# Analysis 2 | Class 11

# Multivariable integrals

① Evaluate:

$$\iint_{[0,1]\times[0,2]} x^3 \cdot \sqrt{y} \, dx\,dy = ?$$

Sol: let $I := [0,1]\times[0,2]$ a two-dimensional interval (a rectangle) and

$f(x,y) := x^3 \cdot \sqrt{y}$ $(x \in \mathbb{R}, y \geq 0)$

Since $f \in C(I) \Rightarrow f \in R(I)$

and

$$\iint_I f = \iint_I f(x,y)\,dx\,dy =$$

$$= \int_0^1 \int_0^2 x^3 \sqrt{y}\,dy\,dx =$$

$$= \int_0^1 x^3 \left( \int_0^2 \sqrt{y}\,dy \right) dx =$$

$$= \int_0^1 x^3 \left[ \frac{y^{3/2}}{3/2} \right]_{y=0}^{y=2} dx =$$

$$= \frac{2}{3} \cdot \int_0^1 x^3 \cdot \left( 2^{3/2} - 0^{3/2} \right) dx$$

$$= \frac{2}{3} \; 2\sqrt{2} \cdot \int_0^1 x^3\,dx$$

$$= \frac{4}{3}\sqrt{2} \cdot \left[\frac{x^4}{4}\right]_0^1 = \frac{4\sqrt{2}}{3} \cdot \frac{1}{4} =$$

$$= \boxed{\frac{\sqrt{2}}{3}} ;$$

OR

$$\iint f(x,y)\,dx\,dy = \int_0^2 \int_0^1 x^3 \sqrt{y}\,dx\,dy$$

$I$

$$= \int_0^2 \sqrt{y} \left( \int_0^1 x^3 \, dx \right) dy =$$

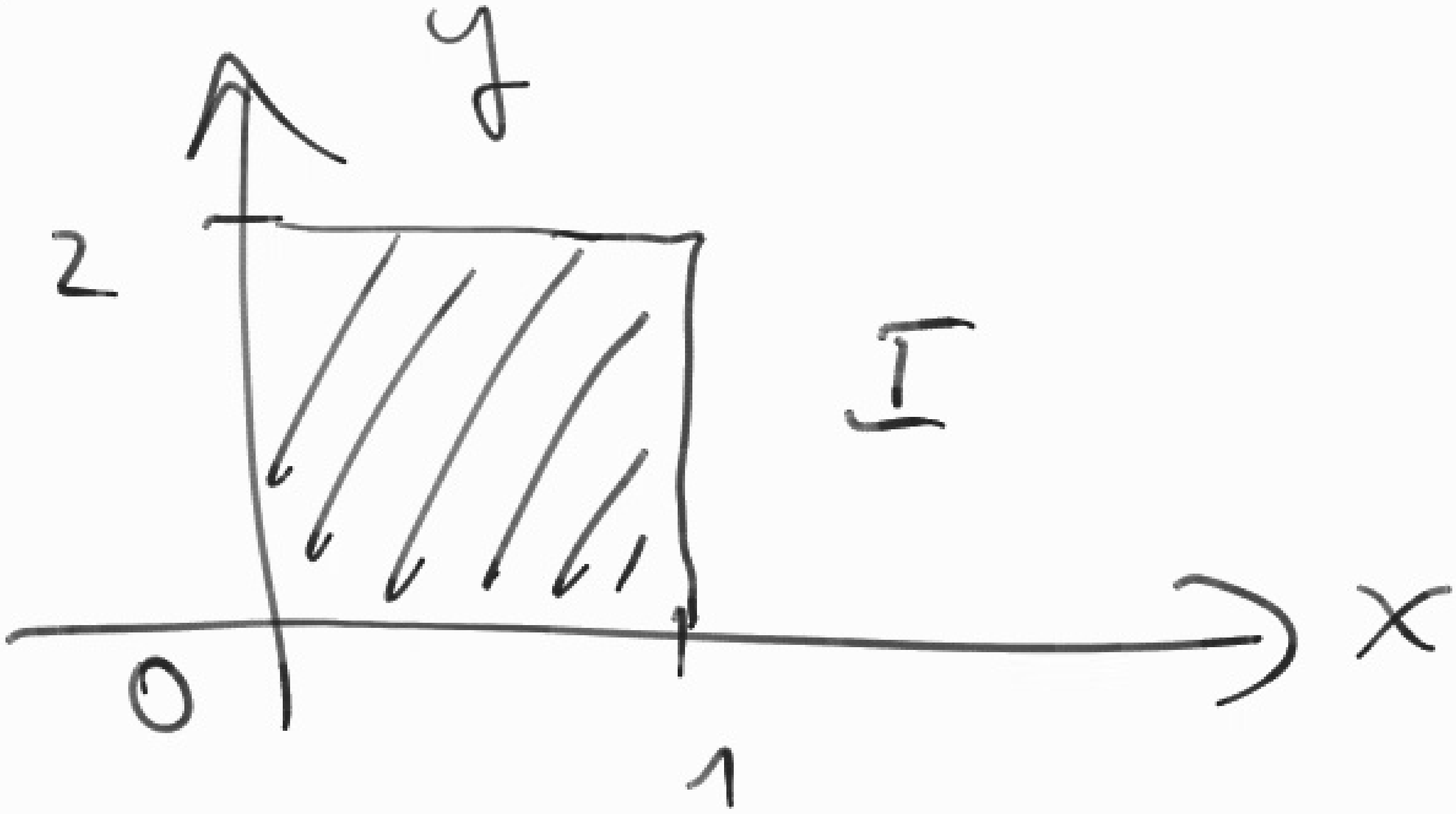
$$= \int_0^2 \sqrt{y} \left[\frac{x^4}{4}\right]_{x=0}^{x=1} dy =$$

$$= \frac{1}{4} \cdot \int_0^2 \sqrt{y}\, dy = \frac{1}{4}\left[\frac{y^{3/2}}{3/2}\right]_0^2 =$$

