Problem set 6.: Complex numbers – complex plane (Gaussian plane), roots of unity

Question 1.
In each of the following examples describe the geometric transformation of the Gaussian plane determined by the given function.
(a) $z \mapsto 3z+2$
(b) $z \mapsto (1+i)z$
(c) $z \mapsto 1/\overline{z}$

Question 2.
In the Gaussian plane the center of a square is at point $K = 1+2i$ and one of the vertices of this square is point $A = 5+4i$. Determine the complex numbers represented by the other three vertices of this square.

Question 3.
Let $z \neq w$ be two complex numbers. Find the complex number represented by the midpoint of the line segment between z and w. Consider the two equilateral triangles having both z and w among their vertices. Find the complex number represented by the third vertex of each of these two equilateral triangles. Find the complex number represented by the centroid (or median point) of each triangle.

Question 4.
Find the vector obtained by rotating the vector $\begin{bmatrix} 2 \\ -2\sqrt{3} \end{bmatrix} \in \mathbb{R}^2$ in the plane by each of the following angles: (a) $34°$ (b) $-176°$.

Question 5.
Consider the sets

$$A = \{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$$
$$B = \{z \in \mathbb{C} \mid \operatorname{Im} z < 2\}$$
$$C = \{z \in \mathbb{C} \mid |z-2| = 3\}$$
$$D = \{z \in \mathbb{C} \mid z^2 - (3+2i)z + (5+5i) = 0\}$$

Represent each of the following sets in the Gaussian plane:

(a) A (b) B (c) C (d) D (e) $A \cap B$ (f) $A \cup B$
(g) $A \cap C$ (h) $B \cup C$ (i) $A \setminus B$ (j) $A \triangle B$ (k) $A \cap D$ (l) $C \setminus \overline{B}$

Question 6.
Represent each of the following sets in the Gaussian plane:
(a) $\{z \in \mathbb{C} \mid |z-i+2| = 10\}$
(b) $\{z \in \mathbb{C} \mid \operatorname{Re} z = \operatorname{Im} z\}$
(c) $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \operatorname{Im} z\}$

(d) $\{z \in \mathbb{C} \mid |z-2| \leq |z+3|\}$
(e) $\{z \in \mathbb{C} \mid 2 < |z+i-2| \leq 4\}$

Question 7.
For each number z below, decide if z is a complex root of unity. If z is a complex root of unity then:

(1) find the order of z;

(2) find all those values $n \in \mathbb{N}^+$ for which z is an n^{th} root of unity and

(3) find those values $n \in \mathbb{N}^+$ for which z is a primitive n^{th} root of unity.

(a) 1	(b) -1	(c) i
(d) $1+i$	(e) $\dfrac{1+i}{\sqrt{2}}$	(f) $\dfrac{1+\sqrt{3}i}{2}$
(g) $\dfrac{-1+\sqrt{3}i}{2}$	(h) $\dfrac{-1+\sqrt{3}i}{2}$	(i) $\cos\left(\sqrt{2}\pi\right)+i\sin\left(\sqrt{2}\pi\right)$
(j) $\cos\left(\dfrac{\pi}{361}\right)+i\sin\left(\dfrac{\pi}{361}\right)$		

Question 8.
Show that if $\varepsilon^4 = i$ then $4 \mid o(\varepsilon)$.

Question 9.
Suppose $o(\varepsilon) = 128$. What is $o(i \cdot \varepsilon)$? Justify your answer.

Question 10.

(a) One of the fourth roots of $z = -1-\sqrt{3}i$ is $w_0 = \dfrac{\sqrt[4]{2}}{2}(\sqrt{3}-i)$. Using a primitive fourth root of unity, find all fourth roots of unity. With the help of these, calculate all fourth roots of z.

(b) One of the sixth roots of $z = -i$ is $w_0 = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$. Using a primitive sixth root of unity, find all sixth roots of unity. With the help of these, calculate all sixth roots of z.

(Please do not use the formula for calculating the n^{th} roots of a complex number in this question.)