

Support Vector Machines: Training with Stochastic Gradient Descent

Machine Learning
Fall 2018



Support vector machines

- Training by maximizing margin
- The SVM objective
- **Solving the SVM optimization problem**
- Support vectors, duals and kernels

SVM objective function

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

Regularization term:

- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

Empirical Loss:

- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

A **hyper-parameter** that controls the tradeoff between a large margin and a small hinge-loss

Support vector machines

- Training by maximizing margin
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- Solving the SVM optimization problem
- Support vectors, duals and kernels

Outline: Training SVM by optimization

1. Review of convex functions and gradient descent
2. Stochastic gradient descent
3. Gradient descent vs stochastic gradient descent
4. Sub-derivatives of the hinge loss
5. Stochastic sub-gradient descent for SVM
6. Comparison to perceptron

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Solving the SVM optimization problem

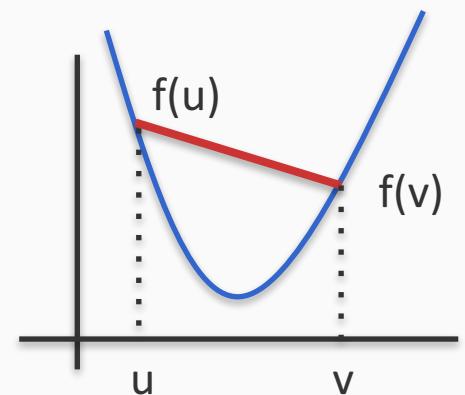
$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

This function is **convex** in \mathbf{w}

Recall: Convex functions

A function f is **convex** if for every u, v in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$



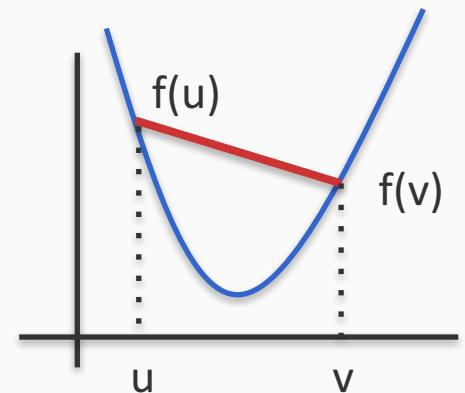
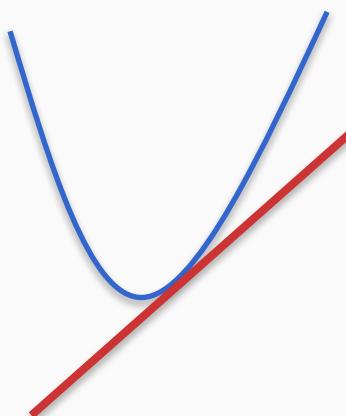
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From geometric perspective

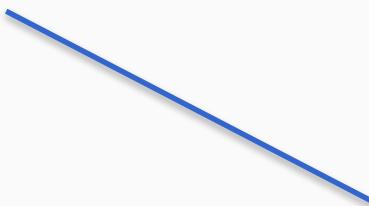
Every tangent plane lies below the function



Convex functions

$$f(x) = -x$$

Linear functions



$$f(x) = x^2$$

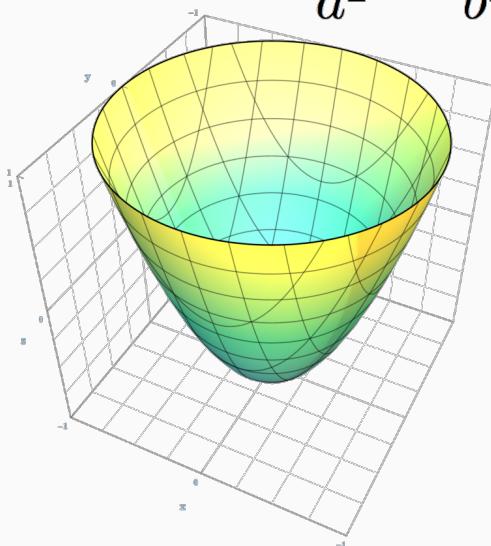


$$f(x) = \max(0, x)$$

max is convex



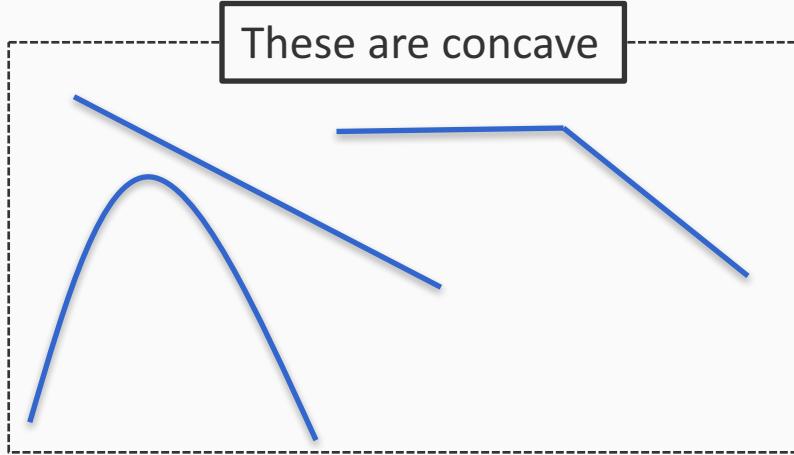
$$f(x_1, x_2) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$$



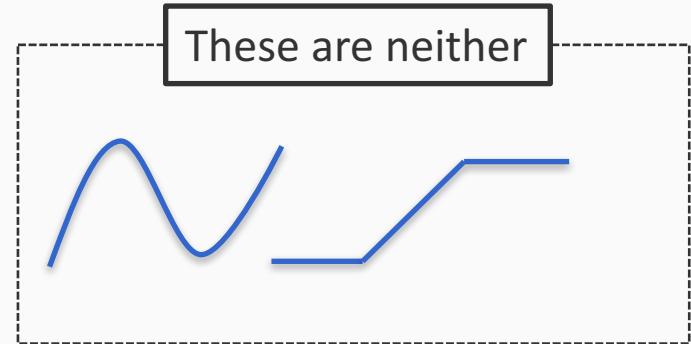
Some ways to show that a function is convex:

1. Using the definition of convexity
2. Showing that the second derivative is positive (for one dimensional functions)
3. Showing that the second derivative is positive semi-definite (for vector functions)

Not all functions are convex



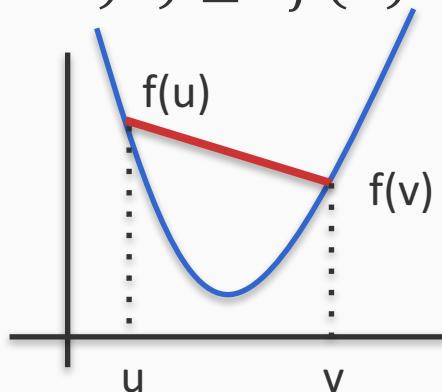
$$f(\lambda \mathbf{u} + (1 - \lambda) \mathbf{v}) \geq \lambda f(\mathbf{u}) + (1 - \lambda) f(\mathbf{v})$$



Convex functions are convenient

A function f is **convex** if for every \mathbf{u}, \mathbf{v} in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}) \leq \lambda f(\mathbf{u}) + (1 - \lambda)f(\mathbf{v})$$



In general: Necessary condition for x to be a minimum for the function f is $\nabla f(x) = 0$

For convex functions, this is both necessary *and* sufficient

Solving the SVM optimization problem

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

This function is convex in \mathbf{w}

- This is a quadratic optimization problem because the objective is quadratic
- Older methods: Used techniques from Quadratic Programming
 - Very slow
- No constraints, can use *gradient descent*
 - Still very slow!

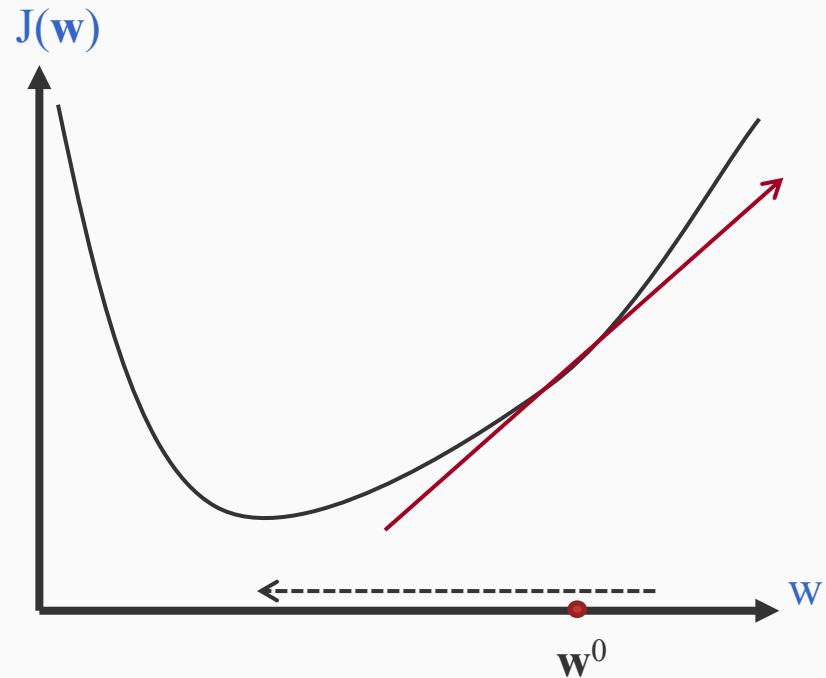
We are trying to minimize

Gradient descent

General strategy for minimizing a function $J(\mathbf{w})$

- Start with an initial guess for \mathbf{w} , say \mathbf{w}^0
- Iterate till convergence:
 - Compute the gradient of the gradient of J at \mathbf{w}^t
 - Update \mathbf{w}^t to get \mathbf{w}^{t+1} by taking a step in the opposite direction of the gradient