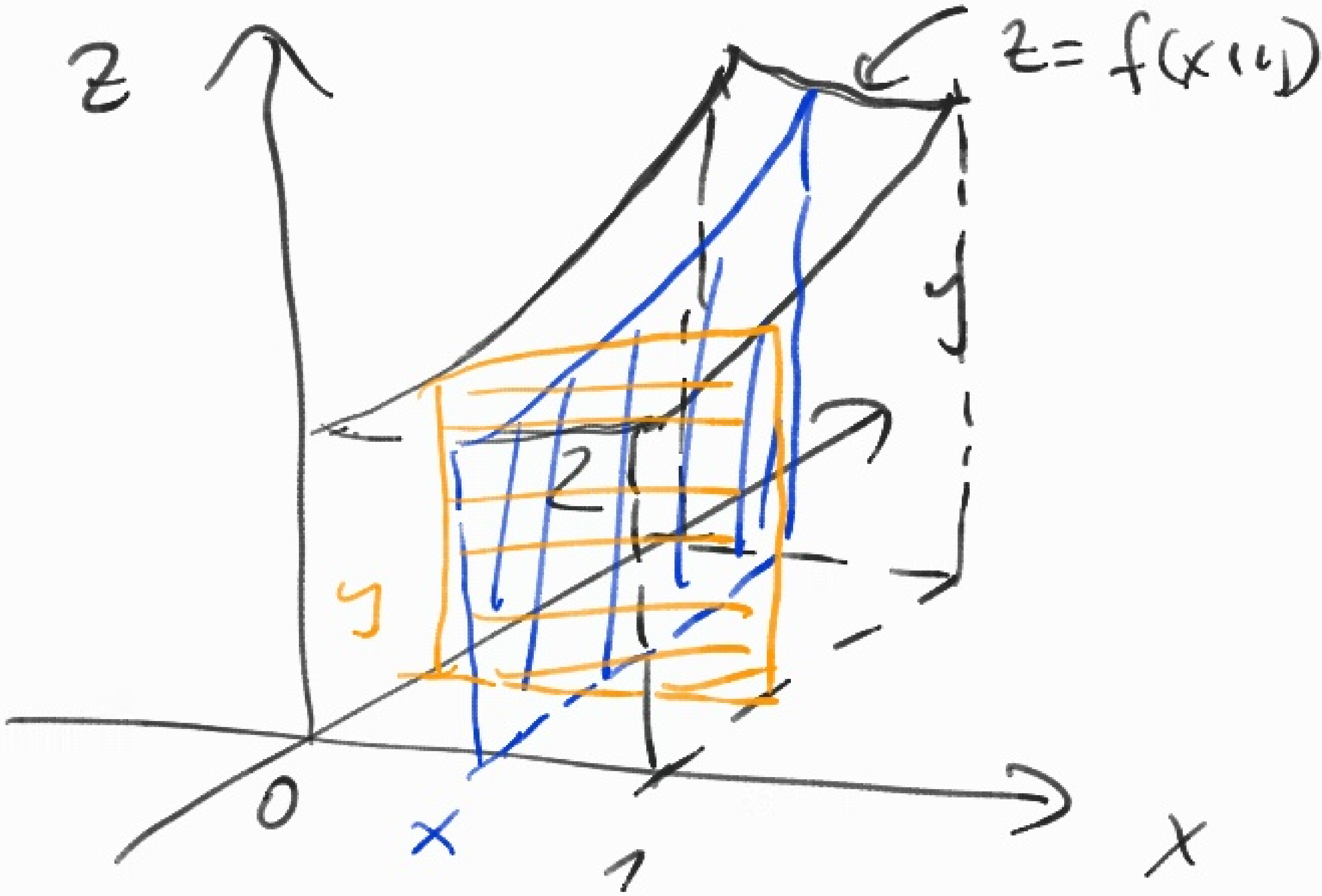
$$= \frac{1}{4} \cdot \frac{2}{3} \cdot 2^{3/2} = \boxed{\frac{\sqrt{2}}{3}}$$

What is this number?

$\underline{I}$:




f on $\underline{I}$:




This area (with $x \in [0,1]$ fixed) is

$$A(x) = \int_0^2 f(x,y)\,dy$$

If we „sum up" $A(x)$ for all $x$ values $\Rightarrow$

$$\int_0^1 A(x)\,dx = \int_0^1 \int_0^2 f(x,y)\,dy\,dx$$

is the volume of this solid „under" $f$.

What is the reversed order? Fix a $y \in [0,2]$ and we get the intersection of area

$$A(y) = \int_0^1 f(x,y)\,dx$$

Summing up these areas $\Rightarrow$

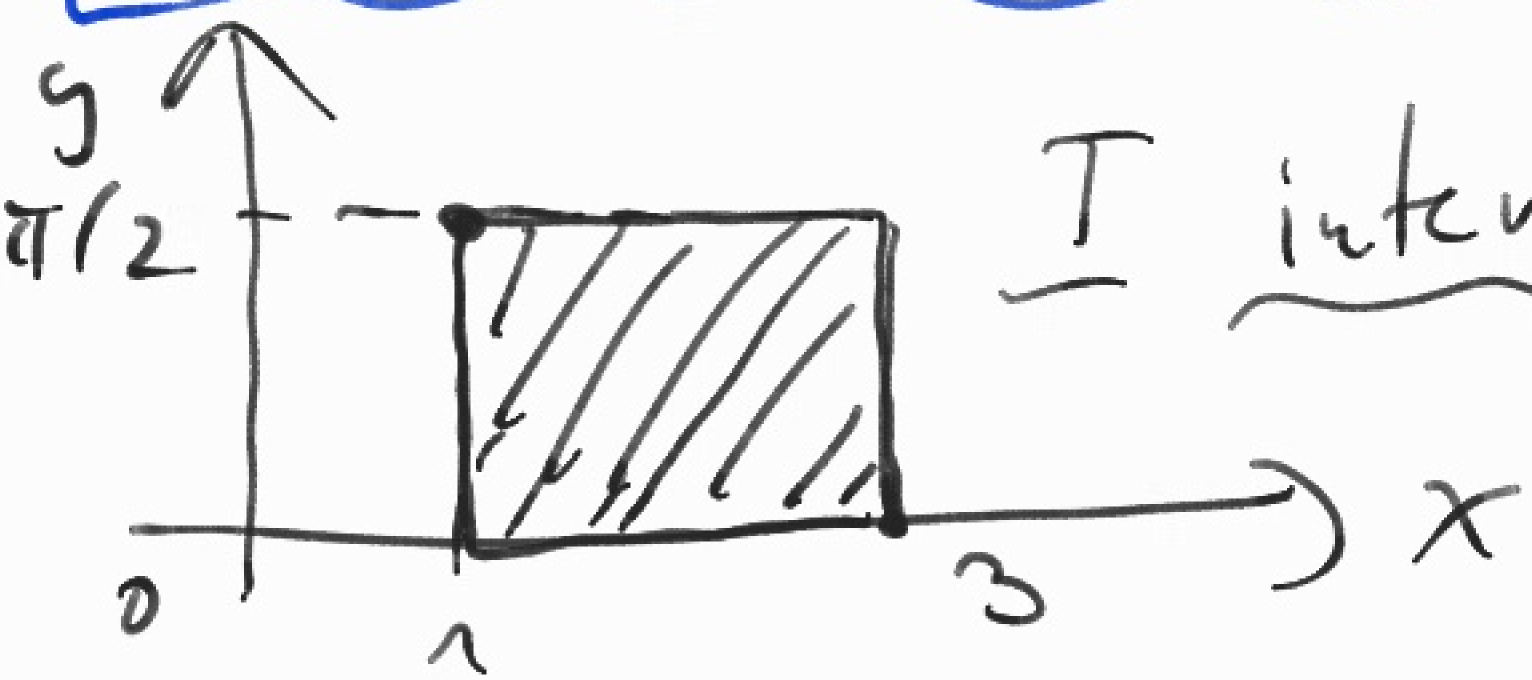
$$\int_0^2 A(y)\,dy = \int_0^2 \left( \int_0^1 f(x,y)\,dx \right) dy$$

