Combinatorics problems

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Permutations, variations, combinations

1. At a running race there are 25 participants. In how many different orders can they finish (if we assume that there are no equal finishers and everyone completes the race)?

2. Consider all 5-digit numbers consisting of the digits 2, 3, 4, 5 and 7 containing each of these digits exactly once.
   (a) How many such numbers exist?
   (b) How many of the above numbers are even?
   (c) How many of the above numbers are divisible by four?

3. How many of the six-digit numbers containing each of the digits 0, 1, 2, 3, 4, 5 exactly once are divisible by 5?

4. A group of ten people organize a prize draw: 4 different books can be won, and each person can win at most one book. How many different outcomes are possible?

5. In a factory, 4% of the 500 locks manufactured during a shift are faulty. In how many different ways can we choose 10 locks out of the 500 locks such that there are

   (a) exactly 5 faulty locks
   (b) at least 2 faulty locks

   among the ones chosen?

6. In how many different ways can 9 people get on a tram of 3 carriages in such a way that exactly 3 people get into each carriage?

7. A party was attended by 9 men and 12 women. During the party 7 different pairs had a dance. In how many different ways could these 7 pairs be formed from the people at the party if each pair consists of a man and a woman and any person can be a member of more than one pairs? The order in which the pairs danced does not matter.

8. A bus ticket has a grid of 9 cells printed on it numbered from 1 to 9. When validating the ticket, the ticket machine punches the ticket through 3 or 4 of these 9 cells. How many different combinations of punch holes can be created this way?

9. (a) Find the sum of all the different 3-digit numbers which contain each of the digits $1, 2, 3$ exactly once.

(b) Find the sum of all the different 6-digit numbers which contain each of the digits $1, 2, 3, 4, 5, 6$ exactly once.
(c) What is the sum of all the different 6-digit numbers which can be formed from the digits $1, 2, 3, 4, 5, 6$ in such a way that not every digit has to be used and any digit can appear more than once?

10. When rolling a die three times, how many possible sequences of results exist which contain 6 at least once?

11. A standard 52-card deck of French playing cards contains 4 aces and 4 kings. In how many ways can the 52 cards be handed out to 4 players in such a way that each player gets exactly 13 cards containing 1 ace and 1 king?

12. In a standard 52-card deck of French playing cards the cards are divided into four suites containing 13 cards each. In each suit there is one ace.
a) How many different ways are there to give 13 cards to each of four players?
b) How many different ways are there to give 13 cards to each of four players if each player needs to get one ace?
c) How many different ways are there to give 13 cards to each of four players if all the aces need to be given to the same player?

13. A deck of Hungarian playing cards consists of 32 cards, which are divided into four suits: acorns, hearts, leaves and bells with 8 cards in each suit. Each suit contains one ace. We draw 5 cards from the 32-card deck. How many different outcomes are possible such that
(a) all cards drawn are hearts;
(b) there is exactly one hearts among the cards drawn;
(c) there is at least one card which is hearts among the ones drawn;
(d) there are 2 hearts and 3 leaves among the cards drawn;
(e) the 5 cards drawn contain at least one card from each suit;
(f) the 5 cards drawn contain exactly 1 ace and 4 hearts;
(g) all cards drawn are ace or hearts?

14. In how many different ways can we place 24 identical balls into 8 different boxes, if
(a) we can leave any of the boxes empty;
(b) we need to put at least 1 ball into each box;
(c) we need to put at least 2 balls into each box?

15. In how many different ways can we distribute 30 balls in 100 compartments if each compartment has to contain exactly 6 or 0 balls and
(a) all the balls are different;
(b) all the balls are different and we take into account the order of the balls inside each compartment;
(c) all the balls are different and we do not take into account the order of the balls inside the compartments?

16. In the football pools players can bet on the results of 14 football matches by filling in a pools ticket. On each ticket there are 14 questions each corresponding to a match to be played

soon. At each question you need to choose one from among three different answers: 1 (home win), 2 (away win) and X (draw). In how many different ways can a ticket be completed?

17. Let $A = \{1, 2, .., 20\}$. How many subsets does A have? How many subsets does A have which
   a) contain 1; **b)** contain both 1 and 2; **c)** contain 1 or 2?

18. How many six-digit numbers exist with the property that
   a) no adjacent digits are identical; **b)** all digits are different;
   c) exactly one of the digits is 0; **d)** at least one of the digits is 0?

19. In how many different ways can you arrange the numbers $1, 2, \dots, n$ into an order such that the numbers 1 and 2 are not adjacent?

20. How many different strings can you form using all letters of the word MISSISSIPPI with the property that
   a) it contains no adjacent letter S's;
   b) not all the 4 letter S's are directly following each other?

