Problem set 10.: Undirected graphs

Question 1.
Draw the following undirected graph: $G = (V, E, \varphi)$, $V = \{A, B, C, D\}$, $E = \{e_1, e_2, e_3, e_4\}$, $\varphi = \{(e_1, \{A, B\}), (e_2, \{B, C\}), (e_3, \{A, C\}), (e_4, \{C, D\})\}$. Determine $d(A)$, $d(B)$, $d(C)$ and $d(D)$. Draw $\overline{G}$, and draw a different graph isomorphic to $\overline{G}$. Are G and $\overline{G}$ isomorphic?

Question 2.
Find all pairwise non-isomorphic simple graphs with 3, 4 and 5 vertices respectively. How many of them are connected (each pair of vertices forms the endpoints of a path)? How many of them are regular?

Question 3.
How many self-complementary graphs (isomorphic to its complement) are there with 3, 4 and 5 vertices respectively?

Question 4.
Prove that the number of vertices of odd degree in any graph G is even.

Question 5.
Does there exist a simple graph with 7 vertices and the following degree sequence?
(a) $4, 4, 3, 3, 2, 2, 1$
(b) $6, 3, 3, 3, 3, 2, 0$
(c) $5, 5, 5, 2, 2, 2, 1$
(d) $2, 2, 2, 2, 2, 2, 2$

Question 6.
Does there exist a simple graph with 9 vertices and the following degree sequence?
(a) $7, 7, 7, 6, 6, 6, 5, 5, 5$
(b) $6, 6, 5, 4, 4, 3, 2, 2, 1$
(c) $2, 2, 3, 5, 6, 6, 6, 8, 8$

Question 7.
Does there exist an 8-vertex graph with degrees 6,6,6,6,3,3,2,2? Does there exist a 8-vertex simple graph with the same degree sequence?

Question 8.
Prove that any simple finite graph has two distinct vertices with the same degree.

Question 9.
Prove that if a connected graph with at least two vertices has fewer edges than vertices, then it has a leaf (a vertex of degree 1).

Question 10.
Given a simple graph and its complement, prove that either of them is always connected.

Question 11.
Prove that if the simple graph G has $2n$ vertices where every vertex has degree at least n, then G is connected. What can we say if every vertex has degree at least $n - 1$?

Question 12.
Let $G = (V, E)$ be a simple graph with $|V| = 6$. Prove that either G or $\overline{G}$ contains a 3-vertex complete graph.