Analysis 2 | Class - 9

I Integration by parts:

Review: $(f\cdot g)' = f'g + f\cdot g' \overset{\int}{\Longrightarrow}$

$\int (f\cdot g)' = \int f'g + \int f\cdot g' \Longrightarrow$

$\int f'g = fg - \int f\cdot g'$ or

$$\boxed{\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx}$$

Find the integrals:

1 a) $\int x e^{2x}\,dx \underset{\uparrow}{=}$

$(x \in \mathbb{R})$ $f'(x) = e^{2x}$ $f(x) = \frac{e^{2x}}{2}$

$g(x) = x$ $g'(x) = 1$

$$= \int x \cdot \left(\frac{e^{2x}}{2}\right)' dx = x \cdot \frac{e^{2x}}{2} -$$

$$- \int (x)' \cdot \frac{e^{2x}}{2} dx = x \cdot \frac{e^{2x}}{2} -$$

$$- \frac{1}{2} \int e^{2x} dx = \frac{x \cdot e^{2x}}{2} - \frac{1}{2} \cdot \frac{e^{2x}}{2} + c$$

$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c \,, \, (c \in \mathbb{R}) \,;$$

$$\boxed{b) \int x^2 e^{-x} dx} =$$

$(x \in \mathbb{R})$ Choose the exponential part to be $f'(x)$

$$= \int x^2 \left(-e^{-x}\right)' dx = \text{i.p.} =$$

$$= -x^2 e^{-x} - \int \underbrace{(x^2)'}_{2x} \cdot \left(-e^{-x}\right) dx$$

$$= -x^2 \cdot e^{-x} + 2 \int x e^{-x} dx =$$

$$= -x^2 e^{-x} + 2 \cdot \int x \left(-e^{-x}\right)' dx$$

$$= -x^2 e^{-x} + 2 \cdot \left[x\left(-e^{-x}\right) - \int (x)'\left(-e^{-x}\right) dx \right]$$

$$= -x^2 e^{-x} - 2x e^{-x} + \int e^{-x} dx =$$

$$= -x^2 e^{-x} - 2x e^{-x} - e^{-x} + C.$$

$(C \in \mathbb{R})$.

$$\boxed{c)\ \int x^2 \cdot \sin 5x \, dx =}$$

$$(x \in \mathbb{R}) = \int x^2 \cdot \left(-\frac{\cos 5x}{5}\right)' dx =$$

i.p.

$$= \frac{-x^2 \cos 5x}{5} - \int \underbrace{(x^2)'}_{2x} \cdot \left(-\frac{\cos 5x}{5}\right) dx =$$

$$= -\frac{x^2 \cdot \cos 5x}{5} + \frac{2}{5} \cdot \int x \cdot \cos 5x \, dx =$$

$$= -\frac{x^2 \cos 5x}{5} + \frac{2}{5} \cdot \int x \cdot \left(\frac{\sin 5x}{5}\right)' dx$$

$$= -\frac{x^2 \cos 5x}{5} + \frac{2}{5} \cdot \left(x \cdot \frac{\sin 5x}{5} - \right.$$

$$\left. - \int 1 \cdot \frac{\sin 5x}{5} dx \right) =$$

$$= -\frac{x^2 \cos 5x}{5} + \frac{2}{25} x \cdot \sin 5x -$$

$$- \frac{1}{5} \int \sin 5x \, dx =$$

$$= \underbrace{-\frac{x^2}{5} \cos 5x + \frac{2}{25} x \sin 5x +}$$

$$\underbrace{+ \frac{1}{5} \cdot \frac{\cos 5x}{5} + c (\in \mathbb{R}).}$$

2. Inverse functions in integrals.

$$\boxed{a) \int \ln x \, dx} = \int 1 \cdot \ln x \, dx$$

$$(x > 0) \quad = \int (x)' \cdot \ln x \, dx =$$

$$\underset{i.p.}{=} x \ln x - \int x \cdot (\ln x)' \, dx =$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx =$$

$$= x \ln x - \int 1 \, dx =$$

$$= x \cdot \ln x - x + c \; (\in \mathbb{R});$$

$$\boxed{b) \int (x^3 + 2x + 2) \cdot \ln x \, dx =}$$

$(x > 0)$ $\quad f'(x)$, $\quad g(x)$

$$= \int \left(\frac{x^4}{4} + x^2 + 2x \right)' \cdot \ln x \, dx \underset{i.p.}{=}$$

$$= \left(\frac{x^4}{4} + x^2 + 2x \right) \cdot \ln x - \int \left(\frac{x^4}{4} + x^2 + 2x \right) \cdot \frac{1}{x} \, dx$$