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$



Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction

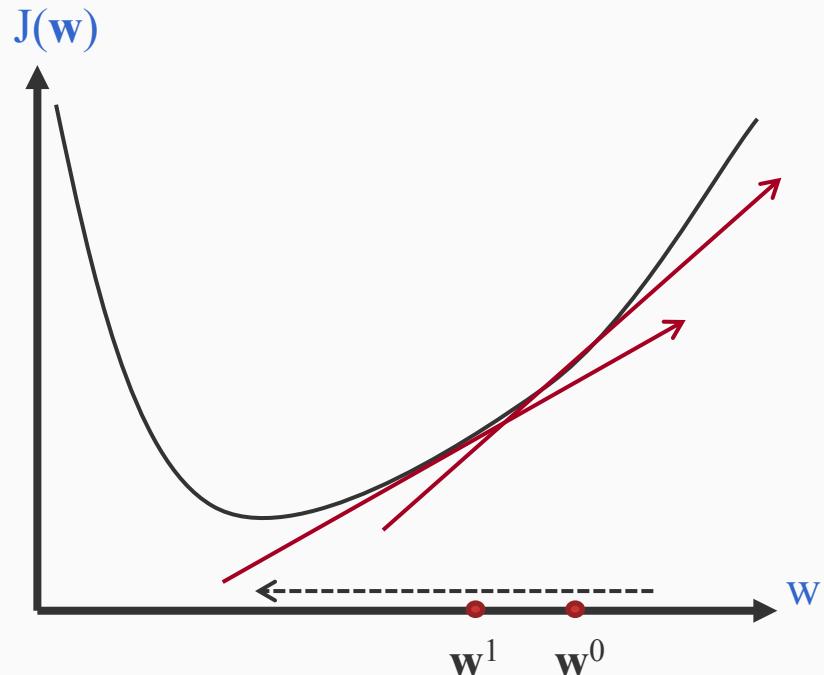
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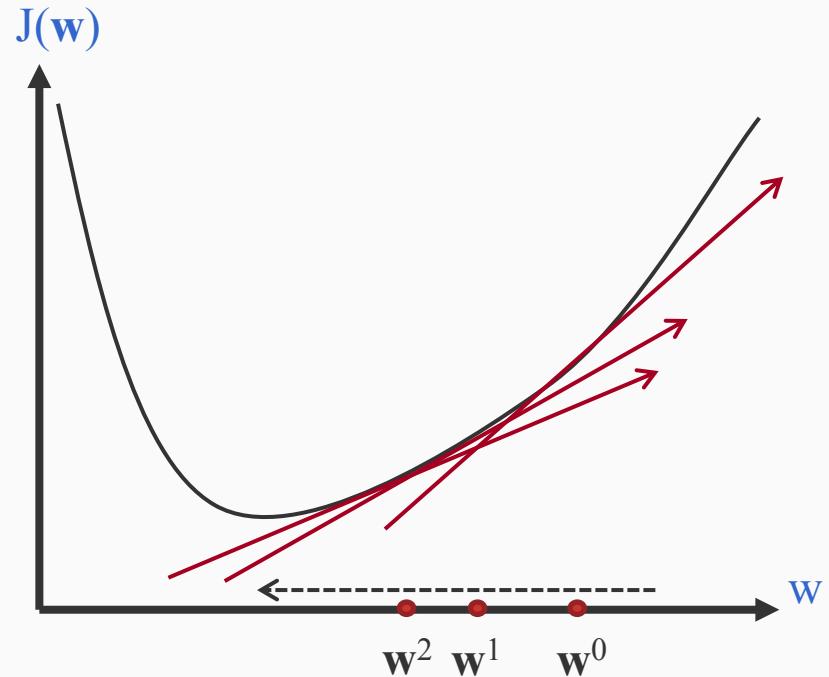
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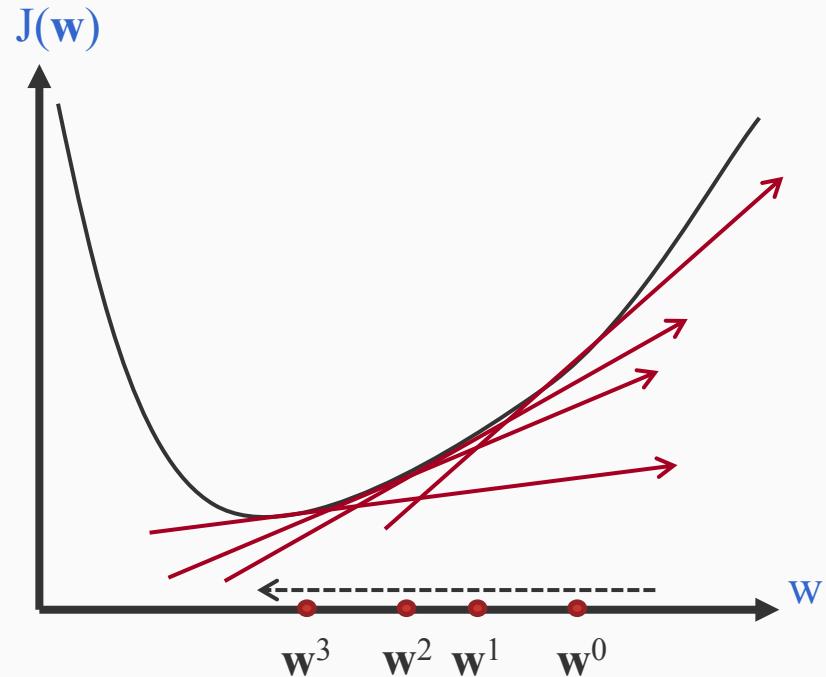
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Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction

Gradient descent for SVM

1. Initialize \mathbf{w}^0
2. For $t = 0, 1, 2, \dots$

1. Compute gradient of $J(\mathbf{w})$ at \mathbf{w}^t . Call it $\nabla J(\mathbf{w}^t)$

2. Update \mathbf{w} as follows:

$$\mathbf{w}^{t+1} = \mathbf{w}^t - r \nabla J(\mathbf{w}^t)$$

r : Called the learning rate .

We are trying to minimize

$$J(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

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Gradient of the SVM objective requires summing over the entire training set

Slow, does not really scale

r : Called the learning rate

$$J(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

Stochastic gradient descent for SVM

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1, 1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 ... T:
 3. Return final \mathbf{w}

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This algorithm is guaranteed to converge to the minimum of J if γ_t is small enough.
Why? The objective $J(\mathbf{w})$ is a **convex** function

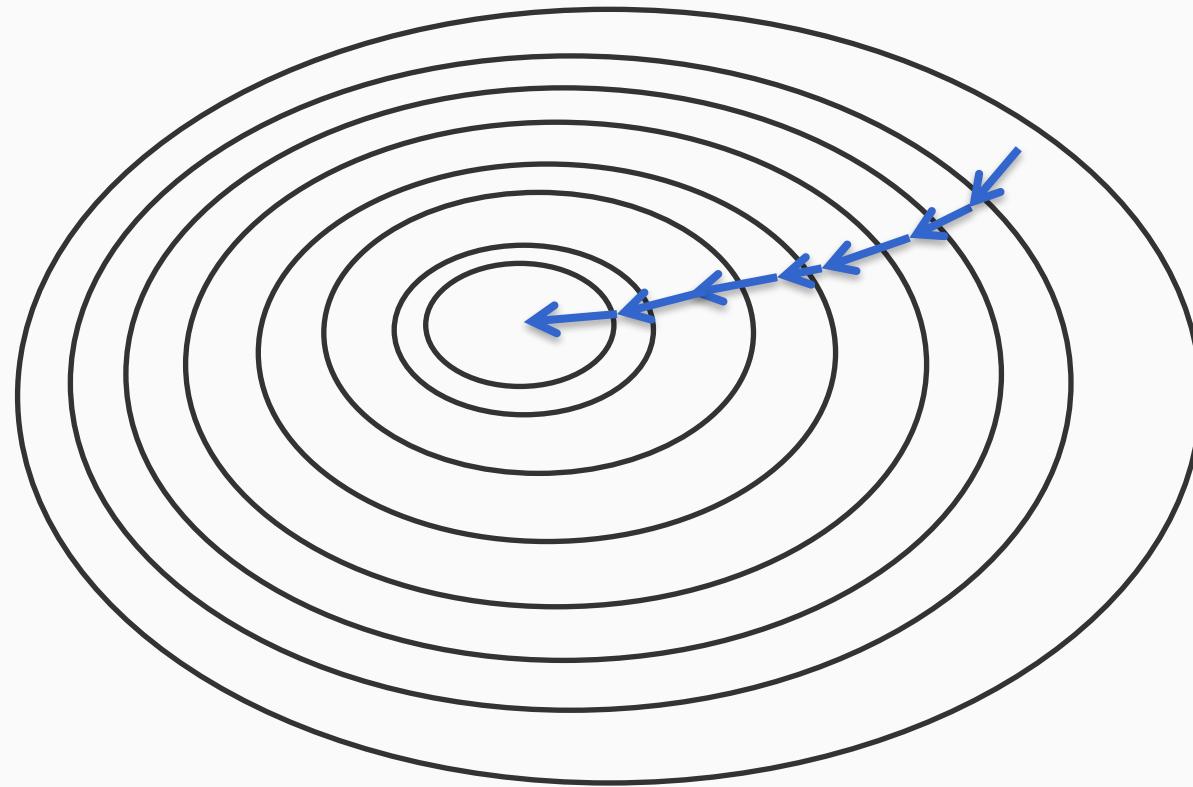
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- ✓ Stochastic gradient descent

3. Gradient descent vs stochastic gradient descent

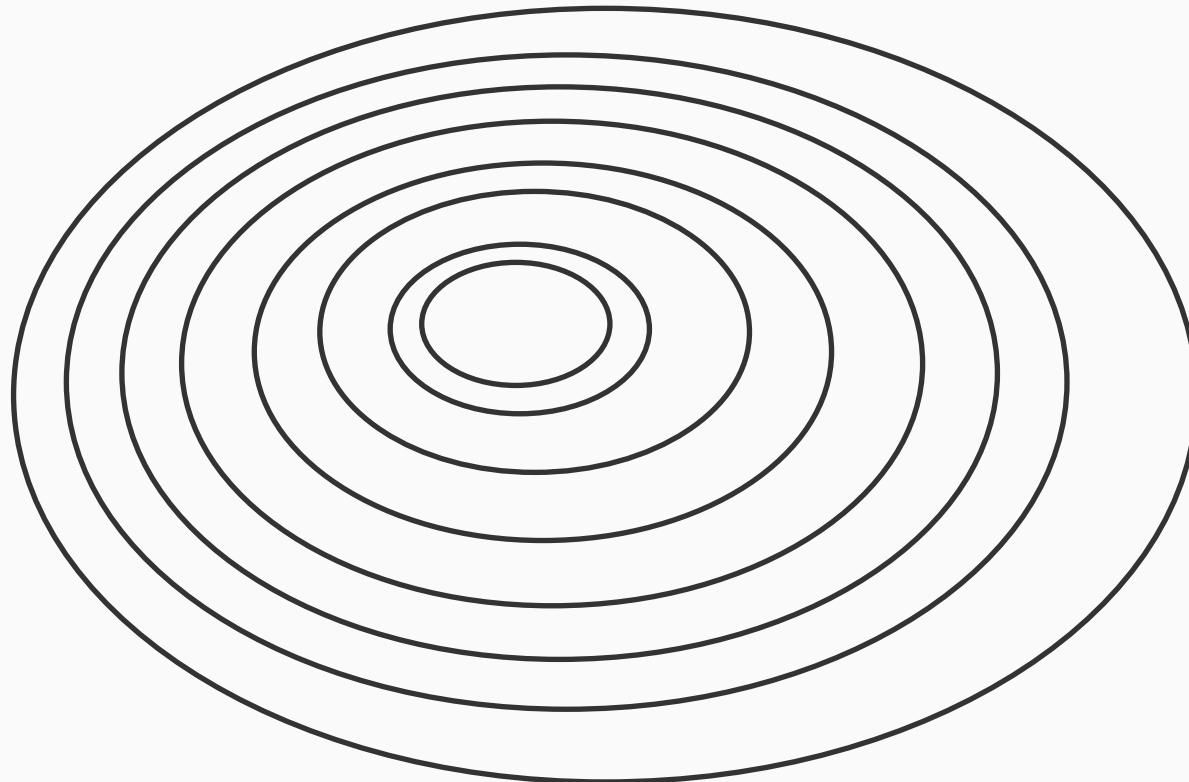
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Gradient Descent vs SGD



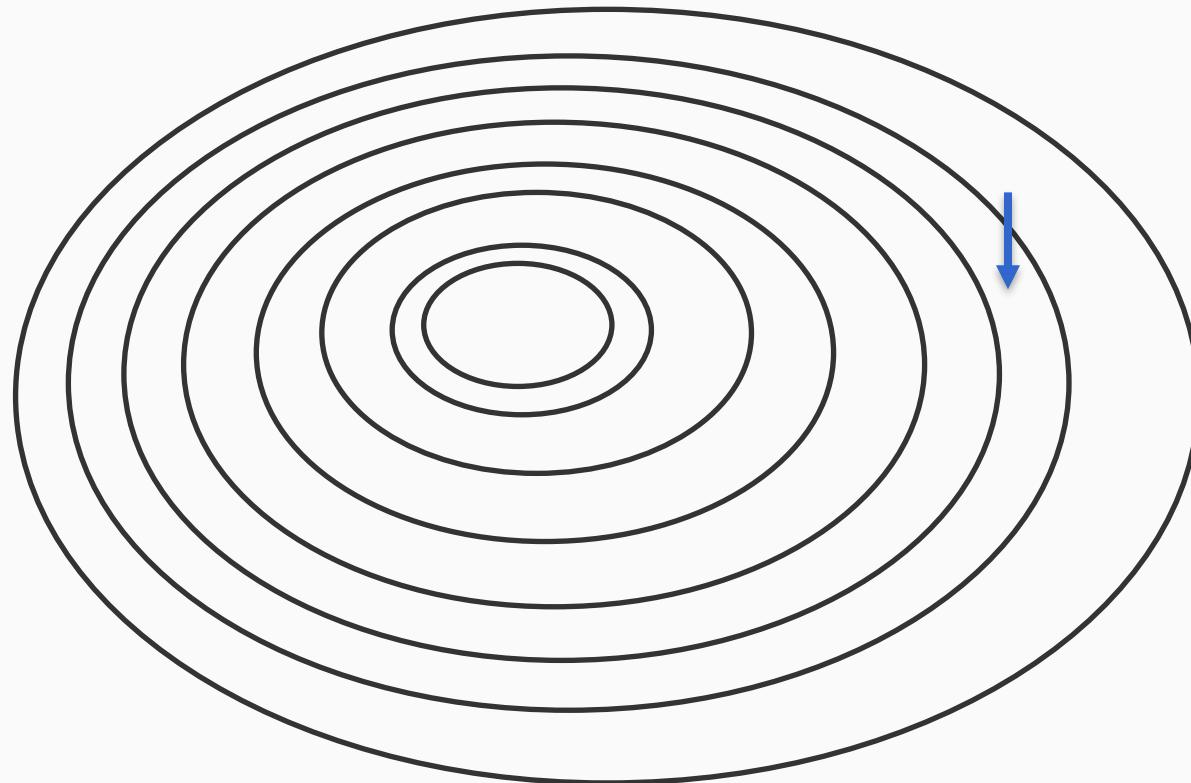
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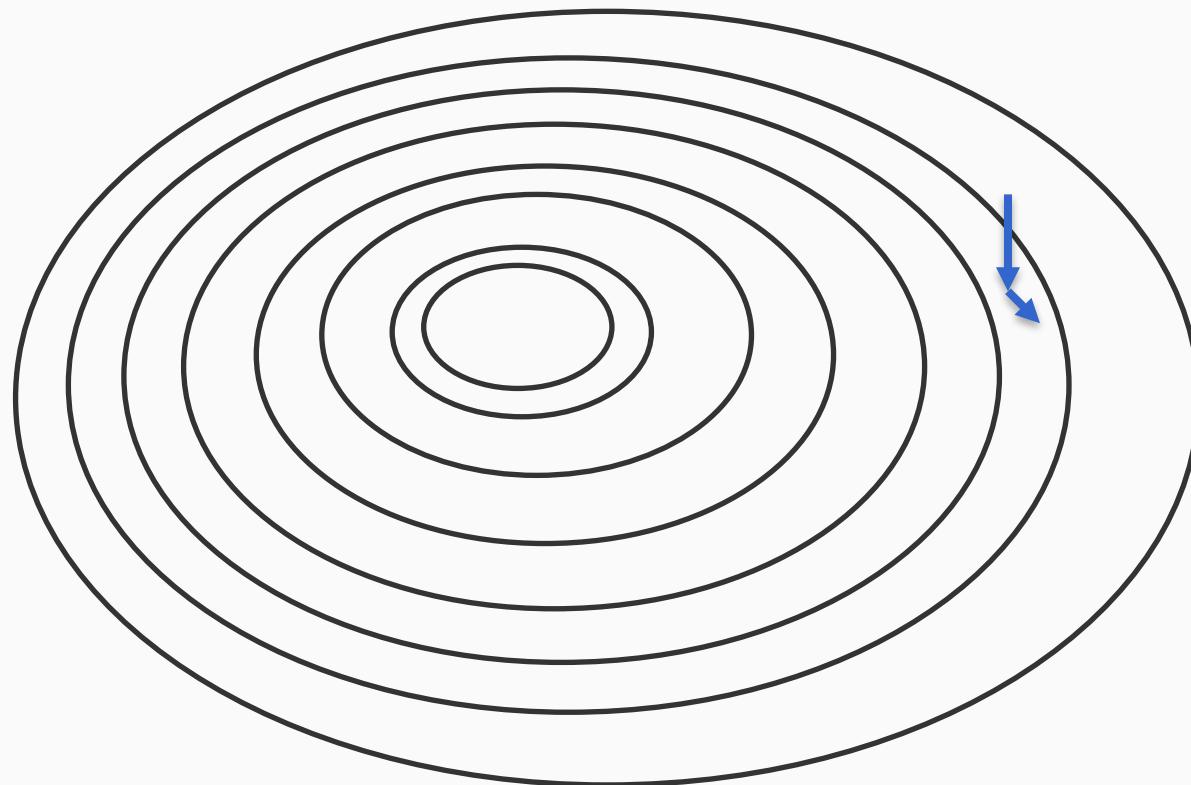
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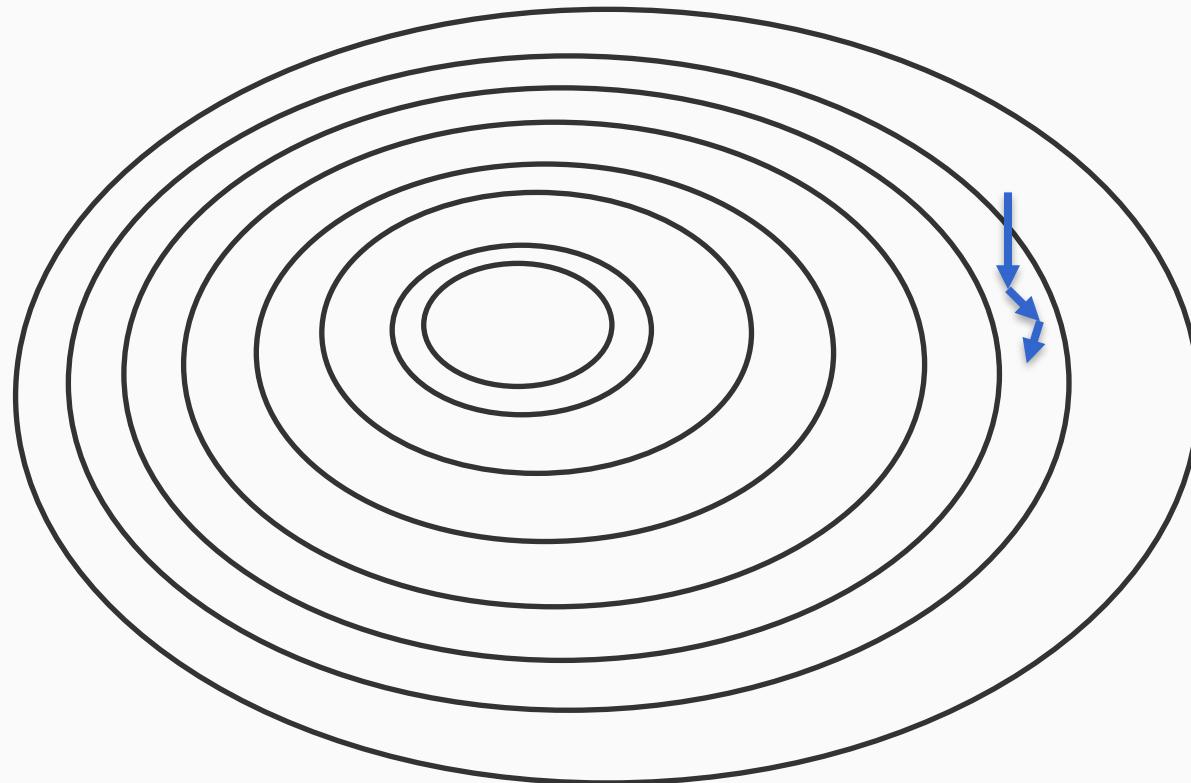
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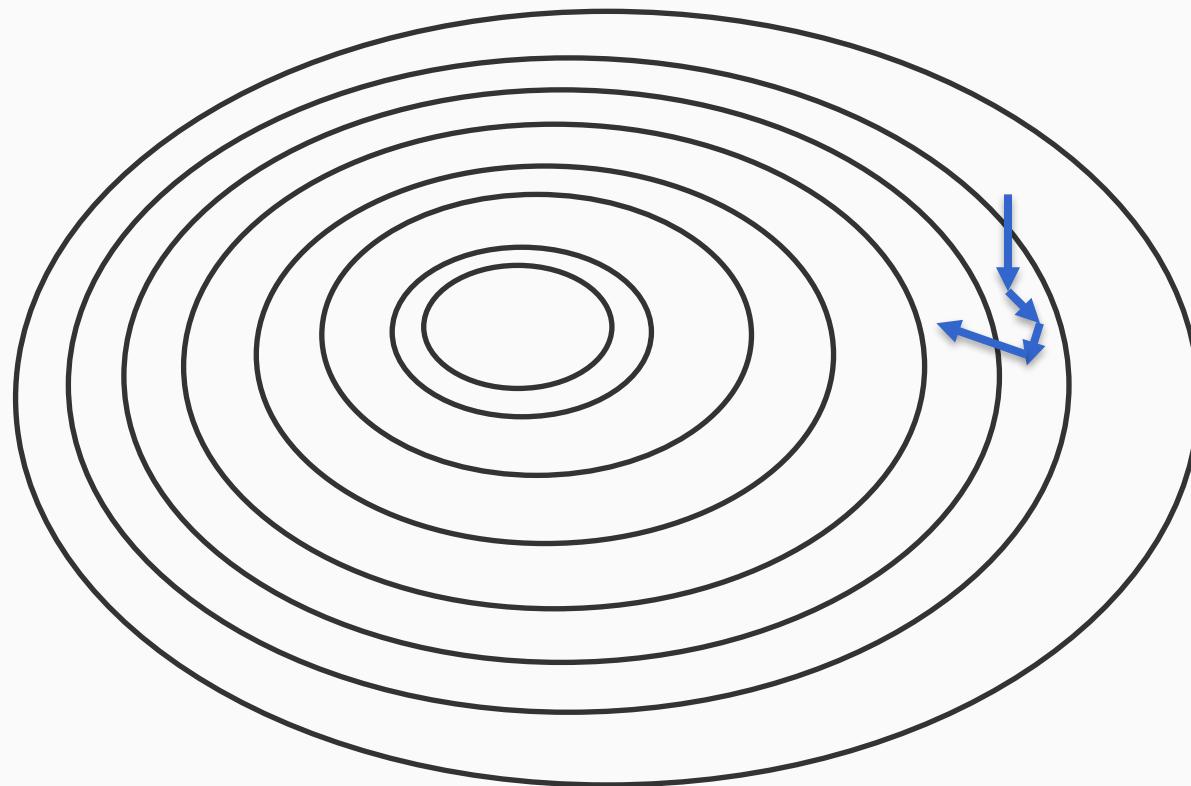