= is the same volume

② Evaluate:

$$\iint_I x \cdot \sin(xy)\,dx\,dy$$

where $I = [1,3] \times [0, \pi/2]$




$I$ interval

$f(x,y) = x \cdot \sin(xy) \quad ((x,y) \in \mathbb{R}^2)$

$\Rightarrow f \in C(\mathbb{R}^2) \Rightarrow f \in C(I)$

$\Rightarrow$ (theorem) $f \in R[I]$ <u>and</u>

$$\iint_I f(x,y)\,dx\,dy =$$

$$= \int_1^3 \left( \int_0^{\pi/2} x \sin(xy)\,dy \right) dx =$$

$$= \text{OR} =$$

$$= \int_0^{\pi/2} \left( \int_1^3 x \sin(xy)\,dx \right) dy.$$

Which way should we do it?

where the inner integral is easier to evaluate

Now:

$$\int_1^3 \left( \int_0^{\pi/2} x \sin(x \cdot y) \, dy \right) dx =$$

$$= \int_1^3 \left[ -\cos(xy) \right]_{y=0}^{y=\pi/2} dx =$$

$$= \int_1^3 \left[ -\cos\frac{\pi x}{2} + \overset{1}{\cos 0} \right] dx$$

$$= \left[ -\frac{\sin\frac{\pi x}{2}}{\pi/2} + x \right]_1^3 =$$

$$= \left(-\frac{2}{\pi}\sin\frac{3\pi}{2} + 3\right) - \left(-\frac{2}{\pi}\sin\frac{\pi}{2} + 1\right) =$$

$$= \boxed{2 + \frac{4}{\pi}};$$ Remark: the other way is more complicated now.

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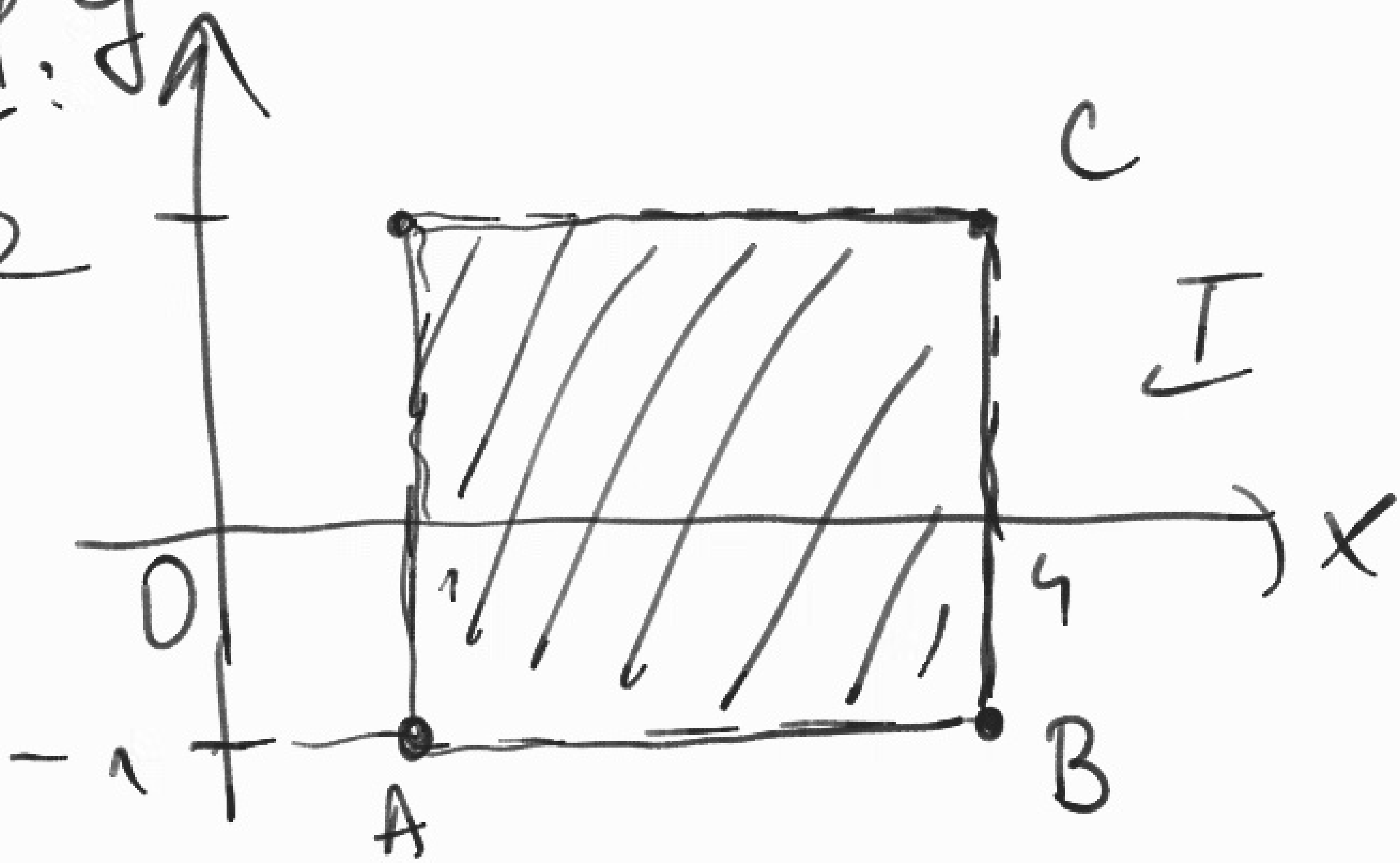
③ Integrate the function $f(x,y) = xy^2 + 3x^2y$ over the interval whose vertices are $A(1,-1)$, $B(4,-1)$, $C(4,2)$, $D(1,2)$.

Sol:




$I = [1,4] \times [-1,2]$

$f \in C(\mathbb{R}^2) \Rightarrow f \in C(I)$

$\Rightarrow f \in R(I)$ <u>and</u>

$\iint_I f(x,y)\,dx\,dy =$

$$= \int_1^4 \int_{-1}^2 (xy^2 + 3x^2y)\,dy\,dx =$$

$$= \int_1^4 \left[ x\frac{y^3}{3} + 3x^2\frac{y^2}{2} \right]_{y=-1}^{y=2} dx$$

$$= \int_1^4 \left[ \frac{8}{3}x + 6x^2 + \frac{x}{3} - \frac{3x^2}{2} \right] dx$$

$$= \int_1^4 \left( 3x + \frac{9}{2}x^2 \right) dx =$$

$$= \left[ \frac{3x^2}{2} + \frac{3}{2}x^3 \right]_1^4 =$$

$$= \frac{3 \cdot 16}{2} + \frac{3}{2} \cdot 64 - 3 =$$

$$= \frac{3}{2} \cdot 16 \cdot (1 + 4) - 3 =$$

$$= 15 \cdot 8 - 3 = 120 - 3 = \underline{\underline{117}}.$$

Reversed order : HW.

