:

Theorem (The polar form of the n^{th} roots of unity)

For any $n \in \mathbb{N}^+$ the n^{th} roots of unity are:

$$\epsilon_k = \epsilon_k^{(n)} = (\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}) \;\; : \;\; k = 0, 1, \ldots, n-1.$$

The 8^{th} roots of unity: