Class 13. Generated Subspace.

① $W := \text{Span}(\underbrace{(1,2,-1)}_{a}; \underbrace{(-3,1,1)}_{b}) \subseteq \mathbb{R}^3$

a) What are the elements of W.

$$\text{Span}(a,b) = \text{def} = \{\alpha a + \beta b \in \mathbb{R}^3 \mid \alpha, \beta \in \mathbb{R}\}$$
$$= \left\{\alpha \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}\right\}$$
$$= \left\{\begin{pmatrix} \alpha - 3\beta \\ 2\alpha + \beta \\ -\alpha + \beta \end{pmatrix} \in \mathbb{R}^3 \middle| \alpha, \beta \in \mathbb{R}\right\}$$

b) Give some example elements of W.

$$\alpha = 1, \beta = 2 \Rightarrow \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} \in W$$

or $\alpha = -1, \beta = 1 \Rightarrow \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \in W$

c) $x := (2,4,0)$ $\quad y := (5,-4,-1)$

$x \in W$? $\quad y \in W$?

$\begin{pmatrix}2\\4\\0\end{pmatrix} \in W \iff \exists \alpha, \beta \in \mathbb{R}.$

so that $\begin{pmatrix}2\\4\\0\end{pmatrix} = \begin{pmatrix}\alpha-3\beta\\2\alpha+\beta\\-\alpha+\beta\end{pmatrix}$

$$\begin{cases}\alpha-3\beta=2\\2\alpha+\beta=4\\-\alpha+\beta=0 \Rightarrow \beta=\alpha.\end{cases}$$

$$\alpha-3\beta=2 \Rightarrow -2\alpha=2.$$

$$\alpha=-1$$

$$3\alpha=4 \quad \alpha=\frac{3}{4}$$

so $x \notin W$.

For $y=\begin{pmatrix}5\\-4\\-1\end{pmatrix} \in W \iff$

$\exists \alpha, \beta \in \mathbb{R};$

$$\begin{pmatrix}5\\-4\\-1\end{pmatrix} = \begin{pmatrix}\alpha-3\beta\\2\alpha+\beta\\-\alpha+\beta\end{pmatrix} \Rightarrow$$

$$\begin{cases} \alpha - 3\beta = 5 & (1) \\ 2\alpha + \beta = -4 & (2) \\ -\alpha + \beta = -1 & (3) \end{cases}$$

(3) + (1)

$$-2\beta = 4$$

$$\beta = -2.$$

$$\alpha = \beta + 1$$

$$\alpha = -1$$

and $2\alpha + \beta = -1 \cdot 2 + -2 = -4$ ✓

so $y \in W$, $y = -1 \cdot a - 2b$.

(2) Consider the vectors

$u = (1, 2, -1)$, $v = (6, 4, 2)$

$x = (9, 2, 7)$, $y = (4, -1, 8)$

a) Compute $-2u + 3v$.

$$-2u + 3v = -2\begin{pmatrix}1\\2\\-1\end{pmatrix} + 3\begin{pmatrix}6\\4\\2\end{pmatrix}$$

$$= \begin{pmatrix} 16 \\ 8 \\ 8 \end{pmatrix}$$

b) $\mathrm{Span}(u, v) = ?$

$$\mathrm{Span}(u,v) = \{ \alpha u + \beta v \mid \alpha, \beta \in \mathbb{R} \}$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

c) Is $x \in \mathrm{Span}(u, v)$?

$x \in \mathrm{Span}(u, v) \Leftrightarrow \exists \alpha, \beta \in \mathbb{R}.$

$x = \alpha u + \beta v \Leftrightarrow$

$$\exists \alpha, \beta \in \mathbb{R} : x = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix}$$

$$\begin{cases} \alpha + 6\beta = 9 & (1) \\ 2\alpha + 4\beta = 2 & (2) \\ -\alpha + 2\beta = 7 & (3) \end{cases}$$

$(1) + (3).$

$$8\beta = 16$$

$$\beta = 2$$

$$2\alpha + 4\beta = 2$$

$$2\alpha + 8 = 2$$

$$2\alpha = -6$$

$$\alpha = -3$$

$$-\alpha + 2\beta = 3 + 4 = 7 \checkmark$$

So $x = -3u + 2v \in \text{Span}(u,v)$

d) is $y \in \text{Span}(u,v)$

$$y \in \text{Span}(u,v) \Rightarrow \exists \alpha, \beta \in \mathbb{R}$$

$$y = \alpha u + \beta v. \iff$$

$$\iff \exists \alpha, \beta \in \mathbb{R}.$$

$$y = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix}$$

$$\begin{cases} \alpha + 6\beta = 4 & (1) \\ 2\alpha + 4\beta = -1 & (2) \\ -\alpha + 2\beta = 8 & (3) \end{cases}$$

