Problem Set 1

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1. For each of the following λ-terms write down an α-equivalent term where all variable have different names:
   a. $\lambda x.(\lambda y.x\ y)\ x \to \lambda a.(\lambda b.\ a\ b)\ a$
   b. $\lambda x.(\lambda x.x)\ x \to \lambda a.(\lambda a.\ a)\ a$
   c. $\lambda x.\lambda y.x\ y \to \lambda a.\lambda b.\ a\ b$
   d. $\lambda x.\ x(\lambda x.\ x) \to \lambda a.\ a(\lambda a.\ a)$
   e. $\lambda x.(\lambda x.x)\ x \to \lambda a.(\lambda a.\ a)\ a$
   f. $(\lambda x.\lambda y.\ y)\ z\ x \to (\lambda a.\lambda b.\ b)\ c\ a$
2. Write down evaluation sequence for the following λ-terms:
   a. $(\lambda x.\ \lambda y.\ x)\ y\ z$
      i. $\to (\lambda x.\ \lambda y_0.\ x)\ y\ z$
      ii. $\to ([x \to y](\lambda y_0.\ x))\ z$
      iii. $\to (\lambda y_0.\ y)\ z$
      iv. $\to y$
   b. $(\lambda x.\ \lambda y.\ x)\ (\lambda z.\ y)\ z\ w$
      i. $\to ([x\ \to\ (\lambda z_0.\ y)])\ (\lambda y_0.\ x)\ z\ w$
      ii. $\to (\lambda y_0\ \lambda z_0.\ y)\ z\ w$
      iii. $\to (\lambda z_0.\ y)\ w$
      iv. $\to y$
   c. $(\lambda b.\lambda f.\lambda t.\ b\ t\ f)(\lambda f.\lambda t.\ t)$
      i. $\to ([b \to (\lambda f.\lambda t.\ t)])(\lambda f.\lambda t.\ b\ t\ f)$
      ii. $\to (\lambda f.\lambda t.\ (\lambda f.\lambda t.t)\ t\ f)$
      iii. $\to \lambda f.\lambda t.f$
   d. $(\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.\ b\ t\ f)\ (\lambda f.\lambda t.\ t)$
      i. $\to (\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)\ (\lambda f.\lambda t.\ t)$
      ii. $\to ([s\ \to\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)])(\lambda z.\ s\ (s\ z))\ (\lambda f.\lambda t.\ t)$
      iii. $\to (\lambda z.\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)\ ((\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)\ z))\ (\lambda f.\lambda t.\ t)$
      iv. $\to (\lambda z.\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)\ (\lambda f_0.\lambda t_0.\ z\ t_0\ f_0))\ (\lambda f.\lambda t.\ t)$

v. $\to$ $(\lambda z.\ (\lambda f_1.\lambda t_1.\ (\lambda f_0.\lambda t_0.\ z\ t_0\ \ f_0)\ t_1\ \ f_1))\ (\lambda f.\lambda t.\ t)$

vi. $\to$ $(\lambda f_1.\lambda t_1.\ (\lambda f_0.\lambda t_0.\ (\lambda f.\lambda t.\ t)\ t_0\ \ f_0)\ t_1\ \ f_1))$

vii. $\to$ $(\lambda f_1.\lambda t_1.\ (\lambda f_0.\lambda t_0.\ f_0)\ t_1\ \ f_1))$

viii. $\to$ $(\lambda f_1.\lambda t_1.\ t_1))$

e. $(\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.\ b\ t\ f)\ (\lambda f.\lambda t.\ t)$

i. $\to(\lambda s_0.\lambda z_0.\ s_0\ (s_0\ z_0))\ (\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)\ (\lambda f.\lambda t.\ t)$

ii. $\to(\lambda z_0.\ (\lambda s.\lambda z.\ s\ (s\ z))((\lambda s.\lambda z.\ s(sz))z_0))(\lambda b_0.\lambda f_0.\lambda t_0.b_0t_0f_0)(\lambda f.\lambda t.\ t)$

iii. $\to((\lambda s.\lambda z.\ s\ (s\ z))((\lambda s.\lambda z.\ s\ (s\ z))\ (\lambda b_0.\lambda f_0.\lambda t_0.\ b_0\ t_0\ f_0)))\ (\lambda f.\lambda t.\ t)$

iv. $\to$
$((\lambda s.\lambda z.s(sz))((\lambda z.(\lambda b_0.\lambda f_0.\lambda t_0.b_0t_0f_0)((\lambda b_0.\lambda f_0.\lambda t_0.b_0t_0f_0)z))))(\lambda f.\lambda t.t)$