$$\int_{-1}^{2} \int_{1}^{4} (xy^2 + 3x^2 y)\, dx\, dy = ?$$

4. $$\iiint_I (xy + xz)\, dx\, dy\, dz = ?$$

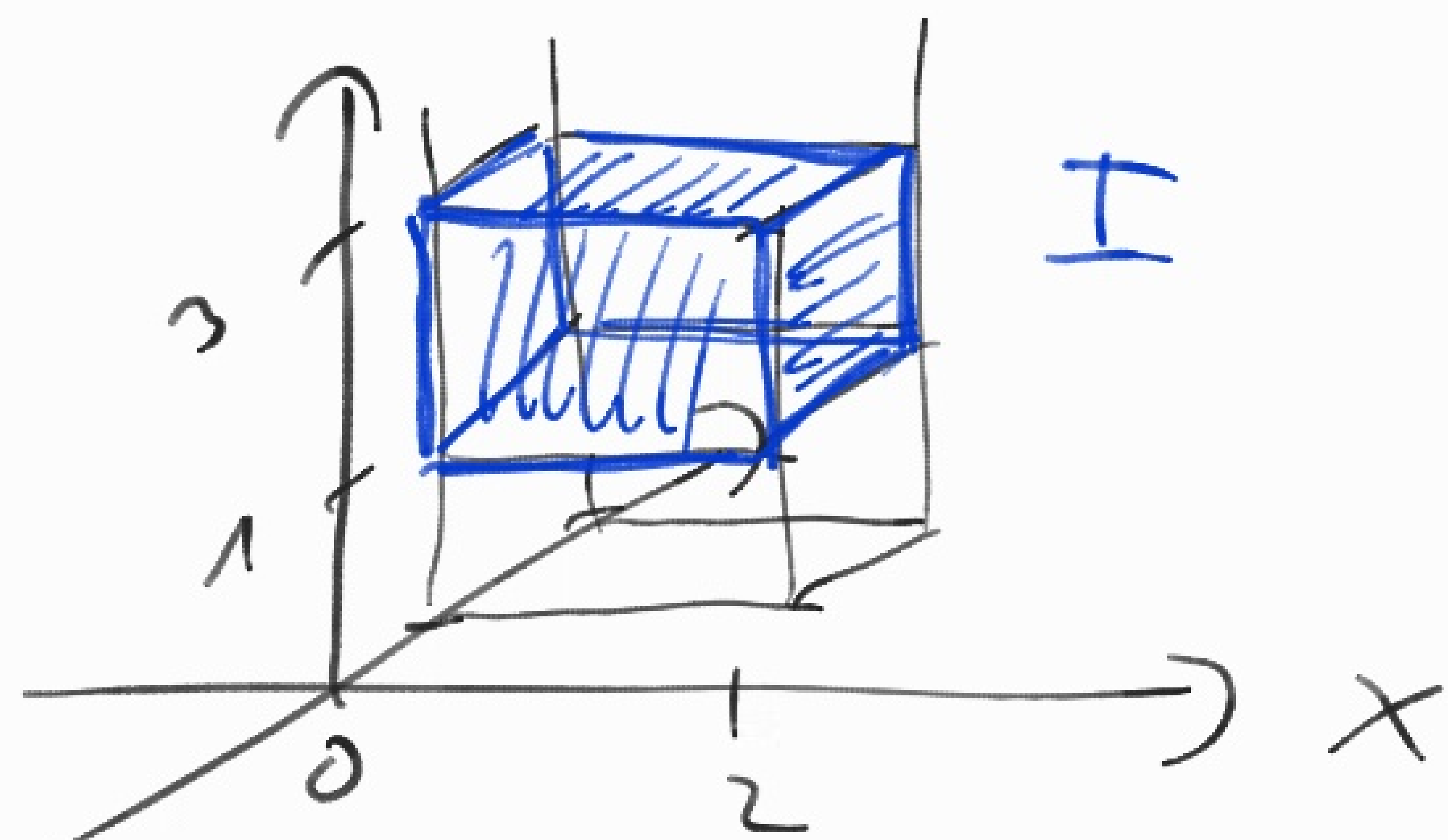
<u>where</u> $I = [0,2] \times [1,2] \times [1,3]$.

Sol: $f(x,y,z) = xy + xz$

$((x,y,z) \in \mathbb{R}^3)$; $f: \mathbb{R}^3 \to \mathbb{R}$;

$I = [0,2] \times [1,2] \times [1,3]$

its a 3-dimensional interval or cuboid.




$$f \in C(I) \Rightarrow f \in R(I)$$

(Integrable by Riemann)

**and** we can integrate in any order. (now we have $3! = 6$ possibilities)

$$\Rightarrow \iiint f(x,y,z)\,dx\,dy\,dz =$$

$$= \int_0^2 \int_1^2 \int_1^3 (xy + xz)\,dz\,dy\,dx =$$

$$= \int_0^2 \int_1^2 \left[ xyz + x\frac{z^2}{2} \right]_{z=1}^{z=3} dy\,dx$$

$$= \int_0^2 \int_1^2 \left[ 3xy + \frac{9}{2}x - xy - \frac{x}{2} \right] dy\,dx$$

$$= \int_0^2 \int_1^2 (2xy + 4x)\, dy\, dx =$$

$$= \int_0^2 \left[ xy^2 + 4xy \right]_{y=1}^{y=2} dx =$$

$$= \int_0^2 \left( 4x + 8x - x - 4x \right) dx =$$

$$= \int_0^2 7x\, dx = 7 \left[ \frac{x^2}{2} \right]_0^2 =$$

$$= 7 \left( \frac{4}{2} - \frac{0}{2} \right) = \boxed{14}$$

For HW evaluate two other orders:, for expl:

$$\int_1^3 \int_0^2 \int_1^2 (xy+xz)\,dy\,dx\,dz$$

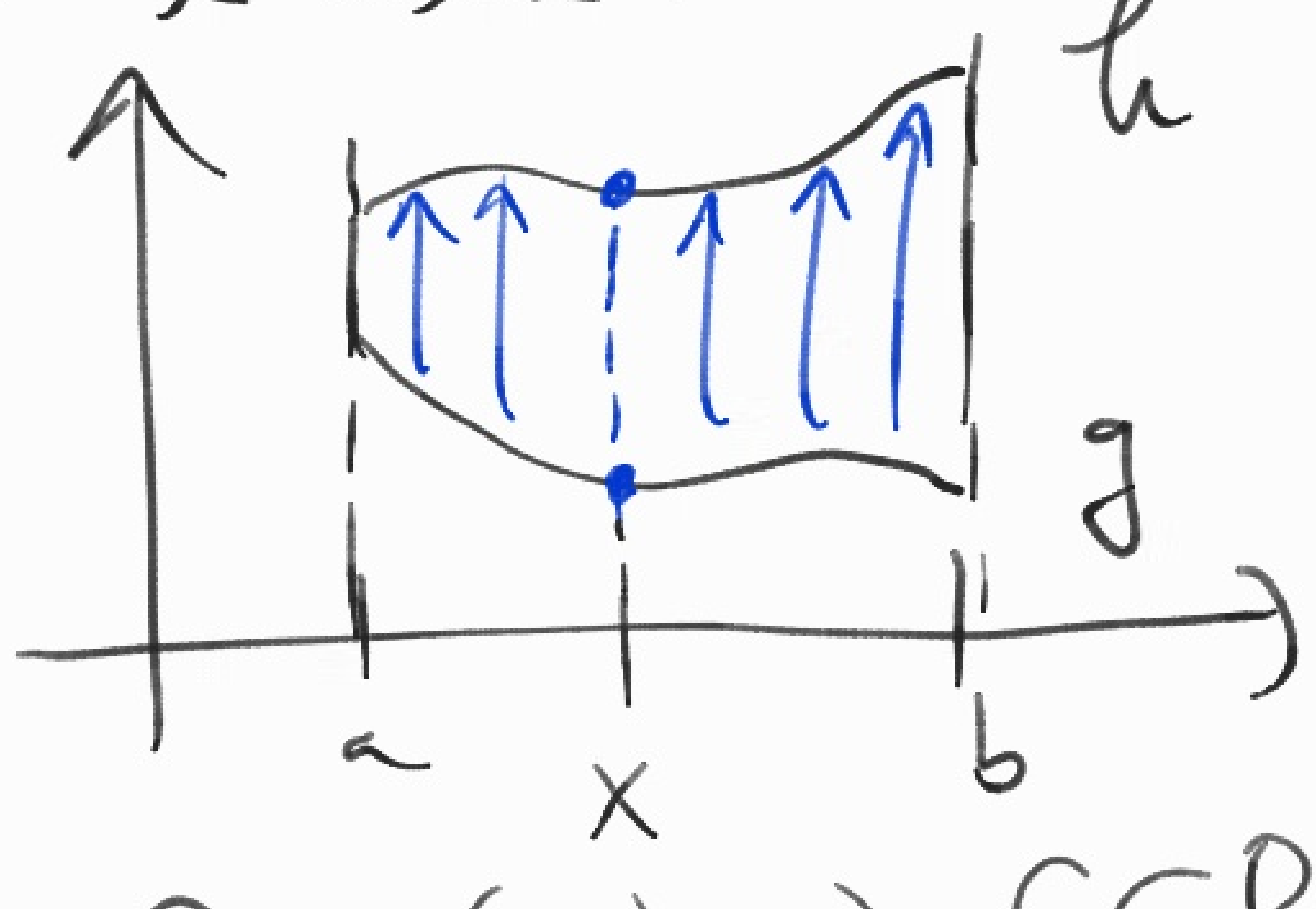
## II Integrating on NORMAL-regions.

Review: Suppose

$g, h: [a, b] \to \mathbb{R}$, $g, h \in C$

$g \leq h.$ $\Longrightarrow$

$N := \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b$ and $g(x) \le y \le h(x)\}$ is a normal-regim related to $x$ axis.




If $f \in C(N) \Rightarrow f \in R(N)$

<u>and</u> $$\iint_N f = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Similar option when we have a normal region related to $y$ axis.

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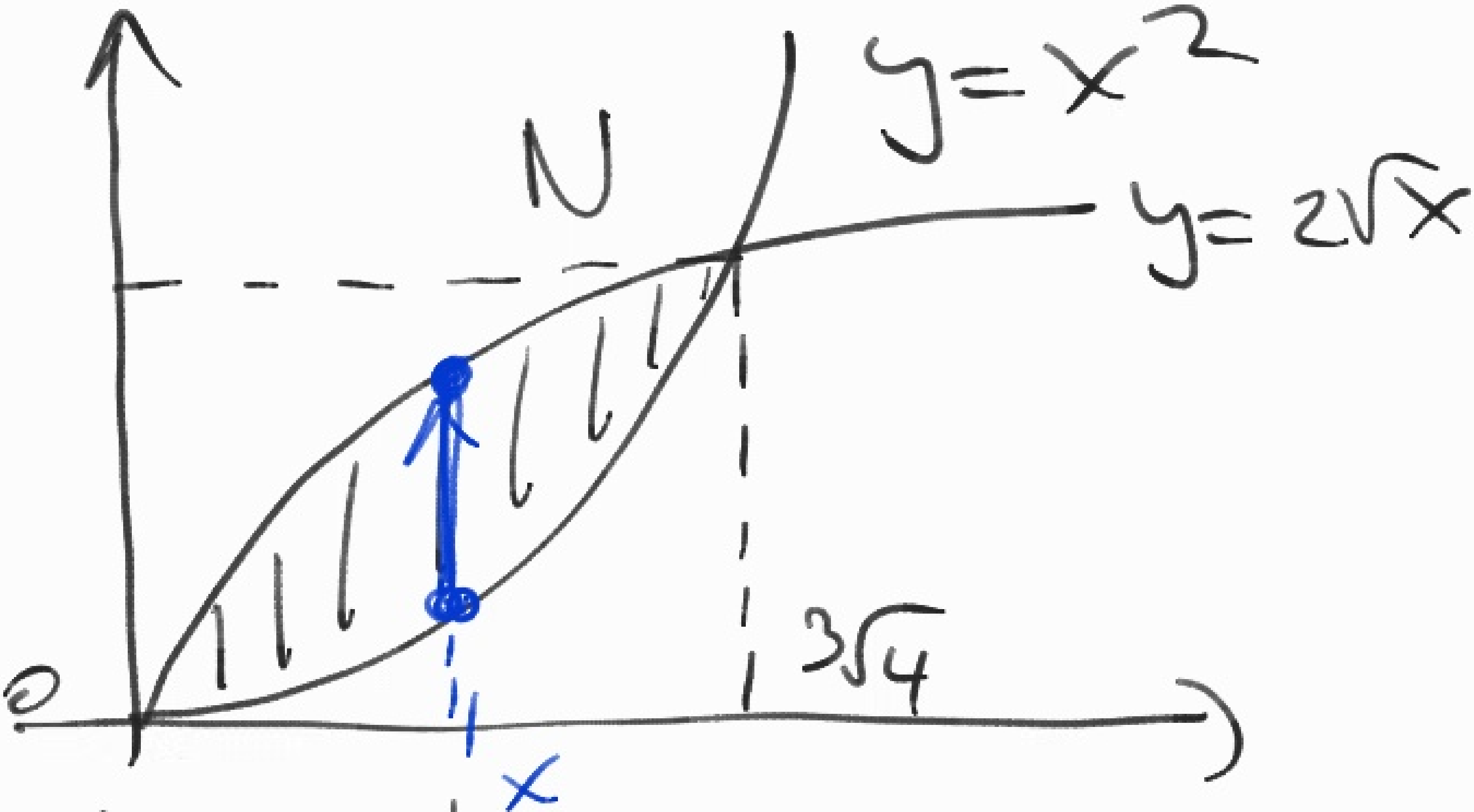
① Evaluate

$$\iint_N xy^2\,dx\,dy$$ where

$N$ is the bounded region by the curves: $y = x^2$ and $y = 2\sqrt{x}$ ;




Intersections:

$$x^2 = 2\sqrt{x} \quad (x \geq 0)$$

$$\Rightarrow \quad x^4 = 4x$$

$$x(x^3-4)=0 \qquad x_1 = 0$$

$$x_2 = \sqrt[3]{4}$$

$$f(x,y) = xy^2 \quad (x,y) \in N$$

$$f \in C(N) \Rightarrow f \in R(N)$$

and

$$\iint_N f(x,y)\,dx\,dy =$$

$$= \int_0^{\sqrt[3]{4}} \int_{x^2}^{2\sqrt{x}} x y^2 \, dy \, dx =$$

$$= \int_0^{\sqrt[3]{4}} x \left[ \frac{y^3}{3} \right]_{y=x^2}^{y=2\sqrt{x}} dx =$$

$$= \frac{1}{3} \int_0^{\sqrt[3]{4}} x \cdot \left( 8x\sqrt{x} - x^6 \right) dx$$

$$= \frac{1}{3} \cdot \int_0^{\sqrt[3]{4}} \left(8x^{\frac{5}{2}} - x^7\right) dx =$$