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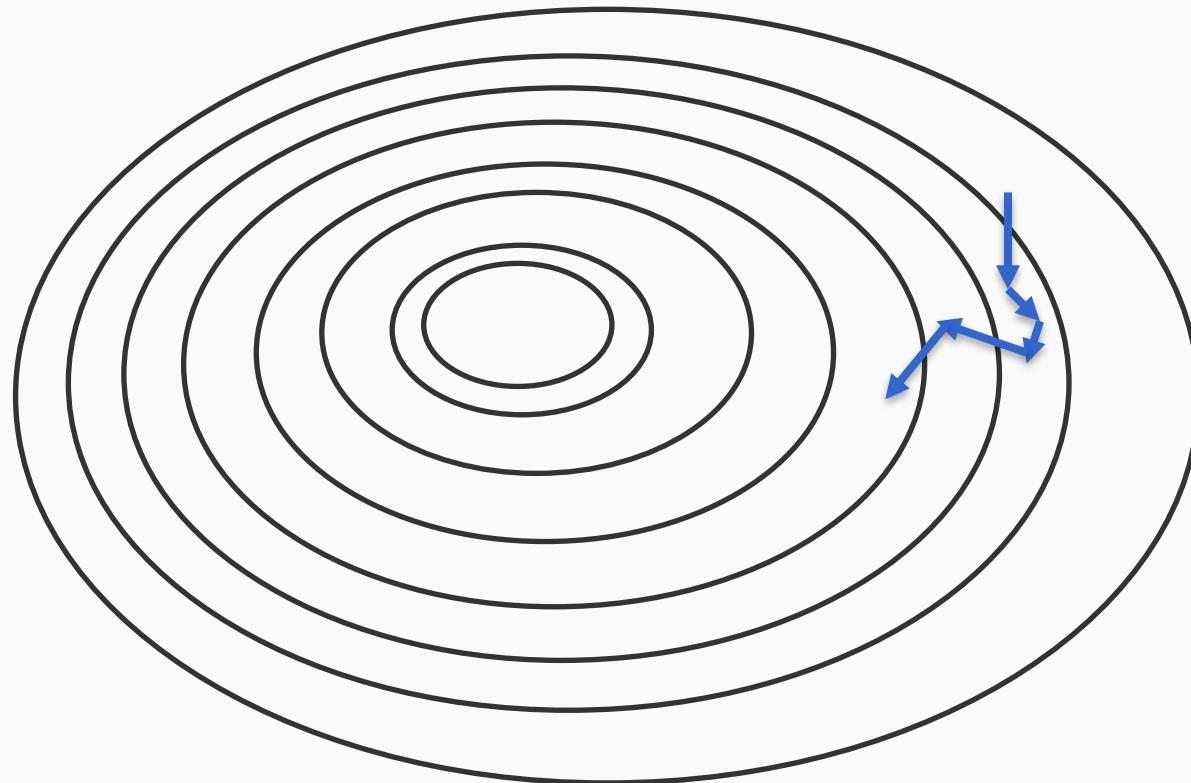
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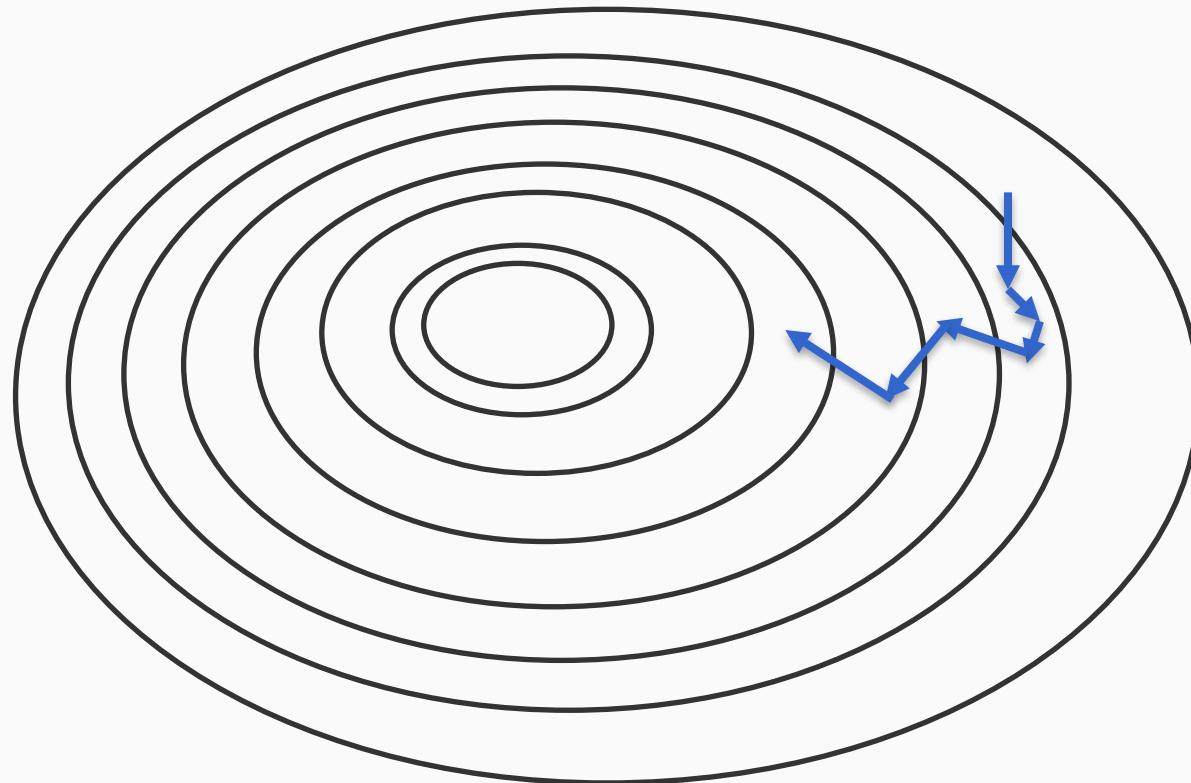
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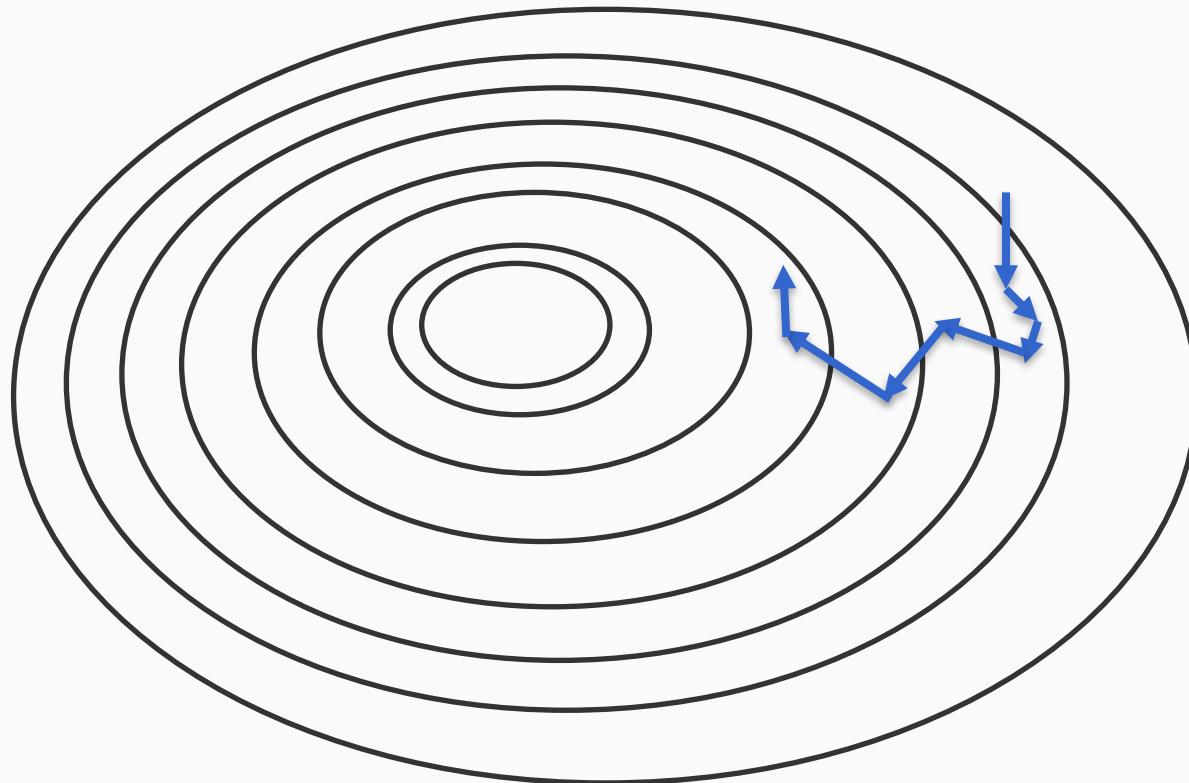
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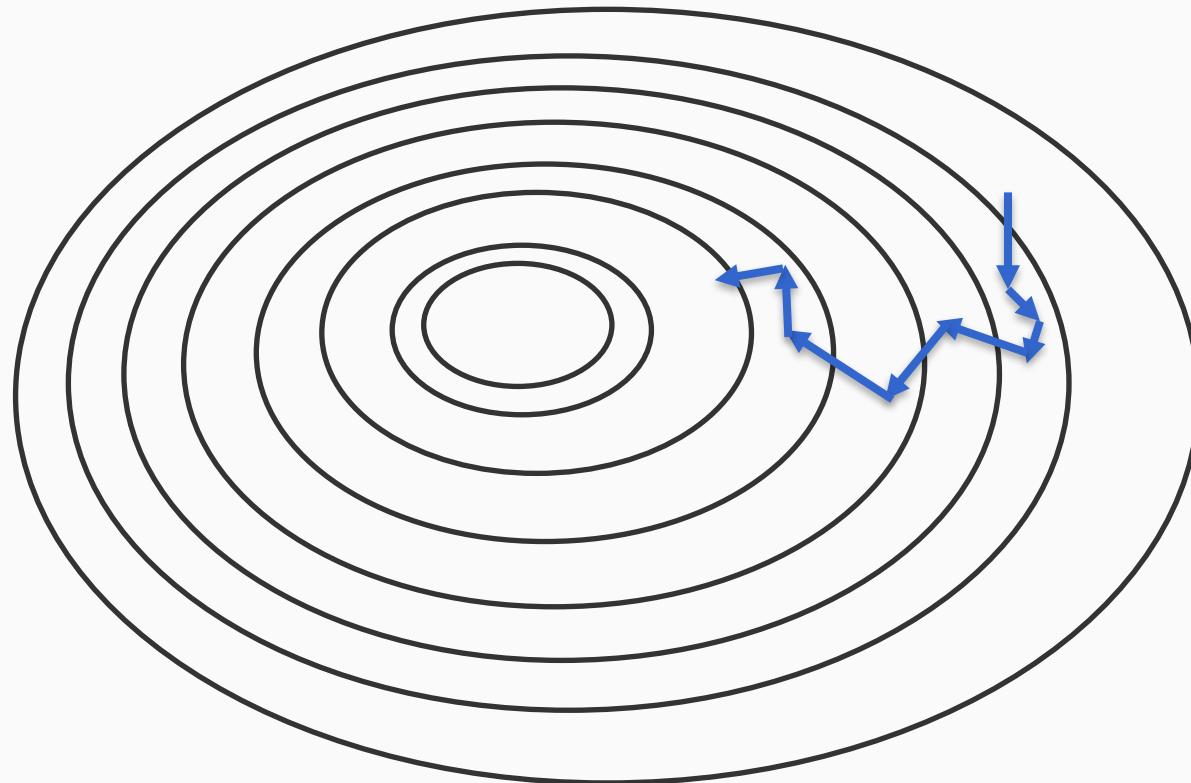