$$=\left(\frac{x^4}{4}+x^2+2x\right)\cdot\ln x-\int\left(\frac{x^3}{4}+x+2\right)dx$$

$$=\left(\frac{x^4}{4}+x^2+2x\right)\cdot\ln x-\frac{x^4}{16}-\frac{x^2}{2}+2x+$$

$$+C\,(\in\mathbb{R});$$

$$\boxed{c)\ \int\arctan x\,dx=}$$

$$(x\in\mathbb{R})=\int 1\cdot\arctan x\,dx$$

$$=\int(x)'\cdot\arctan x\,dx=\text{i.p.}$$

$$=x\cdot\arctan x-\int x\cdot\frac{1}{1+x^2}dx=$$

$$= x \cdot \arctan x - \frac{1}{2} \cdot \int \frac{2x}{1+x^2}\,dx =$$

$$= x \cdot \arctan x - \frac{1}{2} \int \frac{(1+x^2)'}{1+x^2}\,dx$$

$$= x \cdot \arctan x - \frac{1}{2} \ln(x^2+1) + \overset{\in \mathbb{R}}{c}$$

d) Homework: $\int \arcsin x \, dx$

$x \in (-1, 1)$

③ Definite integrals

Riemann integral

Theorem: If $f \in R[a, b]$

(f has a Riemann-integral) and $\int f \neq \emptyset$ (f has an antiderivate) then for any primitive function $F \in \int f$

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a)$$

a) $$\int_0^{\pi/3} \sin(8x)\,dx =$$

$$= \left[-\frac{\cos 8x}{8}\right]_0^{\pi/3} = -\frac{1}{8}\cdot\left[\cos\frac{8\pi}{3} - \cos 0\right] = \frac{1}{8}\left(1 - \cos\frac{2\pi}{3}\right) =$$

$$= \frac{1}{8}\left(1 - \left(-\frac{1}{2}\right)\right) = \frac{1}{8} \cdot \frac{3}{2} = \underline{\underline{\frac{3}{16}}};$$

$$\boxed{\text{5)} \quad \int_0^1 \frac{1}{3x-5}\,dx} = \frac{1}{3} \int_0^1 \frac{(3x-5)'}{3x-5}\,dx$$

$$= \frac{1}{3}\Big[\ln|3x-5|\Big]_0^1 =$$

$$= \frac{1}{3}\left(\ln|-2| - \ln|-5|\right) =$$

$$= \frac{1}{3}\left(\ln 2 - \ln 5\right) = \underline{\underline{\frac{1}{3}\ln\frac{2}{5}}}.$$

$$c)\ \boxed{\int_1^e \frac{\ln^2 x}{x}\,dx =}$$

$$= \int_1^e \frac{1}{x}\cdot(\ln x)^2\,dx =$$

$$= \int_1^e (\ln x)'\cdot(\ln x)^2\,dx =$$

$$= \left[\frac{(\ln x)^3}{3}\right]_1^e = \frac{\ln^3 e}{3} - \frac{\overbrace{\ln^3 1}^{=0}}{3} =$$

$$= \boxed{\frac{1}{3}}$$

$$\boxed{④\ \text{a)}\ \int_0^1 x e^{-x^2}\,dx =}$$

$$= -\frac{1}{2}\int_0^1 (-x^2)' \cdot e^{-x^2}\,dx =$$

$$= -\frac{1}{2}\cdot\left[e^{-x^2}\right]_0^1 =$$

$$= -\frac{1}{2}\left[e^{-1} - e^0\right] =$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right);$$

$$\boxed{\text{b)}\ \int_{1/2}^1 \frac{\sqrt{1-x^2}}{x^2}\,dx =}$$

Substitution technic:

Review: Suppose $F \in \int f \neq 0$ ($\text{so } F' = f$) and $\exists F \circ g \in D$

$\Rightarrow (F \circ g)' = F' \circ g \cdot g' = f \circ g \cdot g'$

$\Rightarrow \int \Rightarrow$

$\int (F \circ g)' = \int f \circ g \cdot g' \Rightarrow$

$\int f \circ g \cdot g' = F \circ g = \int f \circ g$

$$\boxed{\begin{array}{l} \int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + c \\ F \in \int f \qquad \text{RULE 1.} \end{array}}$$

If g has an inverse function (+ conditions ...)

$$\boxed{\int f(x)\,dx \underset{\substack{\uparrow \\ x=g(t)}}{=} \int f(g(t))\cdot g'(t)\,dt \Big/_{t=g^{-1}(x)}}$$

RULE 2.