21. In how many different ways can we move from the bottom left corner to the top right corner of a 3×10 'chessboard' if in each step we can move by 1 field up or by 1 field to the right or to the next field diagonally up and to the right?

22. There are two different straight lines given in the plane: e and f. We are given 5 different points on line e and 7 different points on line f. (All points given are pairwise different, hence we have 12 distinct points in total). How many triangles exist with the property that all vertices of the triangle are among these 12 points?

23. There are two parallel lines e and f given in a plane; on line e there are p different points marked and on line f there are q different points marked. How many triangles exist with the property that all vertices of the triangle are among the points marked?

24. How many 0's does the number $11^{100} - 1$ end in?

25. A flea is jumping on the number line: it makes one jump every second and at each jump it moves 1 unit in the positive or 1 unit in the negative direction.
   a) In how many different ways can the flea get from 0 to 10 with exactly 18 jumps?
   b) In how many different ways can the flea get from 0 to 24 in exactly one minute?

26. In how many different ways can we express
   a) 100 as the sum of 7 positive integers; **b)** 200 as the sum of 12 natural numbers;
   c) 12 as a sum where each term is either 1 or 2

   if the order of the numbers in the sum matters?

27. How many positive integers exist with the property that the sequence of digits read from left to right is
   a) strictly increasing; **b)** strictly decreasing;
   c) increasing; **d)** decreasing?

28. A class of 25 students are electing a committee of 6 from among its members. After the committee has been elected they appoint one chair and a secretary from among the 6 elected members. How many different outcomes are possible if the chair and the secretary cannot be the same person?

29. A class of 32 students are electing a student committee from among its members. The committee consists of 1 secretary and 4 other committee members. Paul Smith is a student in the class. In how many ways can the committee be selected in such a way that

    (a) Paul Smith is the secretary of the committee;
    (b) Paul Smith is a non-secretary member of the committee;
    (c) Paul Smith is a member of the committee?

30. In how many different ways can n identical coins be distributed among k people? How many different ways can the coins be distributed if everyone needs to be given at least one coin?

31. In a café five different kinds of cakes are sold: gerbeaud, apple pie, chestnut heart, cheese cake and mignon. There are at least 20 pieces of each kind in stock. In how many different ways can we eat three cakes

    a) if the order in which we eat the cakes matters;

    b) if the order in which we eat the cakes does not matter?

32. We would like to post 8 different photos to a friend. We have 5 envelopes which are all different. In how many different ways can we put the 8 photos into these envelopes? (We do not necessarily have to use all envelopes.)

33. In how many different ways can we distribute 30 balls in 100 compartments if in each compartment we can put either exactly 0 or 6 balls if

    a) all the balls are identical;
    b) all balls are different and we take into account the order of the balls in each compartment;
    c) all balls are different and we do not take into account the order of the balls in any of the compartments?

34. A box contains 10 red, 20 white and 40 green balls. We take some balls out of the box one after the other, randomly, without replacement. At least how many balls do we need to take out of the box in order to be certain that these contain

    a) a white ball; **b)** 3 balls of different colours; **c)** 3 balls of the same colour;
    d) 5 balls of the same colour; **e)** 15 balls of the same colour;
    f) two green balls taken out immediately one after the other?

35. In how many different ways can a ‘chessboard’ of size 2×12 be covered by dominoes of size 2×1? (Each domino needs to lie horizontally or vertically, covering exactly two fields of the chessboard.)

36. In how many different ways can you list n number of 0’s and k number of 1’s in such a way that there are no 1’s adjacent to each other in the list?

37. There are 12 knights sitting at King Arthur's Round table. Each knight is on bad terms with the two knights sitting directly next to him. King Arthur would like to choose five knights to rescue the princess. In how many different ways can he choose the five knights so that there are no knights of the five are on bad terms with each other? What is the answer if the king needs to choose k knights from among n knights?

Binomial coefficients

38. The coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is C_n^k. The n^{th} row of the **Pascal-triangle** contains the numbers C_n^k where $n, k \in \mathbb{N}$, $k \leq n$.
a) What is the sum of all numbers in the n^{th} row of the Pascal-triangle?

b) What result do we obtain if we add up the elements in the n^{th} row of the Pascal-triangle with *alternating* signs?

39. Solve the equation $0,7 \cdot \binom{25}{x} = \binom{23}{x}$ on the set $\{0, 1, 2, ..., 23\}$ (where $x \in \mathbb{N}$, $0 \leq x \leq 23$ and $\binom{25}{x}$ and $\binom{23}{x}$ denote binomial coefficients).