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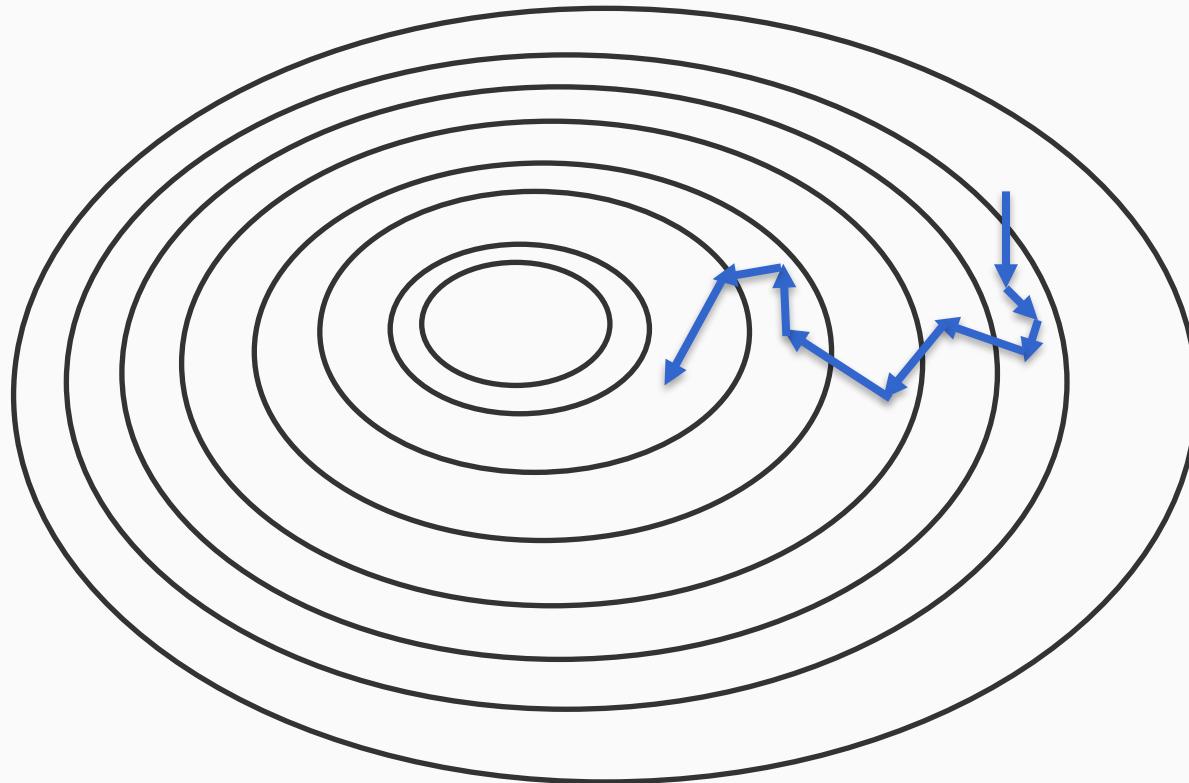
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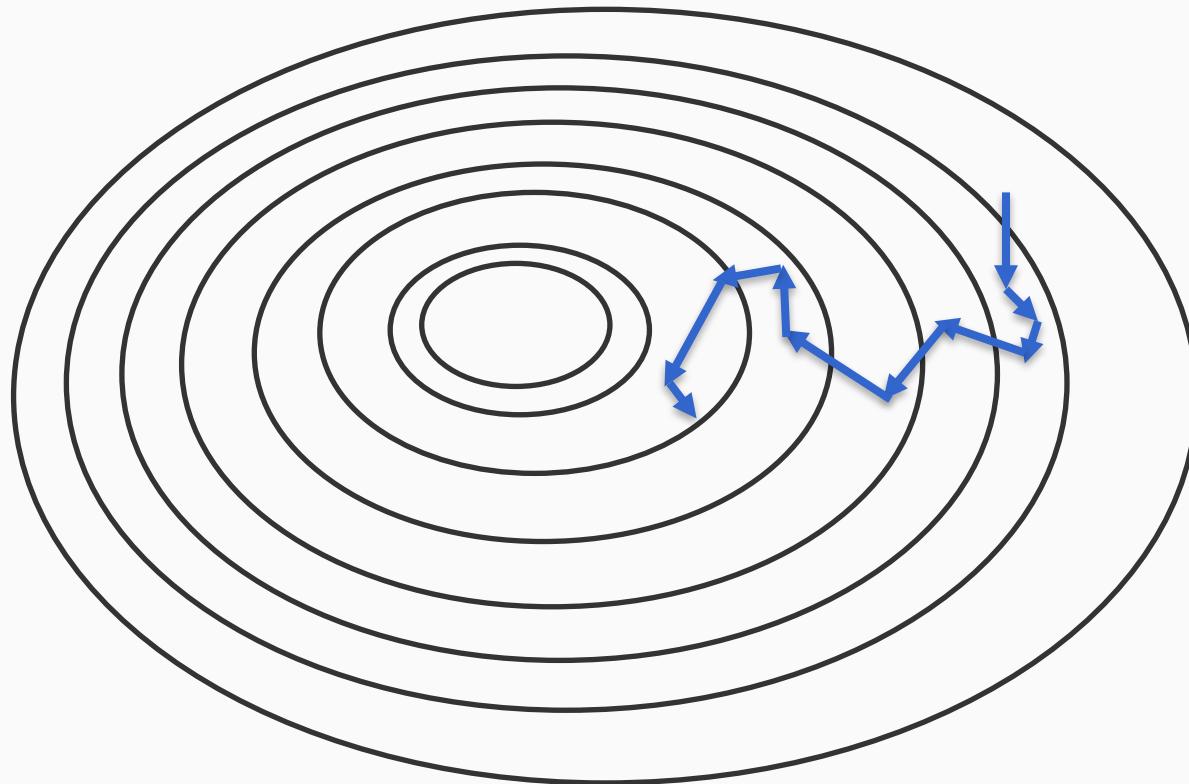
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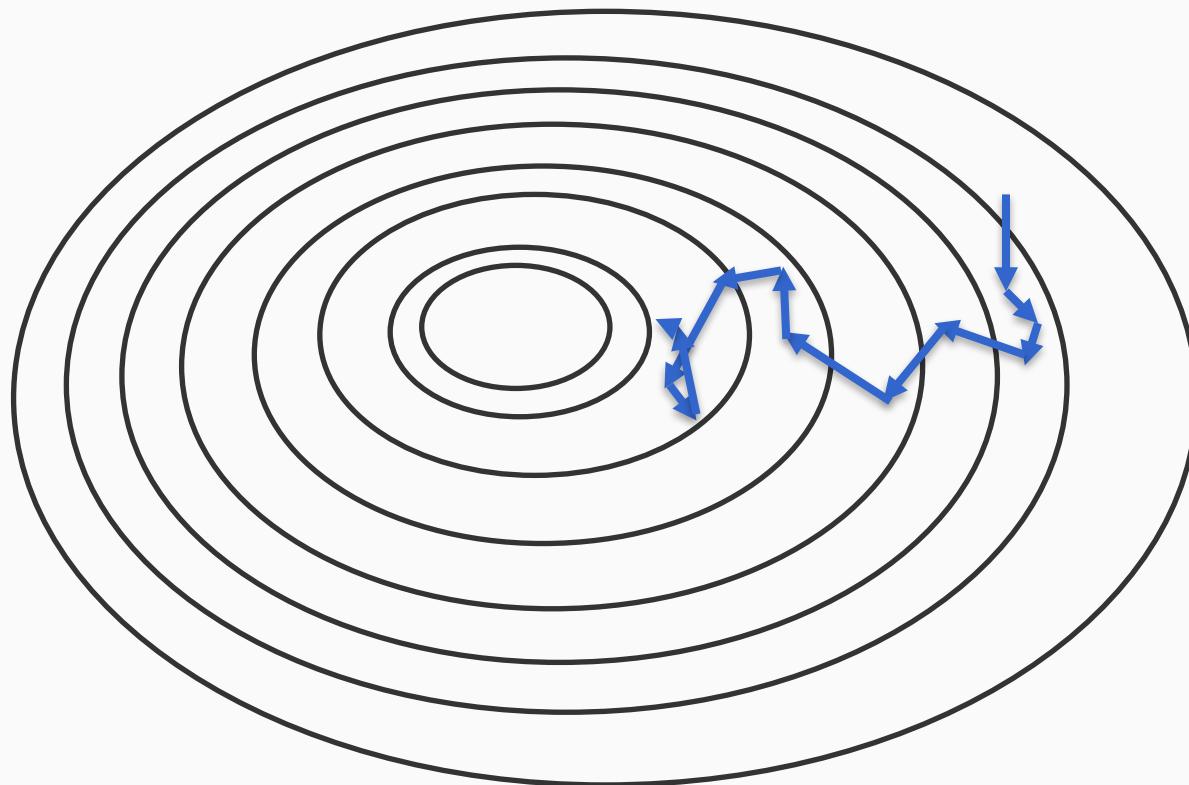
