# Reaction Wheel Pendulum Dynamics

To obtain the equations of motion for the system, a Lagrangian approach is used. First, the energies and virtual work of the system are computed:

$$KE_1 = \frac{1}{2}m_1 \left(\frac{l_1}{2}\dot{q}_1\right)^2 + \frac{1}{2}J_1\dot{q}_1{}^2$$

$$KE_2 = \frac{1}{2}m_2 \left(l_1\dot{q}_1\right)^2 + \frac{1}{2}J_2 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

$$PE_1 = m_1\, g\, \frac{l_1}{2}\cos(q_1)$$

$$PE_2 = m_2\, g\, l_1 \cos(q_1)$$

$$dW = \tau_{motor}\, dq_2$$

where

$KE_1$ = kinetic energy of the link
$KE_2$ = kinetic energy of the flywheel
$PE_1$ = potential energy of the link, link pivot point is at zero height
$PE_2$ = potential energy of the flywheel, link pivot point is at zero height
$m_1$   = mass of the link
$m_2$   = mass of the flywheel
$J_1$   = rotational inertia of the link
$J_2$   = rotational inertia of the flywheel
$q_1$   = angle of the link, tared at the directly inverted position
$q_2$   = angle of the flywheel with respect to the link
$l_1$   = length of the link
$g$   = gravitational acceleration

Then, the following Lagrangians are evaluated in terms of the two generalized coordinates, $q_1$ and $q_2$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \tau$$

From simplifying these equations, the (messy) equations of motion are finally derived:

$$\ddot{q}_1 = \frac{(m_1 g \frac{l_1}{2} + m_2 g l_1) q_1 - \tau}{m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2}$$

$$\ddot{q}_2 = \frac{(m_1 g \frac{l_1}{2} + m_2 g l_1) q_1 - \frac{m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2 + J_2}{J_2} \tau}{-m_1 \frac{l_1^2}{4} - J_1 - m_2 l_1^2}$$

Instead of clearing denominators, we'll make a few substitutions. Letting

$$a = m_1 \frac{l_1^2}{4} + J_1 + m_2 l_1^2 + J_2, \text{ and}$$

$$b = m_1 g \frac{l_1}{2} + m_2 g l_1,$$

the equations simplify to:

$$\ddot{q}_1 = \frac{b q_1 - \tau}{a - J_2}$$

$$\ddot{q}_2 = \frac{b q_1 - \frac{a}{J_2} \tau}{J_2 - a}$$

which, when written in state space form, takes on the following representation:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b}{J_2 - a} & 0 & 0 & 0 \\ \frac{b}{J_2 - a} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{a - J_2} \\ \frac{-a}{J_2(J_2 - a)} \end{pmatrix} \tau_{motor}$$

This state-space form is ultimately used in the controller simulations in Python using the Control Systems Library (link).