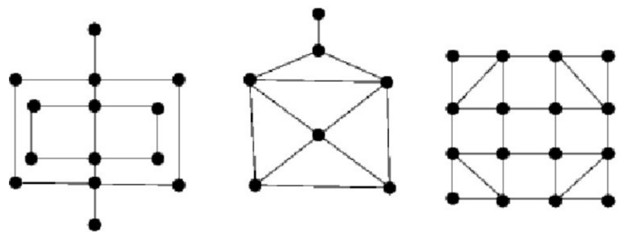
Problem set 11.: Hamiltonian cycles, Eulerian trails

Question 1.
Which of the following graphs can be drawn by a single continuous line if each edge must be drawn only once (i.e. which graph has open/closed Eulerian trail)? Justify your answer.



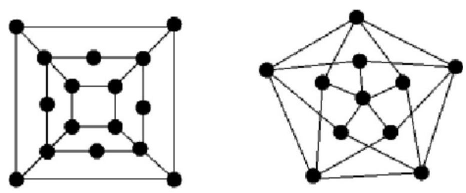
Question 2.
Is it possible for a simple graph with an Eulerian circuit to have even number of vertices and odd number of edges?

***Question 3.**
Prove that if all degrees of a graph equals to 4, then its edges can be coloured by red and blue such that all vertices have two incident red and two blue edges.

Question 4.
Prove that if any single edge is deleted from a graph with a Hamiltonian cycle, then it remains connected. What happens if a vertex is removed instead?

Question 5.
Prove that if a graph has k vertices which, when removed, make the graph fall into more than k components, then the graph has no Hamiltonian cycle.

Question 6.
Do the following graphs have Hamiltonian cycle (or path)? Justify your answer.



Question 7.
Let C be a cycle in a finite connected graph. Prove that if any of its edges, when removed, leaves a longest path in the graph, then C is a Hamiltonian cycle in the graph.

Question 8.
Prove that for any $n \geq 5$, **(a)** there exists an n-vertex G graph such that both G and $\overline{G}$ contains a Hamiltonian cycle; and **(b)** there exists an n-vertex G graph such that neither G nor $\overline{G}$ contains Hamiltonian cycle.

Question 9.
Prove that if a bipartite graph has a Hamiltonian cycle, then the two vertex classes have the same number of vertices.

Question 10.
Is it possible for a knight to visit all squares of a 9×9 chessboard and then return to the initial position without touching any other square twice?

***Question 11.**
Let G be an n-vertex *simple* graph ($n \geq 3$). Prove that if all degrees of the graph are at least $\frac{n}{2}$, then G has a Hamiltonian cycle.

Question 12.
Prove that 100 people can sit around a big round table at least 25 times if no two people sit next to each other more than once.

***Question 13.**
Prove that you can create a cycle from a domino set. (What does that mean?)

***Question 14.**
Prove that the Petersen graph contains no Hamiltonian cycle, but after removing any of its edges, the resulting graph contains one.