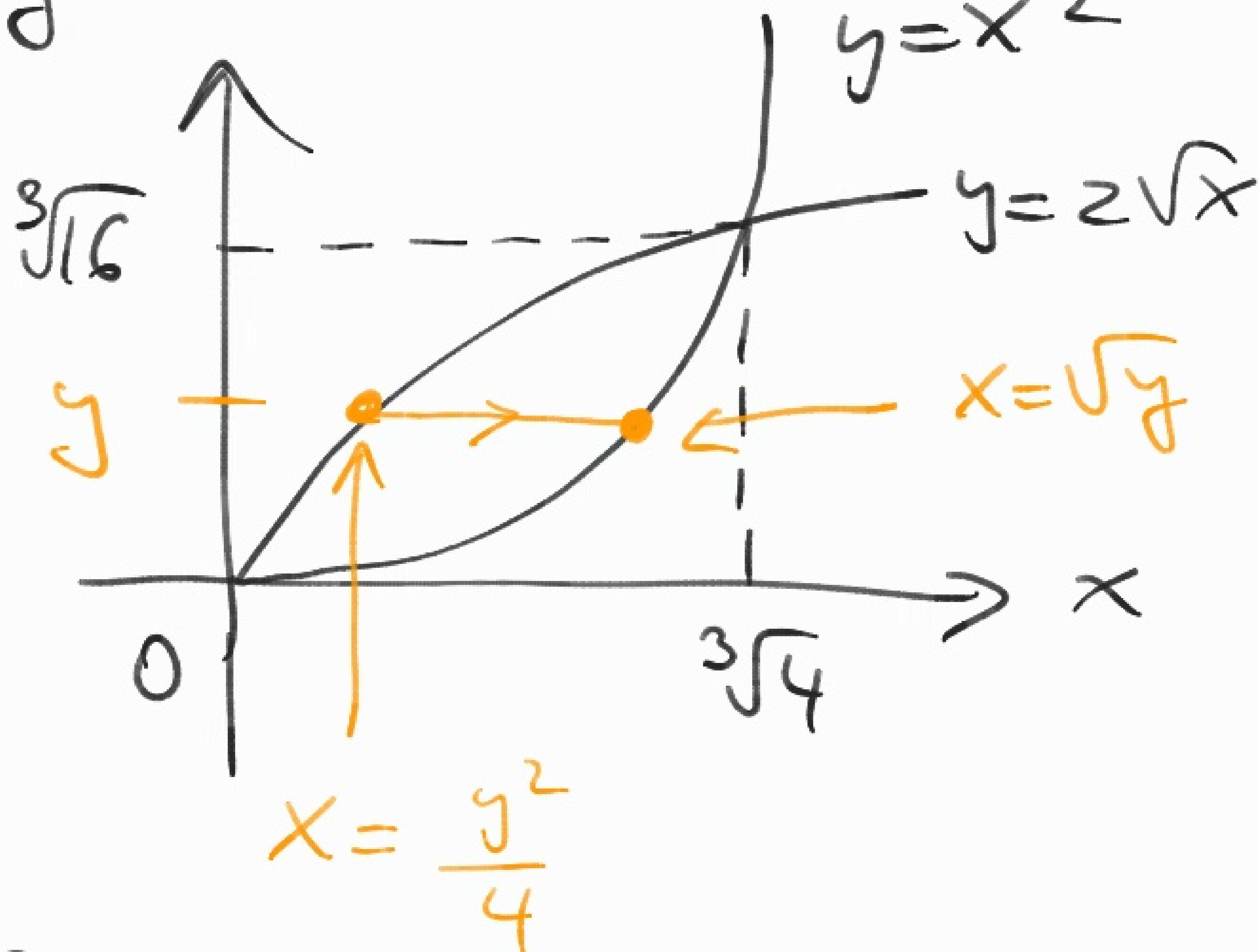
$$= \frac{1}{4} \cdot \left[ \frac{8x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^8}{8} \right]_0^{\sqrt[3]{4}} =$$

$$= \frac{1}{4} \cdot \left[ \frac{16}{7} \cdot 2^{\frac{\cancel{2}}{3} \cdot \frac{7}{\cancel{2}}} - \frac{2^{\frac{2}{3} \cdot 8}}{8} \right]$$

$$= \frac{1}{4} \cdot \left[ \frac{16}{7} 4 \cdot \sqrt[3]{2} - 4 \cdot \sqrt[3]{2} \right] =$$

$$= \sqrt[3]{2} \left( \frac{16}{7} - 1 \right) = \frac{\sqrt[3]{2} \cdot 9}{7}.$$

We can look at N as normal region related to y axis:




So

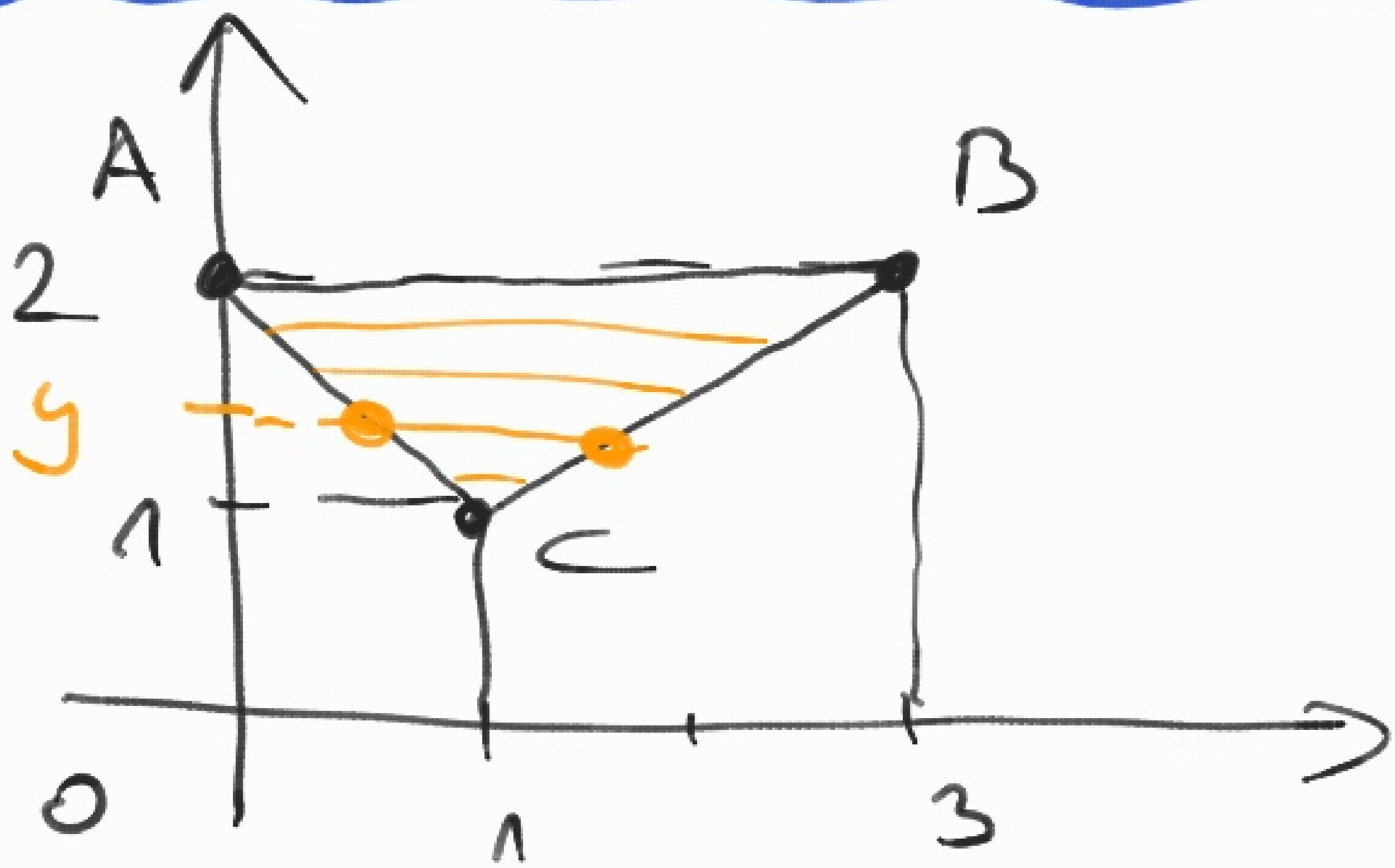
$$\iint_N f(x,y)\,dx\,dy = \int_0^{\sqrt[3]{16}} \int_{y^2/4}^{\sqrt{y}} xy^2\,dx\,dy$$

