Problem set 4.: Functions, partial orders

Question 1.
In each of the following examples decide if relation f is a function. If f is a function then determine the domain and range of f and decide whether f is surjective, injective and/or bijective.
(a) $A = \{1, 2, 3, 4, 5\}, B = \{10, 11, 12, 13, 14\}, f \subseteq A \times B, f = \{(1, 11), (2, 11), (4, 12), (5, 10)\}$
(b) $A = \{1, 2, 3, 4\}, B = \{a, b, c, d, e, f\}, f \subseteq A \times B, f = \{(1, a), (2, c), (3, e), (3, f), (4, a)\}$
(c) $A = \{1, 2, 3, 4, 5\}, B = \{a, b, c, d, e, f\}, f \subseteq A \times B, f = \{(1, a), (4, e), (5, d)\}$
(d) $A = \{1, 2, 3\}, B = \{1, 3, 5\}, f \subseteq A \times B, f = \{(1, 1), (2, 5), (3, 5)\}$

Question 2.
Let $m \in \mathbb{R}^+$ and $A = \{\text{all isosceles triangles with height } m \text{ (from base)}\}$, $B = \mathbb{R}^+$. Define the binary relation $R \subseteq A \times B$ as follows: $aRb, a \in A, b \in B$, if the area of a equals b. Show that R is a function, and examine the properties of f (i.e. decide if f is surjective, injective and/or bijective).

Question 3.
(a) Let $f : \mathbb{R} \to \mathbb{R}, f(x) := 3x - 4$. Prove that function f is bijective, and find the inverse of f.
(b) Let $g : \mathbb{R} \to \mathbb{R}, g(x) := 3 - |x|$. Prove that function g is neither injective, nor surjective.

Question 4.
In each of the following examples decide whether f is a function.
(a) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x \mid y$
(b) P is the set of all prime numbers and $f \subseteq P \times P, xfy \iff x \mid y$
(c) $f \subseteq \{0, 3, 5\} \times \{1, 2, 5\}, xfy \iff xy = 0$
(d) $f \subseteq \{1, 2, 5\} \times \{0, 3, 5\}, xfy \iff xy = 0$
(e) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff$ the set of digits contained by the base-10 form of x equals the set of digits in the base-10 form of y.
(f) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff 2x = y$
(g) $f \subseteq \mathbb{Z} \times \mathbb{Z}, xfy \iff x^2 = y^2$
(h) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x^2 = y^2$
(i) $f \subseteq \mathbb{R} \times \mathbb{R}, xfy \iff x^2 + y^2 = 9$

Question 5.
In each of the following examples decide if the given binary relation is a function.
(a) $f_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 7x = y^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(b) $f_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2 + 6y\} \subseteq \mathbb{R} \times \mathbb{R}$
(c) $f_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 7x^2 - 6 = y\} \subseteq \mathbb{R} \times \mathbb{R}$
(d) $f_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid y = |x|\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(e) $f_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = (x + 4)^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(f) $f_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid 2y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(g) $f_7 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 7 \mid x - y\} \subseteq \mathbb{Z} \times \mathbb{Z}$
(h) $f_8 = \{(x, y) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \mid xy = 1\} \subseteq (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$
(i) $f_9 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(j) $f_{10} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \le 3\} \subseteq \mathbb{Z} \times \mathbb{Z}$

(k) $f_{11} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y(1 - x^2) = x - 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(l) $f_{12} = \{(x, y) \in (\mathbb{R} \setminus \{1, -1\}) \times (\mathbb{R} \setminus \{1, -1\}) \mid y(1 - x^2) = x - 1\} \subseteq (\mathbb{R} \setminus \{1, -1\}) \times (\mathbb{R} \setminus \{1, -1\})$
For each relation that is a function decide if it is injective, surjective and/or bijective. For each relation that is not a function and is a homogeneous relation, decide if it is reflexive, symmetric and/or transitive.

Question 6.
Let $A = \{2, 3, 6, 8, 9, 12, 18\} \subseteq \mathbb{N}^+, R \subseteq A \times A$ and $aRb \iff a \mid b$.
(a) Prove that R is a partial order on set A.
(b) Draw a Hasse-diagram of the partial order R.

Question 7.
(a) Prove that the relation $\leq$ is a partial order on $\mathbb{N}$, where $\leq$ is defined as follows: $\forall n, m \in \mathbb{N} : n \leq m \iff \exists k \in \mathbb{N}$ such that $n + k = m$.
(b) Define the binary relation R on $\mathbb{N}$ as follows: $\forall m_1, m_2, n_1, n_2 \in \mathbb{N} : (m_1, n_1)R(m_2, n_2) \iff m_1 \leq m_2 \wedge n_1 \leq n_2$. Prove that R is a partial order on $\mathbb{N} \times \mathbb{N}$.

Question 8.
In each of the following examples decide if relation R is a partial order on the underlying set.
(a) P is the set of all polynomials with real coefficients and $R \subseteq P \times P, fRy \iff \deg f \leq \deg g$
(b) $R \subseteq \mathbb{Z} \times \mathbb{Z}, aRb \iff |a| \leq |b|$
(c) V is the set of all those vectors in $\mathbb{R}^2$ which are 10 units in length and $R \subseteq V \times V, xRy \iff$ the angle from the positive real axis to vector x is less than or equal to the angle from the positive real axis to vector y (we assume both of these angles to be in the interval $[0; 2\pi[$)
(d) $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2, xRy \iff$ the length of vector x is less than or equal to the length of vector y.

Question 9.
Decide which of the following relations are total orders on the set $A = \{1, 2, 3, 4\}$.
(a) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
(b) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
(c) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 4)\}$