v. $\to$
$((\lambda s.\lambda z.s\ (s\ z))((\lambda z.(\lambda b_0.\lambda f_0.\lambda t_0.b_0\ t_0\ f_0)((\lambda f_0.\lambda t_0.\ z\ t_0\ f_0)))))(\lambda f.\lambda t.\ t)$

vi. $\to$ $((\lambda s.\lambda z.s\ (s\ z))((\lambda z.(\lambda f_1.\lambda t_1.(\lambda f_0.\lambda t_0.\ z\ t_0\ f_0)\ t_1\ f_1))))(\lambda f.\lambda t.\ t)$

vii. $\to$ $((\lambda s.\lambda z.s\ (s\ z))((\lambda z.(\lambda f_1.\lambda t_1.(\lambda t_0.\ z\ t_0\ t_1)\ f_1))))(\lambda f.\lambda t.\ t)$

viii. $\to$ $((\lambda s.\lambda z.s\ (s\ z))((\lambda z.(\lambda f_1.\lambda t_1.\ z\ t_1\ f_1))))(\lambda f.\lambda t.\ t)$

ix. $\to$ $((\lambda z_0.\ (\lambda z.(\lambda f_1.\lambda t_1.\ z\ t_1\ f_1))\ ((\lambda z.(\lambda f_1.\lambda t_1.\ z\ t_1\ f_1))\ z_0)))(\lambda f.\lambda t.\ t)$

x. $\to$ $(\lambda z_0.\ (\lambda z.(\lambda f_1.\lambda t_1.\ z\ t_1\ f_1))\ (\lambda f_1.\lambda t_1.\ z_0\ t_1\ f_1))(\lambda f.\lambda t.\ t)$

xi. $\to$ $(\lambda z_0.\ (\lambda f_1.\lambda t_1.\ (\lambda f_1.\lambda t_1.\ z_0\ t_1\ f_1)\ t_1\ f_1)\ )(\lambda f.\lambda t.\ t)$

xii. $\to$ $(\lambda z_0.\ (\lambda f_1.\lambda t_1.\ (\lambda t_2.\ z_0\ t_2\ t_1)\ f_1)\ )(\lambda f.\lambda t.\ t)$

xiii. $\to$ $(\lambda z_0.\ (\lambda f_1.\lambda t_1.\ z_0\ f_1\ t_1)\ )(\lambda f.\lambda t.\ t)$

xiv. $\to$ $(\lambda f_1.\lambda t_1.\ (\lambda f.\lambda t.\ t)\ f_1\ t_1)$

xv. $\to$ $(\lambda f_1.\lambda t_1.\ (\lambda t.\ t)\ t_1)$

xvi. $\to$ $(\lambda f_1.\lambda t_1.\ t_1)$

3. Recall that the Church booleans we have the following encoding

$$tru = \lambda t.\lambda f.t$$

$$fls = \lambda t.\lambda f.f$$

(a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a λ-term
for logical implication implies of two Church booleans.

$$imp = \lambda x.\lambda y.\ x.\ y.\ tru$$

(b) Verify your implementation of implies by writing down evaluation sequence for the term implies $fls\, tru$.

$$
\begin{aligned}
& imp\ fls\ tru = (\lambda x\ \lambda y.\ x\ y\ tru)\ fls\ tru \\
& \rightarrow (\lambda y.\ fls\ y\ tru)\ tru \\
& \rightarrow fls\ tru\ tru \\
& \rightarrow (\lambda t. \lambda f.\ f)\ tru\ tru \\
& \rightarrow (\lambda f.\ f)\ tru \\
& \rightarrow tru
\end{aligned}
$$

4. Recall that with Church numerals we have the following encoding

$$
\begin{aligned}
& c_0 = \lambda s. \lambda z. z \\
& c_1 = \lambda s. \lambda z. s\ z \\
& c_2 = \lambda s. \lambda z. s\ (s\ z) \\
& c_3 = \lambda s. \lambda z. s\ (s\ (s\ z)) \\
& \ldots
\end{aligned}
$$

(a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a single
λ-term for each of the following functions on natural numbers:

$n \rightarrow 2n + 1$

$n \rightarrow n^2 + 1$

$n \rightarrow 2^n + 1$

$n \rightarrow 2^{n+1}$

(b) Verify each your implementations of the functions above by writing down evaluation sequence for each of them, when applied to c_2