H.W. →

② $$\iint_N y \cdot e^{x} \, dx\,dy$$

where N is the triangle with vertices:

$(0,2), (1,1), (3,2)$



The easiest way is to consider ABC as a normal region related to axis $x$.

We need the equations of lines AC and BC.

<u>AC</u>: $y = -x + 2$

<u>BC</u>: $y = \frac{1}{2}x + \frac{1}{2}$

So if $y \in [1, 2]$ fixed then:

$$2 - y \leq x \leq 2y - 1$$

$$f(x,y) = y \cdot e^x \quad ((x,y) \in \mathbb{R}^2)$$

$$f \in C(\mathbb{R}^2) \Rightarrow f \in C(N)$$

$$\Rightarrow f \in R(N) \Rightarrow$$

$$\iint_N f(x,y)\,dx\,dy =$$

$$= \int_1^2 \int_{2-y}^{2y-1} y e^x \,dx\,dy =$$

$$= \int_1^2 y \cdot \left[e^x\right]_{2-y}^{2y-1} dy =$$

$$= \int_1^2 y \left(e^{2y-1} - e^{2-y}\right) dy$$

$$= \frac{1}{e}\int_1^2 y e^{2y}\,dy - e^2 \int_1^2 y e^{-y}\,dy$$

<u>where</u>

$$\int_1^2 y e^{2y} = \left[ y \cdot \frac{e^{2y}}{2} \right]_1^2 -$$

$$- \int_1^2 \frac{e^{2y}}{2}\,dy = e^4 - \frac{e^2}{2} -$$

$$- \frac{1}{2}\left[ \frac{e^{2y}}{2} \right]_1^2 = \frac{e^2}{4}\left(3e^2 - 1\right)$$

<u>and</u>

$$\int_1^2 y e^{-y}\, dy = \text{i.p.} = \text{HW} =$$

$$= \frac{2e-3}{e^2} \Longrightarrow$$

$$\boxed{\iint_N f = \frac{3}{4}e^3 - \frac{9}{4}e + 3}$$

③ $$\int_0^1 \int_{y^2}^1 y \sin(x^2)\, dx\, dy$$

$= ?$

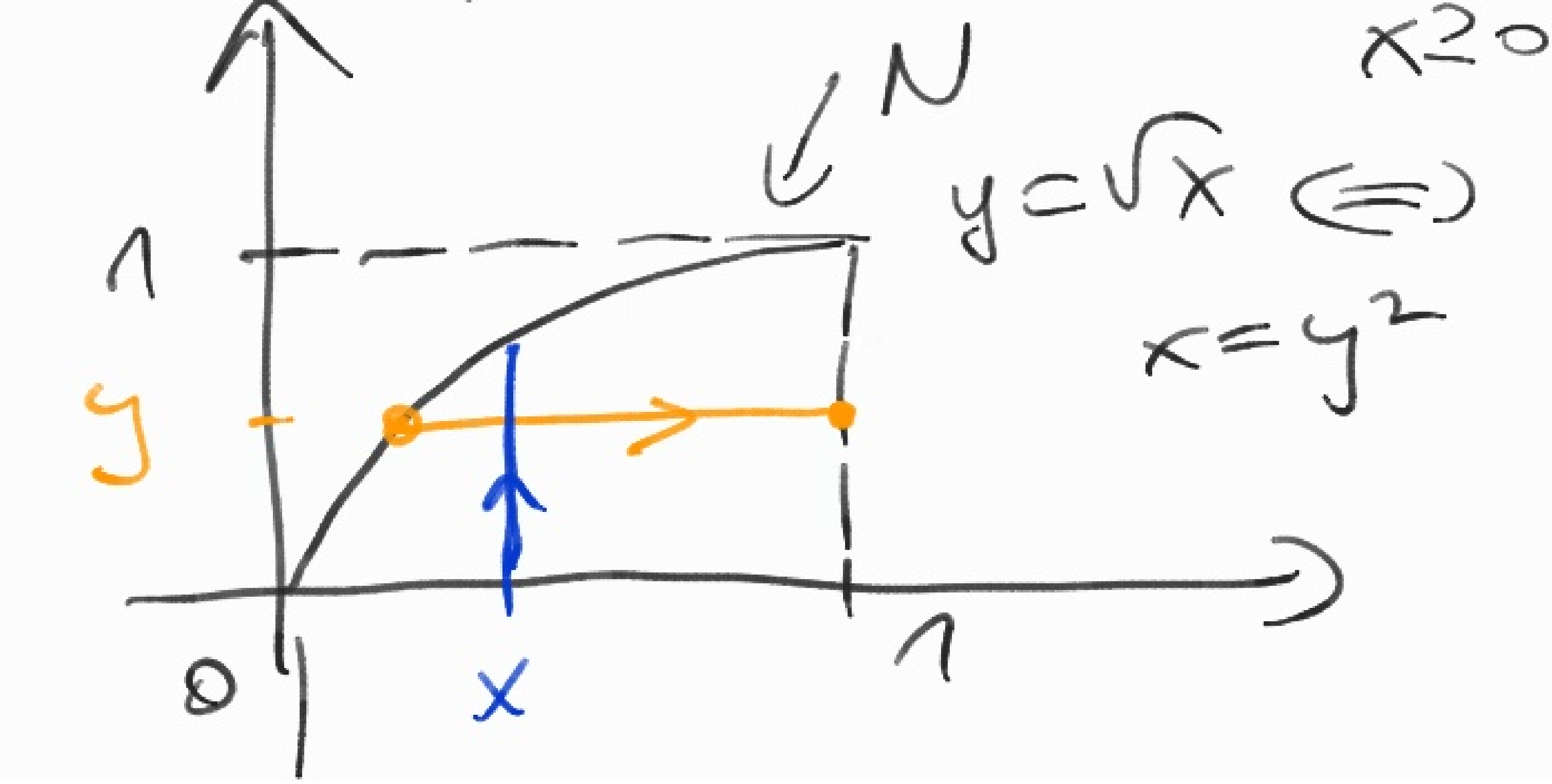
We cannot evaluate the inner integral, because it's antiderivate

$$\int \sin x^2 dx \text{ is not}$$

elementary function...

