Analysis II. practice 1

Continuity of functions

reminder: $f \in \mathbb{R} \to \mathbb{R}$, $a \in \mathcal{D}_f$:

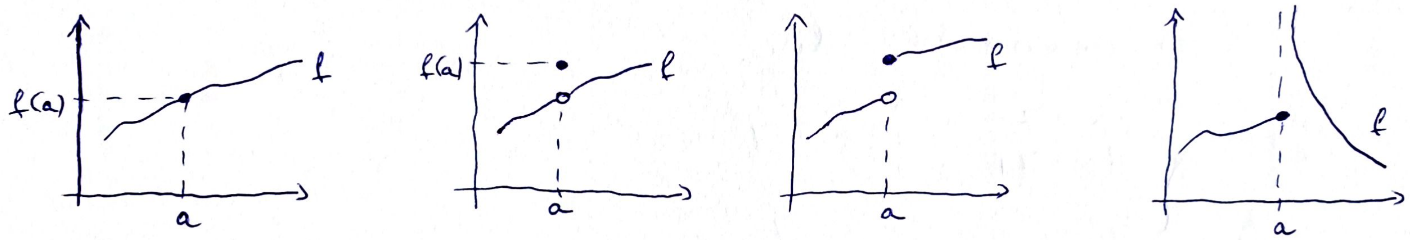
$$f \in C\{a\} \iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in B(a,\delta) \cap \mathcal{D}_f:\ f(x) \in B(f(a),\varepsilon)$$

if $a \in \mathcal{D}_f \setminus \mathcal{D}_f'$ (isolated point): $f \in C\{a\}$

if $a \in \mathcal{D}_f \cap \mathcal{D}_f'$: $f \in C\{a\} \iff \exists \lim\limits_a f = f(a)$

$f \in \mathbb{R} \to \mathbb{R}$, $a \in \mathcal{D}_f \cap \mathcal{D}_f'$, $f \notin C\{a\} \Rightarrow$ discontinuities:

1. $\exists \lim\limits_a f \in \mathbb{R}$, $\lim\limits_a f \neq f(a)$: removable
2. $\exists \lim\limits_{a\pm 0} f \in \mathbb{R}$, $\lim\limits_{a+0} f \neq \lim\limits_{a-0} f$: jump
3. otherwise: second kind



cont. 1. 2. 3.

theorem (operations + cont.):

i) $f, g \in C\{a\} \Rightarrow f \pm g,\ f \cdot g \in C\{a\}$

ii) $f, g \in C\{a\},\ g(a) \neq 0 \Rightarrow f/g \in C\{a\}$

iii) $g \in C\{a\},\ f \in C\{g(a)\} \Rightarrow f \circ g \in C\{a\}$

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1. discuss the continuity

a) $$f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x^2 - 7x + 10}, & x \in \mathbb{R} \setminus \{2, 5\} \\ 0, & x \in \{2, 5\} \end{cases}$$

$x \in \mathbb{R} \setminus \{2,5\}$: $f \in C\{x\}$ by the theorem (polynomials are cont., $x^2 - 7x + 10 \neq 0$)

$x = 2$: $\lim\limits_{2} f = \lim\limits_{x \to 2} \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-5)} = \frac{1}{3}$

so $\exists \lim\limits_{2} f \neq f(2) = 0 \Rightarrow f \notin C\{2\}$

(removable disc.)

$x = 5$: $\lim\limits_{5 \pm 0} f = \lim\limits_{x \to 5 \pm 0} \frac{x-3}{x-5} = \pm\infty$

$\Rightarrow f \notin C\{5\}$ (disc. of second kind)

b, $f(x) = \begin{cases} \frac{3-\sqrt{x}}{9-x}, & x \geq 0,\ x \neq 9 \\ 0, & x = 9 \end{cases}$

$x \in [0, +\infty) \setminus \{9\}$: $f \in C\{x\}$ (theorem; $9 - x \neq 0$)

$x = 9$: $\lim\limits_{9} f = \lim\limits_{x \to 9} \frac{\cancel{3-\sqrt{x}}}{\cancel{(3-\sqrt{x})}(3+\sqrt{x})} = \frac{1}{6} \neq f(9)$

$\Rightarrow$ removable disc.

c, $f(x) = \begin{cases} \frac{x-7}{|x-7|}, & x \neq 7 \\ 5, & x = 7 \end{cases}$

$x > 7$: $f \equiv 1 \Rightarrow f \in C\{x\}$

$x < 7$: $f \equiv -1 \Rightarrow f \in C\{x\}$

$x = 7$: $\lim\limits_{7+0} f = 1$, $\lim\limits_{7-0} f = -1$

$\Rightarrow$ jump disc.

d, $f(x) = \begin{cases} 2x+1, & x \leq -1 \\ 3x, & -1 < x < 1 \\ 2x-1, & x \geq 1 \end{cases}$

homework :)

e, $f(x) = \begin{cases} \frac{1}{x+1}, & x<-1 \\ x, & -1 \leq x < 0 \\ \ln(x+1), & x \geq 0 \end{cases}$