40. Let $n, k \in \mathbb{N}$, $k \leq n$. Prove the following identities:

(a) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

(b) $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$

41. Prove the following identities:
a) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ for every $n, k \in \mathbb{N}$, $k \leq n-1$;
b) $\binom{4}{4} + \binom{5}{4} + \cdots + \binom{n}{4} = \binom{n+1}{5}$ for every $n \in \mathbb{N}$, $n \geq 4$.

42. Bring the following expression to its simplest form: $1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$.

Pigeonhole principle

43. (a) Is it true that in any group of 8 children there are always (at least) 2 who were born on the same day of the week?
(b) At a meeting of 34 people each participant has at most 10 acquaintances present at the meeting. Is it true that there must be 4 people at the meeting who have the same number of acquaintances present?

44. What is the minimum class size at which we can be certain that there must be four students in the class who were born in the same month?

45. At most how many natural numbers can be selected in such a way that there are no two numbers among them with a difference divisible by 8?

46. In the football pools players can bet on the results of 14 football matches by filling in a pools ticket. On each ticket there are 14 questions each corresponding to a match to be played soon. At each question you need to choose one from among three different answers: 1 (home win), 2 (away win) and X (draw). What is the minimum number of pools tickets we need to complete in order to make sure that there will be at least one ticket with at least 5 correct answers?

∗ 47 Prove that for every $m \in \mathbb{N}^+$ there exists $n \in \mathbb{N}^+$ such that all digits in the base-10 form of mn are 0 or 1.

∗ 48 Prove that there exist two different natural numbers n and k such that $2^n - 2^k$ is divisible by 711.

∗ 49 Suppose we are given a square with sides of unit length. We choose 33 arbitrary points inside the square in such a way that no 3 points lie on the same line. Prove that there must be 3 points among the chosen ones such that the triangle determined by these points has an area of at most $\frac{1}{32}$.

∗ 50 Show that there is at least one number among the numbers $\pi, 2\pi, 3\pi, \ldots, 100\pi$ such that its distance from the nearest integer is at most $\frac{1}{101}$. Generalize the statement.

Inclusion-exclusion principle

51. In a class of 30 pupils 12 like maths, 5 like maths and physics, 14 like physics, 4 like maths and chemistry, 13 like chemistry, 7 like physics and chemistry and 3 pupils like all these three subjects. How many pupils in the class do not like any of these three subjects?

52. In a survey, 100 participants were asked about the types of media sources they obtain information from. The number of people mentioning each type of media was as follows: television - 65, radio - 38, newspapers - 39 , television and radio - 20, television and newspapers - 20, radio and newspapers - 9 and television, radio and newspapers - 6. How many of the 100 respondents do not use any of the above sources (television, radio, newspapers)?

53. How many natural numbers exist which are

    a) less than 100 and are not divisible by any of the numbers 2,3 and 5;

    b) less than 1000 and are not divisible by any of the numbers 2,3,5 and 7?

54. A group of 8 friends are going to the movies. In how many different ways can they sit in a row of 8 seats so that neither Anne and Beatrix nor Daniel and Esther are sitting next to each other?

55. Consider all the strings of length six which can be formed using only the digits 0-9 (repetition is allowed and not all of these digits have to be used). How many of these strings are such that they do not contain the ‘substring’ 42, i.e. the digits 4 and 2 in this order next to each other?

Binomial theorem, Polynomial theorem

56. In the expansion of $(a+b)^{22}$ find the coefficient of the term

    a) $a^{14}b^8$;
    b) $a^{17}b^5$.

57. (a) Let $a \in \mathbb{R},\ a \neq 0$. Find the term in the expansion of $\left(\frac{1}{a} + a^2\right)^9$ which does not contain the parameter a.
    (b) In the expansion of $(x^7 + 2x^3)^{27}$ find the coefficient of the x^{97} term.
    (c) In the expansion of $(x^{11} + 5x^4)^{57}$ find the coefficient of the x^{417} term.
    (d) In the expansion of $(6x^8 - 11x^5)^{32}$ find the coefficient of the x^{179} term.

58. Prove that for every real number $x \geq 0$ we have $(x^3+7)^{61} \geq 7^{60} \cdot (7 + 61x^3)$.

59. Find the coefficient of each of the terms below in the expansion of $(x_1 + x_2 + x_3 + x_4)^{73}$:

    (a) $x_1^{10} \cdot x_2^{23} \cdot x_3^{28} \cdot x_4^{12}$;
    (b) $x_1^{9} \cdot x_2^{21} \cdot x_3^{20} \cdot x_4^{23}$;
    (c) $x_1^{52} \cdot x_2^{7} \cdot x_3 \cdot x_4^{13}$;
    (d) $x_1^{37} \cdot x_2^{11} \cdot x_3^{12} \cdot x_4^{14}$.