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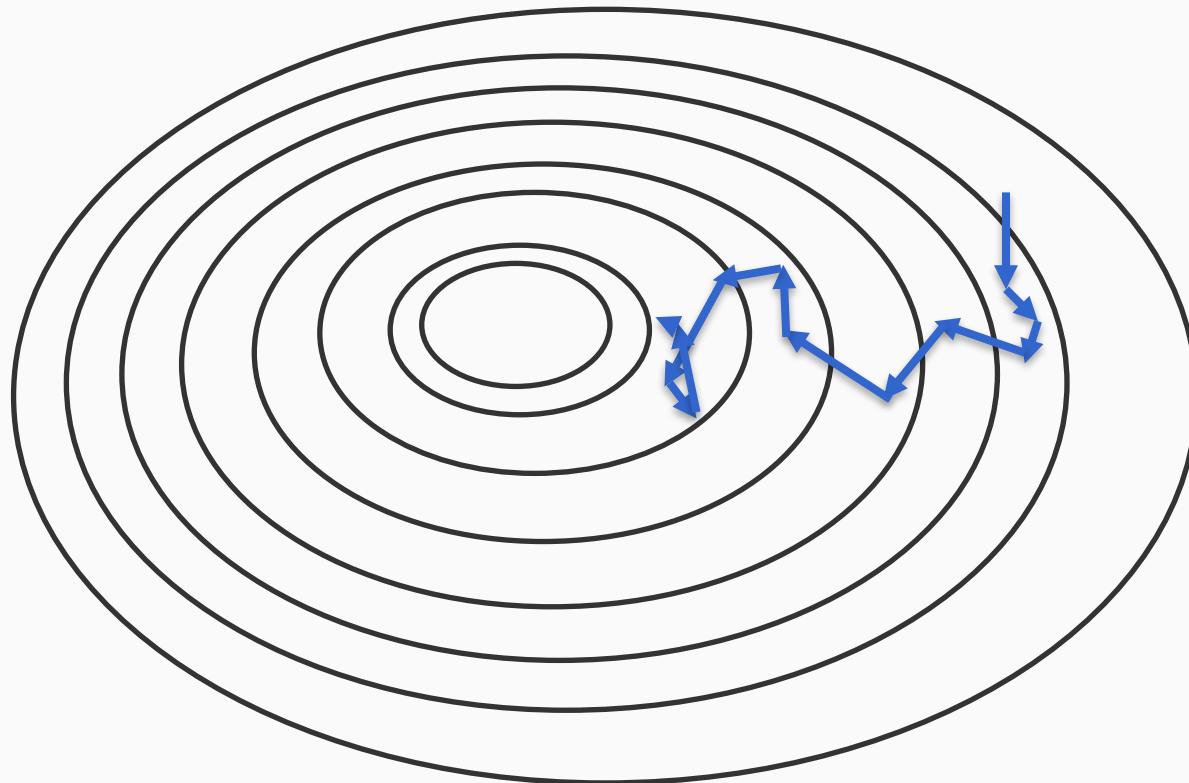
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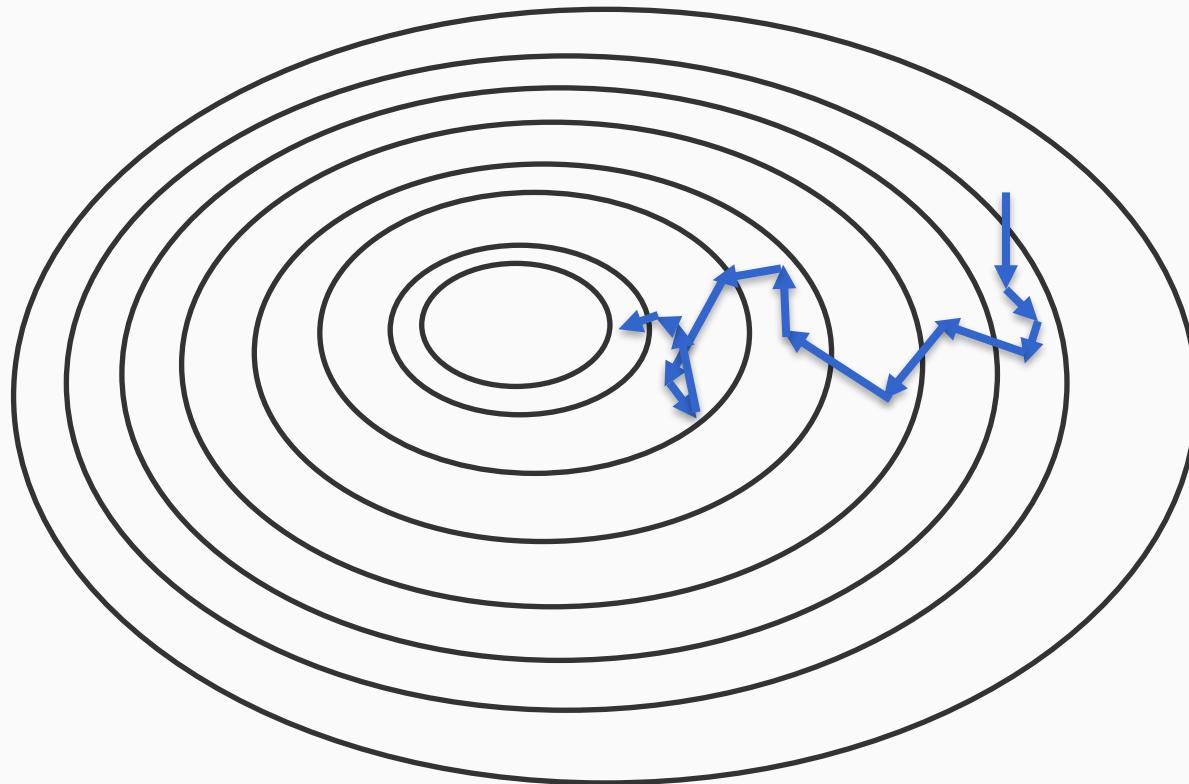
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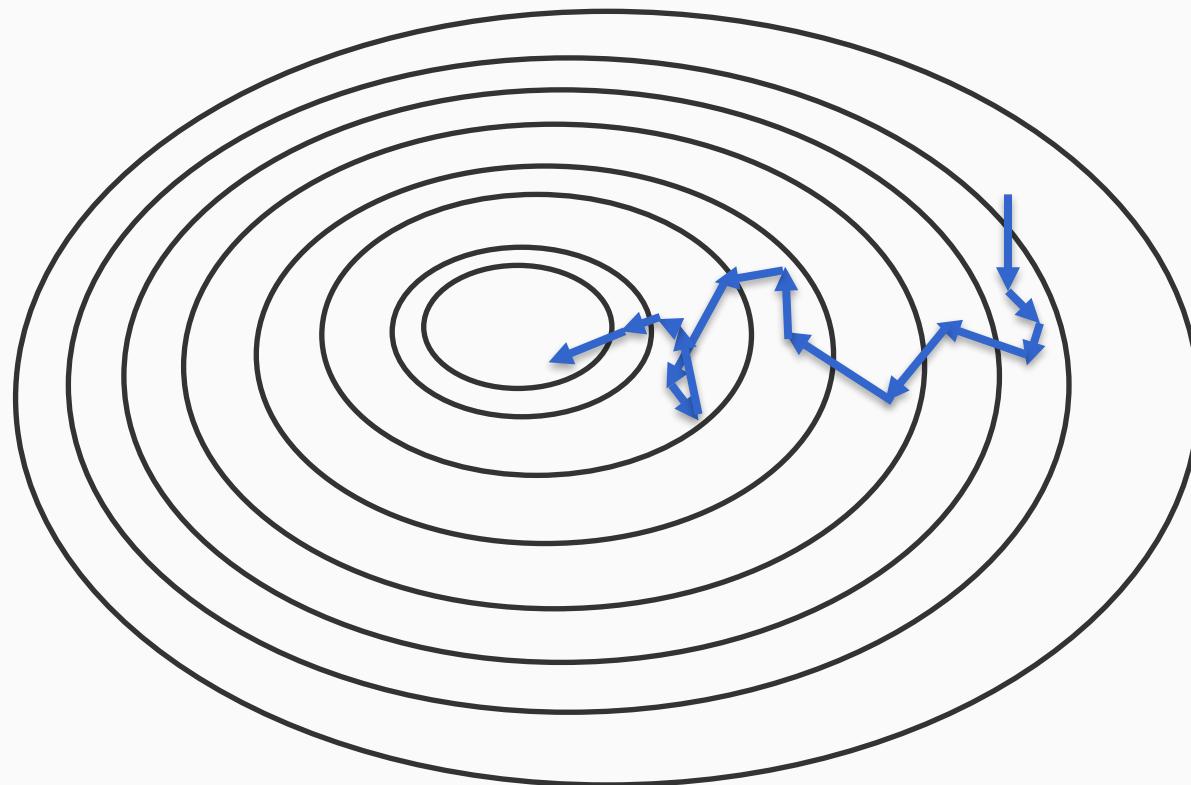
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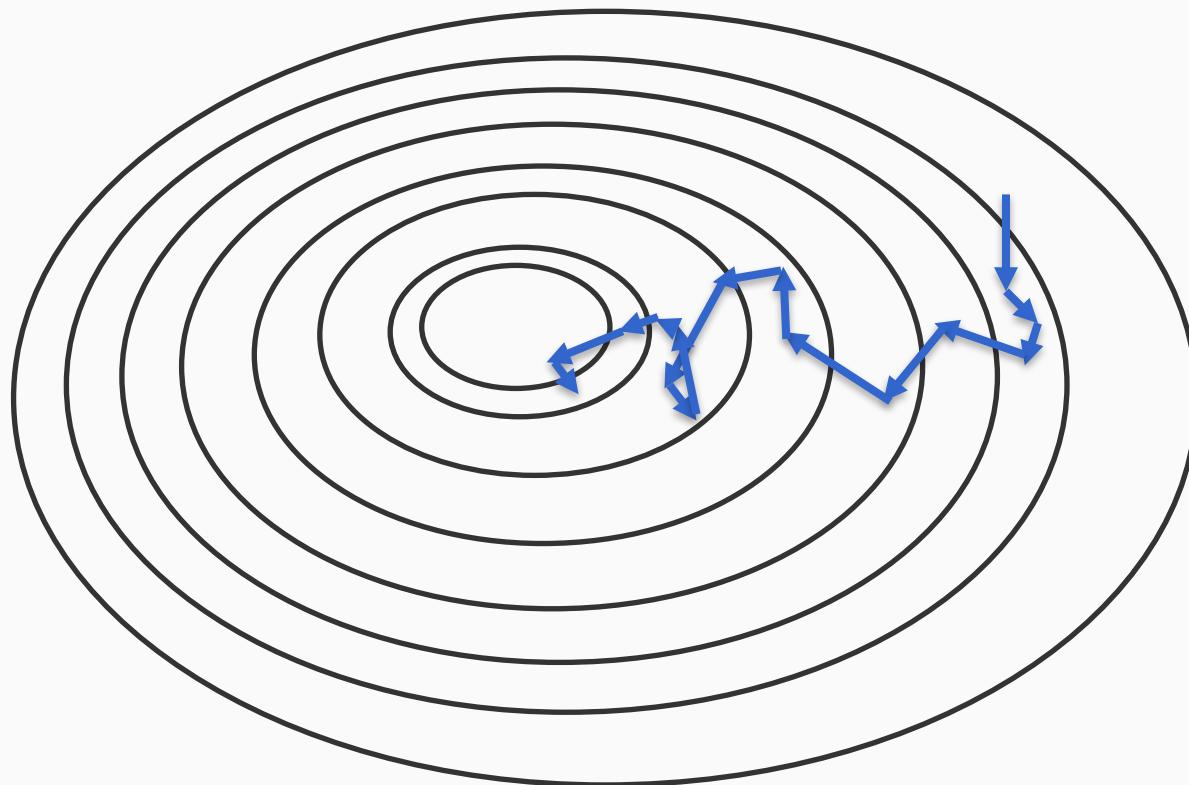
