Problem set 2.: Basic concepts of binary relations and composition of binary relations

Basic concepts of binary relations

Question 1.
Let $A = \{1,2,3,4\}$ and $B = \{5,6,7,8,9\}$. Consider the following binary relation $\rho \subseteq A \times B$: $\rho = \{(1,5),(1,6),(1,7),(3,6),(3,9),(4,5),(4,7),(4,9)\}$.
(a) Find the domain and the range of ρ.
(b) Represent ρ on an arrow diagram.
(c) Let $H_1 = \{1,2,3\}$ and $H_2 = \{4\}$. Determine the restrictions $\rho|_{H_1}$ and $\rho|_{H_2}$ of ρ to sets H_1 and H_2, respectively.
(d) Find the inverse ρ^{-1} of ρ.

Question 2.
Define $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$ as $\rho = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 2b\}$. Determine the domain, range and inverse of ρ.

Question 3.
Determine the image and the inverse image of the set $\{0\}$ under the relation $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y^2 = 2 - x - x^2\}$. Describe those subsets A of $\mathbb{R}$ for which $R(A)$ contains only one element. Describe those subsets A of $\mathbb{R}$ for which $R^{-1}(A)$ contains only one element.

Composition of binary relations

Question 4.
Let $A = \{1,2,3\}, B = \{a,b,c,d,e,f\}, C = \{2,4,6,8\}$ and define $R \subseteq A \times B$ and $S \subseteq B \times C$ as follows: $R = \{(1,a),(1,b),(2,c),(2,f),(3,d),(3,e),(3,f)\}$ and $S = \{(a,2),(a,4),(c,6),(c,8),(d,2),(d,4),(d,6),(f,8)\}$. Find the composition $S \circ R$.

Question 5.
Let $A = \{1,2,3,4,5,6,7,8\}; S, R \subseteq A \times A$. In each of the following cases determine the composition $S \circ R$.
(a) $R = \{(1,2),(1,3),(2,2),(3,3),(3,4),(4,1)\}$ and $S = \{(1,6),(2,3),(2,4),(3,1)\}$
(b) $R = \{(1,3),(1,4),(2,2),(2,4),(3,5),(5,6),(6,7)\}$ and $S = \{(1,2),(1,4),(2,3),(3,1),(3,2),(4,2),(4,6),(5,6),(7,2)\}$
(c) $R = \{(2,2),(2,4),(3,1),(3,4),(4,4),(5,3)\}$ and $S = \{(2,6),(3,7),(5,1),(5,6),(5,8),(6,2),(7,7)\}$
(d) $R = \{(6,1),(6,2),(7,3),(8,7)\}$ és $S = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),(5,1),(5,3),(5,5),(7,1),(7,2)\}$

Is the composition of relations a commutative operation? Hint: Determine for example the composition $R \circ S$ in case (a).

Question 6.
Let $R, S \subseteq \mathbb{R} \times \mathbb{R}$. In each of the following cases determine the compositions $S \circ R$ and $R \circ S$.
(a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 4x = y^2 + 6\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - 1 = y\}$
(b) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = 2y\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3\}$
(c) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \frac{1}{x} = y^2\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \sqrt{x - 2} = 3y\}$
(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - 6x + 5 = y\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y \wedge 2y = x\}$

Harder and optional questions

Question 7.
Let $f \subseteq A \times A$ be a binary relation. Prove that $f = f^{-1}$ is true if and only if $f \subseteq f^{-1}$ holds.

Question 8.
Consider the following relations:
$\rho = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 3\}$, $\varphi = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 6x - 1 = 4y + 5\}$,
$\lambda = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 4 \mid 2x + 3y\}$, $\alpha = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 1,5x - 1,5 \leq y\}$
Determine the compositions below.

$\rho \circ \varphi$ $\qquad$ $\varphi \circ \lambda$ $\qquad$ φ^3 $\qquad$ $\alpha \circ \rho$ $\qquad$ $\rho \circ \alpha$