<u>Now</u>:

$$\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx \underset{\substack{\uparrow \\ x=\sin t=g(t)}}{=} \int_{\pi/6}^{\pi/2} \frac{\sqrt{1-\sin^2 t}}{\sin^2 t}\cdot\cos t\,dt =$$

if $x=\frac{1}{2} \Rightarrow t=\arcsin\frac{1}{2}=\frac{\pi}{6}$

if $x=1 \Rightarrow t=\arcsin 1=\pi/2$

$$= \int_{\pi/6}^{\pi/2} \frac{\overbrace{|\cos t|}^{+}}{\sin^2 t} \cdot \cos t \, dt =$$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos^2 t}{\sin^2 t} dt = \int_{\pi/6}^{\pi/2} \frac{1 - \sin^2 t}{\sin^2 t} dt =$$

$$= \int_{\pi/6}^{\pi/2} \left(\frac{1}{\sin^2 t} - 1 \right) dt =$$

$$= \Big[-\text{cotan}\, t - t \Big]_{\pi/6}^{\pi/2} =$$

$$= - \overbrace{\text{cotan} \frac{\pi}{2}}^{=0} - \frac{\pi}{2} + \overbrace{\text{cotan} \frac{\pi}{6}}^{\sqrt{3}} + \pi/6 =$$

$$= \sqrt{3} - \frac{\pi}{3}$$

$$\boxed{c) \int_{e}^{e^2} x \cdot \ln x \, dx =} \; i.p. =$$

$$= \int_{e}^{e^2} \left(\frac{x^2}{2}\right)' \cdot \ln x \, dx =$$

$$= \left[\frac{x^2}{2} \cdot \ln x\right]_{e}^{e^2} - \int_{e}^{e^2} \frac{x^2}{2} \cdot \frac{1}{x} \, dx =$$

$$= \frac{1}{2} \cdot \left(e^4 \underbrace{\ln e^2}_{2} - e^2 \ln e\right) -$$

$$-\frac{1}{2}\int_e^{e^2} x\,dx =$$

$$= e^4 - \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_e^{e^2} =$$

$$= e^4 - \frac{1}{2}e^2 - \frac{1}{4}\left[e^4 - e^2\right] =$$

$$= e^4 - \frac{1}{2}e^2 - \frac{1}{4}e^4 + \frac{1}{4}e^2 =$$

$$= \frac{3}{4}e^4 - \frac{1}{4}e^2 = \boxed{\frac{e^2(3e^2-1)}{4}}$$

$$\boxed{\text{d)} \quad \int x \cdot \sqrt{3x-1}\, dx = \quad (x > \tfrac{1}{3})}$$

$$\sqrt{3x-1} = t > 0$$

$$\Rightarrow \quad x = \frac{t^2+1}{3} \Rightarrow$$

$$(\ )'$$

$$dx = \frac{2t}{3} dt$$

$$= \int \frac{t^2+1}{3} \cdot t \cdot \frac{2t}{3} dt \Bigg|_{t=\sqrt{3x-1}}$$

$$\Rightarrow$$

The new integral:

$$\frac{2}{9}\int\left(t^4+t^2\right)dt =$$

$$= \frac{2}{9}\left(\frac{t^5}{5}+\frac{t^3}{3}\right)+c$$

Change back to x:

$$\int x\sqrt{3x-1}\,dx = \frac{2}{45}\left(\sqrt{3x-1}\right)^5 +$$

$$+\frac{2}{27}\left(\sqrt{3x-1}\right)^3 + c\,;$$

$$\boxed{e)\ \int \frac{e^{3x}}{1+e^{x}}\,dx = \quad (x\in\mathbb{R})} =$$

$e^{x} = t > 0$

$x = \ln t$

$$\underset{()'}{\Rightarrow}\ dx = \frac{1}{t}\,dt$$

$$\Rightarrow \int \frac{t^{3}}{1+t}\cdot\frac{1}{t}\,dt =$$

$$= \int \frac{t^{2}}{t+1}\,dt = \int \frac{t^{2}-1+1}{1+t}\,dt$$

$$= \int \left(t-1+\frac{1}{t+1}\right)dt =$$

$t > 0 \Rightarrow t+1 > 0$

$$= \frac{t^2}{2} - t + \ln(t+1) + \underset{\in \mathbb{R}}{c}$$

"Back to" x":

$$\boxed{\int \frac{e^{2x}}{1+e^x}\,dx = \frac{e^{2x}}{2} - e^x + \ln(e^x+1) + \underset{\in \mathbb{R}}{c}}$$

THE END