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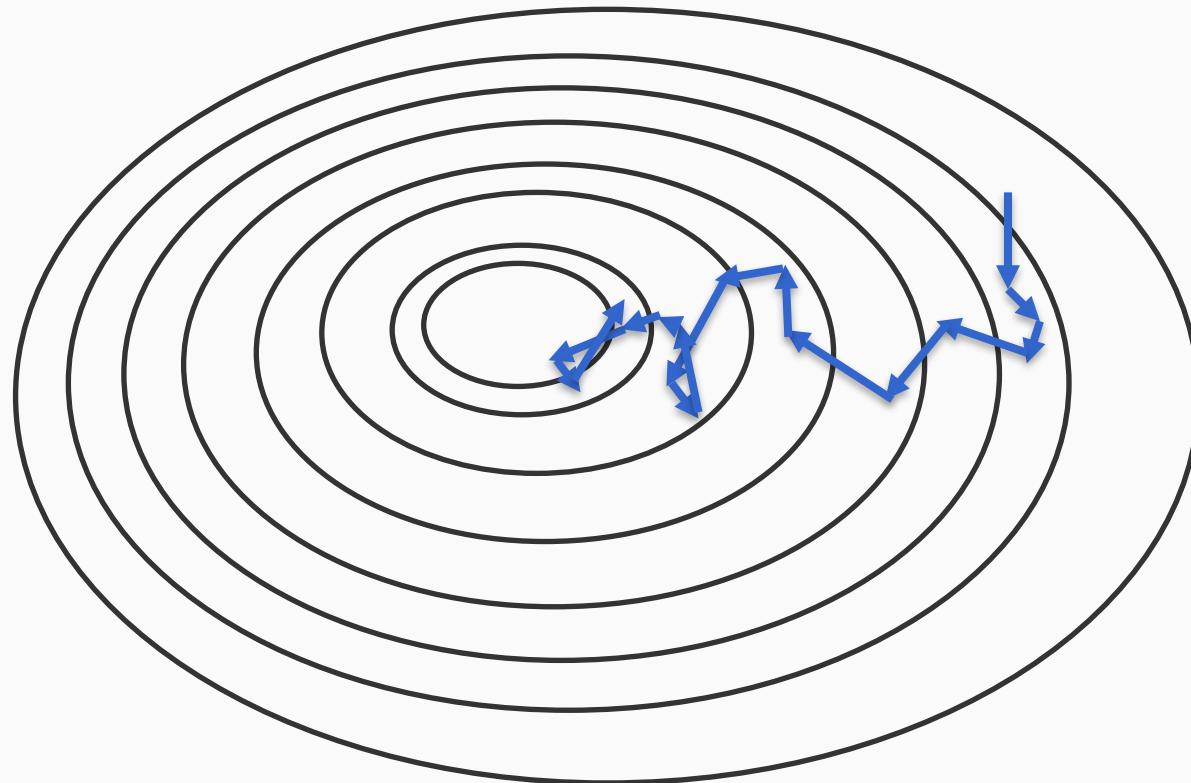
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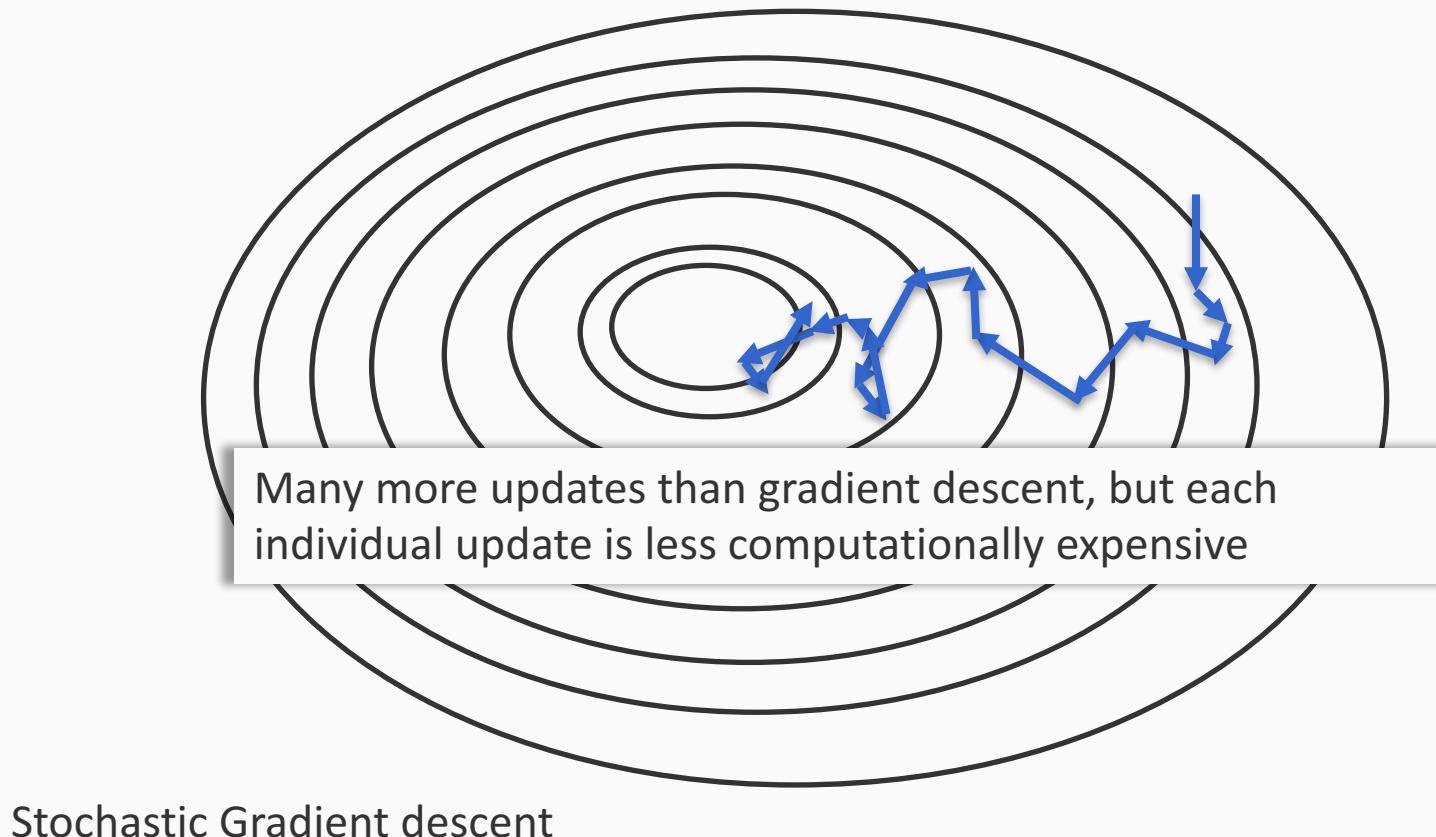
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$$J(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

