Reaction Wheel Pendulum Dynamics

To obtain the equations of motion for the system, a Lagrangian approach is used. First, the energies and virtual work of the system are computed:

$$\begin{aligned}
KE_1 &= \frac{1}{2} m_1 \left(\frac{l_1}{2}\dot{q_1}\right)^2 + \frac{1}{2} J_1 \dot{q_1}^2 \\
KE_2 &= \frac{1}{2} m_2 \left(l_1 \dot{q_1}\right)^2 + \frac{1}{2} J_2 \left(\dot{q_1} + \dot{q_2}\right)^2 \\
PE_1 &= m_1\, g\, \frac{l_1}{2} \cos(q_1) \\
PE_2 &= m_2\, g\, l_1 \cos(q_1) \\
dW &= \tau_{motor}\, dq_2
\end{aligned}$$

where

KE_1 = kinetic energy of the link
KE_2 = kinetic energy of the flywheel
PE_1 = potential energy of the link, link pivot point is at zero height
PE_2 = potential energy of the flywheel, link pivot point is at zero height
m_1 = mass of the link
m_2 = mass of the flywheel
J_1 = rotational inertia of the link
J_2 = rotational inertia of the flywheel
q_1 = angle of the link, tared at the directly inverted position
q_2 = angle of the flywheel with respect to the link
l_1 = length of the link
g = gravitational acceleration

Then, the following Lagrangians are evaluated in terms of the two generalized coordinates, q_1 and q_2:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right) - \frac{\partial L}{\partial q_1} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_2}}\right) - \frac{\partial L}{\partial q_2} &= \tau
\end{aligned}$$

From simplifying these equations, and linearizing about $q_1 = 0$, the (messy) equations of motion are finally derived:

$$
\begin{aligned}
\ddot{q}_1 &= \frac{(m_1 g \frac{l_1}{2} + m_2 g l_1) q_1 - \tau}{m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2} \\
\ddot{q}_2 &= \frac{(m_1 g \frac{l_1}{2} + m_2 g l_1) q_1 - \frac{m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2 + J_2}{J_2} \tau}{-m_1 \frac{l_1^2}{4} - J_1 - m_2 l_1^2}
\end{aligned}
$$

Instead of clearing denominators, we'll make a few substitutions. Letting

$$
\begin{aligned}
a &= m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2 + J_2, \text{ and} \\
b &= m_1 g \frac{l_1}{2} + m_2 g l_1,
\end{aligned}
$$

the equations simplify to:

$$
\begin{aligned}
\ddot{q}_1 &= \frac{b q_1 - \tau}{a - J_2} \\
\ddot{q}_2 &= \frac{b q_1 - \frac{a}{J_2} \tau}{J_2 - a}
\end{aligned}
$$

which, when written in state space form, takes on the following representation:

$$
\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b}{J_2 - a} & 0 & 0 & 0 \\ \frac{b}{J_2 - a} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{a - J_2} \\ \frac{-a}{J_2 (J_2 - a)} \end{pmatrix} \tau_{motor}
$$

This state-space form is ultimately used in the controller simulations in Python using the Control Systems Library (https://pypi.org/project/control/).