$x \in \mathbb{R} \setminus \{-1, 0\}$: $f \in C\{x\}$

$x = -1$: $\lim_{-1-0} f = \lim_{x \to -1-0} \frac{1}{x+1} = -\infty$

$\lim_{-1+0} f = \lim_{-1+0} x = -1 = f(-1)$

$\Rightarrow$ disc. of second kind (but cont. from the right)

$x = 0$: $\left.\begin{array}{l} \lim_{0-0} f = 0 \\ \lim_{0+0} f = 0 = f(0) \end{array}\right\} \Rightarrow f \in C\{0\}$

2. determine $f(0)$ such that $f: \mathbb{R} \to \mathbb{R}$ is continuous:

a, $f(x) := \frac{(1+x)^n - 1}{x}$ $(x \in \mathbb{R} \setminus \{0\}, n \in \mathbb{N})$

if $\exists \lim_0 f =: \alpha \in \mathbb{R}$, then $f(0) := \alpha$ $\Rightarrow f \in C(\mathbb{R})$

$\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{x \to 0} \frac{(1+x-1)((1+x)^{n-1} + \ldots + 1)}{x} =$

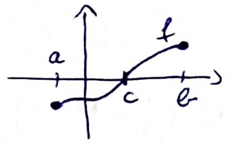
$= \lim_{x \to 0} \left((1+x)^{n-1} + \ldots + (1+x) + 1\right) = n$ $\Rightarrow f(0) := n$

b, $f(x) := \frac{1 - \cos x}{x^2}$ $(x \in \mathbb{R} \setminus \{0\})$

reminder: $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ (see Analysis I.)

$\Rightarrow f(0) := \frac{1}{2}$

reminder: Bolzano theorem: $f \in C[a,b]$, $f(a) \cdot f(b) < 0$
$\Rightarrow \exists c \in (a,b): \quad f(c) = 0$



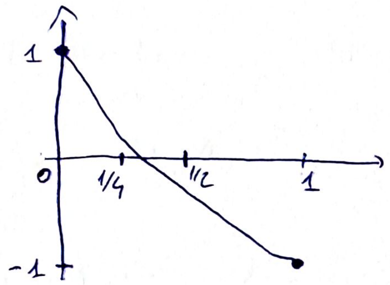
3. prove that there is a solution in the interval I

a, $x^3 - 3x + 1 = 0$, $I = (0,1)$

$f(x) := x^3 - 3x + 1 \quad (x \in \mathbb{R}) \Rightarrow f \in C$

$f(0) = 1 > 0$, $f(1) = -1 < 0 \underset{B.}{\Rightarrow} \exists c \in (0,1): f(c) = 0$

compute the first 3 terms of the sequence that approximates the root; error of approximation?



$[x_0, y_0] = [0, 1]$

half point: $\frac{1}{2}$; $f\left(\frac{1}{2}\right) = -\frac{3}{8} < 0$

$\Rightarrow [x_1, y_1] = \left[0, \frac{1}{2}\right]$

half point: $\frac{1}{4}$; $f\left(\frac{1}{4}\right) = \frac{17}{64} > 0$

$\Rightarrow [x_2, y_2] = \left[\frac{1}{4}, \frac{1}{2}\right]$

approximating sequence:

$(x_n): \quad 0, 0, \frac{1}{4}, \ldots$

(or $(y_n): \quad 1, \frac{1}{2}, \frac{1}{2}, \ldots$)

error: $\left|c - \frac{1}{4}\right| < \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

b, $x^2 = \sqrt{x+1}$, $I = (1,2)$ → homework

c, $\cos x = x$, $I = \left(0, \frac{\pi}{2}\right)$

$f(x) := \cos x - x \quad (x \in \mathbb{R}) \Rightarrow f \in C$

$f(0) = 1 > 0$, $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} < 0$

$\underset{B.}{\Rightarrow} \exists c \in \left(0, \frac{\pi}{2}\right): \quad f(c) = 0$

d, $e^x = 2 - x$, $I = \mathbb{R}$

$f(x) := e^x - 2 + x \quad (x \in \mathbb{R}) \Rightarrow f \in C$

e.g.: $f(0) = -1 < 0$, $f(1) = e - 1 > 0$

$\underset{B.}{\Rightarrow} \exists c \in (0,1): f(c) = 0$

e, $x^5 - x^2 + 2x + 3 = 0$, $I = \mathbb{R}$

$f(x) := x^5 - x^2 + 2x + 3 \quad (x \in \mathbb{R}) \Rightarrow f \in C$

e.g.: $f(0) = 3 > 0$, $f(-1) = -1 < 0$

$\underset{B.}{\Rightarrow} \exists c \in (-1,0): f(c) = 0$