-here $0 \le y \le 1$ and

$$y^2 \le x \le 1$$



## Change the order:

$f(x,y) = y \sin(x^2)$

$(x,y) \in N$

$f \in C(N) \Rightarrow f \in R(N)$

and

$$\iint_N f = \int_0^1 \int_0^{\sqrt{x}} y \sin x^2 \, dy \, dx$$

$$= \int_0^1 \sin x^2 \left( \int_0^{\sqrt{x}} y \, dy \right) dx =$$

$$= \int_0^1 \sin x^2 \cdot \left[ \frac{y^2}{2} \right]_0^{\sqrt{x}} dx$$

$$= \frac{1}{2} \int_0^1 x \sin x^2 \, dx =$$

$$= \frac{1}{4} \cdot \int_0^1 (x^2)' \cdot \sin x^2 \, dx =$$

$$= \frac{1}{4} \left[ -\cos x^2 \right]_0^1 =$$

$$= \frac{1}{4} (1 - \cos 1) = \frac{1 - \cos 1}{4}$$

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$\boxed{4}$ Evaluate:

$$\iiint_N \underbrace{\frac{1}{(1+x+y+z)^3}}_{f(x,y,z)} \, dx\,dy\,dz$$

where N is the phyramid enclosed by the planes
$x+y+z=1, \; x=0, \; y=0,$
$z=0.$

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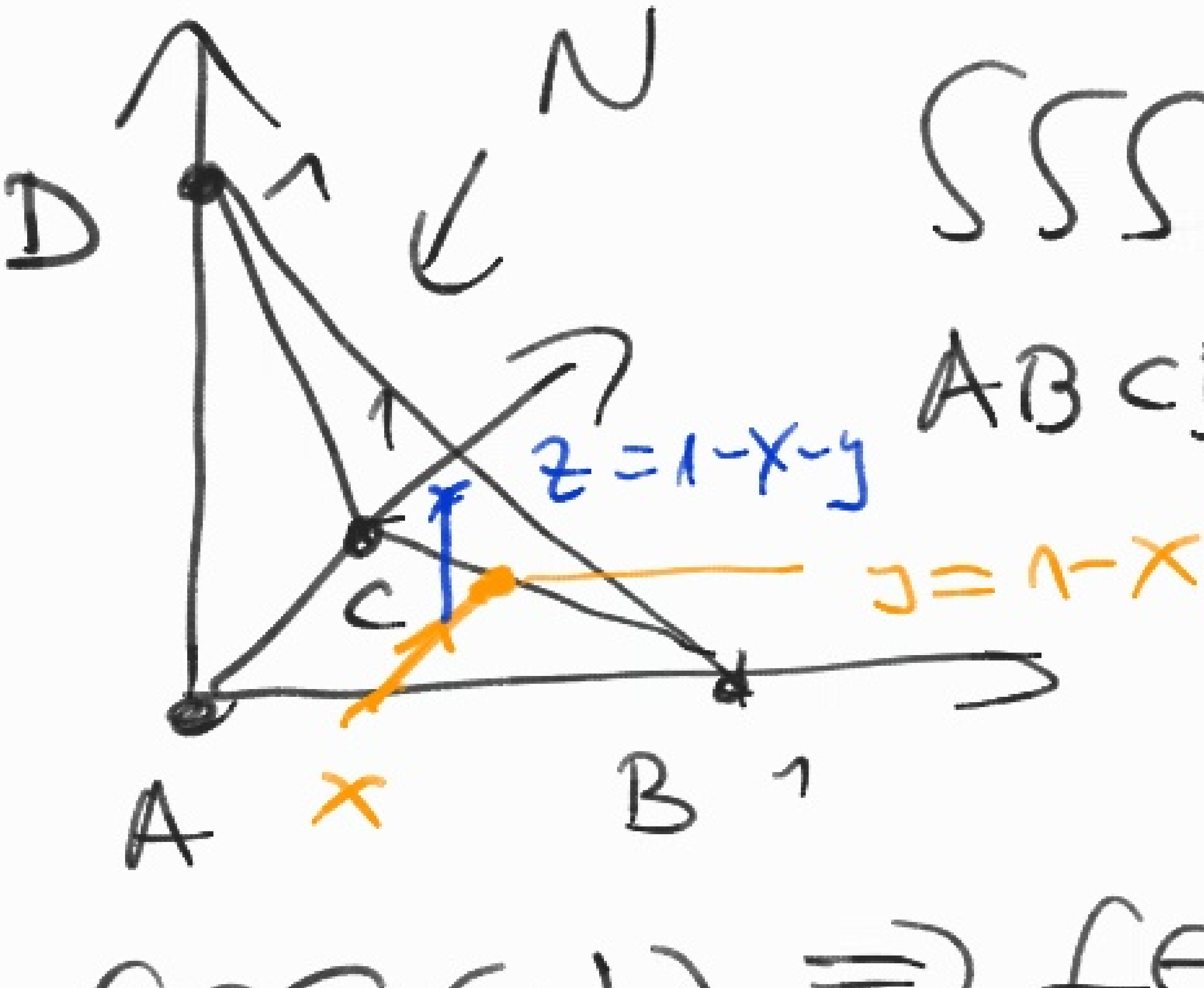


$$\iiint\limits_{ABCD} f = ?$$

$$f \in C(N) \Rightarrow f \in R(N)$$

and

$$\iiint_N f(x,y,z)\,dx\,dy\,dz =$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[ \frac{(1+x+y+z)^{-2}}{-2} \right]_{z=0}^{z=1-x-y} dy\,dx =$$

$$= \int_0^1 \int_0^{1-x} \left( -\frac{1}{8} + \frac{(1+x+y)^{-2}}{2} \right) dy\,dx$$

$$= \int_0^1 \left[ -\frac{y}{8} + \frac{(1+x+y)^{-1}}{-2} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left( \frac{x-1}{8} - \frac{1}{4} + \frac{1}{2(1+x)} \right) dx$$

$$= \int_0^1 \left( \frac{x}{8} - \frac{3}{8} + \frac{1}{2} \cdot \frac{1}{1+x} \right) dx =$$

$$= \left[ \frac{x^2}{16} - \frac{3}{8} x + \frac{1}{2} \ln(1+x) \right]_0^1 =$$

$$= \frac{1}{16} - \frac{3}{8} + \frac{1}{2} \ln 2 =$$

$$= \frac{\ln 2}{2} - \frac{5}{16} ;$$

THE END