Stochastic gradient descent for SVM

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \mathbb{R}^n$, $y \in \{-1, 1\}$

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 2. Treat (\mathbf{x}_i, y_i) as a full dataset and take the *derivative of the SVM objective* at the current \mathbf{w}^{t-1} to be $\nabla J^t(\mathbf{w}^{t-1})$
 3. Update: $\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \gamma_t \nabla J^t(\mathbf{w}^{t-1})$
3. Return final \mathbf{w}

What is the derivative of the hinge loss with respect to w?
(The hinge loss is **not** a differentiable function!)

Hinge loss is **not** differentiable!

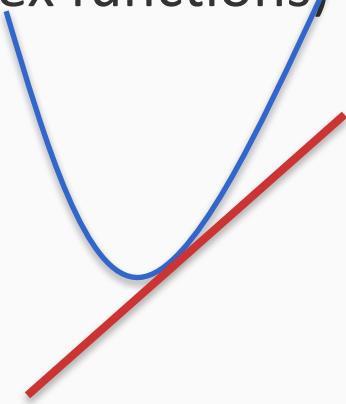
What is the derivative of the hinge loss with respect to \mathbf{w} ?

$$J^t(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

Detour: Sub-gradients

Generalization of gradients to non-differentiable functions

(Remember that every tangent is a hyperplane that lies below the function for convex functions)



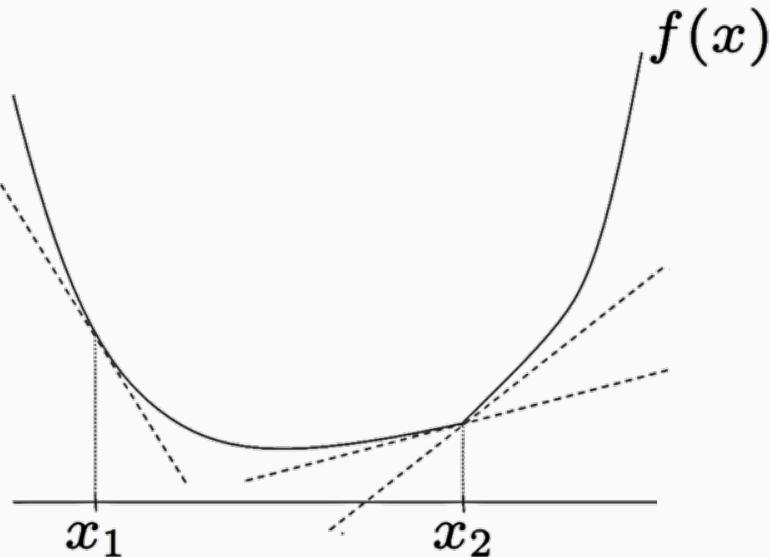
Informally, a sub-tangent at a point is any hyperplane that lies below the function at the point.

A sub-gradient is the slope of that line

Sub-gradients

Formally, a vector g is a subgradient to f at point x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



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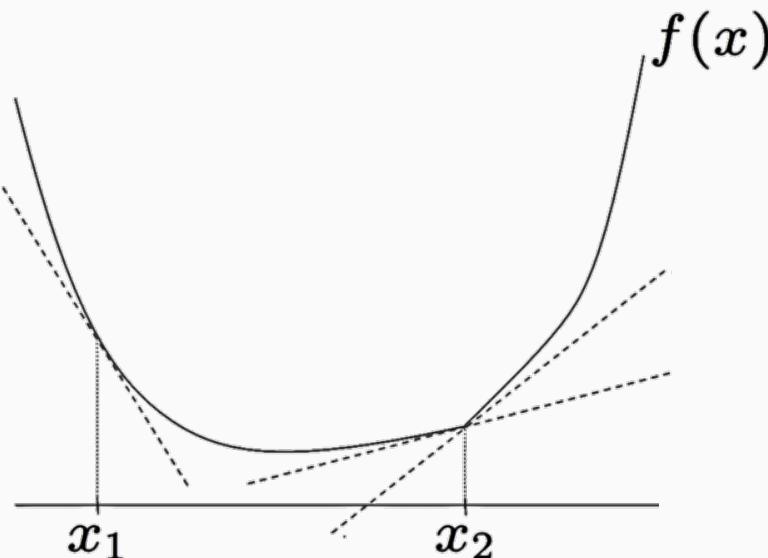
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f is differentiable at x_1

Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

g_1 is a gradient at x_1



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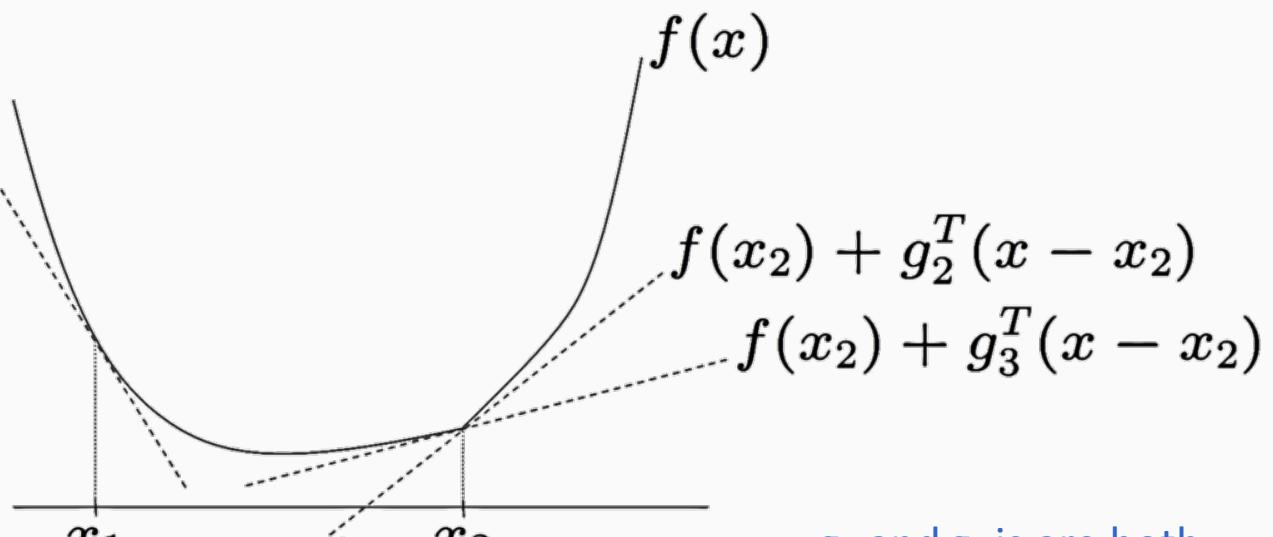
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f is differentiable at x_1

Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

g_1 is a gradient at x_1



g_2 and g_3 are both subgradients at x_2

Sub-gradient of the SVM objective

$$J^t(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

General strategy: First solve the max and compute the gradient for each case

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General strategy: First solve the max and compute the gradient for each case

$$\nabla J^t = \begin{cases} \mathbf{w} & \text{if } \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i) = 0 \\ \mathbf{w} - Cy_i \mathbf{x}_i & \text{otherwise} \end{cases}$$

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Stochastic sub-gradient descent for SVM

$$\nabla J^t = \begin{cases} \mathbf{w} & \text{if } \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i) = 0 \\ \mathbf{w} - Cy_i \mathbf{x}_i & \text{otherwise} \end{cases}$$

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1, 1\}$

1. Initialize $\mathbf{w} = 0 \in \Re^n$

3. Return \mathbf{w}

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Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $\mathbf{w} = 0 \in \mathbb{R}^n$
2. For epoch = 1 ... T:
 1. For each training example $(\mathbf{x}_i, y_i) \in S$:

$$\text{Update } \mathbf{w} \leftarrow \mathbf{w} - \gamma_t \nabla J^t$$

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 1. For each training example $(\mathbf{x}_i, y_i) \in S$:
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γ_t : learning rate, many tweaks possible

Important to shuffle examples at the start of each epoch

3. Return \mathbf{w}

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Convergence and learning rates

With enough iterations, it will converge in expectation

Provided the step sizes are “*square summable, but not summable*”

- Step sizes γ_t are positive
 - Sum of squares of step sizes over $t = 1$ to ∞ is not infinite
 - Sum of step sizes over $t = 1$ to ∞ is infinity
-
- Some examples: $\gamma_t = \frac{\gamma_0}{1 + \frac{\gamma_0 t}{C}}$ or $\gamma_t = \frac{\gamma_0}{1+t}$

Convergence and learning rates

- Number of iterations to get to accuracy within ϵ
- For strongly convex functions, N examples, d dimensional:
 - Gradient descent: $O(Nd \ln(1/\epsilon))$
 - Stochastic gradient descent: $O(d/\epsilon)$
- More subtleties involved, but SGD is generally preferable when the data size is huge

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- Recently, many variants that are based on this general strategy, targeting multilayer neural networks
 - Examples: Adagrad, momentum, Nesterov's accelerated gradient, Adam, RMSProp, etc...

Stochastic sub-gradient descent for SVM

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Outline: Training SVM by optimization

- ✓ Review of convex functions and gradient descent
- ✓ Stochastic gradient descent
- ✓ Gradient descent vs stochastic gradient descent
- ✓ Sub-derivatives of the hinge loss
- ✓ Stochastic sub-gradient descent for SVM

6. Comparison to perceptron

Stochastic sub-gradient descent for SVM

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3. Return \mathbf{w}

Compare with the Perceptron update:
If $y \mathbf{w}^\top \mathbf{x} \leq 0$, update $\mathbf{w} \leftarrow \mathbf{w} + r y \mathbf{x}$

Perceptron vs. SVM

- Perceptron: Stochastic sub-gradient descent for a different loss
 - No regularization though

$$L_{\text{Perceptron}}(y, \mathbf{x}, \mathbf{w}) = \max(0, -y\mathbf{w}^T \mathbf{x})$$

- SVM optimizes the hinge loss
 - With regularization

$$L_{\text{Hinge}}(y, \mathbf{x}, \mathbf{w}) = \max(0, 1 - y\mathbf{w}^T \mathbf{x})$$

SVM summary from optimization perspective

- Minimize regularized hinge loss
- Solve using stochastic gradient descent
 - Very fast, run time does not depend on number of examples
 - Compare with Perceptron algorithm: Perceptron does not maximize margin width
 - Perceptron variants can force a margin
 - Convergence criterion is an issue; can be too aggressive in the beginning and get to a reasonably good solution fast; but convergence is slow for very accurate weight vector
- Other successful optimization algorithms exist
 - Eg: Dual coordinate descent, implemented in `liblinear`

Questions?