$(1) + (3)$

$8\beta = 12$

$$\beta = \frac{3}{2}$$

$2\alpha + 4\beta = -1$

$2\alpha + 6 = -1$

$$\alpha = -\frac{7}{2}$$

$$-\alpha + 2\beta = \frac{7}{2} + 3 = \frac{13}{2} \neq 8.$$

$\Rightarrow y \notin \text{span}(u, v).$

(3) Consider the subspace.
Determine finite generator systems to each of them.

a) $S_5 = \left\{ \begin{pmatrix} x-y \\ 3x \\ 2x+y \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, y \in \mathbb{R} \right\}$

$$\begin{pmatrix} x-y \\ 3x \\ 2x+y \end{pmatrix} = \begin{pmatrix} x \\ 3x \\ 2x \end{pmatrix} + \begin{pmatrix} -y \\ 0 \\ y \end{pmatrix} =$$

$$= x \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{a} + y \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

b

$= x a + y b$.

So $S_5 = \left\{ x \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$

$= \operatorname{Span}((1,3,2),(-1,0,1))$.

So $(1,3,2),(-1,0,1)$ is a generator system in S_5.

b) $S_3 := \{ (x,y,z) \in \mathbb{R}^3 / 2x - 3y + z = 0 \}$

S_3 can be write.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 3y-2x \end{pmatrix} \in \mathbb{R}^3 \middle| x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 0 \\ -2x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 3y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

$= \text{span}\left((1,0,-2), (0,1,3) \right)$

So $(1,0,-2)$, $(0,1,3)$ is a generator system is S_3.

c) $W_1 = \left\{ \begin{pmatrix} x-y+5z \\ 3x-z \\ 2x+y-7z \\ -x \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x,y,z \in \mathbb{R} \right\}$

$W_1 = \left\{ \begin{pmatrix} x \\ 3x \\ 2x \\ -x \end{pmatrix} + \begin{pmatrix} -y \\ 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 5z \\ -z \\ -7z \\ 0 \end{pmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$

$= \left\{ x \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$

$= \text{Span}\left(\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix} \right)$

So generator systes are the vectors

$\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix}$

4. Determine a finite gen. system for the following subspaces in $\mathbb{R}^3$.

a) $w_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; [2,-3,5]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$

The condition shows that

$$2x - 3y + 5z = 0 \Rightarrow y = \frac{2x+5z}{3}$$

$$W_1 = \left\{ \begin{pmatrix} x \\ \frac{2x+5z}{3} \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ \frac{2}{3} \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ \frac{5}{3} \\ 1 \end{pmatrix} \;\middle|\; x, z \in \mathbb{R} \right\} =$$

$$= \operatorname{Span}\left((1, 2/3, 0); (0, 5/3, 1) \right\}$$

$$= \operatorname{Span}\left((3,2,0); (0,5,3) \right).$$

So. gen system : $(3,2,0)$.
$(0,5,3)$.

b) $W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x - 2y + 3z = 0 \\ 2x - z = 0 \end{array} \right\}$$

$\underline{z = 2x}$

$x - 2y + 3z = 0$

$x - 2y + 6x = 0$

$7x - 2y = 0$

$y = \frac{7}{2} x$

$$\Rightarrow W_2 \left\{ \begin{pmatrix} x \\ \frac{7}{2} x \\ 2x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} =$$

$$= \left\{ x \begin{pmatrix} 1 \\ \frac{7}{2} \\ 2 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$= \text{Span}((1, \tfrac{7}{2}, 2))$

$= \text{Span}((2, 7, 4)).$

Gen. System is $(2, 7, 4)$.

c) $W_3 = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 2x - y + 2z = 0 \\ -4x + 2y + 4z = 0 \end{array} \right\}$

Observe that.

$$-4x + 2y + 4z = 0.$$
$$2x - y - 2z = 0$$

So we only have one equation here.

$W_3 = \{ (x, y, z) \in \mathbb{R}^3 \mid 2x - y - 2z = 0 \}$

$y = 2x - 2z.$

$$= \left\{ \begin{pmatrix} x \\ 2x-2z \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$W_3 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right)$$

$\Rightarrow$ gen system $= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$